Representations of the Steenrod group

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Introduction

The ring of cohomology operations on the mod p ordinary cohomology theory is called the Steenrod algebra. J. Lannes developed an elegant theory of unstable modules over the Steenrod algebra which has an application to Sullivan's conjecture ([17]). Since the Steenrod algebra is not commutative, it is difficult to apply knowledge of commutative algebras.

Under certain finiteness conditions, a left module over the Steenrod algebra has a structure of a right comodule over the dual Steenrod algebra ([16]). Hence, roughly speaking, the category of left module over the Steenrod algebra is equivalent to the category of representations of an affine group scheme represented by the dual Steenrod algebra. The aim of this note is a trial of reconstruction of Lanne's theory from the viewpoint of representation theory.

We first collect necessary facts on the category of topological graded rings and the category of topological graded modules in the first section. There, we study topologies on graded algebras and graded modules and give two kinds of standard topologies on graded modules, which are called "cofinite topology" and "skeletal topology". We also define the notion of suspension and completion of topological graded modules. After we review the notion of tensor product and completed tensor product of topological graded modules in section 2, we introduce the space of homomorphisms which is a substitute for the right adjoint of the functor given by tensor product and study on completions of a space of homomorphisms in section 3. In section 4, we investigate some relationships between the tensor products and the spaces of linear maps. In section 5, we consider Hopf algebras and their dual in the category of graded topological modules and show that a certain full subcategory of modules over a Hopf algebra A^* is isomorphic to a category of comodules over the dual of A^* under some conditions on A^*. We study actions of group objects in a cartesian closed category in section 6 and construct the right induction functor which is a right adjoint functor of the restriction functor. In section 7, after reviewing the notion of fibered category, we introduce the notions of fibered category with products and fibered category with exponents to develop a representation theory of group objects in the subsequent sections. The former notion is a generalization of the notion of category with products and the latter notion is a dual of the former notion. By combining these two notions, we have a notion of "cartesian closed fibered category" which generalizes the notion of cartesian closed category in terms of fibered category. In section 8, we formulate the notion of representation of group objects in terms of fibered category and develop a fundamental theory of representation of group objects including constructions of left and right induced representations under the framework of the previous section. In section 9, we develop a general theory on categories enriched by topological spaces, namely, categories with each set of morphisms between two objects has a topology. We introduce a notion of topological affine scheme and give some fundamental properties of topological affine schemes in section 10. We also introduce a notion of topological affine group scheme in section 11 and give some examples. In section 12, we investigate fibered category of modules and give important examples of cartesian closed fibered categories. In section 13, we specialize the results of section 8 and study basic properties of the representations of affine topological group schemes. In section 14, we give several axioms for a topological graded filtered algebra A^* over a field so that the notion of unstable A^*-modules is defined and generalize the notion of unstable modules. In section 16, we consider the dual notion "unstable comodules" of unstable modules and study the relations between the category of unstable modules and unstable comodules.

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Contents

1 Topological graded rings and modules

1.1 Linear topology

Definition 1.1.1 *(1) We say that a graded ring A^* is commutative if $xy = (-1)^{mn}yx$ for any $m, n \in \boldsymbol{Z}$ and $x \in A^m$, $y \in A^n$.*

(2) Let K^* be a graded ring and M^* a graded K^*-module. A submodule of M^* is said to be homogeneous if it is generated by elements of $\bigcup_{n\in\boldsymbol{Z}} M^n$. Similarly, an ideal of K^* is said to be homogeneous if it is generated by elements of $\bigcup_{n\in\boldsymbol{Z}} K^n$.

From now on, "an ideal" of a graded ring always means a homogeneous ideal and "a submodule" of a graded module means a homogeneous submodule unless otherwise stated.

Definition 1.1.2 *(1) For a topological graded ring A^*, we denote by $\mathcal{I}_{A^*}$ the set of open homogeneous two-sided ideals of A^*. If $\mathcal{I}_{A^*}$ is a fundamental system of neighborhoods of 0, A^* is said to be linearly topologized.*

(2) Let A^* and K^* be linearly topologized graded rings and $\eta : K^* \to A^*$ a continuous homomorphism preserving degrees. If $\eta(x)y = (-1)^{mn}y\eta(x)$ holds for any $m, n \in \boldsymbol{Z}$ and $x \in K^m$, $y \in A^n$, (A^*, η) (or A^* for short) is called a topological K^*-algebra.

(3) Let (A^*, η) and (B^*, ι) be topological K^*-algebras. If a continuous homomorphism $f : A^* \to B^*$ preserving degrees satisfies $f\eta = \iota$, we call f a homomorphism of topological K^*-algebras.

(4) For a commutative linearly topologized graded ring K^*, we denote by $\mathcal{TopAlg}_{K^*}$ the category of commutative topological K^*-algebras and homomorphisms of topological K^*-algebras.

Definition 1.1.3 *Let K^* be a commutative graded ring. If every non-zero homogeneous element of K^* is invertible, we call K^* a graded field or a field. If K^* is a graded field, we call a K^*-module a vector space over K^*.*

We note that topology of a linearly topologized graded field is discrete or trivial. A field means a graded field with discrete topology below unless otherwise stated.

Proposition 1.1.4 *Let K^* be a field.*

(1) K^0 is an ungraded field and, if $K^n \neq \{0\}$ for some $n \neq 0$, there exists a homogeneous elememt v of K^* such that $K^* = K^0[v, v^{-1}]$.

(2) Every vector space over K^* has basis.

Proof. (1) The first assertion is obvious. If $K^n \neq \{0\}$ for some $n \neq 0$, take $x \in K^n - \{0\}$. Then, $x^{-1} \in K^{-n}$ is not zero. Hence the set of positive integers n such that $K^n - \{0\}$ is not empty and let d the minimum integer of this set. Take non-zero $v \in K^d$. If $K^n \neq \{0\}$, dividing n by d, we have $n = dm + r$ for $m, r \in \boldsymbol{Z}$ such that $0 \leq r \leq d-1$. For any non-zero $x \in K^n$, since $v^{-m}x \in K^r$ is not zero, we have $K^r \neq \{0\}$ and this implies that $r = 0$ and $x \in K^0[v, v^{-1}]$.

(2) Let M^* be a vector space over K^*. If $K^* = K^0$, choose basis $\{v_{ni} | \, i \in I_n\}$ of M^n over K^0 for each $n \in \boldsymbol{Z}$. Then, $\bigcup_{n\in\boldsymbol{Z}} \{v_{ni} | \, i \in I_n\}$ is a basis of M^* over K^*. If $K^* = K^0[v, v^{-1}]$ for $v \in K^d$, choose basis $\{v_{ni} | \, i \in I_n\}$ of M^n over K^0 for each $n = 0, 1, \ldots, d-1$. Then, $\bigcup_{n=0}^{d-1} \{v_{ni} | \, i \in I_n\}$ is a basis of M^* over K^*. □

Definition 1.1.5 *(1) Let L^*, M^* and N^* be graded abelian groups. A map $\beta : L^* \times M^* \to N^*$ is said to be biadditive if β satisfies the following conditions (i) and (ii).*

(i) $\beta(L^l \times M^m) \subset N^{l+m}$

(ii) $\beta(x + y, z) = \beta(x, z) + \beta(y, z)$, $\beta(x, z + w) = \beta(x, z) + \beta(x, w)$ for any $x, y \in L^*$ and $z, w \in M^*$.

(2) Suppose that K^* is a commutative graded ring and L^*, M^*, N^* are graded left K^*-modules. If $\beta : L^* \times M^* \to N^*$ is biadditive and satisfies the following condition (iii), we say that β is bilinear.

(iii) $\beta(rx, z) = r\beta(x, z)$, $\beta(x, rz) = (-1)^{ln}r\beta(x, z)$ if $r \in K^n$, $x \in L^l$ and $z \in M^*$ for $l, n \in \boldsymbol{Z}$.

Definition 1.1.6 *(1) For a topological graded K^*-module M^*, let us denote by $\mathcal{V}_{M^*}$ the set of homogeneous open submodules of M^*. If $\mathcal{V}_{M^*}$ is a fundamental system of neighborhoods of 0, we say that M^* is linearly topologized.*

(2) Let K^* be a linearly topologized graded ring and M^* a topological graded left (resp.right) K^*-module. If $\{\mathfrak{a}M^* | \mathfrak{a} \in \mathcal{I}_{K^*}\}$ (resp. $\{M^*\mathfrak{a} | \mathfrak{a} \in \mathcal{I}_{K^*}\}$) is a fundamental system of neighborhoods of M^*, we say that the topology of M^* is induced by K^*.

Remark 1.1.7 *(1) Suppose that there is a morphism $\eta : K^* \to A^*$ of graded topological rings. If we regard A^* as a left (right) K^*-module, then the topology of A^* is coarser than the topology induced by K^*.*

(2) Let M^* be a topological graded left (resp. right) A^*-module. We define a left (resp. right) K^*-module M^*_η as follows. $M^*_\eta = M^*$ as graded topological abelian group and the left (resp. right) K^*-module structure on M^*_η is given by $ax = \eta(a)x$ (resp. $xa = x\eta(a)$) for $a \in K^*$, $x \in M^*$. If the topology of M^* is coarser than the topology induced by A^*, then the topology on M^*_η is coarser than the topology induced by K^*.

Definition 1.1.8 *Let L^*, M^* and N^* be linearly topologized abelian groups. We say that a biadditive map $\beta : L^* \times M^* \to N^*$ is strongly continuous if, for any open subgroup U^* of N^*, there exist an open subgroup V^* of L^* and an open subgroup W^* of M^* such that $\beta(V^* \times M^*)$ and $\beta(L^* \times W^*)$ are contained in U^*.*

Remark 1.1.9 *(1) If a biadditive map $\beta : L^* \times M^* \to N^*$ is strongly continuous, it is continuous. In fact, for $(x, y) \in L^* \times M^*$ and an open subgroup U^* of N^*, there exist an open subgroup V^* of L^* and an open subgroup W^* of M^* such that $\beta(V^* \times M^*)$ and $\beta(L^* \times W^*)$ are contained in U^*. Then, β maps $(\{x\} + V^*) \times (\{y\} + W^*)$ into $\{\beta(x, y)\} + U^*$.*

(2) Let $f : A^* \to C^*$ and $g : B^* \to C^*$ be morphisms of $\mathcal{TopAlg}_{K^*}$. Then, a map $\beta : A^* \times B^* \to C^*$ defined by $\beta(x, y) = f(x)g(y)$ is strongly continuous. In particular, the multiplication $\mu : A^* \times A^* \to A^*$ of A^* is strongly continuous.

Proposition 1.1.10 *Let R^* be an object of $\mathcal{TopAlg}_{K^*}$ and N^* a linearly topologized graded K^*-module. Suppose that a right R^*-module with structure $\beta : N^* \times R^* \to N^*$ on N^* is given. We denote by $\mathcal{V}^{R^*}_{N^*}$ the set of open R^*-submodules of N^*. Then, β is strongly continuous if and only if the topology of N^* is coarser than the topology induced by R^* and $\mathcal{V}^{R^*}_{N^*}$ is a fundamental system of neighborhoods of 0 of N^*. In particular, if $R^* = K^*$, β is strongly continuous if and only if the topology of N^* is coarser than the topology induced by K^**

Proof. Suppose that $\beta : N^* \times R^* \to N^*$ is strongly continuous. For $V^* \in \mathcal{V}_{N^*}$, there exist $U^* \in \mathcal{V}_{N^*}$ and $\mathfrak{a} \in \mathcal{I}_{R^*}$ such that $\beta(U^* \times R^*) \cup \beta(N^* \times \mathfrak{a}) \subset V^*$. Let $\bar{U}^*$ be the R^*-submodule of N^* generated by U^*. Since $\bar{U}^*$ is generated by $\beta(U^* \times R^*)$ over K^* and V^* is a K^*-submodule of N^*, we have $\bar{U}^* \subset V^*$. Moreover, since $\beta(N^* \times \mathfrak{a}) \subset V^*$, the topology of N^* is coarser than the topology induced by R^*.

Assume that the topology of N^* is coarser than the topology induced by R^* and $\mathcal{V}^{R^*}_{N^*}$ is a fundamental system of neighborhoods of 0 of N^*. $V^* \in \mathcal{V}_{N^*}$, there exist $U^* \in \mathcal{V}^{R^*}_{N^*}$ and $\mathfrak{a} \in \mathcal{I}_{R^*}$ such that $U^* \subset V^*$ and $N^*\mathfrak{a} \subset V^*$, hence $\beta(U^* \times R^*) \subset U^* \subset V^*$ and $\beta(N^* \times \mathfrak{a}) \subset V^*$. It follows that β is strongly continuous. □

Proposition 1.1.11 *Let K^* be a linearly topologized graded ring and M^*, N^* linearly topologized graded left K^*-modules. If the topology of M^* is finer than the topology induced by K^* and the topology of N^* is coarser than the topology induced by K^*, then a homomorphism $f : M^* \to N^*$ of K^*-modules is continuous.*

Proof. For $W^* \in \mathcal{V}_{N^*}$, there exists $\mathfrak{a} \in \mathcal{I}_{K^*}$ satisfying $\mathfrak{a}N^* \subset W^*$ by the assumption. Hence $f(\mathfrak{a}M^*) = \mathfrak{a}f(M^*) \subset \mathfrak{a}N^* \subset W^*$ and f is continuous. □

We denote by $\mathcal{TopMod}_{K^*}$ the category of linearly topologized graded left K^*-modules and continuous homomorphisms preserving degrees. A full subcategory of $\mathcal{TopMod}_{K^*}$ consisting of linearly topologized graded left K^*-modules whose topologies are coarser than the topology induced by K^* is denoted by $\mathcal{TopMod}^i_{K^*}$. We denote by $\mathrm{Hom}^c_{K^*}(M^*, N^*)$ the set of all morphisms of $\mathcal{TopMod}_{K^*}$ from M^* to N^* instead of $\mathcal{TopMod}_{K^*}(M^*, N^*)$. Note that $\mathcal{TopMod}^i_{K^*} = \mathcal{TopMod}_{K^*}$ if K^* is discrete, especially, K^* is a field.

Proposition 1.1.12 *Submodules and quotient modules of an object of $\mathcal{TopMod}_{K^*}$ (resp. $\mathcal{TopMod}^i_{K^*}$) are objects of $\mathcal{TopMod}_{K^*}$ (resp. $\mathcal{TopMod}^i_{K^*}$).*

Proof. It is clear that submodules and quotient modules of an object of $\mathcal{TopMod}_{K^*}$ are objects of $\mathcal{TopMod}_{K^*}$.

Let M^* be an object of $\mathcal{TopMod}^i_{K^*}$ and N^* a submodule of M^*. For $V^* \in \mathcal{V}_{N^*}$, there exists $W^* \in \mathcal{V}_{M^*}$ such that $V^* = N^* \cap W^*$. By the assumption, there exists $\mathfrak{a} \in \mathcal{I}_{K^*}$ such that $\mathfrak{a}M^* \subset W^*$. Hence we have $\mathfrak{a}N^* \subset N^* \cap W^* = V^*$ and this implies that the topology of N^* is coarser than the topology induced by K^*.

Let us denote by $p : M^* \to M^*/N^*$ the quotient map. For $U^* \in \mathcal{V}_{M^*/N^*}$, there exists $\mathfrak{a} \in \mathcal{I}_{K^*}$ such that $\mathfrak{a}M^* \subset p^{-1}(U^*)$. Then, we have $(\mathfrak{a}M^* + N^*)/N^* \subset U^*$ and this implies that the topology of M^*/N^* is coarser than the topology induced by K^*. □

Proposition 1.1.13 *If $M^* \in \mathrm{Ob}\mathcal{T}op\mathcal{M}od_{K^*}$ is artinian module, then $\mathcal{V}_{M^*}$ has the minimum element. Hence, if K^* is an artinian topological ring and $M^* \in \mathrm{Ob}\mathcal{T}op\mathcal{M}od_{K^*}$ is Hausdorff, then a finitely generated submodule of M^* is discrete.*

Proof. In general, a directed set D satisfying the ascending chain condition has the maximum element. Otherwise, suppose that we have an ascending chain $x_1 < x_2 < \cdots < x_n$ in D, then x_n is not maximum and $y \not\leq x_n$ for some $y \in D$. Hence $y \leq x_{n+1}$ and $x_n \leq x_{n+1}$ for some $x_{n+1} \in D$. Since $y \not\leq x_n$, $x_n < x_{n+1}$. It follows that there exists an ascending chain $x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots$ in D which is not stationary. This contradicts to the assumption. Since $\mathcal{V}_{M^*}^{op}$ is a directed set satisfying the ascending chain condition, $\mathcal{V}_{M^*}^{op}$ has the maximum elemement. □

Proposition 1.1.14 *Suppose that K^* is an artinian topological ring. If $M^* \in \mathrm{Ob}\mathcal{T}op\mathcal{M}od_{K^*}$ is Hausdorff and N^* is a finitely generated submodule of M^*, then there exists an open submodule U^* of M^* such that the composition of the inclusion map $N^* \hookrightarrow M^*$ and the quotient map $M^* \to M^*/U^*$ is injective.*

Proof. Since N^* is finitely generated, N^* is discrete by (1.1.13). There exist $U^* \in \mathcal{V}_{M^*}$ satisfying $U^* \cap N^* = \{0\}$. Then, the composition of the inclusion map $N^* \hookrightarrow M^*$ and the quotient map $M^* \to M^*/U^*$ is injective. □

Proposition 1.1.15 *If $f : M^* \to N^*$ is a quotient map in $\mathcal{T}op\mathcal{M}od_{K^*}$, then f is an open map.*

Proof. Suppose that O is an open subset of M^*. Then, $f^{-1}(f(O)) = O + \mathrm{Ker}\, f = \bigcup\limits_{x \in \mathrm{Ker}\, f} (O + \{x\})$ is open in M^*. Hence f is an open map. □

Lemma 1.1.16 *For a submodule N^* of M^*, the following conditions are equivalent.*

(i) N^* is dense. (ii) 0 is a generic point of M^*/N^*. (iii) The topology of M^*/N^* is trivial.

Proof. Let $p : M^* \to M^*/N^*$ be the quotient map. Suppose that N^* is dense. For $y \in M^*/N^*$ and an open submodule V^* of M^*/N^*, there exists $x \in p^{-1}(y + V^*) \cap N^*$ by the assumption. Hence $0 = p(x) \in y + V^*$ and this shows that $\{0\}$ is a dense subset of M^*/N^*. Suppose that 0 is a generic point of M^*/N^*. For $y \in M^*/N^*$ and an open submodule V^* of M^*/N^*, since $-y + V^*$ is an open set of M^*/N^*, it contains 0. There exists $v \in V^*$ which satisfies $-y + v = 0$, that is, $y = v \in V^*$. Therefore $V^* = M^*/N^*$. Assume that the topology of M^*/N^* is trivial. For $x \in M^*$ and an open submodule U^* of M^*, since $p(x + U^*)$ is a nonempty open subset of M^*/N^* by (refquotients), we have $p(x + U^*) = M^*/N^*$ by the assumption. Hence there exist $u \in U^*$ such that $p(x + u) = 0$, equivalently, $x + u \in N^*$ which shows that $x + U^* \cap N^*$ is not empty. It follows that N^* is dense. □

Clearly, each morphism of $\mathcal{T}op\mathcal{M}od_{K^*}$ (resp. $\mathcal{T}op\mathcal{M}od^i_{K^*}$) has a kernel and it follows from the above result that each morphism of $\mathcal{T}op\mathcal{M}od_{K^*}$ (resp. $\mathcal{T}op\mathcal{M}od^i_{K^*}$) has a cokernel. Howeverever, since the coimage of a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$ (resp. $\mathcal{T}op\mathcal{M}od^i_{K^*}$) is not isomorphic to the image in general, $\mathcal{T}op\mathcal{M}od_{K^*}$ (resp. $\mathcal{T}op\mathcal{M}od^i_{K^*}$) is not an abelian category.

Proposition 1.1.17 *(1) $\mathcal{T}op\mathcal{M}od_{K^*}$ is complete and cocomplete.*
(2) $\mathcal{T}op\mathcal{M}od^i_{K^*}$ is complete and finitely cocomplete.

Proof. (1) Each morphism $f : M^* \to N^*$ has a kernel and a cokernel. In fact, $(\mathrm{Ker}\, f)^n = (\mathrm{Ker}\, f^n : M^n \to N^n)$ and $\mathrm{Ker}\, f$ has the topology induced by M^*. On the other hand, $(\mathrm{Coker}\, f)^n = (\mathrm{Coker}\, f^n : M^n \to N^n)$ and $\mathrm{Coker}\, f$ has the quotient topology.

Let $(M_i^*)_{i \in I}$ be a family of objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. Define $\prod\limits_{i \in I} M_i^*$ and $\coprod\limits_{i \in I} M_i^*$ as follows. Put

$$\left(\prod_{i \in I} M_i^*\right)^n = \prod_{i \in I} M_i^n = \left\{ x : I \to \bigcup_{i \in I} M_i^n \,\middle|\, x(i) \in M_i^n \text{ for any } i \in I. \right\}$$

and define $p_i : \prod\limits_{i \in I} M_i^* \to M_i^*$ so that $p_i^n : \prod\limits_{i \in I} M_i^n \to M_i^n$ is given by $p_i^n(x) = x(i)$. We give $\prod\limits_{i \in I} M_i^*$ the coarsest topology such that every p_i is continuous, that is, neighborhoods of 0 of $\prod\limits_{i \in I} M_i^*$ generated by

$$\left\{ p_i^{-1}(U^*) \,\middle|\, i \in I,\ U^* \in \mathcal{V}_{M_i^*} \right\}.$$

Then, it is easy to verify that $\prod_{i\in I} M_i^*$ is the product of $(M_i^*)_{i\in I}$ in $\mathcal{T}op\mathcal{M}od_{K^*}$. Put

$$\left(\coprod_{i\in I} M_i^*\right)^n = \coprod_{i\in I} M_i^n = \left\{ x : I \to \bigcup_{i\in I} M_i^n \,\middle|\, x(i) \in M_i^n \text{ for any } i \in I,\ \{i \in I \,|\, x(i) \neq 0\} \text{ is a finite set.} \right\}$$

and define $\iota_i : M_i^* \to \coprod_{i\in I} M_i^*$ so that $\iota_i^n : M_i^n \to \coprod_{i\in I} M_i^n$ is given by $(\iota_i^n(a))(j) = \begin{cases} a & j = i \\ 0 & j \neq i \end{cases}$. Give $\coprod_{i\in I} M_i^*$ the finest topology such that every ι_i is continuous, that is, a fundamental system of neighborhoods of 0 is given by $\left\{U^* \,\middle|\, U^* \text{ is a submodule of } \coprod_{i\in I} M_i^* \text{ such that } \iota_i^{-1}(U^*) \in \mathcal{V}_{M_i^*} \text{ for all } i \in I\right\}$. Then, it is easy to verify that $\coprod_{i\in I} M_i^*$ is the sum of $(M_i^*)_{i\in I}$ in $\mathcal{T}op\mathcal{M}od_{K^*}$.

(2) It follows from (1.1.12) that kernels and cokernels of morphisms of $\mathcal{T}op\mathcal{M}od^i_{K^*}$ exist in $\mathcal{T}op\mathcal{M}od^i_{K^*}$.

Let $(M_i^*)_{i\in I}$ be a family of objects of $\mathcal{T}op\mathcal{M}od^i_{K^*}$. For $j \in I$ and $U^* \in \mathcal{V}_{M_j^*}$, there exists $\mathfrak{a} \in \mathcal{I}_{K^*}$ satisfying $\mathfrak{a}M_j^* \subset U^*$. Hence $\mathfrak{a}\prod_{i\in I} M_i^* \subset \mathfrak{a}p_j^{-1}(U^*)$ and it follows that the topology of $\prod_{i\in I} M_i^*$ is coarser than the topology induced by K^*.

Suppose that I is a finite set, $I = \{1, 2, \dots, n\}$ for example. For a submodule U^* of $\coprod_{i\in I} M_i^*$ such that $\iota_i^{-1}(U^*) \in \mathcal{V}_{M_i^*}$ for all $i \in I$, there exist $\mathfrak{a}_j \in \mathcal{I}_{K^*}$ for each $j \in I$ satisfying $\mathfrak{a}_j M_j^* \subset \iota_j^{-1}(U^*)$. Put $\mathfrak{a} = \mathfrak{a}_1\mathfrak{a}_2\cdots\mathfrak{a}_n$, then we have $\mathfrak{a}\coprod_{i\in I} M_i^* \subset U^*$. In fact, $\mathfrak{a}\coprod_{i\in I} M_i^*$ is generated by $\iota_j(\mathfrak{a}M_j^*)$ which is contained in U^*. It follows that the topology of $\coprod_{i\in I} M_i^*$ is coarser than the topology induced by K^*. □

Remark 1.1.18 *For a family $(M_i^*)_{i\in I}$ of objects of $\mathcal{T}op\mathcal{M}od_{K^*}$, a basis of open neighborhood of $\coprod_{i\in I} M_i^*$ is given by $\left\{\coprod_{i\in I} U_i^* \,\middle|\, U_i^* \in \mathcal{V}_{M_i^*}\right\}$. In fact, if U^* is a submodule of $\coprod_{i\in I} M_i^*$ such that $\iota_i^{-1}(U^*) \in \mathcal{V}_{M_i^*}$ for all $i \in I$, $\coprod_{i\in I} \iota_i^{-1}(U^*)$ is contained in U^*. In particular, if every M_i^* is discrete, so is $\coprod_{i\in I} M_i^*$.*

Proposition 1.1.19 *(1) Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$. M^* is Hausdorff if and only if M^n is Hausdorff for every $n \in \boldsymbol{Z}$.*

(2) Let $(M_i^*)_{i\in I}$ be a family of objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. $\coprod_{i\in I} M_i^*$ (resp. $\prod_{i\in I} M_i^*$) is Hausdorff if and only if M_i^* is Hausdorff for every $i \in I$.

Proof. (1) Suppose that M^n is Hausdorff for every $n \in \boldsymbol{Z}$. Let $x = \sum_{i\in \boldsymbol{Z}} x_i$ $(x_i \in M^i)$ be a non-zero element of M^*. Then $x_n \neq 0$ for some $n \in \boldsymbol{Z}$. Since M^n is Hausdorff, there exists $U^* \in \mathcal{V}_{M^*}$ such that $x_n \notin U^* \cap M^n = U^n$. Since U^* is homogeneous, U^* does not contain x. Hence M^* is Hausdorff. The converse is obvious.

(2) Suppose that M_i^* is Hausdorff for every $i \in I$. Let $x = \sum_{i\in I} x_i$ (resp. $x = (x_i)_{i\in I}$) $(x_i \in M_i^*)$ be a non-zero element of $\coprod_{i\in I} M_i^*$ (resp. $\prod_{i\in I} M_i^*$). Then $x_j \neq 0$ for some $j \in I$. Since M_j^* is Hausdorff, there exists $U_j^* \in \mathcal{V}_{M_j^*}$ such that $x_j \notin U_j^*$. Put $U_i^* = M_i^*$ if $i \in I - \{j\}$. Then, $\coprod_{i\in I} U_i^*$ (resp. $\prod_{i\in I} U_i^*$) is an open submodule of $\coprod_{i\in I} M_i^*$ (resp. $\prod_{i\in I} M_i^*$) which does not contain x. Hence $\coprod_{i\in I} M_i^*$ (resp. $\prod_{i\in I} M_i^*$) is Hausdorff. The converse is obvious. □

Proposition 1.1.20 *Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ and $i_1 : M^* \to M^* \oplus N^*$, $i_2 : N^* \to M^* \oplus N^*$, $p_1 : M^* \times N^* \to M^*$, $p_2 : M^* \times N^* \to N^*$ the inclusions and the projections. Then, the unique morphism $\varphi : M^* \oplus N^* \to M^* \times N^*$ satisfying $p_1\varphi i_1 = id_{M^*}$, $p_2\varphi i_2 = id_{N^*}$, $p_2\varphi i_1 = 0$ and $p_1\varphi i_2 = 0$ is an isomorphism.*

Proof. Clearly, φ is a continuous bijection. We identify $M^* \oplus N^*$ with $M^* \times N^*$ as a left K^*-module. Let U^* be an open submodule of $M^* \oplus N^*$. Since $U^* = p_1^{-1}(i_1^{-1}(U^*)) \cap p_2^{-1}(i_2^{-1}(U^*))$, U^* is also open in $M^* \times N^*$. It follows that φ is an open map. □

Corollary 1.1.21 *Let $j_1 : M^* \to L^*$, $j_2 : N^* \to L^*$, $q_1 : L^* \to M^*$, $q_2 : L^* \to N^*$ be morphisms in $\mathcal{T}op\mathcal{M}od_{K^*}$ satisfying $q_1j_1 = id_{M^*}$, $q_2j_2 = id_{N^*}$ and $j_1q_1 + j_2q_2 = id_{L^*}$. Then, the unique morphism $\psi : M^* \oplus N^* \to L^*$ satisfying $\psi i_1 = j_1$ and $\psi i_1 = j_1$ is an isomorphism.*

Proof. Composing q_2 (resp. j_1) on the left (resp. right) of $j_1q_1 + j_2q_2 = id_{L^*}$, we have $q_2j_1 + q_2j_1 = q_2j_1q_1j_1 + q_2j_2q_2j_1 = q_2j_1$, we have $q_2j_1 = 0$. Similarly, we have $q_1j_2 = 0$. Let $\lambda : L^* \to M^* \times N^*$ be the unique morphism satisfying $p_1\lambda = q_1$ and $p_2\lambda = q_2$. Then, $p_1\lambda\psi i_1 = q_1\psi i_1 = q_1j_1 = id_{M^*}$, $p_2\lambda\psi i_2 = q_2\psi i_2 = q_2j_2 = id_{N^*}$, $p_2\lambda\psi i_1 = q_2\psi i_1 = q_2j_1 = 0$ and $p_1\lambda\psi i_2 = q_1\psi i_2 = q_1j_2 = 0$. Thus we have $\lambda\psi = \varphi$ by the uniqueness of φ. Hence $\varphi^{-1}\lambda\psi = id_{L^*}$. Since $i_1p_1\varphi + i_2p_2\varphi = id_{M^*\oplus N^*}$, we have $\psi\varphi^{-1}\lambda = \psi(i_1p_1 + i_2p_2)\lambda = \psi i_1p_1\lambda + \psi i_2p_2\lambda = j_1q_1 + j_2q_2 = id_{L^*}$. Therefore $\varphi^{-1}\lambda$ is the inverse of ψ. □

Definition 1.1.22 *An epimorphism $p : M^* \to N^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$ (resp. $\mathcal{T}op\mathcal{M}od^i_{K^*}$) is said to be regular if p is a cokernel of a morphism of $\mathcal{T}op\mathcal{M}od_{K^*}$ (resp. $\mathcal{T}op\mathcal{M}od^i_{K^*}$).*

The following fact is obvious from the definition.

Proposition 1.1.23 *If $p : M^* \to N^*$ is a regular epimorphism, p induces an isomorphism $M^*/\operatorname{Ker} p \to N^*$.*

Proposition 1.1.24 *Let $p : M^* \to N^*$ be a regular epimorphism and $f : Q^* \to N^*$ a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$ (resp. $\mathcal{T}op\mathcal{M}od^i_{K^*}$). Then, a pull-back $q : P^* \to Q^*$ of p along f in $\mathcal{T}op\mathcal{M}od_{K^*}$ (resp. $\mathcal{T}op\mathcal{M}od^i_{K^*}$) is also a regular epimorphism.*

Proof. We may assume that $N^* = M^*/\operatorname{Ker} p$ and p is the quotient map. P^* can be identified with a submodule of $M^* \oplus Q^*$ consisting of elememts (v, z) such that $p(v) = f(z)$. Then, q is given by $q(v, z) = z$. Let $\bar{f} : P^* \to M^*$ be the map given by $\bar{f}(v, z) = v$. Since p is surjective, it is easy to verify that q is also surjective and its kernel is $\bar{f}^{-1}(\operatorname{Ker} p)$. We show that $q(T^*)$ is open for any $T^* \in \mathcal{V}_{P^*}$. There exist $T^*_1 \in \mathcal{V}_{M^*}$ and $T^*_2 \in \mathcal{V}_{Q^*}$ such that $(T^*_1 \oplus T^*_2) \cap P^* \subset T^*$. We claim that $f^{-1}(p(T^*_1)) \cap T^*_2 \subset q(T^*)$. In fact, for $z \in f^{-1}(p(T^*_1)) \cap T^*_2$, there exist $v \in T^*_1$ such that $f(z) = p(v)$. Then, $(v, z) \in (T^*_1 \oplus T^*_2) \cap P^* \subset T^*$ and $q(v, z) = z$. Since p is an open map by (1.1.15), it follows from $f^{-1}(p(T^*_1)) \cap T^*_2 \subset q(T^*)$ that $q(T^*)$ is open. □

By (1.1.17) and (1.1.24), $\mathcal{T}op\mathcal{M}od_{K^*}$ and $\mathcal{T}op\mathcal{M}od^i_{K^*}$ are "regular categories".

Lemma 1.1.25 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$.*
(1) If N^* is a subgroup of M^* which contains an element of $\mathcal{V}_{M^*}$, then N^* is open.
(2) If U is a subgroup of M^n which is open in M^n, then $U + \sum_{k\neq n} M^k$ is open in M^*.

Proof. (1) If $U^* \subset N^*$ for $U^* \in \mathcal{V}_{M^*}$, then $N^* = \bigcup_{x\in N^*} (x + U^*)$ is open.

(2) Take an open set O of M^* such that $U = O \cap M^n$. Since $0 \in U \subset O$, there exists $N^* \in \mathcal{V}_{M^*}$ such that $N^* \subset O$. Then, $N^n \subset O \cap M^n = U$ which implies $N^* \subset U + \sum_{k\neq n} M^k$. Hence the assertion follows from (1). □

Remark 1.1.26 *Let K^* be a discrete ring such that $K^i = \{0\}$ if $i \neq 0$. For $n \in \boldsymbol{Z}$, let M^*_n be a topological K^*-module given by $M^i_n = \begin{cases} K^0 & i = n \\ \{0\} & i \neq n \end{cases}$ and consider $\coprod_{n\in\boldsymbol{Z}} M^*_n$ and $\prod_{n\in\boldsymbol{Z}} M^*_n$. Then, $\coprod_{n\in\boldsymbol{Z}} M^*_n$ has the discrete topology. However, since $\left\{ \operatorname{Ker}\left(p_s : \prod_{n\in\boldsymbol{Z}} M^*_n \to M^*_s\right) \middle|\, s \in \boldsymbol{Z} \right\}$ forms a subbase of $\prod_{n\in\boldsymbol{Z}} M^*_n$, the topology on $\prod_{n\in\boldsymbol{Z}} M^*_n$ is not discrete. Thus, the unique map $\varphi : \coprod_{n\in\boldsymbol{Z}} M^*_n \to \prod_{n\in\boldsymbol{Z}} M^*_n$ satisfying $p_n\varphi i_n = id_{M^*_n}$ and $p_n\varphi i_m = 0$ if $n \neq m$ is a continuous bijection but it is not an isomorphism. The converse is clear.*

Proposition 1.1.27 *A morphism of $\mathcal{T}op\mathcal{M}od_{K^*}$ is an epimorphism if and only if it is surjective.*

Proof. Let $f^* : M^* \to N^*$ be an epimorphism of $\mathcal{T}op\mathcal{M}od_{K^*}$. We denote by $p : N^* \to N^*/\operatorname{Im} f$ the quotient map. Then, $pf : M^* \to N^*/\operatorname{Im} f$ is the trivial map which is an epimorphism. If we denote by id the identity map of $N^*/\operatorname{Im} f$ and by 0 the trivial map $N^*/\operatorname{Im} f \to N^*/\operatorname{Im} f$, then we have $id(pf) = 0(pf)$ which implies $id = 0$. Thus we have $N^*/\operatorname{Im} f = \{0\}$, that is $N^* = \operatorname{Im} f$. □

1.2 Suspension

Definition 1.2.1 *Let $c_{K^*} : K^* \to K^*$ be a homomorphism of topological graded rings given by $c_{K^*}(r) = (-1)^n r$ if $r \in K^n$. Then, it is clear that c_{K^*} is continuous and $c_{K^*}c_{K^*} = id_{K^*}$. We call c_{K^*} the conjugation of K^*.*

For $m \in \boldsymbol{Z}$ and an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, define an object $\Sigma^m M^*$ of $\mathcal{T}op\mathcal{M}od_{K^*}$ as follows.

Definition 1.2.2 *Put $(\Sigma^m M^*)^i = \{[m]\} \times M^{i-m}$ for $i \in \boldsymbol{Z}$ and give $(\Sigma^m M^*)^i$ the structure of an abelian group such that the projection $\{[m]\} \times M^{i-m} \to M^{i-m}$ onto the second component is an isomorphism of abelian groups.*

If $\alpha : K^* \times M^* \to M^*$ is the K^*-module structure of M^*, we define the K^*-module structure $\alpha^m : K^* \times \Sigma^m M^* \to \Sigma^m M^*$ of $\Sigma^m M^*$ by $\alpha^m(r, ([m], x)) = ([m], \alpha(c_{K^*}^m(r), x))$ for $r \in K^*$ and $x \in M^*$, where $c_{K^*}^m : K^* \to K^*$ is the m times composition of c_{K^*}.

If U^* is an submodule of M^*, we can regard $\Sigma^m U^*$ as a submodule of $\Sigma^m M^*$. We give a linear topology on $\Sigma^m M^*$ such that the set of open submodules of $\Sigma^m M^*$ is given by $\mathcal{V}_{\Sigma^m M^*} = \{\Sigma^m U^* | \, U^* \in \mathcal{V}_{M^*}\}$.

If $f : M^* \to N^*$ is a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$, we denote by $\Sigma^m f : \Sigma^m M^* \to \Sigma^m N^*$ the map which maps $([m], x) \in (\Sigma^m M^*)^i$ to $([m], f(x)) \in (\Sigma^m N^*)^i$. It is easy to verify that $\Sigma^m f$ is a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$. Thus we have a functor $\Sigma^m : \mathcal{T}op\mathcal{M}od_{K^*} \to \mathcal{T}op\mathcal{M}od_{K^*}$. We call $\Sigma^m M^*$ and $\Sigma^m f$ the m-fold suspension of M^* and f, respectively.

Definition 1.2.3 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$.*

(1) For $l \in \boldsymbol{Z}$ and $r \in K^l$, define a map $\mu_{M^*}^r : \Sigma^l M^* \to M^*$ by $\mu_{M^*}^r([l], x) = rx$ for $x \in M^{i-l}$. Then, $\mu_{M^*}^r$ is a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$ which is natural in M^*. Thus we have a natural transformation $\mu^r : \Sigma^l \to id_{\mathcal{T}op\mathcal{M}od_{K^*}}$.

(2) For $m, n \in \boldsymbol{Z}$, define a map $\varepsilon_{m,n,M^*} : \Sigma^{m+n} M^* \to \Sigma^m(\Sigma^n M^*)$ by $\varepsilon_{m,n,M^*}([m+n], x) = ([m], ([n], x))$ for $x \in M^{i-m-n}$. Then, ε_{m,n,M^*} is a isomorphism in $\mathcal{T}op\mathcal{M}od_{K^*}$ which is natural in M^*. Thus we have a natural equivalence $\varepsilon_{m,n} : \Sigma^{m+n} \to \Sigma^m \Sigma^n$.

The following assertions are easily verified.

Proposition 1.2.4 *Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$.*

(1) The following diagram is commutative.

$$\begin{tikzcd}
\Sigma^{k+l+m} M^* \arrow[r, "\varepsilon_{k,l+m,M^*}"] \arrow[d, "\varepsilon_{k+l,m,M^*}"] & \Sigma^k\big(\Sigma^{l+m} M^*\big) \arrow[d, "\Sigma^k \varepsilon_{l,m,M^*}"] \\
\Sigma^{k+l}(\Sigma^m M^*) \arrow[r, "\varepsilon_{k,l,\Sigma^m M^*}"] & \Sigma^k\big(\Sigma^l(\Sigma^m M^*)\big)
\end{tikzcd}$$

(2) For $l, m \in \boldsymbol{Z}$ and $r \in K^l$, the following diagram is commutative.

$$\begin{tikzcd}
\Sigma^{l+m} M^* \arrow[r, "\varepsilon_{l,m,M^*}"] \arrow[d, "\varepsilon_{m,l,M^*}"] & \Sigma^l(\Sigma^m M^*) \arrow[d, "(-1)^{lm}\mu^r_{\Sigma^m M^*}"] \\
\Sigma^m\big(\Sigma^l M^*\big) \arrow[r, "\Sigma^m \mu^r_{M^*}"] & \Sigma^m M^*
\end{tikzcd}$$

(3) Define a map $\sigma_{n,M^*,N^*} : \mathrm{Hom}^c_{K^*}(M^*, N^*) \to \mathrm{Hom}^c_{K^*}(\Sigma^n M^*, \Sigma^n N^*)$ by $\sigma_{n,M^*,N^*}(f) = \Sigma^n f$ for $n \in \boldsymbol{Z}$. Then, $\sigma_{M^*,N^*,n}$ is a natural isomorphism of abelian groups.

Let k be a linearly topologized commutative graded ring such that $k^i = \{0\}$ if $i \neq 0$. We denote by $\mathcal{T}op\mathcal{M}od_k$ the category of ungraded linearly topologized k-modules and continuous homomorphisms. Let $\varepsilon : K^* \to k$ be a morphism in $\mathcal{T}op\mathcal{A}lg^*_k$. For $n \in \boldsymbol{Z}$, define functors $\epsilon_n : \mathcal{T}op\mathcal{M}od_{K^*} \to \mathcal{T}op\mathcal{M}od_k$ and $\iota_n : \mathcal{T}op\mathcal{M}od_k \to \mathcal{T}op\mathcal{M}od_{K^*}$ as follows. We set

$$\epsilon_n(M^*) = M^n, \; \epsilon_n(f : M^* \to N^*) = (f^n : M^n \to N^n), \; \iota_n(V)^i = \begin{cases} V & i = n \\ 0 & i \neq n \end{cases}, \; \iota_n(f : V \to W)^i = \begin{cases} f & i = n \\ 0 & i \neq n \end{cases}.$$

We give $\epsilon_n(M^*) = M^n$ the topology induced by M^* and give $\iota_n(V)$ the topology generated by $\mathcal{V}_{\iota_n(V)} = \{\iota_n(U) | \, U \in \mathcal{V}_V\}$. Since K^* is a k-algebra and $k^i = \{0\}$ if $i \neq 0$, $\epsilon_n(M^*)$ can be regarded as a k-module. Since k is regarded as a K^*-algebra by ε, $\iota_n(V)$ can be regarded as a K^*-module.

Proposition 1.2.5 *If $K^i = \{0\}$ for $i \neq 0$, ι_n is a left and right adjoint of ϵ_n. Hence ϵ_n preserves limits and colimits.*

Proof. Define natural transformations $\bar{c}_n : id_{\mathcal{T}op\mathcal{M}od_k} \to \epsilon_n\iota_n$ and $c_n : \iota_n\epsilon_n \to id_{\mathcal{T}op\mathcal{M}od_{K^*}}$ as follows. For $M^* \in \operatorname{Ob}\mathcal{T}op\mathcal{M}od_{K^*}$, $c_{n\,M^*}(x) = x$ $(x \in M^n)$. For $V \in \operatorname{Ob}\mathcal{T}op\mathcal{M}od_k$, $\bar{c}_{n\,V}(y) = y$ $(y \in V)$. Clearly, $c_{n\,M^*} : \iota_n\epsilon_n(M^*) \to M^*$ is a homeomorphism onto its image and $\bar{u}_{n\,V} : \epsilon_n\iota_n(V) \to V$ and $\bar{c}_{n\,V} : V \to \epsilon_n\iota_n(V)$ can be regarded as identity maps. Then, $\bar{c}_n$ and c_n are the unit and the counit of the adjunction $\iota_n \vdash \epsilon_n$, respectively.

Define natural transformations $u_n : id_{\mathcal{T}op\mathcal{M}od_{K^*}} \to \iota_n\epsilon_n$ and $\bar{u}_n : \epsilon_n\iota_n \to id_{\mathcal{T}op\mathcal{M}od_k}$ as follows. For $M^* \in \operatorname{Ob}\mathcal{T}op\mathcal{M}od_{K^*}$, $u_{n\,M^*}(x) = \begin{cases} x & x \in M^n \\ 0 & x \in M^i,\ i \neq n \end{cases}$. For $V \in \operatorname{Ob}\mathcal{T}op\mathcal{M}od_k$, $\bar{u}_{n\,V}(y) = y$ $(y \in (\epsilon_n\iota_n(V))^n = V)$. It follows from (1.1.25) that $u_{n\,M^*} : M^* \to \iota_n\epsilon_n(M^*)$ is continuous. Clearly, $\bar{u}_{n\,V} : \epsilon_n\iota_n(V) \to V$ can be regarded as identity map. Then, u_n and $\bar{u}_n$ are the unit and the counit of the adjunction $\epsilon_n \vdash \iota_n$ respectively. □

Remark 1.2.6 *(1) For any $M^* \in \operatorname{Ob}\mathcal{T}op\mathcal{M}od_{K^*}$ and $n \in \boldsymbol{Z}$, $u_{n\,M^*} : M^* \to \iota_n\epsilon_n(M^*)$ is an open map.*

(2) We note that $\epsilon_m(\Sigma^n M^*) = \epsilon_{m-n}(M^*)$ and $\Sigma^n\iota_m(V) = \iota_{n+m}(V)$ hold for $M^* \in \operatorname{Ob}\mathcal{T}op\mathcal{M}od_{K^*}$ and $V \in \operatorname{Ob}\mathcal{T}op\mathcal{M}od_k$. Hence we have $\Sigma^n\iota_m\epsilon_m(M^*) = \iota_{m+n}\epsilon_{m+n}(\Sigma^n M^*)$ and $\Sigma^n u_{m,M^*} : \Sigma^n M^* \to \Sigma^n\iota_m\epsilon_m(M^*)$ coincides with $u_{m+n,\Sigma^n M^*} : \Sigma^n M^* \to \iota_{m+n}\epsilon_{m+n}(\Sigma^n M^*)$.

(3) For a morphism $f : M^* \to N^*$ of $\mathcal{T}op\mathcal{M}od_{K^*}$, $\Sigma^n\iota_m\epsilon_m(f) : \Sigma^n\iota_m\epsilon_m(M^*) \to \Sigma^n\iota_m\epsilon_m(N^*)$ coincides with $\iota_{m+n}\epsilon_{m+n}(\Sigma^n f) : \iota_{m+n}\epsilon_{m+n}(\Sigma^n M^*) \to \iota_{m+n}\epsilon_{m+n}(\Sigma^n N^*)$.

(4) For a morphism $f : \Sigma^m M^* \to N^*$ of $\mathcal{T}op\mathcal{M}od_{K^*}$, the following diagrams are commutative.

$$\begin{tikzcd}
\Sigma^m M^* \arrow[r, "\Sigma^m u_{n\,M^*}"] \arrow[dd, "f"] & \Sigma^m\iota_n\epsilon_n(M^*) \arrow[d, equal] \\
 & \iota_{m+n}\epsilon_{m+n}(\Sigma^m M^*) \arrow[d, "\iota_{m+n}\epsilon_{m+n}(f)"] \\
N^* \arrow[r, "u_{m+n\,N^*}"] & \iota_{m+n}\epsilon_{m+n}(N^*)
\end{tikzcd}
\qquad
\begin{tikzcd}
\iota_{m+n}\epsilon_{m+n}(\Sigma^m M^*) \arrow[r, equal] \arrow[d, "\iota_{m+n}\epsilon_{m+n}(f)"] & \Sigma^m\iota_n\epsilon_n(M^*) \arrow[r, "\Sigma^m c_{n\,M^*}"] & \Sigma^m M^* \arrow[d, "f"] \\
\iota_{m+n}\epsilon_{m+n}(N^*) \arrow[rr, "c_{m+n\,N^*}"] & & N^*
\end{tikzcd}$$

1.3 Completion of topological modules

Definition 1.3.1 *We say that an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ is complete if M^n is complete for each $n \in \boldsymbol{Z}$.*

Let be M^* an object of $\mathcal{T}op\mathcal{M}od_{K^*}$. Regarding $\mathcal{V}_{M^*}$ as a category whose morphisms are inclusion maps, consider a functor $D_{M^*} : \mathcal{V}_{M^*} \to \mathcal{T}op\mathcal{M}od_{K^*}$ given by $D_{M^*}(U^*) = M^*/U^*$. We denote by $\widehat{M}^*$ the limit $\varprojlim D_{M^*}$ of D_{M^*}, namely there is a limiting cone $\left(\widehat{M}^* \xrightarrow{\pi_{U^*}} M^*/U^*\right)_{U^* \in \mathcal{V}_{M^*}}$. Since the quotient maps $p_{U^*} : M^* \to M^*/U^*$ $(U^* \in \mathcal{V}_{M^*})$ define a cone of D_{M^*}, there is a unique map $\eta_{M^*} : M^* \to \widehat{M}^*$ satisfying $\pi_{U^*}\eta_{M^*} = p_{U^*}$ for any $U^* \in \mathcal{V}_{M^*}$. We call η_{M^*} the completion map of M^*.

Proposition 1.3.2 *(1) The image of $\eta_{M^*} : M^* \to \widehat{M}^*$ is dense.*

(2) $\eta_{M^*} : M^* \to \widehat{M}^*$ is an open map onto its image.

Proof. (1) First, we note that $\left\{\pi_{U^*}^{-1}(0) \middle| U^* \in \mathcal{V}_{M^*}\right\}$ forms a fundamental system of neighborhood of 0 in $\widehat{M}^*$. For any $x \in \widehat{M}^*$ and $U^* \in \mathcal{V}_{M^*}$, take $v \in M^*$ such that $p_{U^*}(v) = \pi_{U^*}(x)$. Then $\pi_{U^*}(\eta_{M^*}(v) - x) = 0$ which implies $\eta_{M^*}(v) \in \{x\} + \pi_{U^*}^{-1}(0)$.

(2) For $U^* \in \mathcal{V}_{M^*}$, it suffices to show $\eta_{M^*}(U^*) = \eta_{M^*}(M^*) \cap \pi_{U^*}^{-1}(0)$. $\eta_{M^*}(U^*) \subset \eta_{M^*}(M^*) \cap \pi_{U^*}^{-1}(0)$ is clear. If $\eta_{M^*}(x) \in \pi_{U^*}^{-1}(0)$ for $x \in M^*$, then $p_{U^*}(x) = \pi_{U^*}\eta_{M^*}(x) = 0$ hence $x \in U^*$. Thus we have $\eta_{M^*}(x) \in \eta_{M^*}(U^*)$ and this implies $\eta_{M^*}(U^*) \supset \eta_{M^*}(M^*) \cap \pi_{U^*}^{-1}(0)$. □

Proposition 1.3.3 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$.*

(1) M^* is Hausdorff if and only if $\eta_{M^*} : M^* \to \widehat{M}^*$ is injective. Hence if M^* is Hausdorff, η_{M^*} is a homeomorphism onto its image.

(2) $\widehat{M}^*$ is complete Hausdorff.

(3) M^* is complete Hausdorff if and only if $\eta_{M^*} : M^* \to \widehat{M}^*$ is bijective. Hence if M^* is complete Hausdorff, η_{M^*} is an isomorphism.

Proof. (1) Since the map $\rho : \widehat{M}^* \to \prod_{U^* \in \mathcal{V}_{M^*}} M^*/U^*$ induced by the canonical projections $\pi_{U^*} : \widehat{M}^* \to M^*/U^*$ is injective and $\rho\eta_{M^*} : M^* \to \prod_{U^* \in \mathcal{V}_{M^*}} M^*/U^*$ is the map induced by the quotient maps $p_{U^*} : M^* \to M^*/U^*$, the kernel of η_{M^*} is $\bigcap_{U^* \in \mathcal{V}_{M^*}} U^*$. Thus the assertion follows.

(2) Let I is a directed set and $(x_i)_{i\in I}$ a Cauchy sequence in $\widehat{M}^n$. Then, for $U^* \in \mathcal{V}_{M^*}$, $(\pi_{U^*}(x_i))_{i\in I}$ is a Cauchy sequence in $(M^*/U^*)^n$. Since M^*/U^* is discrete, there exists $k(U^*) \in I$ such that $\pi_{U^*}(x_i) = \pi_{U^*}(x_{k(U^*)})$ if $i \geqq k(U^*)$. Put $\bar{\alpha} = \left(\pi_{U^*}\left(x_{k(U^*)}\right)\right)_{U^*\in\mathcal{V}_{M^*}} \in \prod_{U^*\in\mathcal{V}_{M^*}} M^*/U^*$. Then $\lim_{i\in I}\rho(x_i) = \bar{\alpha}$ in $\prod_{U^*\in\mathcal{V}_{M^*}} M^*/U^*$. If $V^* \subset U^*$, take $j \in I$ such that $j \geqq k(U^*)$ and $j \geqq k(V^*)$, then $\pi_{V^*}(x_{k(V^*)}) = \pi_{V^*}(x_j) \in M^*/V^*$ maps to $\pi_{U^*}(x_{k(U^*)}) = \pi_{U^*}(x_j) \in M^*/U^*$ by the map $M^*/V^* \to M^*/U^*$. Hence there exists $\alpha \in \widehat{M}^*$ satisfying $\rho(\alpha) = \bar{\alpha}$. Since ρ is a homeomorphism onto its image, we have $\lim_{i\in I} x_i = \alpha$ in $\widehat{M}^*$. Therefore $\widehat{M}^*$ is complete.

Since each $(M^*/U^*)^n$ is discrete for $U^* \in \mathcal{V}_{M^*}$ and $n \in \boldsymbol{Z}$, $\prod_{U^*\in\mathcal{V}_{M^*}} (M^*/U^*)^n = \left(\prod_{U^*\in\mathcal{V}_{M^*}} M^*/U^*\right)^n$ is Hausdorff. Hence $\prod_{U^*\in\mathcal{V}_{M^*}} M^*/U^*$ is Hausdorff and so is $\widehat{M}^*$ which is homeomorphic to a subspace of $\prod_{U^*\in\mathcal{V}_{M^*}} M^*/U^*$.

(3) Suppose that M^* is complete Hausdorff. Take $x \in \left(\widehat{M}^*\right)^n$. For $U^* \in \mathcal{V}_{M^*}$ and $a, b \in p_{U^*}^{-1}(\pi_{U^*}(x)) \cap M^n$, we have $a - b \in U^*$. It follows that $(p_{U^*}^{-1}(\pi_{U^*}(x)) \cap M^n)_{U^*\in\mathcal{V}_{M^*}}$ is a filter basis of a Cauchy filter in M^n. Hence $(p_{U^*}^{-1}(\pi_{U^*}(x)) \cap M^n)_{U^*\in\mathcal{V}_{M^*}}$ converges to a point $v \in M^n$. Note that $\eta_{M^*}(v) = x$ if and only if $p_{N^*}(v) = \pi_{N^*}(x)$ for any $N^* \in \mathcal{V}_{M^*}$. For any $N^* \in \mathcal{V}_{M^*}$, there exists $U^* \in \mathcal{V}_{M^*}$ such that $p_{U^*}^{-1}(\pi_{U^*}(x)) \cap M^n \subset \{v\} + N^*$. We may assume that $U^* \subset N^*$. Then,

$$\begin{aligned}\pi_{N^*}(x) &= d_{M^*}(U^* \to N^*)\pi_{U^*}(x) \in D_{M^*}(U^* \to N^*)(p_{U^*}(p_{U^*}^{-1}(\pi_{U^*}(x)) \cap M^n)) \\ &= p_{N^*}(p_{U^*}^{-1}(\pi_{U^*}(x)) \cap M^n) \subset p_{N^*}(\{v\} + N^*) = \{p_{N^*}(v)\}.\end{aligned}$$

Thus we have $p_{N^*}(v) = \pi_{N^*}(x)$ for any $N^* \in \mathcal{V}_{M^*}$ and η_{M^*} is surjective.

Conversely, if $\eta_{M^*} : M^* \to \widehat{M}^*$ is bijective, it follows from (1) and (2) of (1.3.2) that η_{M^*} is an isomorphism. Hence M^* is complete Hausdorff by (2) above. □

Let $f : M^* \to N^*$ be a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$. For each $U^* \in \mathcal{V}_{N^*}$, we have a map $f_{U^*} : M^*/f^{-1}(U^*) \to N^*/U^*$ induced by f. Then, $\left(\widehat{M}^* \xrightarrow{f_{U^*}\pi_{f^{-1}(U^*)}} N^*/U^*\right)_{U^*\in\mathcal{V}_{N^*}}$ is a cone of $D_{N^*} : \mathcal{V}_{N^*} \to \mathcal{T}op\mathcal{M}od_{K^*}$. There exists a unique morphism $\hat{f} : \widehat{M}^* \to \widehat{N}^*$ such that $f_{U^*}\pi_{f^{-1}(U^*)} = \pi_{U^*}\hat{f}$ for any $U^* \in \mathcal{V}_{N^*}$. Hence $\pi_{U^*}\hat{f}\eta_{M^*} = f_{U^*}\pi_{f^{-1}(U^*)}\eta_{M^*} = f_{U^*}p_{f^{-1}(U^*)} = p_{U^*}f = \pi_{U^*}\eta_{N^*}f$. Therefore the following diagram commutes.

$$\begin{tikzcd} M^* \arrow[r, "\eta_{M^*}"] \arrow[d, "f"] & \widehat{M}^* \arrow[d, "\hat{f}"] \\ N^* \arrow[r, "\eta_{N^*}"] & \widehat{N}^* \end{tikzcd}$$

Proposition 1.3.4 *Let $f : M^* \to N^*$ a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$ such that N^* is complete Hausdorff. Then, there exists a unique morphism $g : \widehat{M}^* \to N^*$ such that $g\eta_{M^*} = f$.*

Proof. Since N^* is complete Hausdorff, $\eta_{N^*} : N^* \to \widehat{N}^*$ is an isomorphism. Put $g = \eta_{N^*}^{-1}\hat{f}$, then $g\eta_{M^*} = f$ by the commutativity of the above diagram. The uniqueness of g follows from the fact that the image of η_{M^*} is dense. □

Definition 1.3.5 *The limit $\widehat{M}^*$ of $D_{M^*} : \mathcal{V}_{M^*} \to \mathcal{T}op\mathcal{M}od_{K^*}$ is called the completion of M^*.*

Remark 1.3.6 *The following diagram is commutative by the definition of $\hat{\eta}_{M^*} : \widehat{M}^* \to \widehat{\widehat{M}}^*$.*

$$\begin{tikzcd} M^* \arrow[r, "\eta_{M^*}"] \arrow[d, "\eta_{M^*}"] & \widehat{M}^* \arrow[d, "\eta_{\widehat{M}^*}"] \\ \widehat{M}^* \arrow[r, "\hat{\eta}_{M^*}"] & \widehat{\widehat{M}}^* \end{tikzcd}$$

Since the image of $\eta_{\widehat{M}^*} : M^* \to \widehat{M}^*$ is dense by (1.3.2) and $\widehat{\widehat{M}}^*$ is a Hausdorff space by (1.3.3), it follows from the continuity of $\hat{\eta}_{M^*}$ and $\eta_{\widehat{M}^*}$ and the commutativity of the above diagram that we have $\hat{\eta}_{M^*} = \eta_{\widehat{M}^*}$.

Lemma 1.3.7 *Let $F : \mathcal{C} \to \mathcal{D}$ and $D : \mathcal{D} \to \mathcal{E}$ be functors Assume that the comma category $(F\!\downarrow\! i)$ is not empty for any $i \in \mathrm{Ob}\,\mathcal{D}$. If $(X \xrightarrow{\pi_i} D(i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of D and a monomorphic family in $\mathcal{E}$, $(X \xrightarrow{\pi_{F(j)}} DF(j))_{j\in \mathrm{Ob}\,\mathcal{C}}$ is also a monomorphic family.*

Proof. Let $f, g : Y \to X$ be morphisms of $\mathcal{E}$ such that $\pi_{F(j)}f = \pi_{F(j)}g$ for any $j \in \mathrm{Ob}\,\mathcal{C}$. For any object i of $\mathcal{D}$, there exists an object $\langle j, \alpha\rangle$ of $(F\!\downarrow\! i)$. Since $(X \xrightarrow{\pi_i} D(i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of D, we have

$$\pi_i f = D(\alpha)\pi_{F(j)}f = D(\alpha)\pi_{F(j)}g = \pi_i g$$

for any $i \in \mathrm{Ob}\,\mathcal{D}$. Thus $f = g$ by the assumption. Hence $(X \xrightarrow{\pi_{F(j)}} DF(j))_{j\in \mathrm{Ob}\,\mathcal{C}}$ is a monomorphic family. □

Proposition 1.3.8 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that, for any object i of $\mathcal{D}$, the comma category $(F\!\downarrow\! i)$ is non-empty and connected. Let $D : \mathcal{D} \to \mathcal{E}$ be a functor.*

(1) Suppose that $(X \xrightarrow{\pi_j} DF(j))_{j\in \mathrm{Ob}\,\mathcal{C}}$ is a limiting cone of DF. For each object i of $\mathcal{D}$, choose an object $\langle j_i, \alpha_i\rangle$ of $(F\!\downarrow\! i)$ and put $\tilde{\pi}_i = D(\alpha_i)\pi_{j_i}$. Then $(X \xrightarrow{\tilde{\pi}_i} D(i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of D.

(2) If $(X \xrightarrow{\pi_i} D(i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of F, then $(X \xrightarrow{\pi_{F(j)}} DF(j))_{j\in \mathrm{Ob}\,\mathcal{C}}$ is a limiting cone of DF.

Proof. Let $(Y \xrightarrow{\rho_j} DF(j))_{j\in \mathrm{Ob}\,\mathcal{C}}$ be a cone of DF. For an object i of $\mathcal{D}$ and objects $\langle j, \alpha\rangle$, $\langle k, \beta\rangle$ of $(F\!\downarrow\! i)$, it follows from the connectivity of $(F\!\downarrow\! i)$ that there exist objects $\langle l_m, \gamma_m\rangle$ $(m = 1, 2, \ldots, 2n-1)$ and morphisms $\varphi_s : \langle l_{2s-1}, \gamma_{2s-1}\rangle \to \langle l_{2s-2}, \gamma_{2s-2}\rangle$, $\psi_s : \langle l_{2s-1}, \gamma_{2s-1}\rangle \to \langle l_{2s}, \gamma_{2s}\rangle$ $(s = 1, 2, \ldots, n)$ of $(F\!\downarrow\! i)$, where we put $l_0 = j$, $\beta_0 = \alpha$, $l_{2n} = k$, $\beta_{2n} = \beta$. Then, we have $\gamma_{2s-2}F(\varphi_s) = \gamma_{2s-1}$ and $\gamma_{2s}F(\psi_s) = \gamma_{2s-1}$ for $s = 1, 2, \ldots, n$. Since $DF(\varphi_s)\rho_{l_{2s-1}} = \rho_{l_{2s-2}}$ and $DF(\psi_s)\rho_{l_{2s-1}} = \rho_{l_{2s}}$ hold for $s = 1, 2, \ldots, n$, we have the following equalities for $s = 1, 2, \ldots, n$, which imply $D(\alpha)\rho_j = D(\beta)\rho_k$.

$$D(\gamma_{2s-2})\rho_{l_{2s-2}} = D(\gamma_{2s-2})DF(\varphi_s)\rho_{l_{2s-1}} = D(\gamma_{2s-2}F(\varphi_s))\rho_{l_{2s-1}} = D(\gamma_{2s-1})\rho_{l_{2s-1}}$$
$$D(\gamma_{2s})\rho_{l_{2s}} = D(\gamma_{2s})DF(\psi_s)\rho_{l_{2s-1}} = D(\gamma_{2s}F(\psi_s))\rho_{l_{2s-1}} = D(\gamma_{2s-1})\rho_{l_{2s-1}}$$

For an object i of $\mathcal{D}$, we choose an object $\langle j, \alpha\rangle$ of $(F\!\downarrow\! i)$ and define $\tilde{\rho}_i : Y \to D(i)$ by $\tilde{\rho}_i = D(\alpha)\rho_j$. Then, this definition of $\tilde{\rho}_i$ does not depend on the choice of $\langle j, \alpha\rangle$. Let $\tau : i \to i'$ be a morphism of $\mathcal{D}$. Since $\langle j, \tau\alpha\rangle$ is an object of $(F\!\downarrow\! i')$, we have $\tilde{\rho}_{i'} = D(\tau\alpha)\rho_j = D(\tau)D(\alpha)\rho_j = D(\tau)\tilde{\rho}_i$ which shows that $(Y \xrightarrow{\tilde{\rho}_i} D(i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of D.

(1) By the above result, $(X \xrightarrow{\tilde{\pi}_i} D(i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of D. Let $(Y \xrightarrow{\lambda_i} D(i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ be a cone of D. Then, $(Y \xrightarrow{\lambda_{F(j)}} DF(j))_{j\in \mathrm{Ob}\,\mathcal{C}}$ is a cone of DF. Hence there exists unique morphism $f : Y \to X$ of $\mathcal{E}$ that satisfies $\pi_k f = \lambda_{F(k)}$ for any $k \in \mathrm{Ob}\,\mathcal{C}$. For an object i of $\mathcal{D}$, we have $\tilde{\pi}_i f = D(\alpha_i)\pi_{j_i} f = D(\alpha_i)\lambda_{F(j_i)} = \lambda_i$. Suppose that a morphism $g : Y \to X$ satisfies $\tilde{\pi}_i g = \lambda_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. Since $\langle F(k), id_{F(k)}\rangle$ is an object of $(F\!\downarrow\! F(k))$, we have $\tilde{\pi}_{F(k)} = \pi_k$. It follows that $\pi_k g = \tilde{\pi}_{F(k)}g = \lambda_{F(k)}$ for any $k \in \mathrm{Ob}\,\mathcal{C}$. Since $(X \xrightarrow{\pi_j} DF(j))_{j\in \mathrm{Ob}\,\mathcal{C}}$ is a limiting cone of DF, we have $g = f$. Therefore $(X \xrightarrow{\tilde{\pi}_i} D(i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of D.

(2) Let $(Y \xrightarrow{\rho_j} DF(j))_{j\in \mathrm{Ob}\,\mathcal{C}}$ be a cone of DF. Since $(Y \xrightarrow{\tilde{\rho}_i} D(i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of D, there exists unique morphism $f : Y \to X$ that satisfies $\pi_i f = \tilde{\rho}_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. Since $\langle F(j), id_{F(j)}\rangle$ is an object of $(F\!\downarrow\! F(j))$, we have $\tilde{\rho}_{F(j)} = \rho_j$, which implies that $\pi_{F(j)}f = \tilde{\rho}_{F(j)} = \rho_j$ holds for any $j \in \mathrm{Ob}\,\mathcal{C}$. Since $(X \xrightarrow{\pi_{F(j)}} DF(j))_{j\in \mathrm{Ob}\,\mathcal{C}}$ is a monomorphic family by (1.3.7), the uniqueness of the morphism $f : Y \to X$ that satisfies $\pi_{F(j)}f = \rho_j$ for all $j \in \mathrm{Ob}\,\mathcal{C}$ follows. □

Remark 1.3.9 *For an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ and a subset $\mathcal{B}_{M^*}$ of $\mathcal{V}_{M^*}$, we regard the inclusion map $I : \mathcal{B}_{M^*} \to \mathcal{V}_{M^*}$ as a functor. Let U^* be an element of $\mathcal{V}_{M^*}$. Then, $(I\!\downarrow\! U^*)$ is not empty if and only if there exists $V^* \in \mathcal{B}_{M^*}$ which is contained in U^*. If $(I\!\downarrow\! U^*)$ is not empty for any $U^* \in \mathcal{V}_{M^*}$, $(I\!\downarrow\! U^*)$ connected for any $U^* \in \mathcal{V}_{M^*}$. In fact, for objects $\langle V^*, \iota\rangle$ and $\langle W^*, \eta\rangle$ of $(I\!\downarrow\! U^*)$, since $(I\!\downarrow\! V^* \cap W^*)$ not empty, there exists $Z^* \in \mathcal{B}_{M^*}$ which is contained both V^* and W^*.*

If $(I\!\downarrow\! U^*)$ is not empty for any $U^* \in \mathcal{V}_{M^*}$, we say that $\mathcal{B}_{M^*}$ is a cofinal subset of $\mathcal{V}_{M^*}$.

Proposition 1.3.10 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$. If M^* is a 1st countable space, then the completion of M^* is also a 1st countable space.*

Proof. Let $\mathcal{B}_{M^*}$ a countable cofinal subset of $\mathcal{V}_{M^*}$. We regard the inclusion map $I : \mathcal{B}_{M^*} \to \mathcal{V}_{M^*}$ as a functor. Let $\big(\widehat{M}^* \xrightarrow{\pi_{U^*}} M^*/U^*\big)_{U^*\in\mathcal{V}_{M^*}}$ be a limiting cone of D_{M^*}. It follow from (1.3.8) that $\big(\widehat{M}^* \xrightarrow{\pi_{U^*}} M^*/U^*\big)_{U^*\in\mathcal{B}_{M^*}}$ is a limiting cone of D_{M^*}. Since $\mathcal{B}_{M^*}$ is countable, $\prod_{U^*\in\mathcal{B}_{M^*}} M^*/U^*$ is a countable product of 1st countable spaces. Hence $\prod_{U^*\in\mathcal{B}_{M^*}} M^*/U^*$ is a 1st countable space. Since $\widehat{M}^*$ is homeomorphic to a subspace of $\prod_{U^*\in\mathcal{B}_{M^*}} M^*/U^*$, $\widehat{M}^*$ is also a 1st countable space. $\square$

It follows from (1.1.17) that $\widehat{M}^*$ is an object of $\mathcal{TopMod}^i_{K^*}$ if M^* is so. Let us denote by $\mathcal{TopMod}_{cK^*}$ (resp. $\mathcal{TopMod}^i_{cK^*}$) the full subcategory of $\mathcal{TopMod}_{K^*}$ (resp. $\mathcal{TopMod}^i_{K^*}$) consisting of objects of $\mathcal{TopMod}_{K^*}$ (resp. $\mathcal{TopMod}^i_{K^*}$) which is complete Hausdorff. By assigning $\widehat{M}^*$ to M^* and $\hat{f}:\widehat{M}^*\to\widehat{N}^*$ to $f:M^*\to N^*$, we have the completion functor $C:\mathcal{TopMod}_{K^*}\to\mathcal{TopMod}_{cK^*}$ (resp. $C:\mathcal{TopMod}^i_{K^*}\to\mathcal{TopMod}^i_{cK^*}$). The next result is a direct consequence of (1.3.4).

Proposition 1.3.11 *$C:\mathcal{TopMod}_{K^*}\to\mathcal{TopMod}_{cK^*}$ (resp. $C:\mathcal{TopMod}^i_{K^*}\to\mathcal{TopMod}^i_{cK^*}$) is a left adjoint of the inclusion functor $\mathcal{TopMod}_{cK^*}\hookrightarrow\mathcal{TopMod}_{K^*}$ (resp. $\mathcal{TopMod}^i_{cK^*}\hookrightarrow\mathcal{TopMod}^i_{K^*}$). In particular, C preserves epimorphisms and colimits.*

Proposition 1.3.12 *If $f:M^*\to N^*$ is a morphism of $\mathcal{TopMod}_{K^*}$ which is a homeomorphism onto its image, then $\hat{f}:\widehat{M}^*\to\widehat{N}^*$ is also a homeomorphism onto its image.*

Proof. We may assume that M^* is a submodule of N^* and that f is the inclusion map. Since $\hat{f}$ is the morphism induced by a family of monomorphisms $(f_{U^*}:M^*/M^*\cap U^*\to N^*/U^*)_{U^*\in\mathcal{V}_{N^*}}$ of discrete spaces, the assertion follows. $\square$

Proposition 1.3.13 *([14]) For a submodule N^* of M^*, the closure of N^* is $\bigcap_{U^*\in\mathcal{V}_{M^*}}(N^*+U^*)$.*

Proof. $x\in M^*$ belongs to the closure of N^* if and only if $(x+U^*)\cap N^*$ is not empty for any $U^*\in\mathcal{V}_{M^*}$, which is equivalent to $x\in N^*+U^*$ for any $U^*\in\mathcal{V}_{M^*}$. $\square$

Proposition 1.3.14 *([14] Theorem 8.1) Let M^* be an object of $\mathcal{TopMod}_{K^*}$ and N^* a submodule of M^*. We denote by $i:N^*\to M^*$ and $q:M^*\to M^*/N^*$ the inclusion map and the quotient map, respectively. Then, $\hat{i}$ maps $\widehat{N}^*$ isomorphically onto the closure of $\eta_{M^*}(N^*)$ and $0\to\widehat{N}^*\xrightarrow{\hat{i}}\widehat{M}^*\xrightarrow{\hat{q}}\widehat{M^*/N^*}$ is exact.*

Proof. $\operatorname{Im}\hat{i}$ is a closed subset of $\widehat{M}^*$ by the completeness of $\widehat{N}^*$ and (1.3.12). Note that $\eta_{N^*}(N^*)$ is dense in $\widehat{N}^*$. Hence $\overline{\eta_{M^*}(N^*)}=\hat{i}\,\overline{\eta_{N^*}(N^*)}=\operatorname{Im}\hat{i}$. Since a submodule S^* of M^*/N^* is open in M^*/N^* if and only if there exists $U^*\in\mathcal{V}_{M^*}$ satisfying $(U^*+N^*)/N^*=q(U^*)\subset S^*$, $\{q(U^*)\,|\,U^*\in\mathcal{V}_{M^*}\}$ is a cofinal subset of $\mathcal{V}_{M^*/N^*}$. Let $D,D',D'':\mathcal{V}_{M^*}\to\mathcal{TopMod}_{K^*}$ be functors defined by $D(U^*)=M^*/U$, $D'(U^*)=N^*/(N^*\cap U^*)$ and $D''(U^*)=M^*/(N^*+U^*)$. We have limiting cones $\left(\widehat{M}^*\xrightarrow{\pi_{U^*}}M^*/U^*\right)_{U^*\in\mathcal{V}_{M^*}}$, $\left(\widehat{N}^*\xrightarrow{\pi'_{U^*}}N^*/(N^*\cap U^*)\right)_{U^*\in\mathcal{V}_{M^*}}$ and $\left(\widehat{M^*/N^*}\xrightarrow{\pi''_{U^*}}M^*/(N^*+U^*)\right)_{U^*\in\mathcal{V}_{M^*}}$ of D, D' and D'', respectivey. If $x\in\operatorname{Ker}\hat{q}$, choose $x_{U^*}\in M^*$ satisfying $\pi_{U^*}(x)=p_{U^*}(x_{U^*})$ for each $U^*\in\mathcal{V}_{M^*}$. Since $x_{U^*}\in N^*+U^*$ for any $U^*\in\mathcal{V}_{M^*}$, there exist $y_{U^*}\in N^*$ and $z_{U^*}\in U^*$ such that $x_{U^*}=y_{U^*}+z_{U^*}$. Thus $\pi_{U^*}(x)=p_{U^*}(y_{U^*})$ for any $U^*\in\mathcal{V}_{M^*}$, which implies $x\in\operatorname{Im}\hat{i}$. $\square$

Proposition 1.3.15 *The kernel of the morphism $\hat{f}:\widehat{M}^*\to\widehat{N}^*$ induced by a morphism $f:M^*\to N^*$ of $\mathcal{TopMod}_{K^*}$ is $\bigcap_{V^*\in\mathcal{V}_{N^*}}\operatorname{Ker}\pi_{f^{-1}(V^*)}$, where $\left(\widehat{M}^*\xrightarrow{\pi_{U^*}}M^*/U^*\right)_{U^*\in\mathcal{V}_{M^*}}$ is a limiting cone of $D_{M^*}:\mathcal{V}_{M^*}\to\mathcal{TopMod}_{K^*}$.*

Proof. For $V^*\in\mathcal{V}_{N^*}$, let $f_{V^*}:M^*/f^{-1}(V^*)\to N^*/V^*$ be the map satisfying $f_{V^*}p_{f^{-1}(V^*)}=q_{V^*}f$, where $p_{f^{-1}(V^*)}:M^*\to M^*/f^{-1}(V^*)$ and $q_{V^*}:N^*\to N^*/V^*$ are the quotient maps. Then, f_{V^*} is injective and the following diagram commutes by the definition of $\hat{f}$.

$$\begin{tikzcd}
\widehat{M}^* \arrow[r, "\pi_{f^{-1}(V^*)}"] \arrow[d, "\hat{f}"] & M^*/f^{-1}(V^*) \arrow[d, "f_{V^*}"] \\
\widehat{N}^* \arrow[r, "\pi'_{V^*}"] & N^*/V^*
\end{tikzcd}$$

Hence $\bigcap_{V^*\in\mathcal{V}_{N^*}} \operatorname{Ker}\pi_{f^{-1}(V^*)} = \bigcap_{V^*\in\mathcal{V}_{N^*}} \operatorname{Ker}(\pi'_{V^*}\hat{f}) = \bigcap_{V^*\in\mathcal{V}_{N^*}} \hat{f}^{-1}(\operatorname{Ker}(\pi'_{V^*})) = \hat{f}^{-1}\left(\bigcap_{V^*\in\mathcal{V}_{N^*}} \operatorname{Ker}(\pi'_{V^*})\right) = \operatorname{Ker}\hat{f}$. □

Proposition 1.3.16 *Let U^* be an open submodule of M^*. The closure of $\eta_{M^*}(U^*)$ coincides with* $\operatorname{Ker}\pi_{U^*}$.

Proof. Since $\pi_{U^*}\eta_{M^*} = p_{U^*}$, $\eta_{M^*}(U^*) \subset \operatorname{Ker}\pi_{U^*}$. It follows $\eta_{M^*}(U^*) \subset \operatorname{Ker}\pi_{U^*}$. Since $\operatorname{Ker}\pi_{U^*}$ is a closed subset of $\widehat{M}^*$, the closure of $\eta_{M^*}(U^*)$ is contained in $\operatorname{Ker}\pi_{U^*}$. We put $\mathcal{U} = \{V^* \in \mathcal{V}_{M^*} \,|\, V^* \subset U^*\}$. Then, $\mathcal{U}$ is a cofinal subset of $\mathcal{V}_{M^*}$. For $x \in \operatorname{Ker}\pi_{U^*}$, we choose $x_{V^*} \in M^*$ satisfying $\pi_{V^*}(x) = p_{V^*}(x_{V^*})$ for each $V^* \in \mathcal{U}$. Then, $(\eta_{M^*}(x_{V^*}))_{V^*\in\mathcal{U}}$ is a Cauchy sequence converging to x and $x_{U^*} \in U^*$. Since the map $\tau_{V^*,U^*} : M^*/V^* \to M^*/U^*$ satisfying $\tau_{V^*,U^*}p_{V^*} = p_{U^*}$ maps $p_{V^*}(x_{V^*})$ to $p_{U^*}(x_{U^*}) = 0$, it follows that $x_{V^*} \in U^*$ if $V^* \in \mathcal{U}$. Therefore $(\eta_{M^*}(x_{V^*}))_{V^*\in\mathcal{U}}$ is a sequence in $\eta_{M^*}(U^*)$ and this implies x belongs to the closure of $\eta_{M^*}(U^*)$. □

If M^* is complete, so is $\Sigma^l M^*$ by the definition of completeness. Hence the following fact holds.

Proposition 1.3.17 *Let $\eta_{M^*} : M^* \to \widehat{M}^*$ be the completion of M^*. There is a unique isomorphism $t : \Sigma^l\widehat{M}^* \to \widehat{\Sigma^l M^*}$ such that the following diagram commutes.*

$$\begin{tikzcd} \Sigma^l M^* \arrow[r, "\Sigma^l\eta_{M^*}"] \arrow[dr, "\eta_{\Sigma^l M^*}"'] & \Sigma^l\widehat{M}^* \arrow[d, "t"] \\ & \widehat{\Sigma^l M^*} \end{tikzcd}$$

Proposition 1.3.18 *Let $f : M^* \to N^*$ be a morphism of $\mathcal{T}op\mathcal{M}od_{K^*}$ whose image is dense.*
(1) If M^* is a submodule of N^* and f is the inclusion map, then $\hat{f}$ is an isomorphism.
(2) $\hat{f} : \widehat{M}^* \to \widehat{N}^*$ is an epimorphism of $\mathcal{T}op\mathcal{M}od_{cK^*}$.

Proof. (1) We denote by $p : N^* \to N^*/\operatorname{Im} f$ the quotient map. There is an exact sequence $0 \to \widehat{M}^* \xrightarrow{\hat{f}} \widehat{N}^* \xrightarrow{\hat{p}} \widehat{N^*/\operatorname{Im} f}$ by (1.3.14). Since the topology of N^*/M^* is trivial by (1.1.16), $\widehat{N^*/M^*}$ is a trivial module. Hence $\hat{f}$ is an isomorphism by (1.3.12).

(2) Let $f' : M^* \to \operatorname{Im} f$ be the surjection induced by f and denote by $i : \operatorname{Im} f \to N^*$ the inclusion map. By (1.3.11), $\hat{f}' : \widehat{M}^* \to \widehat{\operatorname{Im} f}$ is an epimorphism of $\mathcal{T}op\mathcal{M}od_{cK^*}$. Since $\hat{i} : \widehat{\operatorname{Im} f} \to \widehat{N}^*$ is an isomorphism by (1), $\hat{f} = \hat{i}\hat{f}' : \widehat{M}^* \to \widehat{N}^*$ is an epimorphism of $\mathcal{T}op\mathcal{M}od_{cK^*}$. □

Proposition 1.3.19 *A morphism $f : M^* \to N^*$ of $\mathcal{T}op\mathcal{M}od_{cK^*}$ is an epimorphism if and only if the image of f is dense.*

Proof. Assume that f is an epimorphism. Let L^* be the closure of the image of f and $p : N^* \to N^*/L^*$ the quotient map. If we denote by $0 : \widehat{N^*/L^*} \to \widehat{N^*/L^*}$ be the trivial map, then we have $0\eta_{N^*/L^*}pf = \eta_{N^*/L^*}pf$. We note that $0\eta_{N^*/L^*}pf$ and $\eta_{N^*/L^*}pf$ are both morphisms of $\mathcal{T}op\mathcal{M}od_{cK^*}$. Since f is an epimorphism and p is surjective, it follows that $0\eta_{N^*/L^*} = \eta_{N^*/L^*}$, namely, η_{N^*/L^*} is the trivial map. On the other hand, since N^*/L^* is an Hausdorff module, $\eta_{N^*/L^*} : N^*/L^* \to \widehat{N^*/L^*}$ is injective. Therefore N^*/L^* is the trivial K^*-module, hence the image of f is dense. □

Proposition 1.3.20 *For a morphism $f : M^* \to N^*$ of $\mathcal{T}op\mathcal{M}od_{cK^*}$, a morphism $g : N^* \to Q^*$ is a cokernel of f in $\mathcal{T}op\mathcal{M}od_{cK^*}$ if and only if g is an epimorphism whose kernel is the closure of the image of f.*

Proof. Suppose that g is an epimorphism whose kernel is the closure $\overline{\operatorname{Im} f}$ of the image of f. Let $h : N^* \to L^*$ be a morphism of $\mathcal{T}op\mathcal{M}od_{cK^*}$ which satisfies $hf = 0$. Since L^* is a Hausdorff module, $\operatorname{Ker} h$ is a closed submodule of N^* which contains $\operatorname{Im} f$. Hence $\operatorname{Ker} h$ contains $\overline{\operatorname{Im} f} = \operatorname{Ker} g$ and it follows that there exists a map $\bar{h} : Q^* \to L^*$ which satisfies $\bar{h}g = h$. The uniqueness of $\bar{h}$ satisfying $\bar{h}g = h$ follows form the assumption that g is an epimorphism. Conversely, assume that $g : N^* \to Q^*$ is a cokernel of f in $\mathcal{T}op\mathcal{M}od_{cK^*}$. It is clear that g is an epimorphism. Since Q^* is a Hausdorff module, $\operatorname{Ker} g$ is a closed submodule of N^* which contains $\operatorname{Im} f$. Thus we have $\overline{\operatorname{Im} f} \subset \operatorname{Ker} g$. Let $p : N^* \to N^*/\overline{\operatorname{Im} f}$ be the quotient map and consider a composition $N^* \xrightarrow{p} N^*/\overline{\operatorname{Im} f} \xrightarrow{\eta_{N^*/\overline{\operatorname{Im} f}}} \widehat{N^*/\overline{\operatorname{Im} f}}$. Since $pf = 0$ and $\eta_{N^*/\overline{\operatorname{Im} f}}\,p : N^* \to \widehat{N^*/\overline{\operatorname{Im} f}}$ is a morphism of $\mathcal{T}op\mathcal{M}od_{cK^*}$, there exists unique map $q : N^* \to \widehat{N^*/\overline{\operatorname{Im} f}}$ that satisfies $qg = \eta_{N^*/\overline{\operatorname{Im} f}}\, p$. Since $N^*/\overline{\operatorname{Im} f}$ is a Hausdorff module, $\eta_{N^*/\overline{\operatorname{Im} f}}$ is injective. Thus we have $\operatorname{Ker} g \subset \operatorname{Ker} qg = \operatorname{Ker} \eta_{N^*/\overline{\operatorname{Im} f}}\, p = \operatorname{Ker} p = \overline{\operatorname{Im} f}$. $\square$

Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$. For a sequence $(x_n)_{n\in\boldsymbol{N}}$ of elements of M^*, we say that a series $\sum_{n\in\boldsymbol{N}} x_n$ converges to $\alpha \in M^*$ if the following condition is satisfied.

For any $U^* \in \mathcal{V}_{M^*}$, there exists $N \in \boldsymbol{N}$ which satisfies "$\sum_{n=1}^{m} x_n - \alpha \in U^*$ if $m \geqq N$".

Proposition 1.3.21 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ which is a 1st countable space and D^* a dense submodule of M^*. For $x \in M^*$, there exists a sequence $(x_n)_{n\in\boldsymbol{N}}$ of elements of D^* such that $\sum_{n\in\boldsymbol{N}} x_n$ converges to x.*

Proof. Let $\mathcal{B}_{M^*}$ be a countable cofinal subset of $\mathcal{V}_{M^*}$. We put $\mathcal{B}_{M^*} = \{U_n^* \mid n \in \boldsymbol{N}\}$ and $V_n^* = \bigcap_{k=1}^{n} U_k^*$. Since D^* is dense, there exists $y_n \in D^*$ which satisfies $x - y_n \in V_n^*$ for each $n \in \boldsymbol{Z}$. Put $x_1 = y_1$ and $x_n = y_n - y_{n-1}$ for $n \geqq 2$. Then, $x_n \in D^*$ for any $n \in \boldsymbol{Z}$ and $x - \sum_{n=1}^{m} x_n = x - y_m \in V_m^*$. For any $U^* \in \mathcal{V}_{M^*}$, there exists $n \in \boldsymbol{N}$ such that $U_n^* \subset U^*$. Since $V_m^* \subset U_n^*$ if $m \geqq n$, we have $x - \sum_{n=1}^{m} x_n = x - y_m \in U^*$ if $m \geqq n$. Hence $\sum_{n\in\boldsymbol{N}} x_n$ converges to x. $\square$

Proposition 1.3.22 *Let K^* be a field with discrete topology and M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ which is a Hausdorff space. Assume that a sequence $(x_n)_{n\in\boldsymbol{N}}$ of M^* is linearly independent. If $\sum_{n\in\boldsymbol{N}} a_n x_n = 0$ for $a_n \in K^*$, then $a_n = 0$ for all $n \in \boldsymbol{N}$.*

Proof. For $i \in \boldsymbol{N}$, let N^* be a subspace of M^* spanned by $\{x_n \mid n \neq i\}$ and $p_i : M^* \to M^*/N_i^*$ the quotient map. Since p_i maps the left hand side of $\sum_{n\in\boldsymbol{N}} a_n x_n = 0$ to $a_i p_i(x_i)$ by the continuity of p_i. Since N_i^* does not contain x_i, we have $p_i(x_i) \neq 0$. Hence $a_i = 0$ for all $i \in \boldsymbol{N}$. $\square$

1.4 Topologies on graded modules

Definition 1.4.1 *Let K^* be a linearly topologized graded ring.*

(1) K^* is said to be finite if K^* is an Artinian ring and discrete.

(2) We say that an open ideal $\mathfrak{a}$ of K^* is cofinite if $K^*/\mathfrak{a}$ is artinian. We say that K^* has the cofinite topology if the set of all cofinite ideals of K^* is a fundamental system of the neighborhood of 0.

(3) If the topology of K^* is coarser (resp. finer) than the cofinite topology, we say that K^* is subcofinite (resp. supercofinite). Hence K^* is subcofinite (resp. supercofinite) if and only if every open ideal is cofinite (resp. every cofinite ideal is open).

Definition 1.4.2 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$.*

(1) M^* is said to be finite if M^* is discrete and has a composition series.

(2) We say that a submodule N^* of M^* is cofinite if M^*/N^* is finite. We say that M^* has the cofinite topology if the set of all cofinite submodules of M^* is a fundamental system of the neighborhood of 0.

(3) If the topology of M^* is coarser (resp. finer) than the cofinite topology, we say that M^* is subcofinite (resp. supercofinite). Hence M^* is subcofinite (resp. supercofinite) if and only if every open submodule is cofinite (resp. every cofinite submodule is open).

(4) For a non-negative integer n, let us denote by $M^*[n]$ a submodule of M^* generated by $\bigcup_{|i|\geqq n} M^i$. We say that an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ has a skeletal topology if $\{M^*[n] \mid n = 0, 1, 2, \dots\}$ is a fundamental system of the neighborhood of 0.

(5) If the topology of M^* is coarser (resp. finer) than the skeletal topology, we say that M^* is subskeletal (resp. superskeletal). Hence M^* is subskeletal (resp. superskeletal) if and only if every open submodule contains $M^*[n]$ for some n (resp. every submodule containing $M^*[n]$ for some n is open).

Remark 1.4.3 *(1) Suppose that M^* has finite length. Then, M^* is supercofinite if and only if M^* is discrete.*

(2) Let K^* be a field such that $K^d \neq \{0\}$ for some $d \neq 0$. For a K^*-module M^* and an integer k, since $\sum_{i=0}^{d-1} M^{i+k}$ generates M^*, the skeletal topology on M^* is trivial. Hence we assume that a field K^* satisfies $K^i = \{0\}$ if $i \neq 0$ when we consider the skeletal topology on K^*-modules.

Definition 1.4.4 *Let A^* be a graded K^*-module or a graded ring.*

(1) We say that A^* is n-connected (resp. n-coconnected) if $A^k = \{0\}$ for $k \leq n$ (resp. $A^k = \{0\}$ for $k \geq n$).

(2) We say that A^* is connective (resp. coconnective) if there exists $n \in \boldsymbol{Z}$ such that A^* is n-connected (resp. n-coconnected). We say that A^* is bounded if A^* is both connective and coconnective.

Proposition 1.4.5 *The skeletal topology on a graded K^*-module M^* is Hausdorff if one of the following conditions is satisfied.*

(i) Both K^* and M^* are connective. (ii) Both K^* and M^* are coconnective. (iii) K^* is bounded.

Proof. Suppose that K^* and M^* are connective (resp. coconnective). Take N such that $K^n = \{0\}$ and $M^n = \{0\}$ if $n < N$ (resp. $n > N$). Then, $K^*M^n \subset \sum_{i \geq N+n} M^i$ (resp. $K^*M^n \subset \sum_{i \leq N+n} M^i$). It follows that, if $m > |N|$, $M^*[m] = \sum_{n \geq m} K^*M^n \subset \sum_{i \geq N+m} M^i$ (resp. $M^*[m] = \sum_{n \leq -m} K^*M^n \subset \sum_{i \leq N-m} M^i$). Hence we have $\bigcap_{m \geq 1} M^*[m] = \{0\}$.

Suppose that K^* is bounded. Take N such that $K^n = \{0\}$ if $|n| > N$. Then, $K^*M^n \subset \sum_{i=n-N}^{n+N} M^i$. It follows that, if $m > N$, $M^*[m] = \sum_{|n| \geq m} K^*M^n \subset \sum_{|i| \geq m-N} M^i$. Hence we also have $\bigcap_{m \geq 1} M^*[m] = \{0\}$ in this case. □

Proposition 1.4.6 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ with skeletal topology. Suppose that one of the conditions of (1.4.5) is satisfied. If N^* is a cofinite submodule of M^*, then there exists a positive integer n satisfying $M^*[n] \subset N^*$. Hence the skeletal topology on M^* is supercofinite in this case.*

Proof. We have a descending chain

$$M^*/N^* \supset (M^*/N^*)[1] \supset (M^*/N^*)[2] \supset \cdots \supset (M^*/N^*)[m] \supset (M^*/N^*)[m+1] \supset \cdots$$

of submodules of M^*/N^*. Since M^*/N^* is artinian, there exists n such that $(M^*/N^*)[n] = (M^*/N^*)[m]$ for any $m \geq n$. On the other hand, since one of the conditions of (1.4.5) is also satisfied for M^*/N^*, M^*/N^* is Hausdorff. Therefore we have $(M^*/N^*)[n] = \bigcap_{m \geq 1} (M^*/N^*)[m] = \{0\}$. This implies $M^*[n] \subset N^*$. □

Proposition 1.4.7 *(1) If K^*M^k is of finite length for every $k \in \boldsymbol{Z}$, then $M^*[n]$ is cofinite for every $n \geq 1$. Hence if M^* is supercofinite, M^* is superskeletal in this case. In other words, the skeletal topology on M^* is subcofinite.*

(2) Suppose that, for each integer k, there exists a positive integer m satisfying $K^*M^k \cap M^*[m] = \{0\}$ (for example, K^* is bounded). If $M^*[n]$ is cofinite for every $n \geq 1$, then K^*M^k is of finite length.

Proof. (1) Since $M^* = \sum_{|k| < n} K^*M^k + M^*[n]$, the map $\sum_{|k| < n} K^*M^k \to M^*/M^*[n]$ induced by the inclusion map is surjective. Hence $M^*/M^*[n]$ is of finite length and $M^*[n]$ is cofinite.

(2) If $K^*M^k \cap M^*[m] = \{0\}$, the composition $K^*M^k \hookrightarrow M^* \to M^*/M^*[m]$ is injective. Hence K^*M^k is of finite length. □

Proposition 1.4.8 *Let $f : M^* \to N^*$ be a homomorphism of $\mathcal{T}op\mathcal{M}od_{K^*}$ and Z^* a cofinite submodule of N^*. Then, $f^{-1}(Z^*)$ is a cofinite submodule of M^*.*

Proof. Since f induces an injective homomorphism $M^*/f^{-1}(Z^*) \to N^*/Z^*$ and N^*/Z^* is of finite length, so is $M^*/f^{-1}(Z^*)$. □

Proposition 1.4.9 *Let K^* be a field and M^* a supercofinite vector space over K^*.*

(1) M^* is Hausdorff and every finite dimensional subspace of M^* is discrete.

(2) Let S^* be a finite dimensional subspace of M^* and $i_{S^*} : S^* \to M^*$ the inclusion map. Then, i_{S^*} is a split monomorphism.

Proof. (1) Let S^* be a finite dimensional subspace of M^*. By (1.1.4), there exists a subspace N^* satisfying $S^* \cap N^* = \{0\}$ and $S^* + N^* = M^*$. Then, N^* is cofinite, hence open. Thus $S^* \cap N^* = \{0\}$ is an open subspace of S^* and S^* is discrete. For non-zero $x = \sum_{i=1}^{m} x_i \in M^*$ ($x_i \in M^{n_i}$), let S^* be the subspace generated by $x_1, x_2, \ldots, x_m$. Then, N^* as above is a neighborhood of zero which does not contain x.

(2) Take N^* as in (1) and define $p : M^* \to S^*$ by $p(x) = \begin{cases} x & x \in S^* \\ 0 & x \in N^* \end{cases}$. Then, $pi_{S^*} = id_{S^*}$. Since $\operatorname{Ker} p = N^*$ is cofinite, p is continuous. □

The product of finite vector spaces is not cofinite in general. A counter example is given as follows. Let K_i be a copy of a ungraded discrete field K for each $i \in \boldsymbol{N}$ and consider the product space $V = \prod_{i \in \boldsymbol{N}} K_i$. We denote by $p_i : V \to K_i = K$ the projection onto the i-th component. Let us denote by S the set of all finite subsets of $\boldsymbol{N}$. Then S is a directed set by $\subset$. For $I \in S$, we put $W_I = \bigcap_{i \in I} \operatorname{Ker} p_i$. Then $\{W_I | I \in S\}$ is a fundamental system of neighborhoods of 0. Let e_i be the element of V whose i-th component is 1 and other components are all 0 and e_∞ the element of V whose components are all 1. Put $\mathcal{B}' = \{e_i | i \in \boldsymbol{N}\} \cup \{e_\infty\}$. Then, $\mathcal{B}'$ is linearly independent. Let $\mathcal{B}$ be a basis of V containing $\mathcal{B}'$ and define a linear map $f : V \to K$ by $f(v) = \begin{cases} 1 & v = e_\infty \\ 0 & v \in \mathcal{B} - \{e_\infty\} \end{cases}$. Since $f(e_i) = 0$ for all $i \in \boldsymbol{N}$, we have $f\left(\sum_{i \in I} e_i\right) = 0$ for any $I \in S$. It is clear that $\left(\sum_{i \in I} e_i\right)_{I \in S}$ converges to e_∞, however, $f(e_\infty) = 1 \neq 0$. Therefore f is not continuous, so a subspace $\operatorname{Ker} f$ of codimension 1 is not open in V. Hence V is not cofinite.

Proposition 1.4.10 *If $(M_i^*)_{i \in I}$ is a family of subcofinite K^*-modules, then $\prod_{i \in I} M_i^*$ is also subcofinite.*

Proof. Let $p_j : \prod_{i \in I} M_i^* \to M_j^*$ be the projection. Since $\left\{ \bigcap_{k=1}^{n} p_{j_k}^{-1}(W_{j_k}^*) \,\middle|\, j_k \in I,\ W_{j_k}^* \in \mathcal{V}_{M_{j_k}^*} \right\}$ is a fundamental system of the neighborhood of 0 and each $\bigcap_{k=1}^{n} p_{j_k}^{-1}(W_{j_k}^*)$ is cofinite, the assertion follows. □

Proposition 1.4.11 *(1) If M^* is a subcofinite K^*-module, then each submodule and quotient module of M^* are subcofinite.*

(2) Suppose that K^* is a field. If M^* is a supercofinite vector space over K^*, then each subspace and quotient space of M^* are supercofinite.

(3) If M^* is subcofinite, then the completion $\widehat{M}^*$ is also subcofinite.

Proof. Let N^* be a submodule of M^* and $p : M^* \to M^*/N^*$ be the quotient map.

(1) Let Z^* be an open submodule of N^*. There exists an open submodule U^* of M^* sucn that $Z^* = U^* \cap N^*$. Since U^* is cofinite and there is an injection $N^*/Z^* \to M^*/U^*$, Z^* is cofinite in N^*. Hence Z^* is subcofinite. Let T^* be an open submodule of M^*/N^*. Then, $p^{-1}(T^*)$ is open, hence cofinite. Since p induces an isomorphism $M^*/p^{-1}(T^*) \to (M^*/N^*)/T^*$, T^* is of finite codimension in M^*/N^*. Hence M^*/N^* is subcofinite.

(2) Let Z^* be a subspace of N^* which is cofinite in N^*. Take a subspace T^* of M^* satisfying $T^* + N^* = M^*$ and $T^* \cap N^* = \{0\}$. Then, the inclusion map $i : N^* \to M^*$ induces a bijection $N^*/Z^* \to M^*/(Z^* + T^*)$. Hence $Z^* + T^*$ is a cofinite subspace of M^*, therefore open. Since $N^* \cap (Z^* + T^*) = Z^*$, Z^* is open in N^*. Therefore N^* is supercofinite. Let T^* be a cofinite subspace of M^*/N^*. Then, by (1.4.8), $p^{-1}(T^*)$ is a cofinite subspace M^*. Hence $p^{-1}(T^*)$ is open and so is T^*. Thus M^*/N^* is supercofinite.

(3) If M^* is subcofinite, then $\widehat{M}^*$ is a submodule of product of finite modules which is subcofinite by (1) of (1.4.10) and the above (1). Hence $\widehat{M}^*$ is subcofinite. □

Proposition 1.4.12 *M^* is subcofinite and Hausdorff if and only if M^* is isomorphic to a submodule of product of finite K^*-modules.*

Proof. Assume that M^* is subcofinite and Hausdorff. For $U^* \in \mathcal{V}_{M^*}$, let $p_{U^*} : M^* \to M^*/U^*$ be the quotient map. The map $f : M^* \to \prod_{U^* \in \mathcal{V}_{M^*}} M^*/U^*$ induced by p_{U^*} is a homeomorphism onto its image by (1) of (1.3.3). Since M^*/U^* is finite for any $U^* \in \mathcal{V}_{M^*}$, M^* is isomorphic to a submodule of product of finite K^*-modules.

Conversely, assume that M^* is isomorphic to a submodule of product $\prod_{i \in I} M_i^*$ of finite K^*-modules M_i^* for $i \in I$. Since each M_i^* is discrete and cofinite, hence Hausdorff, $\prod_{i \in I} M_i^*$ is also Hausdorff and cofinite by (1.4.10). Thus all submodules of $\prod_{i \in I} M_i^*$ are Hausdorff and subcofinite by (1) of (1.4.11). □

Proposition 1.4.13 *Let M^* be a left K^*-module whose topology is coarser than the topology induced by K^*. If K^* is subcofinite and M^*/U^* is finitely generated for every open submodule U^* of M^*, then the topology on M^* is subcofinite.*

Proof. Let U^* be an open submodule of M^*. There exists an open ideal $\mathfrak{a}$ of K^* satisfying $\mathfrak{a}M^* \subset U^*$. Hence M^*/U^* is a module over an artinian ring $K^*/\mathfrak{a}$. Moreover, since M^*/U^* is a finitely generated $K^*/\mathfrak{a}$-module, M^*/U^* is of finite length. □

We also observe the following facts.

Proposition 1.4.14 *Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. If "$M^*$ is supercofinite and N^* is subcofinite" or "M^* is superskeletal and N^* is subskeletal", then every linear map from M^* to N^* preserving degrees is continuous.*

Proposition 1.4.15 *(1) If M^* is subskeletal (resp. superskeletal), then each submodule and quotient module of M^* are subskeletal (resp. superskeletal).*

(2) If M^* is superskeletal and one of the conditions of (1.4.5) is satisfied, M^n is discrete for each $n \in \boldsymbol{Z}$ and M^* is complete Hausdorff .

(3) If both M^* and N^* have the skeletal topologies, then $M^* \oplus N^*$ has the skeletal topology.

Remark 1.4.16 *By giving the cofinite (resp. skeletal) topology to a graded K^*-module, we have a fully faithful functor from the category of graded K^*-modules to $\mathcal{T}op\mathcal{M}od_{K^*}$.*

Definition 1.4.17 *An object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ is said to be profinite if M^* is complete Hausdorff and subcofinite.*

2 Tensor products

2.1 Tensor product of topological modules

For objects M^*, N^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, define a graded K^*-module $M^* \otimes_{K^*} N^*$ as follows. Let $F(M^*, N^*)$ be the free K^*-module generated by $M^* \times N^*$ and $R(M^*, N^*)$ a submodule of $F(M^*, N^*)$ generated by

$$\{(x+y,z)-(x,z)-(y,z)\,|\,x,y \in M^*,\ z \in N^*\} \cup \{(x,z+w)-(x,z)-(x,w)\,|\,x \in M^*,\ z,w \in N^*\} \cup$$

$$\{(rx,z)-r(x,z)\,|\,x \in M^*,\ z \in N^*,\ r \in K^*\} \cup \bigcup_{l,m \in \boldsymbol{Z}} \{(x,rz)-(-1)^{lm}r(x,z)\,\big|\,x \in M^m,\ z \in N^*,\ r \in K^l\}$$

We assign degree $m+n$ to $(x,y) \in F(M^*, N^*)$ if $x \in M^m$ and $y \in N^n$ and denote by $x \otimes y$ the equivalence class of (x,y). If $f : M^* \to P^*$ and $g : N^* \to Q^*$ are morphisms in $\mathcal{T}op\mathcal{M}od_{K^*}$, let $f \otimes g : M^* \otimes_{K^*} N^* \to P^* \otimes_{K^*} Q^*$ be the map induced by the map $F(f,g) : F(M^*, N^*) \to F(P^*, Q^*)$ defined by $F(f,g)(x,y) = (f(x), g(y))$.

For a submodule V^* of M^* and a submodule W^* of N^*, let us denote by $p_{V^*} : M^* \to M^*/V^*$, $q_{W^*} : N^* \to N^*/W^*$ the quotient maps and put

$$o(V^*, W^*) = \mathrm{Ker}(p_{V^*} \otimes q_{W^*} : M^* \otimes_{K^*} N^* \to M^*/V^* \otimes_{K^*} N^*/W^*).$$

We give a topology on $M^* \otimes_{K^*} N^*$ so that $\{o(V^*, W^*)\,|\,V^* \in \mathcal{V}_{M^*},\ W^* \in \mathcal{V}_{N^*}\}$ forms a fundamental system of the neighborhood of 0.

We denote by $i_{V^*} : V^* \to M^*$ and $j_{W^*} : W^* \to N^*$ the inclusion maps. Let

$$k_{V^*,W^*} : (V^* \otimes_{K^*} N^*) \oplus (M^* \otimes_{K^*} W^*) \to M^* \otimes_{K^*} N^*$$

be the map induced by $i_{V^*} \otimes 1 : V^* \otimes_{K^*} N^* \to M^* \otimes_{K^*} N^*$ and $1 \otimes j_{W^*} : M^* \otimes_{K^*} W^* \to M^* \otimes_{K^*} N^*$. Then, the following diagram is exact.

$$(V^* \otimes_{K^*} N^*) \oplus (M^* \otimes_{K^*} W^*) \xrightarrow{k_{V^*,W^*}} M^* \otimes_{K^*} N^* \xrightarrow{p_{V^*} \otimes q_{W^*}} M^*/V^* \otimes_{K^*} N^*/W^* \longrightarrow 0$$

Proposition 2.1.1 *If $f_1 : M_1^* \to N_1^*$ and $f_2 : M_2^* \to N_2^*$ are surjective open maps, so is $f_1 \otimes f_2 : M_1^* \otimes_{K^*} M_2^* \to N_1^* \otimes_{K^*} N_2^*$.*

Proof. For open submodules U_l^* $(l = 1, 2)$ of M_l^*, let $p_{U_l^*} : M_l^* \to M_l^*/U_l^*$ and $q_{f_l(U_l^*)} : N_l^* \to N_l^*/f_l(U_l^*)$ be the quotient maps. We denote by $\bar{f}_l : U_l^* \to f_l(U_l^*)$ the map induced by f_l. Then the vertical maps of the following diagram is surjective.

$$\begin{array}{ccccc}
(U_1^* \otimes_{K^*} M_2^*) \oplus (M_1^* \otimes_{K^*} U_2^*) & \xrightarrow{k_{U_1^*,U_2^*}} & M_1^* \otimes_{K^*} M_2^* & \xrightarrow{p_{U_1^*} \otimes p_{U_2^*}} & M_1^*/U_1^* \otimes_{K^*} M_2^*/U_2^* \\
\downarrow{\scriptstyle (\bar{f}_1 \otimes f_2) \oplus (f_1 \otimes \bar{f}_2)} & & \downarrow{\scriptstyle f_1 \otimes f_2} & & \downarrow \\
(f_1(U_1^*) \otimes_{K^*} N_2^*) \oplus (N_1^* \otimes_{K^*} f_2(U_2^*)) & \xrightarrow{k_{f_1(U_1^*),f_2(U_2^*)}} & N_1^* \otimes_{K^*} N_2^* & \xrightarrow{q_{f_1(U_1^*)} \otimes q_{f_2(U_2^*)}} & N_1^*/f_1(U_1^*) \otimes_{K^*} N_2^*/f_2(U_2^*)
\end{array}$$

Hence $f_1 \otimes f_2$ maps $o(U_1^*, U_2^*)$ onto $o(f_1(U_1^*), f_2(U_2^*))$. □

Since the quotient map $p_{V^*} : M^* \to M^*/V^*$ is a surjective open map if V^* is an open submodule of M^*, the above result implies the following.

Corollary 2.1.2 *If V^* and W^* be open submodules of M^* and N^* respectively, $p_{V^*} \otimes q_{W^*} : M^* \otimes_{K^*} N^* \to M^*/V^* \otimes_{K^*} N^*/W^*$ induces an isomorphism $M^* \otimes_{K^*} N^*/o(V^*, W^*) \to M^*/V^* \otimes_{K^*} N^*/W^*$.*

Proposition 2.1.3 *For a morphism $f : M^* \to N^*$ of $\mathcal{T}op\mathcal{M}od_{K^*}$, let $\pi : N^* \to C^*$ be a cokernel of f. Then, for $L^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{K^*}$, $id_{L^*} \otimes_{K^*} \pi : L^* \otimes_{K^*} N^* \to L^* \otimes_{K^*} C^*$ is a cokernel of $id_{L^*} \otimes_{K^*} f : L^* \otimes_{K^*} M^* \to L^* \otimes_{K^*} N^*$.*

Proof. Since $\pi : N^* \to C^*$ is an open map by (1.1.15), $id_{L^*} \otimes_{K^*} \pi : L^* \otimes_{K^*} N^* \to L^* \otimes_{K^*} C^*$ is also an open map by (2.1.1). Hence $id_{L^*} \otimes_{K^*} \pi$ is a quotient map in $\mathcal{T}op\mathcal{M}od_{K^*}$. □

Proposition 2.1.4 *Let N^* be a dense submodule of M^* and denote by $i : N^* \to M^*$ the inclusion map. For a K^*-module L^*, the image of $id_{L^*} \otimes_{K^*} i : L^* \otimes_{K^*} N^* \to L^* \otimes_{K^*} M^*$ is dense.*

Proof. Let $p : M^* \to M^*/N^*$ be the quotient map. Then, p is a cokernel of $i : N^* \to M^*$ and it follows from (2.1.3) that $id_{L^*} \otimes_{K^*} p : L^* \otimes_{K^*} M^* \to L^* \otimes_{K^*} M^*/N^*$ is a cokernel of $id_{L^*} \otimes_{K^*} i : L^* \otimes_{K^*} N^* \to L^* \otimes_{K^*} M^*$. Since the topology of M^*/N^* is trivial by (1.1.16), so is $L^* \otimes_{K^*} M^*/N^*$. Hence $\mathrm{Im}\,(id_{L^*} \otimes_{K^*} i) = \mathrm{Ker}\,(id_{L^*} \otimes_{K^*} p)$ is dense by (1.1.16). □

Proposition 2.1.5 *Let K^* be a field. If a morphism $f : M^* \to N^*$ of $\mathcal{T}op\mathcal{M}od_{K^*}$ is an open map into its image, so is $id_{L^*} \otimes_{K^*} f : L^* \otimes_{K^*} M^* \to L^* \otimes_{K^*} N^*$ for any object L^* of $\mathcal{T}op\mathcal{M}od_{K^*}$. In particular, if f is an isomorphism onto its image, so is $id_{L^*} \otimes_{K^*} f : L^* \otimes_{K^*} M^* \to L^* \otimes_{K^*} N^*$.*

Proof. Let U^* and V^* be open submodules of L^* and M^*, respectively. By the assumption, there exists an open submodule W^* of N^* which satisfies $f(V^*) = W^* \cap f(M^*)$. Let $\{u_i\}_{i \in I_1 \cup I_2}$ be basis of L^* such that $\{u_i\}_{i \in I_1}$ is a basis of U^*. We choose a basis $\{v_j\}_{j \in J_1 \cup J_2}$ of $\mathrm{Ker}\, f$ such that $\{v_j\}_{j \in J_1}$ is a basis of $\mathrm{Ker}\, f \cap V^*$ and choose a family of elements $\{v_j\}_{j \in J_3}$ of V^* and a family of elements $\{v_j\}_{j \in J_4}$ of M^* such that $\{f(v_j)\}_{j \in J_3}$ is a basis of $f(V^*) = W^* \cap f(M^*)$ and $\{f(v_j)\}_{j \in J_3 \cup J_4}$ is a basis of $f(M^*)$ and Then, $\{v_j\}_{j \in J_1 \cup J_3}$ is a basis of V^* and $\{v_j\}_{j \in J_1 \cup J_2 \cup J_3 \cup J_4}$ is a basis of M^*. Finally, choose a family of elements $\{w_k\}_{k \in K_1 \cup K_2}$ of N^* so that $\{f(v_j)\}_{j \in J_3} \cup \{w_k\}_{k \in K_1}$ is a basis of W^* and $\{f(v_j)\}_{j \in J_3 \cup J_4} \cup \{w_k\}_{k \in K_1 \cup K_2}$ is a basis of N^*. It follows from that $\{u_i \otimes v_j\}_{i \in I_1 \cup I_2,\, j \in J_1 \cup J_2 \cup J_3 \cup J_4}$ is a basis of $L^* \otimes_{K^*} M^*$, hence $\{u_i \otimes f(v_j)\}_{i \in I_1 \cup I_2,\, j \in J_3 \cup J_4}$ is a basis of $\mathrm{Im}\,(id_* \otimes_{K^*} f)$. We also have a basis $\{u_i \otimes v_j\}_{i \in I_1 \cup I_2,\, j \in J_1 \cup J_3} \cup \{u_i \otimes v_j\}_{i \in I_1,\, j \in J_2 \cup J_4}$ of $o(U^*, V^*)$ and a basis $\{u_i \otimes f(v_j)\}_{i \in I_1,\, j \in J_3 \cup J_4} \cup \{u_i \otimes f(v_j)\}_{i \in I_2,\, j \in J_3} \cup \{u_i \otimes w_k\}_{i \in I_1,\, k \in K_1 \cup K_2}$ of $o(U^*, W^*)$. Therefore,

$$\{u_i \otimes f(v_j)\}_{i \in I_1 \cup I_2,\, j \in J_3} \cup \{u_i \otimes f(v_j)\}_{i \in I_1,\, j \in J_4} \quad \text{and} \quad \{u_i \otimes f(v_j)\}_{i \in I_1,\, j \in J_3 \cup J_4} \cup \{u_i \otimes f(v_j)\}_{i \in I_2,\, j \in J_3}$$

are basis of $(id_{L^*} \otimes_{K^*} f)(o(U^*, V^*))$ and $o(U^*, W^*) \cap \mathrm{Im}\,(id_* \otimes_{K^*} f)$, respectively. The above basis are identical and we have $(id_{L^*} \otimes_{K^*} f)(o(U^*, V^*)) = o(U^*, W^*) \cap \mathrm{Im}\,(id_{L^*} \otimes_{K^*} f)$ which implies that $id_{L^*} \otimes_{K^*} f$ is an open map into its image. □

Proposition 2.1.6 *Let us define a map $\beta_{M^*,N^*} : M^* \times N^* \to M^* \otimes_{K^*} N^*$ by $\beta_{M^*,N^*}(x, y) = x \otimes y$. Then, for a morphism $f : M^* \otimes_{K^*} N^* \to L^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$, a composition $f\beta_{M^*,N^*} : M^* \times N^* \to L^*$ is a strongly continuous bilinear map. For a strongly continuous bilinear map $B : M^* \times N^* \to L^*$, there exists unique morphism $\widetilde{B} : M^* \otimes_{K^*} N^* \to L^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$ satisfying $\widetilde{B}\beta_{M^*,N^*} = B$.*

Proof. For $U^* \in \mathcal{V}_{L^*}$, there exist $V^* \in \mathcal{V}_{M^*}$ and $W^* \in \mathcal{V}_{N^*}$ such that $f(\mathrm{Ker}(p_{V^*} \otimes q_{W^*})) \subset U^*$ by the continuity of f. Since β_{M^*,N^*} maps both $V^* \times M^*$ and $L^* \times W^*$ to $\mathrm{Ker}(p_{V^*} \otimes q_{W^*})$, it follows that β_{M^*,N^*} is strongly continuous. It is clear that there exists a unique map of K^*-module $\widetilde{B} : M^* \otimes_{K^*} N^* \to L^*$ of satisfying $\widetilde{B}\beta_{M^*,N^*} = B$. For $U^* \in \mathcal{V}_{L^*}$, there exist $V^* \in \mathcal{V}_{M^*}$ and $W^* \in \mathcal{V}_{N^*}$ such that $B(V^* \times M^*) \subset U^*$ and $B(L^* \times W^*) \subset U^*$. Thus we have $\widetilde{B}\beta_{M^*,N^*}(V^* \times M^*) \subset U^*$ and $\widetilde{B}\beta_{M^*,N^*}(L^* \times W^*) \subset U^*$. Since $\mathrm{Ker}(p_{V^*} \otimes q_{W^*})$ is generated by $\beta_{M^*,N^*}(V^* \times M^*)$ and $\beta_{M^*,N^*}(L^* \times W^*)$, it follows that $\widetilde{B}$ maps $\mathrm{Ker}(p_{V^*} \otimes q_{W^*})$ into U^*. Therefore $\widetilde{B}$ is continuous. □

Proposition 2.1.7 *If M^* or N^* has a topology coarser than the topology induced by K^*, the topology on $M^* \otimes_{K^*} N^*$ is coarser than the topology induced by K^*.*

Proof. Suppose that M^* has a topology coarser than the topology induced by K^*. For $V^* \in \mathcal{V}_{M^*}$, there exist an open ideal $\mathfrak{a}$ of K^* satisfying $\mathfrak{a}M^* \subset V^*$. Let $i : \mathfrak{a}M^* \to M^*$ be the inclusion map. Since $\mathrm{Im}(i \otimes id_{N^*} : (\mathfrak{a}M^*) \otimes_{K^*} N^* \to M^* \otimes_{K^*} N^*) = \mathfrak{a}(M^* \otimes_{K^*} N^*)$ and $p_{V^*}i = 0$, $\mathfrak{a}(M^* \otimes_{K^*} N^*)$ is contained in $\mathrm{Ker}(p_{V^*} \otimes q_{W^*})$ for any $W^* \in \mathcal{V}_{N^*}$. □

Proposition 2.1.8 *M^* has a topology coarser than the topology induced by K^* if and only if there is an isomorphism $K^* \otimes_{K^*} M^* \to M^*$.*

Proof. Since the topology on K^* is coarser than the topology induced by K^*, so is $K^* \otimes_{K^*} M^*$ by (2.1.7). Suppose that M^* has a topology coarser than the topology induced by K^*. Then, the K^*-module structure map $\alpha : K^* \times M^* \to M^*$ of M^* is strongly continuous by (1.1.10) and it follows from (2.1.6) that α induces $\tilde{\alpha} : K^* \otimes_{K^*} M^* \to M^*$ satisfying $\tilde{\alpha}\beta_{K^*,M^*} = \alpha$. $\tilde{\alpha}$ is an isomorphism whose inverse is given by $x \mapsto 1 \otimes x$. □

The following assertion is clear.

Proposition 2.1.9 *Let A^* be an object of $\mathcal{T}op\mathcal{A}lg_{K^*}$. Suppose that an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ has a structure of A^*-module with structure map $\alpha : A^* \times M^* \to M^*$. For a subset S of M^*, we denote by A^*S the A^*-submodule of M^* generated by S. Put $\mathcal{V}_{M^*}^{\alpha} = \{A^*U^* \,|\, U^* \in \mathcal{V}_{M^*}\}$. Then α is strongly continuous if and only if $\mathcal{V}_{M^*}^{\alpha}$ is cofinal subset of $\mathcal{V}_{M^*}$ and the topology of M^* is coarser than the topology induced by A^*. Moreover, if α is strongly continuous, the map $\tilde{\alpha} : A^* \otimes_{K^*} M^* \to M^*$ induced by α induces an isomorphism $A^* \otimes_{A^*} M^* \to M^*$.*

Let $\overline{T}_{M^*,N^*} : M^* \times N^* \to N^* \otimes_{K^*} M^*$ be a bilinear map defined by $\overline{T}_{M^*,N^*}(x, y) = (-1)^{mn} y \otimes x$ for $x \in M^m$, $y \in N^n$. By (2.1.6), there is a unique morphism $T_{M^*,N^*} : M^* \otimes_{K^*} N^* \to N^* \otimes_{K^*} M^*$ satisfying $T_{M^*,N^*}\beta_{M^*,N^*} = \overline{T}_{M^*,N^*}$.

Proposition 2.1.10 *Let A^* be an object of $\mathcal{T}op\mathcal{A}lg_{K^*}$. Suppose that M^* and N^* have structures of A^*-modules with structure maps $\alpha_{M^*} : A^* \times M^* \to M^*$, $\alpha_{N^*} : A^* \times N^* \to N^*$ which are both strongly continuous. Let $\tilde{\alpha}_{M^*} : A^* \otimes_{K^*} M^* \to M^*$ and $\tilde{\alpha}_{N^*} : A^* \otimes_{K^*} N^* \to N^*$ be the maps induced by α_{M^*} and α_{N^*} that exist by (2.1.6). Note that both M^* and N^* have fundamental systems of neighborhoods of 0 which consist of open A^*-submodules. Then, $M^* \otimes_{A^*} N^*$ is a cokernel of a map*

$$(\tilde{\alpha}_{M^*} T_{M^*,A^*}) \otimes id_{N^*} - id_{M^*} \otimes \tilde{\alpha}_{N^*} : M^* \otimes_{K^*} A^* \otimes_{K^*} N^* \to M^* \otimes_{K^*} N^*$$

in $\mathcal{T}op\mathcal{M}od_{K^*}$.

Proof. Let us denote by $\pi : M^* \otimes_{K^*} N^* \to M^* \otimes_{A^*} N^*$ the quotient map. Suppose that L^* is a submodule of $M^* \otimes_{A^*} N^*$ such that $\pi^{-1}(L^*)$ is an open submodule of $M^* \otimes_{K^*} N^*$. Then, there exist $V^* \in \mathcal{V}_{M^*}$ and $W^* \in \mathcal{V}_{N^*}$ such that V^* (resp. W^*) is an A^*-submodule of M^* (resp. N^*) and $\mathrm{Ker}(p_{V^*} \otimes q_{W^*} : M^* \otimes_{K^*} N^* \to M^*/V^* \otimes_{K^*} N^*/W^*) \subset \pi^{-1}(L^*)$. Since $\mathrm{Ker}(p_{V^*} \otimes q_{W^*} : M^* \otimes_{K^*} N^* \to M^*/V^* \otimes_{K^*} N^*/W^*)$ is generated by the images of maps $V^* \otimes_{K^*} N^* \to M^* \otimes_{K^*} N^*$, $M^* \otimes_{K^*} W^* \to M^* \otimes_{K^*} N^*$ induced by the inclusion maps, π maps $\mathrm{Ker}(p_{V^*} \otimes q_{W^*} : M^* \otimes_{K^*} N^* \to M^*/V^* \otimes_{K^*} N^*/W^*)$ onto $\mathrm{Ker}(p_{V^*} \otimes q_{W^*} : M^* \otimes_{A^*} N^* \to M^*/V^* \otimes_{A^*} N^*/W^*)$. Hence $\mathrm{Ker}(p_{V^*} \otimes q_{W^*} : M^* \otimes_{A^*} N^* \to M^*/V^* \otimes_{A^*} N^*/W^*) \subset L^*$ and L^* is open. Thus π is a topological quotient map. □

Proposition 2.1.11 *Under the situation of (2.1.10), let $\rho : M^* \otimes_{K^*} N^* \to M^* \otimes_{A^*} N^*$ be the quotient map. There exists unique map $\tilde{\beta} : A^* \otimes_{K^*} M^* \otimes_{A^*} N^* \to M^* \otimes_{A^*} N^*$ that makes the following diagram commute.*

$$\begin{tikzcd}
A^* \otimes_{K^*} M^* \otimes_{K^*} N^* \arrow[r, "\tilde{\alpha}_{M^*} \otimes_{K^*} id_{N^*}"] \arrow[d, "id_{A^*} \otimes_{K^*} \rho"] & M^* \otimes_{K^*} N^* \arrow[d, "\rho"] \\
A^* \otimes_{K^*} M^* \otimes_{A^*} N^* \arrow[r, "\tilde{\beta}"] & M^* \otimes_{A^*} N^*
\end{tikzcd}$$

Proof. Since ρ is a cokernel of $(\tilde{\alpha}_{M^*} T_{M^*,A^*}) \otimes_{K^*} id_{N^*} - id_{M^*} \otimes_{K^*} \tilde{\alpha}_{N^*} : M^* \otimes_{K^*} A^* \otimes_{K^*} N^* \to M^* \otimes_{K^*} N^*$, it follows from (2.1.3) that $id_{A^*} \otimes_{K^*} \rho : A^* \otimes_{K^*} M^* \otimes_{K^*} N^* \to A^* \otimes_{K^*} M^* \otimes_{A^*} N^*$ is a cokernel of

$$id_{A^*} \otimes_{K^*} (\tilde{\alpha}_{M^*} T_{M^*,A^*}) \otimes_{K^*} id_{N^*} - id_{A^*} \otimes_{K^*} id_{M^*} \otimes_{K^*} \tilde{\alpha}_{N^*} : A^* \otimes_{K^*} M^* \otimes_{K^*} A^* \otimes_{K^*} N^* \to A^* \otimes_{K^*} M^* \otimes_{K^*} N^*.$$

Let us denote by $\mu : A^* \otimes_{K^*} A^* \to A^*$ the product of A^*. Since $\rho(id_{M^*} \otimes_{K^*} \tilde{\alpha}_{N^*}) = \rho((\tilde{\alpha}_{M^*} T_{M^*,A^*}) \otimes_{K^*} id_{N^*})$, we have

$$\begin{aligned}
\rho(\tilde{\alpha}_{M^*} \otimes_{K^*} id_{N^*})(id_{A^*} \otimes_{K^*} id_{M^*} \otimes_{K^*} \tilde{\alpha}_{N^*}) &= \rho(id_{M^*} \otimes_{K^*} \tilde{\alpha}_{N^*})(T_{A^*,M^*} \otimes_{K^*} id_{N^*})(id_{A^*} \otimes_{K^*} id_{M^*} \otimes_{K^*} \tilde{\alpha}_{N^*}) \\
&= \rho(id_{M^*} \otimes_{K^*} \tilde{\alpha}_{N^*})(id_{M^*} \otimes_{K^*} id_{A^*} \otimes_{K^*} \tilde{\alpha}_{N^*})(T_{A^*,M^*} \otimes_{K^*} id_{A^*} \otimes_{K^*} id_{N^*}) \\
&= \rho(id_{M^*} \otimes_{K^*} \tilde{\alpha}_{N^*}(id_{A^*} \otimes_{K^*} \tilde{\alpha}_{N^*}))(T_{A^*,M^*} \otimes_{K^*} id_{A^*} \otimes_{K^*} id_{N^*}) \\
&= \rho(id_{M^*} \otimes_{K^*} \tilde{\alpha}_{N^*}(\mu \otimes_{K^*} id_{N^*}))(T_{A^*,M^*} \otimes_{K^*} id_{A^*} \otimes_{K^*} id_{N^*}) \\
&= \rho(id_{M^*} \otimes_{K^*} \tilde{\alpha}_{N^*})(id_{M^*} \otimes_{K^*} \mu \otimes_{K^*} id_{N^*})(T_{A^*,M^*} \otimes_{K^*} id_{A^*} \otimes_{K^*} id_{N^*}) \\
&= \rho((\tilde{\alpha}_{M^*} T_{M^*,A^*}) \otimes_{K^*} id_{N^*})(id_{M^*} \otimes_{K^*} \mu \otimes_{K^*} id_{N^*})(T_{A^*,M^*} \otimes_{K^*} id_{A^*} \otimes_{K^*} id_{N^*}) \\
&= \rho(\tilde{\alpha}_{M^*} \otimes_{K^*} id_{N^*})(T_{M^*,A^*} \otimes_{K^*} id_{N^*})(id_{M^*} \otimes_{K^*} \mu \otimes_{K^*} id_{N^*})(T_{A^*,M^*} \otimes_{K^*} id_{A^*} \otimes_{K^*} id_{N^*}) \\
&= \rho(\tilde{\alpha}_{M^*} \otimes_{K^*} id_{N^*})(\mu \otimes_{K^*} id_{M^*} \otimes_{K^*} id_{N^*})(T_{M^*,A^* \otimes_{K^*} A^*} \otimes_{K^*} id_{N^*})(T_{A^*,M^*} \otimes_{K^*} id_{A^*} \otimes_{K^*} id_{N^*}) \\
&= \rho(\tilde{\alpha}_{M^*}(\mu \otimes_{K^*} id_{M^*}) \otimes_{K^*} id_{N^*})(id_{A^*} \otimes_{K^*} T_{M^*,A^*} \otimes_{K^*} id_{N^*}) \\
&= \rho(\tilde{\alpha}_{M^*}(id_{A^*} \otimes_{K^*} \tilde{\alpha}_{M^*}) \otimes_{K^*} id_{N^*})(id_{A^*} \otimes_{K^*} T_{M^*,A^*} \otimes_{K^*} id_{N^*}) \\
&= \rho(\tilde{\alpha}_{M^*} \otimes_{K^*} id_{N^*})(id_{A^*} \otimes_{K^*} \tilde{\alpha}_{M^*} \otimes_{K^*} id_{N^*})(id_{A^*} \otimes_{K^*} T_{M^*,A^*} \otimes_{K^*} id_{N^*}) \\
&= \rho(\tilde{\alpha}_{M^*} \otimes_{K^*} id_{N^*})(id_{A^*} \otimes_{K^*} (\tilde{\alpha}_{M^*} T_{M^*,A^*}) \otimes_{K^*} id_{N^*}).
\end{aligned}$$

It follows that there exists unique map $\tilde{\beta} : A^* \otimes_{K^*} M^* \otimes_{A^*} N^* \to M^* \otimes_{A^*} N^*$ that makes the diagram in the assertion commute. □

Remark 2.1.12 *It is easy to verify that the above $\tilde{\beta}$ defines a left A^*-module structure of $M^* \otimes_{A^*} N^*$. We note that since the following diagram commutes, left A^*-module structure $\tilde{\alpha}_{N^*}$ defines the same left A^*-module structure on $M^* \otimes_{A^*} N^*$ as $\tilde{\beta}$.*

$$\begin{tikzcd}
A^* \otimes_{K^*} M^* \otimes_{K^*} N^* \arrow[r, "\tilde{\alpha}_{M^*} \otimes_{K^*} id_{N^*}"] \arrow[d, "T_{A^*, M^*} \otimes_{K^*} id_{N^*}"] & M^* \otimes_{K^*} N^* \arrow[r, "\rho"] & M^* \otimes_{A^*} N^* \\
M^* \otimes_{K^*} A^* \otimes_{K^*} N^* \arrow[r, "id_{M^*} \otimes_{K^*} \tilde{\alpha}_{N^*}"] & M^* \otimes_{K^*} N^* \arrow[ur, "\rho"] &
\end{tikzcd}$$

Proposition 2.1.13 *For $M_1^*, M_2^*, N^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{K^*}$, we denote by $\iota_s : M_s^* \to M_1^* \oplus M_2^*$, $\pi_s : M_1^* \oplus M_2^* \to M_s^*$, $i_s : M_s^* \otimes_{K^*} N^* \to (M_1^* \otimes_{K^*} N^*) \oplus (M_2^* \otimes_{K^*} N^*)$, $p_s : (M_1^* \otimes_{K^*} N^*) \oplus (M_2^* \otimes_{K^*} N^*) \to M_s^* \otimes_{K^*} N^*$ $(s = 1, 2)$ the canonical morphisms. Then, the unique morphism $\psi : (M_1^* \otimes_{K^*} N^*) \oplus (M_2^* \otimes_{K^*} N^*) \to (M_1^* \oplus M_2^*) \otimes_{K^*} N^*$ satisfying $\psi i_s = \iota_s \otimes id_{N^*}$ for $s = 1, 2$ is an isomorphism in $\mathcal{T}op\mathcal{M}od_{K^*}$.*

Proof. We put $j_s = \iota_s \otimes id_{N^*}$ and $q_s = \pi_s \otimes id_{N^*}$ for $s = 1, 2$. Then, it is easy to verify that $q_1 j_1 = id_{M^*}$, $q_2 j_2 = id_{N^*}$ and $j_1 q_1 + j_2 q_2 = id_{L^*}$ are satisfied. Thus the assertion follows from (1.1.21). □

For an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, define a morphism $s^m_{M^*} : \Sigma^m M^* \to (\Sigma^m K^*) \otimes_{K^*} M^*$ by $s^m_{M^*}([m], x) = \beta_{\Sigma^m K^*, M^*}(([m], 1), x)$ for $x \in M^{i-m}$. It follows from (1.2.2) that $s^m_{M^*}$ is a homomorphism of K^*-modules. We note that $s^m_{M^*}$ is a natural isomorphism if and only if the topology on M^* is coarser than the topology induced by K^*.

For objects M^* and N^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, define a morphism $\tau^{m,n}_{M^*,N^*} : \Sigma^m M^* \otimes_{K^*} \Sigma^n N^* \to \Sigma^{m+n}(M^* \otimes_{K^*} N^*)$ as follows. Define $\tilde{\tau}^{m,n}_{M^*,N^*} : \Sigma^m M^* \times \Sigma^n N^* \to \Sigma^{m+n}(M^* \otimes_{K^*} N^*)$ by

$$\tilde{\tau}^{m,n}_{M^*,N^*}(([m], x), ([n], y)) = \Big([m+n], (-1)^{n(i-m)} \beta_{M^*,N^*}(x, y)\Big)$$

for $(x, y) \in M^{i-m} \times N^{j-n}$. Then, it is easy to verify that $\tilde{\tau}^{m,n}_{M^*,N^*}$ is bilinear and strongly continuous. Let $\tau^{m,n}_{M^*,N^*}$ be the unique morphism satisfying $\tau^{m,n}_{M^*,N^*} \beta_{\Sigma^m M^*, \Sigma^n N^*} = \tilde{\tau}^{m,n}_{M^*,N^*}$. Clearly, $\tau^{m,n}_{M^*,N^*}$ is a natural isomorphism.

Proposition 2.1.14 *The following diagrams commute.*

$$\begin{tikzcd}
\Sigma^m M^* \otimes_{K^*} \Sigma^n N^* \arrow[r, "s^m_{M^*} \otimes s^n_{N^*}"] \arrow[d, "\tau^{m,n}_{M^*,N^*}"] & (\Sigma^m K^*) \otimes_{K^*} M^* \otimes_{K^*} (\Sigma^n K^*) \otimes_{K^*} N^* \arrow[d, "1 \otimes T_{M^*, \Sigma^n K^*} \otimes 1"] \\
\Sigma^{m+n}(M^* \otimes_{K^*} N^*) \arrow[d, "s^{m+n}_{M^* \otimes_{K^*} N^*}"] & (\Sigma^m K^*) \otimes_{K^*} (\Sigma^n K^*) \otimes_{K^*} M^* \otimes_{K^*} N^* \\
(\Sigma^{m+n} K^*) \otimes_{K^*} M^* \otimes_{K^*} N^* \arrow[r, "\varepsilon_{m,n,K^*} \otimes 1 \otimes 1"] & \Sigma^m(\Sigma^n K^*) \otimes_{K^*} M^* \otimes_{K^*} N^* \arrow[u, "s^m_{\Sigma^n K^*} \otimes 1 \otimes 1"']
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma^m M^* \otimes_{K^*} \Sigma^n N^* \arrow[r, "\tau^{m,n}_{M^*,N^*}"] \arrow[d, "(-1)^{mn} T_{\Sigma^m M^*, \Sigma^n N^*}"] & \Sigma^{m+n}(M^* \otimes_{K^*} N^*) \arrow[d, "\Sigma^{m+n} T_{M^*,N^*}"] \\
\Sigma^n N^* \otimes_{K^*} \Sigma^m M^* \arrow[r, "\tau^{n,m}_{N^*,M^*}"] & \Sigma^{m+n}(N^* \otimes_{K^*} M^*)
\end{tikzcd}$$

Lemma 2.1.15 *If M^* and N^* are K^*-modules of finite length, so is $M^* \otimes_{K^*} N^*$.*

Proof. Let $\{0\} = M_0^* \subset M_1^* \subset \cdots \subset M_l^* = M^*$ be a composition series. We show that $M_i^* \otimes_{K^*} N^*$ is a K^*-module of finite length by induction on i. The assertion is trivial if $i = 0$. Assume that $M_{i-1}^* \otimes_{K^*} N^*$ is of finite length. Since M_i^*/M_{i-1}^* is a simple K^*-module, it is isomorphic to $\Sigma^m K^*/\mathfrak{m}$ for some $m \in \boldsymbol{Z}$ and a maximal ideal $\mathfrak{m}$ of K^*. Then $M_i^*/M_{i-1}^* \otimes_{K^*} N^*$ is a finite dimensional vector space over $K^*/\mathfrak{m}$ and it follows that $M_i^*/M_{i-1}^* \otimes_{K^*} N^*$ is a K^*-module of finite length. Let $j : M_{i-1}^* \to M_i^*$ be the inclusion map and $p : M_i^* \to M_i^*/M_{i-1}^*$ the quotient map. By the exactness of

$$M_{i-1}^* \otimes_{K^*} N^* \xrightarrow{j \otimes 1} M_i^* \otimes_{K^*} N^* \xrightarrow{p \otimes 1} M_i^*/M_{i-1}^* \otimes_{K^*} N^* \longrightarrow 0,$$

$M_i^* \otimes_{K^*} N^*$ is a K^*-module of finite length. □

Proposition 2.1.16 *Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. If M^* and N^* are subcofinite, $M^* \otimes_{K^*} N^*$ is also subcofinite.*

Proof. Let V^* and W^* be open submodues of M^* and N^*, respectively. Then $M^* \otimes_{K^*} N^*/\mathrm{Ker}(p_{V^*} \otimes p_{W^*})$ is isomorphic to $M^*/V^* \otimes_{K^*} N^*/W^*$ by (2.1.1). Since M^* and N^* are subcofinite, V^* and W^* are cofinite. Hence $M^*/V^* \otimes_{K^*} N^*/W^*$ is a K^*-module of finite length by (2.1.15) and it follows that $\mathrm{Ker}(p_{V^*} \otimes p_{W^*})$ is cofinite. □

Proposition 2.1.17 *Let $f : M^* \to N^*$ be a morphism of $\mathcal{T}op\mathcal{M}od_{K^*}$ and L^* an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ such that there exists a cofinal subset $\mathcal{U}$ of $\mathcal{V}_{L^*}$ such that L^*/T^* is a flat K^*-module for any $T^* \in \mathcal{U}$. Suppose that $\{f^{-1}(V^*) \,|\, V^* \in \mathcal{V}_{N^*}\}$ is a cofinal subset of $\mathcal{V}_{M^*}$. Then, $\{(f \otimes_{K^*} id_{L^*})^{-1}(W^*) \,|\, W^* \in \mathcal{V}_{N^* \otimes_{K^*} L^*}\}$ is a cofinal subset of $\mathcal{V}_{M^* \otimes_{K^*} L^*}$.*

Proof. For $Z^* \in \mathcal{V}_{M^* \otimes_{K^*} L^*}$, we take $V^* \in \mathcal{V}_{N^*}$ and $T^* \in \mathcal{U}$ satisfying $Z^* \supset \mathrm{Ker}(p_{f^{-1}(V^*)} \otimes_{K^*} q_{T^*})$. Since the map $\bar{f} : M^*/f^{-1}(V^*) \to N^*/V^*$ induced by f is injective, the lower horizontal map of the following diagram is injective by the assumption.

$$\begin{tikzcd}
M^* \otimes_{K^*} L^* \arrow[rr, "f \otimes_{K^*} id_{L^*}"] \arrow[d, "p_{f^{-1}(V^*)} \otimes_{K^*} q_{T^*}"] & & N^* \otimes_{K^*} L^* \arrow[d, "p'_{V^*} \otimes_{K^*} q_{T^*}"] \\
M^*/f^{-1}(V^*) \otimes_{K^*} L^*/T^* \arrow[rr, "\bar{f} \otimes_{K^*} id_{L^*}"] & & N^*/V^* \otimes_{K^*} L^*/T^*
\end{tikzcd}$$

Hence $Z^* \supset \mathrm{Ker}\,(p_{f^{-1}(V^*)} \otimes_{K^*} q_{T^*}) = \mathrm{Ker}(f \otimes_{K^*} id_{L^*})(p'_{V^*} \otimes_{K^*} q_{T^*}) = (f \otimes_{K^*} id_{L^*})^{-1}(\mathrm{Ker}(p'_{V^*} \otimes_{K^*} q_{T^*}))$. □

Lemma 2.1.18 *Suppose that K^* is a field. Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. For non-zero $w \in N^n$, suppose that the subspace of N^* spanned by w is discrete (say, N^* is Hausdorff for example.). Let $i : \Sigma^n K^* \to N^*$ be the map defined by $i([n], a) = c^n_{K^*}(a)w$. Then, $1 \otimes i : M^* \otimes_{K^*} \Sigma^n K^* \to M^* \otimes_{K^*} N^*$ is a homeomorphism onto its image*

Proof. By the assumption, i is a homeomorphism onto its image. It follows from (2.1.17) that $1 \otimes_{K^*} i : M^* \otimes \Sigma^n K^* \to M^* \otimes_{K^*} N^*$ is a homeomorphism onto its image. □

Proposition 2.1.19 *Let K^* be a field and M^*, N^* objects of $\mathcal{T}op\mathcal{M}od_{K^*}$.*

(1) If M^* and N^* are Hausdorff, so is $M^* \otimes_{K^*} N^*$.

(2) If both M^* and N^* have non-trivial open subspaces and $M^* \otimes_{K^*} N^*$ is subcofinite, so are M^* and N^*.

(3) If N^* contains a one dimensional discrete subspace and $M^* \otimes_{K^*} N^*$ is supercofinite (resp. subskeletal), then M^* is supercofinite (superskeletal). Hence if both M^* and N^* are non-trivial Hausdorff spaces and $M^* \otimes_{K^*} N^*$ is supercofinite (resp. subskeletal), then both M^* and N^* are supercofinite (superskeletal).

Proof. (1) Suppose that $z \in M^* \otimes_{K^*} N^*$ is not zero and $z = \sum_{i=1}^{n} x_i \otimes y_i$ for $x_i \in M^*$, $y_i \in N^*$. Let Z_1^* and Z_2^* be the subspaces of M^* and N^* spanned by $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$, respectively. Since Z_1^* and Z_2^* are finite dimensional, they are discrete by (1.1.13). Hence there exist $U_1^* \in \mathcal{V}_{M^*}$ and $U_2^* \in \mathcal{V}_{N^*}$ satisfying $U_1^* \cap Z_1^* = U_2^* \cap Z_2^* = \{0\}$. Let us denote by $i_{Z_1^*} : Z_1^* \to M^*$, $j_{Z_2^*} : Z_2^* \to N^*$ the inclusion maps and $p_{U_1^*} : M^* \to M^*/U_1^*$, $q_{U_2^*} : N^* \to N^*/U_2^*$ the quotient maps. Since $p_{U_1^*} i_{Z_1^*} : Z_1^* \to M^*/U_1^*$ and $q_{U_2^*} j_{Z_2^*} : Z_2^* \to M^*/U_2^*$ injective and K^* is a field, the composition of $i_{Z_1^*} \otimes j_{Z_2^*} : Z_1^* \otimes_{K^*} Z_2^* \to M^* \otimes_{K^*} N^*$ and $p_{U_1^*} \otimes q_{U_2^*} : M^* \otimes_{K^*} N^* \to M^*/U_1^* \otimes_{K^*} N^*/U_2^*$ is injective. This shows that $z \in Z_1^* \otimes_{K^*} Z_2^*$ is not contained in an open subspace $\mathrm{Ker}(p_{U_1^*} \otimes q_{U_2^*})$ of $M^* \otimes_{K^*} N^*$.

(2) Let V^* and W^* be open subspaces of M^* and N^*, respectively. By the assumption and (2.1.2), $M^*/V^* \otimes_{K^*} N^*/W^*$ is a finite dimensional vector space over K^* and this implies that M^*/V^* and N^*/W^* are also finite dimensional if $V^* \neq M^*$ and $W^* \neq N^*$.

(3) Take non-zero $w \in N^n$ which spans a discrete subspace and let $i : \Sigma^n K^* \to N^*$ be the map defined by $i([n], a) = c^n_{K^*}(a)w$. It follows from (2.1.18) that $M^* \otimes_{K^*} \Sigma^n K^*$ which is homeomorphic to M^*. Hence the first assertion follows from (1) of (1.4.11) (resp. (1) of (1.4.15)). If both M^* and N^* are non-trivial Hausdorff spaces, both of them contain one dimensional discrete subspaces by (1.1.13) thus the second assertion follows. □

Proposition 2.1.20 *Let K^* be an topological ring and M^*, N^* objects of $\mathcal{T}op\mathcal{M}od_{K^*}$.*

(1) If both M^* and N^* are subskeletal, so is $M^* \otimes_{K^*} N^*$. Conversely, if K^* is a field and both M^* and N^* have non-trivial open sets and $M^* \otimes_{K^*} N^*$ is subskeletal, then both M^* and N^* are subskeletal.

(2) Suppose that both M^* and N^* have the skeletal topology. If K^*, M^* and N^* are all connective or all coconnecctive, then $M^* \otimes_{K^*} N^*$ has the skeletal topology.

Proof. For integers $l, m \geq 0$, we denote by $p_l : M^* \to M^*/M^*[l]$ and $q_m : N^* \to N^*/N^*[m]$ the quotient maps.

(1) Since $(M^* \otimes_{K^*} N^*)[m+l] \subset \mathrm{Ker}(p_l \otimes q_m)$, $M^* \otimes_{K^*} N^*$ is subskeletal if M^* and N^* are so. Assume that K^* is a field and that $M^* \otimes_{K^*} N^*$ is subskeletal. Let S^* and T^* be open subspaces of M^* and N^*, respectively such that $S^* \neq M^*$ and $T^* \neq N^*$. Let us denote by $p_{S^*} : M^* \to M^*/S^*$, $q_{T^*} : N^* \to N^*/T^*$ the quotient maps. Then, there exists a non-negative integer n such that $(M^* \otimes_{K^*} N^*)[n] \subset \mathrm{Ker}(p_{S^*} \otimes q_{T^*})$. It follows that $p_{S^*} \otimes q_{T^*} : M^* \otimes_{K^*} N^* \to M^*/S^* \otimes_{K^*} N^*/T^*$ induces a surjection $\rho : M^* \otimes_{K^*} N^*/(M^* \otimes_{K^*} N^*)[n] \to M^*/S^* \otimes_{K^*} N^*/T^*$. Hence we have $(M^*/S^* \otimes_{K^*} N^*/T^*)[n] = \{0\}$. Since there exists an injection $i : \Sigma^k K^* \to N^*/T^*$ for some $k \in \boldsymbol{Z}$, there is an injection $\Sigma^k M^*/S^* \cong M^*/S^* \otimes_{K^*} \Sigma^k K^* \xrightarrow{id_{M^*/S^*} \otimes i} M^*/S^* \otimes_{K^*} N^*/T^*$. Therefore $\left(\Sigma^k M^*/S^*\right)[n] = \{0\}$ and this implies $(M^*/S^*)[n + |k|] = \{0\}$. Hence we have $M^*[n + |k|] \subset S^*$. Similarly, we have $N^*[n + |l|] \subset T^*$ for some $l \in \boldsymbol{Z}$. Thus M^* and N^* are subskeletal.

(2) Suppose $K^i = M^i = N^i = \{0\}$ if $i < k$ for some $k \leq 0$. Then, we have $\mathrm{Ker}(p_{n-2k} \otimes q_{n-2k}) \subset (M^* \otimes_{K^*} N^*)[n]$ for any $n \geq 0$. Similarly, if $K^i = M^i = N^i = \{0\}$ if $i \geq k$ for some $k \geq 0$, then we have $\mathrm{Ker}(p_{n+2k} \otimes q_{n+2k}) \subset (M^* \otimes_{K^*} N^*)[n]$ for any $n \geq 0$. Hence the assertion follows. □

Proposition 2.1.21 *Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ and P^*, Q^* submodules of M^*. Assume that N^* is flat K^*-module. We denote by $i : P^* \to M^*$, $j : Q^* \to M^*$ and $k : P^* \cap Q^* \to M^*$ inclusion maps. Then,* $\mathrm{Im}\,(k \otimes_{K^*} id_{N^*}) = \mathrm{Im}\,(i \otimes_{K^*} id_{N^*}) \cap \mathrm{Im}\,(j \otimes_{K^*} id_{N^*})$.

Proof. It is clear that $\mathrm{Im}\,(k \otimes_{K^*} id_{N^*})$ is contained in $\mathrm{Im}\,(i \otimes_{K^*} id_{N^*}) \cap \mathrm{Im}\,(j \otimes_{K^*} id_{N^*})$. Hence $k \otimes_{K^*} id_{N^*} : (P^* \cap Q^*) \otimes_{K^*} N^* \to M^* \otimes_{K^*} N^*$ defines a map $\bar{k} : (P^* \cap Q^*) \otimes_{K^*} N^* \to \mathrm{Im}\,(i \otimes_{K^*} id_{N^*}) \cap \mathrm{Im}\,(j \otimes_{K^*} id_{N^*})$. We define maps $\psi : P^* \cap Q^* \to P^* \oplus Q^*$ and $\varphi : P^* \oplus Q^* \to M^*$ by $\psi(x) = (x, x)$ and $\varphi(x, y) = x - y$. Let $\bar{i} : P^* \otimes_{K^*} N^* \to \mathrm{Im}\,(i \otimes_{K^*} id_{N^*})$ and $\bar{j} : Q^* \otimes_{K^*} N^* \to \mathrm{Im}\,(j \otimes_{K^*} id_{N^*})$ be the isomorphisms defined from $i \otimes_{K^*} id_{N^*} : P^* \otimes_{K^*} N^* \to M^* \otimes_{K^*} N^*$ and $j \otimes_{K^*} id_{N^*} : Q^* \otimes_{K^*} N^* \to M^* \otimes_{K^*} N^*$, respectively. Define

$$\begin{aligned}
\bar{\psi} &: \mathrm{Im}\,(i \otimes_{K^*} id_{N^*}) \cap \mathrm{Im}\,(j \otimes_{K^*} id_{N^*}) \to (P^* \otimes_{K^*} N^*) \oplus (Q^* \otimes_{K^*} N^*),\\
\bar{\varphi} &: (P^* \otimes_{K^*} N^*) \oplus (Q^* \otimes_{K^*} N^*) \to M^* \otimes_{K^*} N^*,\\
s &: (P^* \oplus Q^*) \otimes_{K^*} N^* \to (P^* \otimes_{K^*} N^*) \oplus (Q^* \otimes_{K^*} N^*)
\end{aligned}$$

by $\bar{\psi}(x) = (\bar{i}^{-1}(x), \bar{j}^{-1}(x))$, $\bar{\varphi}(x, y) = (i \otimes_{K^*} id_{N^*})(x) - (j \otimes_{K^*} id_{N^*})(y)$ and $s((x, y) \otimes z) = (x \otimes z, y \otimes z)$. Then the following diagram is commutative and its lower horizontal row is exact. Note that s is an isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & (P^* \cap Q^*) \otimes_{K^*} N^* \arrow[r, "\psi \otimes_{K^*} id_{N^*}"] \arrow[d, "\bar{k}"] & (P^* \oplus Q^*) \otimes_{K^*} N^* \arrow[r, "\varphi \otimes_{K^*} id_{N^*}"] \arrow[d, "s"] & M^* \otimes_{K^*} N^* \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Im}\,(i \otimes_{K^*} id_{N^*}) \cap \mathrm{Im}\,(j \otimes_{K^*} id_{N^*}) \arrow[r, "\bar{\psi}"] & (P^* \otimes_{K^*} N^*) \oplus (Q^* \otimes_{K^*} N^*) \arrow[r, "\bar{\varphi}"] & M^* \otimes_{K^*} N^*
\end{tikzcd}$$

Since $0 \to P^* \cap Q^* \xrightarrow{\psi} P^* \oplus Q^* \xrightarrow{\varphi} M^*$ is exact by the flatness of N^*, the upper horizontal row is exact. Hence $\bar{k}$ is an isomorphism, which shows the result. □

2.2 Change of rings

For a morphism $\varphi : A^* \to B^*$ in $\mathcal{T}op\mathcal{A}lg_{K^*}$, define functors $\varphi^* : \mathcal{T}op\mathcal{M}od_{A^*} \to \mathcal{T}op\mathcal{M}od_{B^*}$ and $\varphi_* : \mathcal{T}op\mathcal{M}od_{B^*} \to \mathcal{T}op\mathcal{M}od_{A^*}$ as follows.

For an object M^* of $\mathcal{T}op\mathcal{M}od_{A^*}$ with A^*-module structure $\alpha : A^* \times M^* \to M^*$, we set $\varphi^*(M^*) = B^* \otimes_{A^*} M^*$ and the left B^*-module structure $\alpha_\varphi : B^* \times \varphi^*(M^*) \to \varphi^*(M^*)$ is given as follows. Since the multiplication $\mu_{B^*} : B^* \times B^* \to B^*$ is a strongly continuous bilinear map, μ_{B^*} induces a morphism $\bar{\mu}_{B^*} : B^* \otimes_{K^*} B^* \to B^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$. Then, α_φ is the following composition.

$$B^* \times (B^* \otimes_{A^*} M^*) \xrightarrow{\beta_{B^*, B^* \otimes_{A^*} M^*}} B^* \otimes_{K^*} (B^* \otimes_{A^*} M^*) \cong (B^* \otimes_{K^*} B^*) \otimes_{A^*} M^* \xrightarrow{\bar{\mu}_{B^*} \otimes id_{M^*}} B^* \otimes_{A^*} M^*$$

For $\mathfrak{b} \in \mathcal{I}_{B^*}$ and $U^* \in \mathcal{V}_{M^*}$, since α_φ maps $\mathfrak{b} \times (B^* \otimes_{A^*} M^*)$ and $B^* \times o(\mathfrak{b}, U^*)$ into $o(\mathfrak{b}, U^*)$, α_φ is strongly continuous. If $f : M^* \to L^*$ is a morphism in $\mathcal{T}op\mathcal{M}od_{A^*}$, we define $\varphi^*(f) : \varphi^*(M^*) \to \varphi^*(L^*)$ by $\varphi^*(f) = id_{B^*} \otimes f$.

For an object N^* of $\mathcal{T}op\mathcal{M}od_{B^*}$ with B^*-module structure $\alpha : B^* \times N^* \to N^*$, we set $\varphi_*(N^*) = N^*$ and the left A^*-module structure $\alpha^\varphi : A^* \times \varphi_*(N^*) \to \varphi_*(N^*)$ is given by $\alpha^\varphi = \alpha(\varphi \times id_{N^*})$. If $g : N^* \to P^*$ is a morphism in $\mathcal{T}op\mathcal{M}od_{B^*}$, we define $\varphi_*(g) : \varphi_*(N^*) \to \varphi_*(P^*)$ by $\varphi_*(g) = g$.

By the above definitions of the functors φ^*, φ_* and (2.1.8), we have the following facts .

Proposition 2.2.1 *Let $\varphi : A^* \to B^*$ and $\psi : B^* \to C^*$ be morphisms in $\mathcal{T}op\mathcal{A}lg_{K^*}$.*

(1) $(\psi\varphi)^* : \mathcal{T}op\mathcal{M}od_{A^*} \to \mathcal{T}op\mathcal{M}od_{C^*}$ is naturally equivalent to $\psi^*\varphi^* : \mathcal{T}op\mathcal{M}od_{A^*} \to \mathcal{T}op\mathcal{M}od_{C^*}$.

(2) Define a natural transformation $\eta : id_{\mathcal{T}op\mathcal{M}od_{A^*}} \to (id_{A^*})^*$ by $\eta_{M^*}(x) = 1 \otimes x$ for $M^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{A^*}$ and $x \in M^*$. Then, η_{M^*} is an isomorphism if and only if M^* is an object of $\mathcal{T}op\mathcal{M}od^i_{A^*}$.

(3) $(\psi\varphi)_* = \varphi_*\psi_*$ and $(id_{A^*})_* = id_{\mathcal{T}op\mathcal{M}od_{A^*}}$ hold.

(4) The composition of functors $\mathcal{T}op\mathcal{M}od_{A^*} \xrightarrow{\varphi^*} \mathcal{T}op\mathcal{M}od_{B^*} \xrightarrow{\Sigma^n} \mathcal{T}op\mathcal{M}od_{B^*}$ is naturally equivalent to the composition of functors $\mathcal{T}op\mathcal{M}od_{A^*} \xrightarrow{\Sigma^n} \mathcal{T}op\mathcal{M}od_{A^*} \xrightarrow{\varphi^*} \mathcal{T}op\mathcal{M}od_{B^*}$.

(5) The composition of functors $\mathcal{T}op\mathcal{M}od_{B^*} \xrightarrow{\varphi_*} \mathcal{T}op\mathcal{M}od_{A^*} \xrightarrow{\Sigma^n} \mathcal{T}op\mathcal{M}od_{A^*}$ coincides with the composition of functors $\mathcal{T}op\mathcal{M}od_{B^*} \xrightarrow{\Sigma^n} \mathcal{T}op\mathcal{M}od_{B^*} \xrightarrow{\varphi_*} \mathcal{T}op\mathcal{M}od_{A^*}$.

Lemma 2.2.2 *φ^* maps each object of $\mathcal{T}op\mathcal{M}od_{A^*}$ to $\mathcal{T}op\mathcal{M}od^i_{B^*}$ and φ_* maps each object of $\mathcal{T}op\mathcal{M}od^i_{B^*}$ to $\mathcal{T}op\mathcal{M}od^i_{A^*}$.*

Proof. Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{A^*}$. For $\mathfrak{b} \in \mathcal{I}_{B^*}$ and $U^* \in \mathcal{N}_{M^*}$, let us denote by $\iota_{\mathfrak{b}} : \mathfrak{b} \to B^*$ and $i_{U^*} : U^* \to M^*$ the inclusion maps. Then, $\alpha_\varphi : B^* \times (B^* \otimes_{A^*} M^*) \to B^* \otimes_{A^*} M^*$ maps $\mathfrak{b} \times (B^* \otimes_{A^*} M^*)$ into $\mathrm{Im}(\iota_{\mathfrak{b}} \otimes id_{M^*})$ and $B^* \times (\mathrm{Im}(\iota_{\mathfrak{b}} \otimes id_{M^*}) + \mathrm{Im}(id_{B^*} \otimes i_{U^*}))$ into $\mathrm{Im}(\iota_{\mathfrak{b}} \otimes id_{M^*}) + \mathrm{Im}(id_{B^*} \otimes i_{U^*})$. Hence α_φ is strongly continuous and it follows from (1.1.10) that $\varphi^*(M^*)$ is an object of $\mathcal{T}op\mathcal{M}od^i_{B^*}$.

Let N^* be an object of $\mathcal{T}op\mathcal{M}od^i_{B^*}$. For $V^* \in \mathcal{N}_{N^*}$, there exists $\mathfrak{b} \in \mathcal{I}_{B^*}$ satisfying $\mathfrak{b}N^* \subset V^*$. Then, $\varphi^{-1}(\mathfrak{b})\varphi_*(N^*) \subset \mathfrak{b}N^* \subset V^*$ and $\varphi^{-1}(\mathfrak{b}) \in \mathcal{I}_{A^*}$ by the continuity of φ. Hence the topology of $\varphi_*(N^*)$ is coarser than the topology induced by K^*. □

We also denote by $\varphi^* : \mathcal{T}op\mathcal{M}od_{A^*} \to \mathcal{T}op\mathcal{M}od^i_{B^*}$ the functor induced by $\varphi^* : \mathcal{T}op\mathcal{M}od_{A^*} \to \mathcal{T}op\mathcal{M}od_{B^*}$ and by $\varphi_* : \mathcal{T}op\mathcal{M}od^i_{B^*} \to \mathcal{T}op\mathcal{M}od_{A^*}$ the restriction of $\varphi_* : \mathcal{T}op\mathcal{M}od_{B^*} \to \mathcal{T}op\mathcal{M}od_{A^*}$.

Proposition 2.2.3 *$\varphi_* : \mathcal{T}op\mathcal{M}od^i_{B^*} \to \mathcal{T}op\mathcal{M}od_{A^*}$ is a right adjoint of $\varphi^* : \mathcal{T}op\mathcal{M}od_{A^*} \to \mathcal{T}op\mathcal{M}od^i_{B^*}$.*

Proof. Define natural transformations $\eta : id_{\mathcal{T}op\mathcal{M}od_{A^*}} \to \varphi_*\varphi^*$ and $\varepsilon : \varphi^*\varphi_* \to id_{\mathcal{T}op\mathcal{M}od^i_{B^*}}$ as follows. For $M^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{A^*}$, $\eta_{M^*} : M^* \to B^* \otimes_{A^*} M^*$ is defined by $\eta_{M^*}(x) = 1 \otimes x$. For $N^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od^i_{B^*}$, let $\alpha : B^* \times N^* \to N^*$ be the structure map. Then, α is strongly continuous by (1.1.10). Moreover, since α is B^*-bilinear, it is A^*-bilinear if we regard α as a map $B^* \times \varphi_*(N^*) \to \varphi_*(N^*)$. Hence there exists a morphism $\varepsilon_{N^*} : B^* \otimes_{A^*} N^* \to N^*$ induced by α. It is easy to verify equalities $\varepsilon_{\varphi^*(M^*)}\varphi^*(\eta_{M^*}) = id_{\varphi^*(M^*)}$ for $M^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{A^*}$ and $\varphi_*(\varepsilon_{N^*})\eta_{\varphi_*(N^*)} = id_{\varphi_*(N^*)}$ for $N^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od^i_{B^*}$. □

2.3 Completed tensor product

For objects M^* and N^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, let us denote by $M^* \widehat{\otimes}_{K^*} N^*$ the completion of $M^* \otimes_{K^*} N^*$.

We denote by $\hat{\tau}^{m,n}_{M^*,N^*} : \Sigma^m M^* \widehat{\otimes}_{K^*} \Sigma^n N^* \to \Sigma^{m+n}\left(M^* \widehat{\otimes}_{K^*} N^*\right)$ the map induced by $\tau^{m,n}_{M^*,N^*}$ and by $\widehat{T} = \widehat{T}_{M^*,N^*} : M^* \widehat{\otimes}_{K^*} N^* \to N^* \widehat{\otimes}_{K^*} M^*$ the map induced by T_{M^*,N^*}.

The following fact is obvious.

Proposition 2.3.1 *If M^* and N^* are both profinite, so is $M^* \widehat{\otimes}_{K^*} N^*$.*

Proposition 2.3.2 *If $K^i = \{0\}$ for $i \neq 0$ and both M^* and N^* have the skeletal topology, $\left(M^* \widehat{\otimes}_{K^*} N^*\right)^n$ is isomorphic to $\prod\limits_{i+j=n} M^i \otimes_{K^*} N^j$.*

Proof. For $k, l \in \boldsymbol{Z}$, let $p_k : M^* \to M^*/M^*[k]$ and $q_l : N^* \to N^*/N^*[l]$ be the quotient maps. Put $U(k,l) = \mathrm{Ker}(p_k \otimes_{K^*} q_l)$, then $\{U(k,l) |\, k, l \in \boldsymbol{Z}\}$ forms a fundamental system of neighborhoods of 0 in $M^* \otimes_{K^*} N^*$. Hence $M^* \widehat{\otimes}_{K^*} N^*$ is the limit of an inverse system $(M^* \otimes_{K^*} N^*/U(k,l) \to M^* \otimes_{K^*} N^*/U(s,t))_{k \le s,\, l \le t}$. If $k \le s$ and $l \le t$, let $\alpha_{k,s} : M^*/M^*[k] \to M^*/M^*[s]$ and $\beta_{l,t} : N^*/N^*[l] \to N^*/N^*[t]$ be quotient maps. For each $k, l \in \boldsymbol{Z}$, we denote by $r_{k,l} : M^* \otimes_{K^*} N^*/U(k,l) \to M^*/M^*[k] \otimes_{K^*} N^*/N^*[l]$ the isomorphism induced by $p_k \otimes q_l$. Since the following square commutes if $k \le s$ and $l \le t$,

$$\begin{tikzcd}
M^* \otimes_{K^*} N^*/U(k,l) \arrow[r] \arrow[d, "r_{k,l}"] & M^* \otimes_{K^*} N^*/U(s,t) \arrow[d, "r_{s,t}"] \\
M^*/M^*[k] \otimes_{K^*} N^*/N^*[l] \arrow[r, "\alpha_{k,s} \otimes \beta_{l,t}"] & M^*/M^*[s] \otimes_{K^*} N^*/N^*[t]
\end{tikzcd}$$

$M^* \widehat{\otimes}_{K^*} N^*$ is the limit of an inverse system

$$\left(M^*/M^*[s] \otimes_{K^*} N^*/N^*[t] \xrightarrow{\alpha_{k,s}\otimes\beta_{l,t}} M^*/M^*[s] \otimes_{K^*} N^*/N^*[t] \right)_{k\le s,\, l\le t}.$$

It follows from (1.2.5), $\left(M^* \widehat{\otimes}_{K^*} N^*\right)^n$ is the limit of an inverse system

$$\left((M^*/M^*[s] \otimes_{K^*} N^*/N^*[t])^n \xrightarrow{\alpha_{k,s}\otimes\beta_{l,t}} (M^*/M^*[s] \otimes_{K^*} N^*/N^*[t])^n \right)_{k\le s,\, l\le t}.$$

Note that $(p_k \otimes_{K^*} q_l)(M^i \otimes_{K^*} N^j) = \{0\}$ if $|i| \ge k$ or $|j| \ge l$. We define

$$\pi_{k,l} : \prod_{i+j=n} M^i \otimes_{K^*} N^j \to (M^*/M^*[k] \otimes_{K^*} N^*/N^*[l])^n$$

by $\pi_{k,l}((x_i)_{i\in \boldsymbol{Z}}) = \sum\limits_{i\in\boldsymbol{Z}} (p_k \otimes_{K^*} q_l)(x_i)$ $(x_i \in M^i \otimes_{K^*} N^j)$. It is easy to verify that

$$\left(\prod_{i+j=n} M^i \otimes_{K^*} N^j \xrightarrow{\pi_{k,l}} (M^*/M^*[k] \otimes_{K^*} N^*/N^*[l])^n \right)_{k,l\in\boldsymbol{Z}}$$

is a limiting cone of the above inverse system. □

Corollary 2.3.3 *Suppose that $K^i = \{0\}$ for $i \neq 0$ and both M^* and N^* has the skeletal topologies. $M^* \otimes_{K^*} N^*$ is complete if and only if, for each $n \in \boldsymbol{Z}$, $\{i \in \boldsymbol{Z} |\, M^i, N^{n-i} \neq \{0\}\}$ has finitely many elements.*

Proposition 2.3.4 *There is unique isomorphism $\hat{\iota}_2 : \widehat{M}^* \to K^* \widehat{\otimes}_{K^*} M^*$ such that the following diagram commutes.*

$$\begin{array}{ccc} M^* & \xrightarrow{\eta_{M^*}} & \widehat{M}^* \\ \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \hat{\iota}_2} \\ K^* \otimes_{K^*} M^* & \xrightarrow{\eta_{K \otimes_{K^*} M^*}} & K^* \widehat{\otimes}_{K^*} M^* \end{array}$$

Here, $\iota_2 : M^* \to K^* \otimes_{K^*} M^*$ is given by $\iota_2(x) = 1 \otimes x$.

Proposition 2.3.5 *For objects M^* and N^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, $\eta_{M^*} \widehat{\otimes}_{K^*} id_{N^*} : M^* \widehat{\otimes}_{K^*} N^* \to \widehat{M}^* \widehat{\otimes}_{K^*} N^*$ and $id_{M^*} \widehat{\otimes}_{K^*} \eta_{N^*} : M^* \widehat{\otimes}_{K^*} N^* \to M^* \widehat{\otimes}_{K^*} \widehat{N}^*$ are isomorphisms. Hence $\eta_{M^*} \widehat{\otimes}_{K^*} \eta_{N^*} : M^* \widehat{\otimes}_{K^*} N^* \to \widehat{M}^* \widehat{\otimes}_{K^*} \widehat{N}^*$ is an isomorphism.*

Proof. Let us denote by $p_{U^*} : M^* \to M^*/U^*$ $(U^* \in \mathcal{V}_{M^*})$ and $q_{V^*} : N^* \to N^*/V^*$ $(V^* \in \mathcal{V}_{N^*})$ be the quotient maps. Then, there is a limiting cone $\left(M^* \widehat{\otimes}_{K^*} N^* \xrightarrow{\pi_{U^*,V^*}} M^*/U^* \otimes_{K^*} N^*/V^* \right)_{U^*\in\mathcal{V}_{M^*}, V^*\in\mathcal{V}_{N^*}}$ such that $\pi_{U^*,V^*}\eta_{M^*\otimes N^*} = p_{U^*} \otimes q_{V^*}$. Consider a cone $\left(\widehat{M}^* \xrightarrow{\pi_{U^*}} M^*/U^* \right)_{U^*\in\mathcal{V}_{M^*}}$, then we have a limiting cone $\left(\widehat{M}^* \widehat{\otimes}_{K^*} N^* \xrightarrow{\rho_{U^*,V^*}} M^*/U^* \otimes_{K^*} N^*/V^* \right)_{U^*\in\mathcal{V}_{M^*}, V^*\in\mathcal{V}_{N^*}}$ such that $\rho_{U^*,V^*}\eta_{\widehat{M}^*\otimes N^*} = \pi_{U^*} \otimes q_{V^*}$. By the definition of $\eta_{M^*} \widehat{\otimes}\, id_{N^*}$, $\rho_{U^*,V^*}(\eta_{M^*} \widehat{\otimes}\, id_{N^*}) = \pi_{U^*,V^*}$ holds for each $U^* \in \mathcal{V}_{M^*}$ and $V^* \in \mathcal{V}_{N^*}$. Hence $\eta_{M^*} \widehat{\otimes}\, id_{N^*}$ is an isomorphism. The proof of the second assertion is similar. □

Remark 2.3.6 *For objects V^*, W^* and Z^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, the natural map $(V^* \otimes_{K^*} W^*) \otimes_{K^*} Z^* \to V^* \otimes_{K^*} (W^* \otimes_{K^*} Z^*)$ is an isomorphism in $\mathcal{T}op\mathcal{M}od_{K^*}$. This induces a natural isomorphism $(V^* \otimes_{K^*} W^*) \widehat{\otimes}_{K^*} Z^* \to V^* \widehat{\otimes}_{K^*} (W^* \otimes_{K^*} Z^*)$. By the above result, we have a natural isomorphism*

$$\left(V^* \widehat{\otimes}_{K^*} W^*\right) \widehat{\otimes}_{K^*} Z^* \to V^* \widehat{\otimes}_{K^*} \left(W^* \widehat{\otimes}_{K^*} Z^*\right).$$

Definition 2.3.7 *For $U^*, V^*, W^*, Z^* \in \mathrm{Ob}\, \mathcal{T}op\mathcal{M}od_{K^*}$,*

$$\eta_{U^*\otimes_{K^*} W^*} \widehat{\otimes}_{K^*} \eta_{V^*\otimes_{K^*} Z^*} : (U^* \otimes_{K^*} W^*) \widehat{\otimes}_{K^*} (V^* \otimes_{K^*} Z^*) \to (U^* \widehat{\otimes}_{K^*} W^*) \widehat{\otimes}_{K^*} (V^* \widehat{\otimes}_{K^*} Z^*)$$

is an isomorphism by (2.3.5). Let us denote by

$$id_{U^*}\widehat{\otimes}_{K^*}\widehat{T}_{V^*,W^*}\widehat{\otimes}_{K^*}id_{Z^*} : (U^*\widehat{\otimes}_{K^*}V^*)\widehat{\otimes}_{K^*}(W^*\widehat{\otimes}_{K^*}Z^*) \to (U^*\widehat{\otimes}_{K^*}W^*)\widehat{\otimes}_{K^*}(V^*\widehat{\otimes}_{K^*}Z^*)$$

the following composition.

$$\begin{aligned}(U^*\widehat{\otimes}_{K^*}V^*)\widehat{\otimes}_{K^*}(W^*\widehat{\otimes}_{K^*}Z^*) &\xrightarrow{\cong} ((U^*\widehat{\otimes}_{K^*}V^*)\widehat{\otimes}_{K^*}W^*)\widehat{\otimes}_{K^*}Z^* \xrightarrow{\cong} (U^*\widehat{\otimes}_{K^*}(V^*\widehat{\otimes}_{K^*}W^*))\widehat{\otimes}_{K^*}Z^* \\ &\xrightarrow{(id_{U^*}\widehat{\otimes}_{K^*}\widehat{T}_{V^*,W^*})\widehat{\otimes}_{K^*}id_{Z^*}} (U^*\widehat{\otimes}_{K^*}(W^*\widehat{\otimes}_{K^*}V^*))\widehat{\otimes}_{K^*}Z^* \xrightarrow{\cong} ((U^*\widehat{\otimes}_{K^*}W^*)\widehat{\otimes}_{K^*}V^*)\widehat{\otimes}_{K^*}Z^* \\ &\xrightarrow{\cong} (U^*\widehat{\otimes}_{K^*}W^*)\widehat{\otimes}_{K^*}(V^*\widehat{\otimes}_{K^*}Z^*)\end{aligned}$$

Hence there is a unique map

$$sh = sh_{U^*,V^*,W^*,Z^*} : (U^*\widehat{\otimes}_{K^*}V^*)\otimes_{K^*}(W^*\widehat{\otimes}_{K^*}Z^*) \to (U^*\otimes_{K^*}W^*)\widehat{\otimes}_{K^*}(V^*\otimes_{K^*}Z^*)$$

that makes the following diagram commute.

$$\begin{tikzcd}
(U^*\widehat{\otimes}_{K^*}V^*)\otimes_{K^*}(W^*\widehat{\otimes}_{K^*}Z^*) \arrow[r, "\eta_{(U^*\widehat{\otimes}_{K^*}V^*)\otimes_{K^*}(W^*\widehat{\otimes}_{K^*}Z^*)}"] \arrow[d, "sh"] & (U^*\widehat{\otimes}_{K^*}V^*)\widehat{\otimes}_{K^*}(W^*\widehat{\otimes}_{K^*}Z^*) \arrow[d, "id_{U^*}\widehat{\otimes}_{K^*}\widehat{T}_{V^*,W^*}\widehat{\otimes}_{K^*}id_{Z^*}"] \\
(U^*\otimes_{K^*}W^*)\widehat{\otimes}_{K^*}(V^*\otimes_{K^*}Z^*) \arrow[r, "\eta_{U^*\otimes_{K^*}W^*}\widehat{\otimes}_{K^*}\eta_{V^*\otimes_{K^*}Z^*}", "\cong"'] & (U^*\widehat{\otimes}_{K^*}W^*)\widehat{\otimes}_{K^*}(V^*\widehat{\otimes}_{K^*}Z^*)
\end{tikzcd}$$

We call sh the shuffling map.

It is easy to verify the following.

Lemma 2.3.8 *Let $(D_\lambda : \mathcal{D} \to \mathcal{C})_{\lambda\in\Lambda}$ be a family of functors. Suppose that, for each $i \in \mathrm{Ob}\,\mathcal{D}$, a product $\prod_{\lambda\in\Lambda} D_\lambda(i)$ exists and that, for each $\lambda \in \Lambda$, $\left(L_\lambda \xrightarrow{p_{\lambda i}} D_\lambda(i)\right)_{i\in\mathrm{Ob}\,cd}$ is a limiting cone of D_λ. Define a functor $D : \mathcal{D} \to \mathcal{C}$ by $D(i) = \prod_{\lambda\in\Lambda} D_\lambda(i)$. If product $\prod_{\lambda\in\Lambda} L_\lambda$ exists, $\left(\prod_{\lambda\in\Lambda} L_\lambda \xrightarrow{\prod_{\lambda\in\Lambda} p_{\lambda i}} \prod_{\lambda\in\Lambda} D_\lambda(i)\right)_{i\in\mathrm{Ob}\,cd}$ is a limiting cone of D.*

Proposition 2.3.9 *Let $D : \mathcal{D} \to \mathcal{T}op\mathcal{M}od_{K^*}$ be a functor and M^* an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ which is finitely generated and free. If $\left(N^* \xrightarrow{p_i} D(i)\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of D, $\left(M^*\otimes_{K^*}N^* \xrightarrow{p_i} M^*\otimes_{K^*}D(i)\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of a functor $D' : \mathcal{D} \to \mathcal{T}op\mathcal{M}od_{K^*}$ defined by $D'(i) = M^*\otimes_{K^*}D(i)$.*

Proof. The assertion is obvious if M^* is generated by a single element. Since M^* is a finite product of submodules generated by a single element by (1.1.20), the assertion follows from (2.3.8). □

Corollary 2.3.10 *Suppose that K^* is discrete. Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. If $M^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{K^*}$ is finitely generated and free, then there exists a unique isomorphism $\xi : M^*\otimes_{K^*}\widehat{N}^* \to M^*\widehat{\otimes}_{K^*}N^*$ satisfying $\xi(id_{M^*}\otimes\eta_{N^*}) = \eta_{M^*\otimes_{K^*}N^*}$.*

Proof. Applying the above result to the limiting cone $\left(\widehat{N}^* \xrightarrow{p_{U^*}} N^*/U^*\right)_{U^*\in\mathcal{V}_{N^*}}$ of $d_{N^*} : \mathcal{V}_{N^*}^{op} \to \mathcal{T}op\mathcal{M}od_{K^*}$, we see that $\left(M^*\otimes_{K^*}\widehat{N}^* \xrightarrow{id_{M^*}\otimes_{K^*}p_{U^*}} M^*\otimes_{K^*}N^*/U^*\right)_{U^*\in\mathcal{V}_{N^*}}$ is a limiting cone of a functor $U^* \mapsto M^*\otimes_{K^*}N^*/U^*$. Since M^* is discrete, $\{M^*\otimes_{K^*}U^* | U^* \in \mathcal{V}_{N^*}\}$ is cofinal in $\mathcal{V}_{M^*\otimes_{K^*}N^*}$. Thus the result follows. □

Proposition 2.3.11 *Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ and J a cofinal subset of $\mathcal{V}_{N^*}$. If N^*/V^* is finitely generated and free for every $V^* \in J$, then*

$$\left(M^*\widehat{\otimes}_{K^*}N^* \xrightarrow{id_{M^*}\widehat{\otimes}q_{V^*}} M^*\widehat{\otimes}_{K^*}N^*/V^*\right)_{V^*\in J}$$

is a limiting cone of a functor $D : J \to \mathcal{T}op\mathcal{M}od_{K^*}$ given by $D(V^*) = M^*\widehat{\otimes}_{K^*}N^*/V^*$.

Proof. Put $I = \mathcal{V}_{M^*}$ and consider the limiting cone $\left(M^* \widehat{\otimes}_{K^*} N^* \xrightarrow{\rho_{U^*,V^*}} M^*/U^* \otimes_{K^*} N^*/V^*\right)_{(U^*,V^*)\in I\times J}$ of a functor $F : I\times J \to \mathcal{T}op\mathcal{M}od_{K^*}$ given by $F(U^*,V^*) = M^*/U^*\otimes_{K^*} N^*/V^*$. Let $\left(T^* \xrightarrow{f_{V^*}} M^*\widehat{\otimes}_{K^*} N^*/V^*\right)_{V^*\in J}$ be a cone of D. Then, $\left(T^* \xrightarrow{(p_{U^*}\widehat{\otimes}\, id_{N^*/V^*})f_{V^*}} M^*/U^*\otimes_{K^*} N^*/V^*\right)_{(U^*,V^*)\in I\times J}$ is a cone of F. Hence there exists unique map $\varphi : T^* \to M^*\widehat{\otimes}_{K^*}N^*$ satisfying $\rho_{U^*,V^*}\varphi = (p_{U^*}\widehat{\otimes}\, id_{N^*/V^*})f_{V^*}$ for any $(U^*,V^*)\in I\times J$. Since $\left(M^*\widehat{\otimes}_{K^*}N^*/V^* \xrightarrow{p_{U^*}\widehat{\otimes}\, id_{N^*/V^*}} M^*/U^*\otimes_{K^*}N^*/V^*\right)_{U^*\in I}$ is a limiting cone of a functor $I\to \mathcal{T}op\mathcal{M}od_{K^*}$ given by $U^*\mapsto M^*/U^*\otimes_{K^*}N^*/V^*$ for each $V^*\in J$ by (2.3.9), we have $(id_{M^*}\widehat{\otimes}\, q_{V^*})\varphi = f_{V^*}$. Suppose that $\psi : T^*\to M^*\widehat{\otimes}_{K^*}N^*$ satisfying $(id_{M^*}\widehat{\otimes}\, q_{V^*})\psi = f_{V^*}$ for any $V^*\in J$. Then, $(p_{U^*}\widehat{\otimes}\, q_{V^*})\psi = (p_{U^*}\widehat{\otimes}\, id_{N^*/V^*})f_{V^*} = (p_{U^*}\widehat{\otimes}\, q_{V^*})\varphi$ for any $(U^*,V^*)\in I\times J$. Thus we have $\psi = \varphi$ and the assertion follows. □

Proposition 2.3.12 *Let $\xi : R^*\to S^*$ and $\lambda : S^*\to T^*$ be morphisms of $\mathcal{T}op\mathcal{A}lg_{K^*}$. For objects M^* and N^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, suppose that a right R^*-module structure $\alpha : M^*\otimes_{K^*}R^*\to M^*$ and a right S^*-module structure $\beta : N^*\otimes_{K^*}S^*\to N^*$ are given and that a morphism $\varphi : M^*\to N^*$ of $\mathcal{T}op\mathcal{M}od_{K^*}$ makes the following diagram commute.*

$$\begin{tikzcd}
M^*\otimes_{K^*}R^* \arrow[r, "\alpha"] \arrow[d, "\varphi\otimes\xi"] & M^* \arrow[d, "\varphi"] \\
N^*\otimes_{K^*}S^* \arrow[r, "\beta"] & N^*
\end{tikzcd}$$

We denote by $\otimes_\xi : N^*\otimes_{R^*}T^*\to N^*\otimes_{S^*}T^*$ the quotient map and by $\widehat{\otimes}_\xi : N^*\widehat{\otimes}_{R^*}T^*\to N^*\widehat{\otimes}_{S^*}T^*$ the map induced by $\otimes_\xi$. If $\psi : M^*\widehat{\otimes}_{R^*}T^*\to N^*\widehat{\otimes}_{S^*}T^*$ is a continuous homomorphism of right T^*-modules which makes the following diagram commute, then ψ is a composition $M^*\widehat{\otimes}_{R^*}T^*\xrightarrow{\varphi\,\widehat{\otimes}_{R^*} id_{T^*}} N^*\widehat{\otimes}_{R^*}T^*\xrightarrow{\widehat{\otimes}_\xi} N^*\widehat{\otimes}_{S^*}T^*$, where $i_{M^*} : M^*\to M^*\otimes_{R^*}T^*$ and $i_{N^*} : N^*\to N^*\otimes_{S^*}T^*$ are maps defined by $i_{M^*}(x) = x\otimes 1$ and $i_{N^*}(x) = x\otimes 1$, respectively.

$$\begin{tikzcd}
M^* \arrow[r, "i_{M^*}"] \arrow[d, "\varphi"] & M^*\otimes_{R^*}T^* \arrow[r, "\eta_{M^*\otimes_{R^*}T^*}"] & M^*\widehat{\otimes}_{R^*}T^* \arrow[d, "\psi"] \\
N^* \arrow[r, "i_{N^*}"] & N^*\otimes_{S^*}T^* \arrow[r, "\eta_{N^*\otimes_{S^*}T^*}"] & N^*\widehat{\otimes}_{S^*}T^*
\end{tikzcd}$$

Proof. Since the following diagram commutes, we have $\psi\eta_{M^*\otimes_{R^*}T^*}i_{M^*} = \widehat{\otimes}_\xi(\varphi\,\widehat{\otimes}_{R^*} id_{T^*})\eta_{M^*\otimes_{R^*}T^*}i_{M^*}$.

$$\begin{tikzcd}
M^* \arrow[r, "i_{M^*}"] \arrow[d, "\varphi"] & M^*\otimes_{R^*}T^* \arrow[r, "\eta_{M^*\otimes_{R^*}T^*}"] \arrow[d, "\varphi\otimes_{R^*} id_{T^*}"] & M^*\widehat{\otimes}_{R^*}T^* \arrow[d, "\varphi\,\widehat{\otimes}_{R^*} id_{T^*}"] \\
N^* \arrow[r, "i_{N^*}"] \arrow[dr, "i_{N^*}"] & N^*\otimes_{R^*}T^* \arrow[r, "\eta_{N^*\otimes_{R^*}T^*}"] \arrow[d, "\otimes_\xi"] & N^*\widehat{\otimes}_{R^*}T^* \arrow[d, "\widehat{\otimes}_\xi"] \\
 & N^*\otimes_{S^*}T^* \arrow[r, "\eta_{N^*\otimes_{S^*}T^*}"] & N^*\widehat{\otimes}_{S^*}T^*
\end{tikzcd}$$

Since both $\psi\eta_{M^*\otimes_{R^*}T^*}$ and $\widehat{\otimes}_\xi(\varphi\,\widehat{\otimes}_{R^*} id_{T^*})\eta_{M^*\otimes_{R^*}T^*}$ are homomorphisms of right T^*-modules and the image of i_{M^*} generates $M^*\otimes_{R^*}T^*$, we have $\psi\eta_{M^*\otimes_{R^*}T^*} = \widehat{\otimes}_\xi(\varphi\,\widehat{\otimes}_{R^*} id_{T^*})\eta_{M^*\otimes_{R^*}T^*}$. Then, the continuity of ψ implies $\psi = \widehat{\otimes}_\xi(\varphi\,\widehat{\otimes}_{R^*} id_{T^*})$. □

Lemma 2.3.13 *If M^* is a dense submodule of N^*, then for a K^*-module L^*, the inclusion map $i : M^*\to N^*$ induces an isomorphism $id_{L^*}\widehat{\otimes}_{K^*} i : L^*\widehat{\otimes}_{K^*}M^*\to L^*\widehat{\otimes}_{K^*}N^*$.*

Proof. Since $\hat{i} : \widehat{M}^*\to \widehat{N}^*$ is an isomorphism by (1.3.18), so is $id_{L^*}\widehat{\otimes}_{K^*}\hat{i} : L^*\widehat{\otimes}_{K^*}\widehat{M}^*\to L^*\widehat{\otimes}_{K^*}\widehat{N}^*$. Therefore the assertion follows from (2.3.5) and the naturality of η. □

Proposition 2.3.14 *If $f : M^*\to N^*$ is an epimorphism in $\mathcal{T}op\mathcal{M}od_{cK^*}$, then for an object L^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, $id_{L^*}\widehat{\otimes}_{K^*} f : L^*\widehat{\otimes}_{K^*}M^*\to L^*\widehat{\otimes}_{K^*}N^*$ is an epimorphism in $\mathcal{T}op\mathcal{M}od_{cK^*}$.*

Proof. Since the image of $f : M^*\to N^*$ is dense, so is the image of $id_{L^*}\otimes_{K^*} f : L^*\otimes_{K^*}M^*\to L^*\otimes_{K^*}N^*$ by (2.1.4). Hence $id_{L^*}\widehat{\otimes}_{K^*} f$ is an epimorphism of $\mathcal{T}op\mathcal{M}od_{cK^*}$ by (1.3.18). □

Proposition 2.3.15 *Let $\pi : N^* \to C^*$ be a cokernel in $\mathcal{T}op\mathcal{M}od_{cK^*}$ of a morphism $f : M^* \to N^*$ of $\mathcal{T}op\mathcal{M}od_{cK^*}$. Then, for an object L^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, $id_{L^*} \widehat{\otimes}_{K^*} \pi : L^* \widehat{\otimes}_{K^*} N^* \to L^* \widehat{\otimes}_{K^*} C^*$ is a cokernel in $\mathcal{T}op\mathcal{M}od_{cK^*}$ of $id_{L^*} \widehat{\otimes}_{K^*} f : L^* \widehat{\otimes}_{K^*} M^* \to L^* \widehat{\otimes}_{K^*} N^*$.*

Proof. $id_{L^*} \widehat{\otimes}_{K^*} \pi$ is an epimorphism of $\mathcal{T}op\mathcal{M}od_{cK^*}$ by (2.3.14). Let $i : \operatorname{Im} f \to \operatorname{Ker}\pi$ be the inclusion map. It follows from (1.3.20) and (2.1.4) that the image of $id_{L^*} \otimes_{K^*} i : L^* \otimes_{K^*} \operatorname{Im} f \to L^* \otimes_{K^*} \operatorname{Ker}\pi$ is dense. Hence $id_{L^*} \widehat{\otimes}_{K^*} i : L^* \widehat{\otimes}_{K^*} \operatorname{Im} f \to L^* \widehat{\otimes}_{K^*} \operatorname{Ker}\pi$ is an epimorphism of $\mathcal{T}op\mathcal{M}od_{cK^*}$ by (1.3.18). Since the surjection $f' : M^* \to \operatorname{Im} f$ induced by f induces an epimorphism $id_{L^*} \widehat{\otimes}_{K^*} f' : L^* \widehat{\otimes}_{K^*} M^* \to L^* \widehat{\otimes}_{K^*} \operatorname{Im} f$ of $\mathcal{T}op\mathcal{M}od_{cK^*}$ by (1.3.11), $id_{L^*} \widehat{\otimes}_{K^*} if' : L^* \widehat{\otimes}_{K^*} M^* \to L^* \widehat{\otimes}_{K^*} \operatorname{Ker}\pi$ is an epimorphism of $\mathcal{T}op\mathcal{M}od_{cK^*}$. We denote by $j : \operatorname{Ker}\pi \to N^*$ be the inclusion map and by $\pi' : N^* \to \operatorname{Im}\pi$ the surjection induced by π. Since π' is a cokernel of j in $\mathcal{T}op\mathcal{M}od_{K^*}$, $id_{L^*} \otimes_{K^*} \pi' : L^* \otimes_{K^*} N^* \to L^* \otimes_{K^*} \operatorname{Im}\pi$ is a cokernel of $id_{L^*} \otimes_{K^*} j : L^* \otimes_{K^*} \operatorname{Ker}\pi \to L^* \otimes_{K^*} N^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$. Hence $id_{L^*} \widehat{\otimes}_{K^*} \pi' : L^* \widehat{\otimes}_{K^*} N^* \to L^* \widehat{\otimes}_{K^*} \operatorname{Im}\pi$ is a cokernel of $id_{L^*} \widehat{\otimes}_{K^*} j : L^* \widehat{\otimes}_{K^*} \operatorname{Ker}\pi \to L^* \widehat{\otimes}_{K^*} N^*$ in $\mathcal{T}op\mathcal{M}od_{cK^*}$ by (1.3.11). Since $id_{L^*} \widehat{\otimes}_{K^*} if'$ is an epimorphism and $f = jif'$, $id_{L^*} \widehat{\otimes}_{K^*} \pi' : L^* \widehat{\otimes}_{K^*} N^* \to L^* \widehat{\otimes}_{K^*} \operatorname{Im}\pi$ is a cokernel of $id_{L^*} \widehat{\otimes}_{K^*} f : L^* \widehat{\otimes}_{K^*} M^* \to L^* \widehat{\otimes}_{K^*} N^*$ in $\mathcal{T}op\mathcal{M}od_{cK^*}$. Since $\operatorname{Im}\pi$ is a dense submodule of C^*, the inclusion map $k : \operatorname{Im}\pi \to C^*$ induces an isomorphism $id_{L^*} \widehat{\otimes}_{K^*} k : L^* \widehat{\otimes}_{K^*} \operatorname{Im}\pi \to L^* \widehat{\otimes}_{K^*} C^*$ by (2.3.13). We conclude that $id_{L^*} \widehat{\otimes}_{K^*} \pi = id_{L^*} \widehat{\otimes}_{K^*} k\pi'$ is a cokernel of $id_{L^*} \widehat{\otimes}_{K^*} f$ in $\mathcal{T}op\mathcal{M}od_{cK^*}$. □

For objects M^*, N^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ and $a \in M^*$, $b \in N^*$, we put $\eta_{M^* \otimes_{K^*} N^*}(a \otimes b) = a \widehat{\otimes} b$.

Lemma 2.3.16 *Let M^*, N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. Assume that M^* is a flat K^*-module and Hausdorff space. For $y \in N^n$, define a map $\hat{R}_{M^*,y} : \Sigma^n M^* \to M^* \widehat{\otimes}_{K^*} N^*$ by $\hat{R}_{M^*,y}([n], x) = (-1)^{n\deg x} x \widehat{\otimes} y$. If y is K^*-torsion free, $\hat{R}_{M^*,y}$ is a monomorphism.*

Proof. Define a map $R_y : \Sigma^n K^* \to N^*$ by $R_y([n], r) = (-1)^{n\deg r} ry$. Then, R_y is a monomorphism since y is K^*-torsion free and $id_{M^*} \otimes_{K^*} R_y : M^* \otimes_{K^*} \Sigma^n K^* \to M^* \otimes_{K^*} N^*$ is also a monomorphism since M^* is flat over K^*. We define $R_{M^*,y} : \Sigma^n M^* \to M^* \otimes_{K^*} N^*$ to be the following composition.

$$\Sigma^n M^* \xrightarrow{s^n_{M^*}} \Sigma^n K^* \otimes_{K^*} M^* \xrightarrow{T_{\Sigma^n K^*, M^*}} M^* \otimes_{K^*} \Sigma^n K^* \xrightarrow{id_{M^*} \otimes_{K^*} R_y} M^* \otimes_{K^*} N^*$$

Then, $R_{M^*,y}$ is a monomorphism, hence $\widehat{R_{M^*,y}} : \widehat{\Sigma^n M^*} \to M^* \widehat{\otimes}_{K^*} N^*$ is a monomorphism by (1.3.14). Since $\hat{R}_{M^*,y}$ is a composition $\eta_{\Sigma^n M^*} : \Sigma^n M^* \to \widehat{\Sigma^n M^*}$ and $\widehat{R_{M^*,y}}$, $\hat{R}_{M^*,y}$ is a monomorphism. □

Proposition 2.3.17 *Let K^* be a field such that $K^i = \{0\}$ if $i \neq 0$ and M^*, N^* objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. Suppose that M^* and N^* are 1st countable spaces.*

(1) For $x \in M^* \widehat{\otimes}_{K^*} N^*$, there exist sequences $(x_n)_{n\in\boldsymbol{N}}$ of M^* and $(y_n)_{n\in\boldsymbol{N}}$ of and N^* such that $\sum\limits_{n\in\boldsymbol{N}} x_n \widehat{\otimes} y_n$ converges to x.

(2) Suppose that M^* and N^* are Hausdorff spaces. Let $(x_n)_{n\in\boldsymbol{N}}$ be a sequence of M^* and $(y_n)_{n\in\boldsymbol{N}}$ a sequence of and N^*. If y_n's are linearly independent and $\sum\limits_{n\in\boldsymbol{N}} x_n \widehat{\otimes} y_n = 0$, then $x_n = 0$ for all $n \in \boldsymbol{N}$.

Proof. (1) Since $M^* \otimes_{K^*} N^*$ is a 1st countable space by the definition of the topology of $M^* \otimes_{K^*} N^*$, $M^* \widehat{\otimes}_{K^*} N^*$ is a 1st countable space by (1.3.10). Since the image of $\eta_{M^* \otimes_{K^*} N^*}$ is dense by (1.3.2), it follows from (1.3.21) that there exist sequences $(x_n)_{n\in\boldsymbol{N}}$ and $(y_n)_{n\in\boldsymbol{N}}$ of M^* and N^* respectively such that $\sum\limits_{n\in\boldsymbol{N}} x_n \widehat{\otimes} y_n$ converges to x.

(2) Let N^*_n be a subspace of N^* spanned by $\{y_i | \, i \neq n\}$ and $p_n : N^* \to N^*/N^*_n$ the quotient map. Since $id_{M^*} \widehat{\otimes}_{K^*} p_n : M^* \widehat{\otimes}_{K^*} N^* \to M^* \widehat{\otimes}_{K^*} N^*/N^*_n$ maps $\sum\limits_{n\in\boldsymbol{N}} x_n \widehat{\otimes} y_n$ to $x_n \widehat{\otimes}_{K^*} p_n(y_n)$, we have $x_n \widehat{\otimes}_{K^*} p_n(y_n) = 0$. Since $y_n \notin N^*_n$, we have $p_n(y_n) \neq 0$ It follows from (2.3.16) that $x_n = 0$. □

3 Spaces of homomorphisms

3.1 Topology on spaces of homomorphisms

For $r \in K^l$ and a morphism $f : \Sigma^m M^* \to N^*$, we define a morphism $rf : \Sigma^{l+m} M^* \to N^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$ by $(rf)([l+m], x) = rf([m], x)$ for $x \in M^*$. In other words, rf is the following composition.

$$\Sigma^{l+m} M^* \xrightarrow{\varepsilon_{l,m,M^*}} \Sigma^l(\Sigma^m M^*) \xrightarrow{\Sigma^l f} \Sigma^l N^* \xrightarrow{\mu^r_{N^*}} N^*$$

It is easy to verify the following fact from the definition of K^*-module structure on suspensions of K^*-modules.

Lemma 3.1.1 *For $l, m, n \in \boldsymbol{Z}$, $r \in K^l$ and a morphism $f : \Sigma^m M^* \to N^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$, we have the following equality in $\mathrm{Hom}^c_{K^*}(\Sigma^{l+m+n} M^*, N^*)$.*

$$\Sigma^n(rf)\varepsilon_{n,l+m,M^*} = (-1)^{ln}(r\Sigma^n f)\varepsilon_{l+n,m,M^*}$$

Definition 3.1.2 *For objects M^* and N^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, we define an object $\mathcal{H}om^*(M^*, N^*)$ of $\mathcal{T}op\mathcal{M}od_{K^*}$ as follows. Put*

$$\mathcal{H}om^n(M^*, N^*) = (\mathcal{H}om^*(M^*, N^*))^n = \mathrm{Hom}^c_{K^*}(\Sigma^n M^*, N^*).$$

The maps $K^l \times \mathcal{H}om^n(M^*, N^*) \to \mathcal{H}om^{n+l}(M^*, N^*)$ for $l, n \in \boldsymbol{Z}$ given by $(r, f) \mapsto rf$ define a left K^*-module structure of $\mathcal{H}om^*(M^*, N^*)$.

For morphisms $f : M^* \to N^*$, $g : N^* \to L^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$, define maps $f^* : \mathcal{H}om^*(N^*, L^*) \to \mathcal{H}om^*(M^*, L^*)$ and $g_* : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(M^*, L^*)$ by $f^*(\varphi) = \varphi\Sigma^n f$ and $g_*(\psi) = g\psi$ for $\varphi \in \mathcal{H}om^n(N^*, L^*)$ and $\psi \in \mathcal{H}om^m(M^*, N^*)$. It is easy to verify that f^* and g_* are maps of K^*-modules.

For morphisms $f : S^* \to M^*$ and $g : N^* \to Q^*$, we put

$$O(f, g) = \mathrm{Ker}(g_* f^* : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(S^*, Q^*)).$$

In particular, if f is an inclusion map $i_{S^*} : S^* \to M^*$ and g is a quotient map $p_{U^*} : N^* \to N^*/U^*$, we denote $O(f, g)$ by $O(S^*, U^*)$. Let us denote by $\mathcal{F}_{M^*}$ the set of finitely generated submodules of M^*. Define a topology on $\mathcal{H}om^*(M^*, N^*)$ such that $\{O(S^*, U^*) |\, S^* \in \mathcal{F}_{M^*},\ U^* \in \mathcal{V}_{N^*}\}$ forms a fundamental system of neighborhoods of 0. We denote by M^{**} the dual space $\mathcal{H}om^*(M^*, K^*)$.

Remark 3.1.3 *(1) $O(f, g)$ depends only on the image of f and the kernel of g, namely $O(f, g) = O(\mathrm{Im}\, f, \mathrm{Ker}\, g)$.*
(2) For each $n \in \boldsymbol{Z}$, $O(S^*, U^*)^n$ consists of morphisms $f : \Sigma^n M^* \to N^*$ which map $\Sigma^n S^*$ into U^*.

Proposition 3.1.4 *Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$.*
(1) If $f : S^* \to M^*$, $h : T^* \to S^*$, $g : N^* \to Q^*$ and $k : Q^* \to P^*$ are morphisms in $\mathcal{T}op\mathcal{M}od_{K^*}$, then $O(f, g) \subset O(fh, kg)$. If h is an epimorphism and k is a monomorphism, then $O(f, g) = O(fh, kg)$.
(2) If S^* is a submodule of M and U^*, V^* are submodules of N^*, $O(S^*, U^*) \cap O(S^*, V^*) = O(S^*, U^* \cap V^*)$.
(3) If S^*, T^* are submodules of M and U^* is a submodule of N^*, $O(S^*, U^*) \cap O(T^*, U^*) = O(S^* + T^*, U^*)$.
$\left\{ O(K^* x, U^*) \,\middle|\, x \in \bigcup_{n \in \boldsymbol{Z}} M^n,\ U^* \in \mathcal{V}_{N^*} \right\}$ is a subbase of the neighborhoods of zero. In particular, $\mathcal{H}om^0(M^*, N^*)$ has the pointwise convergent topology.

Lemma 3.1.5 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$.*
(1) For $x \in M^*$ and $U^* \in \mathcal{N}_{M^*}$, $(U^* : x) = \{r \in K^* |\, rx \in U^*\}$ is an open ideal of K^*.
(2) For $S^* \in \mathcal{F}_{M^*}$ and $U^* \in \mathcal{N}_{M^*}$, $(U^* : S^*) = \{r \in K^* |\, rS^* \subset U^*\}$ is an open ideal of K^*.

Proof. (1) By the continuity of the structure map $\alpha : K^* \times M^* \to M^*$ of M^*, there exist $\mathfrak{a} \in \mathcal{I}_{K^*}$ and $V^* \in \mathcal{N}_{M^*}$ satisfying $\alpha(\mathfrak{a} \times (\{x\} + V^*)) \subset U^*$ for $x \in M^*$. Hence $(U^* : x)$ contains an open ideal $\mathfrak{a}$ and it is open.
(2) If S^* is generated by $x_1, x_2, \ldots, x_n$, then $(U^* : S^*) = \bigcap_{i=1}^{n} (U^* : x_i)$. Hence $(U^* : S^*)$ is open by (1). □

Proposition 3.1.6 *$\mathcal{H}om^*(M^*, N^*)$ is a topological K^*-module. If the topology of N^* is coarser than the topology induced by K^*, so is the topology of $\mathcal{H}om^*(M^*, N^*)$.*

Proof. Clearly, $\mathcal{H}om^*(M^*, N^*)$ is a topological abelian group. For $r \in K^*$, $f \in \mathcal{H}om^*(M^*, N^*)$, $S^* \in \mathcal{F}_{M^*}$ and $U^* \in \mathcal{N}_{N^*}$, $(U^* \colon f(S^*))$ is an open ideal of K^* by (3.1.5) and it is easy to verify that $(\{r\} + (U^* \colon f(S^*)))(\{f\} + O(S^*, U^*))$ is contained in $\{rf\} + O(S^*, U^*)$. Hence the structure map $K^* \times \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(M^*, N^*)$ is continuous.

Suppose that the topology of N^* is coarser than the topology induced by K^*. For $U^* \in \mathcal{N}_{N^*}$, there exists $\mathfrak{a} \in \mathcal{N}_{N^*}$ satisfying $\mathfrak{a}N^* \subset U^*$. Then, we have $\mathfrak{a}\mathcal{H}om^*(M^*, N^*) \subset O(M^*, U^*)$. □

Proposition 3.1.7 *Let $f : M^* \to N^*$ and $g : N^* \to L^*$ be morphisms in $\mathcal{T}op\mathcal{M}od_{K^*}$ and consider maps $f^* : \mathcal{H}om^*(N^*, L^*) \to \mathcal{H}om^*(M^*, L^*)$ and $g_* : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(M^*, L^*)$. Suppose $S^* \in \mathcal{F}_{M^*}$, $T^* \in \mathcal{F}_{N^*}$ and $U^* \in \mathcal{V}_{L^*}$, $M^* \in \mathcal{V}_{N^*}$.*

(1) $(f^*)^{-1}(O(S^*, U^*)) = O(f(S^*), U^*)$ and $(g_*)^{-1}(O(S^*, U^*)) = O(S^*, g^{-1}(U^*))$ hold. Hence f^* and g_* are continuous.

(2) If f has a continuous left inverse $p : N^* \to M^*$, then $f^*(O(T^*, U^*)) \supset O(p(T^*), U^*)$ holds and f^* is a surjective open map.

(3) If g has a continuous right inverse $s : L^* \to N^*$, then $g_*(O(S^*, U^*)) \supset O(S^*, s^{-1}(U^*))$ holds and g_* is a surjective open map.

(4) If f is surjective, then $f^* : \mathcal{H}om^*(N^*, L^*) \to \mathcal{H}om^*(M^*, L^*)$ is a homeomorphism onto its image.

Proof. (1) is easy.

(2) For $\psi \in O(p(T^*), U^*)^n$, it is clear that $\psi\Sigma^n p \in O(T^*, U^*)$ and $f^*(\psi\Sigma^n p) = \psi$ hold. Thus ψ belongs to $f^*(O(T^*, U^*))$.

(3) If $\psi \in O(S^*, s^{-1}(U^*))^n \subset \mathcal{H}om^n(M^*, L^*)$, then $s\psi(\Sigma^n S^*) = s(s^{-1}(U^*)) \subset U^*$, hence $s\psi \in O(S^*, U^*)^n \subset \mathcal{H}om^n(M^*, N^*)$. Since $g_*(s\psi) = gs\psi = \psi$, $\psi \in g(O(S^*, U^*))$.

(4) For $S^* \in \mathcal{F}_{N^*}$, take $T^* \in \mathcal{F}_{M^*}$ such that $f(T^*) = S^*$. It is clear that $f^*(O(S^*, U^*)) \subset O(T^*, U^*)$ for $U^* \in \mathcal{V}_{L^*}$. Assume that $gf \in O(T^*, U^*)$ for $g \in \mathcal{H}om^*(N^*, L^*)$. Then $g(S^*) = g(f(T^*)) \subset U^*$, namely $g \in O(S^*, U^*)$. It follows that $O(T^*, U^*) \cap \operatorname{Im} f^* \subset f^*(O(S^*, U^*))$. Hence we have $O(T^*, U^*) \cap \operatorname{Im} f^* = f^*(O(S^*, U^*))$ and f^* is an open map onto its image. □

Corollary 3.1.8 *Suppose that K^* is a field.*

(1) If M^* is supercofinite, $i_{S^*}^* : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(S^*, N^*)$ is a surjective open map for $S^* \in \mathcal{F}_{M^*}$.

(2) For $U^* \in \mathcal{V}_{N^*}$, $p_{U^**} : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(M^*, N^*/U^*)$ is a surjective open map.

Proof. (1) Straightforward from (2) of (1.4.9) and (2) of (3.1.7).

(2) Since N^*/U^* is discrete and $p_{U^*} : N^* \to N^*/U^*$ is surjective, p_{U^*} has a continuous right inverse. Hence the assertion follows from (3) of (3.1.7). □

For right R^*-modules M^* and N^*, we denote by $\mathcal{H}om^*_{R^*}(M^*, N^*)$ the subset of $\mathcal{H}om^*(M^*, N^*)$ consisting of homomorphisms of right R^*-modules. We give $\mathcal{H}om^*_{R^*}(M^*, N^*)$ the topology induced by $\mathcal{H}om^*(M^*, N^*)$. Let us denote by $\mathcal{F}^{R^*}_{M^*}$ the set of finitely generated R^*-submodules of M^*. For $S^* \in \mathcal{F}^{R^*}_{M^*}$ and $U^* \in \mathcal{V}^{R^*}_{N^*}$, we put

$$O_{R^*}(S^*, U^*) = \operatorname{Ker}(p_{U^**} i_{S^*}^* : \mathcal{H}om^*_{R^*}(M^*, N^*) \to \mathcal{H}om^*_{R^*}(S^*, N^*/U^*)).$$

Proposition 3.1.9 *Suppose that $\mathcal{V}^{R^*}_{N^*}$ is a fundamental system of neighborhoods of 0 of N^*. Then,*

$$\left\{ O_{R^*}(S^*, U^*) \,\middle|\, S^* \in \mathcal{F}^{R^*}_{M^*},\, U^* \in \mathcal{V}^{R^*}_{N^*} \right\}$$

is a fundamental system of neighborhoods of 0 of $\mathcal{H}om^*_{R^*}(M^*, N^*)$.

Proof. For $S^* \in \mathcal{F}^{R^*}_{M^*}$, we choose generators $x_1, x_2, \ldots, x_n$ of S^* over R^* and let $\tilde{S}^*$ the K^*-submodule of M^* generated by $x_1, x_2, \ldots, x_n$. For $V^* \in \mathcal{V}^{R^*}_{N^*}$ and $f \in O(\tilde{S}^*, V^*) \cap \mathcal{H}om^n_{R^*}(M^*, N^*)$, since $f([n], x_i) \in V^*$ and f is a homomorphism of R^*-modules and V^* is an R^*-submodule of N^*, we have $f(\Sigma^n S^*) \subset V^*$, that is, $f \in O_{R^*}(S^*, V^*)$. Therefore $O(\tilde{S}^*, V^*) \cap \mathcal{H}om^*_{R^*}(M^*, N^*) \subset O_{R^*}(S^*, V^*)$ which shows that $O_{R^*}(S^*, V^*)$ is an open submodule of $\mathcal{H}om^*_{R^*}(M^*, N^*)$.

Suppose $S^* \in \mathcal{F}_{M^*}$ and $U^* \in \mathcal{V}_{N^*}$. Let $\bar{S}^*$ be the R^*-submodule of M^* generated by S^*, then $\bar{S}^* \in \mathcal{F}^{R^*}_{M^*}$. There exists $V^* \in \mathcal{V}^{R^*}_{N^*}$ which is contained in U^* by the assumption. Then, we have $O_{R^*}(\bar{S}^*, V^*) \subset O(S^*, U^*)$. Hence $\left\{ O_{R^*}(S^*, U^*) \,\middle|\, S^* \in \mathcal{F}^{R^*}_{M^*},\, U^* \in \mathcal{V}^{R^*}_{N^*} \right\}$ is cofinal in the set of neighborhood of 0 of $\mathcal{H}om^*_{R^*}(M^*, N^*)$. □

Proposition 3.1.10 *Let L^*, M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. A map*

$$T : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(M^* \otimes_{K^*} L^*, N^* \otimes_{K^*} L^*)$$

defined by $T(f) = f \otimes_{K^*} id_{L^*}$ is continuous.

Proof. Suppose $S^* \in \mathcal{F}_{M^*}$, $T^* \in \mathcal{F}_{L^*}$, $U^* \in \mathcal{V}_{N^*}$ and $V^* \in \mathcal{F}_{L^*}$. Let us denote by $\overline{S^* \otimes_{K^*} T^*}$ and $\overline{U^* \otimes_{K^*} T^*}$ the images of maps $S^* \otimes_{K^*} T^* \to M^* \otimes_{K^*} L^*$ and $U^* \otimes_{K^*} T^* \to N^* \otimes_{K^*} L^*$ induced by inclusion maps. If $f \in O(S^*, U^*)^n$, $T(f)$ maps $\Sigma^n \overline{S^* \otimes_{K^*} T^*}$ into $\overline{U^* \otimes_{K^*} T^*}$ which is contained in $o(U^*, V^*)$. Thus we have $T(O(S^*, U^*)) \subset O(\overline{S^* \otimes_{K^*} T^*}, o(U^*, V^*))$. □

Proposition 3.1.11 *Let $(M_i^*)_{i\in I}$ be a family of objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ and N^* an object of $\mathcal{T}op\mathcal{M}od_{K^*}$. For $j \in I$, we denote by $\iota_j : M_j^* \to \bigoplus_{i\in I} M_i^*$ the injection onto the j-th summand. We define a map*

$$P : \mathcal{H}om^*\Big(\bigoplus_{i\in I} M_i^*, N^*\Big) \to \prod_{i\in I} \mathcal{H}om^*(M_i^*, N^*)$$

by $P(\varphi) = (\varphi\Sigma^n\iota_i)_{i\in I}$ for $\varphi \in \mathcal{H}om^n\Big(\bigoplus_{i\in I} M_i^*, N^*\Big)$. Then, P is an isomorphism.

Proof. It follows from (1.1.17) and (3.1.7) that P is continuous. For $(\varphi_i)_{i\in I} \in \prod_{i\in I} \mathcal{H}om^n(M_i^*, N^*)$, there exists unique map $\varphi : \Sigma^n\Big(\bigoplus_{i\in I} M_i^*\Big) \to N^*$ that satisfies $\varphi\Sigma^n\iota_i = \varphi_i$ for any $i \in I$. Then, the inverse P^{-1} of P is defined by $P^{-1}((\varphi_i)_{i\in I}) = \varphi$. For $(x_i)_{i\in I} \in \bigoplus_{i\in I} M_i^*$ and $U^* \in \mathcal{V}_{N^*}$, since $x_i = 0$ except for finite number of i's, $\prod_{i\in I} O(K^*x_i, U^*)$ is an open set of $\prod_{i\in I} \mathcal{H}om^*(M_i^*, N^*)$. If $\varphi \in O(K^*(x_i)_{i\in I}, U^*)^n$, then $\varphi\Sigma^n\iota_i(x_i) \in U^*$ for each $i \in I$, which means $P(O(K^*(x_i)_{i\in I}, U^*)) \subset \prod_{i\in I} O(K^*x_i, U^*)$. For $(\varphi_i)_{i\in I} \in \prod_{i\in I} O(K^*x_i, U^*)$, put $P^{-1}((\varphi_i)_{i\in I}) = \varphi$. Then, we have $\varphi((x_i)_{i\in I}) = \sum_{i\in I} \varphi_i(x_i) \in U^*$. It follows that $P^{-1}\Big(\prod_{i\in I} O(K^*x_i, U^*)\Big) \subset O(K^*(x_i)_{i\in I}, U^*)$, namely $\prod_{i\in I} O(K^*x_i, U^*) \subset P(O(K^*(x_i)_{i\in I}, U^*))$. Therefore $P(O(K^*(x_i)_{i\in I}, U^*)) = \prod_{i\in I} O(K^*x_i, U^*)$ holds and P is an open map. □

Remark 3.1.12 *For $j \in I$, let $p_j : \bigoplus_{i\in I} M_i^* \to M_j^*$ be a map defined by $p_j((x_i)_{i\in I}) = x_j$. If I is a finite set, P^{-1} is given by $P^{-1}((\varphi_i)_{i\in I}) = \sum_{j\in I} \varphi_j\Sigma^n p_j = \sum_{j\in I} p_j^*(\varphi_j)$ if $(\varphi_i)_{i\in I} \in \prod_{i\in I} \mathcal{H}om^n(M_i^*, N^*)$.*

Proposition 3.1.13 *Let $(N_i^*)_{i\in I}$ be a family of objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ and M^* an object of $\mathcal{T}op\mathcal{M}od_{K^*}$. For $j \in I$, we denote by $\mathrm{pr}_j : \prod_{i\in I} N_i^* \to N_j^*$ the projection onto the j-th component. We define a map*

$$Q : \mathcal{H}om^*\Big(M^*, \prod_{i\in I} N_i^*,\Big) \to \prod_{i\in I} \mathcal{H}om^*(M^*, N_i^*)$$

by $Q(\varphi) = (\mathrm{pr}_i\varphi)_{i\in I}$. Then, Q is an isomorphism.

Proof. It follows from (1.1.17) and (3.1.7) that Q is continuous. For $(\varphi_i)_{i\in I} \in \prod_{i\in I} \mathcal{H}om^n(M^*, N_i^*)$, there exists unique map $\varphi : \Sigma^n M^* \to \prod_{i\in I} N_i^*$ that satisfies $\mathrm{pr}_i\varphi = \varphi_i$ for any $i \in I$. Then, the inverse Q^{-1} of Q is defined by $Q^{-1}((\varphi_i)_{i\in I}) = \varphi$. For $x \in M^*$ and $U_i^* \in \mathcal{V}_{N_i^*}$ such that $U_i^* = N^*$ except for finite number of i's so that $\prod_{i\in I} U_i^*$ is an open set of $\prod_{i\in I} N_i^*$, if $\varphi \in O\Big(K^*x, \prod_{i\in I} U_i^*\Big)^n$, then $\mathrm{pr}_i\varphi(x) \in U_i^*$ for each $i \in I$, which means $Q\Big(O\Big(K^*x, \prod_{i\in I} U_i^*\Big)\Big) \subset \prod_{i\in I} O(K^*x, U_i^*)$. For $(\varphi_i)_{i\in I} \in \prod_{i\in I} O(K^*x, U_i^*)$, put $Q^{-1}((\varphi_i)_{i\in I}) = \varphi$. Then, we have $\varphi((x_i)_{i\in I}) = (\varphi_i(x_i))_{i\in I} \in \prod_{i\in I} U_i^*$. It follows that $Q^{-1}\Big(\prod_{i\in I} O(K^*x, U_i^*)\Big) \subset O\Big(K^*x, \prod_{i\in I} U_i^*\Big)$, namely $\prod_{i\in I} O(K^*x, U_i^*) \subset Q\Big(O\Big(K^*x, \prod_{i\in I} U_i^*\Big)\Big)$. Therefore $Q\Big(O\Big(K^*x, \prod_{i\in I} U_i^*\Big)\Big) = \prod_{i\in I} O(K^*x, U_i^*)$ holds and Q is an open map. □

Let K^* be a topological graded ring such that $K^n = \{0\}$ if $n \neq 0$. Recall that $u_n : id_{\mathcal{T}op\mathcal{M}od_{K^*}} \to \iota_n\epsilon_n$ denotes the unit of the adjunction $\epsilon_n \dashv \iota_n$ and $c_n : \iota_n\epsilon_n \to id_{\mathcal{T}op\mathcal{M}od_{K^*}}$ denotes the counit of the adjunction $\iota_n \dashv \epsilon_n$ (1.2.5).

Proposition 3.1.14 *For an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ and $i \in \boldsymbol{Z}$, there is an isomorphism*

$$\theta_{M^*,i} : \iota_i\epsilon_i(\mathcal{H}om^*(M^*, K^*)) \to \mathcal{H}om^*(\iota_{-i}\epsilon_{-i}(M^*), K^*)$$

which is natural in M^*.

Proof. Since $K^n = \{0\}$ for $n \neq 0$ and $(\Sigma^j(\iota_{-i}\epsilon_{-i}(M^*)))^0 = [j] \times (\iota_{-i}\epsilon_{-i}(M^*))^{-j} = \{0\}$ if $j \neq i$, we have $\mathcal{H}om^j(\iota_{-i}\epsilon_{-i}(M^*), K^*) = \mathrm{Hom}^c_{K^*}(\Sigma^j\iota_{-i}\epsilon_{-i}(M^*), K^*) = \{0\}$ if $j \neq i$. For $f \in \mathcal{H}om^i(M^*, K^*)$, we define $(\theta_{M^*,i}(f) : \Sigma^i\iota_{-i}\epsilon_{-i}(M^*) \to K^*) \in \mathcal{H}om^i(\iota_{-i}\epsilon_{-i}(M^*), K^*)$ to be a composition

$$\Sigma^i\iota_{-i}\epsilon_{-i}(M^*) = \iota_0\epsilon_0(\Sigma^i M^*) \xrightarrow{\iota_0\epsilon_0(f)} \iota_0\epsilon_0(K^*) \xrightarrow{c_{0\,K^*}} K^*.$$

For $g \in \iota_i\epsilon_i(\mathcal{H}om^*(\iota_{-i}\epsilon_{-i}(M^*), K^*))^i = \mathcal{H}om^i(\iota_{-i}\epsilon_{-i}(M^*), K^*)$, let $\theta^{-1}_{M^*,i}(g) \in \mathcal{H}om^i(M^*, K^*)$ be the map defined by

$$\theta^{-1}_{M^*,i}(g)([i], x) = \begin{cases} g([i], x) & x \in M^{-i} \\ 0 & x \in M^j,\ j \neq -i \end{cases}.$$

Then, we see that $\theta^{-1}_{M^*,i} : \mathcal{H}om^i(\iota_{-i}\epsilon_{-i}(M^*), K^*) \to \iota_i\epsilon_i(\mathcal{H}om^*(M^*, K^*))^i$ is the inverse of $\theta_{M^*,i}$. □

Remark 3.1.15 *Then, the following diagram is commutative.*

$$\begin{tikzcd} \mathcal{H}om^*(M^*, K^*) \arrow[r, "u_{i\,\mathcal{H}om^*(M^*,K^*)}"] \arrow[dr, "c^*_{-i\,M^*}"] & \iota_i\epsilon_i(\mathcal{H}om^*(M^*, K^*)) \arrow[d, "\theta_{M^*,i}"] \\ & \mathcal{H}om^*(\iota_{-i}\epsilon_{-i}(M^*), K^*) \end{tikzcd}$$

For an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ and an object R^* of $\mathcal{T}op\mathcal{A}lg_{K^*}$ with multiplication $\mu_{R^*} : R^* \otimes_{K^*} R^* \to R^*$, let $\alpha_{R^*} : (M^* \otimes_{K^*} R^*) \otimes_{K^*} R^* \to M^* \otimes_{K^*} R^*$ be the following composition.

$$(M^* \otimes_{K^*} R^*) \otimes_{K^*} R^* \xrightarrow{\cong} M^* \otimes_{K^*} (R^* \otimes_{K^*} R^*) \xrightarrow{id_{M^*} \otimes_{K^*} \mu_{R^*}} M^* \otimes_{K^*} R^*$$

Then, $M^* \otimes_{K^*} R^*$ is a right R^*-module with structure map α_{R^*}.

Proposition 3.1.16 *Let R^* be an object of $\mathcal{T}op\mathcal{A}lg_{K^*}$ with unit $u : K^* \to R^*$ and N^* a right R^*-module. We denote by N^*_u a right K^*-module N^* with structure map given by $(m, x) \mapsto mu(x)$. For a right K^*-module M^*, we regard $M^* \otimes_{K^*} R^*$ as a right R^*-module. We denote by $i_1 : M^* \to M^* \otimes_{K^*} R^*$ the map defined by $i_1(m) = m \otimes 1$. Define a map*

$$ad^{M^*}_{N^*} : \mathcal{H}om^*_{R^*}(M^* \otimes_{K^*} R^*, N^*) \to \mathcal{H}om^*(M^*, N^*_u)$$

by $ad^{M^*}_{N^*}(f) = fi_1$. If $\mathcal{V}^{R^*}_{N^*}$ is a fundamental system of neighborhoods of 0 of N^* and the topology of N^* is coarser than the topology induced by R^*, $ad^{M^*}_{N^*}$ is an isomorphism.

Proof. For $S^* \in \mathcal{F}_{M^*}$ and $U^* \in \mathcal{V}^{R^*}_{N^*}$, since $S^* \otimes_{K^*} R^* \in \mathcal{F}^{R^*}_{M^* \otimes_{K^*} R^*}$ and $ad^{M^*}_{N^*}$ maps $O_{R^*}(S^* \otimes_{K^*} R^*, U^*)$ into $O(S^*, U^*)$, $ad^{M^*}_{N^*}$ is continuous. The right R^*-module structure map $\beta : N^* \otimes_{K^*} R^* \to N^*$ of N^* is continuous by (1.1.10) and the inverse $(ad^{M^*}_{N^*})^{-1}$ of $ad^{M^*}_{N^*}$ is given by $(ad^{M^*}_{N^*})^{-1}(g) = \beta(g \otimes_{K^*} id_{R^*})$ for $g \in \mathcal{H}om^*(M^*, N^*_u)$. Then, $(ad^{M^*}_{N^*})^{-1}$ is continuous by (3.1.10) and (1) of (3.1.7). □

Definition 3.1.17 *Consider the functor $c_{K^*{}_*} : \mathcal{T}op\mathcal{M}od_{K^*} \to \mathcal{T}op\mathcal{M}od_{K^*}$ induced by the conjugation c_{K^*} of K^* defined in (1.2.1). For an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ and $n \in \boldsymbol{Z}$, we denote by $c^n_{K^*{}_*}(M^*)$ by ${}^nM^*$. We call ${}^1M^*$ the conjugate of M^*. Define a map $c_{M^*} : M^* \to {}^1M^*$ by $c_{M^*}(x) = (-1)^p x$ if $x \in M^p$. Then, it is clear that c_{M^*} is an isomorphism in $\mathcal{T}op\mathcal{M}od_{K^*}$ satisfying $c_{M^*}c_{M^*} = id_{M^*}$. We call c_{M^*} the conjugation of M^*.*

Remark 3.1.18 *(1) It follows from (3) of (2.2.1) that ${}^m({}^nM^*) = {}^{m+n}M^*$ and that ${}^nM^* = M^*$ if n is even and ${}^nM^*$ is the conjugate of M^* if n is odd.*

(2) It follows from (5) of (2.2.1) that ${}^n\Sigma^m M^* = \Sigma^m({}^nM^*)$.

(3) $c^{m-n}_{M^*} : {}^nM^* \to {}^mM^*$ is an isomorphism in $\mathcal{T}op\mathcal{M}od_{K^*}$.

It is easy to verify the following.

Proposition 3.1.19 *For $m, n \in \boldsymbol{Z}$ and $M^*, N^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{K^*}$, let $\tilde{\beta}^{m,n}_{M^*,N^*} : {}^mM^* \times {}^nN^* \to {}^{m+n}(M^* \otimes_{K^*} N^*)$ be the map given by $\tilde{\beta}^{m,n}_{M^*,N^*}(x, y) = c^n_{M^*}(x) \otimes c^m_{N^*}(y)$. Then, $\tilde{\beta}^{m,n}_{M^*,N^*}$ is a bilinear and induces an isomorphism ${}^mM^* \otimes_{K^*} {}^nN^* \to {}^{m+n}(M^* \otimes_{K^*} N^*)$ in $\mathcal{T}op\mathcal{M}od_{K^*}$.*

We denote by $\beta^{m,n}_{M^*,N^*}: {}^mM^* \otimes_{K^*} {}^nN^* \to {}^{m+n}(M^* \otimes_{K^*} N^*)$ the isomorphism induced by $\tilde{\beta}^{m,n}_{M^*,N^*}$.

The following facts are direct consequences of the above definitions and (3.1.1).

Proposition 3.1.20 *Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ and $n \in \boldsymbol{Z}$.*

(1) Define a map $\tilde{\sigma}_{n,M^*,N^*} : \Sigma^n\mathcal{H}om^*(M^*,N^*) \to \mathcal{H}om^*(M^*,\Sigma^nN^*)$ by $\tilde{\sigma}_{n,M^*,N^*}([n],f) = (\Sigma^nf)\varepsilon_{n,k-n,M^*}$ for $f \in \mathcal{H}om^{k-n}(M^*,N^*) = \mathrm{Hom}^c_{K^*}(\Sigma^{k-n}M^*,N^*)$. Then, $\tilde{\sigma}_{n,M^*,N^*}$ is an isomorphism in $\mathcal{T}op\mathcal{M}od_{K^*}$.

(2) Define a map $\bar{\sigma}_{n,M^*,N^*} : \Sigma^n\mathcal{H}om^*(M^*,N^*) \to {}^n\mathcal{H}om^*(\Sigma^{-n}M^*,N^*)$ by $\bar{\sigma}_{n,M^*,N^*}([n],f) = f\varepsilon^{-1}_{k,-n,M^*}$ for $f \in \mathcal{H}om^{k-n}(M^*,N^*) = \mathrm{Hom}^c_{K^*}(\Sigma^{k-n}M^*,N^*)$. Then, $\bar{\sigma}_{n,M^*,N^*}$ is an isomorphism in $\mathcal{T}op\mathcal{M}od_{K^*}$.

Definition 3.1.21 *Define a map $\varepsilon_{M^*} : M^* \to \Sigma^0M^*$ by $\varepsilon_{M^*}(x) = ([0],x)$. Then, it follows from (3.1.20) that the composition*

$$\mathcal{H}om^*(M^*,N^*) \xrightarrow{(\varepsilon^{-1}_{M^*})^*} \mathcal{H}om^*(\Sigma^0M^*,N^*) \xrightarrow{(\varepsilon^{-1}_{-n,n,M^*})^*} \mathcal{H}om^*(\Sigma^{-n}(\Sigma^nM^*),N^*) \xrightarrow{\bar{\sigma}^{-1}_{n,\Sigma^nM^*,N^*}}$$

$$\Sigma^n\mathcal{H}om^*(\Sigma^nM^*,N^*) \xrightarrow{\tilde{\sigma}_{n,\Sigma^nM^*,N^*}} \mathcal{H}om^*(\Sigma^nM^*,\Sigma^nN^*)$$

gives an isomorphism $\sigma_{n,M^*,N^*} : \mathcal{H}om^*(M^*,N^*) \to {}^n\mathcal{H}om^*(\Sigma^nM^*,\Sigma^nN^*)$ which is called the n-fold suspension isomorphism.

For $n \in \boldsymbol{Z}$ and $x \in M^n$, we define maps $e_x : {}^n\Sigma^nK^* \to M^*$ and $E_x : \mathcal{H}om^*(M^*,N^*) \to {}^n\Sigma^{-n}N^*$ by $e_x([n],a) = ax$ for $a \in K^*$ and $E_x(f) = ([-n], f([k],x))$ for $f \in \mathcal{H}om^k(M^*,N^*)$, respectively. Then, e_x and E_x are morphisms in $\mathcal{T}op\mathcal{M}od_{K^*}$. We call E_x the evaluation map at x.

Proposition 3.1.22 *Define a map $ev_n : \mathrm{Hom}^c_{K^*}({}^n\Sigma^nK^*,M^*) \to M^n$ by $ev_n(f) = f([n],1)$. Then, ev_n is an isomorphism of abelian groups.*

Proof. It is clear that ev_n is injective. For $x \in M^n$, since ev_n maps e_x to x, ev_n is surjective. □

Proposition 3.1.23 *For $x \in M^n$, $E_x : \mathcal{H}om^*(M^*,N^*) \to {}^n\Sigma^{-n}N^*$ is the following composition.*

$$\mathcal{H}om^*(M^*,N^*) \xrightarrow{e_x^*} \mathcal{H}om^*({}^n\Sigma^nK^*,N^*) \xrightarrow{E_{([n],1)}} {}^n\Sigma^{-n}N^*$$

Moreover, $E_{([n],1)} : \mathcal{H}om^*({}^n\Sigma^nK^*,N^*) \to {}^n\Sigma^{-n}N^*$ is the following composition.

$$\mathcal{H}om^*({}^n\Sigma^nK^*,N^*) \xrightarrow{\bar{\sigma}^{-1}_{-n,K^*,N^*}} {}^n\Sigma^{-n}\mathcal{H}om^*(K^*,N^*) \xrightarrow{{}^n\Sigma^{-n}E_1} {}^n\Sigma^{-n}N^*$$

The next result immediately follows from (3) of (3.1.4).

Proposition 3.1.24 *Let L^*, M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$.*

(1) For a submodule U^* of N^* and $x \in M^n$, $E_x : \mathcal{H}om^*(M^*,N^*) \to {}^n\Sigma^{-n}N^*$ maps $O(K^*x,U^*)$ into $\Sigma^{-n}U^*$.

(2) The topology on $\mathcal{H}om^*(M^*,N^*)$ is the coarsest topology such that $E_x : \mathcal{H}om^*(M^*,N^*) \to {}^n\Sigma^{-n}N^*$ is continuous for any $n \in \boldsymbol{Z}$ and $x \in M^n$.

(3) A linear map $\varphi : L^* \to \mathcal{H}om^*(M^*,N^*)$ is continuous if and only if $E_x\varphi : L^* \to \Sigma^{-n}N^*$ is continuous for any $n \in \boldsymbol{Z}$ and $x \in M^n$.

(4) A map $\kappa_{N^*} : N^* \to \mathcal{H}om^*(K^*,N^*)$ defined by $(\kappa_{N^*}(x))([n],a) = c^n_{K^*}(a)x$ for $x \in N^n$ and $a \in K^*$ is the inverse of $E_1 : \mathcal{H}om^*(K^*,N^*) \to N^*$. Hence $E_{([n],1)} : \mathcal{H}om^*(\Sigma^nK^*,N^*) \to {}^n\Sigma^{-n}N^*$ is an isomorphism.

Definition 3.1.25 *A graded set is a pair (S,d) of a set S and a map $d : S \to \boldsymbol{Z}$. For graded sets (S,d) and (T,e), we say that a map $f : S \to T$ is a map of graded sets from (S,d) to (T,e) if $ef = d$. We denote by $\mathcal{S}et^*$ the category of graded sets and map of graded sets.*

We define a functor $F : \mathcal{S}et^* \to \mathcal{T}op\mathcal{M}od_{K^*}$ as follows. For a graded set (S,d), we set

$$F(S,d) = \coprod_{x \in S} {}^{d(x)}\Sigma^{d(x)}K^*.$$

If f is a is a map of graded sets from (S,d) to (T,e), $F(f) : F(S,d) \to F(T,e)$ is the unique map satisfying $F(f)\iota_z = \iota_{f(z)}$ for any $z \in S$, where $\iota_z : {}^{d(z)}\Sigma^{d(z)}K^* \to F(S,d)$ and $\iota_w : {}^{e(w)}\Sigma^{e(w)}K^* \to F(T,e)$ are the canonical maps.

For an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, we set $S(M^*) = \{(n,x) \in \boldsymbol{Z} \times M^* \,|\, x \in M^n\}$ and define $d_{M^*} : S(M^*) \to \boldsymbol{Z}$ by $d_{M^*}(n,x) = n$. Thus we have a graded set $U(M^*) = (S(M^*), d_{M^*})$. For a homomorphism $f : M^* \to N^*$, let us define $U(f) : S(M^*) \to S(N^*)$ by $U(f)(n,x) = (n, f(x))$. Then, $U(f)$ is a map of graded sets and we have a functor $U : \mathcal{T}op\mathcal{M}od_{K^*} \to \mathcal{S}et^*$.

Proposition 3.1.26 *$F : \mathcal{S}et^* \to \mathcal{T}op\mathcal{M}od_{K^*}$ is a left adjoint of U.*

Proof. Define a map $\varphi : \mathrm{Hom}^c_{K^*}(F(S,d), M^*) \to \mathcal{S}et^*((S,d), U(M^*))$ by $(\varphi(f))(x) = (d(x), f(\iota_x([d(x)],1)))$ for $x \in S$. Then, φ is bijective. In fact, for a map $g : (S,d) \to U(M^*)$ of graded sets, let $\bar{g} : F(S,d) \to M^*$ be the unique homomorphism satisfying $(\bar{g}\iota_z)([d(z)],1) = p_2 g(z)$ for any $z \in S$, where $p_2 : S(M^*) \to M^*$ is the map given by $p_2(n,x) = x$. Then, $\varphi^{-1} : \mathcal{S}et^*((S,d), U(M^*)) \to \mathrm{Hom}^c_{K^*}(F(S,d), M^*)$ is given by $\varphi^{-1}(g) = \bar{g}$. □

Definition 3.1.27 *We say that an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ is free if there exists a graded set (S,d) such that M^* is isomorphic to $F(S,d)$.*

For a map $d : S \to \boldsymbol{Z}$ and $n \in \boldsymbol{Z}$, let $\Sigma^n d : S \to \boldsymbol{Z}$ be the map given by $\Sigma^n d(x) = d(x) + n$. We note that, if f is a map of graded sets from (S,d) to (T,e), then f is also a map of graded sets from $(S, \Sigma^n d)$ to $(T, \Sigma^n e)$. Define a functor $\Sigma^n : \mathcal{S}et^* \to \mathcal{S}et^*$ for $n \in \boldsymbol{Z}$ by $\Sigma^n(S,d) = (S, \Sigma^n d)$ and $\Sigma^n(f) = f$.

Proposition 3.1.28 *The composition $\mathcal{S}et^* \xrightarrow{\Sigma^n} \mathcal{S}et^* \xrightarrow{F} \mathcal{T}op\mathcal{M}od_{K^*}$ of functors is naturally equivalent to the composition $\mathcal{S}et^* \xrightarrow{F} \mathcal{T}op\mathcal{M}od_{K^*} \xrightarrow{\Sigma^n} \mathcal{T}op\mathcal{M}od_{K^*}$.*

Proof. For $(S,d) \in \mathrm{Ob}\,\mathcal{S}et^*$, let $\Psi^n_{(S,d)} : \coprod_{x\in S} \Sigma^n\left({}^{d(x)}\Sigma^{d(x)}K^*\right) \to \Sigma^n\left(\coprod_{x\in S} {}^{d(x)}\Sigma^{d(x)}K^*\right)$ be the unique morphism satisfying $\Psi^n_{(S,d)}\iota^n_v = \Sigma^n\iota_v$ for any $v \in S$, where $\iota^n_v : \Sigma^n\left({}^{d(v)}\Sigma^{d(v)}K^*\right) \to \coprod_{x\in S} \Sigma^n\left({}^{d(x)}\Sigma^{d(x)}K^*\right)$ is the canonical morphism. Then, $\Psi^n_{(S,d)}$ is an isomorphism. Since $\Sigma^n\left({}^{d(x)}\Sigma^{d(x)}c^n_{K^*}\right) : \Sigma^n\left({}^{d(x)}\Sigma^{d(x)}{}^nK^*\right) \to \Sigma^n\left({}^{d(x)}\Sigma^{d(x)}K^*\right)$ and $\varepsilon_{n,d(x),K^*} : {}^{d(x)+n}\Sigma^{d(x)+n}K^* \to {}^n\Sigma^n\left({}^{d(x)}\Sigma^{d(x)}K^*\right) = \Sigma^n\left({}^{d(x)}\Sigma^{d(x)}{}^nK^*\right)$ are isomorphisms of K^*-modules for each $x \in S$, we have isomorphisms $\coprod_{x\in S} \Sigma^n\left({}^{d(x)}\Sigma^{d(x)}c^n_{K^*}\right) : \coprod_{x\in S} \Sigma^n\left({}^{d(x)}\Sigma^{d(x)}{}^nK^*\right) \to \coprod_{x\in S} \Sigma^n\left({}^{d(x)}\Sigma^{d(x)}K^*\right)$ and $\coprod_{x\in S} \varepsilon_{n,d(x),K^*} : \coprod_{x\in S} {}^{d(x)+n}\Sigma^{d(x)+n}K^* \to \coprod_{x\in S} {}^n\Sigma^n\left({}^{d(x)}\Sigma^{d(x)}K^*\right) = \coprod_{x\in S} \Sigma^n\left({}^{d(x)}\Sigma^{d(x)}{}^nK^*\right)$. Then, a natural equivalence $F\Sigma^n \to \Sigma^n F$ is given by the following composition

$$\begin{aligned}
F\Sigma^n(S,d) = \coprod_{x\in S} {}^{d(x)+n}\Sigma^{d(x)+n}K^* &\xrightarrow{\coprod_{x\in S}\varepsilon_{n,d(x),K^*}} \coprod_{x\in S} {}^n\Sigma^n\left({}^{d(x)}\Sigma^{d(x)}K^*\right) = \coprod_{x\in S} \Sigma^n\left({}^{d(x)}\Sigma^{d(x)}{}^nK^*\right) \\
&\xrightarrow{\coprod_{x\in S}\Sigma^n\left({}^{d(x)}\Sigma^{d(x)}c^n_{K^*}\right)} \coprod_{x\in S} \Sigma^n\left({}^{d(x)}\Sigma^{d(x)}K^*\right) \xrightarrow{\Psi^n_{(S,d)}} \Sigma^n\left(\coprod_{x\in S} {}^{d(x)}\Sigma^{d(x)}K^*\right) = \Sigma^n F(S,d)
\end{aligned}$$

□

For $m, n \in \boldsymbol{Z}$, we denote by $\mu^{m,n} : {}^mK^* \otimes_{K^*} {}^nK^* \to {}^{m+n}K^*$ the composition of the isomorphism $\beta^{m,n}_{K^*,K^*} : {}^mK^* \otimes_{K^*} {}^nK^* \to {}^{m+n}(K^* \otimes_{K^*} K^*)$ given in (3.1.19) and the isomorphism ${}^{m+n}(K^* \otimes_{K^*} K^*) \to {}^{m+n}K^*$ induced by the multiplication of K^*. For graded sets (S,d), (T,e) and $x \in S$, $y \in T$, we have following isomorphisms.

$$\begin{aligned}
\tau^{d(x),e(y)}_{{}^{d(x)}K^*,{}^{e(y)}K^*} &: \Sigma^{d(x)}\left({}^{d(x)}K^*\right) \otimes_{K^*} \Sigma^{e(y)}\left({}^{e(y)}K^*\right) \to \Sigma^{d(x)+e(y)}\left({}^{d(x)}K^* \otimes_{K^*} {}^{e(y)}K^*\right) \\
\Sigma^{d(x)+e(y)}\mu^{d(x),e(y)} &: \Sigma^{d(x)+e(y)}\left({}^{d(x)}K^* \otimes_{K^*} {}^{e(y)}K^*\right) \to \Sigma^{d(x)+e(y)}\left({}^{d(x)+e(y)}K^*\right)
\end{aligned}$$

We denote by $d{*}e : S \times T \to \boldsymbol{Z}$ the map given by $(d{*}e)(x,y) = d(x) + e(y)$. There exists a unique morphism $\gamma_{(S,d),(T,e)} : F(S \times T, d{*}e) \to F(S,d) \otimes_{K^*} F(T,e)$ in $\mathcal{T}op\mathcal{M}od_{K^*}$ that makes the following diagram commute for any $(x,y) \in S \times T$.

$$\begin{tikzcd}
\Sigma^{d(x)+e(y)}\left({}^{d(x)+e(y)}K^*\right) \arrow[rr, "\left(\tau^{d(x),e(y)}_{{}^{d(x)}K^*,{}^{e(y)}K^*}\right)^{-1}\left(\Sigma^{d(x)+e(y)}\mu^{d(x),e(y)}\right)^{-1}"] \arrow[d, "\iota_{(x,y)}"] & & \Sigma^{d(x)}\left({}^{d(x)}K^*\right) \otimes_{K^*} \Sigma^{e(y)}\left({}^{e(y)}K^*\right) \arrow[d, "\iota_x \otimes \iota_y"] \\
F(S \times T, d{*}e) \arrow[rr, "\gamma_{(S,d),(T,e)}"] & & F(S,d) \otimes_{K^*} F(T,e)
\end{tikzcd}$$

The next result follows from (2.1.13).

Proposition 3.1.29 *If both S and T are finite sets, $\gamma_{(S,d),(T,e)} : F(S \times T, d{*}e) \to F(S,d) \otimes_{K^*} F(T,e)$ is an isomorphism.*

Proposition 3.1.30 *For a family $(M_i^*)_{i\in I}$ of objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ and an object N^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, let $\iota^* : \mathcal{H}om^*\left(\coprod_{i\in I} M_i^*, N^*\right) \to \prod_{i\in I} \mathcal{H}om^*(M_i^*, N^*)$ be the map induced by the canonical inclusions $\iota_j : M_j^* \to \coprod_{i\in I} M_i^*$ $(j \in I)$. Then, ι^* is an isomorphism.*

Proof. Clearly, ι^* is bijective. Let S^* be a finitely generated submodule of $\coprod_{i\in I} M_i^*$ and W^* an open submodule of N^*. Suppose that S^* is generated by $x_1, x_2, \dots, x_m$. There exists a finite subset J of I such that $x_k = \sum_{j\in J} x_{kj}$ for $x_{kj} \in M_j^*$ and $k = 1, 2, \dots, m$. Let S_j^* be the submodule of M_j^* generated by $x_{1j}, x_{2j}, \dots, x_{mj}$. Then, S^* is contained in the submodule of $\coprod_{i\in I} M_i^*$ generated by $\sum_{j\in J} \iota_j(S_j^*)$. Let $\mathrm{pr}_j : \prod_{i\in I} \mathcal{H}om^*(M_i^*, N^*) \to \mathcal{H}om^*(M_j^*, N^*)$ be the projection. Suppose $(f_i)_{i\in I} \in \left(\bigcap_{j\in J} \mathrm{pr}_j^{-1}(O(S_j^*, V^*))\right)^n$. Let $f \in \mathcal{H}om^n\left(\coprod_{i\in I} M_i^*, N^*\right)$ be the unique element that maps to $(f_i)_{i\in I}$ by ι^*. Then $f\iota_j(\Sigma^n S_j^*) = f_j(\Sigma^n S_j^*) \subset V^*$ for $j \in J$ and this implies that f maps $\Sigma^n S^*$ into V^*. Thus we see $\iota^*(O(S^*, V^*)) \supset \bigcap_{j\in J} \mathrm{pr}_j^{-1}(O(S_j^*, V^*))$ and ι^* is an open map. □

Corollary 3.1.31 *Let (S, d) be a graded set and N^* an object of $\mathcal{T}op\mathcal{M}od_{K^*}$. Then, a family*

$$\left(\mathcal{H}om^*(F(S,d), N^*) \xrightarrow{E_{\iota_x([d(x)],1)}} {}^{d(x)}\Sigma^{-d(x)} N^*\right)_{x\in S}$$

of evaluation maps induces an isomorphism $\mathcal{H}om^*(F(S,d), N^*) \to \prod_{x\in S} {}^{d(x)}\Sigma^{-d(x)} N^*$.

Proof. The canonical inclusions $\iota_y : {}^{d(y)}\Sigma^{d(y)} K^* \to \coprod_{x\in S} {}^{d(x)}\Sigma^{d(x)} K^* = F(S,d)$ for $y \in S$ induce an isomorphism

$$\iota^* : \mathcal{H}om^*(F(S,d), N^*) \to \prod_{x\in S} \mathcal{H}om^*\left({}^{d(x)}\Sigma^{d(x)} K^*, N^*\right)$$

by (3.1.30). It follows from (4) of (3.1.24) that $E_{([d(x)],1)} : \mathcal{H}om^*\left({}^{d(x)}\Sigma^{d(x)} K^*, N^*\right) \to {}^{d(x)}\Sigma^{-d(x)} N^*$ is an isomorphism for each $x \in S$. It is clear that the composition of ι^* and $\prod_{x\in S} E_{([d(x)],1)} : \prod_{x\in S} \mathcal{H}om^*\left(\Sigma^{d(x)} K^*, N^*\right) \to \prod_{x\in S} {}^{d(x)}\Sigma^{-d(x)} N^*$ and the projection $\prod_{x\in S} {}^{d(x)}\Sigma^{-d(x)} N^* \to {}^{d(x)}\Sigma^{-d(x)} N^*$ coincides with the evaluation map $E_{\iota_x([d(x)],1)} : \mathcal{H}om^*(F(S,d), N^*) \to {}^{d(x)}\Sigma^{-d(x)} N^*$. Thus we have the assertion. □

Remark 3.1.32 *The inverse $\psi^{-1} : \prod_{x\in S} {}^{d(x)}\Sigma^{-d(x)} N^* \to \mathcal{H}om^*(F(S,d), N^*)$ of the isomorphism in (3.1.31) is the map given as follows. For $\alpha \in \left(\prod_{x\in S} {}^{d(x)}\Sigma^{-d(x)} N^*\right)^n$ and $x \in S$, we put $\alpha(x) = ([-d(x)], \zeta(x))$ $(\zeta(x) \in N^{n+d(x)})$. Let $\psi^{-1}(\alpha) : \Sigma^n F(S,d) \to N^*$ be the unique map satisfying $(\psi^{-1}(\alpha))([n], \iota_x([d(x)],1)) = \zeta(x)$ for $x \in S$.*

Lemma 3.1.33 *If M^* is finitely generated K^*-module and N^* is finite, then $\mathcal{H}om^*(M^*, N^*)$ is finite.*

Proof. Since M^* is finitely generated, there exists an epimorphism $p : \coprod_{i=1}^n \Sigma^{k_i} K^* \to M^*$. By (4) of (3.1.7), $p^* : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*\left(\coprod_{i=1}^n \Sigma^{k_i} K^*, N^*\right)$ maps $\mathcal{H}om^*(M^*, N^*)$ onto its image isomorphically. Since $\mathcal{H}om^*\left(\coprod_{i=1}^n \Sigma^{k_i} K^*, N^*\right)$ is isomorphic to $\coprod_{i=1}^n \Sigma^{-k_i} N^*$ by (4) of (3.1.24), $\mathcal{H}om^*(M^*, N^*)$ is finite length and discrete. □

Proposition 3.1.34 *If N^* is subcofinite, so is $\mathcal{H}om^*(M^*, N^*)$.*

Proof. Since, for $S^* \in \mathcal{F}_{M^*}$ and $U^* \in \mathcal{V}_{N^*}$, $i_{S^*}^* p_{U^**} : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(S^*, N^*/U^*)$ induces an injective map $\mathcal{H}om^*(M^*, N^*)/O(S^*, U^*) \to \mathcal{H}om^*(S^*, N^*/U^*)$. Then, the assertion follows from (3.1.33). □

Proposition 3.1.35 *For a non-negative integer m and $S^* \in \mathcal{F}_{M^*}$, take an non-negative integer k such that $S^* \subset \sum\limits_{i=-k}^{k} M^i$. Then, we have $(\mathcal{H}om^*(M^*,N^*))[m+k] \subset O\,(S^*, W^*[m])$. Hence, if N^* is subskeletal, so is $\mathcal{H}om^*(M^*,N^*)$.*

Proof. Suppose $f \in \mathcal{H}om^i(M^*,N^*)$ and $|i| \geq m+k$. Take arbitrary $s \in \boldsymbol{Z}$ and $v \in (\Sigma^i S^*)^s$. If $s < -k+i$ or $s > k+i$, then $v \in (\Sigma^i S^*)^s = S^{s-i} = \{0\}$, therefore $v = 0$. If $-k+i \leq s \leq k+i$, since $k+i \leq -m$ or $-k+i \geq m$, we have $s \leq -m$ or $s \geq m$, hence $f(v) \in W^s \subset W^*[m]$. Thus f maps $\Sigma^n S^*$ into $W^*[m]$. $\square$

Proposition 3.1.36 *If M^* is finite type and N^* is discrete and bounded, $\mathcal{H}om^*(M^*,N^*)$ has the skeletal topology. In particular, the dual $M^{**} = \mathcal{H}om^*(M^*,K^*)$ of M^* has the skeletal topology if K^* is discrete and bounded.*

Proof. Since N^* has the skeletal topology by the assumption, it follows from (3.1.35) that $\mathcal{H}om^*(M^*,N^*)$ is subskeletal. Suppose that $N^i = \{0\}$ for $|i| \geq k$. For a non-negative integer j, let S^* be the submodule of M^* generated by $\sum\limits_{|i| \leq j+k} M^i$. Then, S^* is finitely generated by the assumption and we have $O(S^*,0)^n = 0$ if $|n| \leq j$. Hence $O(S^*,0) \subset \mathcal{H}om^*(M^*,N^*)[j]$ hold. $\square$

3.2 Adjointness

Let M^*, N^*, L^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ and $f : M^* \otimes_{K^*} N^* \to L^*$ a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$. For $x \in M^k$, define a map $f_x : \Sigma^k N^* \to L^*$ by $f_x([k],y) = f(x \otimes y)$ for $([k],y) \in (\Sigma^k N^*)^n$. Clearly, f_x is linear. If $U^* \in \mathcal{V}_{L^*}$, there exist $V^* \in \mathcal{V}_{M^*}$ and $W^* \in \mathcal{V}_{N^*}$ such that $f(\operatorname{Ker}(p_{V^*} \otimes q_{W^*})) \subset U^*$ by the continuity of f, where $p_{V^*} : M^* \to M^*/V^*$ and $q_{W^*} : N^* \to N^*/W^*$ are the quotient maps. Hence, for $([k],y) \in (\Sigma^k W^*)^n$, $f_x([k],y) = f(x \otimes y) \in f(\operatorname{Ker}(p_{V^*} \otimes q_{W^*})) \subset U^*$ and it follows that f_x is continuous. Thus we have a map $(f^a)^k : M^k \to \operatorname{Hom}^c_{K^*}(\Sigma^k N^*, L^*)$ given by $(f^a)^k(x) = f_x$ and a family of linear maps $((f^a)^k)_{k \in \boldsymbol{Z}}$ defines a morphism $f^a : M^* \to \mathcal{H}om^*(N^*,L^*)$ in $\mathcal{T}op\mathcal{M}od_{K^*}$. In fact, for $U^* \in \mathcal{V}_{L^*}$, take $V^* \in \mathcal{V}_{M^*}$ and $W^* \in \mathcal{V}_{N^*}$ such that $f(\operatorname{Ker}(p_{V^*} \otimes q_{W^*})) \subset U^*$. Then, for $x \in S^l$, we have $((f^a)^l(x))([l],y) = f_x([l],y) = f(x \otimes y) \in U^*$ for any $([l],y) \in (\Sigma^l N^*)^n$. It follows that $f^a(V^*) \subset \operatorname{Ker} r_{U^**} \subset O(S^*,U^*)$ for any $S^* \in \mathcal{F}_{N^*}$, where $r_{U^*} : L^* \to L^*/U^*$ is the quotient map. Therefore $f^a : M^* \to \mathcal{H}om^*(N^*,L^*)$ is continuous.

Now we can define a map $\Phi = \Phi_{M^*,N^*,L^*} : \operatorname{Hom}^c_{K^*}(M^* \otimes_{K^*} N^*, L^*) \to \operatorname{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*,L^*))$ by $\Phi(f) = f^a$. Clearly, Φ is injective.

Proposition 3.2.1 *Φ_{M^*,N^*,L^*} is natural in each variable, that is, the following diagrams commute for morphisms $f : P^* \to M^*$, $g : P^* \to N^*$, $h : L^* \to P^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$.*

$$\begin{tikzcd}
\operatorname{Hom}^c_{K^*}(M^* \otimes_{K^*} N^*, L^*) \arrow[r, "\Phi_{M^*,N^*,L^*}"] \arrow[d, "(f \otimes_{K^*} id_{N^*})^*"] & \operatorname{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*,L^*)) \arrow[d, "f^*"] \\
\operatorname{Hom}^c_{K^*}(P^* \otimes_{K^*} N^*, L^*) \arrow[r, "\Phi_{P^*,N^*,L^*}"] & \operatorname{Hom}^c_{K^*}(P^*, \mathcal{H}om^*(N^*,L^*))
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Hom}^c_{K^*}(M^* \otimes_{K^*} N^*, L^*) \arrow[r, "\Phi_{M^*,N^*,L^*}"] \arrow[d, "(id_{M^*} \otimes_{K^*} g)^*"] & \operatorname{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*,L^*)) \arrow[d, "(g^*)_*"] \\
\operatorname{Hom}^c_{K^*}(M^* \otimes_{K^*} P^*, L^*) \arrow[r, "\Phi_{M^*,P^*,L^*}"] & \operatorname{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(P^*,L^*))
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Hom}^c_{K^*}(M^* \otimes_{K^*} N^*, L^*) \arrow[r, "\Phi_{M^*,N^*,L^*}"] \arrow[d, "h_*"] & \operatorname{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*,L^*)) \arrow[d, "(h_*)_*"] \\
\operatorname{Hom}^c_{K^*}(M^* \otimes_{K^*} N^*, P^*) \arrow[r, "\Phi_{M^*,N^*,P^*}"] & \operatorname{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*,P^*))
\end{tikzcd}$$

Definition 3.2.2 *For $M^*, N^* \in \operatorname{Ob} \mathcal{T}op\mathcal{M}od_{K^*}$, a subset S of $\mathcal{H}om^*(M^*,N^*)$ is called an equi-continuous set if, for any $W^* \in \mathcal{V}_{N^*}$, there exists $V^* \in \mathcal{V}_{M^*}$ such that, for all $n \in \boldsymbol{Z}$ and $f \in S \cap \mathcal{H}om^n(M^*,N^*)$, $f : \Sigma^n M^* \to N^*$ maps $\Sigma^n V^*$ into W^*.*

Proposition 3.2.3 *Let $f : M^* \otimes_{K^*} N^* \to L^*$ be a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$. Then, $\operatorname{Im} f^a$ is an equi-continuous set of $\mathcal{H}om^*(N^*,L^*)$.*

Proof. For $U^* \in \mathcal{V}_{L^*}$, there exist $V^* \in \mathcal{V}_{M^*}$ and $W^* \in \mathcal{V}_{N^*}$ such that $f(\mathrm{Ker}(p_{V^*} \otimes q_{W^*})) \subset U^*$, where $p_{V^*} : M^* \to M^*/V^*$ and $q_{W^*} : N^* \to N^*/W^*$ are the quotient maps. Then we have $((f^a)^k(x))([k], y) = f(x \otimes y) \in U^*$ for any $x \in M^k$ and $([k], y) \in (\Sigma^k W^*)^n$. $\square$

Proposition 3.2.4 *For objects M^*, N^* and L^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, we put*

$$E(M^*; N^*, L^*) = \{g \in \mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*, L^*)) \,|\, \mathrm{Im}\, g \textit{ is an equi-continuous set of } \mathcal{H}om^*(N^*, L^*)\}.$$

Then, $\Phi_{M^*,N^*,L^*} : \mathrm{Hom}^c_{K^*}(M^* \otimes_{K^*} N^*, L^*) \to \mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*, L^*))$ is a monomorphism whose image is contained in $E(M^*; N^*, L^*)$. If M^* is discrete or N^* is fintely generated, Φ gives an isomorphism $\mathrm{Hom}^c_{K^*}(M^* \otimes_{K^*} N^*, L^*) \to E(M^*; N^*, L^*)$.

Proof. Let $g : M^* \to \mathcal{H}om^*(N^*, L^*)$ be a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$. Define a map $g_a : M^* \otimes_{K^*} N^* \to L^*$ by $g_a(x \otimes y) = g(x)([n], y)$ for $x \in M^n$ and $([n], y) \in (\Sigma^n N^*)^{n+k}$. Assume that $\mathrm{Im}\, g$ is an equi-continuous set of $\mathcal{H}om^*(N^*, L^*)$. For $U^* \in \mathcal{V}_{L^*}$, there exists $W^* \in \mathcal{V}_{N^*}$ such that $g(x)(\Sigma^n W^*) \subset U^*$ for all $n \in \boldsymbol{Z}$ and $x \in M^n$. Then $g_a(\mathrm{Im}(id_{M^*} \otimes j_{W^*})) \subset U^*$, where $j_{W^*} : W^* \to N^*$ is the inclusion map. Hence g_a is continuous if M^* is discrete. Suppose that N^* is fintely generated. Since $N^* \in \mathcal{F}_{N^*}$, the continuity of g implies that there exists $V^* \in \mathcal{V}_{M^*}$ such that $g(V^*) \subset O(N^*, U^*)$, namely, $g_a(\mathrm{Im}(i_{V^*} \otimes id_{N^*})) \subset U^*$, where $i_{V^*} : V^* \to M^*$ is the inclusion map. $\square$

Remark 3.2.5 *For objects N^* and L^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, define a map $ev^{N^*}_{L^*} : \mathcal{H}om^*(N^*, L^*) \otimes_{K^*} N^* \to L^*$ by $ev^{N^*}_{L^*}(f \otimes x) = f([m], x)$ for $f \in \mathcal{H}om^m(N^*, L^*)$ and $x \in N^n$. Then, for $g \in \mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*, L^*))$, $g_a : M^* \otimes_{K^*} N^* \to L^*$ is a composition $M^* \otimes_{K^*} N^* \xrightarrow{g \otimes_{K^*} id_{N^*}} \mathcal{H}om^*(N^*, L^*) \otimes_{K^*} N^* \xrightarrow{ev^{N^*}_{L^*}} L^*$.*

Suppose that $M^* \otimes_{K^*} N^*$ is supercofinite (resp. superskeletal) and that the topology of L^* is subcofinite (resp. subskeletal). Then, for any morphism $g : M^* \to \mathcal{H}om^*(N^*, L^*)$ in $\mathcal{T}op\mathcal{M}od_{K^*}$, the map $g_a : M^* \otimes_{K^*} N^* \to L^*$ defined above is always continuous by (1.4.14). If the topology on $M^* \otimes_{K^*} N^*$ is finer than the topology induced by K^* and the topology on L^* is coarser than the topology induced by K^* the map $g_a : M^* \otimes_{K^*} N^* \to L^*$ defined above is always continuous by (1.1.11). Hence we have the following result.

Proposition 3.2.6 *If one of the following conditions is satisfied, $\Phi_{M^*,N^*,L^*} : \mathrm{Hom}^c_{K^*}(M^* \otimes_{K^*} N^*, L^*) \to \mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*, L^*))$ is an isomorphism.*

(i) $M^* \otimes_{K^*} N^*$ is supercofinite and L^* is subcofinite.

(ii) $M^* \otimes_{K^*} N^*$ is superskeletal and L^* is subskeletal.

(iii) The topology on $M^* \otimes_{K^*} N^*$ is finer than the topology induced by K^* and the topology on L^* is coarser than the topology induced by K^*.

By the equivalence $\tau^{n,0}_{M^*,N^*} : (\Sigma^n M^*) \otimes_{K^*} (\Sigma^0 N^*) \to \Sigma^n(M^* \otimes_{K^*} N^*)$ and $\varepsilon_{N^*} : N^* \to \Sigma^0 N^*$, we identify $\Sigma^n(M^* \otimes_{K^*} N^*)$ with $(\Sigma^n M^*) \otimes_{K^*} N^*$. Define a morphism $\Phi^*_{M^*,N^*,L^*} : \mathcal{H}om^*(M^* \otimes_{K^*} N^*, L^*) \to \mathcal{H}om^*(M^*, \mathcal{H}om^*(N^*, L^*))$ by

$$\Phi^n_{M^*,N^*,L^*} = \Phi_{\Sigma^n M^*,N^*,L^*} : \mathrm{Hom}^c_{K^*}(\Sigma^n M^* \otimes_{K^*} N^*, L^*) \to \mathrm{Hom}^c_{K^*}(\Sigma^n M^*, \mathcal{H}om^*(N^*, L^*)).$$

Proposition 3.2.7 *Let M^*, N^*, L^* and P^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$.*

(1) If S^*, T^*, U^* are submodules of M^*, N^*, L^*, respectively, then we have

$$\Phi^*_{M^*,N^*,L^*}(O(S^* \otimes_{K^*} T^*, U^*)) = O(S^*, O(T^*, U^*)) \cap \mathrm{Im}, \Phi^*_{M^*,N^*,L^*}.$$

Hence $\Phi^*_{M^*,N^*,L^*} : \mathcal{H}om^*(M^* \otimes_{K^*} N^*, L^*) \to \mathcal{H}om^*(M^*, \mathcal{H}om^*(N^*, L^*))$ is a homeomorphism onto its image.

(2) The following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}^c_{K^*}(P^* \otimes_{K^*} M^* \otimes_{K^*} N^*, L^*) \arrow[r, "\Phi_{P^* \otimes_{K^*} M^*,N^*,L^*}"] \arrow[d, "\Phi_{P^*,M^* \otimes_{K^*} N^*,L^*}"] & \mathrm{Hom}^c_{K^*}(P^* \otimes_{K^*} M^*, \mathcal{H}om^*(N^*, L^*)) \arrow[d, "\Phi_{P^*,M^*,\mathcal{H}om^*(N^*,L^*)}"] \\
\mathrm{Hom}^c_{K^*}(P^*, \mathcal{H}om^*(M^* \otimes_{K^*} N^*, L^*)) \arrow[r, "(\Phi^*_{M^*,N^*,L^*})_*"] & \mathrm{Hom}^c_{K^*}(P^*, \mathcal{H}om^*(M^*, \mathcal{H}om^*(N^*, L^*)))
\end{tikzcd}$$

Proof. (1) For $f \in \mathcal{H}om^n(M^* \otimes_{K^*} N^*, L^*)$ and $x \in S^{k-n}$, $y \in T^{l-k}$, we have $\big(\Phi^*_{M^*,N^*,L^*}(f)([n], x)\big)([k], y) = f(([n], x \otimes y))$ by the definition. Thus $f \in O(S^* \otimes_{K^*} T^*, U^*)$ if and only if $\Phi_{\Sigma^n M^*,N^*,L^*}(f) \in O(S^*, O(T^*, U^*))$.

(2) This is straightforward form the definitions. $\square$

3.3 Homomorphisms

Definition 3.3.1 *Let us define the following maps.*

$$\xi = \xi(L^*; M^*, N^*) : \mathrm{Hom}^c_{K^*}(M^*, N^*) \to \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*, M^*), \mathcal{H}om^*(L^*, N^*))$$
$$\zeta = \zeta(M^*, N^*; L^*) : \mathrm{Hom}^c_{K^*}(M^*, N^*) \to \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(N^*, L^*), \mathcal{H}om^*(M^*, L^*))$$

For $\varphi \in \mathrm{Hom}^c_{K^*}(M^*, N^*)$, $\xi(\varphi)$ maps $f \in \mathcal{H}om^k(L^*, M^*) = \mathrm{Hom}^c_{K^*}(\Sigma^k L^*, M^*)$ to $\varphi f \in \mathcal{H}om^k(L^*, N^*)$ and $\zeta(\varphi)$ maps $g \in \mathcal{H}om^k(N^*, L^*) = \mathrm{Hom}^c_{K^*}(\Sigma^k N^*, L^*)$ to $g\Sigma^k\varphi \in \mathcal{H}om^k(M^*, L^*)$. It follows from (3.1.7) that $\xi(\varphi)$ and $\zeta(\varphi)$ are continuous.

Similarly, we also define the following morphisms in $\mathcal{T}op\mathcal{M}od_{K^*}$.

$$\xi^* = \xi^*(L^*; M^*, N^*) : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(\mathcal{H}om^*(L^*, M^*), \mathcal{H}om^*(L^*, N^*))$$
$$\zeta^* = \zeta^*(M^*, N^*; L^*) : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(\mathcal{H}om^*(N^*, L^*), \mathcal{H}om^*(M^*, L^*))$$

For $\varphi \in \mathcal{H}om^n(M^*, N^*)$, let

$$\xi^*(\varphi) \in \mathcal{H}om^n(\mathcal{H}om^*(L^*, M^*), \mathcal{H}om^*(L^*, N^*)) = \mathrm{Hom}^c_{K^*}(\Sigma^n\mathcal{H}om^*(L^*, M^*), \mathcal{H}om^*(L^*, N^*))$$

be the map defined by $(\xi^*(\varphi))([n], f) = \varphi(\Sigma^n f)\varepsilon_{n,k-n,L^*}$ for $f \in \mathcal{H}om^{k-n}(L^*, M^*)$ and

$$\zeta^*(\varphi) \in \mathcal{H}om^n(\mathcal{H}om^*(N^*, L^*), \mathcal{H}om^*(M^*, L^*)) = \mathrm{Hom}^c_{K^*}(\Sigma^n\mathcal{H}om^*(N^*, L^*), \mathcal{H}om^*(M^*, L^*))$$

the map defined by $(\zeta^*(\varphi))([n], g) = (-1)^{n(k-n)}g\big(\Sigma^{k-n}\varphi\big)\varepsilon_{k-n,n,M^*}$ for $g \in \mathcal{H}om^{k-n}(N^*, L^*)$. It follows from (3.1.7) that $\xi^*(\varphi)$ and $\zeta^*(\varphi)$ are continuous.

Proposition 3.3.2 *$\xi^* : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(\mathcal{H}om^*(L^*, M^*), \mathcal{H}om^*(L^*, N^*))$ and $\zeta^* : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(\mathcal{H}om^*(N^*, L^*), \mathcal{H}om^*(M^*, L^*))$ are continuous.*

Proof. Suppose $U^* \in \mathcal{V}_{N^*}$ and $S^* \in \mathcal{F}_{\mathcal{H}om^*(L^*,M^*)}$, $T^* \in \mathcal{F}_{L^*}$. Let $f_1, f_2, \dots, f_m$ be generators of S^* with $f_i \in S^{k_i}$. We put $P^* = \sum_{i=1}^{m} f_i\left(\Sigma^{k_i}T^*\right)$. For $\varphi \in O(P^*, U^*)^n$ and $1 \leqq i \leqq m$, we have $(\xi^*(\varphi)(f_i))(\Sigma^{k_i+n}T^*) = \varphi\Sigma^n f(\Sigma^{k_i+n}T^*) \subset \varphi(\Sigma^n P^*) \subset U^*$. Thus $\xi^*(\varphi)(f_i) \in O(T^*, U^*)$ for $\varphi \in O(P^*, U^*)^n$ and $i = 1, 2, \dots, m$. This implies $\xi^*(O(P^*, U^*)) \subset O(S^*, O(T^*, U^*))$.

Suppose $S^* \in \mathcal{F}_{\mathcal{H}om^*(N^*,L^*)}$, $T^* \in \mathcal{F}_{M^*}$ and $U^* \in \mathcal{V}_{L^*}$. Let $g_1, g_2, \dots, g_m$ be generators of S^* with $g_i \in S^{k_i}$. Put $R^* = \bigcap_{i=1}^{m} \Sigma^{-k_i}g_i^{-1}(U^*)$. Since $g_i^{-1}(U^*)$ is an open subspace of $\Sigma^{k_i}N^*$, R^* is an open subspace of N^*. For $\varphi \in O(T^*, R^*)^n$, $\zeta^*(\varphi)(g_i) = (-1)^{k_i n}g_i\Sigma^{k_i}\varphi$ maps $\Sigma^{k_i+n}T^*$ into $g_i\left(\Sigma^{k_i}R^*\right) \subset U^*$. Hence $\zeta^*(\varphi)(g_i) \in O(T^*, U^*)$ and it follows that $\zeta^*(\varphi)(S^*) \in O(T^*, U^*)$. Therefore $\zeta^*(O(T^*, R^*)) \subset O(S^*, O(T^*, U^*))$. □

ξ, ζ, ξ^* and ζ^* are natural. In fact, the following fact is easily verified.

Proposition 3.3.3 *Let $f : P^* \to M^*$ and $g : N^* \to Q^*$ be morphisms in $\mathcal{T}op\mathcal{M}od_{K^*}$. The following diagrams commute.*

$$\begin{array}{ccc}
\mathrm{Hom}^c_{K^*}(M^*, N^*) & \xrightarrow{\xi(L^*;M^*,N^*)} & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*, M^*), \mathcal{H}om^*(L^*, N^*)) \\
\downarrow{\scriptstyle f^*g_*} & & \downarrow{\scriptstyle (f_*)^*(g_*)_*} \\
\mathrm{Hom}^c_{K^*}(P^*, Q^*) & \xrightarrow{\xi(L^*;P^*,Q^*)} & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*, P^*), \mathcal{H}om^*(L^*, Q^*))
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Hom}^c_{K^*}(M^*, N^*) & \xrightarrow{\zeta(M^*,N^*;L^*)} & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(N^*, L^*), \mathcal{H}om^*(M^*, L^*)) \\
\downarrow{\scriptstyle f^*g_*} & & \downarrow{\scriptstyle (f^*)_*(g^*)^*} \\
\mathrm{Hom}^c_{K^*}(P^*, Q^*) & \xrightarrow{\zeta(P^*,Q^*;L^*)} & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(Q^*, L^*), \mathcal{H}om^*(P^*, L^*))
\end{array}$$

$$\begin{array}{ccc}
\mathcal{H}om^*(M^*, N^*) & \xrightarrow{\xi^*(L^*;M^*,N^*)} & \mathcal{H}om^*(\mathcal{H}om^*(L^*, M^*), \mathcal{H}om^*(L^*, N^*)) \\
\downarrow{\scriptstyle f^*g_*} & & \downarrow{\scriptstyle (f_*)^*(g_*)_*} \\
\mathcal{H}om^*(P^*, Q^*) & \xrightarrow{\xi^*(L^*;P^*,Q^*)} & \mathcal{H}om^*(\mathcal{H}om^*(L^*, P^*), \mathcal{H}om^*(L^*, Q^*))
\end{array}$$

$$\begin{array}{ccc}
\mathcal{H}om^*(M^*, N^*) & \xrightarrow{\zeta(M^*,N^*;L^*)} & \mathcal{H}om^*(\mathcal{H}om^*(N^*, L^*), \mathcal{H}om^*(M^*, L^*)) \\
\downarrow{\scriptstyle f^*g_*} & & \downarrow{\scriptstyle (f^*)_*(g^*)^*} \\
\mathcal{H}om^*(P^*, Q^*) & \xrightarrow{\zeta(P^*,Q^*;L^*)} & \mathcal{H}om^*(\mathcal{H}om^*(Q^*, L^*), \mathcal{H}om^*(P^*, L^*))
\end{array}$$

Definition 3.3.4 *For $M^*, N^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{K^*}$, let us define a map $\chi_{M^*,N^*} : M^* \to \mathcal{H}om^*(\mathcal{H}om^*(M^*,N^*),N^*)$ to be the following composition.*

$$M^* \xrightarrow{\kappa_{M^*}} \mathcal{H}om^*(K^*,M^*) \xrightarrow{\zeta^*(K^*,M^*;N^*)} \mathcal{H}om^*(\mathcal{H}om^*(M^*,N^*),\mathcal{H}om^*(K^*,N^*))$$
$$\xrightarrow{E_{1*}=(\kappa_{N^*}^{-1})_*} \mathcal{H}om^*(\mathcal{H}om^*(M^*,N^*),N^*)$$

Then, χ_{M^*,N^*} is given as follows. For $x \in M^n$, $\chi_{M^*,N^*}(x) : \Sigma^n\mathcal{H}om^*(M^*,N^*) \to N^*$ is the map defined by $(\chi_{M^*,N^*}(x))([n],f) = (-1)^{n(k-n)}f([k-n],x)$ for $f \in \mathcal{H}om^{k-n}(M^*,N^*) = \mathrm{Hom}^c_{K^*}(\Sigma^{k-n}M^*,N^*)$.

Proposition 3.3.5 *Suppose that K^* is a field which has discrete topology and satisfies $K^i = \{0\}$ for $i \neq 0$. If $M^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{K^*}$ is a T_1-space, $\chi_{M^*,K^*} : M^* \to \mathcal{H}om^*(\mathcal{H}om^*(M^*,K^*),K^*)$ is injective.*

Proof. Suppose that $\chi_{M^*,K^*}(x) = 0$ for some non-zero $x \in M^n$. There exists an open submodule U^* of M^* such that $x \notin U^*$. Let $p : M^* \to M^*/U^*$ the quotient map. Since M^*/U^* has a discrete topology, there exists a continuous linear map $\varphi : (M^*/U^*)^n \to K^0$ which maps $p(x) \to 1$. φ can be extended to a continuous homomorphism $\bar{\varphi} : \Sigma^n(M^*/U^*) \to K^*$. Then, $\bar{\varphi}(\Sigma^{-n}p) : \Sigma^{-n}M^* \to K^*$ maps $([-n],x)$ to 1. Hence $(\chi_{M^*,K^*}(x))([n],\bar{\varphi}(\Sigma^{-n}p)) = (-1)^n\bar{\varphi}(\Sigma^{-n}p)([-n],x) = (-1)^n$ which contradicts the assumption. □

Proposition 3.3.6 *Suppose that K^* is discrete and bounded. If M^* has the skeletal topology and is projective and finite type, $\chi_{M^*,K^*} : M^* \to \mathcal{H}om^*(\mathcal{H}om^*(M^*,K^*),K^*)$ is an isomorphism.*

Proof. It is clear that χ_{M^*,K^*} is bijective. By (3.1.36), $\mathcal{H}om^*(\mathcal{H}om^*(M^*,K^*),K^*)$ has the skeletal topology. □

3.4 Completion and spaces of homomorphisms

We investigate relationships between the completion functor and functor $\mathcal{H}om^*$ in this section.

Proposition 3.4.1 *If N^* is Hausdorff, so is $\mathcal{H}om^*(M^*,N^*)$.*

Proof. Suppose that $f \in \mathcal{H}om^n(M^*,N^*)$ is not zero. Then, $f([n],x) \neq 0$ for some $x \in (\Sigma^nM^*)^k$. By the assumption, there exists $U^* \in \mathcal{V}_{N^*}$ such that $f([n],x) \notin U^*$. Then $f \notin O(K^*x,U^*)$. □

Proposition 3.4.2 *If N^* is complete Hausdorff, then $\eta^*_{M^*} : \mathcal{H}om^*\big(\widehat{M}^*,N^*\big) \to \mathcal{H}om^*(M^*,N^*)$ is an isomorphism in $\mathcal{T}op\mathcal{M}od_{K^*}$.*

Proof. By (1.3.17) and (1.3.4), $\eta^*_{M^*}$ is a continuous bijection. For $S^* \in \mathcal{F}_{M^*}$ and $U^* \in \mathcal{V}_{N^*}$, it follows from (1) of (3.1.7) that $\eta^*_{M^*}(O(\eta_{M^*}(S^*),U^*)) \subset O(S^*,U^*)$. For $f \in O(S^*,U^*)^n \subset \mathrm{Hom}^c_{K^*}(\Sigma^nM^*,N^*)$, let $g \in \mathrm{Hom}^c_{K^*}\big(\Sigma^n\widehat{M}^*,N^*\big)$ be the unique morphism such that $g\Sigma^n\eta_{M^*} = f$. Then, g maps $(\Sigma^n\eta_{M^*})(\Sigma^nS^*) = \Sigma^n\eta_{M^*}(S^*)$ into U^*. In other words, $g \in O(\eta_{M^*}(S^*),U^*)^n \subset \mathcal{H}om^n\big(\widehat{M}^*,N^*\big)$. Thus $\eta^*_{M^*}$ maps $O(\eta_{M^*}(S^*),U^*)$ onto $O(S^*,U^*)$, hence η_{M^*} is an open map. □

Proposition 3.4.3 *Suppose that N^* is complete Hausdorff. If there exists a finitely generated open submodule of M^*, $\mathcal{H}om^*(M^*,N^*)$ is complete Hausdorff.*

Proof. Let U^* be a finitely generated open submodule of M^* generated by $x_1,x_2,\dots,x_k$. Suppose that $(f_\lambda)_{\lambda\in\Lambda}$ is a Cauchy sequence in $\mathcal{H}om^n(M^*,N^*)$. For $([n],x) \in (\Sigma^nM^*)^k$ and $V^* \in \mathcal{V}_{N^*}$, there exists $\nu(x,V^*) \in \Lambda$ such that $f_\lambda - f_\mu \in O(K^*x+U^*,V^*)$ if $\lambda,\mu \geq \nu(x,V^*)$. Hence $f_\lambda([n],x) - f_\mu([n],x) \in V^*$ if $\lambda,\mu \geq \nu(x,V^*)$ and this implies that $(f_\lambda([n],x))_{\lambda\in\Lambda}$ is a Cauchy sequence in N^*. Thus $(f_\lambda([n],x))_{\lambda\in\Lambda}$ converges and let us denote by $f([n],x)$ the limit of $(f_\lambda([n],x))_{\lambda\in\Lambda}$. Thus we have a map $f : \Sigma^nM^* \to N^*$.

For any $([n],x),([n],y) \in (\Sigma^nM^*)^k$, $r,s \in K^*$ and $V^* \in \mathcal{V}_{N^*}$, there exists $\kappa \in \Lambda$ such that $f([n],rx+sy) - f_\lambda([n],rx+sy)$, $-f([n],x)+f_\lambda([n],x)$, $-f([n],y)+f_\lambda([n],y) \in V^*$ if $\lambda \geq \kappa$. Then, since f_λ is a homomorphism, we have

$$\begin{aligned} f([n],rx+sy) - rf([n],x) - sf([n],y) &= (f([n],rx+sy) - f_\lambda([n],rx+sy)) + r(-f([n],x) + f_\lambda([n],x)) \\ &\quad + s(-f([n],y) + f_\lambda([n],y)) \in V^*. \end{aligned}$$

Since N^* is a Hausdorff space, it follows that $f([n],rx+sy) - rf([n],x) - sf([n],y) = 0$. Therefore, f is a homomorphism.

For $V^* \in \mathcal{V}_{N^*}$, there exists $\mu(V^*) \in \Lambda$ such that $f([n], x_i) - f_\lambda([n], x_i) \in V^*$ for $i = 1, 2, \dots, k$ if $\lambda \geq \mu(V^*)$. Then, $f([n], x) - f_\lambda([n], x) \in V^*$ for any $x \in \Sigma^n U^*$ if $\lambda \geq \mu(V^*)$. By the continuity of $f_{\mu(V^*)}$ at 0, there exists $W^* \in \mathcal{V}_{M^*}$ satisfying $f_{\mu(V^*)}(\Sigma^n W^*) \subset V^*$. If $([n], x) \in \Sigma^n(U^* \cap W^*)$, then we have $f([n], x) = \big(f([n], x) - f_{\mu(V^*)}([n], x)\big) + f_{\mu(V^*)}([n], x) \in V^*$, namely $f(\Sigma^n(U^* \cap W^*)) \subset V^*$. Hence f is continuous. □

If L^* is complete Hausdorff, $\eta_{M^* \otimes_{K^*} N^*} : M^* \otimes_{K^*} N^* \to M^* \widehat{\otimes}_{K^*} N^*$ induces an isomorphism $\eta^*_{M^* \otimes_{K^*} N^*} : \mathrm{Hom}^c_{K^*}\left(M^* \widehat{\otimes}_{K^*} N^*, L^*\right) \to \mathrm{Hom}^c_{K^*}(M^* \otimes_{K^*} N^*, L^*)$ by (3.4.2). Hence (3.2.4) implies the following result.

Proposition 3.4.4 *Let M^*, N^* and L^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. If L^* is complete Hausdorff, then*

$$\Phi\eta^*_{M^* \otimes_{K^*} N^*} : \mathrm{Hom}^c_{K^*}\left(M^* \widehat{\otimes}_{K^*} N^*, L^*\right) \to \mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*, L^*))$$

is a monomorphism into $E(M^*; N^*, L^*)$. If M^* is discrete or N^* is finitely generated, $\Phi\eta^*_{M^* \otimes_{K^*} N^*}$ gives an isomorphism $\mathrm{Hom}^c_{K^*}\left(M^* \widehat{\otimes}_{K^*} N^*, L^*\right) \to E(M^*; N^*, L^*)$.

Proposition 3.4.5 *Let $D : \mathcal{D} \to \mathcal{T}op\mathcal{M}od_{K^*}$ be a functor and $(L^* \xrightarrow{\pi_i} D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$ a limiting cone of D. Then, for an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, $(\mathcal{H}om^*(M^*, L^*) \xrightarrow{\pi_{i*}} \mathcal{H}om^*(M^*, D(i)))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of the functor $D^{M^*} : \mathcal{D} \to \mathcal{T}op\mathcal{M}od_{K^*}$ given by $D^{M^*}(i) = \mathcal{H}om^*(M^*, D(i))$ for $i \in \mathrm{Ob}\,\mathcal{D}$ and $D^{M^*}(\theta) = D(\theta)_* : \mathcal{H}om^*(M^*, D(i)) \to \mathcal{H}om^*(M^*, D(j))$ for $\theta \in \mathcal{D}(i, j)$.*

Proof. Since $(\mathcal{H}om^n(M^*, L^*) \xrightarrow{\pi_{i*}} \mathcal{H}om^n(M^*, D(i)))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of abelian groups for each $n \in \boldsymbol{Z}$, it is easy to verify that $(\mathcal{H}om^*(M^*, L^*) \xrightarrow{\pi_{i*}} \mathcal{H}om^*(M^*, D(i)))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of graded K^*-modules. For $U^* \in \mathcal{V}_{L^*}$, there exist $U^*_s \in \mathcal{V}_{D(i_s)}$ ($s = 1, 2, \dots, l$, $i_s \in \mathrm{Ob}\,\mathcal{D}$) such that $U^* \supset \bigcap_{s=1}^{n} \pi_{i_s}^{-1}(U^*_s)$. By (3.1.4) and (3.1.7), we have $O(S^*, U^*) \supset \bigcap_{s=1}^{n} O(S^*, \pi_{i_s}^{-1}(U^*_s)) = \bigcap_{s=1}^{n} \pi_{i_s *}^{-1}(O(S^*, U^*_s))$ for $S^* \in \mathcal{F}_{M^*}$. Thus the topology on $\mathcal{H}om^*(M^*, L^*)$ coincides with the one such that $\left(\mathcal{H}om^*(M^*, L^*) \xrightarrow{\pi_{i*}} \mathcal{H}om^*(M^*, D(i))\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone in $\mathcal{T}op\mathcal{M}od_{K^*}$. □

Corollary 3.4.6 *For $M^*, N^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{K^*}$, let $\pi_{U^*} : \widehat{M}^* \to N^*/U^*$ ($U^* \in \mathcal{V}_{N^*}$) be the canonical projection. Then $\left(\mathcal{H}om^*(M^*, \widehat{N}^*) \xrightarrow{\pi_{U^*}{}_*} \mathcal{H}om^*(M^*, N^*/U^*)\right)_{U^* \in \mathcal{V}_{N^*}}$ is a limiting cone of $(d_{N^*})^{M^*} : \mathcal{V}_{N^*} \to \mathcal{T}op\mathcal{M}od_{K^*}$.*

Proposition 3.4.7 *Regard $\mathcal{F}_{M^*}$ as a partially ordered set. For $M^*, N^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{K^*}$, we define a functor $F_{M^*, N^*} : \mathcal{F}^{op}_{M^*} \to \mathcal{T}op\mathcal{M}od_{K^*}$ as follows. For $S^* \in \mathcal{F}_{M^*}$, put $F_{M^*, N^*}(S^*) = \mathcal{H}om^*(S^*, N^*)$ and $F_{M^*, N^*}(T^*) \to F_{M^*, N^*}(S^*)$ is the map induced by the inclusion map $S^* \to T^*$. If one of the following conditions is satisfied, $\left(\mathcal{H}om^*(M^*, N^*) \xrightarrow{i^*_{S^*}} F_{M^*, N^*}(S^*)\right)_{S^* \in \mathcal{F}_{M^*}}$ is an limiting cone of F_{M^*, N^*}.*

(i) M^* is supercofinite and N^* is subcofinite.
(ii) M^* is superskeletal and N^* is subskeletal.
(iii) The topology on M^* is finer than the topology induced by K^* and the topology on N^* is coarser than the topology induced by K^*.

Proof. By (1.4.14) and (1.1.11), $\left(\mathcal{H}om^*(M^*, N^*) \xrightarrow{i^*_{S^*}} F_{M^*, N^*}(S^*)\right)_{S^* \in \mathcal{F}_{M^*}}$ is an limiting cone of F_{M^*, N^*} in the category of graded K^*-modules. $\{O(S^*, U^*) |\, U^* \in \mathcal{V}_{N^*}\}$ is a basis of the neighborhood of 0 of $\mathcal{H}om^*(S^*, N^*)$ and $\{O(S^*, U^*) |\, S^* \in \mathcal{F}_{M^*},\ U^* \in \mathcal{V}_{N^*}\}$ is a basis of the neighborhood of 0 of $\mathcal{H}om^*(M^*, N^*)$. It is clear that $(i^*_{S^*})^{-1}(O(S^*, U^*)) = O(S^*, U^*) \subset \mathcal{H}om^*(M^*, N^*)$ and this implies the assertion. □

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ a functor. For each $U \in \mathrm{Ob}\,\mathcal{D}$, let $F_U : \mathcal{C} \to \mathcal{E}$ be the functor given by $F_U(S) = F(S, U)$ for $S \in \mathrm{Ob}\,\mathcal{C}$ and $F_U(f) = F(f, id_U)$ for $f \in \mathrm{Mor}, \mathcal{C}$. Suppose that there exists a limiting cone $(X_U \xrightarrow{i_{S,U}} F(S, U))_{S \in \mathrm{Ob}\,\mathcal{C}}$ of F_U for each $U \in \mathrm{Ob}\,\mathcal{D}$. Then, for a morphism $g : U \to V$ in $\mathcal{D}$, there is unique morphism $\bar{g} : X_U \to X_V$ satisfying $i_{S,V}\bar{g} = F(id_S, g)i_{S,U}$ for any $S \in \mathrm{Ob}\,\mathcal{C}$. We define a functor $\overline{F} : \mathcal{D} \to \mathcal{E}$ by $\overline{F}(U) = X_U$ and $\overline{F}(g) = \bar{g}$.

Lemma 3.4.8 *Under the above situation, if $(Y \xrightarrow{\pi_U} \overline{F}(U))_{U \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of $\overline{F}$,*

$$\left(Y \xrightarrow{i_{S,U}\pi_U} F(S, U)\right)_{(S,U) \in \mathrm{Ob}\,\mathcal{C} \times \mathcal{D}}$$

is a limiting cone of F.

Proof. Let $\left(Z \xrightarrow{p_{S,U}} F(S,U)\right)_{(S,U)\in \mathrm{Ob}\,\mathcal{C}\times\mathcal{D}}$ be a cone of F. Since $\left(Z \xrightarrow{p_{S,U}} F(S,U)\right)_{S\in \mathrm{Ob}\,\mathcal{C}}$ is a cone of F_U for each $U \in \mathrm{Ob}\,\mathcal{D}$, there is unique morphism $\varphi_U : Z \to \overline{F}(U)$ satisfying $i_{S,U}\varphi_U = p_{S,U}$. Let $g : U \to V$ be a morphism in $\mathcal{D}$. Then, $p_{S,V} = F(id_S, g)p_{S,U}$ for any $S \in \mathrm{Ob}\,\mathcal{C}$ by the assumption. It follows from the uniqueness of φ_V that $\varphi_V = \overline{F}(g)\varphi_U$, namely, $\left(Z \xrightarrow{\varphi_U} \overline{F}(U)\right)_{U\in\mathrm{Ob}\,\mathcal{D}}$ is a cone. Hence there is unique morphism $\psi : Z \to Y$ satisfying $\pi_U\psi = \varphi_U$ for any $U \in \mathrm{Ob}\,\mathcal{D}$ and we have $i_{S,U}\pi_U\psi = i_{S,U}\varphi_U = p_{S,U}$.

Suppose that $\psi, \psi' : Z \to Y$ satisfy $i_{S,U}\pi_U\psi = i_{S,U}\pi_U\psi' = p_{S,U}$ for any $(S,U) \in \mathrm{Ob}\,\mathcal{C}\times\mathcal{D}$. Since $(X_U \xrightarrow{i_{S,U}} F(S,U))_{S\in\mathrm{Ob}\,\mathcal{C}}$ is a limiting cone of F_U, we have $\pi_U\psi = \pi_U\psi'$ for any $U \in \mathrm{Ob}\,\mathcal{D}$. Moreover, since $(Y \xrightarrow{\pi_U} \overline{F}(U))_{U\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of $\overline{F}$, we have $\psi = \psi'$. □

For $M^*, N^* \in \mathrm{Ob}\,\mathcal{TopMod}_{K^*}$, consider a directed set $\mathcal{F}_{M^*}\times\mathcal{V}_{N^*}^{op}$, that is, $(S^*,U^*) \le (T^*,V^*)$ if and only if $S^* \subset T^*$ and $U^* \supset V^*$. We define a functor $D_{M^*,N^*} : (\mathcal{F}_{M^*}\times\mathcal{V}_{N^*}^{op})^{op} \to \mathcal{TopMod}_{K^*}$ by $D_{M^*,N^*}(S^*,U^*) = \mathcal{Hom}^*(S^*, N^*/U^*)$. If $(S^*,U^*) \le (T^*,V^*)$, the map $D_{M^*,N^*}(T^*,V^*) \to D_{M^*,N^*}(S^*,U^*)$ is the composition of the maps induced by the inclusion map $S^* \to T^*$ and the quotient map $N^*/V^* \to N^*/U^*$. We also define a functor $d_{M^*,N^*} : (\mathcal{F}_{M^*}\times\mathcal{V}_{N^*}^{op})^{op} \to \mathcal{TopMod}_{K^*}$ by $d_{M^*,N^*}(S^*,U^*) = \mathcal{Hom}^*(M^*,N^*)/O(S^*,U^*)$. If $(S^*,U^*) \le (T^*,V^*)$, then $O(S^*,U^*) \supset O(T^*,V^*)$ and the map $d_{M^*,N^*}(T^*,V^*) \to d_{M^*,N^*}(S^*,U^*)$ is the map induced by the identity map of $\mathcal{Hom}^*(M^*,N^*)$. Then, there exists a limiting cone

$$\left(\mathcal{Hom}^*(M^*,N^*)^{\frown} \xrightarrow{\tilde{\pi}_{S^*,U^*}} d_{M^*,N^*}(S^*,U^*)\right)_{(S^*,U^*)\in\mathcal{F}_{M^*}\times\mathcal{V}_{N^*}}$$

of d_{M^*,N^*} such that $\tilde{\pi}_{S^*,U^*}\eta_{\mathcal{Hom}^*(M^*,N^*)} : \mathcal{Hom}^*(M^*,N^*) \to d_{M^*,N^*}(S^*,U^*)$ is the quotient map for each $(S^*,U^*) \in \mathcal{F}_{M^*}\times\mathcal{V}_{N^*}$. Since $O(S^*,U^*)$ is the kernel of $p_{U^*{*}}i_{S^*}^* : \mathcal{Hom}^*(M^*,N^*) \to \mathcal{Hom}^*(S^*,N^*/U^*) = D_{M^*,N^*}(S^*,U^*)$, $p_{U^*}i_{S^*}^*$ induces a natural monomorphism $\iota_{S^*,U^*} : d_{M^*,N^*}(S^*,U^*) \to D_{M^*,N^*}(S^*,U^*)$.

Proposition 3.4.9 *Let $\left(\widehat{N}^* \xrightarrow{\pi_{U^*}} N^*/U^*\right)_{U^*\in\mathcal{V}_{N^*}}$ be the limiting cone of $d_{N^*} : \mathcal{V}_{N^*} \to \mathcal{TopMod}_{K^*}$. If M^* is supercofinite and N^* is subcofinite, then $\left(\mathcal{Hom}^*(M^*,\widehat{N}^*) \xrightarrow{\pi_{U^*{*}}i_{S^*}^*} \mathcal{Hom}^*(S^*,N^*/U^*)\right)_{(S^*,U^*)\in\mathcal{F}_{M^*}\times\mathcal{V}_{N^*}}$ is a limiting cone of D_{M^*,N^*}.*

Proof. For each $U^* \in \mathcal{V}_{N^*}$, $\left(\mathcal{Hom}^*(M^*,N^*/U^*) \xrightarrow{i_{S^*}^*} D_{M^*,N^*}(S^*,U^*)\right)_{S^*\in\mathcal{F}_{M^*}}$ is a limiting cone of $(D_{M^*,N^*})_{U^*}$ by (3.4.7). It follows from (3.4.6) that $\left(\mathcal{Hom}^*(M^*,\widehat{N}^*) \xrightarrow{\pi_{U^*{*}}} \mathcal{Hom}^*(M^*,N^*/U^*)\right)_{U^*\in\mathcal{V}_{N^*}}$ is a limiting cone of $(d_{N^*})^{M^*}$. Therefore the result follows from (3.4.8). □

Lemma 3.4.10 *Suppose that there exists a morphism $\lambda_{M^*,N^*} : \mathcal{Hom}^*(M^*,N^*)^{\frown} \to \mathcal{Hom}^*(M^*,\widehat{N}^*)$ that makes the following diagram commute.*

$$\begin{tikzcd}
\mathcal{Hom}^*(M^*,N^*) \arrow[r, "\eta_{\mathcal{Hom}^*(M^*,N^*)}"] \arrow[dr, "\eta_{N^*{*}}"] & \mathcal{Hom}^*(M^*,N^*)^{\frown} \arrow[d, "\lambda_{M^*,N^*}"] \\
& \mathcal{Hom}^*(M^*,\widehat{N}^*)
\end{tikzcd}$$

(1) λ_{M^*,N^*} is unique and a monomorphism.
(2) The following diagram commute for $(S^*,U^*) \in \mathcal{F}_{M^*}\times\mathcal{V}_{N^*}$.

$$\begin{tikzcd}
\mathcal{Hom}^*(M^*,N^*)^{\frown} \arrow[r, "\iota_{S^*,U^*}\tilde{\pi}_{S^*,U^*}"] \arrow[d, "\lambda_{M^*,N^*}"] & \mathcal{Hom}^*(S^*,N^*/U^*) \\
\mathcal{Hom}^*(M^*,\widehat{N}^*) \arrow[r, "\pi_{U^*{*}}"] & \mathcal{Hom}^*(M^*,N^*/U^*) \arrow[u, "i_{S^*}^*"]
\end{tikzcd}$$

Proof. (1) Since the image of $\eta_{\mathcal{Hom}^*(M^*,N^*)}$ is dense by (1.1.16), the uniqueness of λ_{M^*,N^*} is clear. Suppose that there are morphisms $f, g : L^* \to \mathcal{Hom}^*(M^*,N^*)^{\frown}$ satisfying $\lambda_{M^*,N^*}f = \lambda_{M^*,N^*}g$. Then, we have $\iota_{S^*,U^*}\tilde{\pi}_{S^*,U^*}f = \pi_{U^*{*}}i_{S^*}^*\lambda_{M^*,N^*}f = \pi_{U^*{*}}i_{S^*}^*\lambda_{M^*,N^*}g = \iota_{S^*,U^*}\tilde{\pi}_{S^*,U^*}g$. Since ι_{S^*,U^*} is a monomorphism and

$\left(\mathcal{H}om^*(M^*,N^*)\hat{}\xrightarrow{\tilde{\pi}_{S^*,U^*}} d_{M^*,N^*}(S^*,U^*)\right)_{(S^*,U^*)\in\mathcal{F}_{M^*}\times\mathcal{V}_{N^*}}$ is a limiting cone, it follows that $f = g$, hence λ_{M^*,N^*} is a monomorphism.

(2) Since $i^*_{S^*}\pi_{U^**}\lambda_{M^*,N^*}\eta_{\mathcal{H}om^*(M^*,N^*)} = i^*_{S^*}\pi_{U^**}\eta_{N^**} = i^*_{S^*}p_{U^**} = \iota_{S^*,U^*}\tilde{\pi}_{S^*,U^*}\eta_{\mathcal{H}om^*(M^*,N^*)}$ and the image of $\eta_{\mathcal{H}om^*(M^*,N^*)}$ is dense by (1.1.16), we have $i^*_{S^*}\pi_{U^**}\lambda_{M^*,N^*} = \iota_{S^*,U^*}\tilde{\pi}_{S^*,U^*}$. □

Proposition 3.4.11 *If one of the following conditions (i) or (ii) is satisfied, there exists a unique monomorphism* $\lambda_{M^*,N^*} : \mathcal{H}om^*(M^*,N^*)\hat{}\to \mathcal{H}om^*(M^*,\widehat{N}^*)$ *that makes a diagram*

$$\begin{tikzcd} \mathcal{H}om^*(M^*,N^*) \arrow[r, "\eta_{\mathcal{H}om^*(M^*,N^*)}"] \arrow[dr, "\eta_{N^**}"'] & \mathcal{H}om^*(M^*,N^*)\hat{} \arrow[d, "\lambda_{M^*,N^*}"] \\ & \mathcal{H}om^*(M^*,\widehat{N}^*) \end{tikzcd}$$

commute.

(i) M^* is supercofinite and N^* is subcofinite. *(ii) M^* has a finitely generated open submodule.*

Proof. Suppose that M^* is supercofinite and N^* is subcofinite. Since

$$\left(\mathcal{H}om^*(M^*,N^*)\hat{}\xrightarrow{\iota_{S^*,U^*}\tilde{\pi}_{S^*,U^*}} D_{M^*,N^*}(S^*,U^*)\right)_{(S^*,U^*)\in\mathcal{F}_{M^*}\times\mathcal{V}_{N^*}}$$

is a cone of D_{M^*,N^*}, there exists a unique map $\lambda_{M^*,N^*} : \mathcal{H}om^*(M^*,N^*)\hat{}\to \mathcal{H}om^*(M^*,\widehat{N}^*)$ satisfying $\pi_{U^**}i^*_{S^*}\lambda_{M^*,N^*} = \iota_{S^*,U^*}\tilde{\pi}_{S^*,U^*}$ for any $(S^*,U^*)\in\mathcal{F}_{M^*}\times\mathcal{V}_{N^*}$ by (3.4.9). Since $\iota_{S^*,U^*}\tilde{\pi}_{S^*,U^*}\eta_{\mathcal{H}om^*(M^*,N^*)} = p_{U^**}i^*_{S^*}$, we have $\pi_{U^**}i^*_{S^*}\lambda_{M^*,N^*}\eta_{\mathcal{H}om^*(M^*,N^*)} = \iota_{S^*,U^*}\tilde{\pi}_{S^*,U^*}\eta_{\mathcal{H}om^*(M^*,N^*)} = p_{U^**}i^*_{S^*} = \pi_{U^**}\eta_{N^**}i^*_{S^*} = \pi_{U^**}i^*_{S^*}\eta_{N^**}$ which implies the commutativity of the diagram. By (3.4.10), λ_{M^*,N^*} is unique and a monomorphism.

Suppose that M^* has a finitely generated open submodule. Since $\mathcal{H}om^*(M^*,\widehat{N}^*)$ is complete Hausdorff by (3.4.3), there exists a unique morphism $\lambda_{M^*,N^*} : \mathcal{H}om^*(M^*,N^*)\hat{}\to \mathcal{H}om^*(M^*,\widehat{N}^*)$ that makes the above diagram commute. Then, it follows from (3.4.10) that λ_{M^*,N^*} is unique and a monomorphism. □

Definition 3.4.12 *Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$.*

(1) We say that a pair (M^*,N^*) is nice if there exists a cofinal subset $\mathcal{C}$ of $\mathcal{F}_{M^*}\times\mathcal{V}^{op}_{N^*}$ such that $p_{U^**}i^*_{S^*} : \mathcal{H}om^*(M^*,N^*)\to\mathcal{H}om^*(S^*,N^*/U^*)$ is surjective for each $(S^*,U^*)\in\mathcal{C}$.

(2) We say that a pair (M^*,N^*) is very nice if there exists a cofinal subset $\mathcal{C}$ of $\mathcal{F}_{M^*}\times\mathcal{V}^{op}_{N^*}$ such that $p_{U^**}i^*_{S^*} : \mathcal{H}om^*(M^*,N^*)\to\mathcal{H}om^*(S^*,N^*/U^*)$ is surjective and S^* is projective for each $(S^*,U^*)\in\mathcal{C}$.

Remark 3.4.13 *(1) A pair (M^*,N^*) is nice if one of the following conditions is satisfied.*

(i) M^* is projective and there exists a cofinal subset $\mathcal{M}$ of $\mathcal{V}^{op}_{N^*}$ such that N^*/U^* is injective for every $U^*\in\mathcal{M}$.

(ii) There exists a cofinal subset $\mathcal{S}$ of $\mathcal{F}_{M^*}$ such that every element of $\mathcal{S}$ is a direct summand of M^* and there exists a cofinal subset $\mathcal{M}$ of $\mathcal{V}^{op}_{N^*}$ such that every element of $\mathcal{M}$ is a direct summand of N^*.

(2) (M^*,N^*) is a very nice pair if N^* is injective and there exists a cofinal subset $\mathcal{S}$ of $\mathcal{F}_{M^*}$ such that every element of $\mathcal{S}$ is projective.

(3) It follows from (3.1.8) that the above (ii) is satisfied for $\mathcal{S} = \mathcal{F}_{M^*}$ and $\mathcal{M} = \mathcal{V}^{op}_{N^*}$ if K^* is a field and M^* is supercofinite. In this case, (M^*,N^*) is very nice by (1) of (1.4.9).

Proposition 3.4.14 *Suppose that a pair (M^*,N^*) of objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ is nice.*

(1) $\left(\mathcal{H}om^*(M^*,N^*)\hat{}\xrightarrow{\iota_{S^*,U^*}\tilde{\pi}_{S^*,U^*}}\mathcal{H}om^*(S^*,N^*/U^*)\right)_{(S^*,U^*)\in\mathcal{F}_{M^*}\times\mathcal{V}_{N^*}}$ *is a limiting cone of D_{M^*,N^*}.*

(2) There exists a unique morphism $\mu_{M^*,N^*} : \mathcal{H}om^*(M^*,\widehat{N}^*)\to\mathcal{H}om^*(M^*,N^*)\hat{}$ that makes the following diagram commute for any $(S^*,U^*)\in\mathcal{F}_{M^*}\times\mathcal{V}_{N^*}$.

$$\begin{tikzcd} \mathcal{H}om^*(M^*,\widehat{N}^*) \arrow[r, "\pi_{U^**}"] \arrow[d, "\mu_{M^*,N^*}"] & \mathcal{H}om^*(M^*,N^*/U^*) \arrow[d, "i^*_{S^*}"] \\ \mathcal{H}om^*(M^*,N^*)\hat{} \arrow[r, "\iota_{S^*,U^*}\tilde{\pi}_{S^*,U^*}"] & \mathcal{H}om^*(S^*,N^*/U^*) \end{tikzcd}$$

Proof. (1) Let $\mathcal{C}$ be a cofinal subset $\mathcal{C}$ of $\mathcal{F}_{M^*} \times \mathcal{V}_{N^*}^{op}$ such that $p_{U^**} i_{S^*}^* : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(S^*, N^*/U^*)$ is surjective for each $(S^*, U^*) \in \mathcal{C}$. Assume $(S^*, U^*) \in \mathcal{C}$. Since $\mathcal{H}om^*(S^*, N^*/U^*)$ is discrete, $p_{U^**} i_{S^*}^* : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(S^*, N^*/U^*)$ is an open map, hence a regular epimorphism. Then, the map $\iota_{S^*,U^*} : \mathcal{H}om^*(M^*, N^*)/O(S^*, U^*) \to \mathcal{H}om^*(S^*, N^*/U^*)$ induced by $p_{U^**} i_{S^*}^*$ is an isomorphism by (1.1.23). Since we have a limiting cone $\left(\mathcal{H}om^*(M^*, N^*)^{\frown} \xrightarrow{\tilde{\pi}_{S^*,U^*}} d_{M^*,N^*}(S^*, U^*)\right)_{(S^*,U^*) \in \mathcal{F}_{M^*} \times \mathcal{V}_{N^*}}$, the result follows.

(2) Since $\left(\mathcal{H}om^*(M^*, \widehat{N}^*) \xrightarrow{i_{S^*}^* \pi_{U^**}} D_{M^*,N^*}(S^*, U^*)\right)_{(S^*,U^*) \in \mathcal{F}_{M^*} \times \mathcal{V}_{N^*}}$ is a cone of D_{M^*,N^*}, the assertion follows from (1). $\square$

Theorem 3.4.15 *Suppose that (M^*, N^*) is nice. If one of the following conditions (i) or (ii) is satisfied, then the morphism $\lambda_{M^*,N^*} : \mathcal{H}om^*(M^*, N^*)^{\frown} \to \mathcal{H}om^*(M^*, \widehat{N}^*)$ given in (3.4.11) is an isomorphism whose inverse is μ_{M^*,N^*}.*

(i) M^* is supercofinite and N^* is subcofinite. *(ii) M^* has a finitely generated open submodule.*

Proof. Assume that M^* is supercofinite and N^* is subcofinite. Then, the assertion follows from (3.4.9), (2) of (3.4.10) and (3.4.14).

Assume that M^* has a finitely generated open submodule. We have $\iota_{S^*,U^*} \tilde{\pi}_{S^*,U^*} \mu_{M^*,N^*} \lambda_{M^*,N^*} = \iota_{S^*,U^*} \tilde{\pi}_{S^*,U^*}$ and $i_{S^*}^* \pi_{U^**} \lambda_{M^*,N^*} \mu_{M^*,N^*} = i_{S^*}^* \pi_{U^**}$ for any $(S^*, U^*) \in \mathcal{F}_{M^*} \times \mathcal{V}_{N^*}$ by (2) of (3.4.14) and (3.4.10). It follows from (3.4.14) that $\mu_{M^*,N^*} \lambda_{M^*,N^*} = id_{\mathcal{H}om^*(M^*,N^*)^{\frown}}$. Since

$$\left(\mathcal{H}om^*(M^*, N^*/U^*) \xrightarrow{i_{S^*}^*} \mathcal{H}om^*(S^*, N^*/U^*)\right)_{S^* \in \mathcal{F}_{M^*}}$$

is a monomorphic family, equalities $i_{S^*}^* \pi_{U^**} \lambda_{M^*,N^*} \mu_{M^*,N^*} = i_{S^*}^* \pi_{U^**}$ for $S^* \in \mathcal{F}_{M^*}$ imply $\pi_{U^**} \lambda_{M^*,N^*} \mu_{M^*,N^*} = \pi_{U^**}$. It follows from (3.4.6) that $\lambda_{M^*,N^*} \mu_{M^*,N^*} = id_{\mathcal{H}om^*\left(M^*, \widehat{N}^*\right)}$. $\square$

The above result and (3.1.34) imply the following.

Corollary 3.4.16 *Suppose that (M^*, N^*) is nice. If M^* is supercofinite and N^* is profinite, then $\mathcal{H}om^*(M^*, N^*)$ is profinite.*

Define a map $c_{M^*,N^*} : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(\widehat{M}^*, \widehat{N}^*)$ by $c_{M^*,N^*}(f) = \hat{f} : \Sigma^n \widehat{M}^* = \widehat{\Sigma^n M^*} \to \widehat{N}^*$ for $f \in \mathcal{H}om^n(M^*, N^*)$.

Proposition 3.4.17 *c_{M^*,N^*} is continuous and the following diagrams commute.*

$$\begin{tikzcd}
\mathcal{H}om^*(\widehat{M}^*, N^*) \arrow[r, "\eta_{M^*}^*"] \arrow[dr, "\eta_{N^**}"'] & \mathcal{H}om^*(M^*, N^*) \arrow[r, "\eta_{N^**}"] \arrow[d, "c_{M^*,N^*}"] & \mathcal{H}om^*(M^*, \widehat{N}^*) \\
& \mathcal{H}om^*(\widehat{M}^*, \widehat{N}^*) \arrow[ur, "\eta_{M^*}^*"'] &
\end{tikzcd}$$

Proof. For $\varphi \in \mathcal{H}om^m(\widehat{M}^*, N^*)$ and $\psi \in \mathcal{H}om^n(M^*, N^*)$, the following diagrams are commutative by the definition of $\hat{\varphi}$ and $\hat{\psi}$.

$$\begin{tikzcd}
\Sigma^m \widehat{M}^* \arrow[r, "\varphi"] \arrow[d, "\Sigma^m \eta_{\widehat{M}^*}"] & N^* \arrow[d, "\eta_{N^*}"] \\
\Sigma^m \widehat{\widehat{M}}^* \arrow[r, "\hat{\varphi}"] & \widehat{N}^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^n M^* \arrow[r, "\psi"] \arrow[d, "\Sigma^n \eta_{M^*}"] & N^* \arrow[d, "\eta_{N^*}"] \\
\Sigma^n \widehat{M}^* \arrow[r, "\hat{\psi}"] & \widehat{N}^*
\end{tikzcd}$$

Thus we have $c_{M^*,N^*} \eta_{M^*}^*(\varphi) = c_{M^*,N^*}(\varphi \Sigma^m \eta_{M^*}) = \hat{\varphi} \Sigma^m \hat{\eta}_{M^*} = \hat{\varphi} \Sigma^m \eta_{\widehat{M}^*} = \eta_{N^*} \varphi = \eta_{N^**}(\varphi)$ by (1.3.6) and $\eta_{M^*}^* c_{M^*,N^*}(\psi) = \hat{\psi} \Sigma^n \eta_{M^*} = \eta_{N^*} \psi = \eta_{N^**}(\psi)$.

For $U^* \in \mathcal{V}_{N^*}$, let $\pi_{U^*} : \widehat{N}^* \to N^*/U^*$ be the map induced by the quotient map $p_{U^*} : N^* \to N^*/U^*$. Then, $\left(\mathcal{H}om^*(M^*, \widehat{N}^*) \xrightarrow{\pi_{U^**}} \mathcal{H}om^*(M^*, N^*/U^*)\right)_{U^* \in \mathcal{V}_{N^*}}$ is a limiting cone in $\mathcal{T}op\mathcal{M}od_{K^*}$ by (3.4.5). Since $\pi_{U^**} \eta_{M^*}^* c_{M^*,N^*} = \pi_{U^**} \eta_{N^**} = p_{U^**}$ is continuous for every $U^* \in \mathcal{V}_{N^*}$, $\eta_{M^*}^* c_{M^*,N^*}$ is continuous. Hence the continuity of c_{M^*,N^*} follows from (3.4.2). $\square$

Lemma 3.4.18 *Let $F : \mathcal{C} \to \mathcal{E}$ be a functor preserving limits. Suppose that $\mathcal{E}$ has finite products. For functors $D_1 : \mathcal{D}_1 \to \mathcal{C}$ and $D_2 : \mathcal{D}_2 \to \mathcal{C}$, let $\left(L_1 \xrightarrow{p_i} D_1(i)\right)_{i\in \mathrm{Ob}\,\mathcal{D}_1}$ and $\left(L_2 \xrightarrow{q_j} D_2(j)\right)_{j\in \mathrm{Ob}\,\mathcal{D}_2}$ be limiting cones of D_1 and D_2, respectively. Then, $\left(F(L_1)\times F(L_2) \xrightarrow{F(p_i)\times F(q_j)} FD_1(i)\times FD_2(j)\right)_{(i,j)\in \mathrm{Ob}\,\mathcal{D}_1\times\mathcal{D}_2}$ is a limiting cone of a functor $D : \mathcal{D}_1\times\mathcal{D}_2 \to \mathcal{E}$ defined by $D(i,j) = FD_1(i)\times FD_2(j)$.*

Proof. Let $P_1 : \mathcal{D}_1\times\mathcal{D}_2 \to \mathcal{D}_1$ and $P_2 : \mathcal{D}_1\times\mathcal{D}_2 \to \mathcal{D}_2$ be projection functors. For each $i \in \mathrm{Ob}\,\mathcal{D}_1$, it is clear that $\left(FD_1(i) \xrightarrow{id_{D_1(i)}} FD_1P_1(i,j)\right)_{j\in \mathrm{Ob}\,\mathcal{D}_2}$ is a limiting cone of a functor $D_{1i} : \mathcal{D}_2 \to \mathcal{E}$ given by $D_{1i}(j) = FD_1P_1(i,j) = FD_1(i)$ and $D_{1i}(f) = FD_1P_1(id_i, f) = id_{FD_1(i)}$. Since $\left(F(L_1) \xrightarrow{F(p_i)} FD_1(i)\right)_{i\in \mathrm{Ob}\,\mathcal{D}_1}$ is a limiting cone of FD_1 by the assumption, it follows from (3.4.8) that $\left(F(L_1) \xrightarrow{F(p_i)} FD_1P_1(i,j)\right)_{(i,j)\in \mathrm{Ob}\,\mathcal{D}_1\times\mathcal{D}_2}$ is a limiting cone of FD_1P_1. Similarly, $\left(F(L_2) \xrightarrow{F(q_j)} FD_2P_2(i,j)\right)_{(i,j)\in \mathrm{Ob}\,\mathcal{D}_1\times\mathcal{D}_2}$ is a limiting cone of FD_2P_2.

Suppose that $\left(X \xrightarrow{\pi_{ij}} FD_1(i)\times FD_2(j)\right)_{(i,j)\in \mathrm{Ob}\,\mathcal{D}_1\times\mathcal{D}_2}$ is a cone of D. Let us denote by

$$pr_{ij1} : FD_1(i)\times FD_2(j) \to FD_1(i) = FD_1P_1(i,j), \quad pr_{ij2} : FD_1(i)\times FD_2(j) \to FD_2(j) = FD_2P_2(i,j)$$

projections. Then, $\left(X \xrightarrow{pr_{ij1}\pi_{ij}} FD_1P_1(i,j)\right)_{i\in \mathrm{Ob}\,\mathcal{D}_1}$ and $\left(X \xrightarrow{pr_{ij2}\pi_{ij}} FD_2P_2(i,j)\right)_{j\in \mathrm{Ob}\,\mathcal{D}_2}$ are cones of FD_1P_1 and FD_2P_2, respectively. Hence there exist unique morphisms $\alpha : X \to F(L_1)$, $\beta : X \to F(L_2)$ satisfying $F(p_i)\alpha = pr_{ij1}\pi_{ij}$, $F(q_j)\beta = pr_{ij2}\pi_{ij}$ for any $(i,j) \in \mathrm{Ob}\,\mathcal{D}_1\times\mathcal{D}_2$. Let $\gamma : X \to F(L_1)\times F(L_2)$ be the morphism induced by α and γ. It is easy to see that γ is the unique morphism satisfying $(F(p_i)\times F(q_j))\gamma = \pi_{ij}$ for each $(i,j) \in \mathrm{Ob}\,\mathcal{D}_1\times\mathcal{D}_2$. $\square$

For objects L^*, M^* and N^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, define a map

$$\mu_{L^*,M^*,N^*} : \mathcal{H}om^*(L^*,M^*)\times\mathcal{H}om^*(M^*,N^*) \to \mathcal{H}om^*(L^*,N^*)$$

by $\mu_{L^*,M^*,N^*}(f,g) = g\Sigma^n f$ for $f \in \mathcal{H}om^m(L^*,M^*)$, $g \in \mathcal{H}om^m(M^*,N^*)$.

Proposition 3.4.19 *Suppose that $K^i = \{0\}$ if $i \neq 0$. Let s and t be fixed integers. If L^* is supercofinite, M^* is superskeletal and N^* is profinite, then $\mu_{L^*,M^*,N^*} : \mathcal{H}om^s(L^*,M^*)\times\mathcal{H}om^t(M^*,N^*) \to \mathcal{H}om^{s+t}(L^*,N^*)$ is continuous.*

Proof. For a non-negative integer k, put $M^*\langle k\rangle = \sum_{i=-k}^{k} N^i$. Let $\alpha_k : M^*\langle k\rangle \to M^*/M^*[k+1]$ be the composition of inclusion map $i_k : M^*\langle k\rangle \to M^*$ and quotient map $p_{M^*[k+1]} : M^* \to M^*/M^*[k+1]$. Since $K^i = \{0\}$ if $i \neq 0$, we have $M^*\langle k\rangle \cap M^*[k+1] = \{0\}$. Hence both $M^*\langle k\rangle$ and $M^*/M^*[k+1]$ are discrete since M^* is superskeletal. Therefore α_k is an isomorphism. For $S^* \in \mathcal{F}_{L^*}$, $U^* \in \mathcal{V}_{N^*}$ and a non-negative integer k, define a map

$$\mu_{S^*,k,U^*} : \mathcal{H}om^s(S^*, M^*/M^*[k+1])\times\mathcal{H}om^t(M^*\langle k\rangle, N^*/U^*) \to \mathcal{H}om^{s+t}(S^*, N^*/U^*)$$

by $\mu_{S^*,k,U^*}(f,g) = g\Sigma^t(\alpha_k^{-1}f)$ for $f \in \mathcal{H}om^s(S^*, M^*/M^*[k+1])$, $g \in \mathcal{H}om^t(M^*\langle k\rangle, N^*/U^*)$.
Since both $\mathcal{H}om^s(S^*, M^*/M^*[k+1])$ and $\mathcal{H}om^t(M^*\langle k\rangle, N^*/U^*)$ are discrete, μ_{S^*,k,U^*} is continuous.

For each $(S^*,U^*) \in \mathcal{F}_{L^*}\times\mathcal{V}_{N^*}$, the following diagram commutes if $S^* \subset L^*\langle l\rangle$ and $k \geq l + |s|$.

$$\begin{tikzcd}
\mathcal{H}om^s(L^*,M^*)\times\mathcal{H}om^t(M^*,N^*) \arrow[r, "\mu_{L^*,M^*,N^*}"] \arrow[d, "i^*_{S^*}p_{M^*[k+1]*}\times i^*_{M^*\langle k\rangle}p_{U^**}"] & \mathcal{H}om^{s+t}(L^*,N^*) \arrow[d, "i^*_{S^*}p_{U^**}"] \\
\mathcal{H}om^s(S^*, M^*/M^*[k+1])\times\mathcal{H}om^t(M^*\langle k\rangle, N^*/U^*) \arrow[r, "\mu_{S^*,k,U^*}"] & \mathcal{H}om^{s+t}(S^*, N^*/U^*)
\end{tikzcd}$$

Hence $i^*_{S^*}p_{U^**}\mu_{L^*,M^*,N^*}$ is continuous and

$$\left(\mathcal{H}om^s(L^*,M^*)\times\mathcal{H}om^t(M^*,N^*) \xrightarrow{i^*_{S^*}p_{U^**}\mu_{L^*,M^*,N^*}} \mathcal{H}om^{s+t}(S^*, N^*/U^*)\right)_{(S^*,U^*)\in\mathcal{F}_{L^*}\times\mathcal{V}_{N^*}}$$

is a cone of $F\epsilon_{s+t}D_{L^*,N^*} : (\mathcal{F}_{L^*} \times \mathcal{V}_{N^*}^{op})^{op} \to \mathcal{T}op$, where $F : \mathcal{T}op\mathcal{M}od_{K^*} \to \mathcal{T}op$ denotes the forgetful functor. On the other hand, since N^* is profinite, it follows from (3.4.9) and (1.2.5) that

$$\left(\mathcal{H}om^{s+t}(L^*, N^*) \xrightarrow{i_{S^*}^* p_{U^*}{}_*} \mathcal{H}om^{s+t}(S^*, N^*/U^*)\right)_{(S^*,U^*)\in\mathcal{F}_{L^*}\times\mathcal{V}_{N^*}}$$

is a limiting cone of $F\epsilon_{s+t}D_{L^*,N^*} : (\mathcal{F}_{L^*} \times \mathcal{V}_{N^*}^{op})^{op} \to \mathcal{T}op$. This implies the continuity of $\mu_{L^*,M^*,N^*} : \mathcal{H}om^s(L^*, M^*) \times \mathcal{H}om^t(M^*, N^*) \to \mathcal{H}om^{s+t}(L^*, N^*)$. $\square$

4 Relations between tensor products and spaces of homomorphisms

4.1 Completed tensor products of spaces of linear maps

Let M_s^*, N_s^* $(s = 1, 2)$ be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. We define a map

$$\bar{\phi} : \mathcal{H}om^*(M_1^*, N_1^*) \times \mathcal{H}om^*(M_2^*, N_2^*) \to \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)$$

by $\bar{\phi}(f, g) = (f \otimes_{K^*} g)\big(\tau^{m,n}_{M_1^*, M_2^*}\big)^{-1}$ for $f \in \mathcal{H}om^m(M_1^*, N_1^*)$ and $g \in \mathcal{H}om^n(M_2^*, N_2^*)$. In other words, if $x \in M_1^{i-m}$, $y \in M_2^{j-n}$, $\bar{\phi}(f, g) : \Sigma^{m+n} M_1^* \otimes_{K^*} M_2^* \to N_1^* \otimes_{K^*} N_2^*$ is the map defined by

$$\bar{\phi}(f, g)([m + n], x \otimes y) = (-1)^{n(i-m)} f([m], x) \otimes g([n], y).$$

Then, it is easy to verify that $\bar{\phi}$ is bilinear and it defines a map

$$\phi = \phi(M_1^*, M_2^*; N_1^*, N_2^*) : \mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*) \longrightarrow \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)$$

of graded K^*-modules.

Proposition 4.1.1 *$\phi : \mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*) \longrightarrow \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)$ is continuous.*

Proof. For $T^* \in \mathcal{F}_{M_1^* \otimes_{K^*} M_2^*}$ and $U^* \in \mathcal{V}_{N_1^* \otimes_{K^*} N_2^*}$, there exist $S_i^* \in \mathcal{F}_{M_i^*}$, $U_i^* \in \mathcal{V}_{N_i^*}$ $(i = 1, 2)$ such that $\mathrm{Im}(i_{S_1^*} \otimes i_{S_2^*}) \supset T^*$ and $\mathrm{Im}(j_{U_1^*} \otimes id_{N_2^*}) + \mathrm{Im}(id_{N_1^*} \otimes j_{U_2^*}) \subset U^*$, where $i_{S_l^*} : S_l^* \to M_l^*$ and $j_{U_l^*} : U_l^* \to N_l^*$ $(l = 1, 2)$ are the inclusion maps. Let us denote by $k_{O(S_l^*, U_l^*)} : O(S_l^*, U_l^*) \to \mathcal{H}om^*(M_l^*, N_l^*)$ $(l = 1, 2)$ the inclusion maps. It is clear that ϕ maps $\mathrm{Im}\left(k_{O(S_1^*, U_1^*)} \otimes id_{\mathcal{H}om^*(M_2^*, N_2^*)}\right) + \mathrm{Im}\left(id_{\mathcal{H}om^*(M_1^*, N_1^*)} \otimes k_{O(S_2^*, U_2^*)}\right)$ into $O(\mathrm{Im}(i_{S_1^*} \otimes i_{S_2^*}), \mathrm{Im}(j_{U_1^*} \otimes id_{N_2^*}) + \mathrm{Im}(id_{N_1^*} \otimes j_{U_2^*})) \subset O(T^*, U^*)$. $\square$

We denote by $\iota_1 : M^* \to M^* \otimes_{K^*} K^*$ the map given by $\iota_1(x) = x \otimes 1$. Then, ι_1 is a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$ and it is an isomorphism if the topology on M^* is coarser than the topology induced by K^*.

Suppose that the topology on N^* is coarser than the topology induced by K^*. Then, the K^*-module structure map of N^* induces an isomorphism $\tilde{\alpha} : K^* \otimes_{K^*} N^* \to N^*$ by (2.1.8). Let

$$\varphi^{M^*}_{N^*} : \mathcal{H}om^*(M^*, K^*) \otimes_{K^*} N^* \longrightarrow \mathcal{H}om^*(M^*, N^*)$$

be the following composition of morphisms.

$$\mathcal{H}om^*(M^*, K^*) \otimes_{K^*} N^* \xrightarrow{id_{\mathcal{H}om^*(M^*, K^*)} \otimes_{K^*} \kappa_{N^*}} \mathcal{H}om^*(M^*, K^*) \otimes_{K^*} \mathcal{H}om^*(K^*, N^*) \xrightarrow{\phi(M^*, K^*; K^*, N^*)}$$
$$\mathcal{H}om^*(M^* \otimes_{K^*} K^*, K^* \otimes_{K^*} N^*) \xrightarrow{\iota_1^*} \mathcal{H}om^*(M^*, K^* \otimes_{K^*} N^*) \xrightarrow{\tilde{\alpha}_*} \mathcal{H}om^*(M^*, N^*)$$

We note that $\varphi^{M^*}_{N^*}$ maps $f \otimes y \in \mathcal{H}om^m(M^*, K^*) \otimes_{K^*} N^n$ to a map $\Sigma^{m+n} M^* \to N^*$ given by $([m+n], x) \mapsto (-1)^{n(s-m-n)} f([m], x) y$ for $x \in M^{s-m-n}$.

The next assertion is easily verified from the definitions of Φ and ϕ.

Proposition 4.1.2 *Let $f_i : M_i^* \otimes_{K^*} N_i^* \to Z_i^*$ $(i = 1, 2)$ be morphisms in $\mathcal{T}op\mathcal{M}od_{K^*}$. Then, composition*

$$M_1^* \otimes_{K^*} M_2^* \xrightarrow{\Phi_{M_1^*, N_1^*, Z_1^*}(f_1) \otimes \Phi_{M_2^*, N_2^*, Z_2^*}(f_2)} \mathcal{H}om^*(N_1^*, Z_1^*) \otimes_{K^*} \mathcal{H}om^*(N_2^*, Z_2^*) \xrightarrow{\phi} \mathcal{H}om^*(N_1^* \otimes_{K^*} N_2^*, Z_1^* \otimes_{K^*} Z_2^*)$$

coincides with $\Phi_{M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*, Z_1^* \otimes_{K^*} Z_2^*}((f_1 \otimes_{K^*} f_2)(id_{M_1^*} \otimes_{K^*} T_{M_2^*, N_1^*} \otimes_{K^*} id_{N_2^*}))$.

The following facts are also easily verified.

Proposition 4.1.3 *(1) The following diagram commutes.*

$$\begin{CD}
\mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*) @>{T_{\mathcal{H}om^*(M_1^*, N_1^*), \mathcal{H}om^*(M_2^*, N_2^*)}}>> \mathcal{H}om^*(M_2^*, N_2^*) \otimes_{K^*} \mathcal{H}om^*(M_1^*, N_1^*) \\
@V{\phi}VV @VV{\phi}V \\
\mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*) @>{T_{N_1^*, N_2^* *} T^*_{M_2^*, M_1^*}}>> \mathcal{H}om^*(M_2^* \otimes_{K^*} M_1^*, N_2^* \otimes_{K^*} N_1^*)
\end{CD}$$

(2) For $x \in M_1^p$, $y \in M_2^q$, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*) \arrow[r, "\phi"] \arrow[d, "E_x \otimes E_y"] & \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*) \arrow[d, "E_{x\otimes y}"] \\
{}^p\Sigma^{-p} N_1^* \otimes_{K^*} {}^q\Sigma^{-q} N_2^* \arrow[d, equal] & {}^{p+q}\Sigma^{-p-q}(N_1^* \otimes_{K^*} N_2^*) \arrow[d, equal] \\
\Sigma^{-p}({}^pN_1^*) \otimes_{K^*} \Sigma^{-q}({}^qN_2^*) \arrow[d, "\tau^{-p,-q}_{{}^pN_1^*, {}^qN_2^*}"] & \Sigma^{-p-q}({}^{p+q}(N_1^* \otimes_{K^*} N_2^*)) \arrow[d, "(-1)^{pq}"] \\
\Sigma^{-p-q}({}^pN_1^*) \otimes_{K^*} ({}^qN_2^*) \arrow[r, "\Sigma^{-p-q}\beta^{p,q}_{N_1^*, N_2^*}"] & \Sigma^{-p-q}({}^{p+q}(N_1^* \otimes_{K^*} N_2^*))
\end{tikzcd}$$

Proposition 4.1.4 *Let M_i^* and N_i^* $(i = 1, 2)$ be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. If both M_1^* and M_2^* are finitely generated and projective, then $\phi : \mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*) \to \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)$ is an isomorphism.*

Proof. First, we show the assertion assuming that both M_1^* and M_2^* are finitely generated and free. Then, we may assume that $M_1^* = F(S_1, d_1)$, $M_2^* = F(S_2, d_2)$ for some graded sets (S_1, d_1) and (S_2, d_2) such that S_1 and S_2 are finite sets. By (3.1.31), there are isomorphisms $\varepsilon_i : \mathcal{H}om^*(M_i^*, N_i^*) \to \prod_{x \in S_i} {}^{d_i(x)}\Sigma^{-d_i(x)} N_i^*$ for $i = 1, 2$ and $\varepsilon_3 : \mathcal{H}om^*(F(S_1 \times S_2, d_1 {*} d_2), N_1^* \otimes_{K^*} N_2^*) \to \prod_{(x,y) \in S_1 \times S_2} {}^{d_1(x)+d_2(y)}\Sigma^{-d_1(x)-d_2(y)} N_1^* \otimes_{K^*} N_2^*$. Since S_1 and S_2 are finite sets, we have an isomorphism $\gamma_{(S_1,d_1),(S_2,d_2)} : F(S_1 \times S_2, d_1 {*} d_2) \to F(S_1, d_1) \otimes_{K^*} F(S_2, d_2)$ by (3.1.29). For $(z, w) \in S_1 \times S_2$, we denote by $\beta_{(z,w)} : {}^{d_1(z)}\Sigma^{-d_1(z)} N_1^* \otimes_{K^*} {}^{d_2(w)}\Sigma^{-d_2(w)} N_2^* \to {}^{d_1(z)+d_2(w)}\Sigma^{-d_1(z)-d_2(w)} N_1^* \otimes_{K^*} N_2^*$ the map $(-1)^{d_1(z)d_2(w)} \Sigma^{-d_1(z)-d_2(w)} \beta^{d_1(z), d_2(w)}_{N_1^*, N_2^*} \tau^{-d_1(z), -d_2(w)}_{{}^{d_1(z)}N_1^*, {}^{d_2(w)}N_2^*}$. Let

$$\beta : \left(\prod_{x \in S_1} {}^{d_1(x)}\Sigma^{-d_1(x)} N_1^* \right) \otimes_{K^*} \left(\prod_{y \in S_2} {}^{d_2(y)}\Sigma^{-d_2(y)} N_2^* \right) \longrightarrow \prod_{(x,y) \in S_1 \times S_2} {}^{d_1(x)+d_2(y)}\Sigma^{-d_1(x)-d_2(y)} N_1^* \otimes_{K^*} N_2^*$$

be the unique map that makes the following diagram commute for every $(z, w) \in S_1 \times S_2$.

$$\begin{tikzcd}
\left(\prod_{x \in S_1} {}^{d_1(x)}\Sigma^{-d_1(x)} N_1^* \right) \otimes_{K^*} \left(\prod_{y \in S_2} {}^{d_2(y)}\Sigma^{-d_2(y)} N_2^* \right) \arrow[r, "\beta"] \arrow[d, "\mathrm{pr}_z \otimes \mathrm{pr}_w"] & \prod_{(x,y) \in S_1 \times S_2} {}^{d_1(x)+d_2(y)}\Sigma^{-d_1(x)-d_2(y)} N_1^* \otimes_{K^*} N_2^* \arrow[d, "\mathrm{pr}_{(z,w)}"] \\
{}^{d_1(z)}\Sigma^{-d_1(z)} N_1^* \otimes_{K^*} {}^{d_2(w)}\Sigma^{-d_2(w)} N_2^* \arrow[r, "\beta_{(z,w)}"] & {}^{d_1(z)+d_2(w)}\Sigma^{-d_1(z)-d_2(w)} N_1^* \otimes_{K^*} N_2^*
\end{tikzcd}$$

Here, we denote by $\mathrm{pr}_z : \prod_{x \in S_1} {}^{d_1(x)}\Sigma^{-d_1(x)} N_1^* \to {}^{d_1(z)}\Sigma^{-d_1(z)} N_1^*$, $\mathrm{pr}_w : \prod_{y \in S_2} {}^{d_2(y)}\Sigma^{-d_2(y)} N_2^* \to {}^{d_2(w)}\Sigma^{-d_2(w)} N_2^*$, $\mathrm{pr}_{(z,w)} : \prod_{(x,y) \in S_1 \times S_2} {}^{d_1(x)+d_2(y)}\Sigma^{-d_1(x)-d_2(y)} N_1^* \otimes_{K^*} N_2^* \to {}^{d_1(z)+d_2(w)}\Sigma^{-d_1(z)-d_2(w)} N_1^* \otimes_{K^*} N_2^*$ the projections. Then β is an isomorphism by (1.1.20) and (2.1.13). Since $\mathrm{pr}_z \varepsilon_1 = E_{\iota_z([d_1(z)],1)}$, $\mathrm{pr}_w \varepsilon_2 = E_{\iota_w([d_2(w)],1)}$ and $\mathrm{pr}_{(z,w)} \varepsilon_3 \gamma^*_{(S_1,d_1),(S_2,d_2)} = E_{\iota_z([d_1(z)],1) \otimes \iota_w([d_2(w)],1)}$, it follows from (4.1.3) that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*) \arrow[r, "\phi"] \arrow[d, "\varepsilon_1 \otimes_{K^*} \varepsilon_2"] & \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*) \arrow[d, "\varepsilon_3 \gamma^*_{(S_1,d_1),(S_2,d_2)}"] \\
\left(\prod_{x \in S_1} {}^{d_1(x)}\Sigma^{-d_1(x)} N_1^* \right) \otimes_{K^*} \left(\prod_{y \in S_2} {}^{d_2(y)}\Sigma^{-d_2(y)} N_2^* \right) \arrow[r, "\beta"] & \prod_{(x,y) \in S_1 \times S_2} {}^{d_1(x)+d_2(y)}\Sigma^{-d_1(x)-d_2(y)} N_1^* \otimes_{K^*} N_2^*
\end{tikzcd}$$

Thus we have shown the assertion when M_1^* and M_2^* are finitely generated and free.

Suppose that M_1^* and M_2^* are finitely generated and projective. Then, there exit finitely generated free K^*-modules L_1^* and L_2^* and split epimorphisms $p_1 : L_1^* \to M_1^*$ and $p_2 : L_2^* \to M_2^*$. Let $s_1 : M_1^* \to L_1^*$ and $s_2 : M_2^* \to L_2^*$ be right inverses of p_1 and p_2, respectively. Since the middle horizontal map of the following diagram is isomorphism, $\phi : \mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*) \to \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)$ is a bijection by the commutativity of the diagram.

$$\begin{tikzcd}
\mathcal{H}om^*(M_1^*,N_1^*)\otimes_{K^*}\mathcal{H}om^*(M_2^*,N_2^*) \arrow[r, "\phi"] \arrow[d, "p_1^*\otimes p_2^*"] & \mathcal{H}om^*(M_1^*\otimes_{K^*}M_2^*, N_1^*\otimes_{K^*}N_2^*) \arrow[d, "(p_1\otimes_{K^*}p_2)^*"] \\
\mathcal{H}om^*(L_1^*,N_1^*)\otimes_{K^*}\mathcal{H}om^*(L_2^*,N_2^*) \arrow[r, "\phi"] \arrow[d, "s_1^*\otimes s_2^*"] & \mathcal{H}om^*(L_1^*\otimes_{K^*}L_2^*, N_1^*\otimes_{K^*}N_2^*) \arrow[d, "(s_1\otimes_{K^*}s_2)^*"] \\
\mathcal{H}om^*(M_1^*,N_1^*)\otimes_{K^*}\mathcal{H}om^*(M_2^*,N_2^*) \arrow[r, "\phi"] & \mathcal{H}om^*(M_1^*\otimes_{K^*}M_2^*, N_1^*\otimes_{K^*}N_2^*)
\end{tikzcd}$$

It follows from (2) of (3.1.7) and (2.1.1) that the lower right map of the above diagram is an open map. Thus $\phi : \mathcal{H}om^*(M_1^*,N_1^*)\otimes_{K^*}\mathcal{H}om^*(M_2^*,N_2^*) \to \mathcal{H}om^*(M_1^*\otimes_{K^*}M_2^*, N_1^*\otimes_{K^*}N_2^*)$ is also an open map. Hence the assertion follows. □

By (1.3.4) and (4.1.1),

$$\eta_{\mathcal{H}om^*(M_1^*\otimes_{K^*}M_2^*,N_1^*\otimes_{K^*}N_2^*)}\phi : \mathcal{H}om^*(M_1^*,N_1^*)\otimes_{K^*}\mathcal{H}om^*(M_2^*,N_2^*) \to \mathcal{H}om^*(M_1^*\otimes_{K^*}M_2^*, N_1^*\otimes_{K^*}N_2^*)\hat{}$$

induces a unique morphism

$$\hat{\phi} : \mathcal{H}om^*(M_1^*,N_1^*)\,\widehat{\otimes}_{K^*}\mathcal{H}om^*(M_2^*,N_2^*) \longrightarrow \mathcal{H}om^*(M_1^*\otimes_{K^*}M_2^*, N_1^*\otimes_{K^*}N_2^*)\hat{}$$

that makes the following diagram commute.

$$\begin{tikzcd}
\mathcal{H}om^*(M_1^*,N_1^*)\otimes_{K^*}\mathcal{H}om^*(M_2^*,N_2^*) \arrow[r, "\phi"] \arrow[d, "\eta_{\mathcal{H}om^*(M_1^*,N_1^*)\otimes_{K^*}\mathcal{H}om^*(M_2^*,N_2^*)}"] & \mathcal{H}om^*(M_1^*\otimes_{K^*}M_2^*, N_1^*\otimes_{K^*}N_2^*) \arrow[d, "\eta_{\mathcal{H}om^*(M_1^*\otimes_{K^*}M_2^*,N_1^*\otimes_{K^*}N_2^*)}"] \\
\mathcal{H}om^*(M_1^*,N_1^*)\,\widehat{\otimes}_{K^*}\mathcal{H}om^*(M_2^*,N_2^*) \arrow[r, "\hat{\phi}"] & \mathcal{H}om^*(M_1^*\otimes_{K^*}M_2^*, N_1^*\otimes_{K^*}N_2^*)\hat{}
\end{tikzcd}$$

For $M_i^*, N_i^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{K^*}$ $(i=1,2)$, consider a directed set $\mathcal{F}_{M_1^*}\times\mathcal{V}_{N_1^*}^{op}\times\mathcal{F}_{M_2^*}\times\mathcal{V}_{N_2^*}^{op}$. We define functors $D_{M_1^*,N_1^*,M_2^*,N_2^*}, \widetilde{D}_{M_1^*,N_1^*,M_2^*,N_2^*} : \left(\mathcal{F}_{M_1^*}\times\mathcal{V}_{N_1^*}^{op}\times\mathcal{F}_{M_2^*}\times\mathcal{V}_{N_2^*}^{op}\right)^{op} \to \mathcal{T}op\mathcal{M}od_{K^*}$ by

$$D_{M_1^*,N_1^*,M_2^*,N_2^*}(S_1^*,U_1^*,S_2^*,U_2^*) = \mathcal{H}om^*(S_1^*\otimes_{K^*}S_2^*, N_1^*/U_1^*\otimes_{K^*}N_2^*/U_2^*),$$

$$\widetilde{D}_{M_1^*,N_1^*,M_2^*,N_2^*}(S_1^*,U_1^*,S_2^*,U_2^*) = \mathcal{H}om^*(S_1^*,N_1^*/U_1^*)\otimes_{K^*}\mathcal{H}om^*(S_2^*,N_2^*/U_2^*).$$

If $(T_1^*,Z_1^*,T_2^*,Z_2^*) \le (S_1^*,U_1^*,S_2^*,U_2^*)$, the maps

$$D_{M_1^*,N_1^*,M_2^*,N_2^*}(S_1^*,U_1^*,S_2^*,U_2^*) \to D_{M_1^*,N_1^*,M_2^*,N_2^*}(T_1^*,Z_1^*,T_2^*,Z_2^*)$$

and

$$\widetilde{D}_{M_1^*,N_1^*,M_2^*,N_2^*}(S_1^*,U_1^*,S_2^*,U_2^*) \to \widetilde{D}_{M_1^*,N_1^*,M_2^*,N_2^*}(T_1^*,Z_1^*,T_2^*,Z_2^*)$$

are the composition of the maps induced by the inclusion maps $T_i^* \to S_i^*$ and the quotient maps $N_i^*/U_i^* \to N_i^*/Z_i^*$ $(i=1,2)$.

Since both $D_{M_1^*,N_1^*,M_2^*,N_2^*}$ and $\widetilde{D}_{M_1^*,N_1^*,M_2^*,N_2^*}$ take values in the full subcategory of $\mathcal{T}op\mathcal{M}od_{K^*}$ consisting of discrete spaces, there are unique maps

$$\rho_{S_1^*,U_1^*,S_2^*,U_2^*} : \mathcal{H}om^*(M_1^*\otimes_{K^*}M_2^*, N_1^*\otimes_{K^*}N_2^*)\hat{}\to D_{M_1^*,N_1^*,M_2^*,N_2^*}(S_1^*,U_1^*,S_2^*,U_2^*),$$

$$\pi_{S_1^*,U_1^*,S_2^*,U_2^*} : \mathcal{H}om^*(M_1^*,N_1^*)\,\widehat{\otimes}_{K^*}\mathcal{H}om^*(M_2^*,N_2^*) \to \widetilde{D}_{M_1^*,N_1^*,M_2^*,N_2^*}(S_1^*,U_1^*,S_2^*,U_2^*)$$

satisfying the followings.

$$\rho_{S_1^*,U_1^*,S_2^*,U_2^*}\eta_{\mathcal{H}om^*(M_1^*\otimes_{K^*}M_2^*,N_1^*\otimes_{K^*}N_2^*)} = p_{N_1^*\otimes_{K^*}U_2^*+U_1^*\otimes_{K^*}N_2^*\,*}(i_{S_1^*}\otimes i_{S_2^*})^*$$

$$\pi_{S_1^*,U_1^*,S_2^*,U_2^*}\eta_{\mathcal{H}om^*(M_1^*,N_1^*)\otimes_{K^*}\mathcal{H}om^*(M_2^*,N_2^*)} = p_{U_1^*\,*}i_{S_1^*}^*\otimes p_{U_2^*\,*}i_{S_2^*}^*$$

Lemma 4.1.5 *(1) If both (M_1^*,N_1^*) and (M_2^*,N_2^*) are nice pairs, then*

$$\left(\mathcal{H}om^*(M_1^*,N_1^*)\widehat{\otimes}_{K^*}\mathcal{H}om^*(M_2^*,N_2^*) \xrightarrow{\pi_{S_1^*,U_1^*,S_2^*,U_2^*}} \widetilde{D}_{M_1^*,N_1^*,M_2^*,N_2^*}(S_1^*,U_1^*,S_2^*,U_2^*)\right)_{(S_1^*,U_1^*,S_2^*,U_2^*)\in\mathcal{F}_{M_1^*}\times\mathcal{V}_{N_1^*}\times\mathcal{F}_{M_2^*}\times\mathcal{V}_{N_2^*}}$$

is a limiting cone of $\widetilde{D}_{M_1^*,N_1^*,M_2^*,N_2^*}$.

(2) Suppose that there exists a cofinal subset $\mathcal{C}$ *of* $\mathcal{F}_{M_1^*} \times \mathcal{V}_{N_1^*}^{op} \times \mathcal{F}_{M_2^*} \times \mathcal{V}_{N_2^*}^{op}$ *such that*

$$(p_{U_1^*} \otimes p_{U_2^*})_*(i_{S_1} \otimes i_{S_2})^* : \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*) \to \mathcal{H}om^*(S_1^* \otimes_{K^*} S_2^*, N_1^*/U_1^* \otimes_{K^*} N_2^*/U_2^*)$$

is surjective for each $(S_1^*, U_1^*, S_2^*, U_2^*) \in \mathcal{C}$. *Then,*

$$\left(\mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)^{\wedge} \xrightarrow{\rho_{S_1^*,U_1^*,S_2^*,U_2^*}} D_{M_1^*,M_2^*,N_1^*,N_2^*}(S_1^*, U_1^*, S_2^*, U_2^*)\right)_{(S_1^*,U_1^*,S_2^*,U_2^*) \in \mathcal{F}_{M_1^*} \times \mathcal{V}_{N_1^*} \times \mathcal{F}_{M_2^*} \times \mathcal{V}_{N_2^*}}$$

is a limiting cone of $D_{M_1^*,N_1^*,M_2^*,N_2^*}$.

Proof. (1) By the assumption, there exist cofinal subsets $\mathcal{C}_k$ $(k = 1, 2)$ of $\mathcal{F}_{M_k^*} \times \mathcal{V}_{N_k^*}^{op}$ such that $p_{U_k^* *} i_{S_k^*}^* : \mathcal{H}om^*(M_k^*, N_k^*) \to \mathcal{H}om^*(S_k^*, N_k^*/U_k^*)$ is surjective for each $(S_k^*, U_k^*) \in \mathcal{C}_k$. For $(S_1^*, U_1^*, S_2^*, U_2^*) \in \mathcal{F}_{M_1^*} \times \mathcal{V}_{N_1^*} \times \mathcal{F}_{M_2^*} \times \mathcal{V}_{N_2^*}$, let $O(S_1^*, U_1^*, S_2^*, U_2^*)$ be the open submodule of $\mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*)$ generated by the images of $O(S_1^*, U_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*)$ and $\mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} O(S_2^*, U_2^*)$. Since $\mathcal{H}om^*(M_k^*, N_k^*) \xrightarrow{p_{U_k^* *} i_{S_k^*}^*} \mathcal{H}om^*(S_k^*, N_k^*/U_k^*)$ $(k = 1, 2)$ is a cokernel of the inclusion map $O(S_k^*, U_k^*) \to \mathcal{H}om^*(M_k^*, N_k^*)$ if $(S_k^*, U_k^*) \in \mathcal{C}_k$,

$$\mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*) \xrightarrow{p_{U_1^* *} i_{S_1^*}^* \otimes p_{U_2^* *} i_{S_2^*}^*} \widetilde{D}_{M_1^*,N_1^*,M_2^*,N_2^*}(S_1^*, U_1^*, S_2^*, U_2^*)$$

is a cokernel of the inclusion map $O(S_1^*, U_1^*, S_2^*, U_2^*) \to \mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*)$. Since

$$\{O(S_1^*, U_1^*, S_2^*, U_2^*) |\, (S_1^*, U_1^*, S_2^*, U_2^*) \in \mathcal{C}_1 \times \mathcal{C}_2\}$$

is a cofinal subset of $\mathcal{V}_{\mathcal{H}om^*(M_1^*,N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*,N_2^*)}$, the assertion follows.

(2) If $(S_1^*, U_1^*, S_2^*, U_2^*) \in \mathcal{C}$,

$$(p_{U_1^* *} \otimes p_{U_2^* *})(i_{S_1^*} \otimes i_{S_2^*})^* : \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*) \to D_{M_1^*,N_1^*,M_2^*,N_2^*}(S_1^*, S_2^*, U_1^*, U_2^*)$$

is a cokernel of the inclusion map $O(\mathrm{Im}(i_{S_1^*} \otimes i_{S_2^*}), \mathrm{Ker}(p_{U_1^*} \otimes p_{U_2^*})) \to \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)$. Since

$$\left\{ O\big(\mathrm{Im}\big(i_{S_1^*} \otimes i_{S_2^*}\big), \mathrm{Ker}\big(p_{U_1^*} \otimes p_{U_2^*}\big)\big) \big|\ (S_1^*, U_1^*, S_2^*, U_2^*) \in \mathcal{C} \right\}$$

is a cofinal subset of $\mathcal{V}_{\mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)}$, the assertion follows. □

Lemma 4.1.6 *If both* (M_1^*, N_1^*) *and* (M_2^*, N_2^*) *are very nice pairs, then the condition of (2) of (4.1.5) is satisfied.*

Proof. Let $\mathcal{C}_k$ $(k = 1, 2)$ be cofinal subsets of $\mathcal{F}_{M_k^*} \times \mathcal{V}_{N_k^*}^{op}$ such that $p_{U_k^* *} i_{S_k^*}^* : \mathcal{H}om^*(M_k^*, N_k^*) \to \mathcal{H}om^*(S_k^*, N_k^*/U_k^*)$ is surjective for each $(S_k^*, U_k^*) \in \mathcal{C}_k$. We set $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$. Clearly, $\mathcal{C}$ is a cofinal subset of $\mathcal{F}_{M_1^*} \times \mathcal{V}_{N_1^*}^{op} \times \mathcal{F}_{M_2^*} \times \mathcal{V}_{N_2^*}^{op}$. Consider the following commutative diagram for $((S_1^*, U_1^*), (S_2^*, U_2^*)) \in \mathcal{C}$.

$$\begin{CD}
\mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*) @>{p_{U_1^* *} i_{S_1^*}^* \otimes p_{U_2^* *} i_{S_2^*}^*}>> \mathcal{H}om^*(S_1^*, N_1^*/U_1^*) \otimes_{K^*} \mathcal{H}om^*(S_2^*, N_2^*/U_2^*) \\
@VV{\phi}V @VV{\phi}V \\
\mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*) @>{(p_{U_1^*} \otimes p_{U_2^*})_*(i_{S_1^*} \otimes i_{S_2^*})^*}>> \mathcal{H}om^*(S_1^* \otimes_{K^*} S_2^*, N_1^*/U_1^* \otimes_{K^*} N_2^*/U_2^*)
\end{CD}$$

Since the right vertical map is an isomophism by (4.1.4) and the upper horizontal map is surjective by the assumption, the lower horizontal map is surjective. □

Theorem 4.1.7 *If both* (M_1^*, N_1^*) *and* (M_2^*, N_2^*) *are very nice pairs, then*

$$\hat{\phi} : \mathcal{H}om^*(M_1^*, N_1^*) \,\widehat{\otimes}_{K^*} \mathcal{H}om^*(M_2^*, N_2^*) \to \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)^{\wedge}$$

is an isomorphism.

Proof. Since the following diagram commutes for $(S_1^*, U_1^*, S_2^*, U_2^*) \in \mathcal{F}_{M_1^*} \times \mathcal{V}_{N_1^*} \times \mathcal{F}_{M_2^*} \times \mathcal{V}_{N_2^*}$, the assertion follows from (4.1.6), (4.1.5) and (4.1.4).

$$\begin{array}{ccc}
\mathcal{H}om^*(M_1^*, N_1^*)\,\widehat{\otimes}_{K^*}\,\mathcal{H}om^*(M_2^*, N_2^*) & \xrightarrow{\pi_{S_1^*,U_1^*,S_2^*,U_2^*}} & \mathcal{H}om^*(S_1^*, N_1^*/U_1^*) \otimes_{K^*} \mathcal{H}om^*(S_2^*, N_2^*/U_2^*) \\
\downarrow{\scriptstyle\hat{\phi}} & & \downarrow{\scriptstyle\phi} \\
\mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)^\frown & \xrightarrow{\rho_{S_1^*,U_1^*,S_2^*,U_2^*}} & \mathcal{H}om^*(S_1^* \otimes_{K^*} S_2^*, N_1^*/U_1^* \otimes_{K^*} N_2^*/U_2^*)
\end{array}$$

□

Suppose that the topology on N^* is coarser than the topology induced by K^*. Let

$$\hat{\varphi}_{N^*}^{M^*} : \mathcal{H}om^*(M^*, K^*)\,\widehat{\otimes}_{K^*}\, N^* \to \mathcal{H}om^*(M^*, N^*)^\frown$$

be the unique morphism satisfying $\hat{\varphi}_{N^*}^{M^*}\eta_{\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}N^*} = \eta_{\mathcal{H}om^*(M^*,N^*)}\varphi_{N^*}^{M^*}$. The following results are special cases of (4.1.7).

Corollary 4.1.8 *Let (M^*, K^*) be a very nice pair.*
(1) $\hat{\phi} : \mathcal{H}om^*(M^*, K^*)\,\widehat{\otimes}_{K^*}\,\mathcal{H}om^*(M^*, K^*) \to \mathcal{H}om^*(M^* \otimes_{K^*} M^*, K^*)^\frown$ is an isomorphism.
(2) If M^* and N^* are objects of $\mathcal{T}op\mathcal{M}od^i_{K^*}$, then $\hat{\varphi}_{N^*}^{M^*} : \mathcal{H}om^*(M^*, K^*)\,\widehat{\otimes}_{K^*}\, N^* \to \mathcal{H}om^*(M^*, N^*)^\frown$ is an isomorphism.

Lemma 4.1.9 *If the condition of (2) of (4.1.5) is satisfied, then $(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)$ is a nice pair.*

Proof. Let $S_1^*\otimes_{K^*}S_2^* \xrightarrow{j} \mathrm{Im}\left(i_{S_1^*} \otimes i_{S_2^*}\right) \xrightarrow{\mathrm{inc}} M_1^*\otimes_{K^*}M_2^*$ be the factorization of $i_{S_1^*}\otimes i_{S_2^*} : S_1^*\otimes_{K^*}S_2^* \to M_1^*\otimes_{K^*}M_2^*$ such that j is surjective and inc is the inclusion map. Suppose $(S_1^*, U_1^*, S_2^*, U_2^*) \in \mathcal{C}$. Since

$$(p_{U_1^*} \otimes p_{U_2^*})_*(i_{S_1} \otimes i_{S_2})^* : \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*) \to \mathcal{H}om^*(S_1^* \otimes_{K^*} S_2^*, N_1^*/U_1^* \otimes_{K^*} N_2^*/U_2^*)$$

is surjective and it is the composition of

$$(p_{U_1^*} \otimes p_{U_2^*})_*\mathrm{inc}^* : \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*) \to \mathcal{H}om^*\left(\mathrm{Im}\left(i_{S_1^*}\otimes i_{S_2^*}\right), N_1^*/U_1^* \otimes_{K^*} N_2^*/U_2^*\right)$$

and an injection

$$j^* : \mathcal{H}om^*\left(\mathrm{Im}\left(i_{S_1^*}\otimes i_{S_2^*}\right), N_1^*/U_1^* \otimes_{K^*} N_2^*/U_2^*\right) \to \mathcal{H}om^*(S_1^* \otimes_{K^*} S_2^*, N_1^*/U_1^* \otimes_{K^*} N_2^*/U_2^*),$$

j^* is bijective. Hence $(p_{U_1^*} \otimes p_{U_2^*})_*\mathrm{inc}^*$ is surjective and this shows that $(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)$ is a nice pair. □

Suppose that "$M_1^* \otimes_{K^*} M_2^*$ is supercofinite and both N_1^* and N_2^* are subcofinite" or "$M_1^* \otimes_{K^*} M_2^*$ has a finitely generated open submodule". Then, there is a morphism

$$\lambda_{M_1^*\otimes_{K^*}M_2^*, N_1^*\otimes_{K^*}N_2^*} : \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*)^\frown \to \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^*\,\widehat{\otimes}_{K^*}\, N_2^*)$$

by (3.4.11). Composing $\lambda_{M_1^*\otimes_{K^*}M_2^*, N_1^*\otimes_{K^*}N_2^*}$ and

$$(\eta^*_{M_1^*\otimes_{K^*}M_2^*})^{-1} : \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^*\,\widehat{\otimes}_{K^*}\, N_2^*) \to \mathcal{H}om^*(M_1^*\,\widehat{\otimes}_{K^*}\, M_2^*, N_1^*\,\widehat{\otimes}_{K^*}\, N_2^*)$$

with $\hat{\phi}$, we have a morphism

$$\tilde{\phi} : \mathcal{H}om^*(M_1^*, N_1^*)\,\widehat{\otimes}_{K^*}\,\mathcal{H}om^*(M_2^*, N_2^*) \longrightarrow \mathcal{H}om^*(M_1^*\,\widehat{\otimes}_{K^*}\, M_2^*, N_1^*\,\widehat{\otimes}_{K^*}\, N_2^*).$$

It can be verified from (3.4.17) that the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{H}om^*(M_1^*, N_1^*) \otimes_{K^*} \mathcal{H}om^*(M_2^*, N_2^*) & \xrightarrow{\phi} & \mathcal{H}om^*(M_1^* \otimes_{K^*} M_2^*, N_1^* \otimes_{K^*} N_2^*) \\
\downarrow{\scriptstyle\eta_{\mathcal{H}om^*(M_1^*,N_1^*)\otimes_{K^*}\mathcal{H}om^*(M_2^*,N_2^*)}} & & \downarrow{\scriptstyle c_{M_1^*\otimes_{K^*}M_2^*, N_1^*\otimes_{K^*}N_2^*}} \\
\mathcal{H}om^*(M_1^*, N_1^*)\,\widehat{\otimes}_{K^*}\,\mathcal{H}om^*(M_2^*, N_2^*) & \xrightarrow{\tilde{\phi}} & \mathcal{H}om^*(M_1^*\,\widehat{\otimes}_{K^*}\, M_2^*, N_1^*\,\widehat{\otimes}_{K^*}\, N_2^*)
\end{array}$$

Combining (4.1.7) and (3.4.15), we have the following result.

Corollary 4.1.10 *Suppose that both (M_1^*, N_1^*) and (M_2^*, N_2^*) are very nice pairs. If one of the following conditions is satisfied, then $\tilde{\phi} : \mathcal{H}om^*(M_1^*, N_1^*)\,\widehat{\otimes}_{K^*}\,\mathcal{H}om^*(M_2^*, N_2^*) \to \mathcal{H}om^*\left(M_1^*\,\widehat{\otimes}_{K^*}\, M_2^*, N_1^*\,\widehat{\otimes}_{K^*}\, N_2^*\right)$ is an isomorphism.*

(i) $M_1^* \otimes_{K^*} M_2^*$ is supercofinite and both N_1^* and N_2^* are subcofinite.
(ii) $M_1^* \otimes_{K^*} M_2^*$ has a finitely generated open submodule (e.g. Both M_1^* and M_2^* are finitely generated.).

Proposition 4.1.11 *Let $f_i : M_i^* \otimes_{K^*} N_i^* \to Z_i^*$ $(i = 1, 2)$ be morphisms in $\mathcal{T}op\mathcal{M}od_{K^*}$. Then, composition*

$$M_1^* \widehat{\otimes}_{K^*} M_2^* \xrightarrow{\Phi_{M_1^*,N_1^*,Z_1^*}(f_1)\,\widehat{\otimes}\,\Phi_{M_2^*,N_2^*,Z_2^*}(f_2)} \mathcal{H}om^*(N_1^*, Z_1^*)\,\widehat{\otimes}_{K^*}\,\mathcal{H}om^*(N_2^*, Z_2^*) \xrightarrow{\tilde{\phi}} \mathcal{H}om^*(N_1^*\,\widehat{\otimes}_{K^*}\,N_2^*, Z_1^*\widehat{\otimes}_{K^*}Z_2^*)$$

coincides with $\Phi_{M_1^*\widehat{\otimes}_{K^*}M_2^*, N_1^*\widehat{\otimes}_{K^*}N_2^*, Z_1^*\widehat{\otimes}_{K^*}Z_2^*}((f_1\,\widehat{\otimes}_{K^*}\,f_2)sh_{M_1^*,M_2^*,N_1^*,N_2^*})$.

Proof. Since the following diagram commutes,

$$\begin{tikzcd}
(M_1^*\otimes_{K^*}M_2^*)\otimes_{K^*}(N_1^*\otimes_{K^*}N_2^*) \arrow[r, "id_{M_1^*}\otimes_{K^*}T_{M_2^*,N_1^*}\otimes_{K^*}id_{N_2^*}"] \arrow[d, "\eta_{M_1^*\otimes_{K^*}M_2^*}\otimes_{K^*}\eta_{N_1^*\otimes_{K^*}N_2^*}"] & (M_1^*\otimes_{K^*}N_1^*)\otimes_{K^*}(M_2^*\otimes_{K^*}N_2^*) \arrow[r, "f_1\otimes_{K^*}f_2"] & Z_1^*\otimes_{K^*}Z_2^* \arrow[d, "\eta_{Z_1^*\otimes_{K^*}Z_2^*}"] \\
(M_1^*\,\widehat{\otimes}_{K^*}M_2^*)\otimes_{K^*}(N_1^*\,\widehat{\otimes}_{K^*}N_2^*) \arrow[r, "sh_{M_1^*,M_2^*,N_1^*,N_2^*}"] & (M_1^*\otimes_{K^*}N_1^*)\,\widehat{\otimes}_{K^*}(M_2^*\otimes_{K^*}N_2^*) \arrow[r, "f_1\,\widehat{\otimes}_{K^*}f_2"] & Z_1^*\,\widehat{\otimes}_{K^*}Z_2^*
\end{tikzcd}$$

(3.2.1) and (4.1.2) imply

$$\begin{aligned}
\eta^*_{N_1^*\otimes_{K^*}N_2^*}\Phi((f_1\,\widehat{\otimes}\,f_2)sh_{M_1^*,M_2^*,N_1^*,N_2^*})\eta_{M_1^*\otimes_{K^*}M_2^*} &= \eta_{Z_1^*\otimes_{K^*}Z_2^*}{}_*\Phi((f_1\otimes_{K^*}f_2)(id_{M_1^*}\otimes T_{M_2^*,N_1^*}\otimes_{K^*}id_{N_2^*})) \\
&= \eta_{Z_1^*\otimes_{K^*}Z_2^*}{}_*\phi(\Phi(f_1)\otimes_{K^*}\Phi(f_2)).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\Phi((f_1\,\widehat{\otimes}\,f_2)sh_{M_1^*,M_2^*,N_1^*,N_2^*})\eta_{M_1^*\otimes_{K^*}M_2^*} &= c_{N_1^*\otimes_{K^*}N_2^*,Z_1^*\otimes_{K^*}Z_2^*}\phi(\Phi(f_1)\otimes\Phi(f_2)) \\
&= \tilde{\phi}\eta_{\mathcal{H}om^*(N_1^*,Z_1^*)\otimes_{K^*}\mathcal{H}om^*(N_2^*,Z_2^*)}(\Phi(f_1)\otimes\Phi(f_2)) \\
&= \tilde{\phi}(\Phi(f_1)\,\widehat{\otimes}\,\Phi(f_2))\eta_{M_1^*\otimes_{K^*}M_2^*}.
\end{aligned}$$

□

Lemma 4.1.12 *Suppose that there exists a morphism $\check{\varphi}^{M^*}_{N^*} : \mathcal{H}om^*(M^*, K^*)\,\widehat{\otimes}_{K^*}\,N^* \to \mathcal{H}om^*\big(M^*, \widehat{N}^*\big)$ that makes the following diagram commute.*

$$\begin{tikzcd}
\mathcal{H}om^*(M^*, K^*)\otimes_{K^*}N^* \arrow[r, "\varphi^{M^*}_{N^*}"] \arrow[d, "\eta_{\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}N^*}"] & \mathcal{H}om^*(M^*, N^*) \arrow[d, "\eta_{N^*}{}_*"] \\
\mathcal{H}om^*(M^*, K^*)\,\widehat{\otimes}_{K^*}\,N^* \arrow[r, "\check{\varphi}^{M^*}_{N^*}"] & \mathcal{H}om^*\big(M^*, \widehat{N}^*\big)
\end{tikzcd}$$

Then, $\check{\varphi}^{M^*}_{N^*}$ is unique and if there exists the morphism $\lambda_{M^*,N^*} : \mathcal{H}om^*(M^*, N^*)\widehat{\ } \to \mathcal{H}om^*\big(M^*, \widehat{N}^*\big)$ in (3.4.10), $\check{\varphi}^{M^*}_{N^*}$ is given by $\check{\varphi}^{M^*}_{N^*} = \lambda_{M^*,N^*}\hat{\varphi}^{M^*}_{N^*}$.

Proof. Since the image of $\eta_{\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}N^*}$ is dense by (1.1.16), the uniqueness of $\check{\varphi}^{M^*}_{N^*}$ is clear. The second assertion follows from (3.4.10). □

Remark 4.1.13 *If $\hat{\varphi}^{M^*}_{N^*} : \mathcal{H}om^*(M^*, K^*)\,\widehat{\otimes}_{K^*}\,N^* \to \mathcal{H}om^*(M^*, N^*)\widehat{\ }$ is an isomorphism and $\check{\varphi}^{M^*}_{N^*}$ above exists, then λ_{M^*,N^*} in (3.4.10) exists and given by $\lambda_{M^*,N^*} = \check{\varphi}^{M^*}_{N^*}\big(\hat{\varphi}^{M^*}_{N^*}\big)^{-1}$.*

We also have the following result from (4.1.10).

Corollary 4.1.14 *Suppose that (M^*, K^*) is a very nice pair and that both M^* and N^* are objects of $\mathcal{T}op\mathcal{M}od^i_{K^*}$. If "$M^*$ is supercofinite and N^* is subcofinite." or "M^* has a finitely generated open submodule.", then $\check{\varphi}^{M^*}_{N^*} : M^{**}\,\widehat{\otimes}_{K^*}\,N^* \to \mathcal{H}om^*\big(M^*, \widehat{N}^*\big)$ is an isomorphism.*

Let K^* be a field such that $K^i = \{0\}$ for $i \neq 0$ and M^*, N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ such that both of them are finite type and have the skeletal topologies. Since $\mathcal{H}om^*(M^*, K^*)$ and $\mathcal{H}om^*(N^*, K^*)$ have the skeletal topologies by (3.1.36), it follows from (2.3.2) that $(\mathcal{H}om^*(M^*, K^*)\,\widehat{\otimes}_{K^*}\,\mathcal{H}om^*(N^*, K^*))^n$ and $(\mathcal{H}om^*(M^*, K^*)\,\widehat{\otimes}_{K^*}\,N^*)^n$ are isomorphic to $\prod\limits_{i\in\boldsymbol{Z}}\mathcal{H}om^{-i}(M^*, K^*)\otimes_{K^*}\mathcal{H}om^{n+i}(N^*, K^*)$ and $\prod\limits_{i\in\boldsymbol{Z}}\mathcal{H}om^{-i}(M^*, K^*)\otimes_{K^*}N^{n+i}$, respectively.

We choose a basis $b_{i1}, b_{i2}, \dots, b_{id_i}$ of M^i and let $b^*_{i1}, b^*_{i2}, \dots, b^*_{id_i}$ ($b^*_{ij} \in \mathcal{H}om^{-i}(M^*, K^*)$) be the dual basis of $b_{i1}, b_{i2}, \dots, b_{id_i}$. Similarly, let $c_{i1}, c_{i2}, \dots, c_{ie_i}$ of N^i and $c^*_{i1}, c^*_{i2}, \dots, c^*_{ie_i}$ ($c^*_{ij} \in \mathcal{H}om^{-i}(N^*, K^*)$) the dual basis of $c_{i1}, c_{i2}, \dots, c_{ie_i}$. Define maps

$$\begin{aligned}
&\rho : \mathcal{H}om^*(M^* \otimes_{K^*} N^*, K^*) \to \mathcal{H}om^*(M^*, K^*) \mathbin{\widehat{\otimes}}_{K^*} \mathcal{H}om^*(N^*, K^*) \\
&\psi : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(M^*, K^*) \mathbin{\widehat{\otimes}}_{K^*} N^*
\end{aligned}$$

by $\rho(g) = \sum_{i+k=n} \sum_{l=1}^{e_k} \sum_{j=1}^{d_i} (-1)^{ik} g([-n], b_{ij} \otimes c_{kl}) b^*_{ij} \otimes c^*_{kl}$ for $g \in \mathcal{H}om^{-n}(M^* \otimes_{K^*} N^*, K^*)$ and $\psi(f) = \sum_{k \in \boldsymbol{Z}} \sum_{l=1}^{d_k} (-1)^{k(k-n)} b^*_{kl} \otimes f([-n], b_{kl})$ for $f \in \mathcal{H}om^{-n}(M^*, N^*)$.

Proposition 4.1.15 *ρ is the inverse of $\hat{\phi} : \mathcal{H}om^*(M^*, K^*) \mathbin{\widehat{\otimes}}_{K^*} \mathcal{H}om^*(N^*, K^*) \to \mathcal{H}om^*(M^* \otimes_{K^*} N^*, K^*)$ and ψ is the inverse of $\hat{\varphi}^{M^*}_{N^*} : \mathcal{H}om^*(M^*, K^*) \mathbin{\widehat{\otimes}}_{K^*} N^* \to \mathcal{H}om^*(M^*, N^*)$.*

Proof. For $g \in \mathcal{H}om^{-n}(M^* \otimes_{K^*} N^*, K^*)$ and $f \in \mathcal{H}om^{-n}(M^*, N^*)$, we have the following equalities if $r + t = n$.

$$\begin{aligned}
(\hat{\phi}(\rho(g)))([-n], b_{rs} \otimes c_{tu}) &= \sum_{i+k=n} \sum_{l=1}^{e_k} \sum_{j=1}^{d_i} (-1)^{ik} g([-n], b_{ij} \otimes c_{kl}) (\phi(b^*_{ij} \otimes c^*_{kl}))([-n], b_{rs} \otimes c_{tu}) \\
&= \sum_{i+k=n} \sum_{l=1}^{e_k} \sum_{j=1}^{d_i} (-1)^{k(i+r)} g([-n], b_{ij} \otimes c_{kl}) b^*_{ij}([-i], b_{rs}) c^*_{kl}([-k], c_{tu}) \\
&= g([-n], b_{rs} \otimes c_{tu}) \\
\left(\hat{\varphi}^{M^*}_{N^*}(\psi(f))\right)([-n], b_{ij}) &= \sum_{k \in \boldsymbol{Z}} \sum_{l=1}^{d_k} (-1)^{k(k-n)} \left(\hat{\varphi}^{M^*}_{N^*}(b^*_{kl} \otimes f([-n], b_{kl}))\right)([-n], b_{ij}) \\
&= \sum_{k \in \boldsymbol{Z}} \sum_{l=1}^{d_k} (-1)^{(i+k)(k-n)} b^*_{kl}([-n], b_{ij}) f([-n], b_{kl}) \\
&= f([-n], b_{ij})
\end{aligned}$$

Thus we have $\hat{\phi}(\rho(g)) = g$ and $\hat{\varphi}^{M^*}_{N^*}(\psi(f)) = f$. □

4.2 Commutative diagrams

Definition 4.2.1 *Suppose that the topology on L^* is coarser than the topology induced by K^* and that $\hat{\varphi}^{M^*}_{L^*} : M^{**} \mathbin{\widehat{\otimes}}_{K^*} L^* \to \mathcal{H}om^*(M^*, L^*)^{\wedge}$ is an isomorphism. We denote by*

$$\Lambda = \Lambda_{M^*, N^*, L^*} : \mathrm{Hom}^c_{K^*}(M^* \otimes_{K^*} N^*, L^*) \to \mathrm{Hom}^c_{K^*}(N^*, L^* \mathbin{\widehat{\otimes}}_{K^*} M^{**})$$

the composition of following maps.

$$\begin{aligned}
&\mathrm{Hom}^c_{K^*}\big(M^* \otimes_{K^*} N^*, L^*\big) \xrightarrow{T^*_{N^*, M^*}} \mathrm{Hom}^c_{K^*}\big(N^* \otimes_{K^*} M^*, L^*\big) \xrightarrow{\Phi_{N^*, M^*, L^*}} \mathrm{Hom}^c_{K^*}\big(N^*, \mathcal{H}om^*(M^*, L^*)\big) \\
&\xrightarrow{\eta_{\mathcal{H}om^*(M^*, L^*)*}} \mathrm{Hom}^c_{K^*}\big(N^*, \mathcal{H}om^*(M^*, L^*)^{\wedge}\big) \xrightarrow{(\hat{\varphi}^{M^*}_{L^*})^{-1}_*} \mathrm{Hom}^c_{K^*}\big(N^*, M^{**} \mathbin{\widehat{\otimes}}_{K^*} L^*\big) \\
&\xrightarrow{\widehat{T}_{M^{**}, L^* *}} \mathrm{Hom}^c_{K^*}\big(N^*, L^* \mathbin{\widehat{\otimes}}_{K^*} M^{**}\big)
\end{aligned}$$

For $f \in \mathrm{Hom}^c_{K^*}(M^* \otimes_{K^*} N^*, L^*)$, we call $\Lambda(f) : N^* \to L^* \mathbin{\widehat{\otimes}}_{K^*} M^{**}$ the Milnor map associated with f.

Proposition 4.2.2 *Assume that V^* and L^* are complete Hausdorff and that the topologies on V^* and L^* are coarser than the topology induced by K^*. Let us denote by $\tilde{\alpha}_{V^*} : K^* \otimes_{K^*} V^* \to V^*$ the isomorphism induced by the K^*-module structure map of V^*. If $\check{\varphi}^{M^*}_{L^*} : M^{**} \mathbin{\widehat{\otimes}}_{K^*} L^* \to \mathcal{H}om^*(M^*, L^*)$ in (4.1.12) is defined and it is an isomorphism, the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{Hom}^c_{K^*}(N^*, M^{**}\widehat{\otimes}_{K^*} L^*) \arrow[r, "\zeta"] & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(M^{**}\widehat{\otimes}_{K^*} L^*, V^*), \mathcal{H}om^*(N^*, V^*)) \arrow[d, "\left(\left(\eta^*_{M^{**}\otimes_{K^*}L^*}\right)^*\right)^{-1}"] \\
\mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(M^*, L^*)) \arrow[u, "\left(\check{\varphi}^{M^*}_{L^*}\right)^{-1}_*"] & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(M^{**}\otimes_{K^*} L^*, V^*), \mathcal{H}om^*(N^*, V^*)) \arrow[d, "(\tilde{\alpha}_{V^**}\phi)^*"] \\
\mathrm{Hom}^c_{K^*}(N^*\otimes_{K^*} M^*, L^*) \arrow[u, "\Phi_{N^*,M^*,L^*}"] \arrow[d, "T^*_{M^*,N^*}"] & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(M^{**}, K^*)\otimes_{K^*}\mathcal{H}om^*(L^*, V^*), \mathcal{H}om^*(N^*, V^*)) \arrow[d, "\left(\chi_{M^*,K^*}\otimes_{K^*} id_{\mathcal{H}om^*(L^*,V^*)}\right)^*"] \\
\mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*} N^*, L^*) \arrow[d, "\Phi_{M^*,N^*,L^*}"] & \mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}\mathcal{H}om^*(L^*, V^*), \mathcal{H}om^*(N^*, V^*)) \arrow[d, "\Phi_{M^*,\mathcal{H}om^*(L^*,V^*),\mathcal{H}om^*(N^*,V^*)}"] \\
\mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*, L^*)) \arrow[r, "(\zeta^*)_*"] & \mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(\mathcal{H}om^*(L^*, V^*), \mathcal{H}om^*(N^*, V^*)))
\end{tikzcd}$$

Proof. By the the naturality of ζ, the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(M^*, L^*)) \arrow[r, "\zeta"] & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(\mathcal{H}om^*(M^*, L^*), V^*), \mathcal{H}om^*(N^*, V^*)) \\
\mathrm{Hom}^c_{K^*}(N^*, M^{**}\widehat{\otimes}_{K^*} L^*) \arrow[u, "\left(\check{\varphi}^{M^*}_{L^*}\right)_*"] \arrow[r, "\zeta"] & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(M^{**}\widehat{\otimes}_{K^*} L^*, V^*), \mathcal{H}om^*(N^*, V^*)) \arrow[u, "\left(\left(\check{\varphi}^{M^*}_{L^*}\right)^*\right)^*"]
\end{tikzcd}$$

Since both $\check{\varphi}^{M^*}_{L^*}$ and $\eta^*_{M^{**}\otimes_{K^*}L^*} : \mathcal{H}om^*\big(M^{**}\widehat{\otimes}_{K^*} L^*, V^*\big) \to \mathcal{H}om^*(M^{**}\otimes_{K^*} L^*, V^*)$ are isomorphisms by the assumption and we have $\check{\varphi}^{M^*}_{L^*}\eta_{M^{**}\otimes_{K^*}L^*} = \varphi^{M^*}_{L^*}$, it follows that $\big(\varphi^{M^*}_{L^*}\big)^* : \mathcal{H}om^*(\mathcal{H}om^*(M^*, L^*), V^*) \to \mathcal{H}om^*(M^{**}\otimes_{K^*} L^*, V^*)$ is an isomorphism. Hence it suffices to verify that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(M^*, L^*)) \arrow[r, "\zeta"] & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(\mathcal{H}om^*(M^*, L^*), V^*), \mathcal{H}om^*(N^*, V^*)) \arrow[d, "\left(\left(\left(\varphi^{M^*}_{L^*}\right)^*\right)^*\right)^{-1}"] \\
\mathrm{Hom}^c_{K^*}(N^*\otimes_{K^*} M^*, L^*) \arrow[u, "\Phi_{N^*,M^*,L^*}"] \arrow[dd, "T^*_{M^*,N^*}"] & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(M^{**}\otimes_{K^*} L^*, V^*), \mathcal{H}om^*(N^*, V^*)) \arrow[d, "(\tilde{\alpha}_{V^**}\phi)^*"] \\
 & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(M^{**}, K^*)\otimes_{K^*}\mathcal{H}om^*(L^*, V^*), \mathcal{H}om^*(N^*, V^*)) \arrow[d, "\left(\chi_{M^*,K^*}\otimes_{K^*} id_{\mathcal{H}om^*(L^*,V^*)}\right)^*"] \\
\mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*} N^*, L^*) \arrow[d, "\Phi_{M^*,N^*,L^*}"] & \mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}\mathcal{H}om^*(L^*, V^*), \mathcal{H}om^*(N^*, V^*)) \arrow[d, "\Phi_{M^*,\mathcal{H}om^*(L^*,V^*),\mathcal{H}om^*(N^*,V^*)}"] \\
\mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*, L^*)) \arrow[r, "(\zeta^*)_*"] & \mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(\mathcal{H}om^*(L^*, V^*), \mathcal{H}om^*(N^*, V^*)))
\end{tikzcd}$$

For a morphism $f : N^*\otimes_{K^*} M^* \to L^*$, we have $(\zeta^*)_*\Phi_{M^*,N^*,L^*}T^*_{M^*,N^*}(f) = \zeta^*\Phi_{M^*,N^*,L^*}(fT_{M^*,N^*})$ and, for $x \in M^n$, $(\zeta^*\Phi_{M^*,N^*,L^*}(fT_{M^*,N^*}))(x) : \Sigma^n\mathcal{H}om^*(L^*, V^*) \to \mathcal{H}om^*(N^*, V^*)$ maps $([n], g) \in (\Sigma^n\mathcal{H}om^*(L^*, V^*))^k$ to a map $(-1)^{n(k-n)}g\Sigma^{k-n}(\Phi_{M^*,N^*,L^*}(fT_{M^*,N^*})(x))\varepsilon_{k-n,n,N^*} : \Sigma^k N^* \to V^*$ which maps $([k], y) \in (\Sigma^k N^*)^{k+m}$ to $(-1)^{n(k+m-n)}g([k-n], (\Phi_{N^*,M^*,L^*}(f)(y))([m], x))$. On the other hand, for $x \in M^n$ and $g : \Sigma^{k-n}L^* \to V^*$, we set $\gamma = \Big(\big(\varphi^{M^*}_{L^*}\big)^*\Big)^{-1}(\tilde{\alpha}_{V^*}\phi(\chi_{M^*,K^*}(x)\otimes g)) \in \mathcal{H}om^k(\mathcal{H}om^*(M^*, L^*), V^*)$. Then,

$$\left(\Phi_{M^*,\mathcal{H}om^*(L^*,V^*),\mathcal{H}om^*(N^*,V^*)}\big(\chi_{M^*,K^*}\otimes_{K^*} id_{\mathcal{H}om^*(L^*,V^*)}\big)^*(\tilde{\alpha}_{V^**}\phi)^*\left(\left(\left(\varphi^{M^*}_{L^*}\right)^*\right)^*\right)^{-1}\zeta\Phi_{N^*,M^*,L^*}(f)\right)(x)$$

maps $([n], g) \in \Sigma^n\mathcal{H}om^*(L^*, V^*)$ to a composition $\Sigma^k N^* \xrightarrow{\Sigma^k\Phi_{N^*,M^*,L^*}(f)} \Sigma^k\mathcal{H}om^*(M^*, L^*) \xrightarrow{\gamma} V^*$ which maps $([k], y) \in (\Sigma^k N^*)^{k+m}$ to $\gamma([k], \Phi_{N^*,M^*,L^*}(f)(y))$. The following diagram commutes.

$$\begin{tikzcd}
\Sigma^k(M^{**}\otimes_{K^*} L^*) \arrow[r, "\left(\tau^{n,k-n}_{M^{**},L^*}\right)^{-1}"] \arrow[d, "\Sigma^k\varphi^{M^*}_{L^*}"] & \Sigma^n M^{**}\otimes_{K^*}\Sigma^{k-n}L^* \arrow[r, "\chi_{M^*,K^*}(x)\otimes g"] & K^*\otimes_{K^*} V^* \arrow[d, "\tilde{\alpha}_{V^*}"] \\
\Sigma^k\mathcal{H}om^*(M^*, L^*) \arrow[rr, "\gamma"] & & V^*
\end{tikzcd}$$

Hence, for $h \in \mathcal{H}om^l(M^*, K^*)$ and $z \in Z^{m-l}$, we have

$$\begin{aligned}
\gamma\Big([k], \check{\varphi}^{M^*}_{L^*}\eta_{M^{**}\otimes_{K^*}L^*}(h\otimes z)\Big) &= \gamma\Big([k], \varphi^{M^*}_{L^*}(h\otimes z)\Big) = \gamma\Big(\Sigma^k\varphi^{M^*}_{L^*}\Big)([k], h\otimes z) \\
&= \tilde{\alpha}_{V^*}(\chi_{M^*,K^*}(x)\otimes g)\Big(\tau^{n,k-n}_{\mathcal{H}om^*(M^*,K^*),L^*}\Big)^{-1}([k], h\otimes z) \\
&= (-1)^{kl}h([l],x)g([k-n],z) = (-1)^{n(k-l-n)}g([k-n], h([l],x)z) \\
&= (-1)^{n(k+m-n)}g\Big([k-n], \Big(\varphi^{M^*}_{L^*}(h\otimes z)\Big)([m],x)\Big) \\
&= (-1)^{n(k+m-n)}g\Big([k-n], \Big(\check{\varphi}^{M^*}_{L^*}\eta_{M^{**}\otimes_{K^*}L^*}(h\otimes z)\Big)([m],x)\Big).
\end{aligned}$$

It follows that $\gamma\big([k], \check{\varphi}^{M^*}_{L^*}\eta_{M^{**}\otimes_{K^*}L^*}(w)\big) = (-1)^{n(k+m-n)}g\big([k-n], \big(\check{\varphi}^{M^*}_{L^*}\eta_{M^{**}\otimes_{K^*}L^*}(w)\big)([m],x)\big)$ for any $m \in \boldsymbol{N}$ and $w \in (M^{**}\otimes_{K^*}L^*)^m$. Since the image of $\eta_{M^{**}\otimes_{K^*}L^*}$ is dense and $\check{\varphi}^{M^*}_{L^*}$ is an isomorphism, we have $\gamma([k],\psi) = (-1)^{n(k+m-n)}g([k-n], \psi([m],x))$ for any $m \in \boldsymbol{N}$ and $\psi \in \mathcal{H}om^m(M^*, L^*)$. In particular, we have $\gamma([k], \Phi_{N^*,M^*,L^*}(f)(y)) = (-1)^{n(k+m-n)}g([k-n], \Phi_{N^*,M^*,L^*}(f)(y)([m],x))$ which implies the assertion. $\square$

Remark 4.2.3 *Suppose that M^* and L^* satisfies the assumptions of (3.4.16) and that L^* and V^* are complete and have topologies coarser than the topology induced by K^*. Moreover, if $\hat{\varphi}^{M^*}_{L^*} : M^{**}\,\widehat{\otimes}_{K^*}\,L^* \to \mathcal{H}om^*(M^*, L^*)\widehat{\ }$ is an isomorphism, so is $\check{\varphi}^{M^*}_{L^*} : M^{**}\,\widehat{\otimes}_{K^*}\,L^* \to \mathcal{H}om^*(M^*, L^*)$.*

Proposition 4.2.4 *Under the same condition as in (4.2.2), the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}^c_{K^*}\left(N^*, L^*\,\widehat{\otimes}_{K^*}\,M^{**}\right) \arrow[r, "\zeta"] & \mathrm{Hom}^c_{K^*}\left(\mathcal{H}om^*\left(L^*\,\widehat{\otimes}_{K^*}\,M^{**}, V^*\right), \mathcal{H}om^*(N^*, V^*)\right) \arrow[d, "\left(\left(\eta^*_{L^*\otimes_{K^*}M^{**}}\right)^*\right)^{-1}"] \\
\mathrm{Hom}^c_{K^*}\left(N^*, M^{**}\,\widehat{\otimes}_{K^*}\,L^*\right) \arrow[u, "\widehat{T}_{M^{**},L^*\,*}"] & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*\otimes_{K^*}M^{**}, V^*), \mathcal{H}om^*(N^*, V^*)) \arrow[d, "(\tilde{\alpha}_{V^*\,*}T_{V^*,K^*\,*}\phi)^*"] \\
\mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(M^*, L^*)) \arrow[u, "\left(\check{\varphi}^{M^*}_{L^*}\right)^{-1}_*"] & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*, V^*)\otimes_{K^*}\mathcal{H}om^*(M^{**}, K^*), \mathcal{H}om^*(N^*, V^*)) \arrow[d, "\left(id_{\mathcal{H}om^*(L^*,V^*)}\otimes_{K^*}\chi_{M^*,K^*}\right)^*"] \\
\mathrm{Hom}^c_{K^*}(N^*\otimes_{K^*}M^*, L^*) \arrow[u, "\Phi_{N^*,M^*,L^*}"] \arrow[d, "T^*_{M^*,N^*}"] & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*, V^*)\otimes_{K^*}M^*, \mathcal{H}om^*(N^*, V^*)) \arrow[d, "T^*_{M^*,\mathcal{H}om^*(L^*,V^*)}"] \\
\mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}N^*, L^*) \arrow[d, "\Phi_{M^*,N^*,L^*}"] & \mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}\mathcal{H}om^*(L^*, V^*), \mathcal{H}om^*(N^*, V^*)) \arrow[d, "\Phi_{M^*,\mathcal{H}om^*(L^*,V^*),\mathcal{H}om^*(N^*,V^*)}"] \\
\mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(N^*, L^*)) \arrow[r, "(\zeta^*)_*"] & \mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(\mathcal{H}om^*(L^*, V^*), \mathcal{H}om^*(N^*, V^*)))
\end{tikzcd}$$

Proof. The assertion follows from (4.2.2) and the commutativity of the following diagrams.

$$\begin{tikzcd}
\mathcal{H}om^*(L^*, V^*)\otimes_{K^*}M^* \arrow[d, "id_{\mathcal{H}om^*(L^*,V^*)}\otimes_{K^*}\chi_{M^*,K^*}"] & M^*\otimes_{K^*}\mathcal{H}om^*(L^*, V^*) \arrow[l, "T_{M^*,\mathcal{H}om^*(L^*,V^*)}"'] \arrow[d, "\chi_{M^*,K^*}\otimes_{K^*}id_{\mathcal{H}om^*(L^*,V^*)}"] \\
\mathcal{H}om^*(L^*, V^*)\otimes_{K^*}\mathcal{H}om^*(M^{**}, K^*) \arrow[r, "T_{\mathcal{H}om^*(L^*,V^*),\mathcal{H}om^*(M^{**},K^*)}"] \arrow[d, "\phi"] & \mathcal{H}om^*(M^{**}, K^*)\otimes_{K^*}\mathcal{H}om^*(L^*, V^*) \arrow[d, "\phi"] \\
\mathcal{H}om^*(L^*\otimes_{K^*}M^{**}, V^*\otimes_{K^*}K^*) \arrow[r, "T_{V^*,K^*\,*}T^*_{M^{**},L^*}"] \arrow[d, "\tilde{\alpha}_{V^*\,*}T_{V^*,K^*\,*}"] & \mathcal{H}om^*(M^{**}\otimes_{K^*}L^*, K^*\otimes_{K^*}V^*) \arrow[d, "\tilde{\alpha}_{V^*\,*}"] \\
\mathcal{H}om^*(L^*\otimes_{K^*}M^{**}, V^*) & \mathcal{H}om^*(M^{**}\otimes_{K^*}L^*, V^*) \arrow[l, "T^*_{L^*,M^{**}}"']
\end{tikzcd}$$

$$\begin{tikzcd}
M^{**}\otimes_{K^*}L^* \arrow[r, "\eta_{M^{**}\otimes_{K^*}L^*}"] \arrow[d, "T_{M^{**},L^*}"] & M^{**}\,\widehat{\otimes}_{K^*}\,L^* \arrow[d, "\widehat{T}_{M^{**},L^*}"] \\
L^*\otimes_{K^*}M^{**} \arrow[r, "\eta_{L^*\otimes_{K^*}M^{**}}"] & L^*\,\widehat{\otimes}_{K^*}\,M^{**}
\end{tikzcd}$$

$$\begin{array}{ccc} \mathrm{Hom}^c_{K^*}(N^*, M^{**}\,\widehat{\otimes}_{K^*} L^*) & \xrightarrow{\zeta} & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(M^{**}\,\widehat{\otimes}_{K^*} L^*, V^*), \mathcal{H}om^*(N^*, V^*)) \\ \Big\downarrow{\scriptstyle \widehat{T}_{M^{**},L^*\,*}} & & \Big\downarrow{\scriptstyle (\widehat{T}^*_{M^{**},L^*})^*} \\ \mathrm{Hom}^c_{K^*}(N^*, L^*\,\widehat{\otimes}_{K^*} M^{**}) & \xrightarrow{\zeta} & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*\,\widehat{\otimes}_{K^*} M^{**}, V^*), \mathcal{H}om^*(N^*, V^*)) \end{array}$$

□

It follows from (4.1.8), (3.4.1), (3.2.6) and (3.4.15), we have the following result.

Proposition 4.2.5 *Consider the following conditions.*

(i) The topologies are M^* and L^* are coarser than the topologies induced by K^*.
(ii) (M^*, K^*) is a very nice pair.
(iii) L^* is Hausdorff.
(iv) (M^*, L^*) is a nice pair.
(v) M^* and $M^* \otimes_{K^*} N^*$ are supercofinite and L^* is profinite.

Λ_{M^*,N^*,L^*} is defined if (i) and (ii) are satisfied. Λ_{M^*,N^*,L^*} is injective if (i), (ii) and (iii) are satisfied. Λ_{M^*,N^*,L^*} is an isomorphism if (i), (ii), (iv) and (v) are satisfied.

Remark 4.2.6 *If $\hat{\varphi}^{M^*}_{L^*} : M^{**}\,\widehat{\otimes}_{K^*} L^* \to \mathcal{H}om^*(M^*, L^*)^\frown$ is an isomorphism and $\lambda_{M^*,L^*} : \mathcal{H}om^*(M^*, L^*)^\frown \to \mathcal{H}om^*(M^*, \widehat{L}^*)$ in (3.4.10) exists and it is an isomorphism, then $\check{\varphi}^{M^*}_{L^*} : M^{**}\,\widehat{\otimes}_{K^*} L^* \to \mathcal{H}om^*(M^*, \widehat{L}^*)$ is defined and it is an isomorphism. In this case, Λ_{M^*,N^*,L^*} is the following composition.*

$$\mathrm{Hom}^c_{K^*}\big(M^* \otimes_{K^*} N^*, L^*\big) \xrightarrow{T^*_{N^*,M^*}} \mathrm{Hom}^c_{K^*}\big(N^* \otimes_{K^*} M^*, L^*\big) \xrightarrow{\Phi_{N^*,M^*,L^*}} \mathrm{Hom}^c_{K^*}\big(N^*, \mathcal{H}om^*(M^*, L^*)\big) \xrightarrow{(\eta_{L^*\,*})_*}$$
$$\mathrm{Hom}^c_{K^*}\big(N^*, \mathcal{H}om^*(M^*, \widehat{L}^*)\big) \xrightarrow{(\check{\varphi}^{M^*}_{L^*})^{-1}_*} \mathrm{Hom}^c_{K^*}\big(N^*, M^{**}\,\widehat{\otimes}_{K^*} L^*\big) \xrightarrow{\widehat{T}_{M^{**},L^*\,*}} \mathrm{Hom}^c_{K^*}\big(N^*, L^*\,\widehat{\otimes}_{K^*} M^{**}\big)$$

Proposition 4.2.7 *Let K^* be a field such that $K^i = \{0\}$ if $i \neq 0$. Assume that M^* is finite type, L^* is profinite, For a morphism $\gamma : N^* \to L^*\,\widehat{\otimes}_{K^*} M^*$ of $\mathcal{T}op\mathcal{M}od_{K^*}$, let $\tilde{\gamma} : N^* \to \mathcal{H}om^*(M^{**}, L^*)$ be the following composition.*

$$N^* \xrightarrow{\gamma} L^*\,\widehat{\otimes}_{K^*} M^* \xrightarrow{\widehat{T}_{L^*,M^*}} M^*\,\widehat{\otimes}_{K^*} L^* \xrightarrow{\chi_{M^*,K^*}\,\widehat{\otimes}_{K^*}\, id_{L^*}} \mathcal{H}om^*(M^{**}, K^*)\,\widehat{\otimes}_{K^*} L^* \xrightarrow{\hat{\varphi}^{M^{**}}_{L^*}} \mathcal{H}om^*(M^{**}, L^*)$$

If there exists $\bar{\gamma} \in \mathrm{Hom}^c_{K^*}(N^* \otimes_{K^*} M^{**}, L^*)$ which is mapped to $\tilde{\gamma}$ by

$$\Phi_{N^*,M^{**},L^*} : \mathrm{Hom}^c_{K^*}(N^* \otimes_{K^*} M^{**}, L^*) \to \mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(M^{**}, L^*)),$$

then we have $\Lambda_{M^{**},N^*,L^*}(\bar{\gamma} T_{M^{**},N^*}) = (id_{L^*}\,\widehat{\otimes}_{K^*}\,\chi_{M^*,K^*})\gamma$.

Proof. We note that M^{**} has the skeletal topology by (3.1.36) which coincides with the cofinite topology since M^{**} is finite type. Hence $\mathcal{H}om^*(M^{**}, L^*)$ is complete by (3.4.15) and $\hat{\varphi}^{M^{**}}_{L^*} : \mathcal{H}om^*(M^{**}, K^*)\,\widehat{\otimes}_{K^*} L^* \to \mathcal{H}om^*(M^{**}, L^*)$ is an isomorphism by (4.1.14). It follows from (4.2.5) that

$$\Lambda_{M^{**},N^*,L^*} : \mathrm{Hom}^c_{K^*}(M^{**} \otimes_{K^*} N^*, L^*) \to \mathrm{Hom}^c_{K^*}(N^*, L^*\,\widehat{\otimes}_{K^*} M^{***})$$

is defined and injective. Then, the assertion is verified directly from the definition of Λ_{M^{**},N^*,L^*} and $\bar{\gamma}$. □

Proposition 4.2.8 *Suppose that L^* is Hausdorff and the topology on L^* is coarser than the topology induced by K^*. Assume moreover that $\check{\varphi}^{M^*}_{L^*} : M^{**}\,\widehat{\otimes}_{K^*} L^* \to \mathcal{H}om^*(M^*, \widehat{L}^*)$ is defined and it is an isomorphism. For $f \in \mathrm{Hom}^c_{K^*}(M^* \otimes_{K^*} N^*, L^*)$ and $y \in N^m$, let $(\alpha_i)_{i \in I}$ be a sequence in $L^* \otimes_{K^*} M^{**}$ indexed by a directed set I such that $(\eta_{L^* \otimes_{K^*} M^{**}}(\alpha_i))_{i \in I}$ converges to $\Lambda(f)(y)$. If $\alpha_i = \sum_{j=1}^{\nu_i} z_{ij} \otimes g_{ij}$ for $z_{ij} \in L^{m-d_{ij}}$ and $g_{ij} \in (M^{**})^{d_{ij}}$, we set $\beta_i(x) = \sum_{j=1}^{\nu_i} (-1)^{(m-d_{ij})(n+d_{ij})} g_{ij}([d_{ij}], x) z_{ij}$ for $x \in M^n$. Then $((-1)^{mn}\beta_i(x))_{i \in I}$ converges to $f(x \otimes y)$ in L^*.*

Proof. Since $\eta_{L^*\,*}\Phi_{N^*,M^*,L^*}(fT_{N^*,M^*}) = \check{\varphi}^{M^*}_{L^*}\widehat{T}_{L^*,M^{**}}\Lambda(f)$ by the definition of Λ, we have

$$(-1)^{mn}\eta_{L^*}(f(x \otimes y)) = (\eta_{L^*\,*}\Phi_{N^*,M^*,L^*}(fT_{N^*,M^*})(y)))([m], x) = \Big(\check{\varphi}^{M^*}_{L^*}\widehat{T}_{L^*,M^{**}}\Lambda(f)(y)\Big)([m], x).$$

On the other hand, since

$$\begin{aligned}\left(\check{\varphi}_{L^*}^{M^*}\widehat{T}_{L^*,M^{**}}(\eta_{L^*\otimes_{K^*}M^{**}}(\alpha_i))\right)([m],x) &= \left(\eta_{L^*}\varphi_{L^*}^{M^*}T_{L^*,M^{**}}(\alpha_i)\right)([m],x)\\ &= \sum_{j=1}^{\nu_i}(-1)^{d_{ij}(m-d_{ij})}\eta_{L^*}\varphi_{L^*}^{M^*}(g_{ij}\otimes z_{ij})([m],x)\\ &= \sum_{j=1}^{\nu_i}(-1)^{(m-d_{ij})(n+d_{ij})}\eta_{L^*}g_{ij}([d_{ij}],x)z_{ij}\\ &= \eta_{L^*}(\beta_i(x)),\end{aligned}$$

$(\eta_{L^*}(\beta_i(x)))_{i\in I}$ converges to $(-1)^{mn}\eta_{L^*}(f(x\otimes y))$. Hence the assertion follows from the assumption that L^* is Hausdorff. □

Corollary 4.2.9 *Suppose that $K^i=\{0\}$ for $i\neq 0$ and K^* is discrete. If M^* is finite type and L^* has the skeletal topology, then the dual M^{**} of M^* has the skeletal topology by (3.1.36) and $\left(L^*\mathbin{\widehat{\otimes}}_{K^*}M^{**}\right)^m$ is isomorphic to $\prod_{i\in\boldsymbol{Z}}L^{m-i}\otimes_{K^*}(M^{**})^i$ by (2.3.2). Assume moreover that $\check{\varphi}_{L^*}^{M^*}:M^{**}\mathbin{\widehat{\otimes}}_{K^*}L^*\to\mathcal{H}om^*(M^*,\widehat{L}^*)$ is defined and it is an isomorphism. Let $f\in\mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}N^*,L^*)$ and $y\in N^m$.*

(1) We set $\Lambda(f)(y)=(a_j)_{j\in\boldsymbol{Z}}$ where $a_j=\sum_{k=1}^{\nu_j}z_{jk}\otimes g_{jk}\in L^{m+j}\otimes_{K^*}(M^{**})^{-j}$. Then, for $x\in M^n$, we have $f(x\otimes y)=(-1)^{mn}\sum_{k=1}^{\nu_n}g_{nk}([-n],x)z_{nk}$.

(2) Assume that M^* is a free K^*-module. Let $\{v_{ij}\}_{j\in I_i}$ be a basis of M^i and $\{v^*_{ij}\}_{j\in I_i}$ the dual basis of $\{v_{ij}\}_{j\in I_i}$. If we put $a_i=(-1)^{im}\sum_{j\in I_i}f(v_{ij}\otimes y)\otimes v^*_{ij}\in L^{m+i}\otimes_{K^*}(M^{**})^{-i}$, then $\Lambda(f)(y)=(a_i)_{i\in\boldsymbol{Z}}$.

Proof. (1) Put $a_{ij}=\begin{cases}a_j & |j|\leqq i\\ 0 & |j|>i\end{cases}$ and $\alpha_i=(a_{ij})_{j\in\boldsymbol{Z}}$. Then, $(\alpha_i)_{i\in\boldsymbol{N}}$ converges to $\Lambda(f)(y)$. Since $\beta_i(x)=\sum_{k=1}^{\nu_n}g_{nk}([-n],x)z_{nk}$ if $i\geqq|n|$, the assertion follows from (4.2.8).

(2) We may put $\Lambda(f)(y)=(a_i)_{i\in\boldsymbol{Z}}$ where $a_i=\sum_{k\in I_i}z_{ik}\otimes v^*_{ik}\in L^{m+i}\otimes_{K^*}(M^{**})^{-i}$. By (1), we have $f(v_{ij}\otimes y)=(-1)^{im}\sum_{k\in I_i}v^*_{ik}([-i],v_{ij})z_{ik}=(-1)^{im}z_{ij}$. Hence the result follows. □

Suppose that the topology on L^* is coarser than the topology induced by K^* and that $\hat{\varphi}_{L^*}^{M^*}:M^{**}\mathbin{\widehat{\otimes}}_{K^*}L^*\to\mathcal{H}om^*(M^*,L^*)^{\wedge}$ is an isomorphism. If V^* is complete Hausdorff and the topology on V^* is coarser than the topology induced by K^*, we denote by

$$\Theta=\Theta_{M^*,N^*,L^*,V^*}:\mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}N^*,L^*)\to\mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*,V^*)\otimes_{K^*}M^*,\mathcal{H}om^*(N^*,V^*))$$

the composition of following maps.

$$\begin{aligned}\mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}N^*,L^*) &\xrightarrow{\Lambda_{M^*,N^*,L^*}}\mathrm{Hom}^c_{K^*}(N^*,L^*\mathbin{\widehat{\otimes}}_{K^*}M^{**})\\ &\xrightarrow{\zeta}\mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*\mathbin{\widehat{\otimes}}_{K^*}M^{**},V^*),\mathcal{H}om^*(N^*,V^*))\\ &\xrightarrow{\left(\eta^{*-1}_{L^*\otimes_{K^*}M^{**}}\right)^*}\mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*\otimes_{K^*}M^{**},V^*),\mathcal{H}om^*(N^*,V^*))\\ &\xrightarrow{(\alpha_{V^*\,*}T_{V^*,K^{**}})^*}\mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*\otimes_{K^*}M^{**},V^*\otimes_{K^*}K^*),\mathcal{H}om^*(N^*,V^*))\\ &\xrightarrow{\phi^*}\mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*,V^*)\otimes_{K^*}\mathcal{H}om^*(M^{**},K^*),\mathcal{H}om^*(N^*,V^*))\\ &\xrightarrow{(id_{\mathcal{H}om^*(L^*,V^*)}\otimes_{K^*}\chi_{M^*,K^*})^*}\mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*,V^*)\otimes_{K^*}M^*,\mathcal{H}om^*(N^*,V^*))\end{aligned}$$

Proposition 4.2.10 *Under the assumption of (4.2.8), assume moreover that V^* is complete Hausdorff and the topology on V^* is coarser than the topology induced by K^*. The map $\Theta=\Theta_{M^*,N^*,L^*,V^*}$ is given by $(\Theta(f)(g\otimes x))([k],y)=g([k-n],f(x\otimes y))$ for $f\in\mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}N^*,L^*)$, $g\in\mathcal{H}om^{k-n}(L^*,V^*)$, $x\in M^n$ and $y\in N^m$.*

Proof. We put $h = \left(\eta^{*-1}_{L^*\otimes_{K^*}M^{**}}\right)^*(\zeta(\Lambda(f))) \in \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*\otimes_{K^*}M^{**},V^*),\mathcal{H}om^*(N^*,V^*))$. Then, we have $h\eta^*_{L^*\otimes_{K^*}M^{**}} = \zeta(\Lambda(f))$ and $h(\gamma\Sigma^k\eta_{L^*\otimes_{K^*}M^{**}}) = \gamma\Sigma^k\Lambda(f)$ for $\gamma \in \mathcal{H}om^k(L^*\widehat{\otimes}_{K^*}M^{**},V^*)$. On the other hand, since $\eta^*_{L^*\otimes_{K^*}M^{**}} : \mathcal{H}om^k(L^*\widehat{\otimes}_{K^*}M^{**},V^*) \to \mathcal{H}om^k(L^*\otimes_{K^*}M^{**},V^*)$ is an isomorphism, there is a unique $\gamma \in \mathcal{H}om^k(L^*\widehat{\otimes}_{K^*}M^{**},V^*)$ satisfying $\gamma\Sigma^k\eta_{L^*\otimes_{K^*}M^{**}} = \alpha_{V^*}T_{V^*,K^*}\phi(g\otimes_{K^*}\chi_{M^*,K^*}(x))$. Then,

$$\begin{aligned}\Theta(f)(g\otimes x) &= \big(h\alpha_{V^**}T_{V^*,K^**}\phi(id_{\mathcal{H}om^*(L^*,V^*)}\otimes_{K^*}\chi_{M^*,K^*})\big)(g\otimes x) = h(\alpha_{V^*}T_{V^*,K^*}\phi(g\otimes\chi_{M^*,K^*}(x)))\\ &= h\big(\gamma\Sigma^k\eta_{L^*\otimes_{K^*}M^{**}}\big) = \gamma\Sigma^k\Lambda(f)\end{aligned}$$

and it follows that $(\Theta(f)(g\otimes x))([k],y) = \gamma([k],\Lambda(f)(y))$. Let $(\alpha_i)_{i\in I}$ be a sequence in $L^*\otimes_{K^*}M^{**}$ indexed by a directed set I such that $(\eta_{L^*\otimes_{K^*}M^{**}}(\alpha_i))_{i\in I}$ converges to $\Lambda(f)(y)$. Suppose $\alpha_i = \sum_{j=1}^{\nu_i} z_{ij}\otimes g_{ij}$ for $z_{ij}\in L^{m-d_{ij}}$, $g_{ij}\in (M^{**})^{d_{ij}}$ and put $\beta_i(x) = \sum_{j=1}^{\nu_i}(-1)^{(m-d_{ij})(n+d_{ij})}g_{ij}([d_{ij}],x)z_{ij}$. Then, we have

$$\begin{aligned}\gamma([k],\eta_{L^*\otimes_{K^*}M^{**}}(\alpha_i)) &= \gamma\Sigma^k\eta_{L^*\otimes_{K^*}M^{**}}([k],\alpha_i) = \alpha_{V^*}T_{V^*,K^*}\phi(g\otimes\chi_{M^*,K^*}(x))([k],\alpha_i)\\ &= \sum_{j=1}^{\nu_i}\alpha_{V^*}T_{V^*,K^*}\phi(g\otimes\chi_{M^*,K^*}(x))([k],z_{ij}\otimes g_{ij})\\ &= \sum_{j=1}^{\nu_i}(-1)^{nm}\alpha_{V^*}T_{V^*,K^*}g([k-n],z_{ij})\otimes g_{ij}([d_{ij}],x)\\ &= \sum_{j=1}^{\nu_i}(-1)^{nm+(n+d_{ij})(k-n+m-d_{ij})}g_{ij}([d_{ij}],x)g([k-n],z_{ij})\\ &= \sum_{j=1}^{\nu_i}(-1)^{nm+(n+d_{ij})(m-d_{ij})}g([k-n],g_{ij}([d_{ij}],x)z_{ij}) = (-1)^{nm}g([k-n],\beta_i(x)).\end{aligned}$$

Since $(\beta_i(x))_{i\in I}$ converges to $(-1)^{mn}f(x\otimes y)$ in L^* by (4.2.8), we have $\gamma([k],\Lambda(f)(y)) = g([k-n],f(x\otimes y))$ and this shows the assertion. $\square$

Proposition 4.2.11 *Under the same condition as in (4.2.2), $\Theta = \Theta_{M^*,N^*,L^*,V^*}$ is the unique map that makes the following diagram commutative.*

$$\begin{array}{ccc}\mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}N^*,L^*) & \xrightarrow{\Theta} & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*,V^*)\otimes_{K^*}M^*,\mathcal{H}om^*(N^*,V^*))\\ \downarrow{\scriptstyle\Phi_{M^*,N^*,L^*}} & & \downarrow{\scriptstyle T^*_{M^*,\mathcal{H}om^*(L^*,V^*)}}\\ \mathrm{Hom}^c_{K^*}(M^*,\mathcal{H}om^*(N^*,L^*)) & & \mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}\mathcal{H}om^*(L^*,V^*),\mathcal{H}om^*(N^*,V^*))\\ & \searrow{\scriptstyle(\zeta^*)_*} & \downarrow{\scriptstyle\Phi_{M^*,\mathcal{H}om^*(L^*,V^*),\mathcal{H}om^*(N^*,V^*)}}\\ & & \mathrm{Hom}^c_{K^*}(M^*,\mathcal{H}om^*(\mathcal{H}om^*(L^*,V^*),\mathcal{H}om^*(N^*,V^*)))\end{array}$$

Proof. The commutativity of the diagram is nothing but the assertion of (4.2.4). The uniqueness of Θ follows from the injectivity of $T^*_{M^*,\mathcal{H}om^*(L^*,V^*)}$ and $\Phi_{M^*,\mathcal{H}om^*(L^*,V^*),\mathcal{H}om^*(N^*,V^*)}$. $\square$

Proposition 4.2.12 *Under the same condition as in (4.2.10), the following diagram commutes.*

$$\begin{array}{ccc}\mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}N^*,L^*) & \xrightarrow{\Theta} & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*,V^*)\otimes_{K^*}M^*,\mathcal{H}om^*(N^*,V^*))\\ \downarrow{\scriptstyle\zeta} & & \downarrow{\scriptstyle\Phi_{\mathcal{H}om^*(L^*,V^*),M^*,\mathcal{H}om^*(N^*,V^*)}}\\ \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*,V^*),\mathcal{H}om^*(M^*\otimes_{K^*}N^*,V^*)) & \xrightarrow{(\Phi^*_{M^*,N^*,L^*})_*} & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(L^*,V^*),\mathcal{H}om^*(M^*,\mathcal{H}om^*(N^*,V^*)))\end{array}$$

Proof. Since $(\zeta(f))(g) = g\Sigma^{k-n}f$, $(\Phi^*_{M^*,N^*,L^*})_*(\zeta(f)) : \mathcal{H}om^*(L^*,V^*) \to \mathcal{H}om^*(M^*,\mathcal{H}om^*(N^*,V^*))$ maps g to $\Phi_{\Sigma^{k-n}M^*,N^*,L^*}(g\Sigma^{k-n}f)$. Thus we have

$$\begin{aligned}\big(\big(((\Phi^*_{M^*,N^*,L^*})_*(\zeta(f)))(g)\big)([k-n],x)\big)([k],y) &= (\Phi_{\Sigma^{k-n}M^*,N^*,L^*}(g\Sigma^{k-n}f)([k-n],x))([k],y)\\ &= g([k-n],f(x\otimes y))\end{aligned}$$

On the other hand, $\Phi_{\mathcal{H}om^*(L^*,V^*),M^*,\mathcal{H}om^*(N^*,V^*)}\Theta(f)$ maps g to the map $\psi : \Sigma^{k-n}M^* \to \mathcal{H}om^*(N^*,V^*)$ given by $\psi([k-n],x)([k],y) = (\Theta(f))(g\otimes x)([k],y) = g([k-n], f(x\otimes y))$ by (4.2.10). □

The following facts are easily verified from routine diagram chasing.

Proposition 4.2.13 *The following diagrams are commutative.*

$$\begin{tikzcd}
\mathcal{H}om^*(L^*,U^*)\otimes_{K^*}\mathcal{H}om^*(M^*,V^*)\otimes_{K^*}\mathcal{H}om^*(N^*,W^*)\otimes_{K^*}\mathcal{H}om^*(P^*,Z^*) \arrow[r, "\phi\otimes\phi"] \arrow[d, "1\otimes T_{\mathcal{H}om^*(M^*,V^*),\mathcal{H}om^*(N^*,W^*)}\otimes 1"] & \mathcal{H}om^*(L^*\otimes_{K^*}M^*,U^*\otimes_{K^*}V^*)\otimes_{K^*}\mathcal{H}om^*(N^*\otimes_{K^*}P^*,W^*\otimes_{K^*}Z^*) \arrow[d, "\phi"] \\
\mathcal{H}om^*(L^*,U^*)\otimes_{K^*}\mathcal{H}om^*(N^*,W^*)\otimes_{K^*}\mathcal{H}om^*(M^*,V^*)\otimes_{K^*}\mathcal{H}om^*(P^*,Z^*) \arrow[d, "\phi\otimes\phi"] & \mathcal{H}om^*(L^*\otimes_{K^*}M^*\otimes_{K^*}N^*\otimes_{K^*}P^*,U^*\otimes_{K^*}V^*\otimes_{K^*}W^*\otimes_{K^*}Z^*) \arrow[d, "(1\otimes T_{N^*,M^*}\otimes 1)^*(1\otimes T_{V^*,W^*}\otimes 1)_*"] \\
\mathcal{H}om^*(L^*\otimes_{K^*}N^*,U^*\otimes_{K^*}W^*)\otimes_{K^*}\mathcal{H}om^*(M^*\otimes_{K^*}P^*,V^*\otimes_{K^*}Z^*) \arrow[r, "\phi"] & \mathcal{H}om^*(L^*\otimes_{K^*}N^*\otimes_{K^*}M^*\otimes_{K^*}P^*,U^*\otimes_{K^*}W^*\otimes_{K^*}V^*\otimes_{K^*}Z^*)
\end{tikzcd}$$

$$\begin{tikzcd}
M^*\otimes_{K^*}N^* \arrow[r, "\chi_{M^*,U^*}\otimes\chi_{N^*,V^*}"] \arrow[d, "\chi_{M^*\otimes_{K^*}N^*,U^*\otimes_{K^*}V^*}"] & \mathcal{H}om^*(\mathcal{H}om^*(M^*,U^*),U^*)\otimes_{K^*}\mathcal{H}om^*(\mathcal{H}om(N^*,V^*),V^*) \arrow[d, "\phi"] \\
\mathcal{H}om^*(\mathcal{H}om^*(M^*\otimes_{K^*}N^*,U^*\otimes_{K^*}N^*),U^*\otimes_{K^*}V^*) \arrow[r, "\phi^*"] & \mathcal{H}om^*(\mathcal{H}om^*(M^*,U^*)\otimes_{K^*}\mathcal{H}om^*(N^*,V^*),U^*\otimes_{K^*}V^*)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{H}om^*(L^*,K^*)\otimes_{K^*}\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}N^* \arrow[r, "\varphi^{L^*}_{\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}N^*}"] \arrow[d, "id_{\mathcal{H}om^*(L^*,K^*)}\otimes_{K^*}\varphi^{M^*}_{N^*}"] & \mathcal{H}om^*(L^*,\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}N^*) \arrow[d, "\left(\varphi^{M^*}_{N^*}\right)_*"] \\
\mathcal{H}om^*(L^*,K^*)\otimes_{K^*}\mathcal{H}om^*(M^*,N^*) \arrow[r, "\varphi^{L^*}_{\mathcal{H}om^*(M^*,N^*)}"] & \mathcal{H}om^*(L^*,\mathcal{H}om^*(M^*,N^*))
\end{tikzcd}$$

Proposition 4.2.14 *Suppose that the topologies on V^* and W^* are coaser than the topologies induced by K^*. The following diagrams commutes. $\tilde{\mu}_{K^*} : K^*\otimes_{K^*}K^* \to K^*$ denotes the map induced by the product of K^*.*

$$\begin{tikzcd}
\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}V^*\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)\otimes_{K^*}W^* \arrow[r, "\varphi^{M^*}_{V^*}\otimes\varphi^{N^*}_{W^*}"] \arrow[d, "id_{\mathcal{H}om^*(M^*,K^*)}\otimes_{K^*}T_{V^*,\mathcal{H}om^*(N^*,K^*)}\otimes_{K^*}id_{W^*}"] & \mathcal{H}om^*(M^*,V^*)\otimes_{K^*}\mathcal{H}om^*(N^*,W^*) \arrow[d, "\phi"] \\
\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)\otimes_{K^*}V^*\otimes_{K^*}W^* \arrow[d, "\phi\otimes_{K^*}id_{V^*}\otimes_{K^*}id_{W^*}"] & \mathcal{H}om^*(M^*\otimes_{K^*}N^*,V^*\otimes_{K^*}W^*) \\
\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*\otimes_{K^*}K^*)\otimes_{K^*}V^*\otimes_{K^*}W^* \arrow[r, "\tilde{\mu}_{K^*}{}_*\otimes_{K^*}id_{V^*}\otimes_{K^*}id_{V^*}"] & \mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*)\otimes_{K^*}V^*\otimes_{K^*}W^* \arrow[u, "\varphi^{M^*\otimes_{K^*}N^*}_{V^*\otimes_{K^*}W^*}"]
\end{tikzcd}$$

$$\begin{tikzcd}
M^{**}\otimes_{K^*}N^{**}\otimes_{K^*}V^* \arrow[r, "\phi\otimes_{K^*}id_{V^*}"] \arrow[d, "id_{M^{**}}\otimes_{K^*}\varphi^{N^*}_{V^*}"] & \mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*\otimes_{K^*}K^*)\otimes_{K^*}V^* \arrow[d, "\tilde{\mu}_{K^*}{}_*\otimes_{K^*}id_{V^*}"] \\
M^{**}\otimes_{K^*}\mathcal{H}om^*(N^*,V^*) \arrow[d, "\varphi^{M^*}_{\mathcal{H}om^*(N^*,V^*)}"] & \mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*)\otimes_{K^*}V^* \arrow[d, "\varphi^{M^*\otimes_{K^*}N^*}_{V^*}"] \\
\mathcal{H}om^*(M^*,\mathcal{H}om^*(N^*,V^*)) & \mathcal{H}om^*(M^*\otimes_{K^*}N^*,V^*) \arrow[l, "\Phi^*_{M^*,N^*,V^*}"']
\end{tikzcd}$$

$$\begin{tikzcd}
V^*\otimes_{K^*}\mathcal{H}om^*(M^*,K^*) \arrow[rr, "\chi_{V^*,K^*}\otimes 1"] \arrow[d, "T_{V^*,\mathcal{H}om^*(M^*,K^*)}"] & & \mathcal{H}om^*(\mathcal{H}om^*(V^*,K^*),K^*)\otimes_{K^*}\mathcal{H}om^*(M^*,K^*) \arrow[d, "\varphi^{\mathcal{H}om^*(V^*,K^*)}_{\mathcal{H}om^*(M^*,K^*)}"] \\
\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}V^* \arrow[r, "\varphi^{M^*}_{V^*}"] & \mathcal{H}om^*(M^*,V^*) \arrow[r, "\zeta^*"] & \mathcal{H}om^*(\mathcal{H}om^*(V^*,K^*),\mathcal{H}om^*(M^*,K^*))
\end{tikzcd}$$

5 Algebras and coalgebras

5.1 Algebras, coalgebras and duality

We denote by $\tilde{\mu}_{K^*} : K^* \otimes_{K^*} K^* \to K^*$ the isomorphism induced by the product $\mu_{K^*} : K^* \times K^* \to K^*$ of K^* and we often identify $K^* \otimes_{K^*} K^*$ with K^* by this map.

Proposition 5.1.1 *Let K^* be a linearly topologized graded ring which is complete Hausdorff. Suppose that morphisms $\delta : C^* \to C^* \widehat{\otimes}_{K^*} C^*$ and $\varepsilon : C^* \to K^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$ are given. We define $\tilde{\delta} : C^{**} \otimes_{K^*} C^{**} \to C^{**}$ and $\tilde{\varepsilon} : K^* \to C^{**}$ to be the following compositions.*

$$\begin{aligned}
C^{**} \otimes_{K^*} C^{**} = \mathcal{H}om^*(C^*, K^*) \otimes_{K^*} \mathcal{H}om^*(C^*, K^*) &\xrightarrow{\phi(C^*,C^*;K^*,K^*)} \mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^* \otimes_{K^*} K^*) \\
&\xrightarrow{\tilde{\mu}_{K^*{*}}} \mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^*) \xrightarrow{\left(\eta^*_{C^* \otimes_{K^*} C^*}\right)^{-1}} \mathcal{H}om^*(C^* \widehat{\otimes}_{K^*} C^*, K^*) \xrightarrow{\delta^*} \mathcal{H}om^*(C^*, K^*) = C^{**} \\
K^* \xrightarrow{\kappa_{K^*}} \mathcal{H}om^*(K^*, K^*) &\xrightarrow{\varepsilon^*} \mathcal{H}om^*(C^*, K^*) = C^{**}
\end{aligned}$$

(1) The right diagram below commutes if the left one does.

$$\begin{tikzcd}
C^* \arrow[r, "\delta"] \arrow[d, "\delta"] & C^* \widehat{\otimes}_{K^*} C^* \arrow[d, "\delta \widehat{\otimes}_{K^*} 1"] \\
C^* \widehat{\otimes}_{K^*} C^* \arrow[r, "1 \widehat{\otimes}_{K^*} \delta"] & C^* \widehat{\otimes}_{K^*} C^* \widehat{\otimes}_{K^*} C^*
\end{tikzcd}
\qquad
\begin{tikzcd}
C^{**} \otimes_{K^*} C^{**} \otimes_{K^*} C^{**} \arrow[r, "1 \otimes_{K^*} \tilde{\delta}"] \arrow[d, "\tilde{\delta}"] & C^{**} \otimes_{K^*} C^{**} \arrow[d, "\tilde{\delta} \otimes_{K^*} 1"] \\
C^{**} \otimes_{K^*} C^{**} \arrow[r, "\tilde{\delta}"] & C^{**}
\end{tikzcd}$$

(2) The lower diagram below commutes if the upper one does.

$$\begin{tikzcd}
& C^* \arrow[dl, "\eta_{C^* \otimes_{K^*} K^*} i_1"'] \arrow[d, "\delta"] \arrow[dr, "\eta_{K^* \otimes_{K^*} C^*} i_2"] & \\
C^* \widehat{\otimes}_{K^*} K^* & C^* \widehat{\otimes}_{K^*} C^* \arrow[l, "1 \widehat{\otimes}_{K^*} \varepsilon"'] \arrow[r, "\varepsilon \widehat{\otimes}_{K^*} 1"] & K^* \widehat{\otimes}_{K^*} C^*
\end{tikzcd}$$

$$\begin{tikzcd}
C^{**} \otimes_{K^*} K^* \arrow[r, "1 \otimes_{K^*} \tilde{\varepsilon}"] \arrow[dr, "i_1^{-1}"'] & C^{**} \otimes_{K^*} C^{**} \arrow[d, "\tilde{\delta}"] & K^* \otimes_{K^*} C^{**} \arrow[l, "\tilde{\varepsilon} \otimes_{K^*} 1"'] \arrow[dl, "i_2^{-1}"] \\
& C^{**} &
\end{tikzcd}$$

(3) The right diagram below commutes if the left one does.

$$\begin{tikzcd}
C^* \arrow[r, "\delta"] \arrow[dr, "\delta"'] & C^* \widehat{\otimes}_{K^*} C^* \arrow[d, "\widehat{T}"] \\
& C^* \widehat{\otimes}_{K^*} C^*
\end{tikzcd}
\qquad
\begin{tikzcd}
C^{**} \otimes_{K^*} C^{**} \arrow[r, "\tilde{\delta}"] \arrow[d, "T"] & C^{**} \\
C^{**} \otimes_{K^*} C^{**} \arrow[ur, "\tilde{\delta}"'] &
\end{tikzcd}$$

Proof. (1) The commutativity of the right diagram follows from the commutativity of the following diagram. Here, $\otimes$ means $\otimes_{K^*}$ and $\widehat{\otimes}$ means $\widehat{\otimes}_{K^*}$.

$$\begin{tikzcd}
C^{**} \otimes C^{**} \otimes C^{**} \arrow[r, "1 \otimes \phi"] \arrow[d, "\phi \otimes 1"] & C^{**} \otimes \mathcal{H}om^*(C^* \otimes C^*, K^*) \arrow[d, "\phi"] & C^{**} \otimes \mathcal{H}om^*(C^* \widehat{\otimes} C^*, K^*) \arrow[l, "1 \otimes \eta^*_{C^* \otimes C^*}"'] \arrow[r, "1 \otimes \delta^*"] \arrow[d, "\phi"] & C^{**} \otimes C^{**} \arrow[d, "\phi"] \\
\mathcal{H}om^*(C^* \otimes C^*, K^*) \otimes C^{**} \arrow[r, "\phi"] & \mathcal{H}om^*(C^* \otimes C^* \otimes C^*, K^*) & \mathcal{H}om^*(C^* \otimes (C^* \widehat{\otimes} C^*), K^*) \arrow[l, "(1 \otimes \eta_{C^* \otimes C^*})^*"'] \arrow[r, "(1 \otimes \delta)^*"] & \mathcal{H}om^*(C^* \otimes C^*, K^*) \\
\mathcal{H}om^*(C^* \widehat{\otimes} C^*, K^*) \otimes C^{**} \arrow[r, "\phi"] \arrow[u, "\eta^*_{C^* \otimes C^*} \otimes 1"] \arrow[d, "\delta^* \otimes 1"] & \mathcal{H}om^*((C^* \widehat{\otimes} C^*) \otimes C^*, K^*) \arrow[u, "(\eta_{C^* \otimes C^*} \otimes 1)^*"] \arrow[d, "(\delta \otimes 1)^*"] & \mathcal{H}om^*(C^* \widehat{\otimes} C^* \widehat{\otimes} C^*, K^*) \arrow[l, "\eta^*_{(C^* \widehat{\otimes} C^*) \otimes C^*}"'] \arrow[r, "(1 \widehat{\otimes} \delta)^*"] \arrow[u, "\eta^*_{C^* \otimes (C^* \widehat{\otimes} C^*)}"] \arrow[d, "(\delta^* \widehat{\otimes} 1)^*"] & \mathcal{H}om^*(C^* \widehat{\otimes} C^*, K^*) \arrow[u, "\eta^*_{C^* \otimes C^*}"] \arrow[d, "\delta^*"] \\
C^{**} \otimes C^{**} \arrow[r, "\phi"] & \mathcal{H}om^*(C^* \otimes C^*, K^*) & \mathcal{H}om^*(C^* \widehat{\otimes} C^*, K^*) \arrow[l, "\eta^*_{C^* \otimes C^*}"'] \arrow[r, "\delta^*"] & C^{**}
\end{tikzcd}$$

(2) The commutativity of the lower diagram follows from the commutativity of the following diagram.

$$
\begin{tikzcd}
C^{**} \otimes_{K^*} K^* \arrow[r, "1 \otimes_{K^*} \tilde{\varepsilon}"] \arrow[d, "\phi"] & C^{**} \otimes_{K^*} C^{**} \arrow[d, "\phi"] & K^* \otimes_{K^*} C^{**} \arrow[l, "\tilde{\varepsilon} \otimes_{K^*} 1"'] \arrow[d, "\phi"] \\
\mathcal{H}om^*(C^* \otimes_{K^*} K^*, K^*) \arrow[r, "(1 \otimes_{K^*} \varepsilon)^*"] & \mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^*) & \mathcal{H}om^*(K^* \otimes_{K^*} C^*, K^*) \arrow[l, "(\varepsilon \otimes_{K^*} 1)^*"'] \\
\mathcal{H}om^*(C^* \widehat{\otimes}_{K^*} K^*, K^*) \arrow[r, "(1 \widehat{\otimes}_{K^*} \varepsilon)^*"] \arrow[u, "\eta^*_{C^* \otimes_{K^*} K^*}"] \arrow[dr, "(\eta_{C^* \otimes_{K^*} K^*} i_1)^*"'] & \mathcal{H}om^*(C^* \widehat{\otimes}_{K^*} C^*, K^*) \arrow[u, "\eta^*_{C^* \otimes_{K^*} C^*}"] \arrow[d, "\delta^*"] & \mathcal{H}om^*(K^* \widehat{\otimes}_{K^*} C^*, K^*) \arrow[l, "(\varepsilon \widehat{\otimes}_{K^*} 1)^*"'] \arrow[u, "\eta^*_{K^* \otimes_{K^*} C^*}"] \arrow[dl, "(\eta_{K^* \otimes_{K^*} C^*} i_2)^*"] \\
 & C^{**} &
\end{tikzcd}
$$

(3) The commutativity of the right diagram follows from the commutativity of the following diagram.

$$
\begin{tikzcd}
C^{**} \otimes_{K^*} C^{**} \arrow[r, "\phi"] \arrow[d, "T"] & \mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^*) \arrow[d, "T^*"] & \mathcal{H}om^*(C^* \widehat{\otimes}_{K^*} C^*, K^*) \arrow[l, "\eta_{C^* \otimes_{K^*} C^*}"'] \arrow[r, "\delta^*"] \arrow[d, "\widehat{T}^*"] & C^{**} \\
C^{**} \otimes_{K^*} C^{**} \arrow[r, "\phi"] & \mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^*) & \mathcal{H}om^*(C^* \widehat{\otimes}_{K^*} C^*, K^*) \arrow[l, "\eta_{C^* \otimes_{K^*} C^*}"'] \arrow[ur, "\delta^*"'] &
\end{tikzcd}
$$

□

Definition 5.1.2 *Let $\delta : C^* \to C^* \widehat{\otimes}_{K^*} C^*$ and $\varepsilon : C^* \to K^*$ be morphisms in $\mathcal{T}op\mathcal{M}od_{K^*}$. If the left diagrams of (1) and (2) of (5.1.1) commute, we call a triple $(C^*, \delta, \varepsilon)$ a coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$. Moreover, if the left diagram of (3) of (5.1.1) commutes, we say that C^* is cocomutative. We call C^{**} the dual algebra with product $\tilde{\delta} : C^{**} \otimes_{K^*} C^{**} \to C^{**}$ and unit $\tilde{\varepsilon} : K^* \to C^{**}$.*

Remark 5.1.3 *The following diagram is commutative by (3.4.17).*

$$
\begin{tikzcd}
\mathcal{H}om^*(C^* \widehat{\otimes}_{K^*} C^*, K^*) \arrow[r, "\eta^*_{C^* \otimes_{K^*} C^*}"] \arrow[dr, "\eta_{K^* *}", "\cong"'] & \mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^*) \arrow[d, "c_{C^* \otimes_{K^*} C^*, K^*}"] \\
 & \mathcal{H}om^*(C^* \widehat{\otimes}_{K^*} C^*, \widehat{K}^*)
\end{tikzcd}
$$

Hence $\tilde{\delta}$ coincides with a composition

$$
C^{**} \otimes_{K^*} C^{**} \xrightarrow{\phi(C^*, C^*; K^*, K^*)} \mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^* \otimes_{K^*} K^*) \xrightarrow{\tilde{\mu}_{K^* *}} \mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^*) \xrightarrow{c_{C^* \otimes_{K^*} C^*, K^*}}
$$
$$
\mathcal{H}om^*(C^* \widehat{\otimes}_{K^*} C^*, \widehat{K}^*) \xrightarrow{(\eta_{K^* *})^{-1}} \mathcal{H}om^*(C^* \widehat{\otimes}_{K^*} C^*, K^*) \xrightarrow{\delta^*} \mathcal{H}om^*(C^*, K^*).
$$

Definition 5.1.4 *We say that an object M^* is proper if $\hat{\phi} : M^{**} \widehat{\otimes}_{K^*} M^{**} \to \mathcal{H}om^*(M^* \otimes_{K^*} M^*, K^*)^{\wedge}$ is an isomorphism.*

Assume that A^* is proper (*e.g.* (A^*, K^*) is a very nice pair (4.1.8)). For morphisms $\mu : A^* \otimes_{K^*} A^* \to A^*$ and $\eta : K^* \to A^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$, we define morphisms $\hat{\mu} : A^{**} \to A^{**} \widehat{\otimes}_{K^*} A^{**}$ and $\hat{\eta} : A^{**} \to K^*$ to be the following compositions, respectively.

$$
A^{**} = \mathcal{H}om^*(A^*, K^*) \xrightarrow{\mu^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \xrightarrow{\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)}} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^{\wedge} \xrightarrow{\hat{\phi}^{-1}}
$$
$$
\mathcal{H}om^*(A^*, K^*) \widehat{\otimes}_{K^*} \mathcal{H}om^*(A^*, K^*) = A^{**} \widehat{\otimes}_{K^*} A^{**}
$$
$$
A^{**} = \mathcal{H}om^*(A^*, K^*) \xrightarrow{\eta^*} \mathcal{H}om^*(K^*, K^*) \xrightarrow{E_1} K^*
$$

The following result shows that a K^*-algebra A^* defines a coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ if A^* is proper.

Proposition 5.1.5 *(1) The right diagram below commutes if the left one does.*

$$
\begin{tikzcd}
A^* \otimes_{K^*} A^* \otimes_{K^*} A^* \arrow[r, "\mu \otimes_{K^*} 1"] \arrow[d, "1 \otimes_{K^*} \mu"] & A^* \otimes_{K^*} A^* \arrow[d, "\mu"] \\
A^* \otimes_{K^*} A^* \arrow[r, "\mu"] & A^*
\end{tikzcd}
\qquad
\begin{tikzcd}
A^{**} \arrow[r, "\hat{\mu}"] \arrow[d, "\hat{\mu}"] & A^{**} \widehat{\otimes}_{K^*} A^{**} \arrow[d, "\hat{\mu} \widehat{\otimes}_{K^*} 1"] \\
A^{**} \widehat{\otimes}_{K^*} A^{**} \arrow[r, "1 \widehat{\otimes}_{K^*} \hat{\mu}"] & A^{**} \widehat{\otimes}_{K^*} A^{**} \widehat{\otimes}_{K^*} A^{**}
\end{tikzcd}
$$

(2) Assume that the topology of A^* is coarser than the topology induced by K^* and we denote by $\tilde{\alpha}_{A^*} : K^*\otimes_{K^*} A^* \to A^*$ the isomorphism induced by the K^*-module structure of A^*. The right diagram below commutes if the left one does.

$$\begin{tikzcd}
A^*\otimes_{K^*} K^* \arrow[r, "1\otimes\eta"] \arrow[dr, "\cong", "\tilde{\alpha}_{A^*}T_{A^*,K^*}"'] & A^*\otimes_{K^*} A^* \arrow[d, "\mu"] & K^*\otimes_{K^*} A^* \arrow[l, "\eta\otimes 1"'] \arrow[dl, "\cong"', "\tilde{\alpha}_{A^*}"] \\
& A^* &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A^* \arrow[dl, "\eta_{A^{**}\otimes i_1}"'] \arrow[d, "\hat{\mu}"] \arrow[dr, "\eta_{K^*\otimes A^{**}} i_2"] & \\
A^{**}\widehat{\otimes}_{K^*} K^* & A^{**}\widehat{\otimes}_{K^*} A^{**} \arrow[l, "1\,\widehat{\otimes}\,\hat{\eta}"'] \arrow[r, "\hat{\eta}\,\widehat{\otimes}\,1"] & K^*\widehat{\otimes}_{K^*} A^{**}
\end{tikzcd}$$

Proof. (1) Let $\hat{\hat{\phi}} : \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^{\wedge}\, \widehat{\otimes}_{K^*} A^{**} \to \mathcal{H}om^*(A^* \otimes_{K^*} A^* \otimes_{K^*} A^*, K^*)^{\wedge}$ be the map satisfying $\hat{\phi} = \hat{\hat{\phi}}(\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}\,\widehat{\otimes}\,1) : \mathcal{H}om^*(A^*\otimes_{K^*}A^*, K^*)\, \widehat{\otimes}_{K^*} A^{**} \to \mathcal{H}om^*(A^*\otimes_{K^*}A^*\otimes_{K^*}A^*, K^*)^{\wedge}$ (see (2.3.5)). The assertion follows from the commutativity of the following diagram.

$$\begin{tikzcd}
A^{**} \arrow[r, "\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}\mu^*"] \arrow[d, "\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}\mu^*"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^{\wedge} \arrow[d, "(\mu\otimes 1)^{*\wedge}"] & A^{**}\, \widehat{\otimes}_{K^*} A^{**} \arrow[l, "\hat{\phi}"'] \arrow[d, "\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}\mu^*\,\widehat{\otimes}\,1"'] \\
\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^{\wedge} \arrow[r, "(1\otimes\mu)^{*\wedge}"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^* \otimes_{K^*} A^*, K^*)^{\wedge} & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^{\wedge}\, \widehat{\otimes}_{K^*} A^{**} \arrow[l, "\hat{\hat{\phi}}"'] \\
A^{**}\, \widehat{\otimes}_{K^*} A^{**} \arrow[u, "\hat{\phi}"] \arrow[r, "1\,\widehat{\otimes}\,\mu^*"] & A^{**}\, \widehat{\otimes}_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^{\wedge} \arrow[u, "\hat{\phi}"] & A^{**}\, \widehat{\otimes}_{K^*} A^{**}\, \widehat{\otimes}_{K^*} A^{**} \arrow[l, "1\,\widehat{\otimes}\,\hat{\phi}"'] \arrow[u, "\hat{\phi}\,\widehat{\otimes}\,1"']
\end{tikzcd}$$

(2) We first note that the following diagrams commute.

$$\begin{tikzcd}
A^{**} \otimes_{K^*} \mathcal{H}om^*(K^*, K^*) \arrow[r, "1\otimes E_1"] \arrow[d, "\phi"] & A^{**} \otimes_{K^*} K^* \\
\mathcal{H}om^*(A^* \otimes_{K^*} K^*, K^*) \arrow[r, "i_1^*"] & A^{**} \arrow[u, "i_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H}om^*(K^*, K^*) \otimes_{K^*} A^{**} \arrow[r, "E_1\otimes 1"] \arrow[d, "\phi"] & K^* \otimes_{K^*} A^{**} \\
\mathcal{H}om^*(K^* \otimes_{K^*} A^*, K^*) \arrow[r, "i_2^*"] & A^{**} \arrow[u, "i_2"']
\end{tikzcd}$$

$$\begin{tikzcd}
A^{**} \arrow[r, "i_1"] \arrow[d, "\eta_{A^{**}}"] & A^{**} \otimes_{K^*} K^* \arrow[d, "\eta_{A^{**}\otimes_{K^*}K^*}"] \\
\widehat{A^{**}} \arrow[r, "\hat{i}_1"] & A^{**}\, \widehat{\otimes}_{K^*} K^*
\end{tikzcd}
\qquad
\begin{tikzcd}
A^{**} \arrow[r, "i_2"] \arrow[d, "\eta_{A^{**}}"] & K^* \otimes_{K^*} A^{**} \arrow[d, "\eta_{K^*\otimes_{K^*}A^{**}}"] \\
\widehat{A^{**}} \arrow[r, "\hat{i}_2"] & K^*\, \widehat{\otimes}_{K^*} A^{**}
\end{tikzcd}$$

Since $i_1 : A^* \to A^* \otimes_{K^*} K^*$ (resp. $i_2 : A^* \to K^* \otimes_{K^*} A^*$) is the inverse of $\tilde{\alpha}_{A^*}T_{A^*,K^*}$ (resp. $\tilde{\alpha}_{A^*}$), $\mu(1 \otimes \eta) = \tilde{\alpha}_{A^*}T_{A^*,K^*}$ (resp. $\mu(\eta \otimes 1) = \tilde{\alpha}_{A^*}$) implies that $\mu(1 \otimes \eta)i_1 = id_{A^*}$ (resp. $\mu(\eta \otimes 1)i_2 = id_{A^*}$). Thus we have $i_1^*(1 \otimes \eta)^*\mu^* = id_{A^{**}}$ (resp. $i_2^*(\eta \otimes 1)^*\mu^* = id_{A^{**}}$). Then, the assertions follows from the following commutative diagrams.

$$\begin{tikzcd}
A^{**} \arrow[r, "\mu^*"] \arrow[d, "\hat{\mu}"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \arrow[d, "\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}"] & A^{**} \arrow[d, "\mu^*"] \\
A^{**}\, \widehat{\otimes}_{K^*} A^{**} \arrow[r, "\hat{\phi}"] \arrow[d, "1\,\widehat{\otimes}\,\eta^*"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^{\wedge} \arrow[d, "(1\otimes\eta)^{*\wedge}"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \arrow[l, "\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}"'] \arrow[d, "(1\otimes\eta)^*"] \\
A^{**}\, \widehat{\otimes}_{K^*} \mathcal{H}om^*(K^*, K^*) \arrow[r, "\hat{\phi}"] \arrow[d, "1\,\widehat{\otimes}\,E_1"] & \mathcal{H}om^*(A^* \otimes_{K^*} K^*, K^*)^{\wedge} \arrow[d, "\widehat{i_1^*}"] & \mathcal{H}om^*(A^* \otimes_{K^*} K^*, K^*) \arrow[l, "\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}K^*,K^*)}"'] \arrow[d, "i_1^*"] \\
A^{**}\, \widehat{\otimes}_{K^*} K^* & \widehat{A^{**}} \arrow[l, "\hat{i}_1"'] & A^{**} \arrow[l, "\eta_{A^{**}}"']
\end{tikzcd}$$

$$\begin{tikzcd}
A^{**} \arrow[r, "\mu^*"] \arrow[d, "\hat{\mu}"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \arrow[d, "\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}"] & A^{**} \arrow[d, "\mu^*"] \\
A^{**}\, \widehat{\otimes}_{K^*} A^{**} \arrow[r, "\hat{\phi}"] \arrow[d, "\eta^*\,\widehat{\otimes}\,1"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^{\wedge} \arrow[d, "(\eta\otimes 1)^{*\wedge}"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \arrow[l, "\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}"'] \arrow[d, "(\eta\otimes 1)^*"] \\
\mathcal{H}om^*(K^*, K^*)\, \widehat{\otimes}_{K^*} A^{**} \arrow[r, "\hat{\phi}"] \arrow[d, "E_1\,\widehat{\otimes}\,1"] & \mathcal{H}om^*(K^* \otimes_{K^*} A^*, K^*)^{\wedge} \arrow[d, "\widehat{i_2^*}"] & \mathcal{H}om^*(K^* \otimes_{K^*} A^*, K^*) \arrow[l, "\eta_{\mathcal{H}om^*(K^*\otimes_{K^*}A^*,K^*)}"'] \arrow[d, "i_2^*"] \\
K^*\, \widehat{\otimes}_{K^*} A^{**} & \widehat{A^{**}} \arrow[l, "\hat{i}_2"'] & A^{**} \arrow[l, "\eta_{A^{**}}"']
\end{tikzcd}$$

$\square$

Proposition 5.1.6 *Suppose that K^* is complete Hausdorff and that A^* is proper. If a diagram*

$$\begin{tikzcd}
A^* \otimes_{K^*} A^* \arrow[rr, "\mu"] \arrow[d, "\delta\otimes_{K^*}\delta"] & & A^* \arrow[d, "\delta"] \\
(A^* \widehat{\otimes}_{K^*} A^*) \otimes_{K^*} (A^* \widehat{\otimes}_{K^*} A^*) \arrow[r, "sh"] & (A^* \otimes_{K^*} A^*) \widehat{\otimes}_{K^*} (A^* \otimes_{K^*} A^*) \arrow[r, "\mu \widehat{\otimes}_{K^*} \mu"] & A^* \widehat{\otimes}_{K^*} A^*
\end{tikzcd}$$

is commutative, the following diagram is commutative.

$$\begin{tikzcd}
A^{**} \otimes_{K^*} A^{**} \arrow[rr, "\tilde{\delta}"] \arrow[d, "\hat{\mu}\otimes_{K^*}\hat{\mu}"] & & A^{**} \arrow[d, "\hat{\mu}"] \\
(A^{**} \widehat{\otimes}_{K^*} A^{**}) \otimes_{K^*} (A^{**} \widehat{\otimes}_{K^*} A^{**}) \arrow[r, "sh"] & (A^{**} \otimes_{K^*} A^{**}) \widehat{\otimes}_{K^*} (A^{**} \otimes_{K^*} A^{**}) \arrow[r, "\tilde{\delta} \widehat{\otimes}_{K^*} \tilde{\delta}"] & A^{**} \widehat{\otimes}_{K^*} A^{**}
\end{tikzcd}$$

Proof. Since $\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)} \widehat{\otimes}\, \eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)} : \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \widehat{\otimes}_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \to \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\wedge \widehat{\otimes}_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\wedge$ is an isomorphism by (2.3.5), we define a map

$$\check{\phi} : \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\wedge \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\wedge \to \mathcal{H}om^*((A^* \otimes_{K^*} A^*) \otimes_{K^*} (A^* \otimes_{K^*} A^*), K^*)^\wedge$$

to be the following composition.

$$\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\wedge \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\wedge \xrightarrow{\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)^\wedge \otimes_{K^*} \mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)^\wedge}}$$
$$\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\wedge \widehat{\otimes}_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\wedge \xrightarrow{\left(\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)} \widehat{\otimes}\, \eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}\right)^{-1}}$$
$$\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \widehat{\otimes}_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \xrightarrow{\hat{\phi}} \mathcal{H}om^*((A^* \otimes_{K^*} A^*) \otimes_{K^*} (A^* \otimes_{K^*} A^*), K^*)^\wedge$$

Then, by the commutativity of diagram 3 and diagram 4 of the next page, the upper middle rectangle of diagram 5 commutes. By the assumption, the lower left rectangle of the diagram 5 commutes. By the definition of the map $sh : (A^* \widehat{\otimes}_{K^*} A^*) \otimes_{K^*} (A^* \widehat{\otimes}_{K^*} A^*) \to (A^* \otimes_{K^*} A^*) \widehat{\otimes}_{K^*} (A^* \otimes_{K^*} A^*)$ (2.3.7), the following diagram 1 commutes.

$$\begin{tikzcd}
\mathcal{H}om^*((A^* \otimes_{K^*} A^*) \widehat{\otimes}_{K^*} (A^* \otimes_{K^*} A^*), K^*) \arrow[r, "\eta^*_{(A^*\otimes_{K^*}A^*)\otimes_{K^*}(A^*\otimes_{K^*}A^*)}", "\cong"'] \arrow[d, "sh^*"] & \mathcal{H}om^*((A^* \otimes_{K^*} A^*) \otimes_{K^*} (A^* \otimes_{K^*} A^*), K^*) \arrow[d, "(1\otimes T_{A^*,A^*}\otimes 1)^*"] \\
\mathcal{H}om^*((A^* \widehat{\otimes}_{K^*} A^*) \otimes_{K^*} (A^* \widehat{\otimes}_{K^*} A^*), K^*) \arrow[r, "(\eta_{A^*\otimes_{K^*}A^*} \otimes \eta_{A^*\otimes_{K^*}A^*})^*"] & \mathcal{H}om^*((A^* \otimes_{K^*} A^*) \otimes_{K^*} (A^* \otimes_{K^*} A^*), K^*) \\
\mathcal{H}om^*((A^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{K^*} (A^* \widehat{\otimes}_{K^*} A^*), K^*) \arrow[r, "(\eta_{A^*\otimes_{K^*}A^*} \widehat{\otimes}\, \eta_{A^*\otimes_{K^*}A^*})^*", "\cong"'] \arrow[u, "\eta^*_{(A^*\widehat{\otimes}_{K^*}A^*)\otimes_{K^*}(A^*\widehat{\otimes}_{K^*}A^*)}", "\cong"'] & \mathcal{H}om^*((A^* \otimes_{K^*} A^*) \widehat{\otimes}_{K^*} (A^* \otimes_{K^*} A^*), K^*) \arrow[u, "\eta^*_{(A^*\otimes_{K^*}A^*)\otimes_{K^*}(A^*\otimes_{K^*}A^*)}", "\cong"']
\end{tikzcd}$$

diagram 1

It is easy to verify that the following diagram 2 also commutes.

$$\begin{tikzcd}
(A^{**} \otimes_{K^*} A^{**}) \otimes_{K^*} (A^{**} \otimes_{K^*} A^{**}) \arrow[r, "1\otimes T_{A^{**},A^{**}}\otimes 1"] \arrow[d, "\phi\otimes\phi"] & (A^{**} \otimes_{K^*} A^{**}) \otimes_{K^*} (A^{**} \otimes_{K^*} A^{**}) \arrow[d, "\phi\otimes\phi"] \\
\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \arrow[d, "\phi"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \arrow[d, "\phi"] \\
\mathcal{H}om^*((A^* \otimes_{K^*} A^*) \otimes_{K^*} (A^* \otimes_{K^*} A^*), K^*) \arrow[r, "(1\otimes T_{A^*,A^*}\otimes 1)^*"] & \mathcal{H}om^*((A^* \otimes_{K^*} A^*) \otimes_{K^*} (A^* \otimes_{K^*} A^*), K^*)
\end{tikzcd}$$

diagram 2

$$
\begin{tikzcd}
\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \arrow[r, "\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)}"] \arrow[d, "\phi"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \widehat{\otimes}_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \arrow[d, "\hat{\phi}"] \\
\mathcal{H}om^*((A^* \otimes_{K^*} A^*) \otimes_{K^*} (A^* \otimes_{K^*} A^*), K^*) \arrow[r, "\eta_{\mathcal{H}om^*((A^* \otimes_{K^*} A^*) \otimes_{K^*} (A^* \otimes_{K^*} A^*), K^*)}"] & \mathcal{H}om^*((A^* \otimes_{K^*} A^*) \otimes_{K^*} (A^* \otimes_{K^*} A^*), K^*)^\frown
\end{tikzcd}
$$

diagram 3

$$
\begin{tikzcd}
\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \arrow[r, "\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)} \otimes \eta_{\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)}"] \arrow[d, "\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)}"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\frown \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\frown \arrow[d, "\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\frown \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\frown}"'] \\
\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \widehat{\otimes}_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \arrow[d, "\hat{\phi}"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\frown \widehat{\otimes}_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\frown \arrow[l, "(\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)} \widehat{\otimes} \eta_{\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)})^{-1}"', "\cong"] \\
\mathcal{H}om^*((A^* \otimes_{K^*} A^*) \otimes_{K^*} (A^* \otimes_{K^*} A^*), K^*)^\frown & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\frown \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\frown \arrow[l, "\check{\phi}"'] \arrow[u, "\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\frown \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^\frown}"']
\end{tikzcd}
$$

diagram 4

$$
\begin{tikzcd}
A^{**} \otimes A^{**} \arrow[r, "\mu^* \otimes \mu^*"] \arrow[d, "\phi"] & \mathcal{H}om^*(A^* \otimes A^*, K^*) \otimes \mathcal{H}om^*(A^* \otimes A^*, K^*) \arrow[r, "\eta_{\mathcal{H}om^*(A^* \otimes A^*, K^*)} \otimes \eta_{\mathcal{H}om^*(A^* \otimes A^*, K^*)}"] \arrow[d, "\phi"] & \mathcal{H}om^*(A^* \otimes A^*, K^*)^\frown \otimes \mathcal{H}om^*(A^* \otimes A^*, K^*)^\frown \arrow[d, "\check{\phi}"] & (A^{**} \widehat{\otimes} A^{**}) \otimes (A^{**} \widehat{\otimes} A^{**}) \arrow[l, "\hat{\phi} \otimes \hat{\phi}"', "\cong"] \arrow[d, "sh"] \\
\mathcal{H}om^*(A^* \otimes A^*, K^*) \arrow[r, "(\mu \otimes \mu)^*"] & \mathcal{H}om^*((A^* \otimes A^*) \otimes (A^* \otimes A^*), K^*) \arrow[r, "\eta_{\mathcal{H}om^*((A^* \otimes A^*) \otimes (A^* \otimes A^*), K^*)}"] & \mathcal{H}om^*((A^* \otimes A^*) \otimes (A^* \otimes A^*), K^*)^\frown & (A^{**} \otimes A^{**}) \widehat{\otimes} (A^{**} \otimes A^{**}) \arrow[d, "\phi \widehat{\otimes} \phi"] \\
\mathcal{H}om^*(A^* \widehat{\otimes} A^*, K^*) \arrow[r, "(\mu \widehat{\otimes} \mu)^*"] \arrow[u, "\eta^*_{A^* \otimes A^*}", "\cong"'] \arrow[dd, "\delta^*"] & \mathcal{H}om^*((A^* \otimes A^*) \widehat{\otimes} (A^* \otimes A^*), K^*) \arrow[r, "\eta_{\mathcal{H}om^*((A^* \otimes A^*) \widehat{\otimes} (A^* \otimes A^*), K^*)}"] \arrow[u, "\eta^*_{(A^* \otimes A^*) \otimes (A^* \otimes A^*)}", "\cong"'] \arrow[d, "sh^*"] & \mathcal{H}om^*((A^* \otimes A^*) \widehat{\otimes} (A^* \otimes A^*), K^*)^\frown \arrow[u, "\hat{\eta}^*_{(A^* \otimes A^*) \otimes (A^* \otimes A^*)}", "\cong"'] \arrow[d, "\widehat{sh^*}"] & \mathcal{H}om^*(A^* \otimes A^*, K^*) \widehat{\otimes} \mathcal{H}om^*(A^* \otimes A^*, K^*) \\
 & \mathcal{H}om^*((A^* \widehat{\otimes} A^*) \otimes (A^* \widehat{\otimes} A^*), K^*) \arrow[r, "\eta_{\mathcal{H}om^*((A^* \widehat{\otimes} A^*) \otimes (A^* \widehat{\otimes} A^*), K^*)}"] \arrow[d, "(\delta \otimes \delta)^*"] & \mathcal{H}om^*((A^* \widehat{\otimes} A^*) \otimes (A^* \widehat{\otimes} A^*), K^*)^\frown \arrow[d, "(\delta \otimes \delta)^{*\frown}"] & \mathcal{H}om^*(A^* \widehat{\otimes} A^*, K^*) \widehat{\otimes} \mathcal{H}om^*(A^* \widehat{\otimes} A^*, K^*) \arrow[l, "\hat{\phi}"', "\cong"] \arrow[u, "\eta^*_{A^* \otimes A^*} \widehat{\otimes} \eta^*_{A^* \otimes A^*}"', "\cong"] \arrow[d, "\delta^* \widehat{\otimes} \delta^*"] \\
A^{**} \arrow[r, "\mu^*"] & \mathcal{H}om^*(A^* \otimes A^*, K^*) \arrow[r, "\eta_{\mathcal{H}om^*(A^* \otimes A^*, K^*)}"] & \mathcal{H}om^*(A^* \otimes A^*, K^*)^\frown & A^{**} \widehat{\otimes} A^{**} \arrow[l, "\hat{\phi}"', "\cong"]
\end{tikzcd}
$$

diagram 5

It follows that

$$
\begin{tikzcd}
\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \arrow[d, "\phi"] & (A^{**} \otimes_{K^*} A^{**}) \otimes_{K^*} (A^{**} \otimes_{K^*} A^{**}) \arrow[l, "\phi\otimes\phi"'] \arrow[d, "1\otimes T_{A^{**},A^{**}}\otimes 1"] \\
\mathcal{H}om^*((A^* \otimes_{K^*} A^*) \otimes_{K^*} (A^* \otimes_{K^*} A^*), K^*) & (A^{**} \otimes_{K^*} A^{**}) \otimes_{K^*} (A^{**} \otimes_{K^*} A^{**}) \arrow[d, "\phi\otimes\phi"] \\
\mathcal{H}om^*((A^* \otimes_{K^*} A^*) \widehat{\otimes}_{K^*} (A^* \otimes_{K^*} A^*), K^*) \arrow[u, "\eta^*_{(A^*\otimes_{K^*}A^*)\otimes(A^*\otimes_{K^*}A^*)}", "\cong"'] \arrow[d, "sh^*"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \otimes_{K^*} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \\
\mathcal{H}om^*((A^* \widehat{\otimes}_{K^*} A^*) \otimes_{K^*} (A^* \widehat{\otimes}_{K^*} A^*), K^*) & \mathcal{H}om^*(A^* \widehat{\otimes}_{K^*} A^*, K^*) \otimes_{K^*} \mathcal{H}om^*(A^* \widehat{\otimes}_{K^*} A^*, K^*) \arrow[l, "\phi"'] \arrow[u, "\eta^*_{A^*\otimes A^*}\otimes\eta^*_{A^*\otimes A^*}"']
\end{tikzcd}
$$

commutes and this implies that the upper right rectangle of the diagram 5 commutes. The other rectangles of the diagram 5 commute by the naturality of ϕ, $\hat{\phi}$ and the completion maps. Hence the assertion follows. $\square$

Definition 5.1.7 *Let A^* be a proper algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$. If A^* has a structure of colagebra and satisfies the condition of (5.1.6), we call A^* an Hopf algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$. Then, the dual A^{**} of A^* has a structure of Hopf algebra with multiplication $\tilde{\delta} : A^{**} \otimes_{K^*} A^{**} \to A^{**}$ and comultiplication $\hat{\mu} : A^{**} \to A^{**} \widehat{\otimes}_{K^*} A^{**}$. We call this the dual Hopf algebra of A^*.*

Lemma 5.1.8 *Let $\alpha : U^* \otimes_{K^*} V^* \to X^*$, $\beta : V^* \otimes_{K^*} W^* \to Y^*$, $\gamma : X^* \otimes_{K^*} W^* \to Z^*$ and $\delta : U^* \otimes_{K^*} Y^* \to Z^*$ be morphisms in $\mathcal{T}op\mathcal{M}od_{K^*}$. A diagram*

$$
\begin{tikzcd}
U^* \otimes_{K^*} V^* \otimes_{K^*} W^* \arrow[r, "\alpha\otimes_{K^*}1"] \arrow[d, "1\otimes_{K^*}\beta"] & X^* \otimes_{K^*} W^* \arrow[d, "\gamma"] \\
U^* \otimes_{K^*} Y^* \arrow[r, "\delta"] & Z^*
\end{tikzcd}
$$

commutes if and only if the following diagram commutes.

$$
\begin{tikzcd}
U^* \arrow[r, "\Phi(\alpha)"] \arrow[d, "\Phi(\delta)"] & \mathcal{H}om^*(V^*, X^*) \arrow[r, "\Phi(\gamma)_*"] & \mathcal{H}om^*(V^*, \mathcal{H}om^*(W^*, Z^*)) \\
\mathcal{H}om^*(Y^*, Z^*) \arrow[r, "\beta^*"] & \mathcal{H}om^*(V^* \otimes_{K^*} W^*, Z^*) \arrow[ur, "\Phi^*_{V^*,W^*,Z^*}"'] &
\end{tikzcd}
$$

Proof. By (3.2.7) and the naturality of Φ's, the following diagram commutes.

$$
\begin{tikzcd}
 & \mathrm{Hom}^c_{K^*}(U^* \otimes_{K^*} Y^*, Z^*) \arrow[r, "\Phi"] \arrow[d, "(1\otimes\beta)^*"] & \mathrm{Hom}^c_{K^*}(U^*, \mathcal{H}om^*(Y^*, Z^*)) \arrow[d, "(\beta^*)_*"] \\
\mathrm{Hom}^c_{K^*}(X^* \otimes_{K^*} W^*, Z^*) \arrow[r, "(\alpha\otimes 1)^*"] \arrow[d, "\Phi"] & \mathrm{Hom}^c_{K^*}(U^* \otimes_{K^*} V^* \otimes_{K^*} W^*, Z^*) \arrow[r, "\Phi"] \arrow[d, "\Phi"] & \mathrm{Hom}^c_{K^*}(U^*, \mathcal{H}om^*(V^* \otimes_{K^*} W^*, Z^*)) \arrow[d, "(\Phi^*)_*"] \\
\mathrm{Hom}^c_{K^*}(X^*, \mathcal{H}om^*(W^*, Z^*)) \arrow[r, "\alpha^*"] & \mathrm{Hom}^c_{K^*}(U^* \otimes_{K^*} V^*, \mathcal{H}om^*(W^*, Z^*)) \arrow[r, "\Phi"] & \mathrm{Hom}^c_{K^*}(U^*, \mathcal{H}om^*(V^*, \mathcal{H}om^*(W^*, Z^*))) \\
 & \mathrm{Hom}^c_{K^*}(U^* \otimes_{K^*} V^*, X^*) \arrow[r, "\Phi"] \arrow[u, "\Phi(\gamma)_*"] & \mathrm{Hom}^c_{K^*}(U^*, \mathcal{H}om^*(V^*, X^*)) \arrow[u, "(\Phi(\gamma)_*)_*"]
\end{tikzcd}
$$

Then, we have the following equalities.

$$\Phi\Phi(\gamma(\alpha \otimes 1)) = \Phi\Phi((\alpha \otimes 1)^*(\gamma)) = \Phi(\alpha^*(\Phi(\gamma))) = \Phi(\Phi(\gamma)\alpha) = \Phi(\Phi(\gamma)_*(\alpha)) = (\Phi(\gamma)_*)_*(\Phi(\alpha)) = \Phi(\gamma)_*\Phi(\alpha)$$
$$\Phi\Phi(\delta(1 \otimes \beta)) = \Phi\Phi((1 \otimes \beta)^*(\delta)) = (\Phi^*)_*\Phi((1 \otimes \beta)^*(\delta)) = (\Phi^*)_*((\beta^*)_*(\Phi(\delta))) = \Phi^*\beta^*\Phi(\delta)$$

Since Φ's are injective, we have the result. $\square$

5.2 Milnor coaction

Lemma 5.2.1 *Let A^*, B^*, V^* and W^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. Assume that the topologies on V^* and W^* are coarser than the topology induced by K^* and that $\hat{\varphi}^{B^*}_{W^*} : B^{**}\,\widehat{\otimes}_{K^*} W^* \to \mathcal{H}om^*(B^*, W^*)^\wedge$ is an isomorphism. Then, for a morphism $f : V^* \to \mathcal{H}om^*(B^*, W^*)$ in $\mathcal{T}op\mathcal{M}od_{K^*}$, the following diagram commutes.*

$$
\begin{tikzcd}
A^{**}\,\widehat{\otimes}_{K^*} V^* \arrow[r, "1\,\widehat{\otimes}\, f"] \arrow[dd, "\varphi^{A^*}_{V^*}"] & A^{**}\,\widehat{\otimes}_{K^*} \mathcal{H}om^*(B^*, W^*) \arrow[r, "1\,\widehat{\otimes}\,\eta_{\mathcal{H}om^*(B^*,W^*)}"] & A^{**}\,\widehat{\otimes}_{K^*} \mathcal{H}om^*(B^*, W^*)^\wedge \arrow[d, "1\,\widehat{\otimes}\left(\hat{\varphi}^{B^*}_{W^*}\right)^{-1}"] \\
 & & A^{**}\,\widehat{\otimes}_{K^*} B^{**}\,\widehat{\otimes}_{K^*} W^* \arrow[d, "\hat{\phi}\,\widehat{\otimes}\,1"] \\
\mathcal{H}om^*(A^*, V^*)^\wedge \arrow[r, "\widehat{f_*}"] & \mathcal{H}om^*(A^*, \mathcal{H}om^*(B^*, W^*))^\wedge & \mathcal{H}om^*(A^*\otimes_{K^*} B^*, K^*)^\wedge\,\widehat{\otimes}_{K^*} W^* \arrow[d, "\left(\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}B^*,K^*)}\,\widehat{\otimes}\,1\right)^{-1}"] \\
 & \mathcal{H}om^*(A^*\otimes_{K^*} B^*, W^*)^\wedge \arrow[u, "\widehat{\Phi}^*_{A^*,B^*,V^*}"] & \mathcal{H}om^*(A^*\otimes_{K^*} B^*, K^*)\,\widehat{\otimes}_{K^*} W^* \arrow[l, "\hat{\varphi}^{A^*\otimes_{K^*}B^*}_{W^*}"']
\end{tikzcd}
$$

Proof. By the naturality of completion of modules and the commutativity of the second diagram of (4.2.14) every rectangle of diagram 6 commutes except for the central rectangle. Hence we have

$$
\begin{aligned}
&\widehat{\Phi}^*_{A^*,B^*,W^*}\hat{\varphi}^{A^*\otimes_{K^*}B^*}_{W^*}\left(\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}B^*,K^*)}\widehat{\otimes}1\right)^{-1}(\hat{\phi}\widehat{\otimes}1)\left(1\widehat{\otimes}\hat{\varphi}^{B^*}_{W^*}\right)^{-1}\left(1\widehat{\otimes}\eta_{\mathcal{H}om^*(B^*,W^*)}\right)\left(1\widehat{\otimes}\varphi^{B^*}_{W^*}\right)\eta_{A^{**}\otimes_{K^*}B^{**}\otimes_{K^*}W^*}\\
&\quad= \widehat{\Phi}^*_{A^*,B^*,W^*}\hat{\varphi}^{A^*\otimes_{K^*}B^*}_{W^*}\left(\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}B^*,K^*)}\,\widehat{\otimes}\,1\right)^{-1}(\hat{\phi}\,\widehat{\otimes}\,1)\left(1\,\widehat{\otimes}\,\eta_{B^{**}\otimes_{K^*}W^*}\right)\eta_{A^{**}\otimes_{K^*}B^{**}\otimes_{K^*}W^*}\\
&\quad= \widehat{\Phi}^*_{A^*,B^*,W^*}\hat{\varphi}^{A^*\otimes_{K^*}B^*}_{W^*}\,\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}B^*,K^*)\otimes_{K^*}W^*}(\phi\otimes 1)\\
&\quad= \widehat{\Phi}^*_{A^*,B^*,W^*}\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}B^*,W^*)}\varphi^{A^*\otimes_{K^*}B^*}_{W^*}(\phi\otimes 1)\\
&\quad= \eta_{\mathcal{H}om^*(A^*,\mathcal{H}om^*(B^*,W^*))}\Phi^*_{A^*,B^*,W^*}\varphi^{A^*\otimes_{K^*}B^*}_{W^*}(\phi\otimes 1)\\
&\quad= \eta_{\mathcal{H}om^*(A^*,\mathcal{H}om^*(B^*,W^*))}\varphi^{A^*}_{\mathcal{H}om^*(B^*,W^*)}\left(1\otimes\varphi^{B^*}_{W^*}\right)\\
&\quad= \hat{\varphi}^{A^*}_{\mathcal{H}om^*(B^*,W^*)}\eta_{A^{**}\otimes_{K^*}\mathcal{H}om^*(B^*,W^*)}\left(1\otimes\varphi^{B^*}_{W^*}\right)\\
&\quad= \hat{\varphi}^{A^*}_{\mathcal{H}om^*(B^*,W^*)}\left(1\,\widehat{\otimes}\,\varphi^{B^*}_{W^*}\right)\eta_{A^{**}\otimes_{K^*}B^{**}\otimes_{K^*}W^*}.
\end{aligned}
$$

Since $\eta_{A^{**}\otimes_{K^*}B^{**}\otimes_{K^*}W^*}$ is an epimorphism in $\mathcal{T}op\mathcal{M}od_{K^*}$ and $1\,\widehat{\otimes}\,\varphi^{B^*}_{W^*}$ is an isomorphism, It follows that

$$
\hat{\varphi}^{A^*}_{\mathcal{H}om^*(B^*,W^*)} = \widehat{\Phi}^*_{A^*,B^*,W^*}\hat{\varphi}^{A^*\otimes_{K^*}B^*}_{W^*}\left(\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}B^*,K^*)}\,\widehat{\otimes}\,1\right)^{-1}(\hat{\phi}\,\widehat{\otimes}\,1)\left(1\,\widehat{\otimes}\,\hat{\varphi}^{B^*}_{W^*}\right)^{-1}\left(1\,\widehat{\otimes}\,\eta_{\mathcal{H}om^*(B^*,W^*)}\right).
$$

This implies that

$$
\begin{aligned}
&\widehat{\Phi}^*_{A^*,B^*,W^*}\hat{\varphi}^{A^*\otimes_{K^*}B^*}_{W^*}\left(\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}B^*,K^*)}\,\widehat{\otimes}\,1\right)^{-1}(\hat{\phi}\,\widehat{\otimes}\,1)\left(1\,\widehat{\otimes}\,\hat{\varphi}^{B^*}_{W^*}\right)^{-1}\left(1\,\widehat{\otimes}\,\eta_{\mathcal{H}om^*(B^*,W^*)}\right)(1\,\widehat{\otimes}\,f)\eta_{A^{**}\otimes_{K^*}V^*}\\
&\quad= \hat{\varphi}^{A^*}_{\mathcal{H}om^*(B^*,W^*)}(1\,\widehat{\otimes}\,f)\eta_{A^{**}\otimes_{K^*}V^*} = \hat{\varphi}^{A^*}_{\mathcal{H}om^*(B^*,W^*)}\eta_{A^{**}\otimes_{K^*}\mathcal{H}om^*(B^*,W^*)}(1\otimes f)\\
&\quad= \eta_{\mathcal{H}om^*(A^*,\mathcal{H}om^*(B^*,W^*))}\varphi^{A^*}_{\mathcal{H}om^*(B^*,W^*)}(1\otimes f) = \eta_{\mathcal{H}om^*(A^*,\mathcal{H}om^*(B^*,W^*))}f_*\varphi^{A^*}_{V^*}\\
&\quad= \widehat{f}_*\eta_{\mathcal{H}om^*(A^*,V^*)}\varphi^{A^*}_{V^*} = \widehat{f}_*\hat{\varphi}^{A^*}_{V^*}\eta_{A^{**}\otimes_{K^*}V^*}.
\end{aligned}
$$

Since $\eta_{A^{**}\otimes_{K^*}V^*}$ is an epimorphism in $\mathcal{T}op\mathcal{M}od_{K^*}$, the assertion follows. $\square$

Let V^*, W^* and Z^* be objects in $\mathcal{T}op\mathcal{M}od_{K^*}$. Suppose that the topology on Z^* is coarser than the topology induced by K^* and that $\hat{\varphi}^{W^*}_{Z^*} : W^{**}\,\widehat{\otimes}_{K^*} Z^* \to \mathcal{H}om^*(W^*, Z^*)^\wedge$ is an isomorphism. We define a map $\Xi : \mathrm{Hom}^c_{K^*}(V^*\otimes_{K^*} W^*, Z^*) \to \mathrm{Hom}^c_{K^*}(V^*, W^{**}\,\widehat{\otimes}_{K^*} Z^*)$ to be the the following composition.

$$
\begin{aligned}
&\mathrm{Hom}^c_{K^*}(V^*\otimes_{K^*} W^*, Z^*) \xrightarrow{\Phi_{V^*,W^*,Z^*}} \mathrm{Hom}^c_{K^*}(V^*, \mathcal{H}om^*(W^*, Z^*)) \xrightarrow{\eta_{\mathcal{H}om^*(W^*,Z^*)*}} \mathrm{Hom}^c_{K^*}(V^*, \mathcal{H}om^*(W^*, Z^*)^\wedge)\\
&\qquad\xrightarrow{\left(\hat{\varphi}^{W^*}_{Z^*}\right)^{-1}_*} \mathrm{Hom}^c_{K^*}(V^*, W^{**}\,\widehat{\otimes}_{K^*} Z^*)
\end{aligned}
$$

$$
\begin{tikzcd}
A^{**} \otimes_{K^*} \mathcal{H}om^*(B^*, W^*)^\frown \arrow[r, "\eta_{A^{**} \otimes_{K^*} \mathcal{H}om^*(B^*,W^*)^\frown}"] & A^{**} \widehat{\otimes}_{K^*} \mathcal{H}om^*(B^*, W^*)^\frown & A^{**} \widehat{\otimes}_{K^*} B^{**} \widehat{\otimes}_{K^*} W^* \arrow[l, "1 \widehat{\otimes} \hat{\varphi}^{B^*}_{W^*}"', "\cong"] \arrow[r, "\hat{\phi} \widehat{\otimes} 1", "\cong"'] & \mathcal{H}om^*(A^* \otimes_{K^*} B^*, K^*)^\frown \widehat{\otimes}_{K^*} W^* \\
A^{**} \otimes_{K^*} \mathcal{H}om^*(B^*, W^*) \arrow[u, "1 \otimes \eta_{\mathcal{H}om^*(B^*,W^*)}"'] \arrow[r, "\eta_{A^{**} \otimes_{K^*} \mathcal{H}om^*(B^*,W^*)}"] & A^{**} \widehat{\otimes}_{K^*} \mathcal{H}om^*(B^*, W^*) \arrow[u, "\cong", "1 \widehat{\otimes} \eta_{\mathcal{H}om^*(B^*,W^*)}"'] & A^{**} \widehat{\otimes}_{K^*} (B^{**} \otimes_{K^*} W^*) \arrow[l, "1 \widehat{\otimes} \varphi^{B^*}_{W^*}"', "\cong"] \arrow[u, "1 \widehat{\otimes} \eta_{B^{**} \otimes_{K^*} W^*}", "\cong"'] & \mathcal{H}om^*(A^* \otimes_{K^*} B^*, K^*) \widehat{\otimes}_{K^*} W^* \arrow[u, "\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} B^*, K^*)} \widehat{\otimes} 1", "\cong"'] \\
A^{**} \otimes_{K^*} V^* \arrow[u, "1 \otimes f"'] \arrow[r, "\eta_{A^{**} \otimes_{K^*} V^*}"] \arrow[d, "\varphi^{A^*}_{V^*}"'] & A^{**} \widehat{\otimes}_{K^*} V^* \arrow[u, "1 \widehat{\otimes} f"'] \arrow[d, "\hat{\varphi}^{A^*}_{V^*}"'] & A^{**} \otimes_{K^*} B^{**} \otimes_{K^*} W^* \arrow[u, "\eta_{A^{**} \otimes_{K^*} B^{**} \otimes_{K^*} W^*}"] \arrow[r, "\phi \otimes 1"] \arrow[d, "1 \otimes \varphi^{B^*}_{W^*}"] & \mathcal{H}om^*(A^* \otimes_{K^*} B^*, K^*) \otimes_{K^*} W^* \arrow[u, "\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} B^*, K^*) \otimes_{K^*} W^*}"] \arrow[ddd, "\varphi^{A^* \otimes_{K^*} B^*}_{W^*}"] \\
\mathcal{H}om^*(A^*, V^*) \arrow[r, "\eta_{\mathcal{H}om^*(A^*,V^*)}"] \arrow[d, "f_*"'] & \mathcal{H}om^*(A^*, V^*)^\frown \arrow[d, "\widehat{f_*}"'] & A^{**} \otimes_{K^*} \mathcal{H}om^*(B^*, W^*) \arrow[d, "\varphi^{A^*}_{\mathcal{H}om^*(B^*,W^*)}"] & \\
\mathcal{H}om^*(A^*, \mathcal{H}om^*(B^*, W^*)) \arrow[r, "\eta_{\mathcal{H}om^*(A^*,\mathcal{H}om^*(B^*,W^*))}"] & \mathcal{H}om^*(A^*, \mathcal{H}om^*(B^*, W^*))^\frown & \mathcal{H}om^*(A^*, \mathcal{H}om^*(B^*, W^*)) \arrow[l, "\eta_{\mathcal{H}om^*(A^*,\mathcal{H}om^*(B^*,W^*))}"'] & \\
& \mathcal{H}om^*(A^* \otimes_{K^*} B^*, W^*)^\frown \arrow[u, "\widehat{\Phi}^*_{A^*,B^*,W^*}"'] & \mathcal{H}om^*(A^* \otimes_{K^*} B^*, W^*) \arrow[l, "\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} B^*, W^*)}"'] \arrow[u, "\Phi^*_{A^*,B^*,W^*}"] &
\end{tikzcd}
$$

diagram 6

Proposition 5.2.2 *Let V^* and A^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ such that V^* is Hausdorff and that the topology on V^* is coarser than the topology induced by K^*. Let $\beta : V^* \otimes_{K^*} A^* \to V^*$ be a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$. Suppose that A^* is proper and that the following morphisms are isomorphisms.*

$$\hat{\varphi}^{A^*}_{V^*} : A^{**} \widehat{\otimes}_{K^*} V^* \to \mathcal{H}om^*(A^*, V^*)^{\wedge}, \quad \hat{\varphi}^{A^* \otimes_{K^*} A^*}_{V^*} : \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \widehat{\otimes}_{K^*} V^* \to \mathcal{H}om^*(A^* \otimes_{K^*} A^*, V^*)^{\wedge}$$

(1) For a morphism $\mu : A^* \otimes_{K^*} A^* \to A^*$, the following right diagram commutes if and only if the left one does.

$$\begin{tikzcd}
V^* \otimes_{K^*} A^* \otimes_{K^*} A^* \arrow[r, "\beta \otimes_{K^*} id_{A^*}"] \arrow[d, "id_{V^*} \otimes_{K^*} \mu"] & V^* \otimes_{K^*} A^* \arrow[d, "\beta"] \\
V^* \otimes_{K^*} A^* \arrow[r, "\beta"] & V^*
\end{tikzcd}
\qquad
\begin{tikzcd}
V^* \arrow[r, "\Xi(\beta)"] \arrow[d, "\Xi(\beta)"] & A^{**} \widehat{\otimes}_{K^*} V^* \arrow[d, "id_{A^{**}} \widehat{\otimes}_{K^*} \Xi(\beta)"] \\
A^{**} \widehat{\otimes}_{K^*} V^* \arrow[r, "\hat{\mu} \widehat{\otimes}_{K^*} id_{V^*}"] & A^{**} \widehat{\otimes}_{K^*} V^{**} \widehat{\otimes}_{K^*} V^*
\end{tikzcd}$$

(2) For a morphism $\eta : K^* \to A^*$, the following left diagram commutes if and only if the right one does.

$$\begin{tikzcd}
V^* \otimes_{K^*} K^* \arrow[r, "id_{V^*} \otimes_{K^*} \eta"] \arrow[d, "T_{V^*, K^*}"] & V^* \otimes_{K^*} A^* \arrow[d, "\beta"] \\
K^* \otimes_{K^*} V^* \arrow[r, "\tilde{\alpha}_{V^*}"] & V^*
\end{tikzcd}
\qquad
\begin{tikzcd}
V^* \arrow[r, "\Xi(\beta)"] \arrow[d, "\eta_{K^* \otimes_{K^*} V^*} i_2"] & \mathcal{H}om^*(A^*, K^*) \widehat{\otimes}_{K^*} V^* \arrow[d, "\eta^* \widehat{\otimes}_{K^*} id_{V^*}"] \\
K^* \widehat{\otimes}_{K^*} V^* \arrow[r, "\kappa_{K^*} \widehat{\otimes}_{K^*} id_{V^*}"] & \mathcal{H}om^*(K^*, K^*) \widehat{\otimes}_{K^*} V^*
\end{tikzcd}$$

Proof. (1) We denote by $\hat{\varphi}^2 : \mathcal{H}om^*(A^* \otimes_{K^*} V^*, K^*) \widehat{\otimes}_{K^*} V^* \to \mathcal{H}om^*(A^*, \mathcal{H}om^*(A^*, V^*))^{\wedge}$ the composition

$$\mathcal{H}om^*(A^* \otimes_{K^*} V^*, K^*) \widehat{\otimes}_{K^*} V^* \xrightarrow{\hat{\varphi}^{A^* \otimes_{K^*} V^*}_{V^*}} \mathcal{H}om^*(A^* \otimes_{K^*} V^*, V^*)^{\wedge} \xrightarrow{\widehat{\Phi}^*_{A^*, A^*, V^*}} \mathcal{H}om^*(A^*, \mathcal{H}om^*(A^*, V^*))^{\wedge}.$$

It follows from (3.2.7) and (1.3.12) that $\hat{\varphi}^2$ is injective. Since $\Xi(\beta) : V^* \to A^{**} \widehat{\otimes}_{K^*} V^*$ is a composition

$$V^* \xrightarrow{\Phi_{V^*, A^*, V^*}(\beta)} \mathcal{H}om^*(A^*, V^*) \xrightarrow{\eta_{\mathcal{H}om^*(A^*, V^*)}} \mathcal{H}om^*(A^*, V^*)^{\wedge} \xrightarrow{\left(\hat{\varphi}^{A^*}_{V^*}\right)^{-1}} A^{**} \widehat{\otimes}_{K^*} V^*,$$

it follows from (5.2.1) that the top rectangle of diagram 7 commutes. By (5.1.8), $\beta(1 \otimes \mu) = \beta(\beta \otimes 1)$ holds if and only if the trapezoid in the left middle of the diagram 7 commutes. The other rectangles, triangles and trapezoid commute.

$$\begin{tikzcd}
A^{**} \widehat{\otimes}_{K^*} A^{**} \widehat{\otimes}_{K^*} V^* \arrow[r, "\hat{\phi} \widehat{\otimes} 1"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^{\wedge} \widehat{\otimes}_{K^*} V^* \arrow[r, "\left(\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)} \widehat{\otimes} 1\right)^{-1}"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \widehat{\otimes}_{K^*} V^* \arrow[d, "\hat{\varphi}^2"] \\
A^{**} \widehat{\otimes}_{K^*} V^* \arrow[u, "1 \widehat{\otimes} \Xi(\beta)"] \arrow[r, "\hat{\varphi}^{A^*}_{V^*}", "\cong"'] & \mathcal{H}om^*(A^*, V^*)^{\wedge} \arrow[r, "\widehat{\Phi(\beta)_*}"] & \mathcal{H}om^*(A^*, \mathcal{H}om^*(A^*, V^*))^{\wedge} \\
V^* \arrow[u, "\Xi(\beta)"] \arrow[r, "\Phi(\beta)"] \arrow[d, "\Phi(\beta)"] & \mathcal{H}om^*(A^*, V^*) \arrow[u, "\eta_{\mathcal{H}om^*(A^*, V^*)}"] \arrow[r, "\Phi(\beta)_*"] & \mathcal{H}om^*(A^*, \mathcal{H}om^*(A^*, V^*)) \arrow[u, "\eta_{\mathcal{H}om^*(A^*, \mathcal{H}om^*(A^*, V^*))}"'] \arrow[dd, "\eta_{\mathcal{H}om^*(A^*, \mathcal{H}om^*(A^*, V^*))}"'] \\
\mathcal{H}om^*(A^*, V^*) \arrow[r, "\mu^*"] \arrow[d, "\eta_{\mathcal{H}om^*(A^*, V^*)}"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, V^*) \arrow[ur, "\Phi^*_{A^*, A^*, V^*}"] \arrow[d, "\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} A^*, V^*)}"] & \\
\mathcal{H}om^*(A^*, V^*)^{\wedge} \arrow[r, "\widehat{\mu^*}"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, V^*)^{\wedge} \arrow[r, "\widehat{\Phi}^*_{A^*, A^*, V^*}"] & \mathcal{H}om^*(A^*, \mathcal{H}om^*(A^*, V^*))^{\wedge} \\
A^{**} \widehat{\otimes}_{K^*} V^* \arrow[u, "\cong", "\hat{\varphi}^{A^*}_{V^*}"'] \arrow[r, "\mu^* \widehat{\otimes} 1"] \arrow[ddr, "\hat{\mu} \widehat{\otimes} 1"] & \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \widehat{\otimes}_{K^*} V^* \arrow[u, "\cong", "\hat{\varphi}^{A^* \otimes_{K^*} A^*}_{V^*}"'] \arrow[ur, "\hat{\varphi}^2"] & (A^{**} \otimes_{K^*} A^{**}) \widehat{\otimes}_{K^*} V^* \arrow[l, "\phi \widehat{\otimes} 1"'] \arrow[ddl, "\cong", "\eta_{A^{**} \otimes_{K^*} A^{**}} \widehat{\otimes} 1"'] \\
& \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)^{\wedge} \widehat{\otimes}_{K^*} V^* \arrow[u, "\left(\eta_{\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*)} \widehat{\otimes} 1\right)^{-1}"'] & \\
& A^{**} \widehat{\otimes}_{K^*} A^{**} \widehat{\otimes}_{K^*} V^* \arrow[u, "\hat{\phi} \widehat{\otimes} 1"'] &
\end{tikzcd}$$

diagram 7

Therefore we have

$$\hat{\varphi}^2\left(\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}\widehat{\otimes}1\right)^{-1}(\hat{\phi}\widehat{\otimes}1)(1\widehat{\otimes}\Xi(\beta))\Xi(\beta) = \eta_{\mathcal{H}om^*(A^*,\mathcal{H}om^*(A^*,V^*))}\Phi(\beta)_*\Phi(\beta)$$
$$\hat{\varphi}^2\left(\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}\widehat{\otimes}1\right)^{-1}(\hat{\phi}\widehat{\otimes}1)(\hat{\mu}\widehat{\otimes}1)\Xi(\beta) = \eta_{\mathcal{H}om^*(A^*,\mathcal{H}om^*(A^*,V^*))}\Phi^*_{A^*,A^*,V^*}\mu^*\Phi(\beta)$$

Since $\hat{\varphi}^2\left(\eta_{\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}\widehat{\otimes}1\right)^{-1}(\hat{\phi}\widehat{\otimes}1)$ and $\eta_{\mathcal{H}om^*(A^*,\mathcal{H}om^*(A^*,V^*))}$ are injective, $(1\widehat{\otimes}\Xi(\beta))\Xi(\beta) = (\hat{\mu}\widehat{\otimes}1)\Xi(\beta)$ holds if and only if the trapezoid in the left middle of the diagram 7 commutes.

(2) We first claim that $(\kappa_{V^*})^{-1}\eta^*\Phi(\beta) = \beta(1\otimes\eta)T_{K^*,V^*}i_2$. In fact, for $x \in V^n$, we have

$$\begin{aligned}\Big((\kappa_{V^*})^{-1}\eta^*\Phi(\beta)\Big)(x) &= (\kappa_{V^*})^{-1}(\Phi(\beta)(x)\Sigma^n\eta) = (\Phi(\beta)(x))\Sigma^n\eta([n],1) = \beta(x\otimes\eta(1))\\ &= (\beta(1\otimes\eta))(x\otimes 1) = \beta(1\otimes\eta)T_{K^*,V^*}i_2(x).\end{aligned}$$

Since the following diagram commutes, it follows that

$$\eta_{V^*}\beta(1\otimes\eta)T_{K^*,V^*}i_2 = \eta_{V^*}\left(\kappa_{V^*}\right)^{-1}\eta^*\Phi(\beta) = \widehat{\tilde{\alpha}_{V^*}}\Big((\kappa_{K^*})^{-1}\widehat{\otimes}1\Big)(\eta^*\widehat{\otimes}1)\Xi(\beta).$$

$$\begin{tikzcd}
V^* \arrow[r, "\Xi(\beta)"] \arrow[d, "\Phi(\beta)"] & \mathcal{H}om^*(A^*,K^*)\widehat{\otimes}_{K^*}V^* \arrow[r, "\eta^*\widehat{\otimes}1"] \arrow[d, "\hat{\varphi}^{A^*}_{V^*}"] & \mathcal{H}om^*(K^*,K^*)\widehat{\otimes}_{K^*}V^* \arrow[r, "(\kappa_{K^*})^{-1}\widehat{\otimes}1", "\cong"'] \arrow[d, "\hat{\varphi}^{K^*}_{V^*}"] & K^*\widehat{\otimes}_{K^*}V^* \arrow[d, "\widehat{\tilde{\alpha}_{V^*}}"] \\
\mathcal{H}om^*(A^*,V^*) \arrow[r, "\eta_{\mathcal{H}om^*(A^*,V^*)}"] \arrow[drr, "\eta^*"] & \mathcal{H}om^*(A^*,V^*)^\frown \arrow[r, "\widehat{\eta^*}"] & \mathcal{H}om^*(K^*,V^*)^\frown \arrow[r, "(\widehat{\kappa_{V^*}})^{-1}", "\cong"'] & \widehat{V}^* \\
 & & \mathcal{H}om^*(K^*,V^*) \arrow[r, "(\kappa_{V^*})^{-1}", "\cong"'] \arrow[u, "\eta_{\mathcal{H}om^*(K^*,V^*)}"] & V^* \arrow[u, "\eta_{V^*}"']
\end{tikzcd}$$

Moreover, since the map $\tilde{\alpha}_{V^*}$ induced by the K^*-module structure map $\alpha_{V^*} : K^*\times V^* \to V^*$ is the inverse of the map $i_2 : V^* \to K^*\otimes_{K^*}V^*$, we have

$$\eta_{K^*\otimes_{K^*}V^*}i_2\beta(1\otimes\eta)T_{K^*,V^*}i_2 = \hat{i}_2\eta_{V^*}\beta(1\otimes\eta)T_{K^*,V^*}i_2 = \Big((\kappa_{K^*})^{-1}\widehat{\otimes}1\Big)(\eta^*\widehat{\otimes}1)\Xi(\beta).$$

Therefore $(\eta^*\widehat{\otimes}1)\Xi(\beta) = \Big(\kappa_{K^*}\widehat{\otimes}1\Big)\eta_{K^*\otimes_{K^*}V^*}i_2\beta(1\otimes\eta)T_{K^*,V^*}i_2$. If $\beta(1\otimes\eta) = \tilde{\alpha}_{V^*}T_{V^*,K^*}$, then the right hand side of the above equality is $\Big(\kappa_{K^*}\widehat{\otimes}1\Big)\eta_{K^*\otimes_{K^*}V^*}i_2$. Since i_2, $\eta_{K^*\otimes_{K^*}V^*}$ and $\kappa_{K^*}\widehat{\otimes}1$ are monomorphisms, the above equality implies $\beta(1\otimes\eta)T_{K^*,V^*}i_2 = id_{V^*}$, namely $\beta(1\otimes\eta) = \tilde{\alpha}_{V^*}T_{V^*,K^*}$. □

Remark 5.2.3 *Suppose that K^* is complete Hausdorff, M^* is profinite and that the topologies of M^* and A^* are coarser than the topology induced by K^*. We also assume that A^* is supercofinite and (A^*, M^*) is a very nice pair. Then $\hat{\varphi}^{A^*}_{M^*} : A^{**}\widehat{\otimes}_{K^*}M^* \to \mathcal{H}om^*(A^*,M^*)^\frown$ is an isomorphism and the following diagram is commutative by (4.2.4).*

$$\begin{tikzcd}
\mathrm{Hom}^c_{K^*}(A^*\otimes_{K^*}M^*,M^*) \arrow[r, "\Lambda_{A^*,M^*,M^*}"] \arrow[d, "\Phi_{A^*,M^*,M^*}"] & \mathrm{Hom}^c_{K^*}\left(M^*,A^{**}\widehat{\otimes}_{K^*}M^*\right) \arrow[d, "\widehat{T}_{A^{**},M^**}"] \\
\mathrm{Hom}^c_{K^*}(A^*,\mathcal{H}om^*(M^*,M^*)) \arrow[d, "(\zeta^*)_*"] & \mathrm{Hom}^c_{K^*}\left(M^*,M^*\widehat{\otimes}_{K^*}A^{**}\right) \arrow[d, "\zeta"] \\
\mathrm{Hom}^c_{K^*}(A^*,\mathcal{H}om^*(M^{**},M^{**})) & \mathrm{Hom}^c_{K^*}\left(\mathcal{H}om^*\left(M^*\widehat{\otimes}_{K^*}A^{**},K^*\right),M^{**}\right) \arrow[d, "\left(\left(\eta^*_{M^*\otimes_{K^*}A^{**}}\right)^*\right)^{-1}"] \\
\mathrm{Hom}^c_{K^*}(A^*\otimes_{K^*}M^{**},M^{**}) \arrow[u, "\Phi_{A^*,M^{**},M^{**}}"] & \mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(M^*\otimes_{K^*}A^{**},K^*),M^{**}) \arrow[d, "(\tilde{\alpha}_{K^**}T_{K^*,K^**}\phi)^*"] \\
\mathrm{Hom}^c_{K^*}(M^{**}\otimes_{K^*}A^*,M^{**}) \arrow[u, "T^*_{A^*,M^{**}}"] & \mathrm{Hom}^c_{K^*}(M^{**}\otimes_{K^*}\mathcal{H}om^*(A^{**},K^*),M^{**}) \arrow[l, "\left(id_{M^{**}}\otimes_{K^*}\chi_{A^*,K^*}\right)^*"']
\end{tikzcd}$$

If K^* is a field such that $K^i = \{0\}$ for $i \neq 0$, both A^* and M^* are finite type and A^* has skeletal topology, then $\Phi_{A^*,M^{**},M^{**}}$, χ_{A^*,K^*}, ψ and ζ in the above diagram are all isomorphisms. Hence, in the case that A^* is the mod p Steenrod algebra $\mathcal{A}^*_p$ and M^* is the mod p cohomology $H^*(X)$ of a space X of finite type, the image of the action $\mathcal{A}^*_p\otimes_{\boldsymbol{F}_p}H^*(X)\to H^*(X)$ by $\Lambda_{\mathcal{A}^*_p,H^*(X),H^*(X)}$ coincides with the homomorphism "λ^*" given in [16] which is called the Milnor coaction.

Proposition 5.2.4 *Let $\alpha : A^* \otimes_{K^*} V^* \to V^*$ be a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$.*

(1) Under the assumption of (5.2.2), the following right diagram commutes if and only if the left one does.

$$\begin{tikzcd}
A^* \otimes_{K^*} A^* \otimes_{K^*} V^* \arrow[r, "id_{A^*} \otimes_{K^*} \alpha"] \arrow[d, "\mu \otimes_{K^*} id_{V^*}"] & A^* \otimes_{K^*} V^* \arrow[d, "\alpha"] \\
A^* \otimes_{K^*} V^* \arrow[r, "\alpha"] & V^*
\end{tikzcd}
\qquad
\begin{tikzcd}
V^* \arrow[r, "\Lambda(\alpha)"] \arrow[d, "\Lambda(\alpha)"] & V^* \widehat{\otimes}_{K^*} A^{**} \arrow[d, "\Lambda(\alpha) \widehat{\otimes}_{K^*} id_{A^{**}}"] \\
V^* \widehat{\otimes}_{K^*} A^{**} \arrow[r, "id_{V^*} \widehat{\otimes}_{K^*} \hat{\mu}"] & V^* \widehat{\otimes}_{K^*} A^{**} \widehat{\otimes}_{K^*} A^{**}
\end{tikzcd}$$

(2) Under the assumption of (5.2.2), the following left diagram commutes if and only if the right one does.

$$\begin{tikzcd}
K^* \otimes_{K^*} V^* \arrow[r, "\eta \otimes_{K^*} id_{V^*}"] \arrow[dr, "\tilde{\alpha}_{V^*}", "\cong"'] & A^* \otimes_{K^*} V^* \arrow[d, "\alpha"] \\
 & V^*
\end{tikzcd}
\qquad
\begin{tikzcd}
V^* \arrow[r, "\Lambda(\alpha)"] \arrow[d, "\eta_{V^* \otimes_{K^*} K^*} i_1", "\cong"'] & V^* \widehat{\otimes}_{K^*} \mathcal{H}om^*(A^*, K^*) \arrow[d, "id_{V^*} \widehat{\otimes}_{K^*} \eta^*"] \\
V^* \widehat{\otimes}_{K^*} K^* \arrow[r, "id_{V^*} \widehat{\otimes}_{K^*} \kappa_{K^*}", "\cong"'] & V^* \widehat{\otimes}_{K^*} \mathcal{H}om^*(K^*, K^*)
\end{tikzcd}$$

Proof. Put $\beta = \alpha T_{V^*,A^*} : V^* \otimes_{K^*} A^* \to V^*$ and $\nu = \mu T_{A^*,A^*} : A^* \otimes_{K^*} A^* \to A^*$. Then, $\Lambda(\alpha) = \widehat{T}_{A^{**},V^*}\Xi(\beta)$.

(1) The lower right rectangle of the upper diagram below commutes if and only if the outer rectangle commutes. Note that the outer rectangle is nothing but the lower one below.

$$\begin{tikzcd}
V^* \otimes_{K^*} A^* \otimes_{K^*} A^* \arrow[r, "T_{V^*,A^*} \otimes 1"] \arrow[d, "1 \otimes T_{A^*,A^*}"] & A^* \otimes_{K^*} V^* \otimes_{K^*} A^* \arrow[r, "\alpha \otimes 1"] \arrow[d, "T_{A^* \otimes V^*,A^*}"] & V^* \otimes_{K^*} A^* \arrow[d, "T_{V^*,A^*}"] \\
V^* \otimes_{K^*} A^* \otimes_{K^*} A^* \arrow[r, "T_{V^*,A^* \otimes_{K^*} A^*}"] \arrow[d, "1 \otimes \mu"] & A^* \otimes_{K^*} A^* \otimes_{K^*} V^* \arrow[r, "1 \otimes \alpha"] \arrow[d, "\mu \otimes 1"] & A^* \otimes_{K^*} V^* \arrow[d, "\alpha"] \\
V^* \otimes_{K^*} A^* \arrow[r, "T_{V^*,A^*}"] & A^* \otimes_{K^*} V^* \arrow[r, "\alpha"] & V^*
\end{tikzcd}$$

$$\begin{tikzcd}
V^* \otimes_{K^*} A^* \otimes_{K^*} A^* \arrow[r, "\beta \otimes 1"] \arrow[d, "1 \otimes \nu"] & V^* \otimes_{K^*} A^* \arrow[d, "\beta"] \\
V^* \otimes_{K^*} A^* \arrow[r, "\beta"] & V^*
\end{tikzcd}$$

On the other hand, the upper left rectangle of the uppoer diagram below commutes if and only if the outer rectangle commutes. Note that the outer rectangle is nothing but the lower one below.

$$\begin{tikzcd}
V^* \arrow[r, "\Xi(\beta)"] \arrow[d, "\Xi(\beta)"] & A^{**} \widehat{\otimes}_{K^*} V^* \arrow[r, "\widehat{T}_{A^{**},V^*}"] \arrow[d, "1 \widehat{\otimes} \Xi(\beta)"] & V^* \widehat{\otimes}_{K^*} A^{**} \arrow[d, "\Xi(\beta) \widehat{\otimes} 1"] \\
A^{**} \widehat{\otimes}_{K^*} V^* \arrow[r, "\hat{\nu} \widehat{\otimes} 1"] \arrow[d, "\widehat{T}_{A^{**},V^*}"] & A^{**} \widehat{\otimes}_{K^*} A^{**} \widehat{\otimes}_{K^*} V^* \arrow[r, "\widehat{T}_{A^{**},A^{**} \widehat{\otimes}_{K^*} V^*}"] \arrow[d, "\widehat{T}_{A^{**} \widehat{\otimes}_{K^*} A^{**},V^*}"] & A^{**} \widehat{\otimes}_{K^*} V^* \widehat{\otimes}_{K^*} A^{**} \arrow[d, "\widehat{T}_{A^{**},V^*} \widehat{\otimes} 1"] \\
V^* \widehat{\otimes}_{K^*} A^{**} \arrow[r, "1 \widehat{\otimes} \hat{\nu}"] & V^* \widehat{\otimes}_{K^*} A^{**} \widehat{\otimes}_{K^*} A^{**} \arrow[r, "1 \widehat{\otimes} \widehat{T}_{A^{**},A^{**}}"] & V^* \widehat{\otimes}_{K^*} A^{**} \widehat{\otimes}_{K^*} A^{**}
\end{tikzcd}$$

$$\begin{tikzcd}
V^* \arrow[r, "\Lambda(\alpha)"] \arrow[d, "\Lambda(\alpha)"] & V^* \widehat{\otimes}_{K^*} A^{**} \arrow[d, "\Lambda(\alpha) \widehat{\otimes} 1"] \\
V^* \widehat{\otimes}_{K^*} A^{**} \arrow[r, "1 \widehat{\otimes} \hat{\mu}"] & V^* \widehat{\otimes}_{K^*} A^{**} \widehat{\otimes}_{K^*} A^{**}
\end{tikzcd}$$

Applying (5.2.2), the first assertion follows.

(2) It is clear that $\alpha(\eta \otimes 1) = \tilde{\alpha}_{V^*}$ holds if and only if $\beta(1 \otimes \eta) = \tilde{\alpha}_{V^*} T_{V^*,K^*}$ and that $(1 \widehat{\otimes} \eta^*)\Lambda(\alpha) = \left(1 \widehat{\otimes} \kappa_{K^*}\right) \eta_{V^* \otimes_{K^*} K^*} i_1$ holds if and only if $(\eta^* \widehat{\otimes} 1)\Xi(\beta) = \left(\kappa_{K^*} \widehat{\otimes} 1\right)\eta_{K^* \otimes_{K^*} V^*} i_2$. Hence the second assertion follows from (5.2.2). □

Proposition 5.2.5 *Let V^*, W^* and A^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ such that V^* and W^* are Hausdorff. Assume that $\hat{\varphi}^{A^*}_{V^*} : A^{**} \widehat{\otimes}_{K^*} V^* \to \mathcal{H}om^*(A^*, V^*)^{\wedge}$ and $\hat{\varphi}^{A^*}_{W^*} : A^{**} \widehat{\otimes}_{K^*} W^* \to \mathcal{H}om^*(A^*, W^*)^{\wedge}$ are isomorphisms. For a morphism $f : V^* \to W^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$, the following right diagram commutes if and only if the left one does.*

$$\begin{tikzcd} A^* \otimes_{K^*} V^* \arrow[r, "\alpha"] \arrow[d, "1\otimes f"] & V^* \arrow[d, "f"] \\ A^* \otimes_{K^*} W^* \arrow[r, "\beta"] & W^* \end{tikzcd} \qquad \begin{tikzcd} V^* \arrow[r, "\Lambda(\alpha)"] \arrow[d, "f"] & V^* \widehat{\otimes}_{K^*} A^{**} \arrow[d, "f\,\widehat{\otimes}\,1"] \\ W^* \arrow[r, "\Lambda(\beta)"] & W^* \widehat{\otimes}_{K^*} A^{**} \end{tikzcd}$$

Proof. Since the switching maps $T_{V^*,A^*}: V^* \otimes_{K^*} A^* \to A^* \otimes_{K^*} V^*$ and $T_{W^*,A^*}: W^* \otimes_{K^*} A^* \to A^* \otimes_{K^*} W^*$ are isomorphisms, the above left diagram commutes if and only if $f\alpha T_{V^*,A^*} = \beta T_{W^*,A^*}(f\otimes 1)$ holds. Since $\Phi: \mathrm{Hom}^c_{K^*}(V^* \otimes_{K^*} A^*, W^*) \to \mathrm{Hom}^c_{K^*}(V^*, \mathcal{H}om^*(A^*, W^*))$ is injective, it follows from (3.2.1) that the above equality is equivalent to $f_*\Phi(\alpha T_{V^*,A^*}) = \Phi(f\alpha T_{V^*,A^*}) = \Phi(\beta T_{W^*,A^*}(f\otimes 1)) = \Phi(\beta T_{W^*,A^*})f$. Hence the above left diagram commutes if and only if the left rectangle of the following commute.

$$\begin{tikzcd} V^* \arrow[r, "\Phi(\alpha T_{V^*,A^*})"] \arrow[d, "f"] & \mathcal{H}om^*(A^*, V^*) \arrow[r, "\eta_{\mathcal{H}om^*(A^*,V^*)}"] \arrow[d, "f_*"] & \mathcal{H}om^*(A^*, V^*)^\frown \arrow[r, "(\hat{\varphi}^{A^*}_{V^*})^{-1}"] \arrow[d, "\widehat{f_*}"] & A^{**}\,\widehat{\otimes}_{K^*} V^* \arrow[r, "\widehat{T}_{A^{**},V^*}"] \arrow[d, "1\,\widehat{\otimes}\,f"] & V^*\,\widehat{\otimes}_{K^*} A^{**} \arrow[d, "f\,\widehat{\otimes}\,1"] \\ W^* \arrow[r, "\Phi(\beta T_{W^*,A^*})"] & \mathcal{H}om^*(A^*, W^*) \arrow[r, "\eta_{\mathcal{H}om^*(A^*,W^*)}"] & \mathcal{H}om^*(A^*, W^*)^\frown \arrow[r, "(\hat{\varphi}^{A^*}_{W^*})^{-1}"] & A^{**}\,\widehat{\otimes}_{K^*} W^* \arrow[r, "\widehat{T}_{A^{**},W^*}"] & W^*\,\widehat{\otimes}_{K^*} A^{**} \end{tikzcd}$$

Since the other rectangles commute and maps $\eta_{\mathcal{H}om^*(A^*,W^*)}$, $(\hat{\varphi}^{A^*}_{W^*})^{-1}$ and $\widehat{T}_{A^{**},W^*}$ are monomorphisms, the commutativity of the left rectangle is equivalent to the commutativity of the outer rectangle. Thus the assertion follows from the definition of Λ. $\square$

Definition 5.2.6 *Let C^* be a coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with coproduct $\delta : C^* \to C^*\,\widehat{\otimes}_{K^*} C^*$ and counit $\varepsilon : C^* \to K^*$. A right C^*-comodule in $\mathcal{T}op\mathcal{M}od_{K^*}$ is a pair (M^*, λ) of an object M^* and a morphism $\lambda : M^* \to M^*\,\widehat{\otimes}_{K^*} C^*$ of $\mathcal{T}op\mathcal{M}od_{K^*}$ such that the following diagrams commute.*

$$\begin{tikzcd} M^* \arrow[r, "\lambda"] \arrow[d, "\lambda"] & M^*\,\widehat{\otimes}_{K^*} C^* \arrow[d, "\lambda\,\widehat{\otimes}\,1"] \\ M^*\,\widehat{\otimes}_{K^*} C^* \arrow[r, "1\,\widehat{\otimes}\,\delta"] & M^*\,\widehat{\otimes}_{K^*} C^*\,\widehat{\otimes}_{K^*} C^* \end{tikzcd} \qquad \begin{tikzcd} M^* \arrow[r, "\lambda"] \arrow[d, "i_1"] & M^*\,\widehat{\otimes}_{K^*} C^* \arrow[d, "1\,\widehat{\otimes}_{K^*}\,\varepsilon"] \\ M^* \otimes_{K^*} K^* \arrow[r, "\eta_{M^*\otimes_{K^*}K^*}"] & M^*\,\widehat{\otimes}_{K^*} K^* \end{tikzcd}$$

Let A^* be a proper Hopf algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with multiplication $\mu : A^* \otimes_{K^*} A^* \to A^*$ and comultiplication $\delta : A^* \to A^*\,\widehat{\otimes}_{K^*} A^*$. The following result shows that a left A^*-algebra has a structure of a right A^{**}-comodule algebra if $\hat{\varphi}^{A^*}_{M^*} : A^{**}\,\widehat{\otimes}_{K^*} M^* \to \mathcal{H}om^*(A^*, M^*)^\frown$ is an isomorphism.

Proposition 5.2.7 *Let M^* be a Hausdorff left A^*-module with structure map $\alpha : A^* \otimes_{K^*} M^* \to M^*$. Suppose that M^* has a structure of K^*-algebra with multiplication $\nu : M^* \otimes_{K^*} M^* \to M^*$ which is a homomorphism of left A^*-modules. If $\hat{\varphi}^{A^*}_{M^*} : A^{**}\,\widehat{\otimes}_{K^*} M^* \to \mathcal{H}om^*(A^*, M^*)^\frown$ is an isomorphism, $\Lambda(\alpha) : M^* \to M^*\,\widehat{\otimes}_{K^*} A^{**}$ is a K^*-algebra homomorphism. Namely, the following diagram commutes.*

$$\begin{tikzcd} M^* \otimes_{K^*} M^* \arrow[r, "\Lambda(\alpha)\otimes\Lambda(\alpha)"] \arrow[d, "\nu"] & (M^*\,\widehat{\otimes}_{K^*} A^{**}) \otimes_{K^*} (M^*\,\widehat{\otimes}_{K^*} A^{**}) \arrow[r, "sh"] & (M^* \otimes_{K^*} M^*)\,\widehat{\otimes}_{K^*} (A^{**} \otimes_{K^*} A^{**}) \arrow[d, "1\,\widehat{\otimes}\,\tilde{\mu}_{K^*{}_*}\phi"] \\ M^* \arrow[r, "\Lambda(\alpha)"] & M^*\,\widehat{\otimes}_{K^*} A^{**} & (M^* \otimes_{K^*} M^*)\,\widehat{\otimes}_{K^*}\,\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \arrow[l, "\nu\,\widehat{\otimes}\,\delta^*"'] \end{tikzcd}$$

Proof. The following diagram commutes by the assumption.

$$\begin{tikzcd} A^* \otimes_{K^*} M^* \arrow[d, "\alpha"] & A^* \otimes_{K^*} (M^* \otimes_{K^*} M^*) \arrow[l, "1_{A^*}\otimes\nu"'] \arrow[r, "\delta\otimes 1_{M^*\otimes_{K^*}M^*}"] & (A^* \otimes_{K^*} A^*) \otimes_{K^*} (M^* \otimes_{K^*} M^*) \arrow[d, "1_{A^*}\otimes T_{A^*,M^*}\otimes 1_{M^*}"] \\ M^* & M^* \otimes_{K^*} M^* \arrow[l, "\nu"'] & (A^* \otimes_{K^*} M^*) \otimes_{K^*} (A^* \otimes_{K^*} M^*) \arrow[l, "\alpha\otimes\alpha"'] \end{tikzcd}$$

$$\begin{tikzcd} A^*\,\widehat{\otimes}_{K^*} (M^*\,\widehat{\otimes}_{K^*} M^*) \arrow[r, "\delta\,\widehat{\otimes}\,1_{M^*\,\widehat{\otimes}_{K^*}\,M^*}"] \arrow[d, "1_{A^*}\,\widehat{\otimes}\,\nu"] & (A^*\,\widehat{\otimes}_{K^*} A^*)\,\widehat{\otimes}_{K^*} (M^*\,\widehat{\otimes}_{K^*} M^*) \arrow[d, "1_{A^*}\,\widehat{\otimes}\,\widehat{T}_{A^*,M^*}\,\widehat{\otimes}\,1_{M^*}"] \\ A^*\,\widehat{\otimes}_{K^*} M^* \arrow[d, "\hat{\alpha}"] & (A^*\,\widehat{\otimes}_{K^*} M^*)\,\widehat{\otimes}_{K^*} (A^*\,\widehat{\otimes}_{K^*} M^*) \arrow[d, "\hat{\alpha}\,\widehat{\otimes}\,\hat{\alpha}"] \\ M^* & M^*\,\widehat{\otimes}_{K^*} M^* \arrow[l, "\hat{\nu}"'] \end{tikzcd}$$

By (3.2.1), (4.1.2) and the commutativity of the above diagram, we have

$$\begin{aligned}
\Phi(\alpha T_{M^*,A^*})\nu &= \Phi(\alpha T_{M^*,A^*}(\nu\otimes 1_{A^*})) = \Phi(\alpha(1_{A^*}\otimes\nu)T_{M^*\otimes_{K^*}M^*,A^*})\\
&= \Phi(\nu(\alpha\otimes\alpha)(1_{A^*}\otimes T_{A^*,M^*}\otimes 1_{M^*})(\delta\otimes 1_{M^*\otimes_{K^*}M^*})T_{M^*\otimes_{K^*}M^*,A^*})\\
&= \nu_*\Phi((\alpha\otimes\alpha)(1_{A^*}\otimes T_{A^*,M^*}\otimes 1_{M^*})T_{M^*\otimes_{K^*}M^*,A^*\otimes_{K^*}A^*}(1_{M^*\otimes_{K^*}M^*}\otimes\delta))\\
&= \nu_*\delta^*\Phi((\alpha T_{M^*,A^*}\otimes\ \alpha T_{M^*,A^*})(1_{M^*}\otimes T_{M^*,A^*}\otimes 1_{A^*}))\\
&= \nu_*\delta^*\phi(\Phi(\alpha T_{M^*,A^*})\otimes\Phi(\alpha T_{M^*,A^*})).
\end{aligned}$$

Since $\Lambda(\alpha)=\widehat{T}_{A^{**},M^*}\left(\hat{\varphi}^{A^*}_{M^*}\right)^{-1}\eta_{\mathcal{H}om^*(A^*,M^*)}\Phi(\alpha T_{M^*,A^*})$, it follows that

$$\begin{aligned}
\Lambda(\alpha)\nu &= \widehat{T}_{A^{**},M^*}\left(\hat{\varphi}^{A^*}_{M^*}\right)^{-1}\eta_{\mathcal{H}om^*(A^*,M^*)}\nu_*\delta^*\phi(\Phi(\alpha T_{M^*,A^*})\otimes\Phi(\alpha T_{M^*,A^*}))\\
&= \widehat{T}_{A^{**},M^*}\left(\hat{\varphi}^{A^*}_{M^*}\right)^{-1}\hat{\nu}_*\hat{\delta}^*\hat{\phi}\eta_{\mathcal{H}om^*(A^*,M^*)\otimes_{K^*}\mathcal{H}om^*(A^*,M^*)}(\Phi(\alpha T_{M^*,A^*})\otimes\Phi(\alpha T_{M^*,A^*}))\\
&= \widehat{T}_{A^{**},M^*}\left(\hat{\varphi}^{A^*}_{M^*}\right)^{-1}\hat{\nu}_*\hat{\delta}^*\hat{\phi}(\eta_{\mathcal{H}om^*(A^*,M^*)}\,\widehat{\otimes}\,\eta_{\mathcal{H}om^*(A^*,M^*)})^{-1}\eta_{\mathcal{H}om^*(A^*,M^*)^\frown\otimes_{K^*}\mathcal{H}om^*(A^*,M^*)^\frown}\\
&\quad(\eta_{\mathcal{H}om^*(A^*,M^*)}\otimes\eta_{\mathcal{H}om^*(A^*,M^*)})(\Phi(\alpha T_{M^*,A^*})\otimes\Phi(\alpha T_{M^*,A^*}))\\
&= \widehat{T}_{A^{**},M^*}\left(\hat{\varphi}^{A^*}_{M^*}\right)^{-1}\hat{\nu}_*\hat{\delta}^*\hat{\phi}(\eta_{\mathcal{H}om^*(A^*,M^*)}\,\widehat{\otimes}\,\eta_{\mathcal{H}om^*(A^*,M^*)})^{-1}\eta_{\mathcal{H}om^*(A^*,M^*)^\frown\otimes_{K^*}\mathcal{H}om^*(A^*,M^*)^\frown}\\
&\quad\left(\hat{\varphi}^{A^*}_{M^*}\widehat{T}_{M^*,A^{**}}\otimes\hat{\varphi}^{A^*}_{M^*}\widehat{T}_{M^*,A^{**}}\right)(\Lambda(\alpha)\otimes\Lambda(\alpha))\\
&= \widehat{T}_{A^{**},M^*}\left(\hat{\varphi}^{A^*}_{M^*}\right)^{-1}\hat{\nu}_*\hat{\delta}^*\hat{\phi}(\eta_{\mathcal{H}om^*(A^*,M^*)}\,\widehat{\otimes}\,\eta_{\mathcal{H}om^*(A^*,M^*)})^{-1}\left(\hat{\varphi}^{A^*}_{M^*}\,\widehat{\otimes}\,\hat{\varphi}^{A^*}_{M^*}\right)\\
&\quad\left(\widehat{T}_{M^*,A^{**}}\,\widehat{\otimes}\,\widehat{T}_{M^*,A^{**}}\right)\eta_{(M^*\,\widehat{\otimes}_{K^*}\,A^{**})\otimes_{K^*}(M^*\,\widehat{\otimes}_{K^*}\,A^{**})}(\Lambda(\alpha)\otimes\Lambda(\alpha))\quad\cdots(*)
\end{aligned}$$

The following diagram 8 commutes by the naturality of $\hat{\varphi}^{A^*}$.

$$\begin{tikzcd}
(M^*\otimes_{K^*}M^*)\,\widehat{\otimes}_{K^*}\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*) \arrow[r, "\nu\,\widehat{\otimes}\,\delta^*"] \arrow[d, "\widehat{T}_{M^*\otimes_{K^*}M^*,\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}"] & M^*\,\widehat{\otimes}_{K^*}A^{**} \arrow[r, "\widehat{T}_{M^*,A^{**}}"] & A^{**}\,\widehat{\otimes}_{K^*}M^* \arrow[d, "\hat{\varphi}^{A^*}_{M^*}"] \\
\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)\,\widehat{\otimes}_{K^*}(M^*\otimes_{K^*}M^*) \arrow[r, "\hat{\varphi}^{A^*\otimes_{K^*}A^*}_{M^*\otimes_{K^*}M^*}"] & \mathcal{H}om^*(A^*\otimes_{K^*}A^*,M^*\otimes_{K^*}M^*)^\frown \arrow[r, "\hat{\nu}_*\hat{\delta}^*"] & \mathcal{H}om^*(A^*,M^*)^\frown
\end{tikzcd}$$

diagram 8

It follows from the definition (2.3.7) of sh and the commutativity of the first diagram of (4.2.14) that following diagram 9 commutes.

$$\begin{tikzcd}
(M^*\,\widehat{\otimes}_{K^*}A^{**})\otimes_{K^*}(M^*\,\widehat{\otimes}_{K^*}A^{**}) \arrow[r, "sh"] \arrow[d, "\eta_{(M^*\,\widehat{\otimes}_{K^*}\,A^{**})\otimes_{K^*}(M^*\,\widehat{\otimes}_{K^*}\,A^{**})}"] & (M^*\otimes_{K^*}M^*)\,\widehat{\otimes}_{K^*}(A^{**}\otimes_{K^*}A^{**}) \arrow[d, "\eta_{M^*\otimes_{K^*}M^*}\,\widehat{\otimes}\,\eta_{A^{**}\otimes_{K^*}A^{**}}"] \\
(M^*\,\widehat{\otimes}_{K^*}A^{**})\,\widehat{\otimes}_{K^*}(M^*\,\widehat{\otimes}_{K^*}A^{**}) \arrow[r, "1_{M^*}\,\widehat{\otimes}\,\widehat{T}_{A^{**},M^*}\,\widehat{\otimes}\,1_{A^{**}}"] \arrow[d, "\widehat{T}_{M^*,A^{**}}\,\widehat{\otimes}\,\widehat{T}_{M^*,A^{**}}"] & (M^*\,\widehat{\otimes}_{K^*}M^*)\,\widehat{\otimes}_{K^*}(A^{**}\,\widehat{\otimes}_{K^*}A^{**}) \arrow[d, "1_{M^*\,\widehat{\otimes}_{K^*}\,M^*}\,\widehat{\otimes}\,\hat{\phi}"] \\
(A^{**}\,\widehat{\otimes}_{K^*}M^*)\,\widehat{\otimes}_{K^*}(A^{**}\,\widehat{\otimes}_{K^*}M^*) \arrow[d, "\hat{\varphi}^{A^*}_{M^*}\,\widehat{\otimes}\,\hat{\varphi}^{A^*}_{M^*}"] & (M^*\,\widehat{\otimes}_{K^*}M^*)\,\widehat{\otimes}_{K^*}\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*\otimes_{K^*}K^*)^\frown \arrow[d, "1_{M^*\,\widehat{\otimes}_{K^*}\,M^*}\,\widehat{\otimes}\,\widehat{\tilde{\mu}_{K^*}}_*"] \\
\mathcal{H}om^*(A^*,M^*)^\frown\,\widehat{\otimes}_{K^*}\mathcal{H}om^*(A^*,M^*)^\frown \arrow[d, "(\eta_{\mathcal{H}om^*(A^*,M^*)}\,\widehat{\otimes}\,\eta_{\mathcal{H}om^*(A^*,M^*)})^{-1}"] & (M^*\,\widehat{\otimes}_{K^*}M^*)\,\widehat{\otimes}_{K^*}\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)^\frown \arrow[d, "\left(\eta_{M^*\otimes_{K^*}M^*}\,\widehat{\otimes}\,\eta_{\mathcal{H}om^*(A^*\otimes_{K^*},M^*)}\right)^{-1}"] \\
\mathcal{H}om^*(A^*,M^*)\,\widehat{\otimes}_{K^*}\mathcal{H}om^*(A^*,M^*) \arrow[d, "\hat{\phi}"] & (M^*\otimes_{K^*}M^*)\,\widehat{\otimes}_{K^*}\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*) \arrow[d, "\widehat{T}_{M^*\otimes_{K^*}M^*,\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}"] \\
\mathcal{H}om^*(A^*\otimes_{K^*}A^*,M^*\otimes_{K^*}M^*)^\frown & \mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)\,\widehat{\otimes}_{K^*}(M^*\otimes_{K^*}M^*) \arrow[l, "\hat{\varphi}^{A^*\otimes_{K^*}A^*}_{M^*\otimes_{K^*}M^*}"']
\end{tikzcd}$$

diagram 9

Therefore we have

$$
\begin{aligned}
(*) &= \widehat{T}_{A^{**},M^*}\left(\hat{\varphi}^{A^*}_{M^*}\right)^{-1}\hat{\nu}_*\hat{\delta}^*\hat{\varphi}^{A^*\otimes_{K^*}A^*}_{M^*\otimes_{K^*}M^*}\widehat{T}_{M^*\otimes_{K^*}M^*,\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}\left(\eta_{M^*\otimes_{K^*}M^*}\widehat{\otimes}\,\eta_{\mathcal{H}om^*(A^*\otimes_{K^*},M^*)}\right)^{-1} \\
&\quad \left(1_{M^*\widehat{\otimes}_{K^*}M^*}\widehat{\otimes}\,\widehat{\tilde{\mu}_{K^*_*}}\right)\left(1_{M^*\widehat{\otimes}_{K^*}M^*}\widehat{\otimes}\,\hat{\phi}\right)\left(\eta_{M^*\otimes_{K^*}M^*}\widehat{\otimes}\,\eta_{A^{**}\otimes_{K^*}A^{**}}\right)sh(\Lambda(\alpha)\otimes\Lambda(\alpha)) \\
&= \widehat{T}_{A^{**},M^*}\left(\hat{\varphi}^{A^*}_{M^*}\right)^{-1}\hat{\nu}_*\hat{\delta}^*\hat{\varphi}^{A^*\otimes_{K^*}A^*}_{M^*\otimes_{K^*}M^*}\widehat{T}_{M^*\otimes_{K^*}M^*,\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)}\left(1_{M^*\otimes_{K^*}M^*}\widehat{\otimes}\,\tilde{\mu}_{K^*_*}\phi\right)sh(\Lambda(\alpha)\otimes\Lambda(\alpha)) \\
&= \left(\nu\,\widehat{\otimes}\,\delta^*\right)\left(1_{M^*\otimes_{K^*}M^*}\widehat{\otimes}\,\tilde{\mu}_{K^*_*}\phi\right)sh(\Lambda(\alpha)\otimes\Lambda(\alpha))
\end{aligned}
$$

and this completes the proof. □

For an algebra A^* (resp. coalgebra C^*) in $\mathcal{T}op\mathcal{M}od_{K^*}$, let us denote by $\mathcal{M}od(A^*)$ (resp. $\mathcal{C}omod(C^*)$) the category of left A^*-modules (resp. right C^*-comodules).

Suppose that A^* is an algebra in $\mathcal{T}op\mathcal{M}od^i_{K^*}$ such that both (A^*,K^*) and $(A^*\otimes_{K^*}A^*,K^*)$ are very nice pairs (for example, K^* is a field and both A^* and $A^*\otimes_{K^*}A^*$ are supercofinite). If an object M^* of $\mathcal{T}op\mathcal{M}od^i_{K^*}$ is a left A^*-module with structure map $\alpha : A^*\otimes_{K^*}M^*\to M^*$, it follows from (5.2.4) that $\Lambda(\alpha) : M^*\to M^*\widehat{\otimes}_{K^*}A^{**}$ is a structure map of right A^{**}-comodule. We denote by $\mathcal{M}od_i(A^*)$ (resp. $\mathcal{C}omod_i(C^*)$) the full subcategory of $\mathcal{M}od(A^*)$ (resp. $\mathcal{C}omod(C^*)$) consisting of objects whose topologies are coarser than the topology induced by K^*. Then, we have a functor $\Gamma : \mathcal{M}od_i(A^*)\to\mathcal{C}omod_i(A^{**})$ defined by $\Gamma(M^*,\alpha) = (M^*,\Lambda(\alpha))$ and $\Gamma(f) = f$. Let us denote by $\mathcal{M}od_\Lambda(A^*)$ (resp. $\mathcal{C}omod_\Lambda(C^*)$) the full subcategory of $\mathcal{M}od_i(A^*)$ (resp. $\mathcal{C}omod_i(C^*)$) consisting of objects (M^*,α) (resp. (M^*,φ)) such that $\Lambda : \mathrm{Hom}^c_{K^*}(A^*\otimes_{K^*}M^*,M^*)\to\mathrm{Hom}^c_{K^*}(M^*,M^*\widehat{\otimes}_{K^*}A^{**})$ is an isomorphism. Clearly, Γ induces $\Gamma_\Lambda : \mathcal{M}od_\Lambda(A^*)\to\mathcal{C}omod_\Lambda(A^{**})$.

Proposition 5.2.8 *Let A^* be an algebra in $\mathcal{T}op\mathcal{M}od^i_{K^*}$ such that A^* and $A^*\otimes_{K^*}A^*$ are supercofinite. Then, $\Gamma_\Lambda : \mathcal{M}od_\Lambda(A^*)\to\mathcal{C}omod_\Lambda(A^{**})$ is an isomorphism of categories.*

Proof. Define a functor $\mathcal{C}omod_\Lambda(A^{**})\to\mathcal{M}od_\Lambda(A^*)$ by $(M^*,\gamma)\mapsto(M^*,\Lambda^{-1}(\gamma))$. This is the inverse of Γ_Λ. □

Suppose that K^* is a field such that $K^i = \{0\}$ for $i\neq 0$ and that A^* is coconnective, finite type and superskeletal. Then, $A^*\otimes_{K^*}A^*$ is superskeletal by (2.1.20), hence A^* and $A^*\otimes_{K^*}A^*$ are supercofinite by (1.4.6). Let us denote by $\mathcal{M}od_{cfs}(A^*)$ (resp. $\mathcal{C}omod_{cfs}(C^*)$) the full subcategory of $\mathcal{M}od_i(A^*)$ (resp. $\mathcal{C}omod_i(C^*)$) consisting of objects (M^*,α) (resp. (M^*,φ)) such that M^* is coconnective and finite type and has the skeletal topology. Then, $\mathcal{M}od_{cfs}(A^*)$ (resp. $\mathcal{C}omod_{cfs}(C^*)$) is a full subcategory of $\mathcal{M}od_\Lambda(A^*)$ (resp. $\mathcal{C}omod_\Lambda(C^*)$) by (4.2.5) and Γ induces $\Gamma_{cfs} : \mathcal{M}od_{cfs}(A^*)\to\mathcal{C}omod_{cfs}(A^{**})$. Thus we have the following result.

Theorem 5.2.9 *Let A^* be an algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ such that A^* is coconnective and superskeletal. Then, $\Gamma_{cfs} : \mathcal{M}od_{cfs}(A^*)\to\mathcal{C}omod_{cfs}(A^{**})$ is an isomorphism of categories.*

For a Hopf algebra A^* in $\mathcal{T}op\mathcal{M}od_{K^*}$, we denote by $\mathcal{A}(A^*)$ (resp. $\mathcal{CA}(A^*)$) the category of left A^*-algebras (resp. right A^*-comodule algebras). $\mathcal{A}_i(A^*)$ (resp. $\mathcal{CA}_i(A^*)$) denotes the full subcategory of $\mathcal{A}(A^*)$ (resp. $\mathcal{CA}(A^*)$) consisting of objects whose topologies are coarser than the topology induced by K^*.

We assume again that K^* is a field such that $K^i = \{0\}$ for $i\neq 0$ and that A^* is coconnective, finite type and superskeletal. If a M^* is a left A^*-algebra with structure map $\alpha : A^*\otimes_{K^*}M^*\to M^*$, it follows from (5.2.7) that $\Lambda(\alpha) : M^*\to M^*\widehat{\otimes}_{K^*}A^{**}$ is a structure map of right A^{**}-comodule algebra. Thus we have a functor $\Gamma : \mathcal{A}_i(A^*)\to\mathcal{CA}_i(A^{**})$ defined by $\Gamma(M^*,\alpha) = (M^*,\Lambda(\alpha))$ and $\Gamma(f) = f$.

Moreover, $\mathcal{A}_{cfs}(A^*)$ (resp. $\mathcal{CA}_{cfs}(A^*)$) denotes the full subcategory of $\mathcal{A}_i(A^*)$ (resp. $\mathcal{CA}_i(A^*)$) consisting of objects (M^*,α) (resp. (M^*,φ)) such that M^* is coconnective and finite type and has the skeletal topology. Then, (4.2.5) imply the following result.

Theorem 5.2.10 *Let A^* be a Hopf algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ which is coconnective and has the the skeletal topology. Then, the functor $\mathcal{A}_{cfs}(A^*)\to\mathcal{CA}_{cfs}(A^{**})$ given by $(M^*,\alpha)\mapsto(M^*,\Lambda(\alpha))$ induces an isomorphism of categories from $\mathcal{A}_{cfs}(A^*)$ to $\mathcal{CA}_{cfs}(A^*)$.*

Proposition 5.2.11 *Let A^* be a Hopf algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ such that the coproduct $\hat{\delta} : A^*\to A^*\widehat{\otimes}_{K^*}A^*$ lifts to $\delta : A^*\to A^*\otimes_{K^*}A^*$ and $\hat{\varphi} : M^*\to M^*\widehat{\otimes}_{K^*}A^*$ a right A^*-comodule such that $\hat{\varphi}$ lifts to $\varphi : M^*\to M^*\otimes_{K^*}A^*$. If M^* is a finitely generated free K^*-module, there exists a finitely generated Hopf subalgebra B^* of A^* such that M^* is a right B^*-comodule, that is, there there exists a map $\psi : M^*\to M^*\widehat{\otimes}_{K^*}B^*$ satisfying $(id_{M^*}\widehat{\otimes}_{K^*}\iota)\psi = \varphi$, where $\iota : B^*\to A^*$ denotes the inclusion map.*

Proof. Let $v_1, v_2, \ldots, v_n$ be a basis of M^*. Put $\varphi(v_j) = \sum_{i=1}^{n} v_i \otimes a_{ij}$ and B^* be the subalgebra of A^* genarated by $\{a_{ij} |\, i, j = 1, 2, \ldots, n\}$. Since

$$\sum_{i=1}^{n} v_i \otimes \delta(a_{ij}) = (1_{M^*} \otimes \delta)\varphi(v_j) = (\varphi \otimes 1_{A^*})\varphi(v_j) = \sum_{k=1}^{n} \varphi(v_k) \otimes a_{kj} = \sum_{i=1}^{n} v_i \otimes \left(\sum_{k=1}^{n} a_{ik} \otimes a_{kj}\right),$$

we have $\delta(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj}$. Hence B^* is a Hopf subalgebra and M^* is a right B^*-comodule. □

Remark 5.2.12 *If M^* is a finitely generated free K^*-module and A^* is complete, it follows from (2.3.9) that $M^* \otimes_{K^*} A^*$ is complete. Hence a comodule structure of M^* always lifts to $M^* \to M^* \otimes_{K^*} A^*$.*

6 Study on fibered categories

6.1 Fibered categories

First, we review the notion of fibered category.

Let $p : \mathcal{F} \to \mathcal{T}$ be a functor. For an object X of $\mathcal{T}$, we denote by $\mathcal{F}_X$ the subcategory of $\mathcal{F}$ consiting of objects M of $\mathcal{F}$ satisfying $p(M) = X$ and morphisms φ satisfying $p(\varphi) = id_X$. For a morphism $f : X \to Y$ in $\mathcal{T}$ and $M \in \mathrm{Ob}\,\mathcal{F}_X$, $N \in \mathrm{Ob}\,\mathcal{F}_Y$, we put $\mathcal{F}_f(M,N) = \{\varphi \in \mathcal{F}(M,N) \,|\, p(\varphi) = f\}$.

Definition 6.1.1 *([9], p.161 Définition 5.1.) Let $\alpha : M \to N$ be a morphism in $\mathcal{F}$ and set $X = p(M)$, $Y = p(N)$, $f = p(\alpha)$. We call α a cartesian morphism if, for any $M' \in \mathrm{Ob}\,\mathcal{F}_X$, the map $\mathcal{F}_X(M', M) \to \mathcal{F}_f(M', N)$ defined by $\varphi \mapsto \alpha\varphi$ is bijective.*

The following assertion is immediate from the definition.

Proposition 6.1.2 *Let $\alpha_i : M_i \to N_i$ $(i = 1, 2)$ be morphisms in $\mathcal{F}$ such that $p(M_1) = p(M_2)$, $p(N_1) = p(N_2)$, $p(\alpha_1) = p(\alpha_2)$ and $\lambda : N_1 \to N_2$ a morphism in $\mathcal{F}$ such that $p(\lambda) = id_{p(N_1)}$. If α_2 is cartesian, there is a unique morphism $\mu : M_1 \to M_2$ such that $p(\mu) = id_{p(M_1)}$ and $\alpha_2\mu = \lambda\alpha_1$.*

Corollary 6.1.3 *If $\alpha_i : M_i \to N$ $(i = 1, 2)$ are cartesian morphisms in $\mathcal{F}$ such that $p(M_1) = p(M_2)$ and $p(\alpha_1) = p(\alpha_2)$, there is a unique morphism $\mu : M_1 \to M_2$ such that $\alpha_1 = \alpha_2\mu$ and $p(\mu) = id_{p(M_1)}$. Moreover, μ is an isomorphism.*

Definition 6.1.4 *([9], p.162 Définition 5.1.) Let $f : X \to Y$ be a morphism in $\mathcal{T}$ and $N \in \mathrm{Ob}\,\mathcal{F}_Y$. If there exists a cartesian morphism $\alpha : M \to N$ such that $p(\alpha) = f$, M is called an inverse image of N by f. We denote M by $f^*(N)$ and α by $\alpha_f(N) : f^*(N) \to N$. By (6.1.3), $f^*(N)$ is unique up to isomorphism.*

Remark 6.1.5 *For an identity morphism id_X of $X \in \mathrm{Ob}\,\mathcal{T}$ and $N \in \mathrm{Ob}\,\mathcal{F}_X$, the identity morphism id_N of N is obviously cartesian. Hence the inverse image of N by the identity morphism of X always exists and $\alpha_{id_X}(N) : id_X^*(N) \to N$ can be chosen as the identity morphism of N. By the uniqueness of $id_X^*(N)$ up to isomorphism, $\alpha_{id_X}(N) : id_X^*(N) \to N$ is an isomorphism for any choice of $id_X^*(N)$.*

The following assertion is also immediate.

Proposition 6.1.6 *Let $f : X \to Y$ be a morphism in $\mathcal{T}$. If, for any $N \in \mathrm{Ob}\,\mathcal{F}_Y$, there exists a cartesian morphism $\alpha_f(N) : f^*(N) \to N$, $N \mapsto f^*(N)$ defines a functor $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ such that, for any morphism $\varphi : N \to N'$ in $\mathcal{F}_Y$, the following square commutes.*

$$\begin{tikzcd}
f^*(N) \arrow[r, "\alpha_f(N)"] \arrow[d, "f^*(\varphi)"] & N \arrow[d, "\varphi"] \\
f^*(N') \arrow[r, "\alpha_f(N')"] & N'
\end{tikzcd}$$

Definition 6.1.7 *([9], p.162 Définition 5.1.) If the assumption of (6.1.6) is satisfied, we say that the functor of the inverse image by f exists.*

Definition 6.1.8 *([9], p.164 Définition 6.1.) If a functor $p : \mathcal{F} \to \mathcal{T}$ satisfies the following condition (i), p is called a prefibered category and if p satisfies both (i) and (ii), p is called a fibered category or p is fibrant.*

(i) For any morphism f in $\mathcal{T}$, the functor of the inverse image by f exists.
(ii) The composition of cartesian morphisms is cartesian.

Example 6.1.9 *Let Δ^1 be a category given by $\mathrm{Ob}\,\Delta^1 = \{0, 1\}$ and $\mathrm{Mor}\,\Delta^1 = \{id_0, id_1, 0 \to 1\}$. For a category $\mathcal{E}$, we set $\mathcal{E}^{(2)} = \mathrm{Funct}(\Delta^1, \mathcal{E})$. Then, an object of $\mathcal{E}^{(2)}$ is identified with a morphism $(R \xrightarrow{\eta} A)$ in $\mathcal{E}$ and a morphism from $(R \xrightarrow{\eta} A)$ to $(S \xrightarrow{\iota} B)$ in $\mathcal{E}^{(2)}$ is identified with a pair (f, φ) of morphisms $f : R \to S$ and $\varphi : A \to B$ in $\mathcal{E}$ satisfying $\iota f = \varphi\eta$.*

(1) Let $p : \mathcal{E}^{(2)} \to \mathcal{E}$ be the evaluation functor E_0 at 0. For a morphism $f : R \to S$ in $\mathcal{E}$, consider the functor $f^* : \mathcal{E}^{(2)}_S \to \mathcal{E}^{(2)}_R$ given by $f^*(S \xrightarrow{\eta} B) = (R \xrightarrow{\eta f} B)$ and $f^*(id_S, \varphi) = (id_R, \varphi)$. We define a morphism $\alpha_f(S \xrightarrow{\eta} B) : f^*(S \xrightarrow{\eta} B) \to (S \xrightarrow{\eta} B)$ to be (f, id_B). Then, for $(R \xrightarrow{\iota} A) \in \mathrm{Ob}\,\mathcal{E}^{(2)}_R$, the map

$\mathcal{E}^{(2)}_R((R \xrightarrow{\iota} A), f^*(S \xrightarrow{\eta} B)) \to \mathcal{E}^{(2)}_f((R \xrightarrow{\iota} A), (S \xrightarrow{\eta} B))$ *given by* $(id_R, \varphi) \mapsto \alpha_f(S \xrightarrow{\eta} B)(id_R, \varphi) = (f, \varphi)$ *is bijective. Hence* $\alpha_f(S \xrightarrow{\eta} B)$ *is cartesian. Let* $g : Q \to R$ *be a morphism in* $\mathcal{E}$*. Then,*

$$\alpha_f(S \xrightarrow{\eta} B)\alpha_g(f^*(S \xrightarrow{\eta} B)) = (f, id_B)(g, id_B) = (fg, id_B) = \alpha_{fg}(S \xrightarrow{\eta} B),$$

hence $c_{f,g}(S \xrightarrow{\eta} B)$ *is the identity morphism of* $g^*f^*(S \xrightarrow{\eta} B) = (Q \xrightarrow{\eta fg} B) = (fg)^*(S \xrightarrow{\eta} B)$*. Thus* $p : \mathcal{E}^{(2)} \to \mathcal{E}$ *is a fibered category.*

(2) Suppose that $\mathcal{E}$ *has finite limits. Let* $p : \mathcal{E}^{(2)} \to \mathcal{E}$ *be the evaluation functor* E_1 *at* 1*. For* $(f : X \to Y) \in \operatorname{Mor}\mathcal{E}$ *and* $(N \xrightarrow{\pi} Y) \in \operatorname{Ob}\mathcal{E}^{(2)}_Y$*, consider the following cartesian square.*

$$\begin{tikzcd} N \times_Y X \arrow[r, "f_\pi"] \arrow[d, "\pi_f"] & N \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then, $(f, f_\pi) : (N \times_Y X \xrightarrow{\pi_f} X) \to (N \xrightarrow{\pi} Y)$ *induces a bijection*

$$\mathcal{E}^{(2)}_X((M \xrightarrow{\rho} X), (N \times_Y X \xrightarrow{\pi_f} X)) \to \mathcal{E}^{(2)}_f((M \xrightarrow{\rho} X), (N \xrightarrow{\pi} Y)).$$

Hence (f, f_π) *is a cartesian morphism and we have a functor* $f^* : \mathcal{E}^{(2)}_Y \to \mathcal{E}^{(2)}_X$ *which is given by* $f^*(N \xrightarrow{\pi} Y) = (N \times_Y X \xrightarrow{\pi_f} X)$ *and* $f^*(id_Y, \varphi) = (id_X, \varphi \times_Y id_X)$*, where* $(id_Y, \varphi) : (N \xrightarrow{\pi} Y) \to (N' \xrightarrow{\pi'} Y)$ *is a morphism of* $\mathcal{E}^{(2)}_Y$ *and* $\varphi \times_Y id_X : N \times_Y X \to N' \times_Y X$ *is the unique morphism that satisfies* $\pi'_f(\varphi \times_Y id_X) = \pi_f$ *and* $f_\pi(\varphi \times_Y id_X) = f_{\pi'}\varphi$*. For morphisms* $f : X \to Y$*,* $g : Z \to X$ *in* $\mathcal{E}$ *and an object* $N \xrightarrow{\pi} Y$ *of* $\mathcal{E}^{(2)}$*,*

$$c_{f,g}(N \xrightarrow{\pi} Y) : (fg)^*(N \xrightarrow{\pi} Y) \to g^*f^*(N \xrightarrow{\pi} Y)$$

is the isomorphism induced by $(id_N \times_Y g, pr_2) : N \times_Y Z \to (N \times_Y X) \times_X Z$*. Hence* $p : \mathcal{E}^{(2)} \to \mathcal{E}$ *is a fibered category.*

Definition 6.1.10 *([9], p.170 Définition 7.1.) Let* $p : \mathcal{F} \to \mathcal{T}$ *be a functor. A map*

$$\kappa : \operatorname{Mor}\mathcal{T} \to \coprod_{X,Y \in \operatorname{Ob}\mathcal{T}} \operatorname{Funct}(\mathcal{F}_Y, \mathcal{F}_X)$$

is called a cleavage if $\kappa(f)$ *is an inverse image functor* $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ *for* $(f : X \to Y) \in \operatorname{Mor}\mathcal{T}$*. A cleavage* κ *is said to be normalized if* $\kappa(id_X) = id_{\mathcal{F}_X}$ *for any* $X \in \operatorname{Ob}\mathcal{T}$*. A category* $\mathcal{F}$ *over* $\mathcal{T}$ *is called a cloven prefibered category (resp. normalized cloven prefibered category) if a cleavage (resp. normalized cleavage) is given.*

$p : \mathcal{F} \to \mathcal{T}$ has a cleavage if and only if p is prefibered. If p is prefibered, p has a normalized cleavage by (6.1.5).

Let $f : X \to Y$, $g : Z \to X$ be morphisms in $\mathcal{T}$ and N an object of $\mathcal{F}_Y$. If $p : \mathcal{F} \to \mathcal{T}$ is a prefibered category, there is a unique morphism $c_{f,g}(N) : g^*f^*(N) \to (fg)^*(N)$ such that the following square commutes and $p(c_{f,g}(N)) = id_Z$.

$$\begin{tikzcd} g^*f^*(N) \arrow[r, "\alpha_g(f^*(N))"] \arrow[d, "c_{f,g}(N)"] & f^*(N) \arrow[d, "\alpha_f(N)"] \\ (fg)^*(N) \arrow[r, "\alpha_{fg}(N)"] & N \end{tikzcd}$$

Then, we see the following.

Proposition 6.1.11 *For a morphism* $\varphi : M \to N$ *in* $\mathcal{F}_Y$*, the following square commutes.*

$$\begin{tikzcd} g^*f^*(M) \arrow[r, "c_{f,g}(M)"] \arrow[d, "g^*f^*(\varphi)"] & (fg)^*(M) \arrow[d, "(fg)^*(\varphi)"] \\ g^*f^*(N) \arrow[r, "c_{f,g}(N)"] & (fg)^*(N) \end{tikzcd}$$

In other words, $c_{f,g}$ gives a natural transformation $g^*f^* \to (fg)^*$ of functors from $\mathcal{F}_Y$ to $\mathcal{F}_Z$.

Proof. In fact,

$$\begin{aligned}\alpha_{fg}(N)(fg)^*(\varphi)c_{f,g}(M) &= \varphi\alpha_{fg}(M)c_{f,g}(M) = \varphi\alpha_f(M)\alpha_g(f^*(M)) = \alpha_f(N)f^*(\varphi)\alpha_g(f^*(M)) \\ &= \alpha_f(N)\alpha_g(f^*(N))g^*f^*(\varphi) = \alpha_{fg}(N)c_{f,g}(N)g^*f^*(\varphi).\end{aligned}$$

Since $\alpha_{fg}(N)$ is cartesian and $p((fg)^*(\varphi)c_{f,g}(M)) = p(c_{f,g}(N)g^*f^*(\varphi)) = id_Z$, the assertion follows. □

Remark 6.1.12 *Suppose that $p : \mathcal{F} \to \mathcal{T}$ is a normalized prefibered category. For a morphism $f : X \to Y$ of $\mathcal{T}$ and $N \in \operatorname{Ob}\mathcal{F}_Y$, since $\alpha_{id_X}(f^*(N))$ is the identity morphism of $f^*(N)$, $c_{f,id_X}(N)$ is also the identity morphism of $f^*(N)$. Similarly, since $\alpha_{id_Y}(N)$ is the identity morphism of N, $c_{id_Y,f}(N)$ is the identity morphism of $f^*(N)$.*

Proposition 6.1.13 *([9], p.172 Proposition 7.2.) Let $p : \mathcal{F} \to \mathcal{T}$ be a cloven prefibered category. Then, p is a fibered category if and only if $c_{f,g}(N)$ is an isomorphism for any $Z \xrightarrow{g} X \xrightarrow{f} Y$ and $N \in \operatorname{Ob}\mathcal{F}_Y$.*

Proof. Suppose that p is a fibered category. Then, both $\alpha_{fg}(N)$ and $\alpha_f(N)\alpha_g(f^*(N))$ are cartesian morphisms such that $p(\alpha_{fg}(N)) = p(\alpha_f(N)\alpha_g(f^*(N))) = fg$. Hence by (6.1.3), $c_{f,g}(N)$ is an isomorphism.

Conversely, suppose that $c_{f,g}(N)$ is an isomorphism for any $Z \xrightarrow{g} X \xrightarrow{f} Y$ and $N \in \operatorname{Ob}\mathcal{F}_X$. Let $\alpha : M \to N$ and $\beta : L \to M$ be a cartesian morphisms in $\mathcal{F}$. Put $p(M) = X$, $p(N) = Y$, $p(L) = Z$, $p(\alpha) = f$ and $p(\beta) = g$. There is a unique morphism $\zeta : L \to (fg)^*(N)$ such that $\alpha_{fg}(N)\zeta = \alpha\beta$ and $p(\zeta) = id_Z$. There are isomorphisms $\psi : M \to f^*(N)$ and $\xi : L \to g^*(M)$ such that $\alpha = \alpha_f(N)\psi$, $\beta = \alpha_g(M)\xi$ and $p(\psi) = id_X$, $p(\xi) = id_Z$. By (6.1.6), $\alpha_g(f^*(N))g^*(\psi) = \psi\alpha_g(M)$. Hence $\alpha_{fg}(N)c_{f,g}(N)g^*(\psi)\xi = \alpha_f(N)\alpha_g(f^*(N))g^*(\psi)\xi = \alpha_f(N)\psi\alpha_g(M)\xi = \alpha\beta$ and $p(c_{f,g}(N)g^*(\psi)\xi) = id_Z$. By the uniqueness of ζ, $c_{f,g}(N)g^*(\psi)\xi = \zeta$. Thus ζ is an isomorphism and it follows that $\alpha\beta$ is cartesian. □

Proposition 6.1.14 *([9], p.172 Proposition 7.4.) Let $p : \mathcal{F} \to \mathcal{T}$ be a cloven prefibered category. For a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathcal{T}$ and an object M of $\mathcal{F}_W$, we have $c_{h,id_Z}(M) = \alpha_{id_Z}(id_Z^*h^*(M))$, $c_{id_W,h}(M) = h^*(\alpha_{id_W}(M))$ and the following diagram commutes.*

$$\begin{tikzcd}
(f^*g^*)h^*(M) \arrow[r, "c_{g,f}(h^*(M))"] \arrow[d, equal] & (gf)^*h^*(M) \arrow[r, "c_{h,gf}(M)"] & (h(gf))^*(M) \arrow[d, equal] \\
f^*(g^*h^*)(M) \arrow[r, "f^*(c_{h,g}(M))"] & f^*(hg)^*(M) \arrow[r, "c_{hg,f}(M)"] & ((hg)f)^*(M)
\end{tikzcd}$$

Proof. By the definition of $c_{h,id_Z}(M)$, we have $\alpha_h(M)c_{h,id_Z}(M) = \alpha_h(M)\alpha_{id_Z}(id_Z^*h^*(M))$. On the other hand, $\alpha_h(M)c_{id_W,h}(M) = \alpha_{id_W}(M)\alpha_h(id_W^*(M))) = \alpha_h(M)h^*(\alpha_{id_W}(M))$ by the definition of $c_{id_W,h}(M)$ and $h^*(\alpha_{id_W}(M))$. Since $\alpha(M)$ is cartesian and

$$p(c_{h,id_Z}(M)) = p(\alpha_{id_Z}(id_Z^*h^*(M))) = p(c_{id_W,h}(M)) = p(h^*(\alpha_{id_W}(M))) = id_Z,$$

it follows $c_{h,id_Z}(M) = \alpha_{id_Z}(id_Z^*h^*(M))$ and $c_{id_W,h}(M) = h^*(\alpha_{id_W}(M))$. Similarly, since

$$\begin{aligned}\alpha_{hgf}(M)c_{h,gf}(M)c_{g,f}(h^*(M)) &= \alpha_h(M)\alpha_{gf}(h^*(M))c_{g,f}(h^*(M)) = \alpha_h(M)\alpha_g(h^*(M))\alpha_f(g^*h^*(M)) \\ &= \alpha_{hg}(M)c_{h,g}(M)\alpha_f(g^*h^*(M)) = \alpha_{hg}(M)\alpha_f((hg)^*(M))f^*(c_{h,g}(M)) \\ &= \alpha_{hgf}(M)c_{hg,f}(M)f^*(c_{h,g}(M)),\end{aligned}$$

we have $c_{h,gf}(M)c_{g,f}(h^*(M)) = c_{hg,f}(M)f^*(c_{h,g}(M))$. □

Let $p : \mathcal{F} \to \mathcal{E}$, $q : \mathcal{G} \to \mathcal{C}$ be normalized cloven fibered categories and $F : \mathcal{E} \to \mathcal{C}$, $\Phi : \mathcal{F} \to \mathcal{G}$ functors such that $q\Phi = Fp$. For a morphism $f : X \to Y$ of $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, since $\alpha_{F(f)}(\Phi(M)) : F(f)^*(\Phi(M)) \to \Phi(M)$ is a cartesian morphism mapped to $F(f)$ by q and $\Phi(\alpha_f(M)) : \Phi(f^*(M)) \to \Phi(M)$ also mapped to $F(f)$ by q, there exists unique morphism $c_{f,\Phi}(M) : \Phi(f^*(M)) \to F(f)^*(\Phi(M))$ of $\mathcal{G}_{F(X)}$ that makes the following diagram commute.

$$\begin{tikzcd}
\Phi(f^*(M)) \arrow[r, "\Phi(\alpha_f(M))"] \arrow[d, "c_{f,\Phi}(M)"] & \Phi(M) \\
F(f)^*(\Phi(M)) \arrow[ur, "\alpha_{F(f)}(\Phi(M))"'] &
\end{tikzcd}$$

We note that Φ preserves cartesian morphisms if and only if $c_{f,\Phi}(M)$ is an isomorphism for any morphism $f : X \to Y$ of $\mathcal{E}$ and any object M of $\mathcal{F}_Y$.

Proposition 6.1.15 *For a morphism $\varphi : M \to N$ of $\mathcal{F}_Y$, the following digram is commutative.*

$$\begin{tikzcd}
\Phi(f^*(M)) \arrow[r, "\Phi(f^*(\varphi))"] \arrow[d, "c_{f,\Phi}(M)"] & \Phi(f^*(N)) \arrow[d, "c_{f,\Phi}(N)"] \\
F(f)^*(\Phi(M)) \arrow[r, "F(f)^*(\Phi(\varphi))"] & F(f)^*(\Phi(N))
\end{tikzcd}$$

Proof. It follows from (6.1.6) that the lower middle rectangle and the outer trapezoid of the following diagram are commutative. The triangles of the both sides are also commutative by the definition of $c_{f,\Phi}(M)$ and $c_{f,\Phi}(N)$.

$$\begin{tikzcd}
F(f)^*(\Phi(M)) \arrow[rrr, "F(f)^*(\Phi(\varphi))"] \arrow[ddr, "\alpha_{F(f)}(\Phi(M))"'] & & & F(f)^*(\Phi(N)) \arrow[ddl, "\alpha_{F(f)}(\Phi(N))"] \\
& \Phi(f^*(M)) \arrow[r, "\Phi(f^*(\varphi))"] \arrow[ul, "c_{f,\Phi}(M)"'] \arrow[d, "\Phi(\alpha_f(M))"] & \Phi(f^*(N)) \arrow[ur, "c_{f,\Phi}(N)"] \arrow[d, "\Phi(\alpha_f(N))"'] & \\
& \Phi(M) \arrow[r, "\Phi(\varphi)"] & \Phi(N) &
\end{tikzcd}$$

Hence we have

$$\alpha_{F(f)}(\Phi(M))c_{f,\Phi}(N)\Phi(f^*(\varphi)) = \alpha_{F(f)}(\Phi(M))F(f)^*(\Phi(\varphi))c_{f,\Phi}(M).$$

Since both $c_{f,\Phi}(N)\Phi(f^*(\varphi))$ and $F(f)^*(\Phi(\varphi))c_{f,\Phi}(M)$ are morphisms of $\mathcal{G}_{F(X)}$ and $\alpha_{F(f)}(\Phi(M))$ is a cartesian morphism, the above equality implies the result. $\square$

Proposition 6.1.16 *For morphisms $f : X \to Y$, $k : V \to X$ of $\mathcal{E}$ and $M \in \operatorname{Ob}\mathcal{F}_Y$, the following diagram is commutative.*

$$\begin{tikzcd}
\Phi(k^*(f^*(M))) \arrow[r, "c_{k,\Phi}(f^*(M))"] \arrow[d, "\Phi(c_{f,k}(M))"] & F(k)^*(\Phi(f^*(M))) \arrow[r, "F(k)^*(c_{f,\Phi}(M))"] & F(k)^*(F(f)^*(\Phi(M))) \arrow[d, "c_{F(f),F(k)}(\Phi(M))"] \\
\Phi((fk)^*(M)) \arrow[rr, "c_{fk,\Phi}(M)"] & & F(fk)^*(\Phi(M))
\end{tikzcd}$$

Proof. The inner triangles are all commutative by (6.1.6) and definitions of $c_{f,k}(M)$, $c_{k,\Phi}(f^*(M))$, $c_{f,\Phi}(M)$, $c_{F(f),F(k)}(\Phi(M))$, $c_{fk,\Phi}(M)$.

$$\begin{tikzcd}
\Phi(k^*(f^*(M))) \arrow[rr, "c_{k,\Phi}(f^*(M))"] \arrow[dr, "\Phi(\alpha_k(f^*(M)))"] \arrow[ddd, "\Phi(c_{f,k}(M))"] & & F(k)^*(\Phi(f^*(M))) \arrow[dl, "\alpha_{F(k)}(\Phi(f^*(M)))"] \arrow[dr, "F(k)^*(c_{f,\Phi}(M))"] & \\
& \Phi(f^*(M)) \arrow[r, "c_{f,\Phi}(M)"] \arrow[dr, "\Phi(\alpha_f(M))"] & F(f)^*(\Phi(M)) \arrow[d, "\alpha_{F(f)^*}(\Phi(M))"] & F(k)^*(F(f)^*(\Phi(M))) \arrow[l, "\alpha_{F(k)}(F(f)^*(\Phi(M)))"'] \arrow[ddl, "c_{F(f),F(k)}(\Phi(M))"] \\
& & \Phi(M) & \\
\Phi((fk)^*(M)) \arrow[rr, "c_{fk,\Phi}(M)"] \arrow[urr, "\Phi(\alpha_{fk}(M))"] & & F(fk)^*(\Phi(M)) \arrow[u, "\alpha_{F(fk)}(\Phi(M))"] &
\end{tikzcd}$$

Thus we have the following equality.

$$\alpha_{F(fk)}(\Phi(M))c_{F(f),F(k)}(\Phi(M))F(k)^*(c_{f,\Phi}(M))c_{k,\Phi}(f^*(M)) = \alpha_{F(fk)}(\Phi(M))c_{fk,\Phi}(M)\Phi(c_{f,k}(M))$$

Since both $c_{F(f),F(k)}(\Phi(M))F(k)^*(c_{f,\Phi}(M))c_{k,\Phi}(f^*(M))$ and $c_{fk,\Phi}(M)\Phi(c_{f,k}(M))$ are morphisms of $\mathcal{G}_{F(V)}$ and $\alpha_{F(fk)}(\Phi(M))$ is a cartesian morphism, the assertion follows from the above equality. $\square$

Let $p : \mathcal{F} \to \mathcal{T}$ be a normalized cloven fibered category. Assume that $\mathcal{T}$ has a terminal object 1. We denote by $o_X : X \to 1$ the unique morphism of $\mathcal{T}$ for $X \in \operatorname{Ob}\mathcal{T}$. Define a functor $F_X : \mathcal{F}_1^{op} \times \mathcal{F}_1 \to \mathcal{S}et$ by

$F_X(M,N) = \mathcal{F}_X(o_X^*(M), o_X^*(N))$ for $M, N \in \operatorname{Ob}\mathcal{F}_1$ and $F_X(\varphi,\psi) = o_X^*(\varphi)^* o_X^*(\psi)_*$ for $\varphi,\psi \in \operatorname{Mor}\mathcal{F}_1$. For a morphism $f : Y \to X$ of $\mathcal{T}$ and $M, N \in \operatorname{Ob}\mathcal{F}_1$, let $f^\sharp_{M,N} : F_X(M,N) \to F_Y(M,N)$ be the following composition.

$$\begin{aligned}F_X(M,N) = \mathcal{F}_X(o_X^*(M), o_X^*(N)) &\xrightarrow{f^*} \mathcal{F}_Y(f^*(o_X^*(M)), f^*(o_X^*(N))) \xrightarrow{(c_{o_X,f}(M)^{-1})^*} \mathcal{F}_Y((o_Xf)^*(M), f^*(o_X^*(N))) \\ &\xrightarrow{c_{o_X,f}(N)_*} \mathcal{F}_Y((o_Xf)^*(M), (o_Xf)^*(N)) = \mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) = F_Y(M,N)\end{aligned}$$

Let $\varphi : M \to L$ and $\psi : P \to N$ be morphisms of $\mathcal{F}_1$. Since the following diagram is commutative by (6.1.11), $f^\sharp_{M,N}$ is natural in M, N and we have a natural transformation $f^\sharp : F_X \to F_Y$.

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(L), o_X^*(P)) \arrow[r, "f^*"] \arrow[d, "o_X^*(\varphi)^* o_X^*(\psi)_*"] & \mathcal{F}_Y(f^*(o_X^*(L)), f^*(o_X^*(P))) \arrow[r, "c_{o_X,f}(P)_*(c_{o_X,f}(L)^{-1})^*"] \arrow[d, "f^*(o_X^*(\varphi))^* f^*(o_X^*(\psi))_*"] & \mathcal{F}_Y((o_Xf)^*(L), (o_Xf)^*(P)) \arrow[d, "(o_Xf)^*(\varphi)^*(o_Xf)^*(\psi)_*"'] \\
\mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[r, "f^*"] & \mathcal{F}_Y(f^*(o_X^*(M)), f^*(o_X^*(N))) \arrow[r, "c_{o_X,f}(N)_*(c_{o_X,f}(M)^{-1})^*"] & \mathcal{F}_Y((o_Xf)^*(M), (o_Xf)^*(N))
\end{tikzcd}$$

Proposition 6.1.17 *Let $f : Y \to X$ be a morphism of $\mathcal{T}$ and L, M, N objects of $\mathcal{F}_1$.*
(1) For $\zeta \in \mathcal{F}_X(o_X^*(L), o_X^*(M))$ and $\xi \in \mathcal{F}_X(o_X^*(M), o_X^*(N))$, we have $f^\sharp_{L,N}(\xi\zeta) = f^\sharp_{M,N}(\xi) f^\sharp_{L,M}(\zeta)$.

(2) A composition $\mathcal{F}_1(M,N) \xrightarrow{o_X^*} \mathcal{F}_X(o_X^*(M), o_X^*(N)) \xrightarrow{f^\sharp_{M,N}} \mathcal{F}_Y(o_Y^*(M), o_Y^*(N))$ coincides with the map $o_Y^* : \mathcal{F}_1(M,N) \to \mathcal{F}_Y(o_Y^*(M), o_Y^*(N))$. In particular, $f^\sharp_{M,M} : \mathcal{F}_X(o_X^*(M), o_X^*(M)) \to \mathcal{F}_Y(o_Y^*(M), o_Y^*(M))$ maps the identity morphism of $o_X^*(M)$ to the identity morphism of $o_Y^*(M)$.

Proof. (1) The assertion follows from

$$\begin{aligned}f^\sharp_{M,N}(\xi) f^\sharp_{L,M}(\zeta) &= c_{o_X,f}(N) f^*(\xi) c_{o_X,f}(M)^{-1} c_{o_X,f}(M) f^*(\zeta) c_{o_X,f}(L)^{-1} = c_{o_X,f}(N) f^*(\xi) f^*(\zeta) c_{o_X,f}(L)^{-1} \\ &= c_{o_X,f}(N) f^*(\xi\zeta) c_{o_X,f}(L)^{-1} = f^\sharp_{L,N}(\xi\zeta).\end{aligned}$$

(2) The assertion follows from the definition of $k^\sharp$ and (6.1.11). □

Proposition 6.1.18 *For morphisms $f : Y \to X$ and $g : Z \to Y$ of $\mathcal{T}$, $(fg)^\sharp = g^\sharp f^\sharp$.*

Proof. For $M, N \in \operatorname{Ob}\mathcal{F}_1$ and $\xi \in F_X(M,N)$, it follows from (6.1.11) and (6.1.14) that

$$\begin{aligned}g^\sharp_{M,N} f^\sharp_{M,N}(\xi) &= c_{o_Y,g}(N) g^*(c_{o_X,f}(N) f^*(\xi) c_{o_X,f}(M)^{-1}) c_{o_Y,g}(M)^{-1} \\ &= c_{o_Y,g}(N) g^*(c_{o_X,f}(N)) g^*(f^*(\xi)) g^*(c_{o_X,f}(M)^{-1}) c_{o_Y,g}(M)^{-1} \\ &= c_{o_Y,g}(N) g^*(c_{o_X,f}(N)) c_{f,g}(o_X^*(N))^{-1} (fg)^*(\xi) c_{f,g}(o_X^*(M)) g^*(c_{o_X,f}(M)^{-1}) c_{o_Y,g}(M)^{-1} \\ &= c_{o_Y,g}(N) g^*(c_{o_X,f}(N)) c_{f,g}(o_X^*(N))^{-1} (fg)^*(\xi) (c_{o_Y,g}(M) g^*(c_{o_X,f}(M)) c_{f,g}(o_X^*(M))^{-1})^{-1} \\ &= c_{o_X,fg}(N) (fg)^*(\xi) c_{o_X,fg}(M)^{-1} = (fg)^\sharp_{M,N}(\xi).\end{aligned}$$

Hence we have $g^\sharp_{M,N} f^\sharp_{M,N} = (fg)^\sharp_{M,N}$ for any $M, N \in \operatorname{Ob}\mathcal{F}_1$. □

Let $p : \mathcal{F} \to \mathcal{E}$, $q : \mathcal{G} \to \mathcal{C}$ be normalized cloven fibered categories and $F : \mathcal{E} \to \mathcal{C}$, $\Phi : \mathcal{F} \to \mathcal{G}$ functors such that $q\Phi = Fp$. Assume that F preserves terminal objects and Φ preserves cartesian morphisms. For object X of $\mathcal{E}$ and objects M, N of $\mathcal{F}_1$, we denote by $\Phi^X_{M,N}$ a composition

$$\begin{aligned}\mathcal{F}_X(o_X^*(M), o_X^*(N)) &\xrightarrow{\Phi} \mathcal{G}_{F(X)}(\Phi(o_X^*(M)), \Phi(o_X^*(N))) \xrightarrow{(c_{o_X,\Phi}(M)^{-1})^*} \mathcal{G}_{F(X)}(o_{F(X)}^*(\Phi(M)), \Phi(o_X^*(N))) \\ &\xrightarrow{c_{o_X,\Phi}(N)_*} \mathcal{G}_{F(X)}(o_{F(X)}^*(\Phi(M)), o_{F(X)}^*(\Phi(N))).\end{aligned}$$

Proposition 6.1.19 *Let X be an object of $\mathcal{E}$ and M, N, L objects of $\mathcal{F}_1$. For $\varphi \in \mathcal{F}_X(o_X^*(M), o_X^*(N))$ and $\psi \in \mathcal{F}_X(o_X^*(N), o_X^*(L))$, $\Phi^X_{M,L}(\psi\varphi) = \Phi^X_{N,L}(\psi)\Phi^X_{M,N}(\varphi)$ holds.*

Proof. The assertion follows from

$$\begin{aligned}\Phi^X_{N,L}(\psi)\Phi^X_{M,N}(\varphi) &= c_{o_X,\Phi}(L)\Phi(\psi) c_{o_X,\Phi}(N)^{-1} c_{o_X,\Phi}(N)\Phi(\varphi) c_{o_X,\Phi}(M)^{-1} = c_{o_X,\Phi}(L)\Phi(\psi)\Phi(\varphi) c_{o_X,\Phi}(M)^{-1} \\ &= c_{o_X,\Phi}(L)\Phi(\psi\varphi) c_{o_X,\Phi}(M)^{-1} = \Phi^X_{M,L}(\psi\varphi)\end{aligned}$$

□

Proposition 6.1.20 *For a morphism $k : V \to X$ of $\mathcal{E}$ and objects M, N of $\mathcal{F}_1$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[r, "k^\sharp_{M,N}"] \arrow[d, "\Phi^X_{M,N}"] & \mathcal{F}_V(o_V^*(M), o_V^*(N)) \arrow[d, "\Phi^V_{M,N}"] \\
\mathcal{G}_{F(X)}(o^*_{F(X)}(\Phi(M)), o^*_{F(X)}(\Phi(N))) \arrow[r, "F(k)^\sharp_{\Phi(M),\Phi(N)}"] & \mathcal{G}_{F(V)}(o^*_{F(V)}(\Phi(M)), o^*_{F(V)}(\Phi(N)))
\end{tikzcd}$$

Proof. The following diagram is commutative by (6.1.16), (6.1.6) and the definition of $c_{k,\Phi}(o_X^*(M))$.

$$\begin{tikzcd}
\Phi(o_V^*(M)) \arrow[dd, "c_{o_V,\Phi}(M)"] & \Phi(k^*(o_X^*(M))) \arrow[l, "\Phi(c_{o_X,k}(M))"'] \arrow[d, "c_{k,\Phi}(o_X^*(M))"] \arrow[dr, "\Phi(\alpha_k(o_X^*(M)))"] & \\
 & F(k)^*(\Phi(o_X^*(M))) \arrow[r, "\alpha_{F(k)}(\Phi(o_X^*(M)))"] \arrow[d, "F(k)^*(c_{o_X,\Phi}(M))"] & \Phi(o_X^*(M)) \arrow[d, "c_{o_X,\Phi}(M)"] \\
o^*_{F(V)}(\Phi(M)) & F(k)^*(o^*_{F(X)}(\Phi(M))) \arrow[l, "c_{o_{F(X)},F(k)}(\Phi(M))"'] \arrow[r, "\alpha_{F(k)}(o^*_{F(X)}(\Phi(M)))"] & o^*_{F(X)}(\Phi(M))
\end{tikzcd}$$

Hence we have the following equality.

$$\Phi(\alpha_k(o_X^*(M))c_{o_X,k}(M)^{-1})c_{o_V,\Phi}(M)^{-1} = c_{o_X,\Phi}(M)^{-1}\alpha_{F(k)}(o^*_{F(X)}(\Phi(M)))c_{o_{F(X)},F(k)}(\Phi(M))^{-1} \quad \cdots (*)$$

Consider the cartesian morphism $\alpha_{o_{F(V)}}(\Phi(N)) : o^*_{F(V)}(\Phi(N)) \to \Phi(N)$. For $\varphi \in \mathcal{F}_X(o_X^*(M), o_X^*(N))$, we have

$$\begin{aligned}
\alpha_{o_{F(V)}}(\Phi(N))\Phi^V_{M,N}(k^\sharp_{M,N}(\varphi)) &= \alpha_{o_{F(V)}}(\Phi(N))c_{o_V,\Phi}(N)\Phi(k^\sharp_{M,N}(\varphi))c_{o_V,\Phi}(M)^{-1} \\
&= \Phi(\alpha_{o_V}(N))\Phi(k^\sharp_{M,N}(\varphi))c_{o_V,\Phi}(M)^{-1} \\
&= \Phi(\alpha_{o_V}(N)c_{o_X,k}(N)k^*(\varphi)c_{o_X,k}(M)^{-1})c_{o_V,\Phi}(M)^{-1} \\
&= \Phi(\alpha_{o_X}(N)\alpha_k(o_X^*(N))k^*(\varphi)c_{o_X,k}(M)^{-1})c_{o_V,\Phi}(M)^{-1} \\
&= \Phi(\alpha_{o_X}(N))\Phi(\varphi\alpha_k(o_X^*(M))c_{o_X,k}(M)^{-1})c_{o_V,\Phi}(M)^{-1} \\
&= \alpha_{o_{F(X)}}(\Phi(N))c_{o_X,\Phi}(N)\Phi(\varphi)\Phi(\alpha_k(o_X^*(M))c_{o_X,k}(M)^{-1})c_{o_V,\Phi}(M)^{-1} \\
\alpha_{o_{F(V)}}(\Phi(N))F(k)^\sharp_{\Phi(M),\Phi(N)}(\Phi^X_{M,N}(\varphi)) &= \alpha_{o_{F(V)}}(\Phi(N))F(k)^\sharp_{\Phi(M),\Phi(N)}(c_{o_X,\Phi}(N)\Phi(\varphi)c_{o_X,\Phi}(M)^{-1}) \\
&= \alpha_{o_{F(V)}}(\Phi(N))c_{o_{F(X)},F(k)}(\Phi(N))F(k)^*(c_{o_X,\Phi}(N)\Phi(\varphi)c_{o_X,\Phi}(M)^{-1})c_{o_{F(X)},F(k)}(\Phi(M))^{-1} \\
&= \alpha_{o_{F(X)}}(\Phi(N))\alpha_{F(k)}(o^*_{F(X)}(\Phi(N)))F(k)^*(c_{o_X,\Phi}(N)\Phi(\varphi)c_{o_X,\Phi}(M)^{-1})c_{o_{F(X)},F(k)}(\Phi(M))^{-1} \\
&= \alpha_{o_{F(X)}}(\Phi(N))c_{o_X,\Phi}(N)\Phi(\varphi)c_{o_X,\Phi}(M)^{-1}\alpha_{F(k)}(o^*_{F(X)}(\Phi(M)))c_{o_{F(X)},F(k)}(\Phi(M))^{-1}.
\end{aligned}$$

Then, $(*)$ implies $\alpha_{o_{F(V)}}(\Phi(N))(\Phi^V_{M,N}k^\sharp_{M,N}(\varphi)) = \alpha_{o_{F(V)}}(\Phi(N))F(k)^\sharp_{\Phi(M),\Phi(N)}(\Phi^X_{M,N}(\varphi))$. Therefore we have $\Phi^V_{M,N}k^\sharp_{M,N}(\varphi) = F(k)^\sharp_{\Phi(M),\Phi(N)}\Phi^X_{M,N}(\varphi)$. □

6.2 Bifibered category

We briefly review the notion of bifibered category following section 10 of [19].

Definition 6.2.1 *Let $p : \mathcal{F} \to \mathcal{E}$ be a functor and $\alpha : M \to N$ a morphism in $\mathcal{F}$. Set $X = p(M)$, $Y = p(N)$, $f = p(\alpha)$. We call α a cocartesian morphism if, for any $N' \in \operatorname{Ob}\mathcal{F}_Y$, the map $\mathcal{F}_X(N, N') \to \mathcal{F}_f(M, N')$ defined by $\varphi \mapsto \varphi\alpha$ is bijective.*

The following assertion is the dual of (6.1.2).

Proposition 6.2.2 *If $\alpha_i : M \to N_i$ $(i = 1, 2)$ are cocartesian morphisms in $\mathcal{F}$ such that $p(N_1) = p(N_2)$ and $p(\alpha_1) = p(\alpha_2)$, there is a unique morphism $\psi : N_1 \to N_2$ such that $\alpha_1 = \alpha_2\psi$ and $p(\psi) = id_{p(N_1)}$. Moreover, ψ is an isomorphism.*

Definition 6.2.3 *Let $f : X \to Y$ be a morphism in $\mathcal{E}$ and $M \in \operatorname{Ob}\mathcal{F}_X$. If there exists a cocartesian morphism $\alpha : M \to N$ such that $p(\alpha) = f$, N is called a direct image of M by f. We denote M by $f_*(N)$ and α by $\alpha^f(M) : M \to f_*(M)$. By (6.2.2), $f_*(N)$ is unique up to isomorphism.*

Proposition 6.2.4 *Let $\alpha : M \to N$, $\alpha' : M' \to N'$ be morphisms in $\mathcal{F}$ such that $p(M) = p(M')$, $p(N) = p(N')$, $p(\alpha) = p(\alpha')(= f)$ and $\lambda : M \to M'$ a morphism in $\mathcal{F}$ such that $p(\lambda) = id_{p(M)}$. If α' is cocartesian, there is a unique morphism $\mu : N \to N'$ such that $p(\mu) = id_{p(N)}$ and $\alpha'\mu = \lambda\alpha$.*

Corollary 6.2.5 *Let $f : X \to Y$ be a morphism in $\mathcal{E}$. If, for any $M \in \operatorname{Ob}\mathcal{F}_X$, there exists a cocartesian morphism $\alpha^f(M) : M \to f_*(M)$, $M \mapsto f_*(M)$ defines a functor $f_* : \mathcal{F}_X \to \mathcal{F}_Y$.*

Definition 6.2.6 *If the assumption of (6.2.5) is satisfied, we say that the functor of the direct image by f exists.*

Definition 6.2.7 *If a functor $p : \mathcal{F} \to \mathcal{E}$ sadisfies the following condition (i), p is called a precofibered category and if p satisfies both (i) and (ii), p is called a cofibered category or p is cofibrant.*

(i) For any morphism f in $\mathcal{E}$, the functor of the direct image by f exists.
(ii) The composition of cocartesian morphisms is cocartesian.

In other words, $p : \mathcal{F} \to \mathcal{E}$ is a precofibered (resp. cofibered) category if and only if $p : \mathcal{F}^{op} \to \mathcal{E}^{op}$ is a prefibered (resp. fibered) category.

Let $p : \mathcal{F} \to \mathcal{E}$ be a functor. A map $\kappa : \operatorname{Mor}\mathcal{E} \to \coprod_{X,Y \in \operatorname{Ob}\mathcal{E}} \operatorname{Funct}(\mathcal{F}_X, \mathcal{F}_Y)$ is called a cocleavage if $\kappa(f)$ is a direct image functor $f_* : \mathcal{F}_X \to \mathcal{F}_Y$ for $(f : X \to Y) \in \operatorname{Mor}\mathcal{E}$. A cocleavage κ is said to be normalized if $\kappa(id_X) = id_{\mathcal{F}_X}$ for any $X \in \operatorname{Ob}\mathcal{E}$. A category $\mathcal{F}$ over $\mathcal{E}$ is called a cloven precofibered category (resp. normalized cloven precofibered category) if a cocleavage (resp. normalized cocleavage) is given.

$p : \mathcal{F} \to \mathcal{E}$ has a cocleavage if and only if p is precofibered. If p is precofibered, p has a normalized cocleavage.

Let $f : X \to Y$, $g : Z \to X$ be morphisms in $\mathcal{E}$ and M an object of $\mathcal{F}_Z$. If $p : \mathcal{F} \to \mathcal{E}$ is a precofibered category, there is a unique morphism $c^{f,g}(M) : (fg)_*(M) \to f_*g_*(M)$ such that the following square commutes and $p(c_{f,g}(M)) = id_Z$.

$$\begin{tikzcd}
M \arrow[r, "\alpha^{fg}(M)"] \arrow[d, "\alpha^g(M)"'] & (fg)_*(M) \arrow[d, "c^{f,g}(M)"] \\
g_*(M) \arrow[r, "\alpha^f(g_*(M))"] & f_*g_*(M)
\end{tikzcd}$$

The following is the dual of (6.1.10).

Proposition 6.2.8 *Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven precofibered category. Then, p is a cofibered category if and only if $c^{f,g}(M)$ is an isomorphism for any $Z \xrightarrow{g} X \xrightarrow{f} Y$ and $M \in \operatorname{Ob}\mathcal{F}_Z$.*

Proposition 6.2.9 *Let $p : \mathcal{F} \to \mathcal{E}$ be a functor and $f : X \to Y$ a morphism in $\mathcal{E}$.*

(1) Suppose that the functor of the inverse image by f exists. Then, the inverse image $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ by f has a left adjoint if and only if the functor of the direct image by f exists.

(2) Suppose that the functor of the direct image by f exists. Then, the direct image $f_* : \mathcal{F}_X \to \mathcal{F}_Y$ by f has a right adjoint if and only if the functor of the inverse image by f exists.

Proof. (1) Suppose that the functor of the inverse image by f exists and that it has a left adjoint $f_* : \mathcal{F}_X \to \mathcal{F}_Y$. We denote by $\eta : id_{\mathcal{F}_X} \to f^*f_*$ the unit of the adjunction $f_* \dashv f^*$. For $M \in \operatorname{Ob}\mathcal{F}_X$, set $\alpha^f(M) = \alpha_f(f_*(M))\eta_M : M \to f_*(M)$. By the assumption, the following composition is bijective for any $M \in \operatorname{Ob}\mathcal{F}_X$, $N \in \operatorname{Ob}\mathcal{F}_Y$.

$$\mathcal{F}_Y(f_*(M), N) \xrightarrow{f^*} \mathcal{F}_X(f^*f_*(M), f^*(N)) \xrightarrow{\eta_M^*} \mathcal{F}_X(M, f^*(N)) \xrightarrow{\alpha_f(N)_*} \mathcal{F}_f(M, N)$$

We note that, since $\alpha_f(N)f^*(\varphi) = \varphi\alpha_f(f_*(M))$ for $\varphi \in \mathcal{F}_Y(f_*(M), N)$, the above composition coincides with the map $\alpha^f(M)^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_f(M, N)$ induced by $\alpha^f(M)$. This shows that the functor of the direct image by f exists.

Conversely, assume that the functor of the direct image by f exists. For $M \in \operatorname{Ob}\mathcal{F}_X$, let us denote by $\alpha^f(M) : M \to f_*(M)$ a cocartesian morphism. Then, we have bijections $\alpha^f(M)^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_f(M, N)$ and $\alpha_f(M)_* : \mathcal{F}_X(M, f^*(N)) \to \mathcal{F}_f(M, N)$ given by $\psi \mapsto \psi\alpha^f(M)$ and $\varphi \mapsto \alpha_f(M)\varphi$, which are natural in $M \in \operatorname{Ob}\mathcal{F}_X$ and $N \in \operatorname{Ob}\mathcal{F}_Y$. Thus we have a natural bijection $\mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$.

(2) Suppose that the functor of the direct image by f exists and that it has a right adjoint $f^*: \mathcal{F}_Y \to \mathcal{F}_X$. We denote by $\varepsilon : f_*f^* \to id_{\mathcal{F}_Y}$ the counit of the adjunction $f_* \dashv f^*$. For $N \in \operatorname{Ob}\mathcal{F}_Y$, set $\alpha_f(N) = \varepsilon_N\alpha^f(f^*(N)) : f^*(N) \to N$. By the assumption, the following composition is bijective for any $M \in \operatorname{Ob}\mathcal{F}_X$, $N \in \operatorname{Ob}\mathcal{F}_Y$.

$$\mathcal{F}_X(M, f^*(N)) \xrightarrow{f_*} \mathcal{F}_Y(f_*(M), f_*f^*(N)) \xrightarrow{\varepsilon_{N*}} \mathcal{F}_Y(f_*(M), N) \xrightarrow{\alpha^f(M)^*} \mathcal{F}_f(M, N)$$

We note that, since $f_*(\varphi)\alpha^f(M) = \alpha^f(f^*(N))\varphi$ for $\varphi \in \mathcal{F}_X(M, f^*(N))$, the above composition coincides with the map $\alpha_f(N)_* : \mathcal{F}_X(M, f^*(N)) \to \mathcal{F}_f(M, N)$ induced by $\alpha_f(N)$. This shows that the functor of the inverse image by f exists.

Conversely, assume that the functor of the inverse image by f exists. For $N \in \operatorname{Ob}\mathcal{F}_Y$, let us denote by $\alpha_f(N) : f^*(N) \to N$ a cartesian morphism. Then, we have bijections $\alpha_f(N)_* : \mathcal{F}_X(M, f^*(N)) \to \mathcal{F}_f(M, N)$ and $\alpha^f(M)^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_f(M, N)$ given by $\varphi \mapsto \alpha_f(N)\varphi$ and $\psi \mapsto \psi\alpha^f(M)\varphi$, which are natural in $M \in \operatorname{Ob}\mathcal{F}_X$ and $N \in \operatorname{Ob}\mathcal{F}_Y$. Thus we have a natural bijection $\mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$. □

Remark 6.2.10 *Let $p : \mathcal{F} \to \mathcal{E}$ be a functor and $f : X \to Y$ a morphism in $\mathcal{E}$ such that the functors of the inverse and direct images by f exist. For $M \in \operatorname{Ob}\mathcal{F}_X$ and $N \in \mathcal{F}_Y$, since there exist a cartesian morphism $\alpha_f(N) : f^*(N) \to N$ and a cocartesian morphism $\alpha^f(M) : M \to f_*(M)$, there is a bijection $ad_f(M, N) : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$ which satisfies $\alpha_f(N)ad_f(M, N)(\varphi) = \varphi\alpha^f(M)$ for any $\varphi \in \mathcal{F}_Y(f_*(M), N)$. Hence the unit $\eta : id_{\mathcal{F}_X} \to f^*f_*$ of the adjunction $f_* \dashv f^*$ is the unique natural transformation satisfying $\alpha_f(f_*(M))\eta_M = \alpha^f(M)$ for any $M \in \operatorname{Ob}\mathcal{F}_X$. Dually, the counit $\varepsilon : f_*f^* \to id_{\mathcal{F}_Y}$ is the unique natural transformation satisfying $\varepsilon_N\alpha^f(f^*(N)) = \alpha_f(N)$ for any $N \in \operatorname{Ob}\mathcal{F}_Y$.*

Proposition 6.2.11 *([19], p.182 Proposition 10.1.) Let $p : \mathcal{E} \to \mathcal{F}$ be a prefibered and precofibered category. Then, it is a fibered category if and only if it is a cofibered category.*

Proof. For a morphism $f : X \to Y$ in $\mathcal{E}$, we denote by $\eta^f : id_{\mathcal{F}_X} \to f^*f_*$ the unit of the adjunction $f_* \dashv f^*$. Let $f : X \to Y$, $g : Z \to X$ be morphisms in $\mathcal{E}$. For $M \in \operatorname{Ob}\mathcal{F}_Z$ and $N \in \operatorname{Ob}\mathcal{F}_Y$, we claim that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}_X(f^*f_*g_*(M), f^*(N)) \arrow[d, "\eta^{f*}_{g_*(M)}"] & \mathcal{F}_Y(f_*g_*(M), N) \arrow[l, "f^*"'] \arrow[r, "c^{f,g}(M)^*"] & \mathcal{F}_Y((fg)_*(M), N) \arrow[d, "(fg)^*"] \\
\mathcal{F}_X(g_*(M), f^*(N)) \arrow[d, "g^*"] & & \mathcal{F}_Z((fg)^*(fg)_*(M), (fg)^*(N)) \arrow[d, "\eta^{fg*}_M"] \\
\mathcal{F}_Z(g^*g_*(M), g^*f^*(N)) \arrow[r, "\eta^{g*}_M"] & \mathcal{F}_Z(M, g^*f^*(N)) \arrow[r, "c_{f,g}(M)_*"] & \mathcal{F}_Z(M, (fg)^*(N))
\end{tikzcd}$$

Let $\psi : f_*g_*(M) \to N$ be a morphism in $\mathcal{F}_Y$. Then we have

$$\begin{aligned}
\alpha_{fg}(N)\eta^{fg*}_M(fg)^*c^{f,g}(M)^*(\psi) &= \alpha_{fg}(N)(fg)^*(\psi)(fg)^*(c^{f,g}(M))\eta^{fg}_M = \psi\alpha_{fg}(f_*g_*(M))(fg)^*(c^{f,g}(M))\eta^{fg}_M \\
&= \psi c^{f,g}(M)\alpha_{fg}((fg)_*(M))\eta^{fg}_M = \psi c^{f,g}(M)\alpha^{fg}(M) = \psi\alpha^f(g_*(M))\alpha^g(M) \\
&= \psi\alpha_f(f_*g_*(M))\eta^f_{g_*(M)}\alpha_g(g_*(M))\eta^g_M \\
&= \alpha_f(N)f^*(\psi)\alpha_g(f^*f_*g_*(M))g^*(\eta^f_{g_*(M)})\eta^g_M \\
&= \alpha_f(N)\alpha_g(f^*(N))g^*f^*(\psi)g^*(\eta^f_{g_*(M)})\eta^g_M \\
&= \alpha_{fg}(N)c_{f,g}(N)g^*f^*(\psi)g^*(\eta^f_{g_*(M)})\eta^g_M = \alpha_{fg}(N)c_{f,g}(N)_*\eta^{g*}_M g^*\eta^{f*}_{g_*(M)}(\psi).
\end{aligned}$$

Since $\alpha_{fg}(N) : (fg)^*(N) \to N$ is cartesian and both $\eta^{fg*}_M(fg)^*c^{f,g}(M)^*(\psi)$ and $c_{f,g}(N)_*\eta^{g*}_M g^*\eta^{f*}_{g_*(M)}(\psi)$ are morphisms in $\mathcal{F}_Y$, we see that the above diagram commutes. Note that the compositions $\eta^{f*}_M f^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$, $\eta^{g*}_M g^* : \mathcal{F}_X(g_*(M), N) \to \mathcal{F}_Z(M, g^*(N))$ and $\eta^{fg*}_M(fg)^* : \mathcal{F}_Y((fg)_*(M), N) \to \mathcal{F}_Z(M, (fg)^*(N))$ are bijective. Hence, by the commutativity of the above diagram, $c_{f,g}(N)_*$ is bijective if and only if $c^{f,g}(M)^*$ is so. Then the assertion follows from (6.1.10) and (6.2.8). □

Definition 6.2.12 *We call a functor $p : \mathcal{F} \to \mathcal{E}$ a bifibered category if it is a fibered and cofibered category.*

Example 6.2.13 *Let $p : \mathcal{E}^{(2)} \to \mathcal{E}$ be the fibered category given in (2) of (6.1.9). For a morphism $f : X \to Y$ in $\mathcal{E}$, define a functor $f_* : \mathcal{E}^{(2)}_X \to \mathcal{E}^{(2)}_Y$ by $f_*(E \xrightarrow{\pi} X) = (E \xrightarrow{f\pi} Y)$ and*

$$f_*((\varphi, id_X) : (E \xrightarrow{\pi} X) \to (G \xrightarrow{\rho} X)) = ((\varphi, id_Y) : (E \xrightarrow{f\pi} Y) \to (G \xrightarrow{f\rho} Y)).$$

For $(G \xrightarrow{\rho} Y) \in \operatorname{Ob}\mathcal{E}^{(2)}_Y$, let $G \xleftarrow{f_\rho} G \times_Y X \xrightarrow{\rho_f} X$ be a limit of a diagram $G \xrightarrow{\rho} Y \xleftarrow{f} X$. Then,

$$\mathcal{E}^{(2)}_Y(f_*(E \xrightarrow{\pi} X), (G \xrightarrow{\rho} Y)) = \{\varphi \in \mathcal{E}(E, G) \,|\, \rho\varphi = f\pi\}$$
$$\mathcal{E}^{(2)}_X((E \xrightarrow{\pi} X), f^*(G \xrightarrow{\rho} Y)) = \{\psi \in \mathcal{E}(E, G \times_Y X) \,|\, \rho_f\psi = \pi\}$$

and define a map $\Psi : \mathcal{E}^{(2)}_X((E \xrightarrow{\pi} X), f^*(G \xrightarrow{\rho} Y)) \to \mathcal{E}^{(2)}_Y(f_*(E \xrightarrow{\pi} X), (G \xrightarrow{\rho} Y))$ by $\Psi(\psi) = f_\rho\psi$. It is easily seen that Ψ is bijective and f_* is a left adjoint of f^*.

6.3 Fibered category with products

For $X \in \operatorname{Ob}\mathcal{T}$ and $M \in \operatorname{Ob}\mathcal{F}_1$, define a presheaf $F_{X,M} : \mathcal{F}_1 \to \mathcal{S}et$ on $\mathcal{F}_1^{op}$ by

$$F_{X,M}(N) = F_X(M, N) = \mathcal{F}_X(o_X^*(M), o_X^*(N))$$

for $N \in \operatorname{Ob}\mathcal{F}_1$ and $F_{X,M}(\varphi) = o_X^*(\varphi)_*$ for $\varphi \in \operatorname{Mor}\mathcal{F}_1$.

Suppose that $F_{X,M}$ is representable for $X \in \operatorname{Ob}\mathcal{T}$ and $M \in \operatorname{Ob}\mathcal{F}_1$. We choose an object $X \times M$ of $\mathcal{F}_1$ such that there exists a natural equivalence $P_X(M) : F_{X,M} \to \hat{h}_{X\times M}$, where $\hat{h}_{X\times M}$ is the presheaf on $\mathcal{F}_1^{op}$ represented by $X \times M$. Since $o_1^* : \mathcal{F}_1 \to \mathcal{F}_1$ is the identity functor of $\mathcal{F}_1$, we take M as $1 \times M$. Hence $P_1(M)_N$ is the identity map of $\mathcal{F}_1(M, N)$. Let us denote by $\iota_X(M) : o_X^*(M) \to o_X^*(X \times M)$ the morphism of $\mathcal{F}_X$ which is mapped to the identity morphism of $X \times M$ by $P_X(M)_{X\times M} : \mathcal{F}_X(o_X^*(M), o_X^*(X \times M)) \to \mathcal{F}_1(X \times M, X \times M)$.

Remark 6.3.1 *If $o_X^* : \mathcal{F}_1 \to \mathcal{F}_X$ has a left adjoint $o_{X*} : \mathcal{F}_X \to \mathcal{F}_1$, $F_{X,M} : \mathcal{F}_1 \to \mathcal{S}et$ is representable for any object M of $\mathcal{F}_1$. In fact, $X \times M$ is defined to be $o_{X*}o_X^*(M)$ in this case. If we denote by $(\mathrm{ad}_X)_{P,N} : \mathcal{F}_1(o_{X*}(P), N) \to \mathcal{F}_X(P, o_X^*(N))$ the bijection which is natural in $P \in \operatorname{Ob}\mathcal{F}_X$ and $N \in \operatorname{Ob}\mathcal{F}_1$, we have $P_X(M)_N = (\mathrm{ad}_X)^{-1}_{o_X^*(M),N} : \mathcal{F}_X(o_X^*(M), o_X^*(N)) \to \mathcal{F}_1(o_{X*}o_X^*(M), N)$. Let us denote by $\eta_X : id_{\mathcal{F}_X} \to o_X^*o_{X*}$ the unit of the adjunction $o_{X*} \dashv o_X^*$. We have $\iota_X(M) = (\eta_X)_{o_X^*(M)} : o_X^*(M) \to o_X^*o_{X*}o_X^*(M) = o_X^*(X \times M)$.*

Proposition 6.3.2 *The inverse of $P_X(M)_N : \mathcal{F}_X(o_X^*(M), o_X^*(N)) \to \mathcal{F}_1(X\times M, N)$ is given by the map defined by $\varphi \mapsto o_X^*(\varphi)\iota_X(M)$.*

Proof. For $\varphi \in \mathcal{F}_1(X \times M, N)$, the following diagram commutes by naturality of $P_X(M)$.

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(M), o_X^*(X \times M)) \arrow[r, "o_X^*(\varphi)_*"] \arrow[d, "P_X(M)_{X\times M}"] & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[d, "P_X(M)_N"] \\
\mathcal{F}_1(X \times M, X \times M) \arrow[r, "\varphi_*"] & \mathcal{F}_1(X \times M, N)
\end{tikzcd}$$

It follows that $P_X(M)_N$ maps $o_X^*(\varphi)\iota_X(M)$ to φ. □

For a morphism $\varphi : L \to M$ of $\mathcal{F}_1$, define a natural transformation $F_{X,\varphi} : F_{X,M} \to F_{X,L}$ by

$$(F_{X,\varphi})_N = o_X^*(\varphi)^* : F_{X,M}(N) = \mathcal{F}_X(o_X^*(M), o_X^*(N)) \to \mathcal{F}_X(o_X^*(L), o_X^*(N)) = F_{X,L}(N).$$

It is clear that $F_{X,\psi\varphi} = F_{X,\varphi}F_{X,\psi}$ for morphisms $\psi : M \to P$ and $\varphi : L \to M$ of $\mathcal{F}_1$. If $F_{X,L}$ is also representable, we define $X \times \varphi : X \times L \to X \times M$ by

$$X \times \varphi = P_X(L)_{X\times M}((F_{X,\varphi})_{X\times M}(\iota_X(M))) = P_X(L)_{X\times M}(\iota_X(M)o_X^*(\varphi)) \in \hat{h}_{X\times L}(X \times M).$$

Proposition 6.3.3 *Let $\varphi : L \to M$ be a morphism of $\mathcal{F}_1$.*

(1) The following diagrams commute for any $N \in \operatorname{Ob}\mathcal{F}_1$.

$$\begin{tikzcd}
o_X^*(L) \arrow[r, "o_X^*(\varphi)"] \arrow[d, "\iota_X(L)"] & o_X^*(M) \arrow[d, "\iota_X(M)"] \\
o_X^*(X \times L) \arrow[r, "o_X^*(X\times\varphi)"] & o_X^*(X \times M)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[r, "o_X^*(\varphi)^*"] \arrow[d, "P_X(M)_N"] & \mathcal{F}_X(o_X^*(L), o_X^*(N)) \arrow[d, "P_X(L)_N"] \\
\mathcal{F}_1(X \times M, N) \arrow[r, "(X\times\varphi)^*"] & \mathcal{F}_1(X \times L, N)
\end{tikzcd}$$

(2) For morphisms $\psi : M \to K$ and $\varphi : L \to M$ of $\mathcal{F}_1$, we have $X \times (\psi\varphi) = (X \times \psi)(X \times \varphi)$.

(3) If $o_X^* : \mathcal{F}_1 \to \mathcal{F}_X$ preserves epimorphisms (o_X^* has a right adjoint, for example) and $\varphi : L \to M$ is an epimorphism, so is $X \times \varphi : X \times L \to X \times M$.

Proof. (1) We have $P_X(L)_{X\times M}(\iota_X(M)o_X^*(\varphi)) = X \times \varphi$ by the definition of $X \times \varphi$. On the other hand, $P_X(L)_{X\times M}(o_X^*(X \times \varphi)\iota_X(L)) = X \times \varphi$ by (6.3.2). Since $P_X(L)_{X\times M}$ is bijective, the left diagram commutes.

For $\psi \in \mathcal{F}_1(X \times M, N)$, it follows from (6.3.2) and commutativity of the left diagram that we have

$$\begin{aligned} o_X^*(\varphi)^* P_X(M)_N^{-1}(\psi) &= o_X^*(\psi)\iota_X(M)o_X^*(\varphi) = o_X^*(\psi)o_X^*(X \times \varphi)\iota_X(L) = o_X^*(\psi(X \times \varphi))\iota_X(L) \\ &= P_X(L)_N^{-1}(\psi(X \times \varphi)) = P_X(L)_N^{-1}(X \times \varphi)^*(\psi). \end{aligned}$$

Hence the right diagram commutes.

(2) The following diagram commutes by (1).

$$\begin{tikzcd} \mathcal{F}_X(o_X^*(K), o_X^*(X \times K)) \arrow[r, "o_X^*(\psi)^*"] \arrow[d, "P_X(K)_{X\times K}"] & \mathcal{F}_X(o_X^*(M), o_X^*(X \times K)) \arrow[r, "o_X^*(\varphi)^*"] \arrow[d, "P_X(M)_{X\times K}"] & \mathcal{F}_X(o_X^*(L), o_X^*(X \times K))) \arrow[d, "P_X(L)_{X\times K}"] \\ \mathcal{F}_1(X \times K, X \times K) \arrow[r, "(X\times\psi)^*"] & \mathcal{F}_1(X \times M, X \times K) \arrow[r, "(X\times\varphi)^*"] & \mathcal{F}_1(X \times L, X \times K) \end{tikzcd}$$

Hence, by the definition of $X \times (\psi\varphi)$ we have

$$\begin{aligned} (X \times \psi)(X \times \varphi) &= (X \times \varphi)^*(X \times \psi)^*(id_{X\times K}) = (X \times \varphi)^*(X \times \psi)^* P_X(K)_{X\times K}(\iota_X(K)) \\ &= P_X(L)_{X\times K} o_X^*(\varphi)^* o_X^*(\psi)^*(\iota_X(K)) = P_X(L)_{X\times K}(\iota_X(K)o_X^*(\varphi\psi)) = X \times (\psi\varphi). \end{aligned}$$

(3) is a direct consequence of (1). □

Remark 6.3.4 *If $o_X^* : \mathcal{F}_1 \to \mathcal{F}_X$ has a left adjoint $o_{X*} : \mathcal{F}_X \to \mathcal{F}_1$, for a morphism $\varphi : L \to M$ of $\mathcal{F}_1$, we have $X \times \varphi = o_{X*}o_X^*(\varphi) : X \times L = o_{X*}o_X^*(L) \to o_{X*}o_X^*(M) = X \times M$. In fact, if we denote by $\varepsilon_X : o_X^* o_{X*} \to id_{\mathcal{F}_X}$ the counit of the adjunction $o_{X*} \dashv o_X^*$, we have $X \times \varphi = P_X(L)_{X\times M}(\iota_X(M)o_X^*(\varphi)) = (\mathrm{ad}_X)^{-1}_{o_X^*(L), X\times M}((\eta_X)_{o_X^*(M)}o_X^*(\varphi)) = (\varepsilon_X)_{o_{X*}o_X^*(M)}o_{X*}((\eta_X)_{o_X^*(M)})o_{X*}o_X^*(\varphi) = o_{X*}o_X^*(\varphi)$.*

Lemma 6.3.5 *Let $\xi : o_X^*(L) \to o_X^*(M)$, $\zeta : o_X^*(N) \to o_X^*(K)$ be morphisms of $\mathcal{F}_X$ for morphisms $\varphi : L \to N$ and $\psi : M \to K$ of $\mathcal{F}_1$. We put $\hat{\xi} = P_X(L)_M(\xi)$ and $\hat{\zeta} = P_X(N)_K(\zeta)$. The following left diagram commutes if and only if the right one commutes.*

$$\begin{tikzcd} o_X^*(L) \arrow[r, "\xi"] \arrow[d, "o_X^*(\varphi)"] & o_X^*(M) \arrow[d, "o_X^*(\psi)"] \\ o_X^*(N) \arrow[r, "\zeta"] & o_X^*(K) \end{tikzcd} \qquad \begin{tikzcd} X \times L \arrow[r, "\hat{\xi}"] \arrow[d, "X\times\varphi"] & M \arrow[d, "\psi"] \\ X \times N \arrow[r, "\hat{\zeta}"] & K \end{tikzcd}$$

Proof. The following diagram is commutative by (6.3.3).

$$\begin{tikzcd} \mathcal{F}_X(o_X^*(L), o_X^*(M)) \arrow[r, "o_X^*(\psi)_*"] \arrow[d, "P_X(L)_M"] & \mathcal{F}_X(o_X^*(L), o_X^*(K)) \arrow[d, "P_X(L)_K"] & \mathcal{F}_X(o_X^*(N), o_X^*(K)) \arrow[l, "o_X^*(\varphi)^*"'] \arrow[d, "P_X(N)_K"] \\ \mathcal{F}_1(X \times L, M) \arrow[r, "\psi_*"] & \mathcal{F}_1(X \times L, K) & \mathcal{F}_1(X \times N, K) \arrow[l, "(X\times\varphi)^*"'] \end{tikzcd}$$

Since $\hat{\xi} = P_X(L)_M(\xi)$, $\hat{\zeta} = P_X(N)_K(\zeta)$ and $P_X(L)_K$ is bijective, $o_X^*(\psi)\xi = o_X^*(\psi)_*(\xi) = o_X^*(\varphi)^*(\zeta) = \zeta o_X^*(\varphi)$ if and only if $\psi\hat{\xi} = \psi_*(\hat{\xi}) = (X \times \varphi)^*(\hat{\zeta}) = \hat{\zeta}(X \times \varphi)$. □

For $X, Y \in \mathrm{Ob}\,\mathcal{T}$ and $M \in \mathrm{Ob}\,\mathcal{F}_1$, suppose that $F_{X,M}$ and $F_{Y,M}$ are representable. For a morphism $f : X \to Y$ of $\mathcal{T}$, we define a morphism $f \times M : X \times M \to Y \times M$ of $\mathcal{F}_1$ by

$$f \times M = P_X(M)_{Y\times M}(f^\sharp_{M,Y\times M}(\iota_Y(M))).$$

Since $F_{1,M}$ is represented by M, we identify M with $1 \times M$ and o_X induces $o_X \times M : X \times M \to M$.

Proposition 6.3.6 *(1) The following diagram commutes for any $N \in \mathrm{Ob}\,\mathcal{F}_1$.*

$$
\begin{array}{ccc}
\mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) & \xrightarrow{f^\sharp_{M,N}} & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \\
\downarrow{\scriptstyle P_Y(M)_N} & & \downarrow{\scriptstyle P_X(M)_N} \\
\mathcal{F}_1(Y \times M, N) & \xrightarrow{(f\times M)^*} & \mathcal{F}_1(X \times M, N)
\end{array}
$$

(2) For $X, Y, Z \in \operatorname{Ob}\mathcal{T}$ and $M \in \operatorname{Ob}\mathcal{F}_1$, suppose that $F_{X,M}$, $F_{Y,M}$ and $F_{Z,M}$ are representable. For morphisms $f : X \to Y$ and $g : Y \to Z$ of $\mathcal{T}$, we have $gf \times M = (g \times M)(f \times M)$.

(3) The image of the identity morphism of $o_X^*(M)$ by $P_X(M)_M$ is $o_X \times M : X \times M \to 1 \times M = M$.

(4) A compisition $o_X^*(M) \xrightarrow{\iota_X(M)} o_X^*(X \times M) \xrightarrow{o_X^*(o_X \times M)} o_X^*(1 \times M) = o_X^*(M)$ is the identity morphism of $o_X^*(M)$.

Proof. (1) For $\varphi \in \mathcal{F}_1(Y \times M, N)$, it follows from the naturality of $f^\sharp_{M,N}$ and (6.3.2) that we have

$$
\begin{aligned}
f^\sharp_{M,N} P_Y(M)_N^{-1}(\varphi) &= f^\sharp_{M,N}(o_Y^*(\varphi)\iota_Y(M)) = f^\sharp_{M,N} o_Y^*(\varphi)_*(\iota_Y(M)) = o_X^*(\varphi)_* f^\sharp_{M,Y\times M}(\iota_Y(M)) \\
&= o_X^*(\varphi)_* P_X(M)^{-1}_{Y\times M}(f \times M) = o_X^*(\varphi) o_Y^*(f \times M)\iota_X(M) = o_X^*(\varphi(f \times M))\iota_X(M) \\
&= o_X^*((f \times M)^*(\varphi))\iota_X(M) = P_X(M)_N^{-1}(f \times M)^*(\varphi).
\end{aligned}
$$

(2) The following diagram commutes by (1). Hence the assertion follows from (6.1.18).

$$
\begin{array}{ccccc}
\mathcal{F}_Z(o_Y^*(M), o_Y^*(N)) & \xrightarrow{g^\sharp_{M,N}} & \mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) & \xrightarrow{f^\sharp_{M,N}} & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \\
\downarrow{\scriptstyle P_Z(M)_N} & & \downarrow{\scriptstyle P_Y(M)_N} & & \downarrow{\scriptstyle P_X(M)_N} \\
\mathcal{F}_1(Z \times M, N) & \xrightarrow{(g\times M)^*} & \mathcal{F}_1(Y \times M, N) & \xrightarrow{(f\times M)^*} & \mathcal{F}_1(X \times M, N)
\end{array}
$$

(3) Apply (1) for $N = M$, $Y = 1$ and $f = o_X : X \to 1$.

(4) It follows from (6.3.2) that $P_X(M)_M : \mathcal{F}_X(o_X^*(M), o_X^*(M)) \to \mathcal{F}_1(X \times M, M)$ maps $o_X^*(o_X \times M)\iota_X(M)$ to $o_X \times M : X \times M \to M$. Thus the assertion follows from (3). □

Remark 6.3.7 *Suppose that the inverse image functors $o_X^* : \mathcal{F}_1 \to \mathcal{F}_X$ and $o_Y^* : \mathcal{F}_1 \to \mathcal{F}_Y$ have left adjoints $o_{X*} : \mathcal{F}_X \to \mathcal{F}_1$ and $o_{Y*} : \mathcal{F}_Y \to \mathcal{F}_1$, respectively.*

(1) Since $f^\sharp_{M,Y\times M}(\iota_Y(M)) = c_{o_Y,f}(Y \times M) f^*\big((\eta_Y)_{o_Y^*(M)}\big) c_{o_X,f}(M)^{-1}$ by (6.3.1) and

$$
P_X(M)_{Y\times M} = (\mathrm{ad}_X)^{-1}_{o_X^*(M), Y\times M} : \mathcal{F}_X(o_X^*(M), o_X^*(Y \times M)) \to \mathcal{F}_1(X \times M, Y \times M)
$$

maps $\varphi \in \mathcal{F}_X(o_X^*(M), o_X^*(Y \times M))$ to $(\varepsilon_X)_{Y\times M} o_{X*}(\varphi)$, $f \times M : X \times M \to Y \times M$ coincides with the following composition.

$$
\begin{aligned}
X \times M = o_{X*}o_X^*(M) &\xrightarrow{o_{X*}(c_{o_X,f}(M))^{-1}} o_{X*}f^*o_Y^*(M) \xrightarrow{o_{X*}f^*\left((\eta_Y)_{o_Y^*(M)}\right)} o_{X*}f^*o_Y^*o_{Y*}o_Y^*(M) = o_{X*}f^*o_Y^*(Y \times M) \\
&\xrightarrow{o_{X*}(c_{o_X,f}(Y\times M))} o_{X*}o_X^*(Y \times M) \xrightarrow{(\varepsilon_X)_{Y\times M}} Y \times M
\end{aligned}
$$

(2) The following diagram commutes by (6.3.6).

$$
\begin{array}{ccc}
\mathcal{F}_1(o_{1*}(o_1^*(M)), M) & \xrightarrow{(o_X\times M)^*} & \mathcal{F}_1(o_{X*}(o_X^*(M)), M) \\
\downarrow{\scriptstyle \mathrm{ad}^1_{o_1^*(M),M}} & & \downarrow{\scriptstyle \mathrm{ad}^X_{o_X^*(M),M}} \\
\mathcal{F}_1(o_1^*(M), o_1^*(M)) & \xrightarrow{(o_X^\sharp)_{M,M}} & \mathcal{F}_X(o_X^*(M), o_X^*(M))
\end{array}
$$

Since o_1^* is the identity functor of $\mathcal{F}_1$, so is o_{1*}. Hence $o_X \times M : o_{X*}o_X^*(M) = X \times M \to 1 \times M = M$ is identified with the counit $(\varepsilon_X)_M : o_{X*}o_X^*(M) \to M$ of the adjunction $o_{X*} \dashv o_X^*$ by the above diagram.

Lemma 6.3.8 *For a morphism $f : X \to Y$ of $\mathcal{T}$ and an object M of $\mathcal{F}_1$,*

$$
f^\sharp_{M,Y\times M} : \mathcal{F}_Y(o_Y^*(M), o_Y^*(Y \times M)) \to \mathcal{F}_X(o_X^*(M), o_X^*(Y \times M))
$$

maps $\iota_Y(M)$ to $o_X^*(f \times M)\iota_X(M)$.

Proof. The following diagram commutes by (1) of (6.3.6).

$$\begin{tikzcd}
\mathcal{F}_Y(o_Y^*(M), o_Y^*(Y \times M)) \arrow[r, "f^\sharp_{M,Y\times M}"] \arrow[d, "P_Y(M)_{Y\times M}"] & \mathcal{F}_X(o_X^*(M), o_X^*(Y \times M)) \arrow[d, "P_X(M)_{Y\times M}"] \\
\mathcal{F}_1(Y \times M, Y \times M) \arrow[r, "(f\times M)^*"] & \mathcal{F}_1(X \times M, Y \times M)
\end{tikzcd}$$

The assertion follows from (6.3.2). □

Proposition 6.3.9 *For a morphism $f : X \to Y$ of $\mathcal{T}$ and a morphism $\varphi : L \to M$ of $\mathcal{F}_1$, the following diagram commutes.*

$$\begin{tikzcd}
X \times L \arrow[r, "f\times L"] \arrow[d, "X\times\varphi"] & Y \times L \arrow[d, "Y\times\varphi"] \\
X \times M \arrow[r, "f\times M"] & Y \times M
\end{tikzcd}$$

Proof. The following diagram commutes by the naturality of $f^\sharp$.

$$\begin{tikzcd}
\mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) \arrow[r, "f^\sharp_{M,N}"] \arrow[d, "o_Y^*(\varphi)^*"] & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[d, "o_X^*(\varphi)^*"] \\
\mathcal{F}_Y(o_Y^*(L), o_Y^*(N)) \arrow[r, "f^\sharp_{L,N}"] & \mathcal{F}_X(o_X^*(L), o_X^*(N))
\end{tikzcd}$$

Then, it follows from the commutativity of four diagrams

$$\begin{tikzcd}
\mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) \arrow[r, "P_Y(M)_N"] \arrow[d, "o_Y^*(\varphi)^*"] & \mathcal{F}_1(Y \times M, N) \arrow[d, "(Y\times\varphi)^*"] \\
\mathcal{F}_Y(o_Y^*(L), o_Y^*(N)) \arrow[r, "P_Y(L)_N"] & \mathcal{F}_1(Y \times L, N)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[r, "P_X(M)_N"] \arrow[d, "o_X^*(\varphi)^*"] & \mathcal{F}_1(X \times M, N) \arrow[d, "(X\times\varphi)^*"] \\
\mathcal{F}_X(o_X^*(L), o_X^*(N)) \arrow[r, "P_X(L)_N"] & \mathcal{F}_1(X \times L, N)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) \arrow[r, "P_Y(M)_N"] \arrow[d, "f^\sharp_{M,N}"] & \mathcal{F}_1(Y \times M, N) \arrow[d, "(f\times M)^*"] \\
\mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[r, "P_X(M)_N"] & \mathcal{F}_1(X \times M, N)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_Y(o_Y^*(L), o_Y^*(N)) \arrow[r, "P_Y(L)_N"] \arrow[d, "f^\sharp_{L,N}"] & \mathcal{F}_1(Y \times L, N) \arrow[d, "(f\times L)^*"] \\
\mathcal{F}_X(o_X^*(L), o_X^*(N)) \arrow[r, "P_X(L)_N"] & \mathcal{F}_1(Y \times L, N)
\end{tikzcd}$$

and the fact that $P_Y(M)_N : \mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) \to \mathcal{F}_1(Y \times M, N)$ is bijective that the following diagram commutes for any $N \in \operatorname{Ob}\mathcal{F}_1$.

$$\begin{tikzcd}
\mathcal{F}_1(Y \times M, N) \arrow[r, "(f\times M)^*"] \arrow[d, "(Y\times\varphi)^*"] & \mathcal{F}_1(X \times M, N) \arrow[d, "(X\times\varphi)^*"] \\
\mathcal{F}_1(Y \times L, N) \arrow[r, "(f\times L)^*"] & \mathcal{F}_1(X \times L, N)
\end{tikzcd}$$

Thus the assertion follows. □

Remark 6.3.10 *We denote by $f \times \varphi : X \times L \to Y \times M$ the composition $(f \times M)(X \times \varphi) = (Y \times \varphi)(f \times L)$. It follows from (6.3.9) that $f \times (g \times M) = (Y \times (g \times M))(f \times (Z \times M)) = (f \times (W \times M))(X \times (g \times M))$ for $g \in \mathcal{T}(Z, W)$.*

For $X \in \operatorname{Ob}\mathcal{T}$ and $M \in \operatorname{Ob}\mathcal{F}_1$, we define a morphism $\delta_{X,M} : X \times M \to X \times (X \times M)$ of $\mathcal{F}_1$ to be the image of $\iota_X(X \times M)\iota_X(M) \in \mathcal{F}_X(o_X^*(M), o_X^*(X \times (X \times M)))$ by

$$P_X(M)_{X\times(X\times M)} : \mathcal{F}_X(o_X^*(M), o_X^*(X \times (X \times M))) \to \mathcal{F}_1(X \times M, X \times (X \times M)).$$

Proposition 6.3.11 *The following diagram commutes for any $N \in \operatorname{Ob}\mathcal{F}_1$.*

$$\begin{array}{ccc}\mathcal{F}_X(o_X^*(X\times M),o_X^*(N)) & \xrightarrow{\iota_X(M)^*} & \mathcal{F}_X(o_X^*(M),o_X^*(N))\\ \downarrow{\scriptstyle P_X(X\times M)_N} & & \downarrow{\scriptstyle P_X(M)_N}\\ \mathcal{F}_1(X\times(X\times M),N) & \xrightarrow{\delta_{X,M}^*} & \mathcal{F}_1(X\times M,N)\end{array}$$

Proof. For $\varphi\in\mathcal{F}_1(X\times(X\times M),N)$, by the definition of $\delta_{X,M}$ and the naturality of $P_X(M)$, we have

$$\begin{aligned}\iota_X(M)^*P_X(X\times M)_N^{-1}(\varphi)&=o_X^*(\varphi)\iota_X(X\times M)\iota_X(M)=o_X^*(\varphi)_*P_X(M)^{-1}_{X\times(X\times M)}(\delta_{X,M})\\&=P_X(M)_N^{-1}\varphi_*(\delta_{X,M})=P_X(M)_N^{-1}\delta_{X,M}^*(\varphi).\end{aligned}$$

□

We note that $\delta_{X,M}:X\times M\to X\times(X\times M)$ is the unique morphism that makes the diagram of (6.3.11) commute for any $N\in\operatorname{Ob}\mathcal{F}_1$.

Remark 6.3.12 *If $o_X^*:\mathcal{F}_1\to\mathcal{F}_X$ has a left adjoint $o_{X*}:\mathcal{F}_X\to\mathcal{F}_1$, the following diagram is commutative for any $N\in\operatorname{Ob}\mathcal{F}_1$ by the naturality of ad_X.*

$$\begin{array}{ccc}\mathcal{F}_X(o_X^*o_{X*}o_X^*(M),o_X^*(N)) & \xrightarrow{(\eta_X)^*_{o_X^*(M)}} & \mathcal{F}_X(o_X^*(M),o_X^*(N))\\ \downarrow{\scriptstyle (\mathrm{ad}_X)^{-1}_{o_X^*o_{X*}o_X^*(M),N}} & & \downarrow{\scriptstyle (\mathrm{ad}_X)^{-1}_{o_X^*(M),N}}\\ \mathcal{F}_1(o_{X*}o_X^*o_{X*}o_X^*(M),N) & \xrightarrow{o_{X*}\left((\eta_X)_{o_X^*(M)}\right)^*} & \mathcal{F}_1(o_{X*}o_X^*(M),N)\end{array}$$

It follows that $\delta_{X,M}=o_{X*}\big((\eta_X)_{o_X^*(M)}\big)$.

Proposition 6.3.13 *For a morphism $f:X\to Y$ of $\mathcal{T}$ and a morphism $\varphi:L\to M$ of $\mathcal{F}_1$, the following diagrams are commutative.*

$$\begin{array}{ccc}X\times L & \xrightarrow{\delta_{X,L}} & X\times(X\times L)\\ \downarrow{\scriptstyle X\times\varphi} & & \downarrow{\scriptstyle X\times(X\times\varphi)}\\ X\times M & \xrightarrow{\delta_{X,M}} & X\times(X\times M)\end{array}\qquad\begin{array}{ccc}X\times M & \xrightarrow{\delta_{X,M}} & X\times(X\times M)\\ \downarrow{\scriptstyle f\times M} & & \downarrow{\scriptstyle f\times(f\times M)}\\ Y\times M & \xrightarrow{\delta_{Y,M}} & Y\times(Y\times M)\end{array}$$

Proof. The following diagram is commutative for any $N\in\operatorname{Ob}\mathcal{F}_1$ by (1) of (6.3.3).

$$\begin{array}{ccc}\mathcal{F}_X(o_X^*(X\times M),o_X^*(N)) & \xrightarrow{\iota_X(M)^*} & \mathcal{F}_X(o_X^*(M),o_X^*(N))\\ \downarrow{\scriptstyle o_X^*(X\times\varphi)^*} & & \downarrow{\scriptstyle o_X^*(\varphi)^*}\\ \mathcal{F}_X(o_X^*(X\times L),o_X^*(N)) & \xrightarrow{\iota_X(L)^*} & \mathcal{F}_X(o_X^*(L),o_X^*(N))\end{array}$$

Hence the following diagram commutes by (6.3.11) and (1) of (6.3.3).

$$\begin{array}{ccc}\mathcal{F}_1(X\times(X\times M),N) & \xrightarrow{\delta_{X,M}^*} & \mathcal{F}_1(X\times M,N)\\ \downarrow{\scriptstyle (X\times(X\times\varphi))^*} & & \downarrow{\scriptstyle (X\times\varphi)^*}\\ \mathcal{F}_1(X\times(X\times L),N) & \xrightarrow{\delta_{X,L}^*} & \mathcal{F}_1(X\times L,N)\end{array}$$

Thus the left diagram is commutative.

For $N\in\operatorname{Ob}\mathcal{F}_1$ and $\xi\in\mathcal{F}_Y(o_Y^*(Y\times M),o_Y^*(N))$, it follows from (6.3.8) and (6.1.17) that we have

$$f^\sharp_{Y\times M,N}(\xi)o_X^*(f\times M)\iota_X(M)=f^\sharp_{Y\times M,N}(\xi)f^\sharp_{M,Y\times M}(\iota_Y(M))=f^\sharp_{M,N}(\xi\iota_Y(M)).$$

This shows that the following diagram commutes.

$$\begin{array}{ccc}\mathcal{F}_Y(o_Y^*(Y\times M),o_Y^*(N)) & \xrightarrow{\iota_Y(M)^*} & \mathcal{F}_Y(o_Y^*(M),o_Y^*(N))\\ \downarrow{\scriptstyle o_X^*(f\times M)^*f^\sharp_{Y\times M,N}} & & \downarrow{\scriptstyle f^\sharp_{M,N}}\\ \mathcal{F}_X(o_X^*(X\times M),o_X^*(N)) & \xrightarrow{\iota_X(M)^*} & \mathcal{F}_X(o_X^*(X\times M),o_X^*(N))\end{array}$$

The following diagram commutes by (1) of (6.3.3) and (6.3.6).

$$\begin{tikzcd}
\mathcal{F}_Y(o_Y^*(Y\times M),o_Y^*(N)) \arrow[r, "f^\sharp_{Y\times M,N}"] \arrow[d, "P_Y(Y\times M)_N"] & \mathcal{F}_X(o_X^*(Y\times M),o_X^*(N)) \arrow[r, "o_X^*(f\times M)^*"] \arrow[d, "P_X(Y\times M)_N"] & \mathcal{F}_X(o_X^*(X\times M),o_X^*(N)) \arrow[d, "P_X(X\times M)_N"] \\
\mathcal{F}_1(Y\times(Y\times M),N) \arrow[r, "(f\times(Y\times M))^*"] & \mathcal{F}_1(X\times(Y\times M),N) \arrow[r, "(X\times(f\times M))^*"] & \mathcal{F}_1(X\times(X\times M),N)
\end{tikzcd}$$

Since $f\times(f\times M)=(Y\times(f\times M))(f\times(X\times M))$, it follows from (6.3.11) and (1) of (6.3.6) that the following diagram commutes for any $N\in\operatorname{Ob}\mathcal{F}_1$.

$$\begin{tikzcd}
\mathcal{F}_1(Y\times(Y\times M),N) \arrow[r, "\delta^*_{Y,M}"] \arrow[d, "(f\times(f\times M))^*"] & \mathcal{F}_1(Y\times M,N) \arrow[d, "(f\times M)^*"] \\
\mathcal{F}_1(X\times(X\times M),N) \arrow[r, "\delta^*_{X,M}"] & \mathcal{F}_1(X\times M,N)
\end{tikzcd}$$

Thus the right diagram is also commutative. $\square$

Proposition 6.3.14 *The following diagrams are commutative.*

$$\begin{tikzcd}
o_X^*(M) \arrow[r, "\iota_X(M)"] \arrow[d, "\iota_X(M)"] & o_X^*(X\times M) \arrow[d, "\iota_X(X\times M)"] \\
o_X^*(X\times M) \arrow[r, "o_X^*(\delta_{X,M})"] & o_X^*(X\times(X\times M))
\end{tikzcd}
\qquad
\begin{tikzcd}
X\times M \arrow[r, "\delta_{X,M}"] \arrow[d, "\delta_{X,M}"] & X\times(X\times M) \arrow[d, "\delta_{X,X\times M}"] \\
X\times(X\times M) \arrow[r, "X\times\delta_{X,M}"] & X\times(X\times(X\times M))
\end{tikzcd}$$

Proof. It follows from the definition of $\delta_{X,M}$ and (6.3.2) that

$$\iota_X(X\times M)\iota_X(M)=P_X(M)^{-1}_{X\times(X\times M)}(\delta_{X,M})=o_X^*(\delta_{X,M})\iota_X(M).$$

Hence the following diagram commutes for $N\in\operatorname{Ob}\mathcal{F}_1$.

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(X\times(X\times M)),o_X^*(N)) \arrow[r, "o_X^*(\delta_{X,M})^*"] \arrow[d, "\iota_X(X\times M)^*"] & \mathcal{F}_X(o_X^*(X\times M),o_X^*(N)) \arrow[d, "\iota_X(M)^*"] \\
\mathcal{F}_X(o_X^*(X\times M),o_X^*(N)) \arrow[r, "\iota_X(M)^*"] & \mathcal{F}_X(o_X^*(M),o_X^*(N))
\end{tikzcd}$$

Therefore the following diagram commutes by (6.3.11) and (1) of (6.3.3).

$$\begin{tikzcd}
\mathcal{F}_1(X\times(X\times(X\times M)),N) \arrow[r, "(X\times\delta_{X,M})^*"] \arrow[d, "\delta^*_{X,X\times M}"] & \mathcal{F}_1(X\times(X\times M),N) \arrow[d, "\delta^*_{X,M}"] \\
\mathcal{F}_1(X\times(X\times M),N) \arrow[r, "\delta^*_{X,M}"] & \mathcal{F}_1(X\times M,N)
\end{tikzcd}$$

$\square$

Proposition 6.3.15 *The following compositions coincide with the identity morphism of $X\times M$.*

$$X\times M\xrightarrow{\delta_{X,M}}X\times(X\times M)\xrightarrow{o_X\times(X\times M)}1\times(X\times M)=X\times M$$

$$X\times M\xrightarrow{\delta_{X,M}}X\times(X\times M)\xrightarrow{X\times(o_X\times M)}X\times(1\times M)=X\times M$$

Proof. The following diagram commutes for any $N\in\operatorname{Ob}\mathcal{F}_1$ by (1) of (6.3.6) and (6.3.11).

$$\begin{tikzcd}
\mathcal{F}_1(o_1^*(X\times M),o_1^*(N)) \arrow[r, "(o_X)^\sharp_{X\times M,N}"] \arrow[d, "P_1(X\times M)_N"] & \mathcal{F}_X(o_X^*(X\times M),o_X^*(N)) \arrow[r, "\iota_X(M)^*"] \arrow[d, "P_X(X\times M)_N"] & \mathcal{F}_X(o_X^*(M),o_X^*(N)) \arrow[d, "P_X(M)_N"] \\
\mathcal{F}_1(1\times(X\times M),N) \arrow[r, "(o_X\times(X\times M))^*"] & \mathcal{F}_1(X\times(X\times M),N) \arrow[r, "\delta^*_{X,M}"] & \mathcal{F}_1(X\times M,N)
\end{tikzcd}$$

It follows from (6.3.2) that $\delta^*_{X,M}(o_X \times (X \times M))^* : \mathcal{F}_1(X \times M, N) = \mathcal{F}_1(1 \times (X \times M), N) \to \mathcal{F}_1(X \times M, N)$ is the identity map of $\mathcal{F}_1(X \times M, N)$.

The following diagram commutes by (1) of (6.3.3) and and (6.3.11).

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(1 \times M), o_X^*(N)) \arrow[r, "o_X^*(o_X \times M)^*"] \arrow[d, "P_X(1\times M)_N"] & \mathcal{F}_X(o_X^*(X \times M), o_X^*(N)) \arrow[r, "\iota_X(M)^*"] \arrow[d, "P_X(X\times M)_N"] & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[d, "P_X(M)_N"] \\
\mathcal{F}_1(X \times (1 \times M), N) \arrow[r, "(X\times(o_X\times M))^*"] & \mathcal{F}_1(X \times (X \times M), N) \arrow[r, "\delta^*_{X,M}"] & \mathcal{F}_1(X \times M, N)
\end{tikzcd}$$

Since the composition of the upper horizontal maps of the above diagram coincides with the identity map of $\mathcal{F}_X(o_X^*(M), o_X^*(N))$ by (4) of (6.3.6), the composition of the lower horizontal maps of the above diagram is the identity map of $\mathcal{F}_1(X \times M, N)$. $\square$

Lemma 6.3.16 *If the following left diagram in $\mathcal{T}$ is commutative, then the following right diagram in $\mathcal{F}_1$ is commutative for $M \in \operatorname{Ob}\mathcal{F}_1$.*

$$\begin{tikzcd}
 & & X \arrow[dl, "g"'] \arrow[dr, "f"] & & \\
 & Z \arrow[dl, "j"'] \arrow[dr, "q"] & & Y \arrow[dl, "p"'] \arrow[dr, "i"] & \\
V & & W & & U
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times M \arrow[r, "(f\times(jg\times M))\delta_{X,M}"] \arrow[d, "(if\times(g\times M))\delta_{X,M}"] & Y \times (V \times M) \arrow[d, "(i\times(p\times(V\times M)))\delta_{Y,V\times M}"] \\
U \times (Z \times M) \arrow[r, "U\times((q\times(j\times M))\delta_{Z,M})"] & U \times (W \times (V \times M))
\end{tikzcd}$$

Proof. The following diagram is commutative by (6.3.14), (6.3.13), (6.3.9), (6.3.3) and (6.3.6).

$$\begin{tikzcd}
X \times M \arrow[r, "\delta_{X,M}"] \arrow[d, "\delta_{X,M}"] & X \times (X \times M) \arrow[r, "f\times(X\times M)"] \arrow[d, "\delta_{X,X\times M}"] & Y \times (X \times M) \arrow[r, "Y\times(jg\times M)"] \arrow[d, "\delta_{Y,X\times M}"] & Y \times (V \times M) \arrow[d, "\delta_{Y,V\times M}"] \\
X\times(X\times M) \arrow[r, "X\times\delta_{X,M}"] \arrow[d, "X\times(g\times M)"] & X\times(X\times(X\times M)) \arrow[r, "f\times(f\times(X\times M))"] \arrow[d, "X\times(g\times(g\times M))"] & Y\times(Y\times(X\times M)) \arrow[r, "Y\times(Y\times(jg\times M))"] \arrow[d, "Y\times(p\times(jg\times M))"] & Y\times(Y\times(V\times M)) \arrow[dl, "Y\times(p\times(V\times M))"] \\
X\times(Z\times M) \arrow[r, "X\times\delta_{Z,M}"] \arrow[d, "if\times(Z\times M)"] & X\times(Z\times(Z\times M)) \arrow[r, "f\times(q\times(j\times M))"] \arrow[d, "if\times(Z\times(Z\times M))"] & Y\times(W\times(V\times M)) \arrow[d, "i\times(W\times(W\times M))"] & \\
U\times(Z\times M) \arrow[r, "U\times\delta_{Z,M}"] & U\times(Z\times(Z\times M)) \arrow[r, "U\times(q\times(j\times M))"] & U\times(W\times(V\times M)) &
\end{tikzcd}$$

Hence the asserion follows. $\square$

For $X \in \operatorname{Ob}\mathcal{T}$, let us denote by $\mathrm{pr}_X : X \times 1 \to X$ and $\mathrm{pr}_2 : X \times 1 \to 1$ the projections. Similarly, for $Y \in \operatorname{Ob}\mathcal{T}$, let us denote by $\mathrm{pr}_1 : 1 \times Y \to 1$ and $\mathrm{pr}_Y : 1 \times Y \to Y$ the projections. We note that pr_X and pr_Y are isomorphisms.

Proposition 6.3.17 *Suppose that $p : \mathcal{F} \to \mathcal{T}$ is a normalized cloven fibered category. For $X, Y \in \operatorname{Ob}\mathcal{T}$ and $M \in \operatorname{Ob}\mathcal{F}_1$, the following diagrams commutes.*

$$\begin{tikzcd}
(X \times 1) \times M \arrow[r, "\delta_{X\times 1,M}"] \arrow[d, "\mathrm{pr}_X\times M"] & (X \times 1) \times ((X \times 1) \times M) \arrow[d, "\mathrm{pr}_X\times(\mathrm{pr}_2\times M)"] \\
X \times M \arrow[r, equal] & X \times (1 \times M)
\end{tikzcd}
\qquad
\begin{tikzcd}
(1 \times Y) \times M \arrow[r, "\delta_{1\times Y,M}"] \arrow[d, "\mathrm{pr}_Y\times M"] & (1 \times Y) \times ((1 \times Y) \times M) \arrow[d, "\mathrm{pr}_1\times(\mathrm{pr}_Y\times M)"] \\
Y \times M \arrow[r, equal] & 1 \times (Y \times M)
\end{tikzcd}$$

Proof. Since $\mathrm{pr}_2 = o_{X\times 1} : X \times 1 \to 1$ and $\mathrm{pr}_1 = o_{1\times Y} : 1 \times Y \to 1$, it follows from (6.3.15) that the following diagrams commutes.

$$\begin{tikzcd}
 & (X \times 1) \times ((X \times 1) \times M) \arrow[d, "(X\times 1)\times(\mathrm{pr}_2\times M)"] \\
(X \times 1) \times M \arrow[ur, "\delta_{X\times 1,M}"] \arrow[r, equal] \arrow[d, "\mathrm{pr}_X\times M"] & (X \times 1) \times (1 \times M) \arrow[d, "\mathrm{pr}_X\times(1\times M)"] \\
X \times M \arrow[r, equal] & X \times (1 \times M)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (1 \times Y) \times ((1 \times Y) \times M) \arrow[d, "\mathrm{pr}_1\times((1\times Y)\times M)"] \\
(1 \times Y) \times M \arrow[ur, "\delta_{1\times Y,M}"] \arrow[r, equal] \arrow[d, "\mathrm{pr}_Y\times M"] & 1 \times ((1 \times Y) \times M) \arrow[d, "1\times(\mathrm{pr}_Y\times M)"] \\
Y \times M \arrow[r, equal] & 1 \times (Y \times M)
\end{tikzcd}$$

Hence the assertions follows. $\square$

Let X be an object of $\mathcal{T}$ and L, M, N objects of $\mathcal{F}_1$. We define a map

$$\gamma_{X,L,M,N} : \mathcal{F}_1(X \times L, M) \times \mathcal{F}_1(X \times M, N) \to \mathcal{F}_1(X \times L, N)$$

as follows. For $\varphi \in \mathcal{F}_1(X \times L, M)$ and $\psi \in \mathcal{F}_1(X \times M, N)$, let $\gamma_{X,L,M,N}(\varphi, \psi)$ be the following composition.

$$X \times L \xrightarrow{\delta_{X,L}} X \times (X \times L) \xrightarrow{X \times \varphi} X \times M \xrightarrow{\psi} N$$

Proposition 6.3.18 *The following diagram is commutative.*

$$\begin{array}{ccc} \mathcal{F}_X(o_X^*(L), o_X^*(M)) \times \mathcal{F}_X(o_X^*(M), o_X^*(N)) & \xrightarrow{\text{composition}} & \mathcal{F}_X(o_X^*(L), o_X^*(N)) \\ \downarrow{\scriptstyle P_X(L)_M \times P_X(M)_N} & & \downarrow{\scriptstyle P_X(L)_N} \\ \mathcal{F}_1(X \times L, M) \times \mathcal{F}_1(X \times M, N) & \xrightarrow{\gamma_{X,L,M,N}} & \mathcal{F}_1(X \times L, N) \end{array}$$

Proof. For $\zeta \in \mathcal{F}_X(o_X^*(L), o_X^*(M))$ and $\xi \in \mathcal{F}_X(o_X^*(M), o_X^*(N))$, we put $\varphi = P_X(L)_M(\zeta)$ and $\psi = P_X(M)_N(\xi)$. Then, we have $\psi(X \times \varphi) = P_X(X \times L)_N(\xi o_X^*(\varphi))$ by (6.3.3). It follows from (6.3.11) and (6.3.2) that

$$\psi(X \times \varphi)\delta_{X,L} = \delta_{X,L}^* P_X(X \times L)_N(\xi o_X^*(\varphi)) = P_X(L)_N(\xi o_X^*(\varphi)\iota_X(L)) = P_X(L)_N(\xi\zeta).$$

Thus the result follows. $\square$

Definition 6.3.19 *Let $p : \mathcal{F} \to \mathcal{T}$ be a normalized cloven fibered category. We say that $p : \mathcal{F} \to \mathcal{T}$ is a fibered category with products if the presheaf $F_{X,M}$ on $\mathcal{F}_1^{op}$ is representable for any $X \in \operatorname{Ob}\mathcal{T}$ and $M \in \operatorname{Ob}\mathcal{F}_1$.*

Let $p : \mathcal{F} \to \mathcal{T}$ be a fibered category with products. Suppose that $\mathcal{T}$ has finite products. For $X, Y \in \operatorname{Ob}\mathcal{T}$, we denote by $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ the projections. For $M \in \operatorname{Ob}\mathcal{F}_1$, we define a morphism $\theta_{X,Y}(M) : (X \times Y) \times M \to X \times (Y \times M)$ of $\mathcal{F}_1$ to be the following composition.

$$(X \times Y) \times M \xrightarrow{\delta_{X \times Y, M}} (X \times Y) \times ((X \times Y) \times M) \xrightarrow{\mathrm{pr}_X \times (\mathrm{pr}_Y \times M)} X \times (Y \times M)$$

Proposition 6.3.20 *For morphisms $f : X \to Z$, $g : Y \to W$ of $\mathcal{T}$ and a morphism $\varphi : L \to M$ of $\mathcal{F}_1$, the following diagrams are commutative.*

$$\begin{array}{ccc} (X \times Y) \times L & \xrightarrow{\theta_{X,Y}(L)} & X \times (Y \times L) \\ \downarrow{\scriptstyle (X \times Y) \times \varphi} & & \downarrow{\scriptstyle X \times (Y \times \varphi)} \\ (X \times Y) \times M & \xrightarrow{\theta_{X,Y}(M)} & X \times (Y \times M) \end{array} \qquad \begin{array}{ccc} (X \times Y) \times M & \xrightarrow{\theta_{X,Y}(M)} & X \times (Y \times M) \\ \downarrow{\scriptstyle (f \times g) \times M} & & \downarrow{\scriptstyle f \times (g \times M)} \\ (Z \times W) \times M & \xrightarrow{\theta_{Z,W}(M)} & Z \times (W \times M) \end{array}$$

Proof. The following diagrams commute by (6.3.13), (6.3.9), (6.3.3) and (6.3.6).

$$\begin{array}{ccccc} (X \times Y) \times L & \xrightarrow{\delta_{X \times Y, L}} & (X \times Y) \times ((X \times Y) \times L) & \xrightarrow{\mathrm{pr}_X \times (\mathrm{pr}_Y \times L)} & X \times (Y \times L) \\ \downarrow{\scriptstyle (X \times Y) \times \varphi} & & \downarrow{\scriptstyle (X \times Y) \times ((X \times Y) \times \varphi)} & & \downarrow{\scriptstyle X \times (Y \times \varphi)} \\ (X \times Y) \times M & \xrightarrow{\delta_{X \times Y, M}} & (X \times Y) \times ((X \times Y) \times M) & \xrightarrow{\mathrm{pr}_X \times (\mathrm{pr}_Y \times M)} & X \times (Y \times M) \end{array}$$

$$\begin{array}{ccccc} (X \times Y) \times M & \xrightarrow{\delta_{X \times Y, M}} & (X \times Y) \times ((X \times Y) \times M) & \xrightarrow{\mathrm{pr}_X \times (\mathrm{pr}_Y \times M)} & X \times (Y \times M) \\ \downarrow{\scriptstyle (f \times g) \times M} & & \downarrow{\scriptstyle (f \times g) \times ((f \times g) \times M)} & & \downarrow{\scriptstyle f \times (g \times M)} \\ (Z \times W) \times M & \xrightarrow{\delta_{Z \times W, M}} & (Z \times W) \times ((Z \times W) \times M) & \xrightarrow{\mathrm{pr}_Z \times (\mathrm{pr}_W \times M)} & Z \times (W \times M) \end{array}$$

Hence the assertion follows. $\square$

Proposition 6.3.21 *For $X, Y, Z \in \operatorname{Ob}\mathcal{T}$ and $M \in \operatorname{Ob}\mathcal{F}_1$, the following diagram is commutative.*

$$\begin{array}{ccc} (X \times Y \times Z) \times M & \xrightarrow{\theta_{X \times Y, Z}(M)} & (X \times Y) \times (Z \times M) \\ \downarrow{\scriptstyle \theta_{X, Y \times Z}(M)} & & \downarrow{\scriptstyle \theta_{X,Y}(Z \times M)} \\ X \times ((Y \times Z) \times M) & \xrightarrow{X \times \theta_{Y,Z}(M)} & X \times (Y \times (Z \times M)) \end{array}$$

Proof. Let us denote by $p_{X\times Y} : X\times Y\times Z \to X\times Y$, $p_{Y\times Z} : X\times Y\times Z\to Y\times Z$, $\mathrm{pr}_X : X\times Y\to X$, $\mathrm{pr}_Y : X\times Y\to Y$, $\mathrm{pr}'_Y : Y\times Z\to Y$ and $\mathrm{pr}'_Z : Y\times Z\to Z$ the projections. Since

$$\begin{aligned}
\theta_{X\times Y,Z}(M) &= (p_{X\times Y}\times(\mathrm{pr}'_Z p_{Y\times Z}\times M))\delta_{X\times Y\times Z,M} : (X\times Y\times Z)\times M\to(X\times Y)\times(Z\times M)\\
\theta_{X,Y\times Z}(M) &= (\mathrm{pr}_X p_{X\times Y}\times(p_{Y\times Z}\times M))\delta_{X\times Y\times Z,M} : (X\times Y\times Z)\times M\to X\times((Y\times Z)\times M)\\
\theta_{X,Y}(Z\times M) &= (\mathrm{pr}_X\times(\mathrm{pr}_Y\times(Z\times M)))\delta_{X\times Y,Z\times M} : (X\times Y)\times(Z\times M)\to X\times(Y\times(Z\times M))\\
X\times\theta_{Y,Z}(M) &= X\times((\mathrm{pr}'_Y\times(\mathrm{pr}'_Z\times M))\delta_{Y\times Z,M}) : X\times((Y\times Z)\times M)\to X\times(Y\times(Z\times M)),
\end{aligned}$$

the assertion follows by applying (6.3.16) for $f = p_{X\times Y}$, $g = p_{Y\times Z}$, $p = \mathrm{pr}_Y$, $q = \mathrm{pr}'_Y$, $i = \mathrm{pr}_X$ and $j = \mathrm{pr}'_Z$. □

Proposition 6.3.22 *For objects X, Y of $\mathcal{T}$ and an object M of $\mathcal{F}_1$, $\theta_{X,1}(M) : (X\times 1)\times M\to X\times(1\times M) = X\times M$ is identified with $\mathrm{pr}_X\times M : (X\times 1)\times M\to X\times M$ and $\theta_{1,Y}(M) : (1\times Y)\times M\to X\times M$ is identified with $\mathrm{pr}_Y\times M : (1\times Y)\times M\to Y\times M$.*

Proof. This is a direct consequence of (6.3.17). □

Lemma 6.3.23 *For objects X, Y of $\mathcal{T}$ and an object M of $\mathcal{F}_1$, the following diagram is commutative.*

$$\begin{tikzcd}
o^*_{X\times Y}(M) \arrow[r, "\iota_{X\times Y}(M)"] \arrow[d, "\mathrm{pr}^\sharp_Y(\iota_Y(M))"'] & o^*_{X\times Y}((X\times Y)\times M) \arrow[d, "o^*_{X\times Y}(\theta_{X,Y}(M))"] \\
o^*_{X\times Y}(Y\times M) \arrow[r, "\mathrm{pr}^\sharp_X(\iota_X(Y\times M))"] & o^*_{X\times Y}(X\times(Y\times M))
\end{tikzcd}$$

Proof. It follows from (6.3.8) and (1) of (6.3.3) that we have

$$\begin{aligned}
\mathrm{pr}^\sharp_X(\iota_X(Y\times M))\mathrm{pr}^\sharp_Y(\iota_Y(M)) &= o^*_{X\times Y}(\mathrm{pr}_X\times(Y\times M))\iota_{X\times Y}(Y\times M)o^*_{X\times Y}(\mathrm{pr}_Y\times M)\iota_{X\times Y}(M)\\
&= o^*_{X\times Y}(\mathrm{pr}_X\times(Y\times M))o^*_{X\times Y}((X\times Y)\times(\mathrm{pr}_Y\times M))\iota_{X\times Y}((X\times Y)\times M)\iota_{X\times Y}(M)\\
&= o^*_{X\times Y}(\mathrm{pr}_X\times(\mathrm{pr}_Y\times M))\iota_{X\times Y}((X\times Y)\times M)\iota_{X\times Y}(M)
\end{aligned}$$

By the naturality of $P_{X\times Y}(M)$ and the definition of $\delta_{X\times Y,M}$, the above equality implies that

$$P_{X\times Y}(M)_{X\times(Y\times M)} : \mathcal{F}_{X\times Y}(o^*_{X\times Y}(M), o^*_{X\times Y}(X\times(Y\times M)))\to\mathcal{F}_1((X\times Y)\times M, X\times(Y\times M))$$

maps $\mathrm{pr}^\sharp_X(\iota_X(Y\times M))\mathrm{pr}^\sharp_Y(\iota_Y(M))$ to $(\mathrm{pr}_X\times(\mathrm{pr}_Y\times M))\delta_{X\times Y,M} = \theta_{X,Y}(M)$. By (6.3.2), $P_{X\times Y}(M)_{X\times(Y\times M)}$ also maps $o^*_{X\times Y}(\theta_{X,Y}(M))\iota_{X\times Y}(M)$ to $\theta_{X,Y}(M)$. □

Let us denote by $\Delta_X : X\to X\times X$ the diagonal morphism of $X\in\mathrm{Ob}\,\mathcal{T}$.

Proposition 6.3.24 *For $X\in\mathrm{Ob}\,\mathcal{T}$ and $M\in\mathrm{Ob}\,\mathcal{F}_1$, we have $\theta_{X,X}(M)(\Delta_X\times M) = \delta_{X,M}$.*

Proof. We denote by $\mathrm{pr}_i : X\times X\to X$ the projection to i th component for $i = 1,2$. It follows from the commutativity of the right diagram of (6.3.13) that

$$\begin{aligned}
\theta_{X,X}(M)(\Delta_X\times M) &= (\mathrm{pr}_1\times(\mathrm{pr}_2\times M))\delta_{X\times X,M}(\Delta_X\times M) = (\mathrm{pr}_1\times(\mathrm{pr}_2\times M))(\Delta_X\times(\Delta_X\times M))\delta_{X,M}\\
&= (\mathrm{pr}_1\Delta_X\times(\mathrm{pr}_2\Delta_X\times M))\delta_{X,M} = \delta_{X,M}
\end{aligned}$$

since $\mathrm{pr}_1\Delta_X = \mathrm{pr}_2\Delta_X = id_X$. □

Definition 6.3.25 *Let $p : \mathcal{F}\to\mathcal{T}$ be a fibered category with products. Suppose that $\mathcal{T}$ has finite products. If $\theta_{X,Y}(M) : (X\times Y)\times M\to X\times(Y\times M)$ is an isomorphism for any $X, Y\in\mathrm{Ob}\,\mathcal{T}$ and $M\in\mathrm{Ob}\,\mathcal{F}_1$, we say that $p : \mathcal{F}\to\mathcal{T}$ is an associative fibered category with products.*

6.4 Fibered category with exponents

Let $p : \mathcal{F} \to \mathcal{T}$ be a normalized cloven fibered category. Assume that $\mathcal{T}$ has a terminal object 1.

For $X \in \operatorname{Ob}\mathcal{T}$ and $N \in \operatorname{Ob}\mathcal{F}_1$, define a presheaf $F_N^X : \mathcal{F}_1^{op} \to \mathcal{S}et$ on $\mathcal{F}_1$ by

$$F_N^X(M) = F_X(M, N) = \mathcal{F}_X(o_X^*(M), o_X^*(N))$$

for $M \in \operatorname{Ob}\mathcal{F}_1$ and $F_N^X(\varphi) = o_X^*(\varphi)^*$ for $\varphi \in \operatorname{Mor}\mathcal{F}_1$. Suppose that F_N^X is representable for $X \in \operatorname{Ob}\mathcal{T}$ and $N \in \operatorname{Ob}\mathcal{F}_1$. We choose an object N^X of $\mathcal{F}_1$ such that there exists a natural equivalence $E_X(N) : F_N^X \to h_{N^X}$, where h_{N^X} is the presheaf represented by N^X. Since $o_1^* : \mathcal{F}_1 \to \mathcal{F}_1$ is the identity functor of $\mathcal{F}_1$, we take N as N^1. Hence $E_1(N)_M$ is the identity morphism of $\mathcal{F}_1(M, N)$. Let us denote by $\pi_X(N) : o_X^*(N^X) \to o_X^*(N)$ the morphism of $\mathcal{F}_X$ which is mapped to the identity morphism of N^X by $E_X(N)_{N^X} : \mathcal{F}_X(o_X^*(N^X), o_X^*(N)) \to \mathcal{F}_1(N^X, N^X)$.

Remark 6.4.1 *If $o_X^* : \mathcal{F}_1 \to \mathcal{F}_X$ has a right adjoint $o_{X!} : \mathcal{F}_X \to \mathcal{F}_1$, $F_N^X : \mathcal{F}_1^{op} \to \mathcal{S}et$ is representable for any object N of $\mathcal{F}_1$. In fact, N^X is defined to be $o_{X!}o_X^*(N)$ in this case. If we denote by $\mathrm{ad}_{M,P}^X : \mathcal{F}_X(o_X^*(M), P) \to \mathcal{F}_1(M, o_{X!}(P))$ the bijection which is natural in $M \in \operatorname{Ob}\mathcal{F}_1$ and $P \in \operatorname{Ob}\mathcal{F}_X$, we have $E_X(N)_M = \mathrm{ad}_{M,o_X^*(N)}^X : \mathcal{F}_X(o_X^*(M), o_X^*(N)) \to \mathcal{F}_1(M, o_{X!}o_X^*(N))$. Let us denote by $\varepsilon^X : o_X^* o_{X!} \to id_{\mathcal{F}_X}$ the counit of the adjunction $o_X^* \dashv o_{X!}$. We have $\pi_X(N) = \varepsilon_{o_X^*(N)}^X : o_X^*(N^X) = o_X^* o_{X!} o_X^*(N) \to o_X^*(N)$.*

Proposition 6.4.2 *The inverse of $E_X(N)_M : \mathcal{F}_X(o_X^*(M), o_X^*(N)) \to \mathcal{F}_1(M, N^X)$ is given by the map defined by $\varphi \mapsto \pi_X(N)o_X^*(\varphi)$.*

Proof. For $\varphi \in \mathcal{F}_1(M, N^X)$, the following diagram commutes by naturality of $E_X(N)$.

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(N^X), o_X^*(N)) \arrow[r, "o_X^*(\varphi)^*"] \arrow[d, "E_X(N)_{N^X}"] & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[d, "E_X(N)_M"] \\
\mathcal{F}_1(N^X, N^X) \arrow[r, "\varphi^*"] & \mathcal{F}_1(M, N^X)
\end{tikzcd}$$

It follows that $E_X(N)_M$ maps $\pi_X(N)o_X^*(\varphi)$ to φ. $\square$

For a morphism $\varphi : L \to N$ of $\mathcal{F}_1$, define a natural transformation $F_\varphi^X : F_L^X \to F_N^X$ by

$$(F_\varphi^X)_M = o_X^*(\varphi)_* : F_L^X(M) = \mathcal{F}_X(o_X^*(M), o_X^*(L)) \to \mathcal{F}_X(o_X^*(M), o_X^*(N)) = F_N^X(M).$$

It is clear that $F_{\psi\varphi}^X = F_\psi^X F_\varphi^X$ for morphisms $\psi : N \to P$ and $\varphi : L \to N$ of $\mathcal{F}_1$. We define $\varphi^X : L^X \to N^X$ by $\varphi^X = E_X(N)_{L^X}((F_\varphi^X)_{L^X}(\pi_X(L))) = E_X(N)_{L^X}(o_X^*(\varphi)\pi_X(L)) \in h_{N^X}(L^X)$.

Proposition 6.4.3 *(1) The following diagrams commute for any $M \in \operatorname{Ob}\mathcal{F}_1$.*

$$\begin{tikzcd}
o_X^*(L^X) \arrow[r, "o_X^*(\varphi^X)"] \arrow[d, "\pi_X(L)"] & o_X^*(N^X) \arrow[d, "\pi_X(N)"] \\
o_X^*(L) \arrow[r, "o_X^*(\varphi)"] & o_X^*(N)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_X(o_X^*(M), o_X^*(L)) \arrow[r, "o_X^*(\varphi)_*"] \arrow[d, "E_X(L)_M"] & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[d, "E_X(N)_M"] \\
\mathcal{F}_1(M, L^X) \arrow[r, "\varphi_*^X"] & \mathcal{F}_1(M, N^X)
\end{tikzcd}$$

(2) For morphisms $\psi : N \to P$ and $\varphi : L \to N$ of $\mathcal{F}_1$, we have $(\psi\varphi)^X = \psi^X\varphi^X$.

(3) If $o_X^* : \mathcal{F}_1 \to \mathcal{F}_X$ preserves monomorphisms (o_X^* has a left adjoint, for example) and $\varphi : L \to N$ is a monomorphism, so is $\varphi^X : L^X \to N^X$.

Proof. (1) We have $E_X(N)_{L^X}(o_X^*(\varphi)\pi_X(L)) = \varphi^X$ by the definition of φ^X. On the other hand, it follows from (6.4.2) that $E_X(N)_{L^X}(\pi_X(N)o_X^*(\varphi^X)) = \varphi^X$. Since $E_X(N)_{L^X}$ is bijective, the left diagram commutes.

For $\psi \in \mathcal{F}_1(M, L^X)$, it follows from 6.4.2 and commutativity of the left diagram that we have

$$\begin{aligned}
o_X^*(\varphi)_* E_X(L)_M^{-1}(\psi) &= o_X^*(\varphi)\pi_X(L)o_X^*(\psi) = \pi_X(N)o_X^*(\varphi^X)o_X^*(\psi) = \pi_X(N)o_X^*(\varphi^X\psi) \\
&= E_X(N)_M^{-1}(\varphi^X\psi) = E_X(N)_M^{-1}\varphi_*^X(\psi).
\end{aligned}$$

Hence the right diagram commutes.

(2) The following diagram commutes by (1).

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(L^X), o_X^*(L)) \arrow[r, "o_X^*(\varphi)_*"] \arrow[d, "E_X(L)_{L^X}"] & \mathcal{F}_X(o_X^*(L^X), o_X^*(N)) \arrow[r, "o_X^*(\psi)_*"] \arrow[d, "E_X(N)_{L^X}"] & \mathcal{F}_X(o_X^*(L^X), o_X^*(P))) \arrow[d, "E_X(P)_{L^X}"] \\
\mathcal{F}_1(L^X, L^X) \arrow[r, "\varphi_*^X"] & \mathcal{F}_1(L^X, N^X) \arrow[r, "\psi_*^X"] & \mathcal{F}_1(L^X, P^X)
\end{tikzcd}$$

Hence $\psi^X\varphi^X = \psi_*^X\varphi_*^X(id_{L^X}) = E_X(P)_{L^X}(o_X^*(\psi)o_X^*(\varphi)\pi_X(L)) = E_X(P)_{L^X}(o_X^*(\psi\varphi)\pi_X(L)) = (\psi\varphi)^X$.

(3) is a direct consequence of (1). □

Remark 6.4.4 *Suppose that $o_X^* : \mathcal{F}_1 \to \mathcal{F}_X$ has a right adjoint $o_{X!} : \mathcal{F}_X \to \mathcal{F}_1$. For a morphism $\varphi : L \to N$ of $\mathcal{F}_1$, we have $\varphi^X = o_{X!}o_X^*(\varphi) : L^X = o_{X!}o_X^*(L) \to o_{X!}o_X^*(N) = N^X$. In fact, if we denote by $\eta^X : id_{\mathcal{F}_X} \to o_{X!}o_X^*$ the unit of the adjunction $o_X^* \dashv o_{X!}$, we have $\varphi^X = E_X(N)_{L^X}(o_X^*(\varphi)\pi_X(L)) = \mathrm{ad}^X_{L^X, o_X^*(N)}\big(o_X^*(\varphi)\varepsilon^X_{o_X^*(L)}\big) = o_{X!}o_X^*(\varphi)o_{X!}\big(\varepsilon^X_{o_X^*(L)}\big)\eta^X_{o_{X!}o_X^*(L)} = o_{X!}o_X^*(\varphi)$.*

Lemma 6.4.5 *Let $\xi : o_X^*(L) \to o_X^*(M)$, $\zeta : o_X^*(N) \to o_X^*(K)$ be morphisms of $\mathcal{F}_X$ for $K, L, M, N \in \mathrm{Ob}\,\mathcal{F}_1$ and $\varphi : L \to N$, $\psi : M \to K$ morphisms of $\mathcal{F}_1$. We put $\check{\xi} = E_X(L)_M(\xi)$ and $\check{\zeta} = E_X(K)_N(\zeta)$. The following left diagram commutes if and only if the right one commutes.*

$$\begin{tikzcd}
o_X^*(L) \arrow[r, "\xi"] \arrow[d, "o_X^*(\varphi)"] & o_X^*(M) \arrow[d, "o_X^*(\psi)"] \\
o_X^*(N) \arrow[r, "\zeta"] & o_X^*(K)
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, "\check{\xi}"] \arrow[d, "\varphi"] & M^X \arrow[d, "\psi^X"] \\
N \arrow[r, "\check{\zeta}"] & K^X
\end{tikzcd}$$

Proof. The following diagram is commutative by (6.4.3) and the naturality of $E_X(K)$.

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(L), o_X^*(M)) \arrow[r, "o_X^*(\psi)_*"] \arrow[d, "E_X(M)_L"] & \mathcal{F}_X(o_X^*(L), o_X^*(K)) \arrow[d, "E_X(K)_L"] & \mathcal{F}_X(o_X^*(N), o_X^*(K)) \arrow[l, "o_X^*(\varphi)^*"'] \arrow[d, "E_X(K)_N"] \\
\mathcal{F}_1(L, M^X) \arrow[r, "\psi_*^X"] & \mathcal{F}_1(L, K^X) & \mathcal{F}_1(N, K^X) \arrow[l, "\varphi^*"']
\end{tikzcd}$$

Since $\check{\xi} = E_X(L)_M(\xi)$, $\check{\zeta} = E_X(K)_N(\zeta)$ and $E_X(K)_L$ is bijective, $o_X^*(\psi)\xi = o_X^*(\psi)_*(\xi) = o_X^*(\varphi)^*(\zeta) = \zeta o_X^*(\varphi)$ if and only if $\psi^X\check{\xi} = \psi_*^X(\check{\xi}) = \varphi^*(\check{\zeta}) = \check{\zeta}\varphi$. □

For $X, Y \in \mathrm{Ob}\,\mathcal{T}$ and $N \in \mathrm{Ob}\,\mathcal{F}_1$, suppose that F_N^X and F_N^Y are representable. For a morphism $f : X \to Y$ of $\mathcal{T}$, we define a morphism $N^f : N^Y \to N^X$ of $\mathcal{F}_1$ by

$$N^f = E_X(N)_{N^Y}(f^\sharp_{N^Y,N}(\pi_Y(N))) \in \mathcal{F}_1(N^Y, N^X).$$

Since F_N^1 is represented by N, we identify N with N^1 and o_X induces $N^{o_X} : N \to N^X$.

Proposition 6.4.6 *(1) The following diagram commutes for any $M \in \mathrm{Ob}\,\mathcal{F}_1$.*

$$\begin{tikzcd}
\mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) \arrow[r, "f^\sharp_{M,N}"] \arrow[d, "E_Y(N)_M"] & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[d, "E_X(N)_M"] \\
\mathcal{F}_1(M, N^Y) \arrow[r, "N_*^f"] & \mathcal{F}_1(M, N^X)
\end{tikzcd}$$

(2) For morphisms $f : X \to Y$ and $g : Y \to Z$ of $\mathcal{T}$, $N^{gf} = N^f N^g$.
(3) The image of the identity morphism of $o_X^*(N)$ by $E_X(N)_N$ is $N^{o_X} : N = N^1 \to N^X$.
(4) A composition $o_X^*(N) = o_X^*(N^1) \xrightarrow{o_X^*(N^{o_X})} o_X^*(N^X) \xrightarrow{\pi_X(N)} o_X^*(N)$ is the identity morphism of $o_X^*(N)$.

Proof. (1) For $\varphi \in \mathcal{F}_1(M, N^Y)$, it follows from the naturality of $f^\sharp_{M,N}$ and (6.4.2) that we have

$$\begin{aligned}
f^\sharp_{M,N}E_Y(N)_M^{-1}(\varphi) &= f^\sharp_{M,N}(\pi_Y(N)o_Y^*(\varphi)) = f^\sharp_{M,N}o_Y^*(\varphi)^*(\pi_Y(N)) = o_X^*(\varphi)^* f^\sharp_{N^Y,N}(\pi_Y(N)) \\
&= o_X^*(\varphi)^* E_X(N)_{N^Y}^{-1}(N^f) = \pi_X(N)o_Y^*(N^f)o_X^*(\varphi) = \pi_X(N)o_X^*(N^f\varphi) \\
&= \pi_X(N)o_X^*((N^f)_*(\varphi)) = E_X(N)_M^{-1}(N^f)_*(\varphi).
\end{aligned}$$

(2) The following diagram commutes by (1).

$$\begin{tikzcd}
\mathcal{F}_X(o_Z^*(N^Z), o_Z^*(N)) \arrow[r, "g^\sharp_{N^Z,N}"] \arrow[d, "E_Z(N)_{N^Z}"] & \mathcal{F}_Y(o_Y^*(N^Z), o_Y^*(N)) \arrow[r, "f^\sharp_{N^Z,N}"] \arrow[d, "E_Y(N)_{N^Z}"] & \mathcal{F}_X(o_X^*(N^Z), o_X^*(N)) \arrow[d, "E_X(N)_{N^Z}"] \\
\mathcal{F}_1(N^Z, N^Z) \arrow[r, "N^g_*"] & \mathcal{F}_1(N^Z, N^Y) \arrow[r, "N^f_*"] & \mathcal{F}_1(N^Z, N^X)
\end{tikzcd}$$

Hence $N^f N^g = N^f_* N^g_*(id_{N^Z}) = E_Z(N)_{N^Z}(f^\sharp_{N^Z,N} g^\sharp_{N^Z,N}(\pi_Y(N))) = E_X(N)_{N^Z}((gf)^\sharp_{N^Z,N}(\pi_Z(N))) = N^{gf}$.

(3) Apply (1) for $M = N$, $Y = 1$ and $f = o_X : X \to 1$.

(4) It follows from (6.4.2) that $E_X(N)_N : \mathcal{F}_X(o_X^*(N), o_X^*(N)) \to \mathcal{F}_1(N, N^X)$ maps $\pi_X(N) o_X^*(N^{o_X})$ to $N^{o_X} : N \to N^X$. Thus the assertion follows from (3). $\square$

Remark 6.4.7 *Suppose that the inverse image functors $o_X^* : \mathcal{F}_1 \to \mathcal{F}_X$ and $o_Y^* : \mathcal{F}_1 \to \mathcal{F}_Y$ have right adjoints $o_{X!} : \mathcal{F}_X \to \mathcal{F}_1$ and $o_{Y!} : \mathcal{F}_Y \to \mathcal{F}_1$, respectively.*

(1) Since $f^\sharp_{N^Y,N}(\pi_Y(N)) = c_{o_X,f}(N) f^*\big(\varepsilon^Y_{o_Y^*(N)}\big) c_{o_X,f}(N^Y)^{-1}$ by (6.4.1) and

$$E_X(N)_{N^Y} = \mathrm{ad}^X_{N^Y, o_X^*(N)} : \mathcal{F}_X(o_X^*(N^Y), o_X^*(N)) \to \mathcal{F}_1(N^Y, N^X)$$

maps $\varphi \in \mathcal{F}_X(o_X^*(N^Y), o_X^*(N))$ to $o_{X!}(\varphi)\eta^X_{N^Y}$, $N^f : N^Y \to N^X$ coincides with the following composition.

$$N^Y \xrightarrow{\eta^X_{N^Y}} o_{X!}o_X^*(N^Y) \xrightarrow{o_{X!}(c_{o_X,f}(N^Y))^{-1}} o_{X!}f^*o_Y^*(N^Y) = o_{X!}f^*o_Y^*o_{Y!}o_Y^*(N) \xrightarrow{o_{X!}f^*\left(\varepsilon^Y_{o_Y^*(N)}\right)} o_{X!}f^*o_Y^*(N)$$
$$\xrightarrow{o_{X!}(c_{o_X,f}(N))} o_{X!}o_X^*(N) = N^X$$

(2) The following diagram commutes by (6.4.6).

$$\begin{tikzcd}
\mathcal{F}_1(N, o_{1!}(o_1^*(N))) \arrow[r, "N^{o_X}_*"] \arrow[d, "(\mathrm{ad}^1_{N,o_1^*(N)})^{-1}"] & \mathcal{F}_1(N, o_{X!}(o_X^*(N))) \arrow[d, "(\mathrm{ad}^X_{N,o_X^*(N)})^{-1}"] \\
\mathcal{F}_1(o_1^*(N), o_1^*(N)) \arrow[r, "(o_X^\sharp)_{N,N}"] & \mathcal{F}_X(o_X^*(N), o_X^*(N))
\end{tikzcd}$$

Since o_1^* is the identity functor of $\mathcal{F}_1$, so is $o_{1!}$. Hence $N^{o_X} : N = N^1 \to N^X = o_{X!}o_X^*(N)$ is identified with the unit $\eta^X_N : N \to o_{X!}o_X^*(N)$ of the adjunction $o_X^* \dashv o_{X!}$ by the above diagram.

Lemma 6.4.8 *For a morphism $f : X \to Y$ of $\mathcal{T}$ and an object N of $\mathcal{F}_1$, $f^\sharp_{N^Y,N} : \mathcal{F}_Y(o_Y^*(N^Y), o_Y^*(N)) \to \mathcal{F}_X(o_X^*(N^Y), o_X^*(N))$ maps $\pi_Y(N)$ to $\pi_X(N) o_X^*(N^f)$.*

Proof. The following diagram commutes by (1) of (6.4.6).

$$\begin{tikzcd}
\mathcal{F}_Y(o_Y^*(N^Y), o_Y^*(N)) \arrow[r, "f^\sharp_{N^Y,N}"] \arrow[d, "E_Y(N)_{N^Y}"] & \mathcal{F}_X(o_X^*(N^Y), o_X^*(N)) \arrow[d, "E_X(N)_{N^Y}"] \\
\mathcal{F}_1(N^Y, N^Y) \arrow[r, "N^f_*"] & \mathcal{F}_1(N^Y, N^X)
\end{tikzcd}$$

The assertion follows from (6.4.2). $\square$

Proposition 6.4.9 *For a morphism $\varphi : L \to N$ of $\mathcal{F}_1$ and a morphism $f : Y \to X$ of $\mathcal{T}$, the following diagram commutes.*

$$\begin{tikzcd}
L^X \arrow[r, "\varphi^X"] \arrow[d, "L^f"] & N^X \arrow[d, "N^f"] \\
L^Y \arrow[r, "\varphi^Y"] & N^Y
\end{tikzcd}$$

Proof. The following diagram commutes by the naturality of $f^\sharp$.

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(M), o_X^*(L)) \arrow[r, "f^\sharp_{M,L}"] \arrow[d, "o_X^*(\varphi)_*"] & \mathcal{F}_Y(o_Y^*(M), o_Y^*(L)) \arrow[d, "o_Y^*(\varphi)_*"] \\
\mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[r, "f^\sharp_{M,N}"] & \mathcal{F}_Y(o_Y^*(M), o_Y^*(N))
\end{tikzcd}$$

Then, it follows from the commutativity of four diagrams

$$\begin{tikzcd} \mathcal{F}_X(o_X^*(M), o_X^*(L)) \arrow[r, "E_X(L)_M"] \arrow[d, "o_X^*(\varphi)_*"] & \mathcal{F}_1(M, L^X) \arrow[d, "\varphi_*^X"] \\ \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[r, "E_X(N)_M"] & \mathcal{F}_1(M, N^X) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}_Y(o_Y^*(M), o_Y^*(L)) \arrow[r, "E_Y(L)_M"] \arrow[d, "o_Y^*(\varphi)_*"] & \mathcal{F}_1(M, L^Y) \arrow[d, "\varphi_*^Y"] \\ \mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) \arrow[r, "E_Y(N)_M"] & \mathcal{F}_1(M, N^Y) \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F}_X(o_X^*(M), o_X^*(L)) \arrow[r, "E_X(L)_M"] \arrow[d, "f^\sharp_{M,L}"] & \mathcal{F}_1(M, L^X) \arrow[d, "L_*^f"] \\ \mathcal{F}_Y(o_Y^*(M), o_Y^*(L)) \arrow[r, "E_Y(L)_M"] & \mathcal{F}_1(M, L^Y) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[r, "E_X(N)_M"] \arrow[d, "f^\sharp_{M,N}"] & \mathcal{F}_1(M, N^X) \arrow[d, "N_*^f"] \\ \mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) \arrow[r, "E_Y(N)_M"] & \mathcal{F}_1(M, N^Y) \end{tikzcd}$$

and the fact that $E_X(L)_M : \mathcal{F}_X(o_X^*(M), o_X^*(L)) \to \mathcal{F}_1(M, L^X)$ is bijective that the following diagram commutes for any $M \in \operatorname{Ob}\mathcal{F}_1$.

$$\begin{tikzcd} \mathcal{F}_1(M, L^X) \arrow[r, "\varphi_*^X"] \arrow[d, "L_*^f"] & \mathcal{F}_1(M, N^X) \arrow[d, "N_*^f"] \\ \mathcal{F}_1(M, L^Y) \arrow[r, "\varphi_*^Y"] & \mathcal{F}_1(M, N^Y) \end{tikzcd}$$

Thus the assertion follows. □

Remark 6.4.10 *We denote by* $\varphi^f : L^X \to N^Y$ *the composition* $N^f\varphi^X = \varphi^Y L^f$*. It follows from (6.4.9) that* $(N^f)^g = (N^Y)^g(N^f)^Z = (N^f)^W(N^X)^g$ *for* $g \in \mathcal{T}(W, Z)$*.*

For $X \in \operatorname{Ob}\mathcal{T}$ and $N \in \operatorname{Ob}\mathcal{F}_1$, we define a morphism $\epsilon_N^X : (N^X)^X \to N^X$ of $\mathcal{F}_1$ to be the image of $\pi_X(N)\pi_X(N^X) \in \mathcal{F}_X(o_X^*((N^X)^X), o_X^*(N))$ by

$$E_X(N)_{(N^X)^X} : \mathcal{F}_X(o_X^*((N^X)^X), o_X^*(N)) \to \mathcal{F}_1((N^X)^X, N^X).$$

Proposition 6.4.11 *The following diagram commutes for any* $M \in \operatorname{Ob}\mathcal{F}_1$*.*

$$\begin{tikzcd} \mathcal{F}_X(o_X^*(M), o_X^*(N^X)) \arrow[r, "\pi_X(N)_*"] \arrow[d, "E_X(N^X)_M"] & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[d, "E_X(N)_M"] \\ \mathcal{F}_1(M, (N^X)^X) \arrow[r, "\epsilon^X_{N*}"] & \mathcal{F}_1(M, N^X) \end{tikzcd}$$

Proof. For $\varphi \in \mathcal{F}_1(M, (N^X)^X)$, by the definition of ϵ_N^X and the naturality of $E_X(N)$, we have

$$\begin{aligned} \pi_X(N)_* E_X(N^X)_M^{-1}(\varphi) &= \pi_X(N)\pi_X(N^X)o_X^*(\varphi) = o_X^*(\varphi)^* E_X(N)^{-1}_{(N^X)^X}(\epsilon_N^X) = E_X(N)_M^{-1}\varphi^*(\epsilon_N^X) \\ &= E_X(N)_M^{-1}\epsilon^X_{N*}(\varphi). \end{aligned}$$

□

We note that $\epsilon_N^X : (N^X)^X \to N^X$ is the unique morphism that makes the diagram of (6.4.11) commute for any $M \in \operatorname{Ob}\mathcal{F}_1$.

Remark 6.4.12 *If* $o_X^* : \mathcal{F}_1 \to \mathcal{F}_X$ *has a right adjoint* $o_{X!} : \mathcal{F}_X \to \mathcal{F}_1$*, the following diagram is commutative for any* $M \in \operatorname{Ob}\mathcal{F}_1$ *by the naturality of* ad^X*.*

$$\begin{tikzcd} \mathcal{F}_X(o_X^*(M), o_X^* o_{X!} o_X^*(N)) \arrow[r, "\varepsilon^X_{o_X^*(N)*}"] \arrow[d, "\mathrm{ad}^X_{M, o_X^* o_{X!} o_X^*(N)}"] & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[d, "\mathrm{ad}^X_{M, o_X^*(N)}"] \\ \mathcal{F}_1(M, o_{X!} o_X^* o_{X!} o_X^*(N)) \arrow[r, "o_{X!}\left(\varepsilon^X_{o_X^*(N)}\right)_*"] & \mathcal{F}_1(M, o_{X!} o_X^*(N)) \end{tikzcd}$$

It follows that $\epsilon_N^X = o_{X!}\left(\varepsilon^X_{o_X^*(N)}\right)$*.*

Lemma 6.4.13 *For a morphism* $f : Y \to X$ *of* $\mathcal{T}$ *and a morphism* $\varphi : M \to N$ *of* $\mathcal{F}_1$*, the following diagrams are commutative.*

$$\begin{array}{ccc} (M^X)^X & \xrightarrow{\epsilon_M^X} & M^X \\ \downarrow{\scriptstyle (\varphi^X)^X} & & \downarrow{\scriptstyle \varphi^X} \\ (N^X)^X & \xrightarrow{\epsilon_N^X} & N^X \end{array} \qquad \begin{array}{ccc} (N^X)^X & \xrightarrow{\epsilon_N^X} & N^X \\ \downarrow{\scriptstyle (N^f)^f} & & \downarrow{\scriptstyle N^f} \\ (N^Y)^Y & \xrightarrow{\epsilon_N^Y} & N^Y \end{array}$$

Proof. The following diagram is commutative by (1) of (6.4.3) for any $L \in \operatorname{Ob}\mathcal{F}_1$.

$$\begin{array}{ccc} \mathcal{F}_X(o_X^*(L), o_X^*(M^X)) & \xrightarrow{\pi_X(M)_*} & \mathcal{F}_X(o_X^*(L), o_X^*(M)) \\ \downarrow{\scriptstyle o_X^*(\varphi^X)_*} & & \downarrow{\scriptstyle o_X^*(\varphi)_*} \\ \mathcal{F}_X(o_X^*(L), o_X^*(N^X)) & \xrightarrow{\pi_X(N)_*} & \mathcal{F}_X(o_X^*(L), o_X^*(N)) \end{array}$$

Hence the following diagram commutes by (6.4.11) and (1) of (6.4.3).

$$\begin{array}{ccc} \mathcal{F}_1(L, (M^X)^X) & \xrightarrow{\epsilon_{M*}^X} & \mathcal{F}_1(L, M^X) \\ \downarrow{\scriptstyle (\varphi^X)_*^X} & & \downarrow{\scriptstyle \varphi_*^X} \\ \mathcal{F}_1(L, (N^X)^X) & \xrightarrow{\epsilon_{N*}^X} & \mathcal{F}_1(L, N^X) \end{array}$$

Thus the left diagram is commutative.

For $M \in \operatorname{Ob}\mathcal{F}_1$ and $\xi \in \mathcal{F}_X(o_X^*(M), o_Y^*(N^X))$, it follows from (6.4.8) and (6.1.17) that we have

$$\pi_Y(N) o_Y^*(N^f) f_{M,N^X}^\sharp(\xi) = f_{N^X,N}^\sharp(\pi_X(N)) f_{M,N^X}^\sharp(\xi) = f_{M,N}^\sharp(\pi_X(N)\xi).$$

This shows that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}_X(o_X^*(M), o_X^*(N^X)) & \xrightarrow{\pi_X(N)_*} & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \\ \downarrow{\scriptstyle o_Y^*(N^f)_* f_{M,N^X}^\sharp} & & \downarrow{\scriptstyle f_{M,N}^\sharp} \\ \mathcal{F}_Y(o_Y^*(M), o_Y^*(N^Y)) & \xrightarrow{\pi_Y(N)_*} & \mathcal{F}_Y(o_Y^*(M), o_Y^*(N)) \end{array}$$

The following diagram commutes by (1) of (6.4.3) and (6.4.6).

$$\begin{array}{ccccc} \mathcal{F}_X(o_X^*(M), o_X^*(N^X)) & \xrightarrow{f_{M,N^X}^\sharp} & \mathcal{F}_Y(o_Y^*(M), o_Y^*(N^X)) & \xrightarrow{o_Y^*(N^f)_*} & \mathcal{F}_Y(o_Y^*(M), o_Y^*(N^Y)) \\ \downarrow{\scriptstyle E_X(N^X)_M} & & \downarrow{\scriptstyle E_Y(N^X)_M} & & \downarrow{\scriptstyle E_Y(N)_M} \\ \mathcal{F}_1(M, (N^X)^X) & \xrightarrow{(N^X)_*^f} & \mathcal{F}_1(M, (N^X)^Y) & \xrightarrow{(N^f)_*^Y} & \mathcal{F}_1(M, (N^Y)^Y) \end{array}$$

Since $(N^f)^f = (N^f)^Y(N^X)^f$, it follows from (6.4.11) and (1) of (6.4.6) that the following diagram commutes for any $M \in \operatorname{Ob}\mathcal{F}_1$.

$$\begin{array}{ccc} \mathcal{F}_1(M, (N^X)^X) & \xrightarrow{\epsilon_{N*}^X} & \mathcal{F}_1(M, N^X) \\ \downarrow{\scriptstyle (N^f)_*^f} & & \downarrow{\scriptstyle N_*^f} \\ \mathcal{F}_1(M, (N^Y)^Y) & \xrightarrow{\epsilon_{N*}^Y} & \mathcal{F}_1(M, N^Y) \end{array}$$

Thus the right diagram is also commutative. □

Proposition 6.4.14 *The following diagrams are commutative.*

$$\begin{array}{ccc} o_X^*((N^X)^X) & \xrightarrow{o_X^*(\epsilon_N^X)} & o_X^*(N^X) \\ \downarrow{\scriptstyle \pi_X(N^X)} & & \downarrow{\scriptstyle \pi_X(N)} \\ o_X^*(N^X) & \xrightarrow{\pi_X(N)} & o_X^*(N) \end{array} \qquad \begin{array}{ccc} ((N^X)^X)^X & \xrightarrow{(\epsilon_N^X)^X} & (N^X)^X \\ \downarrow{\scriptstyle \epsilon_{N^X}^X} & & \downarrow{\scriptstyle \epsilon_N^X} \\ (N^X)^X & \xrightarrow{\epsilon_N^X} & N^X \end{array}$$

Proof. Since the following diagram commutes, we have $E_X(N)_{(N^X)^X}(\pi_X(N)\pi_X(N^X)) = \epsilon_N^X$.

$$\begin{tikzcd}
\mathcal{F}^X(o_X^*((N^X)^X), o_X^*(N^X)) \arrow[r, "\pi_X(N)_*"] \arrow[d, "E_X(N^X)_{(N^X)^X}"] & \mathcal{F}_X(o_X^*((N^X)^X), o_X^*(N)) \arrow[d, "E_X(N)_{(N^X)^X}"] \\
\mathcal{F}_1((N^X)^X, (N^X)^X) \arrow[r, "\epsilon^X_{N*}"] & \mathcal{F}_1((N^X)^X, N^X)
\end{tikzcd}$$

It follows from (6.4.2) that $\pi_X(N)\pi_X(N^X) = E_X(N)^{-1}_{(N^X)^X}(\epsilon^X_N) = \pi_X(N)o_X^*(\epsilon^X_N)$. Hence the following diagram commutes for $M \in \operatorname{Ob}\mathcal{F}_1$.

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(M), o_X^*((N^X)^X)) \arrow[r, "\pi_X(N^X)_*"] \arrow[d, "o_X^*(\epsilon^X_N)_*"] & \mathcal{F}_X(o_X^*(M), o_X^*(N^X)) \arrow[d, "\pi_X(N)_*"] \\
\mathcal{F}_X(o_X^*(M), o_X^*(N^X)) \arrow[r, "\pi_X(N)_*"] & \mathcal{F}_X(o_X^*(M), o_X^*(N))
\end{tikzcd}$$

Therefore the following diagram commutes by (6.4.11) and (1) of (6.4.3).

$$\begin{tikzcd}
\mathcal{F}_1(M, ((N^X)^X)^X) \arrow[r, "\epsilon^X_{N^X*}"] \arrow[d, "(\epsilon^X_N)^X_*"] & \mathcal{F}_1(M, (N^X)^X) \arrow[d, "\epsilon^X_{N*}"] \\
\mathcal{F}_1(M, (N^X)^X) \arrow[r, "\epsilon^X_{N*}"] & \mathcal{F}_1(M, N^X)
\end{tikzcd}$$

□

Proposition 6.4.15 *The following compositions coincide with the identity morphism of N^X.*

$$N^X = (N^X)^1 \xrightarrow{(N^X)^{o_X}} (N^X)^X \xrightarrow{\epsilon^X_N} N^X, \qquad N^X = (N^1)^X \xrightarrow{(N^{o_X})^X} (N^X)^X \xrightarrow{\epsilon^X_N} N^X$$

Proof. The following diagram commutes for any $M \in \operatorname{Ob}\mathcal{F}_1$ by (1) of (6.4.6) and (6.4.11).

$$\begin{tikzcd}
\mathcal{F}_1(o_1^*(M), o_1^*(N^X)) \arrow[r, "(o_X^\sharp)_{M,N^X}"] \arrow[d, "E_1(N^X)_M"] & \mathcal{F}_X(o_X^*(M), o_X^*(N^X)) \arrow[r, "\pi_X(N)_*"] \arrow[d, "E_X(N^X)_N"] & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[d, "E_X(N)_M"] \\
\mathcal{F}_1(M, (N^X)^1) \arrow[r, "(N^X)^{o_X}_*"] & \mathcal{F}_1(M, (N^X)^X) \arrow[r, "\epsilon^X_{N*}"] & \mathcal{F}_1(M, N^X)
\end{tikzcd}$$

It follows from (6.4.2) that $\epsilon^X_{N*}(N^X)^{o_X}_* : \mathcal{F}_1(M, N^X) = \mathcal{F}_1(M, (N^X)^1) \to \mathcal{F}_1(M, N^X)$ is the identity map of $\mathcal{F}_1(M, N^X)$.

The following diagram commutes by (1) of (6.4.3) and (6.4.11).

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(M), o_X^*(N^1)) \arrow[r, "o_X^*(N^{o_X})_*"] \arrow[d, "E_X(N^1)_M"] & \mathcal{F}_X(o_X^*(M), o_X^*(N^X)) \arrow[r, "\pi_X(N)_*"] \arrow[d, "E_X(N^X)_M"] & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[d, "E_X(N)_M"] \\
\mathcal{F}_1(M, (N^1)^X) \arrow[r, "(N^{o_X})^X_*"] & \mathcal{F}_1(M, (N^X)^X) \arrow[r, "\epsilon^X_{N*}"] & \mathcal{F}_1(M, N^X)
\end{tikzcd}$$

Since the composition of the upper horizontal maps of the above diagram coincides with the identity map of $\mathcal{F}_X(o_X^*(M), o_X^*(N))$ by (4) of (6.4.6), the composition of the lower horizontal maps of the above diagram is the identity map of $\mathcal{F}_1(M, N^X)$. □

Lemma 6.4.16 *If the following left diagram in $\mathcal{T}$ is commutative, then the following right diagram in $\mathcal{F}_1$ is commutative for $N \in \operatorname{Ob}\mathcal{F}_1$.*

$$\begin{tikzcd}
 & & X \arrow[dl, "g"'] \arrow[dr, "f"] & & \\
 & Z \arrow[dl, "j"'] \arrow[dr, "q"] & & Y \arrow[dl, "p"'] \arrow[dr, "i"] & \\
V & & W & & U
\end{tikzcd}
\qquad
\begin{tikzcd}
((N^V)^W)^U \arrow[r, "\epsilon^Y_{N^V}((N^V)^p)^i"] \arrow[d, "(\epsilon^Z_N(N^j)^q)^U"] & (N^V)^Y \arrow[d, "\epsilon^X_N(N^{jg})^f"] \\
(N^Z)^U \arrow[r, "\epsilon^X_N(N^g)^{if}"] & N^X
\end{tikzcd}$$

Proof. The following diagram is commutative by (6.4.14), (6.4.13), (6.4.9), (6.4.3) and (6.4.6).

$$\begin{tikzcd}
 & & ((N^V)^Y)^Y \arrow[r, "\epsilon^Y_{N^V}"] \arrow[d, "((N^{jg})^Y)^Y"] & (N^V)^Y \arrow[d, "(N^{jg})^Y"] \\
((N^V)^W)^U \arrow[r, "((N^V)^W)^i"] \arrow[d, "((N^j)^q)^U"] & ((N^V)^W)^Y \arrow[ur, "((N^V)^p)^Y"] \arrow[r, "((N^{jg})^p)^Y"] \arrow[d, "((N^j)^q)^f"] & ((N^X)^Y)^Y \arrow[r, "\epsilon^Y_{N^X}"] \arrow[d, "((N^X)^f)^f"] & (N^X)^Y \arrow[d, "(N^X)^f"] \\
((N^Z)^Z)^U \arrow[r, "((N^Z)^Z)^{if}"] \arrow[d, "(\epsilon^Z_N)^U"] & ((N^Z)^Z)^X \arrow[r, "((N^g)^g)^X"] \arrow[d, "(\epsilon^Z_N)^X"] & ((N^X)^X)^X \arrow[r, "\epsilon^X_{N^X}"] \arrow[d, "(\epsilon^X_N)^X"] & (N^X)^X \arrow[d, "\epsilon^X_N"] \\
(N^Z)^U \arrow[r, "(N^Z)^{if}"] & (N^Z)^X \arrow[r, "(N^g)^X"] & (N^X)^X \arrow[r, "\epsilon^X_N"] & N^X
\end{tikzcd}$$

Hence the asserion follows from (6.4.3) and (6.4.6). □

Proposition 6.4.17 *Suppose that $p : \mathcal{F} \to \mathcal{T}$ is a normalized cloven fibered category. For $X, Y \in \operatorname{Ob}\mathcal{T}$ and $N \in \operatorname{Ob}\mathcal{F}_1$, the following diagrams commutes.*

$$\begin{tikzcd}
(N^X)^1 \arrow[r, equal] \arrow[d, "(N^{\mathrm{pr}_X})^{\mathrm{pr}_2}"] & N^X \arrow[d, "N^{\mathrm{pr}_X}"] \\
(N^{X\times 1})^{X\times 1} \arrow[r, "\epsilon^{X\times 1}_N"] & N^{X\times 1}
\end{tikzcd}
\qquad
\begin{tikzcd}
(N^1)^Y \arrow[r, equal] \arrow[d, "(N^{\mathrm{pr}_1})^{\mathrm{pr}_Y}"] & N^Y \arrow[d, "N^{\mathrm{pr}_Y}"] \\
(N^{1\times Y})^{1\times Y} \arrow[r, "\epsilon^{1\times Y}_N"] & N^{1\times Y}
\end{tikzcd}$$

Proof. Since $\mathrm{pr}_2 = o_{X\times 1} : X \times 1 \to 1$ and $\mathrm{pr}_1 = o_{1\times Y} : 1 \times Y \to 1$, it follows from (6.4.15) that the following diagrams commutes.

$$\begin{tikzcd}
(N^X)^1 \arrow[r, equal] \arrow[d, "(N^{\mathrm{pr}_X})^1"] & N^X \arrow[d, "N^{\mathrm{pr}_X}"] \\
(N^{X\times 1})^1 \arrow[r, equal] \arrow[d, "(N^{X\times 1})^{\mathrm{pr}_2}"] & N^{X\times 1} \\
(N^{X\times 1})^{X\times 1} \arrow[ur, "\epsilon^{X\times 1}_N"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(N^1)^Y \arrow[r, equal] \arrow[d, "(N^1)^{\mathrm{pr}_Y}"] & N^Y \arrow[d, "N^{\mathrm{pr}_Y}"] \\
(N^1)^{1\times Y} \arrow[r, equal] \arrow[d, "(N^{\mathrm{pr}_1})^{1\times Y}"] & N^{1\times Y} \\
(N^{1\times Y})^{1\times Y} \arrow[ur, "\epsilon^{1\times Y}_N"'] &
\end{tikzcd}$$

Hence the assertions follows. □

Let X be an object of $\mathcal{T}$ and L, M, N objects of $\mathcal{F}_1$. We define a map

$$\gamma^X_{L,M,N} : \mathcal{F}_1(L, M^X) \times \mathcal{F}_1(M, N^X) \to \mathcal{F}_1(L, N^X)$$

as follows. For $\varphi \in \mathcal{F}_1(L, M^X)$ and $\psi \in \mathcal{F}_1(M, N^X)$, let $\gamma^X_{L,M,N}(\varphi, \psi)$ be the following composition.

$$L \xrightarrow{\varphi} M^X \xrightarrow{\psi^X} (N^X)^X \xrightarrow{\epsilon^X_N} N^X$$

Proposition 6.4.18 *The following diagram is commutative.*

$$\begin{tikzcd}
\mathcal{F}_X(o^*_X(L), o^*_X(M)) \times \mathcal{F}_X(o^*_X(M), o^*_X(N)) \arrow[r, "composition"] \arrow[d, "E_X(M)_L \times E_X(N)_M"] & \mathcal{F}_X(o^*_X(L), o^*_X(N)) \arrow[d, "E_X(N)_L"] \\
\mathcal{F}_1(L, M^X) \times \mathcal{F}_1(M, N^X) \arrow[r, "\gamma^X_{L,M,N}"] & \mathcal{F}_1(L, N^X)
\end{tikzcd}$$

Proof. For $\zeta \in \mathcal{F}_X(o^*_X(L), o^*_X(M))$ and $\xi \in \mathcal{F}_X(o^*_X(M), o^*_X(N))$, we put $\varphi = E_X(M)_L(\zeta)$ and $\psi = E_X(N)_M(\xi)$. Then, we have $\psi^X\varphi = E_X(N^X)_L(o^*_X(\psi)\zeta)$ by (6.4.3). It follows from (6.4.11) and (6.4.2) that

$$\epsilon^X_N\psi^X\varphi = \epsilon^X_{N*}E_X(N^X)_L(o^*_X(\psi)\zeta) = E_X(N)_L(\pi_X(N)o^*_X(\psi)\zeta) = E_X(N)_L(\xi\zeta).$$

Thus the result follows. □

Definition 6.4.19 *Let $p : \mathcal{F} \to \mathcal{T}$ be a normalized cloven fibered category. We say that $p : \mathcal{F} \to \mathcal{T}$ is a fibered category with exponentials if the presheaf F^X_N on $\mathcal{F}_1$ is representable for any $X \in \operatorname{Ob}\mathcal{T}$ and $N \in \operatorname{Ob}\mathcal{F}_1$.*

Let $p : \mathcal{F} \to \mathcal{T}$ be a fibered category with exponentials. Assume that $\mathcal{T}$ has finite products. For $X, Y \in \operatorname{Ob}\mathcal{T}$ and $N \in \operatorname{Ob}\mathcal{F}_1$, we define a morphism $\theta^{X,Y}(N) : (N^X)^Y \to N^{X\times Y}$ of $\mathcal{F}_1$ to be the following composition.

$$(N^X)^Y \xrightarrow{(N^{\mathrm{pr}_X})^{\mathrm{pr}_Y}} (N^{X\times Y})^{X\times Y} \xrightarrow{\epsilon_N^{X\times Y}} N^{X\times Y}$$

Proposition 6.4.20 *For morphisms $f : X \to Z$, $g : Y \to W$ of $\mathcal{T}$ and a morphism $\varphi : M \to N$ of $\mathcal{F}_1$, the following diagrams are commutative.*

$$\begin{tikzcd}
(M^X)^Y \arrow[r, "\theta^{X,Y}(M)"] \arrow[d, "(\varphi^X)^Y"] & M^{X\times Y} \arrow[d, "\varphi^{X\times Y}"] \\
(N^X)^Y \arrow[r, "\theta^{X,Y}(N)"] & N^{X\times Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
(N^Z)^W \arrow[r, "\theta^{Z,W}(N)"] \arrow[d, "(N^f)^g"] & N^{Z\times W} \arrow[d, "N^{f\times g}"] \\
(N^X)^Y \arrow[r, "\theta^{X,Y}(N)"] & N^{X\times Y}
\end{tikzcd}$$

Proof. The following diagrams commute by (6.4.13), (6.4.9), (6.4.3) and (6.4.6).

$$\begin{tikzcd}
(M^X)^Y \arrow[r, "(M^{\mathrm{pr}_X})^{\mathrm{pr}_Y}"] \arrow[d, "(\varphi^X)^Y"] & (M^{X\times Y})^{X\times Y} \arrow[r, "\epsilon_M^{X\times Y}"] \arrow[d, "(\varphi^{X\times Y})^{X\times Y}"] & M^{X\times Y} \arrow[d, "\varphi^{X\times Y}"] \\
(N^X)^Y \arrow[r, "(N^{\mathrm{pr}_X})^{\mathrm{pr}_Y}"] & (N^{X\times Y})^{X\times Y} \arrow[r, "\epsilon_N^{X\times Y}"] & N^{X\times Y}
\end{tikzcd}
\quad
\begin{tikzcd}
(N^Z)^W \arrow[r, "(N^{\mathrm{pr}_Z})^{\mathrm{pr}_W}"] \arrow[d, "(N^f)^g"] & (N^{Z\times W})^{Z\times W} \arrow[r, "\epsilon_N^{Z\times W}"] \arrow[d, "(N^{f\times g})^{f\times g}"] & N^{Z\times W} \arrow[d, "N^{f\times g}"] \\
(N^X)^Y \arrow[r, "(N^{\mathrm{pr}_X})^{\mathrm{pr}_Y}"] & (N^{X\times Y})^{X\times Y} \arrow[r, "\epsilon_N^{X\times Y}"] & N^{X\times Y}
\end{tikzcd}$$

Hence the assertion follows. □

Proposition 6.4.21 *For $X, Y, Z \in \operatorname{Ob}\mathcal{T}$ and $N \in \operatorname{Ob}\mathcal{F}_1$, the following diagram is commutative.*

$$\begin{tikzcd}
((N^X)^Y)^Z \arrow[r, "\theta^{Y,Z}(N^X)"] \arrow[d, "\theta^{X,Y}(N)^Z"] & (N^X)^{Y\times Z} \arrow[d, "\theta^{X,Y\times Z}(N)"] \\
(N^{X\times Y})^Z \arrow[r, "\theta^{X\times Y,Z}(N)"] & N^{X\times Y\times Z}
\end{tikzcd}$$

Proof. Let us denote by $p_{X\times Y} : X\times Y\times Z \to X\times Y$, $p_{Y\times Z} : X\times Y\times Z \to Y\times Z$, $\mathrm{pr}_X : X\times Y \to X$, $\mathrm{pr}_Y : X\times Y \to Y$, $\mathrm{pr}'_Y : Y\times Z \to Y$ and $\mathrm{pr}'_Z : Y\times Z \to Z$ the projections. Since

$$\begin{aligned}
\theta^{X\times Y,Z}(N) &= \epsilon_N^{X\times Y\times Z}(N^{p_{X\times Y}})^{\mathrm{pr}'_Z p_{Y\times Z}} : (N^{X\times Y})^Z \to N^{X\times Y\times Z} \\
\theta^{X,Y\times Z}(N) &= \epsilon_N^{X\times Y\times Z}(N^{\mathrm{pr}_X p_{X\times Y}})^{p_{Y\times Z}} : (N^X)^{Y\times Z} \to N^{X\times Y\times Z} \\
\theta^{Y,Z}(N^X) &= \epsilon_{N^X}^{Y\times Z}((N^X)^{\mathrm{pr}'_Y})^{\mathrm{pr}'_Z} : ((N^X)^Y)^Z \to (N^X)^{Y\times Z} \\
\theta^{X,Y}(N)^Z &= (\epsilon_N^{X\times Y}(N^{\mathrm{pr}_X})^{\mathrm{pr}_Y})^Z : ((N^X)^Y)^Z \to (N^{X\times Y})^Z,
\end{aligned}$$

the assertion follows by applying (6.4.16) for $f = p_{Y\times Z}$, $g = p_{X\times Y}$, $p = \mathrm{pr}'_Y$, $q = \mathrm{pr}_Y$, $i = \mathrm{pr}'_Z$ and $j = \mathrm{pr}_X$. □

Proposition 6.4.22 *For an object X, Y of $\mathcal{T}$ and an object N, $\theta^{X,1}(N) : N^X = (N^X)^1 \to N^{X\times 1}$ is identified with $N^{\mathrm{pr}_X} : N^X \to N^{X\times 1}$ and $\theta^{1,Y}(N) : N^Y = (N^1)^Y \to N^{1\times Y}$ is identified with $N^{\mathrm{pr}_Y} : N^Y \to N^{1\times Y}$.*

Proof. This is a direct consequence of (6.4.17). □

Lemma 6.4.23 *For objects X, Y of $\mathcal{T}$ and an object N of $\mathcal{F}_1$, the following diagram is commutative.*

$$\begin{tikzcd}
o^*_{X\times Y}((N^X)^Y) \arrow[r, "\mathrm{pr}_Y^\sharp(\pi_Y(N^X))"] \arrow[d, "o^*_{X\times Y}(\theta^{X,Y}(N))"] & o^*_{X\times Y}(N^X) \arrow[d, "\mathrm{pr}_X^\sharp(\pi_X(N))"] \\
o^*_{X\times Y}(N^{X\times Y}) \arrow[r, "\pi_{X\times Y}(N)"] & o^*_{X\times Y}(N)
\end{tikzcd}$$

Proof. It follows from (6.4.8) and (1) of (6.4.3) that we have

$$\begin{aligned}
\mathrm{pr}_X^\sharp(\pi_X(N))\mathrm{pr}_Y^\sharp(\pi_Y(N^X)) &= \pi_{X\times Y}(N)o^*_{X\times Y}(N^{\mathrm{pr}_X})\pi_{X\times Y}(N^X)o^*_{X\times Y}((N^X)^{\mathrm{pr}_Y}) \\
&= \pi_{X\times Y}(N)\pi_{X\times Y}(N^{X\times Y})o^*_{X\times Y}((N^{\mathrm{pr}_X})^{X\times Y})o^*_{X\times Y}((N^X)^{\mathrm{pr}_Y}) \\
&= \pi_{X\times Y}(N)\pi_{X\times Y}(N^{X\times Y})o^*_{X\times Y}((N^{\mathrm{pr}_X})^{\mathrm{pr}_Y})
\end{aligned}$$

By the naturality of $E_{X\times Y}(N)$ and the definition of $\epsilon_N^{X\times Y}$,

$$E_{X\times Y}(N)_{(N^X)^Y} : \mathcal{F}_{X\times Y}(o^*_{X\times Y}((N^X)^Y), o^*_{X\times Y}(N)) \to \mathcal{F}_1((N^X)^Y, N^{X\times Y})$$

maps $\mathrm{pr}_X^\sharp(\pi_X(N))\mathrm{pr}_Y^\sharp(\pi_Y(N^X))$ to $\epsilon_N^{X\times Y}(N^{\mathrm{pr}_X})^{\mathrm{pr}_Y} = \theta^{X,Y}(N)$. On the other hand, $E_{X\times Y}(N)_{(N^X)^Y}$ also maps $\pi_{X\times Y}(N)o^*_{X\times Y}(\theta^{X\times Y}(N))$ to $\theta^{X\times Y}(N)$ by (6.4.2). □

Proposition 6.4.24 *For $X \in \mathrm{Ob}\,\mathcal{T}$ and $N \in \mathrm{Ob}\,\mathcal{F}_1$, we have $N^{\Delta_X}\theta^{X,X}(N) = \epsilon_N^X$.*

Proof. We denote by $\mathrm{pr}_i : X \times X \to X$ the projection to i th component for $i = 1, 2$. It follows from the commutativity of the right diagram of (6.4.13) that

$$N^{\Delta_X}\theta^{X,X}(N) = N^{\Delta_X}\epsilon_N^{X\times X}(N^{\mathrm{pr}_1})^{\mathrm{pr}_2} = \epsilon_N^X(N^{\Delta_X})^{\Delta_X}(N^{\mathrm{pr}_1})^{\mathrm{pr}_2} = \epsilon_N^X(N^{\mathrm{pr}_1\Delta_X})^{\mathrm{pr}_2\Delta_X} = \epsilon_N^X$$

since $\mathrm{pr}_1\Delta_X = \mathrm{pr}_2\Delta_X = id_X$. □

Definition 6.4.25 *Let $p : \mathcal{F} \to \mathcal{T}$ be a fibered category with exponents. Suppose that $\mathcal{T}$ has finite products. If $\theta^{X,Y}(N) : (N^X)^Y \to N^{X\times Y}$ is an isomorphism for any $X, Y \in \mathrm{Ob}\,\mathcal{T}$ and $N \in \mathrm{Ob}\,\mathcal{F}_1$, we say that $p : \mathcal{F} \to \mathcal{T}$ is an associative fibered category with exponents.*

6.5 Cartesian closed fibered category

Proposition 6.5.1 *Let $\mathcal{E}$ be a category with finite limits and a terminal object 1. Let $p : \mathcal{E}^{(2)} \to \mathcal{E}$ be the fibered category given in (2) of (6.1.9). For objects X and Z of $\mathcal{E}$, define a functor $G_Z^X : \mathcal{E}_1^{(2)op} \to \mathcal{S}et$ by $G_Z^X(Y \xrightarrow{o_Y} 1) = \mathcal{E}_X^{(2)}(o_X^*(Y \xrightarrow{o_Y} 1), o_X^*(Z \xrightarrow{o_Z} 1))$ and $G_Z^X(f) = (f \times id_X)^*$. Then, $\mathcal{E}$ is cartesian closed if and only if G_Z^X is representable for any $X, Z \in \mathrm{Ob}\,\mathcal{E}$.*

Proof. For $X, Y, Z \in \mathrm{Ob}\,\mathcal{E}$, let us denote by $q_{Y,X} : Y \times X \to X$, $q_{Z,X} : Z \times X \to X$ and $p_{Z,X} : Z \times X \to Z$ the projections. Since $o_X^*(Y \xrightarrow{o_Y} 1) = (Y \times X \xrightarrow{q_{Y,X}} X)$, we have

$$G_Z^X(Y \xrightarrow{o_Y} 1) = \mathcal{E}_X^{(2)}(o_X^*(Y \xrightarrow{o_Y} 1), o_X^*(Z \xrightarrow{o_Z} 1)) = \{f \in \mathcal{E}(Y \times X, Z \times X) \,|\, q_{Z,X}f = q_{Y,X}\}.$$

Define a map $\Phi : \mathcal{E}_X^{(2)}(o_X^*(Y \xrightarrow{o_Y} 1), o_X^*(Z \xrightarrow{o_Z} 1)) \to \mathcal{E}(Y \times X, Z)$ by $\Phi(f) = p_{Z,X}f$. It is clear that Φ is bijective and natural in Y.

If G_Z^X is representable for any $X, Z \in \mathrm{Ob}\,\mathcal{E}$, there exist $(W \xrightarrow{o_W} 1) \in \mathrm{Ob}\,\mathcal{E}_1^{(2)}$ and a bijection

$$G_Z^X(Y \xrightarrow{o_Y} 1) = \mathcal{E}_X^{(2)}(o_X^*(Y \xrightarrow{o_Y} 1), o_X^*(Z \xrightarrow{o_Z} 1)) \to \mathcal{E}_1^{(2)}((Y \xrightarrow{o_Y} 1), (W \xrightarrow{o_W} 1))$$

which is natural in Y. Since $\mathcal{E}_1^{(2)}((Y \xrightarrow{o_Y} 1), (W \xrightarrow{o_W} 1))$ is identified with $\mathcal{E}(Y, W)$, we have a bijection $\mathcal{E}(Y \times X, Z) \to \mathcal{E}(Y, W)$ which is natural in Y. Conversely, assume that $\mathcal{E}$ is cartesian closed. For $X, Z \in \mathrm{Ob}\,\mathcal{E}$, since $\mathcal{E}_1^{(2)}((Y \xrightarrow{o_Y} 1), (Z^X \xrightarrow{o_{Z^X}} 1))$ is identified with $\mathcal{E}(Y, Z^X)$ and there is a bijection $\mathcal{E}(Y, Z^X) \to \mathcal{E}(Y \times X, Z)$ which is natural in Y, G_Z^X is representable. □

Lemma 6.5.2 *Let X be an object of $\mathcal{T}$ and $\varphi : M \to N$ a morphism of $\mathcal{F}_1$.*

(1) Suppose that the presheaf F_N^X on $\mathcal{F}_1$ is representable. If φ is an epimorphism,

$$o_X^*(\varphi)^* : \mathcal{F}_X(o_X^*(N), o_X^*(N)) \to \mathcal{F}_X(o_X^*(M), o_X^*(N))$$

is injective. If φ is a coequalizer of morphisms $\alpha, \beta : L \to M$ of $\mathcal{F}_1$, $o_X^*(\varphi)^*$ is an equalizer of

$$o_X^*(\alpha)^*, o_X^*(\beta)^* : \mathcal{F}_X(o_X^*(M), o_X^*(N)) \to \mathcal{F}_X(o_X^*(L), o_X^*(N)).$$

(2) Suppose that the presheaf $F_{K,M}$ on $\mathcal{F}_1^{op}$ is representable. If φ is an monomorphism,

$$o_X^*(\varphi)_* : \mathcal{F}_X(o_X^*(N), o_X^*(N)) \to \mathcal{F}_X(o_X^*(M), o_X^*(N))$$

is injective. If φ is an equalizer of morphisms $\alpha, \beta : N \to L$ of $\mathcal{F}_1$, $o_X^*(\varphi)_*$ is an equalizer of

$$o_X^*(\alpha)_*, o_X^*(\beta)_* : \mathcal{F}_X(o_X^*(M), o_X^*(N)) \to \mathcal{F}_X(o_X^*(M), o_X^*(L)).$$

Proof. (1) Suppose that φ is an epimorphism. We have the following commutative diagram by the assumption.

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(N), o_X^*(N)) \arrow[r, "o_X^*(\varphi)^*"] \arrow[d, "E_X(N)_N"] & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[d, "E_X(N)_M"] \\
\mathcal{F}_1(N, N^X) \arrow[r, "\varphi^*"] & \mathcal{F}_1(M, N^X)
\end{tikzcd}$$

Since both φ^* and $E_X(N)_N$ are injective, so is $o_X^*(\varphi)^*$.

Suppose that φ is a coequalizer of $\alpha, \beta : L \to M$. Then, $\varphi^* : \mathcal{F}_1(N, N^X) \to \mathcal{F}_1(M, N^X)$ is an equalizer of $\alpha^*, \beta^* : \mathcal{F}_1(M, N^X) \to \mathcal{F}_1(L, N^X)$. The following diagram is commutative for $\psi = \alpha, \beta$.

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[r, "o_X^*(\psi)^*"] \arrow[d, "E_X(N)_M"] & \mathcal{F}_X(o_X^*(L), o_X^*(N)) \arrow[d, "E_X(N)_L"] \\
\mathcal{F}_1(M, N^X) \arrow[r, "\psi^*"] & \mathcal{F}_1(L, N^X)
\end{tikzcd}$$

Since the vertical maps of the above diagram are bijective, $o_X^*(\varphi)^*$ is an equalizer of $o_X^*(\alpha)^*$, $o_X^*(\beta)^*$.

(2) Suppose that φ is a monomorphism. We have the following commutative diagram by the assumption.

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(M), o_X^*(M)) \arrow[r, "o_X^*(\varphi)_*"] \arrow[d, "P_X(M)_M"] & \mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[d, "P_X(M)_N"] \\
\mathcal{F}_1(X \times M, M) \arrow[r, "\varphi_*"] & \mathcal{F}_1(X \times M, N)
\end{tikzcd}$$

Since both φ_* and $P_X(M)_M$ are injective, so is $o_X^*(\varphi)_*$.

Suppose that φ is an equalizer of $\alpha, \beta : N \to L$. Then, $\varphi_* : \mathcal{F}_1(X \times M, M) \to \mathcal{F}_1(X \times M, N)$ is an equalizer of $\alpha_*, \beta_* : \mathcal{F}_1(X \times M, N) \to \mathcal{F}_1(X \times M, L)$.

$$\begin{tikzcd}
\mathcal{F}_X(o_X^*(M), o_X^*(N)) \arrow[r, "o_X^*(\psi)_*"] \arrow[d, "P_X(M)_M"] & \mathcal{F}_X(o_X^*(M), o_X^*(L)) \arrow[d, "P_X(M)_L"] \\
\mathcal{F}_1(X \times M, N) \arrow[r, "\psi_*"] & \mathcal{F}_1(X \times M, L)
\end{tikzcd}$$

Since the vertical maps of the above diagram are bijective, $o_X^*(\varphi)_*$ is an equalizer of $o_X^*(\alpha)_*$, $o_X^*(\beta)_*$. □

Proposition 6.5.3 *Let $p : \mathcal{F} \to \mathcal{T}$ be a cloven fibered category and $X \in \operatorname{Ob}\mathcal{T}$, $(\varphi : M \to N) \in \operatorname{Mor}\mathcal{F}_1$.*

(1) Suppose that the presheaf F_K^X on $\mathcal{F}_1$ is representable for any $K \in \operatorname{Ob}\mathcal{F}_1$ and that the presheaves $F_{X,M}$ and $F_{X,N}$ on $\mathcal{F}_1^{op}$ are representable. If $\varphi : M \to N$ is an epimorphism, so is $X \times \varphi : X \times M \to X \times N$.

(2) Suppose that the presheaf $F_{X,K}$ on $\mathcal{F}_1^{op}$ is representable for any $K \in \operatorname{Ob}\mathcal{F}_1$ and that the presheaves F_M^X and F_N^X on $\mathcal{F}_1$ are representable. If $\varphi : M \to N$ is a monomorphism, so is $\varphi^X : M^X \to N^X$.

Proof. (1) The following diagram commutes by (6.3.3) and the naturality of $E_X(K)$.

$$\begin{tikzcd}
\mathcal{F}_1(X \times N, K) \arrow[d, "(X\times\varphi)^*"] & \mathcal{F}_X(o_X^*(N), o_X^*(K)) \arrow[l, "P_X(N)_K"'] \arrow[r, "E_X(K)_N"] \arrow[d, "o_X^*(\varphi)^*"] & \mathcal{F}_1(N, K^X) \arrow[d, "\varphi^*"] \\
\mathcal{F}_1(X \times M, K) & \mathcal{F}_X(o_X^*(M), o_X^*(K)) \arrow[l, "P_X(M)_K"'] \arrow[r, "E_X(K)_M"] & \mathcal{F}_1(M, K^X)
\end{tikzcd}$$

Since $\varphi^* : \mathcal{F}_1(N, K^X) \to \mathcal{F}_1(M, K^X)$ is injective by the assumption, it follows from the above diagram that $(X \times \varphi)^* : \mathcal{F}_1(X \times N, K) \to \mathcal{F}_1(X \times M, K)$ is also injective.

(2) The following diagrams commute by (6.4.3) and the naturality of $P_X(K)$.

$$\begin{tikzcd}
\mathcal{F}_1(K, M^X) \arrow[d, "\varphi_*^X"] & \mathcal{F}_X(o_X^*(K), o_X^*(M)) \arrow[l, "E_X(M)_K"'] \arrow[r, "P_X(K)_M"] \arrow[d, "o_X^*(\varphi)_*"] & \mathcal{F}_1(X \times K, M) \arrow[d, "\varphi_*"] \\
\mathcal{F}_1(K, N^X) & \mathcal{F}_X(o_X^*(K), o_X^*(N)) \arrow[l, "E_X(N)_K"'] \arrow[r, "P_X(K)_N"] & \mathcal{F}_1(X \times K, N)
\end{tikzcd}$$

Since $\varphi_* : \mathcal{F}_1(X \times K, M) \to \mathcal{F}_1(X \times K, N)$ is injective by the assumption, it follows from the above diagram that $\varphi^X : \mathcal{F}_1(K, M^X) \to \mathcal{F}_1(K, N^X)$ is also injective. □

Proposition 6.5.4 *Let $p : \mathcal{F} \to \mathcal{T}$ be a cloven fibered category and $X \in \operatorname{Ob}\mathcal{T}$, $L, M, N \in \operatorname{Ob}\mathcal{F}_1$.*

(1) Suppose that the presheaf F^X_K on $\mathcal{F}_1$ is representable for any $K \in \operatorname{Ob}\mathcal{F}_1$ and that the presheaves $F_{X,L}$, $F_{X,M}$, $F_{X,N}$ on $\mathcal{F}_1^{op}$ are representable. If $\lambda : N \to L$ is a coequalizer of morphisms $\varphi, \psi : M \to N$ of $\mathcal{F}_1$, then $X \times \lambda : X \times N \to X \times L$ is a coequalizer of morphisms $X \times \varphi, X \times \psi : X \times M \to X \times N$.

(2) Suppose that the presheaf $F_{X,K}$ on $\mathcal{F}_1^{op}$ is representable for any $K \in \operatorname{Ob}\mathcal{F}_1$ and that the presheaves F^X_L, F^X_M, F^X_N on $\mathcal{F}_1$ are representable. If $\lambda : L \to M$ is an equalizer of morphisms $\varphi, \psi : M \to N$ of $\mathcal{F}_1$, then $\lambda^X : L^X \to M^X$ is an equalizer of morphisms $\varphi^X, \psi^X : M^X \to N^N$.

Proof. (1) The following diagrams commute by (6.3.3) and the naturality of $E_X(K)$.

$$\begin{tikzcd}
\mathcal{F}_1(X \times N, K) \arrow[d, "(X\times\varphi)^*"] & \mathcal{F}_X(o_X^*(N), o_X^*(K)) \arrow[l, "P_X(N)_K"'] \arrow[r, "E_X(K)_N"] \arrow[d, "o_X^*(\varphi)^*"] & \mathcal{F}_1(N, K^X) \arrow[d, "\varphi^*"] \\
\mathcal{F}_1(X \times M, K) & \mathcal{F}_X(o_X^*(M), o_X^*(K)) \arrow[l, "P_X(M)_K"'] \arrow[r, "E_X(K)_M"] & \mathcal{F}_1(M, K^X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_1(X \times N, K) \arrow[d, "(X\times\psi)^*"] & \mathcal{F}_X(o_X^*(N), o_X^*(K)) \arrow[l, "P_X(N)_K"'] \arrow[r, "E_X(K)_N"] \arrow[d, "o_X^*(\psi)^*"] & \mathcal{F}_1(N, K^X) \arrow[d, "\psi^*"] \\
\mathcal{F}_1(X \times M, K) & \mathcal{F}_X(o_X^*(M), o_X^*(K)) \arrow[l, "P_X(M)_K"'] \arrow[r, "E_X(K)_M"] & \mathcal{F}_1(M, K^X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_1(X \times L, K) \arrow[d, "(X\times\lambda)^*"] & \mathcal{F}_X(o_X^*(L), o_X^*(K)) \arrow[l, "P_X(L)_K"'] \arrow[r, "E_X(K)_L"] \arrow[d, "o_X^*(\lambda)^*"] & \mathcal{F}_1(L, K^X) \arrow[d, "\lambda^*"] \\
\mathcal{F}_1(X \times N, K) & \mathcal{F}_X(o_X^*(N), o_X^*(K)) \arrow[l, "P_X(N)_K"'] \arrow[r, "E_X(K)_N"] & \mathcal{F}_1(N, K^X)
\end{tikzcd}$$

Since $\lambda^* : \mathcal{F}_1(L, K^X) \to \mathcal{F}_1(N, K^X)$ is an equalizer of maps $\varphi^*, \psi^* : \mathcal{F}_1(N, K^X) \to \mathcal{F}_1(M, K^X)$, it follows from the above diagrams that $(X \times \lambda)^* : \mathcal{F}_1(X \times L, K) \to \mathcal{F}_1(X \times N, K)$ is an equalizer of maps $(X \times \varphi)^*, (X \times \psi)^* : \mathcal{F}_1(X \times N, K) \to \mathcal{F}_1(X \times M, K)$.

(2) The following diagrams commute by (6.4.3) and the naturality of $P_X(K)$.

$$\begin{tikzcd}
\mathcal{F}_1(K, M^X) \arrow[d, "\varphi^X_*"] & \mathcal{F}_X(o_X^*(K), o_X^*(M)) \arrow[l, "E_X(M)_K"'] \arrow[r, "P_X(K)_M"] \arrow[d, "o_X^*(\varphi)_*"] & \mathcal{F}_1(X \times K, M) \arrow[d, "\varphi_*"] \\
\mathcal{F}_1(K, N^X) & \mathcal{F}_X(o_X^*(K), o_X^*(N)) \arrow[l, "E_X(N)_K"'] \arrow[r, "P_X(K)_N"] & \mathcal{F}_1(X \times K, N)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_1(K, M^X) \arrow[d, "\psi^X_*"] & \mathcal{F}_X(o_X^*(K), o_X^*(M)) \arrow[l, "E_X(M)_K"'] \arrow[r, "P_X(K)_M"] \arrow[d, "o_X^*(\psi)_*"] & \mathcal{F}_1(X \times K, M) \arrow[d, "\psi_*"] \\
\mathcal{F}_1(K, N^X) & \mathcal{F}_X(o_X^*(K), o_X^*(N)) \arrow[l, "E_X(N)_K"'] \arrow[r, "P_X(K)_N"] & \mathcal{F}_1(X \times K, N)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_1(K, M^X) \arrow[d, "\lambda^X_*"] & \mathcal{F}_X(o_X^*(K), o_X^*(M)) \arrow[l, "E_X(M)_K"'] \arrow[r, "P_X(K)_M"] \arrow[d, "o_X^*(\lambda)_*"] & \mathcal{F}_1(X \times K, M) \arrow[d, "\lambda_*"] \\
\mathcal{F}_1(K, N^X) & \mathcal{F}_X(o_X^*(K), o_X^*(N)) \arrow[l, "E_X(N)_K"'] \arrow[r, "P_X(K)_N"] & \mathcal{F}_1(X \times K, N)
\end{tikzcd}$$

Since $\lambda_* : \mathcal{F}_1(X \times K, L) \to \mathcal{F}_1(X \times K, M)$ is an equalizer of maps $\varphi_*, \psi_* : \mathcal{F}_1(X \times K, M) \to \mathcal{F}_1(X \times K, N)$, it follows from the above diagrams that $\lambda_* : \mathcal{F}_1(K, L^X) \to \mathcal{F}_1(K, M^X)$ is an equalizer of maps $\varphi^X_*, \psi^X_* : \mathcal{F}_1(K, M^X) \to \mathcal{F}_1(K, N^X)$. □

Proposition 6.5.5 *For $X, Y \in \operatorname{Ob}\mathcal{T}$ and $M, N \in \operatorname{Ob}\mathcal{F}_1$, the following diagram is commutative.*

$$
\begin{tikzcd}
\mathcal{F}_1(X \times (Y \times M), N) \arrow[r, "\theta_{X,Y}(M)^*"] \arrow[d, "P_X(Y\times M)_N^{-1}"] & \mathcal{F}_1((X \times Y) \times M, N) \arrow[r, "P_{X\times Y}(M)_N^{-1}"] & \mathcal{F}_{X\times Y}(o^*_{X\times Y}(M), o^*_{X\times Y}(N)) \arrow[d, "E_{X\times Y}(N)_M"] \\
\mathcal{F}_X(o^*_X(Y \times M), o^*_X(N)) \arrow[d, "E_X(N)_{Y\times M}"] & & \mathcal{F}_1(M, N^{X\times Y}) \\
\mathcal{F}_1(Y \times M, N^X) \arrow[r, "P_Y(M)_{N^X}^{-1}"] & \mathcal{F}_Y(o^*_Y(M), o^*_Y(N^X)) \arrow[r, "E_Y(N^X)_M"] & \mathcal{F}_1(M, (N^X)^Y) \arrow[u, "\theta^{X,Y}(N)_*"']
\end{tikzcd}
$$

Proof. For $\varphi \in \mathcal{F}_1(X \times (Y \times M), N)$, we put $\psi = E_X(N)_{Y\times M} P_X(Y\times M)_N^{-1}(\varphi)$ and $\xi = E_Y(N^X)_M P_Y(M)_{N^X}^{-1}(\psi)$. It follows from (6.3.2) and (6.4.2) that the following diagrams commute.

$$
\begin{tikzcd}
o^*_X(Y \times M) \arrow[r, "\iota_X(Y\times M)"] \arrow[d, "o^*_X(\psi)"] & o^*_X(X \times (Y \times M)) \arrow[d, "o^*_X(\varphi)"] \\
o^*_X(N^X) \arrow[r, "\pi_X(N)"] & o^*_X(N)
\end{tikzcd}
\qquad
\begin{tikzcd}
o^*_Y(M) \arrow[r, "\iota_Y(M)"] \arrow[d, "o^*_Y(\xi)"] & o^*_Y(Y \times M) \arrow[d, "o^*_Y(\psi)"] \\
o^*_Y((N^X)^Y) \arrow[r, "\pi_Y(N^X)"] & o^*_Y(N^X)
\end{tikzcd}
$$

By applying $\mathrm{pr}_X^\sharp$ to the above left diagram and $\mathrm{pr}_Y^\sharp$ to the right one, we have the following commutative diagram by (6.1.17).

$$
\begin{tikzcd}
o^*_{X\times Y}(M) \arrow[r, "\mathrm{pr}_Y^\sharp(\iota_Y(M))"] \arrow[d, "o^*_{X\times Y}(\xi)"] & o^*_{X\times Y}(Y \times M) \arrow[r, "\mathrm{pr}_X^\sharp(\iota_X(Y\times M))"] \arrow[d, "o^*_{X\times Y}(\psi)"] & o^*_{X\times Y}(X \times (Y \times M)) \arrow[d, "o^*_{X\times Y}(\varphi)"] \\
o^*_{X\times Y}((N^X)^Y) \arrow[r, "\mathrm{pr}_Y^\sharp(\pi_Y(N^X))"] & o^*_{X\times Y}(N^X) \arrow[r, "\mathrm{pr}_X^\sharp(\pi_X(N))"] & o^*_{X\times Y}(N)
\end{tikzcd}
$$

Hence, by (6.3.23) and (6.4.23), the following diagram commutes.

$$
\begin{tikzcd}
o^*_{X\times Y}(M) \arrow[r, "o^*_{X\times Y}(\theta_{X,Y}(M))\iota_{X\times Y}(M)"] \arrow[d, "o^*_{X\times Y}(\xi)"] & o^*_{X\times Y}(X \times (Y \times M)) \arrow[d, "o^*_{X\times Y}(\varphi)"] \\
o^*_{X\times Y}((N^X)^Y) \arrow[r, "\pi_{X\times Y}(N)o^*_{X\times Y}(\theta^{X,Y}(N))"] & o^*_{X\times Y}(N)
\end{tikzcd}
$$

By (6.3.2) and (6.4.2), we have

$$
\begin{aligned}
P_{X\times Y}(M)_N(o^*_{X\times Y}(\varphi)o^*_{X\times Y}(\theta_{X,Y}(M))\iota_{X\times Y}(M)) &= P_{X\times Y}(M)_N(o^*_{X\times Y}(\varphi\theta_{X,Y}(M))\iota_{X\times Y}(M)) = \varphi\theta_{X,Y}(N) \\
E_{X\times Y}(N)_M(\pi_{X\times Y}(N)o^*_{X\times Y}(\theta^{X,Y}(N))o^*_{X\times Y}(\xi)) &= E_{X\times Y}(N)_M(\pi_{X\times Y}(N)o^*_{X\times Y}(\theta^{X,Y}(N)\xi)) = \theta^{X,Y}(N)\xi.
\end{aligned}
$$

This shows that $P_{X\times Y}(M)_N^{-1}(\varphi\theta_{X,Y}(N)) = E_{X\times Y}(N)_M^{-1}(\theta^{X,Y}(N)\xi)$, which implies the result. $\square$

Remark 6.5.6 *The above result implies that $\theta_{X,Y}(M) : (X \times Y) \times M \to X \times (Y \times M)$ is an isomorphism for all $M \in \mathrm{Ob}\,\mathcal{F}_1$ if and only if $\theta^{X,Y}(N) : (N^X)^Y \to N^{X\times Y}$ is an isomorphism for all $N \in \mathrm{Ob}\,\mathcal{F}_1$.*

Definition 6.5.7 *A normalized cloven fibered category $p : \mathcal{F} \to \mathcal{T}$ is called a cartesian closed fibered category if the following conditions are satisfied.*

(i) $\mathcal{T}$ has finite products with a terminal object 1.
(ii) $p : \mathcal{F} \to \mathcal{T}$ is an associative fibered category with products.
(iii) $p : \mathcal{F} \to \mathcal{T}$ is an associative fibered category with exponents.

7 Quasi-topological category

7.1 Quasi-topological category and continuous functor

We denote by $\mathcal{T}op$ the category of topological spaces and continuous maps. Let X and Y be topological spaces. For $x \in X$, we denote by $ev_x : \mathcal{T}op(X,Y) \to Y$ the map defined by $ev_x(f) = f(x)$. For $O \subset Y$, put $W(x,O) = ev_x^{-1}(O) = \{f \in \mathcal{T}op(X,Y) | \, f(x) \in O\}$. We give $\mathcal{T}op(X,Y)$ the pointwise convergent topology generated by $\{W(x,O) | \, x \in X,\, O$ is an open set of $Y\}$. In other words, the pointwise convergent topology on $\mathcal{T}op(X,Y)$ is the coarsest topology that ev_x is continuous for every $x \in X$.

Proposition 7.1.1 *Let X, Y and Z be topological spaces.*
(1) A map $\varphi : Z \to \mathcal{T}op(X,Y)$ is continuous if and only if $ev_x\varphi : Z \to Y$ is continuous for any $x \in X$.
(2) For a continuous map $f : X \to Y$, the maps $f^* : \mathcal{T}op(Y,Z) \to \mathcal{T}op(X,Z)$ and $f_* : \mathcal{T}op(Z,X) \to \mathcal{T}op(Z,Y)$ induced by f are continuous.

Proof. (1) Since $\{ev_x^{-1}(O) | \, x \in X,\, O$ is an open set of $Y\}$ is a subbasis of the topology of $\mathcal{T}op(X,Y)$, the assertion is straightforward.
(2) Since $ev_x f^* = ev_{f(x)}$ for any $x \in X$ and $ev_z f_* = f ev_z$ for any $z \in Z$, the assertion follows from 1). $\square$

Definition 7.1.2 *A category $\mathcal{T}$ is called a quasi-topological category if the following conditions are satisfied.*
(1) For each $R, S \in \mathrm{Ob}\,\mathcal{T}$, $\mathcal{T}(R,S)$ is a topological space.
(2) For any morphism $f : R \to S$ in $\mathcal{T}$ and $Z \in \mathrm{Ob}\,\mathcal{T}$, the maps $f_* : \mathcal{T}(Z,R) \to \mathcal{T}(Z,S)$ and $f^* : \mathcal{T}(S,Z) \to \mathcal{T}(R,Z)$ are continuous.

It follows from (2) of (7.1.1) that $\mathcal{T}op$ is a quasi-topological category.

Condition 7.1.3 *Let $\mathcal{T}$ be a quasi-topological category and $D : \mathcal{D} \to \mathcal{T}$ a functor. For an object X of $\mathcal{T}$, define functors $D_X : \mathcal{D} \to \mathcal{T}op$ and $D^X : \mathcal{D}^{op} \to \mathcal{T}op$ by $D_X(i) = \mathcal{T}(X, D(i))$, $D_X(\tau) = D(\tau)_*$ and $D^X(i) = \mathcal{T}(D(i), X)$, $D^X(\tau) = D(\tau)^*$ for $i \in \mathrm{Ob}\,\mathcal{D}$ and $\tau \in \mathrm{Mor}\,\mathcal{D}$. We consider the following conditions for D and X.*
(L) If $\left(L \xrightarrow{\pi_i} D(i)\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of D, $\left(\mathcal{T}(X,L) \xrightarrow{\pi_{i*}} \mathcal{T}(X,D(i))\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of D_X.
(C) If $\left(D(i) \xrightarrow{\iota_i} C\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of D, $\left(\mathcal{T}(C,X) \xrightarrow{\iota_i^*} \mathcal{T}(D(i),X)\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of D^X.

Proposition 7.1.4 *The conditions (L) and (C) above are satisfied for any functor $D : \mathcal{D} \to \mathcal{T}op$ and topological space X.*

Proof. Let $\left(L \xrightarrow{\pi_i} D(i)\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ be a limiting cone of D. Suppose that $x \in X$ and O is an open set of L. For any $f \in W(x,O)$, there exist $i_1, i_2, \dots, i_n \in \mathrm{Ob}\,\mathcal{C}$ and open sets O_k of $D(i_k)$ $(k = 1, 2, \dots, n)$ such that $f(x) \in \bigcap_{k=1}^{n} \pi_{i_k}^{-1}(O_k) \subset O$. Hence $f \in \bigcap_{k=1}^{n} W(x, \pi_{i_k}^{-1}(O_k)) \subset W(x,O)$. Since $W(x, \pi_{i_k}^{-1}(O_k)) = \pi_{i_k *}^{-1}(W(x,O_k))$, we have $f \in \bigcap_{k=1}^{n} \pi_{i_k *}^{-1}(W(x,O_k)) \subset W(x,O)$ and it follows that the coarsest topology on $\mathcal{T}op(X,L)$ such that every $\pi_{i*} : \mathcal{T}op(X,L) \to \mathcal{T}op(X,D(i))$ is continuous is finer than the pointwise convergent topology on $\mathcal{T}op(X,L)$. On the other hand, every $\pi_{i*} : \mathcal{T}op(X,L) \to \mathcal{T}op(X,D(i))$ is continuous by (2) of (7.1.1). Thus the condtion (L) is satisfied.

Let $\left(D(i) \xrightarrow{\iota_i} C\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ be a colimiting cone of D. Suppose that $x \in L$ and O is an open set of X. There exist $j \in \mathrm{Ob}\,\mathcal{D}$ and $w \in D(j)$ such that $\iota_j(w) = x$. For any $f \in W(x,O)$, we have $f \in (\iota_j^*)^{-1}(W(w,O)) = W(x,O)$ and it follows that the coarsest topology on $\mathcal{T}op(L,X)$ such that every $\iota_i^* : \mathcal{T}op(L,X) \to \mathcal{T}op(D(i),X)$ is continuous is finer than the pointwise convergent topology on $\mathcal{T}op(L,X)$. On the other hand, every $\iota_i^* : \mathcal{T}op(L,X) \to \mathcal{T}op(D(i),X)$ is continuous by 2) of (7.1.1). Thus the condtion (C) is satisfied. $\square$

Definition 7.1.5 *Let $\mathcal{C}$ and $\mathcal{T}$ be quasi-topological categories. We say that a functor $F : \mathcal{C} \to \mathcal{T}$ is continuous if $F : \mathcal{C}(R,S) \to \mathcal{T}(F(R), F(S))$ is continuous for any $R, S \in \mathrm{Ob}\,\mathcal{C}$. We denote by $\mathrm{Funct}_c(\mathcal{C},\mathcal{T})$ the full subcategory of $\mathrm{Funct}(\mathcal{C},\mathcal{T})$ consiting of continuous functors.*

Proposition 7.1.6 *Let $\mathcal{T}$ be a quasi-topological category and R an object of $\mathcal{T}$. Then, the functor $h_R : \mathcal{T} \to \mathcal{T}op$ represented by R (i.e. the functor given by $h_R(S) = \mathcal{T}(R,S)$) is continuous.*

Proof. Let S and T be objects of $\mathcal{T}$. For $\varphi \in h_R(S)$ and an open set O of $h_R(T)$. The inverse image of $W(\varphi, O)$ by $h_R : \mathcal{T}(S,T) \to \mathcal{T}op(h_R(S), h_R(T))$ coincides with the inverse inverse image of O by $\varphi^* : \mathcal{T}(S,T) \to \mathcal{T}(R,T)$. Since φ^* is continuous, $h_R^{-1}(W(\varphi, O))$ is open. Hence h_R is continuous. $\square$

Let $\mathcal{C}$ and $\mathcal{T}$ be categories. For $R \in \operatorname{Ob}\mathcal{C}$, define an evaluation functor $E_R : \operatorname{Funct}(\mathcal{C}, \mathcal{T}) \to \mathcal{T}$ at R by $E_R(F) = F(R)$ and $E_R(\varphi) = \varphi_R$.

Proposition 7.1.7 *Let $\mathcal{C}$ and $\mathcal{T}$ be quasi-topological categories and $D : \mathcal{D} \to \operatorname{Funct}(\mathcal{C}, \mathcal{T})$ a functor such that $D(i)$ is a continuous functor for every $i \in \operatorname{Ob}\mathcal{C}$.*

(1) If $\left(L \xrightarrow{\pi_i} D(i)\right)_{i \in \operatorname{Ob}\mathcal{D}}$ a cone of D such that $\left(\mathcal{T}(L(R), L(S)) \xrightarrow{(\pi_i)_{S*}} \mathcal{T}(L(R), D(i)(S))\right)_{i \in \operatorname{Ob}\mathcal{D}}$ is a limiting cone of the functor $(E_S D)_{L(R)} : \mathcal{D} \to \mathcal{T}op$ for any $R, S \in \operatorname{Ob}\mathcal{C}$, then L is a continuous functor.

(2) If $\left(D(i) \xrightarrow{\iota_i} L\right)_{i \in \operatorname{Ob}\mathcal{D}}$ a cone of D such that $\left(\mathcal{T}(L(R), L(S)) \xrightarrow{(\iota_i)^*_R} \mathcal{T}(D(i)(R), L(S))\right)_{i \in \operatorname{Ob}\mathcal{D}}$ is a limiting cone of the functor $(E_R D)^{L(S)} : \mathcal{D}^{op} \to \mathcal{T}op$ for any $R, S \in \operatorname{Ob}\mathcal{C}$, L is a continuous functor.

Proof. (1) Since the following diagram commutes for $i \in \operatorname{Ob}\mathcal{D}$, $(\pi_i)_{S*}L : \mathcal{C}(R,S) \to \mathcal{T}(L(R), D(i)(S))$ is continuous.

$$\begin{tikzcd}
\mathcal{C}(R,S) \arrow[r, "L"] \arrow[d, "D(i)"] & \mathcal{T}(L(R), L(S)) \arrow[d, "(\pi_i)_{S*}"] \\
\mathcal{T}(D(i)(R), D(i)(S)) \arrow[r, "(\pi_i)^*_R"] & \mathcal{T}(L(R), D(i)(S))
\end{tikzcd}$$

Hence L is continuous by the assumption.

(2) Since the following diagram commutes for $i \in \operatorname{Ob}\mathcal{D}$, $(\iota_i)^*_R L : \mathcal{C}(R,S) \to \mathcal{T}(D(i)(R), L(S))$ is continuous.

$$\begin{tikzcd}
\mathcal{C}(R,S) \arrow[r, "L"] \arrow[d, "D(i)"] & \mathcal{T}(L(R), L(S)) \arrow[d, "(\iota_i)^*_R"] \\
\mathcal{T}(D(i)(R), D(i)(S)) \arrow[r, "(\iota_i)_{R*}"] & \mathcal{T}(D(i)(R), L(S))
\end{tikzcd}$$

Hence L is continuous by the assumption. $\square$

By the above result and (7.1.4), we have the following result.

Corollary 7.1.8 *Let $\mathcal{C}$ be a quasi-topological category and $D : \mathcal{D} \to \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)$ a functor such that $D(i)$ is a continuous functor for every $i \in \operatorname{Ob}\mathcal{C}$.*

(1) If $\left(L \xrightarrow{\pi_i} D(i)\right)_{i \in \operatorname{Ob}\mathcal{D}}$ is a limiting cone of D, L is a continuous functor.

(2) If $\left(D(i) \xrightarrow{\iota_i} L\right)_{i \in \operatorname{Ob}\mathcal{D}}$ is a colimiting cone of D, L is a continuous functor.

Let us denote by $\mathcal{S}et$ the category of sets and maps and by $\Phi : \mathcal{T}op \to \mathcal{S}et$ the forgetful functor.

Corollary 7.1.9 *For a quasi-topological category $\mathcal{C}$, the composition $\widetilde{\Phi} : \operatorname{Funct}_c(\mathcal{C}, \mathcal{T}op) \to \operatorname{Funct}(\mathcal{C}, \mathcal{S}et)$ of the inclusion functor $\operatorname{Funct}_c(\mathcal{C}, \mathcal{T}op) \to \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)$ and the functor $\Phi_* : \operatorname{Funct}(\mathcal{C}, \mathcal{T}op) \to \operatorname{Funct}(\mathcal{C}, \mathcal{S}et)$ induced by Φ creates limits and colimits. Hence $\operatorname{Funct}_c(\mathcal{C}, \mathcal{T}op)$ is complete and cocomplete.*

Proof. Let $D : \mathcal{D} \to \operatorname{Funct}_c(\mathcal{C}, \mathcal{T}op)$ be a functor. Suppose that $\left(L \xrightarrow{\pi_i} \widetilde{\Phi} D(i)\right)_{i \in \operatorname{Ob}\mathcal{D}}$ (resp. $\left(\widetilde{\Phi} D(i) \xrightarrow{\iota_i} C\right)_{i \in \operatorname{Ob}\mathcal{D}}$) is a limiting cone (resp. colimiting cone) of $\widetilde{\Phi} D : \mathcal{D} \to \operatorname{Funct}(\mathcal{C}, \mathcal{S}et)$. For each $R \in \operatorname{Ob}\mathcal{C}$, we give $L(R)$ (resp. $C(R)$) the topology such that $\left(L(R) \xrightarrow{(\pi_i)_R} D(i)(R)\right)_{i \in \operatorname{Ob}\mathcal{D}}$ (resp. $\left(D(i)(R) \xrightarrow{(\iota_i)_R} C(R)\right)_{i \in \operatorname{Ob}\mathcal{D}}$) is a limiting cone (resp. colimiting cone) of a functor $E_R D : \mathcal{D} \to \mathcal{T}op$. Let $f : R \to S$ be a morphism in $\mathcal{C}$. Since the following diagrams commutes for any $i \in \operatorname{Ob}\mathcal{D}$, $L(f) : L(R) \to L(S)$ and $C(f) : C(R) \to C(S)$ are continuous.

$$\begin{tikzcd}
L(R) \arrow[r, "(\pi_i)_R"] \arrow[d, "L(f)"] & D(i)(R) \arrow[d, "D(i)(f)"] \\
L(S) \arrow[r, "(\pi_i)_S"] & D(i)(S)
\end{tikzcd}
\qquad
\begin{tikzcd}
D(i)(R) \arrow[r, "(\iota_i)_R"] \arrow[d, "D(i)(f)"] & C(R) \arrow[d, "C(f)"] \\
D(i)(S) \arrow[r, "(\iota_i)_S"] & L(S)
\end{tikzcd}$$

Hence L and C are regarded as functors from $\mathcal{C}$ to $\mathcal{T}op$ and the assertion follows from (7.1.8). $\square$

Proposition 7.1.10 *Let $\mathcal{C}$, $\mathcal{D}$ be quasi-topological categories. Suppose a functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint $G : \mathcal{D} \to \mathcal{C}$ and let us denote by $\mathrm{ad}_{X,W} : \mathcal{D}(F(X), W) \to \mathcal{C}(X, G(W))$ the natural bijection for $X \in \mathrm{Ob}\,\mathcal{C}$ and $W \in \mathrm{Ob}\,\mathcal{D}$.*

(1) If $\mathrm{ad}^{-1}_{X,F(Y)} : \mathcal{C}(X, G(F(Y))) \to \mathcal{D}(F(X), F(Y))$ is continuous for any $X, Y \in \mathrm{Ob}\,\mathcal{C}$, F is a continuous functor.

(2) If $\mathrm{ad}_{G(Z),W} : \mathcal{D}(F(G(Z)), W) \to \mathcal{C}(G(Z), G(W))$ is continuous for any $Z, W \in \mathrm{Ob}\,\mathcal{D}$, G is a continuous functor.

(3) If $F : \mathcal{C}(X, G(W)) \to \mathcal{C}(F(X), FG(W))$ is continuous for $X \in \mathrm{Ob}\,\mathcal{C}$ and $W \in \mathrm{Ob}\,\mathcal{D}$, $\mathrm{ad}^{-1}_{X,W} : \mathcal{C}(X, G(W)) \to \mathcal{D}(F(X), W)$ is continuous.

(4) If $G : \mathcal{D}(F(X), W) \to \mathcal{C}(GF(X), G(W))$ is continuous for $X \in \mathrm{Ob}\,\mathcal{C}$ and $W \in \mathrm{Ob}\,\mathcal{D}$, $\mathrm{ad}_{X,W} : \mathcal{D}(F(X), W) \to \mathcal{C}(X, G(W))$ is continuous .

Proof. Let us denote by $\eta : id_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to id_{\mathcal{D}}$ the unit and counit of the adjunction, respectively.

(1) Since the composition

$$\mathcal{C}(X, Y) \xrightarrow{\eta_X^*} \mathcal{C}(X, G(F(Y))) \xrightarrow{\mathrm{ad}^{-1}_{X,F(Y)}} \mathcal{D}(F(X), F(Y))$$

coincide with $F : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$, the assertion follows.

(2) Since the composition

$$\mathcal{D}(Z, W) \xrightarrow{\varepsilon_Z^*} \mathcal{C}(FG(Z), W) \xrightarrow{\mathrm{ad}_{G(Z),W}} \mathcal{D}(G(Z), G(W))$$

coincide with $G : \mathcal{D}(Z, W) \to \mathcal{C}(F(Z), F(W))$, the assertion follows.

(3) Since the composition

$$\mathcal{C}(X, G(W)) \xrightarrow{F} \mathcal{D}(F(X), F(G(W))) \xrightarrow{\varepsilon_{W*}} \mathcal{D}(F(X), W)$$

coincide with $\mathrm{ad}^{-1}_{X,W} : \mathcal{C}(X, G(W)) \to \mathcal{D}(F(X), W)$, the assertion follows.

(4) Since the composition

$$\mathcal{D}(F(X), W) \xrightarrow{G} \mathcal{C}(GF(X), G(W)) \xrightarrow{\eta_X^*} \mathcal{D}(X, G(W))$$

coincide with $\mathrm{ad}_{X,W} : \mathcal{C}(X, G(W)) \to \mathcal{C}(F(X), W)$, the assertion follows. $\square$

7.2 Yoneda's lemma

Definition 7.2.1 *Let $\mathcal{T}$ be a quasi-topological category. For $F, G \in \mathrm{Ob}\,\mathrm{Funct}(\mathcal{C}, \mathcal{T})$, we give $\mathrm{Funct}(\mathcal{C}, \mathcal{T})(F, G)$ the coarsest topology such that $E_R : \mathrm{Funct}(\mathcal{C}, \mathcal{T})(F, G) \to \mathcal{T}(F(R), G(R))$ is continuous for any object R of $\mathcal{C}$. If $\mathcal{F}$ is a subcategory of $\mathrm{Funct}(\mathcal{C}, \mathcal{T})$ (e.g. $\mathcal{C}$ is also a quasi-topological category and $\mathcal{F} = \mathrm{Funct}_c(\mathcal{C}, \mathcal{T})$) and $F, G \in \mathrm{Ob}\,\mathcal{F}$, we give $\mathcal{F}(F, G)$ the topology such that $\mathcal{F}(F, G)$ is a subspace of $\mathrm{Funct}(\mathcal{C}, \mathcal{T})(F, G)$.*

Remark 7.2.2 *(1) Since $\{E_R^{-1}(O) |\, R \in \mathrm{Ob}\,\mathcal{C},\ O$ is an open set of $\mathcal{T}(F(R), G(R))\}$ is a basis of the topology on $\mathrm{Funct}(\mathcal{C}, \mathcal{T})(F, G)$, a map $f : Z \to \mathrm{Funct}(\mathcal{C}, \mathcal{T})(F, G)$ is continuous if and only if $E_R f : Z \to \mathcal{T}(F(R), G(R))$ is continuous for any $R \in \mathrm{Ob}\,\mathcal{C}$.*

(2) If $\mathcal{T} = \mathcal{S}et$ or $\mathcal{T}op$, for a functor $F : \mathcal{C} \to \mathcal{T}$, we denote by $\mathcal{C}_F$ the category of F-models, that is, $\mathcal{C}_F$ is given by $\mathrm{Ob}\,\mathcal{C}_F = \{(R, x) |\, R \in \mathrm{Ob}\,\mathcal{C},\ x \in F(R)\}$ and $\mathcal{C}_F((R, x), (Y, y)) = \{f \in \mathcal{C}(R, Y) |\, F(f)(x) = y\}$. Since $\{E_R^{-1}(W(x, O)) |\, (R, x) \in \mathrm{Ob}\,\mathcal{C}_F,\ O$ is an open set of $G(R)\}$ is a subbasis of the topology on $\mathrm{Funct}(\mathcal{C}, \mathcal{T}op)(F, G)$, a map $f : Z \to \mathrm{Funct}(\mathcal{C}, \mathcal{T}op)(F, G)$ is continuous if and only if $ev_x E_R f : Z \to G(R)$ is continuous for any $(R, x) \in \mathrm{Ob}\,\mathcal{C}_F$.

Proposition 7.2.3 *Let F, G, H be functors from $\mathcal{C}$ to a quasi-topological category $\mathcal{T}$ and $f : F \to G$ a natural transformation. Then, maps $f^* : \mathrm{Funct}(\mathcal{C}, \mathcal{T})(G, H) \to \mathrm{Funct}(\mathcal{C}, \mathcal{T})(F, H)$ and $f_* : \mathrm{Funct}(\mathcal{C}, \mathcal{T})(H, F) \to \mathrm{Funct}(\mathcal{C}, \mathcal{T})(H, G)$ are continuous. Hence $\mathrm{Funct}(\mathcal{C}, \mathcal{T})$ is a quasi-topological category.*

Proof. Since the following diagrams commute for any $R \in \mathrm{Ob}\,\mathcal{C}$, the assertion follows from (7.2.2).

$$\begin{array}{ccc}
\mathrm{Funct}(\mathcal{C},\mathcal{T})(G,H) & \xrightarrow{f^*} & \mathrm{Funct}(\mathcal{C},\mathcal{T})(F,H) \\
\Big\downarrow{\scriptstyle E_R} & & \Big\downarrow{\scriptstyle E_R} \\
\mathcal{T}(G(R),H(R)) & \xrightarrow{f_R^*} & ct(F(R),H(R))
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Funct}(\mathcal{C},\mathcal{T})(H,F) & \xrightarrow{f_*} & \mathrm{Funct}(\mathcal{C},\mathcal{T})(H,G) \\
\Big\downarrow{\scriptstyle E_R} & & \Big\downarrow{\scriptstyle E_R} \\
\mathcal{T}(H(R),F(R)) & \xrightarrow{f_{R*}} & \mathcal{T}(H(R),G(R))
\end{array}$$

□

Proposition 7.2.4 *Let $\mathcal{T}$ be a quasi-topological category and $F:\mathcal{C}\to\mathcal{T}$ a functor.*

(1) Suppose that $\left(L \xrightarrow{\pi_i} D(i)\right)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of a functor $D:\mathcal{D}\to \mathrm{Funct}(\mathcal{C},\mathcal{T})$ and that, for any $R\in\mathrm{Ob}\,\mathcal{C}$, $E_RD:\mathcal{D}\to\mathcal{T}$ and $F(R)\in\mathrm{Ob}\,\mathcal{T}$ satisfy the condition (L) of (7.1.3) ($\mathcal{T}=\mathcal{T}op$, for example). Then,

$$\left(\mathrm{Funct}(\mathcal{C},\mathcal{T})(F,L) \xrightarrow{\pi_{i*}} \mathrm{Funct}(\mathcal{C},\mathcal{T})(F,D(i))\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$$

is a limiting cone of a functor $D_F:\mathcal{D}\to\mathcal{T}op$ defined by $D_F(i)=\mathrm{Funct}(\mathcal{C},\mathcal{T})(F,D(i))$ and $D_F(\tau)=D(\tau)_*$ for $i\in\mathrm{Ob}\,\mathcal{D}$, $\tau\in\mathrm{Mor}\,\mathcal{D}$. In other words, the condition (L) of (7.1.3) is satisfied for D and F.

(2) Suppose that $\left(D(i)\xrightarrow{\iota_i} C\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of a functor $D:\mathcal{D}\to\mathrm{Funct}(\mathcal{C},\mathcal{T})$ and that, for any $R\in\mathrm{Ob}\,\mathcal{C}$, $E_RD:\mathcal{D}\to\mathcal{T}$ and $F(R)\in\mathrm{Ob}\,\mathcal{T}$ satisfy the condition (C) of (7.1.3) ($\mathcal{T}=\mathcal{T}op$, for example). Then,

$$\left(\mathrm{Funct}(\mathcal{C},\mathcal{T})(C,F) \xrightarrow{\iota_i^*} \mathrm{Funct}(\mathcal{C},\mathcal{T})(D(i),F)\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$$

is a limiting cone of a functor $D^F:\mathcal{D}^{op}\to\mathcal{T}op$ defined by $D^F(i)=\mathrm{Funct}(\mathcal{C},\mathcal{T})(D(i),F)$ and $D^F(\tau)=D(\tau)^*$ for $i\in\mathrm{Ob}\,\mathcal{D}$, $\tau\in\mathrm{Mor}\,\mathcal{D}$. In other words, the condition (C) of (7.1.3) is satisfied for D and F.

Proof. (1) It is clear that $\left(\mathrm{Funct}(\mathcal{C},\mathcal{T})(F,L)\xrightarrow{\pi_{i*}}\mathrm{Funct}(\mathcal{C},\mathcal{T})(F,D(i))\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone in the category of sets. Let O be an open set of $\mathrm{Funct}(\mathcal{C},\mathcal{T})(F,L)$ and $\varphi\in O$. There exists $R_1,R_2,\dots,R_n\in\mathrm{Ob}\,\mathcal{C}$ and open sets O_s of $\mathcal{T}(L(R_s),F(R_s))$ $(s=1,2,\dots,n)$ such that $\varphi\in\bigcap\limits_{s=1}^{n}E_{R_s}^{-1}(O_s)\subset O$. Since O_s is open in $\mathcal{T}(L(R_s),F(R_s))$ and $\left(\mathcal{T}(F(R_s),L(R_s))\xrightarrow{(\pi_i)_{R_s*}}\mathcal{T}(F(R_s),D(i)(R_s))\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of $(E_{R_s}D)_{F(R_s)}:\mathcal{D}\to\mathcal{T}op$, there exist open sets O_{ks} $(k=1,2,\dots,\nu_s)$ of $\mathcal{T}(F(R_s),D(i_{ks})(R_s))$ such that $\varphi_{R_s}\in\bigcap\limits_{k=1}^{\nu_s}(\pi_{i_{ks}})_{R_s*}^{-1}(O_{ks})\subset O_s$. Hence $\varphi\in\bigcap\limits_{k=1}^{\nu_s}E_{R_s}^{-1}((\pi_{i_{ks}})_{R_s*}^{-1}(O_{ks}))=\bigcap\limits_{k=1}^{\nu_s}\pi_{i_{ks}*}^{-1}(E_{R_s}^{-1}(O_{ks}))$ and we have $\varphi\in\bigcap\limits_{s=1}^{n}\bigcap\limits_{k=1}^{\nu_s}\pi_{i_{ks}*}^{-1}(E_{R_s}^{-1}(O_{ks}))\subset O$, which implies the assertion.

(2) It is clear that $\left(\mathrm{Funct}(\mathcal{C},\mathcal{T})(C,F)\xrightarrow{\iota_i^*}\mathrm{Funct}(\mathcal{C},\mathcal{T})(D(i),F)\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone in the category of sets. Let O be an open set of $\mathrm{Funct}(\mathcal{C},\mathcal{T})(C,F)$ and $\varphi\in O$. There exists $R_1,R_2,\dots,R_n\in\mathrm{Ob}\,\mathcal{C}$ and open sets O_s of $\mathcal{T}(C(R_s),F(R_s))$ such that $\varphi\in\bigcap\limits_{s=1}^{n}E_{R_s}^{-1}(O_s)\subset O$. Since O_s is open in $\mathcal{T}(F(R_s),C(R_s))$ and $\left(\mathcal{T}(C(R_s),F(R_s))\xrightarrow{(\iota_i)_{R_s}^*}\mathcal{T}(D(i)(R_s),F(R_s))\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of $(E_{R_s}D)^{F(R_s)}:\mathcal{D}^{op}\to\mathcal{T}op$, there exist open sets O_{ks} $(k=1,2,\dots,\nu_s)$ of $\mathcal{T}(D(i_{ks})(R_s),F(R_s))$ such that $\varphi_{R_s}\in\bigcap\limits_{k=1}^{\nu_s}((\iota_{i_{ks}})_{R_s}^*)^{-1}(O_{ks})\subset O_s$. Hence $\varphi\in\bigcap\limits_{k=1}^{\nu_s}E_{R_s}^{-1}(((\iota_{i_{ks}})_{R_s}^*)^{-1}(O_{ks}))=\bigcap\limits_{k=1}^{\nu_s}(\iota_{i_{ks}}^*)^{-1}(E_{R_s}^{-1}(O_{ks}))$ and we have $\varphi\in\bigcap\limits_{s=1}^{n}\bigcap\limits_{k=1}^{\nu_s}(\iota_{i_{ks}}^*)^{-1}(E_{R_s}^{-1}(O_{ks}))\subset O$, which implies the assertion. □

Definition 7.2.5 *Let $\mathcal{C}$ be a quasi-topological category and $F:\mathcal{C}\to\mathcal{T}op$ a functor.*

(1) For $(R,x)\in\mathrm{Ob}\,\mathcal{C}_F$ and $S\in\mathrm{Ob}\,\mathcal{C}$, we define a map $(\varphi(F)_{(R,x)})_S:h_R(S)\to F(S)$ to be the following composition of maps.

$$h_R(S)=\mathcal{C}(R,S)\xrightarrow{F}\mathcal{T}op(F(R),F(S))\xrightarrow{ev_x}F(S)$$

If F is continuous, $(\varphi(F)_{(R,x)})_S$ is continuous and we have a morphism $\varphi(F)_{(R,x)}:h_R\to F$ in $\mathrm{Funct}(\mathcal{C},\mathcal{T}op)$.

(2) For $R\in\mathrm{Ob}\,\mathcal{C}$, we define a map $\theta_R(F):\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(h_R,F)\to F(R)$ to be the following composition of maps. It is clear that $\theta_R(F)$ is continuous.

$$\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(h_R,F)\xrightarrow{E_R}\mathcal{T}op(h_R(R),F(R))\xrightarrow{ev_{id_R}}F(R)$$

The following assertion can be easily verified.

Lemma 7.2.6 *(1) The following diagram commutes for any $(R,x) \in \operatorname{Ob}\mathcal{C}_F$.*

$$\begin{tikzcd}
\mathrm{Funct}(\mathcal{C},\mathcal{Top})(F,G) \arrow[r, "E_R"] \arrow[d, "\varphi(F)^*_{(R,x)}"] & \mathcal{Top}(F(R),G(R)) \arrow[d, "ev_x"] \\
\mathrm{Funct}(\mathcal{C},\mathcal{Top})(h_R,G) \arrow[r, "\theta_R(G)"] & G(R)
\end{tikzcd}$$

(2) The following diagram commutes for any $f \in h_R(S)$.

$$\begin{tikzcd}
\mathrm{Funct}(\mathcal{C},\mathcal{Top})(h_R,G) \arrow[r, "E_S"] \arrow[d, "\theta_R(G)"] & \mathcal{Top}(h_R(S),G(S)) \arrow[d, "ev_f"] \\
G(R) \arrow[r, "G(f)"] & G(S)
\end{tikzcd}$$

Proposition 7.2.7 *For an object R of $\mathcal{C}$ and a functor $G : \mathcal{C} \to \mathcal{Top}$, the following topologies $\mathcal{O}$, $\mathcal{O}_1$ and $\mathcal{O}_2$ on $\mathrm{Funct}(\mathcal{C},\mathcal{Top})(h_R,G)$ are the same. $\mathcal{O}$ is the topology given in (7.2.1). $\mathcal{O}_1$ the coarsest topology on $\mathrm{Funct}(\mathcal{C},\mathcal{Top})(h_R,G)$ such that $\theta_R(G) : \mathrm{Funct}(\mathcal{C},\mathcal{Top})(h_R,G) \to G(R)$ is continuous, $\mathcal{O}_2$ is the coarsest topology such that $E_R : \mathrm{Funct}(\mathcal{C},\mathcal{Top})(h_R,G) \to \mathcal{Top}(h_R(R),G(R))$ is continuous.*

Proof. It is clear that $\mathcal{O}$ is finer than $\mathcal{O}_2$ and that $\mathcal{O}_2$ is finer than $\mathcal{O}_1$ by the definition of $\theta_R(G)$. With topology $\mathcal{O}_1$ on $\mathrm{Funct}(\mathcal{C},\mathcal{Top})(h_R,G)$, it follows from 2) of (7.2.6) that $ev_fE_S : \mathrm{Funct}(\mathcal{C},\mathcal{Top})(h_R,G) \to G(S)$ is continuous for any $S \in \operatorname{Ob}\mathcal{C}$ and $f \in h_S(R)$. Since $\mathcal{Top}(h_R(S),G(S))$ has the coarsest topology such that $ev_f : \mathcal{Top}(h_R(S),G(S)) \to G(S)$ is continuous for any $f \in h_R(S)$, $E_S : \mathrm{Funct}(\mathcal{C},\mathcal{Top})(h_R,G) \to \mathcal{Top}(h_R(S),G(S))$ is continuous. Therefore $\mathcal{O}$ is coarser than $\mathcal{O}_1$. □

Corollary 7.2.8 *A map $f : X \to \mathrm{Funct}(\mathcal{C},\mathcal{Top})(h_R,G)$ is continuous if and only if one of the following conditions is satisfied.*

(1) $E_Sf : X \to \mathcal{Top}(h_R(S),G(S))$ is continuous for any $S \in \operatorname{Ob}\mathcal{C}$.
(2) $E_Rf : X \to \mathcal{Top}(h_R(R),G(R))$ is continuous.
(3) $\theta_R(G)f : X \to G(R)$ is continuous,

Corollary 7.2.9 *A functor $h : \mathcal{C}^{op} \to \mathrm{Funct}_c(\mathcal{C},\mathcal{Top})$ defined by $h(R) = h_R$ and $h(f) = h_f$ is continuous.*

Proof. For $R,S \in \operatorname{Ob}\mathcal{C}$, $\mathcal{C}(S,R) \xrightarrow{h} \mathrm{Funct}_c(\mathcal{C},\mathcal{Top})(h_R,h_S) \xrightarrow{\theta_R(h_S)} h_S(R)$ maps $f \in \mathcal{C}(S,R)$ to $(h_f)_R(id_R) = f$, namely $\theta_R(h_S)h$ is the identity map of $\mathcal{C}(S,R) = h_S(R)$. Hence $\theta_R(h_S)h$ is continuous and the assertion follows from (7.2.8). □

The following is the Yoneda's lemma for continuous functors.

Proposition 7.2.10 *Let $\mathcal{C}$ be a quasi-topological category and $F : \mathcal{C} \to \mathcal{Top}$ a continuous functor. Then, $\theta_R(F) : \mathrm{Funct}(\mathcal{C},\mathcal{Top})(h_R,F) \to F(R)$ is a homeomorphism.*

Proof. $\theta_R(F)$ is continuous by (7.2.7). It is easy to verify that a correspondance $x \mapsto \varphi(F)_{(R,x)}$ gives the inverse of $\theta_R(F)$ by (7.2.5). It follows from (7.2.8) that $\theta_R(F)^{-1} : F(R) \to \mathrm{Funct}(\mathcal{C},\mathcal{Top})(h_R,F)$ is continuous. □

For a functor $F : \mathcal{C} \to \mathcal{Set}$, let $D(F) : \mathcal{C}_F^{op} \to \mathrm{Funct}(\mathcal{C},\mathcal{Set})$ be a functor defined by $D(F)(R,x) = h_R$ and $D(F)(f) = h_f$. If $\mathcal{C}$ is a quasi-topological category, we denote by $D_{top}(F) : \mathcal{C}_F^{op} \to \mathrm{Funct}_c(\mathcal{C},\mathcal{Top})$ a functor given by $D_{top}(F)(R,x) = h_R$ and $D_{top}(F)(f) = h_f$.

Proposition 7.2.11 *Let $\mathcal{C}$ be a quasi-topological category. The functor $\widetilde{\Phi} : \mathrm{Funct}_c(\mathcal{C},\mathcal{Top}) \to \mathrm{Funct}(\mathcal{C},\mathcal{Set})$ given in (7.1.9) has a left adjoint.*

Proof. For $F \in \operatorname{Ob}\mathrm{Funct}(\mathcal{C},\mathcal{Set})$, by the ordianry Yoneda's lemma, $\left(D(F)(R,x) \xrightarrow{\varphi(F)_{(R,x)}} F\right)_{(R,x)\in\operatorname{Ob}\mathcal{C}_F}$ is a colimiting cone of $D(F) : \mathcal{C}_F^{op} \to \mathrm{Funct}(\mathcal{C},\mathcal{Set})$, where $\varphi(F)_{(R,x)} : D(F)(R,x) \to F$ is the morphism given in (7.2.5). By giving $F(S)$ the topology for each $S \in \operatorname{Ob}\mathcal{C}$ such that $\left(D(F)(R,x)(S) \xrightarrow{(\varphi(F)_{(R,x)})_S} F(S)\right)_{(R,x)\in\operatorname{Ob}\mathcal{C}_F}$

is a colimiting cone in $\mathcal{T}op$, namely a subset O of $F(S)$ is open if and only if $(\varphi(F)_{(R,x)})_S^{-1}(O)$ is an open set of $D(F)(R,x)(S)=\mathcal{C}(R,S)$ for any $(R,x)\in \mathrm{Ob}\,\mathcal{C}_F$, we have a functor $\check{F}:\mathcal{C}\to\mathcal{T}op$ satisfying $\widetilde{\Phi}(\check{F})=F$. Then,

$$\left(D_{top}(F)(R,x)\xrightarrow{\varphi(F)_{(R,x)}}\check{F}\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}\quad\cdots(*)$$

is a colimiting cone of $D_{top}(F)$. It follows from (7.1.6) and (2) of (7.1.8) that $\check{F}$ is continuous. We set $\Psi(F)=\check{F}$.

For functors $F,G:\mathcal{C}\to\mathcal{S}et$ and a natural transformation $\lambda:F\to G$, define a functor $\lambda_\sharp:\mathcal{C}_F\to\mathcal{C}_G$ by $\lambda_\sharp(R,x)=(R,\lambda_R(x))$ and $\lambda_\sharp(f:(R,x)\to(S,y))=(f:(R,\lambda_R(x))\to(S,\lambda_S(y)))$. (Note that $y=F(f)(x)$, thus $\lambda_S(y)=G(f)(\lambda_R(x))$.) Then, $D_{top}(G)\lambda_\sharp=D_{top}(F)$ and

$$\left(D_{top}(F)(R,x)=D_{top}(G)\lambda_\sharp(R,x)\xrightarrow{\varphi(G)_{\lambda_\sharp(R,x)}}\check{G}\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$$

is a cone of $D_{top}(F)$ and the following diagram commutes.

$$\begin{array}{ccc}
\widetilde{\Phi}D_{top}(F)(R,x) & \xrightarrow{\widetilde{\Phi}(\varphi(F)_{(R,x)})} & \widetilde{\Phi}(\check{F}) \\
\| & & \downarrow{\scriptstyle\lambda} \\
\widetilde{\Phi}D_{top}(G)\lambda_\sharp(R,x) & \xrightarrow{\widetilde{\Phi}(\varphi(G)_{\lambda_\sharp(R,x)})} & \widetilde{\Phi}(\check{G})
\end{array}$$

Since λ is the unique morphism that makes the above diagram commute and $(*)$ is a colimiting cone of $D_{top}(F)$, $\lambda_S:F(S)\to G(S)$ is continuous for each $S\in \mathrm{Ob}\,\mathcal{C}$ and this implies that λ induces a unique natural transformation $\check{\lambda}:\check{F}\to\check{G}$ satisfying $\check{\lambda}\varphi(F)_{(R,x)}=\varphi(G)_{\lambda_\sharp(R,x)}$ for any $(R,x)\in \mathrm{Ob}\,\mathcal{C}_F$. We set $\Psi(\lambda)=\check{\lambda}$.

It follows from (7.2.5) that, for $F\in \mathrm{Ob}\,\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$, $\left(D_{top}(F)(R,x)\xrightarrow{\varphi(F)_{(R,x)}}F\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$ is a cone in $\mathrm{Funct}(\mathcal{C},\mathcal{T}op)$, hence there is a natural transformation $\rho:\Psi\widetilde{\Phi}\to id_{\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)}$ such that $\widetilde{\Phi}(\rho_F)=id_{\widetilde{\Phi}(F)}$ for $F\in \mathrm{Ob}\,\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$. It is clear that $\widetilde{\Phi}\Psi=id_{\mathrm{Funct}(\mathcal{C},\mathcal{S}et)}$. Moreover, for $F\in \mathrm{Ob}\,\mathrm{Funct}(\mathcal{C},\mathcal{S}et)$, since $\mathcal{C}_{\widetilde{\Phi}(\check{F})}=\mathcal{C}_F$, $D(F)=D(\widetilde{\Phi}(\check{F}))$ and $\varphi(F)_{(R,x)}=\varphi(\widetilde{\Phi}(\check{F}))_{(R,x)}$, we have $\rho_{\check{F}}=id_{\check{F}}$. It follows that Ψ is a left adjoint of $\tilde{\Phi}$. □

7.3 Left adjoint of the Yoneda embedding

For a quasi-topological category $\mathcal{C}$ and a functor $D:\mathcal{D}^{op}\to\mathcal{C}$, we denote by $h_D:\mathcal{D}\to\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$ the composition of functors $D^{op}:\mathcal{D}\to\mathcal{C}^{op}$ and $h:\mathcal{C}^{op}\to\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$ defined in (7.2.9).

Proposition 7.3.1 *Let $\mathcal{C}$ be a quasi-topological category. If $F:\mathcal{C}\to\mathcal{T}op$ is a colimit of representable functors, then $\left(D(F)(R,x)\xrightarrow{\varphi(F)_{(R,x)}}F\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$ is a colimiting cone of the functor $D(F):\mathcal{C}_F^{op}\to\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$. Hence F is in the image of the functor $\Psi:\mathrm{Funct}(\mathcal{C},\mathcal{S}et)\to\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$ given in the proof of (7.2.11).*

Proof. Suppose that $\left(h_D(i)\xrightarrow{\iota_i}F\right)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of h_D. It follows from (7.1.6) and (7.1.8) that F is a continuous functor. Since the map $F(D(\xi)):F(D(j))\to F(D(i))$ induced by a morphism $\xi:i\to j$ in $\mathcal{D}$ maps $(\iota_j)_{D(j)}\left(id_{D(j)}\right)$ to $(\iota_i)_{D(i)}\left(id_{D(i)}\right)$, we can define a functor $\widetilde{D}:\mathcal{D}\to\mathcal{C}_F^{op}$ by

$$\widetilde{D}(i)=\left(D(i),(\iota_i)_{D(i)}\left(id_{D(i)}\right)\right)\text{ and }\widetilde{D}(\xi:i\to j)=D(\xi):\left(D(j),(\iota_j)_{D(j)}\left(id_{D(j)}\right)\right)\to\left(D(i),(\iota_i)_{D(i)}\left(id_{D(i)}\right)\right).$$

Consider the functor $D(F):\mathcal{C}_F^{op}\to\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$ defined in the proof of (7.2.11). Then, we have $D(F)\widetilde{D}=h_D$. Suppose that $\left(D(F)(R,x)\xrightarrow{\psi_{(R,x)}}G\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$ is a cone of $D(F)$. Since $\left(h_D(i)=D(F)\widetilde{D}(i)\xrightarrow{\psi_{\widetilde{D}(i)}}G\right)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of h_D, there exists a unique morphism $f:F\to G$ satisfying $f\iota_i=\psi_{\widetilde{D}(i)}$ for any $i\in \mathrm{Ob}\,\mathcal{C}$. For any $(R,x)\in \mathrm{Ob}\,\mathcal{C}_F$, there exist $i\in \mathrm{Ob}\,\mathcal{D}$ and $\alpha\in h_{D(i)}(R)$ satisfying $(\iota_i)_R(\alpha)=x$. Then, for any $S\in \mathrm{Ob}\,\mathcal{C}$ and $\beta\in h_R(S)=D(F)(R,x)(S)$, we have

$$(\iota_ih_\alpha)_S(\beta)=(\iota_i)_S(h_\alpha)_S(\beta)=(\iota_i)_S(\beta\alpha)=(\iota_i)_Sh_{D(i)}(\beta)(\alpha)=F(\beta)(\iota_i)_R(\alpha)=F(\beta)(x)=(\varphi(F)_{(R,x)})_S(\beta).$$

Hence $\iota_ih_\alpha=\varphi(F)_{(R,x)}$ and it follows from $F(\alpha)\left((\iota_i)_{D(i)}\left(id_{D(i)}\right)\right)=(\iota_i)_Rh_{D(i)}(\alpha)\left(id_{D(i)}\right)=(\iota_i)_R(\alpha)=x$ that $f\varphi(F)_{(R,x)}=f\iota_ih_\alpha=\psi_{\widetilde{D}(i)}D(F)(\alpha)=\psi_{(R,x)}$. Therefore the assertion follows. □

Proposition 7.3.2 *Let $\mathcal{C}$ be a quasi-topological category and $D : \mathcal{D}^{op} \to \mathcal{C}$ a functor. If $F : \mathcal{C} \to \mathcal{T}op$ is a colimit of representable functors and $\left(h_D(i) \xrightarrow{\iota_i} F\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a cone of h_D such that $\left(\widetilde{\Phi}(h_D(i)) \xrightarrow{\widetilde{\Phi}(\iota_i)} \widetilde{\Phi}(F)\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $\widetilde{\Phi}h_D$. Then $\left(h_D(i) \xrightarrow{\iota_i} F\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of h_D.*

Proof. Let $\left(h_D(i) \xrightarrow{\eta_i} G\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ be a cone of h_D. Then, there exists a unique morphism $f : \widetilde{\Phi}(F) \to \widetilde{\Phi}(G)$ in $\mathrm{Funct}(\mathcal{C}, \mathcal{S}et)$ satisfying $f\widetilde{\Phi}(\iota_i) = \widetilde{\Phi}(\eta_i)$ for each $i \in \mathrm{Ob}\,\mathcal{D}$. For $(R, x) \in \mathrm{Ob}\,\mathcal{C}_F$, we choose $i \in \mathrm{Ob}\,\mathcal{D}$ and $\alpha \in h_{D(i)}(R)$ satisfying $(\iota_i)_R(\alpha) = x$. For $S \in \mathrm{Ob}\,\mathcal{C}$ and $\beta \in h_R(S)$, $(\iota_i)_S(h_\alpha)_S(\beta) = (\iota_i)_S(\beta\alpha) = (\iota_i)_S h_{D(i)}(\beta)(\alpha) = F(\beta)(\iota_i)_R(\alpha) = F(\beta)(x) = \left(\varphi(F)_{(R,x)}\right)_S(\beta)$ by the naturality of ι_i. Hence we have $(\iota_i)_S(h_\alpha)_S = \left(\varphi(F)_{(R,x)}\right)_S$. Then, $f_S\left(\varphi(F)_{(R,x)}\right)_S = f_S(\iota_i)_S(h_\alpha)_S = (\eta_i)_S(h_\alpha)_S$ and, since $(\eta_i)_S(h_\alpha)_S : h_R(S) \to G(S)$ is continuous for any $i \in \mathrm{Ob}\,\mathcal{C}$, it follows that $f_S\left(\varphi(F)_{(R,x)}\right)_S$ is continuous for any $(R, x) \in \mathrm{Ob}\,\mathcal{C}_F$. Since $\left(D(F)(R, x) \xrightarrow{\varphi(F)_{(R,x)}} F\right)_{(R,x) \in \mathrm{Ob}\,\mathcal{C}_F}$ is a colimiting cone of $D(F) : \mathcal{C}_F^{op} \to \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)$ by (7.3.1), $f_S : F(S) \to G(S)$ is continuous for any $S \in \mathrm{Ob}\,\mathcal{C}$. Thus f is regarded as a morphism in $\mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)$ and this proves the assertion. □

Proposition 7.3.3 *Let $\mathcal{C}$ be a quasi-topological category and $D : \mathcal{D}^{op} \to \mathcal{C}$ a functor. Suppose that F is a colimit of $h_D : \mathcal{D} \to \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)$ and that L is a limit of D. Then, L is a limit of the functor $\widehat{D}(F) : \mathcal{C}_F \to \mathcal{C}$ defined by $\widehat{D}(F)(R, x) = R$ and $\widehat{D}(F)(f) = f$.*

Proof. Suppose that $\left(h_D(i) \xrightarrow{\iota_i} F\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of h_D and that $\left(L \xrightarrow{\pi_i} D(i)\right)_{i \in \mathcal{D}}$ is a limiting cone of $D : \mathcal{D}^{op} \to \mathcal{C}$. For $R \in \mathrm{Ob}\,\mathcal{C}$, assume that $i, j \in \mathrm{Ob}\,\mathcal{D}$ and $\alpha \in h_{D(i)}(R)$, $\beta \in h_{D(j)}(R)$ satisfy $(\iota_i)_R(\alpha) = (\iota_j)_R(\beta)$. Then there exist objects $i_1, i_2, \ldots, i_{2n-1}$, morphisms $\tau_{2s-1} : i_{2s-1} \to i_{2s-2}$, $\tau_{2s} : i_{2s-1} \to i_{2s}$ $(s = 1, 2, \ldots, n,\ i_0 = i,\ i_{2n} = j)$ of $\mathcal{D}$ and $\alpha_s \in h_{D(i_s)}(R)$ $(s = 1, 2, \ldots, 2n-1)$ such that $(h_{D(\tau_{2s-1})})_R(\alpha_{2s-1}) = \alpha_{2s-2}$, $(h_{D(\tau_{2s})})_R(\alpha_{2s-1}) = \alpha_{2s}$ for $s = 1, 2, \ldots, n$, where we set $\alpha_0 = \alpha$, $\alpha_{2n} = \beta$. Hence we have $\alpha_{2s-2}\pi_{i_{2s-2}} = \alpha_{2s-1}D(\tau_{2s-1})\pi_{i_{2s-2}} = \alpha_{2s-1}\pi_{i_{2s-1}}$, $\alpha_{2s}\pi_{i_{2s}} = \alpha_{2s-1}D(\tau_{2s})\pi_{i_{2s}} = \alpha_{2s-1}\pi_{i_{2s-1}}$ for $s = 1, 2, \ldots, n$. It follows $\alpha\pi_i = \alpha_0\pi_{i_0} = \alpha_1\pi_{i_1} = \alpha_2\pi_{i_2} = \cdots = \alpha_{2n}\pi_{i_{2n}} = \beta\pi_j$.

For $(R, x) \in \mathrm{Ob}\,\mathcal{C}_F$, take $i \in \mathrm{Ob}\,\mathcal{D}$ and $\alpha \in h_{D(i)}(R)$ satisfying $(\iota_i)_R(\alpha) = x$ and we set $\pi_{(R,x)} = \alpha\pi_i : L \to R = \widehat{D}(F)(R, x)$. By the above argument, this definition of $\pi_{(R,x)}$ does not depend on the choice of $i \in \mathrm{Ob}\,\mathcal{D}$ and $\alpha \in h_{D(i)}(R)$ satisfying $(\iota_i)_R(\alpha) = x$. For a morphism $f : (R, x) \to (S, y)$ of $\mathcal{C}_F$, we take $i \in \mathrm{Ob}\,\mathcal{D}$ and $\alpha \in h_{D(i)}(R)$ satisfying $(\iota_i)_R(\alpha) = x$, then $y = F(f)(x) = F(f)((\iota_i)_R(\alpha)) = (\iota_i)_S(h_{D(i)}(f)(\alpha)) = (\iota_i)_S(f\alpha)$. Hence we have $\pi_{(S,y)} = f\alpha\pi_i = \widehat{D}(F)(f)\pi_{(R,x)}$ and this shows that $\left(L \xrightarrow{\pi_{(R,x)}} \widehat{D}(F)(R, x)\right)_{(R,x) \in \mathrm{Ob}\,\mathcal{C}_F}$ is a cone of $\widehat{D}(F)$. Suppose that $\left(A \xrightarrow{\rho_{(R,x)}} \widehat{D}(F)(R, x)\right)_{(R,x) \in \mathrm{Ob}\,\mathcal{C}_F}$ is a cone of $\widehat{D}(F)$. For a morphism $\tau : i \to j$ of $\mathcal{D}$, since $F(D(\tau))\left((\iota_j)_{D(j)}\left(id_{D(j)}\right)\right) = (\iota_j)_{D(i)}h_{D(j)}(D(\tau))\left(id_{D(j)}\right) = (\iota_j)_{D(i)}(D(\tau)) = (\iota_j)_{D(i)}h_{D(\tau)}\left(id_{D(i)}\right) = (\iota_i)_{D(i)}\left(id_{D(i)}\right)$, $D(\tau) : D(j) \to D(i)$ defines a morphism $\left(D(j), (\iota_j)_{D(j)}\left(id_{D(j)}\right)\right) \to \left(D(i), (\iota_i)_{D(i)}\left(id_{D(i)}\right)\right)$. Therefore $\left(A \xrightarrow{\rho_{\left(D(i), (\iota_i)_{D(i)}\left(id_{D(i)}\right)\right)}} \widehat{D}(F)\left(D(i), (\iota_i)_{D(i)}\left(id_{D(i)}\right)\right) = D(i)\right)_{i \in \mathrm{Ob}\,cd}$ is a cone of D and there exists a unique morphism $\lambda : A \to L$ satisfying $\pi_i\lambda = \rho_{\left(D(i), (\iota_i)_{D(i)}\left(id_{D(i)}\right)\right)}$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. For $(R, x) \in \mathcal{C}_F$, take $i \in \mathrm{Ob}\,\mathcal{D}$ and $\alpha \in h_{D(i)}(R)$ satisfying $(\iota_i)_R(\alpha) = x$. Then, since $F(\alpha)\left((\iota_i)_{D(i)}\left(id_{D(i)}\right)\right) = (\iota_i)_R h_{D(i)}(\alpha)\left(\left(id_{D(i)}\right)\right) = (\iota_i)_R(x)$, α is regarded as a morphism $\left(D(i), (\iota_i)_{D(i)}\left(id_{D(i)}\right)\right) \to (R, x)$. It follows that $\pi_{(R,x)}\lambda = \alpha\pi_i\lambda = \widehat{D}(F)(\alpha)\rho_{\left(D(i), (\iota_i)_{D(i)}\left(id_{D(i)}\right)\right)} = \rho_{(R,x)}$. Assume that $\mu : A \to L$ also satisfies $\pi_{(R,x)}\mu = \rho_{(R,x)}$ for any $(R, x) \in \mathrm{Ob}\,\mathcal{C}_F$. Then, $\pi_i\mu = \pi_{\left(D(i), (\iota_i)_{D(i)}\left(id_{D(i)}\right)\right)}\mu = \rho_{\left(D(i), (\iota_i)_{D(i)}\left(id_{D(i)}\right)\right)} = \pi_i\lambda$ for any $i \in \mathrm{Ob}\,\mathcal{D}$ and this implies $\mu = \lambda$. Hence $\left(L \xrightarrow{\pi_{(R,x)}} \widehat{D}(F)(R, x)\right)_{(R,x) \in \mathrm{Ob}\,\mathcal{C}_F}$ is a limiting cone of $\widehat{D}(F)$. □

Let $\mathcal{C}$ be a quasi-topological category and $\psi : F \to G$ a morphism of $\mathrm{Funct}(\mathcal{C}, \mathcal{T}op)$. Suppose that limits of funtors $\widehat{D}(F) : \mathcal{C}_F \to \mathcal{C}$ and $\widehat{D}(G) : \mathcal{C}_G \to \mathcal{C}$ defined in (7.3.3) exist. Let $\left(L(F) \xrightarrow{\pi_{(R,x)}} \widehat{D}(F)(R, x)\right)_{(R,x) \in \mathrm{Ob}\,\mathcal{C}_F}$

and $\left(L(G) \xrightarrow{\pi'_{(R,y)}} \widehat{D}(G)(R,y)\right)_{(R,y)\in \mathrm{Ob}\,\mathcal{C}_G}$ be limiting cones of $\widehat{D}(F)$ and $\widehat{D}(G)$, respectively. Since

$$\left(L(G) \xrightarrow{\pi'_{(R,\psi_R(x))}} \widehat{D}(G)(R,\psi_R(x)) = \widehat{D}(F)(R,x)\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$$

is a cone of $\widehat{D}(F)$, there exists a unique morphism $L(\psi) : L(G) \to L(F)$ satisfying $\pi_{(R,x)}L(\psi) = \pi'_{(R,\psi_R(x))}$ for any $(R,x) \in \mathrm{Ob}\,\mathcal{C}_F$. Hence, if a limit of $\widehat{D}(F)$ exists for any $F \in \mathrm{Ob}\,\mathrm{Funct}(\mathcal{C},\mathcal{T}op)$, we have a functor $L : \mathrm{Funct}(\mathcal{C},\mathcal{T}op) \to \mathcal{C}^{op}$.

For an object A of $\mathcal{C}$, we define a map $\Theta_{F,A} : \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_A) \to \mathcal{C}(A,L(F)) = \mathcal{C}^{op}(L(F),A)$ as follows. Clearly, $\left(D(F)(R,x) \xrightarrow{\varphi(F)_{(R,x)}} F\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$ is a cone of $D(F) : \mathcal{C}_F^{op} \to \mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$, hence

$$\left(\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_A) \xrightarrow{\varphi(F)^*_{(R,x)}} \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(F)(R,x),h_A)\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$$

is a cone of $(D(F))^{h_A} : \mathcal{C}_F \to \mathcal{T}op$ given by $(D(F))^{h_A}(R,x) = \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(F)(R,x),h_A)$, $(D(F))^{h_A}(f) = D(F)(f)^*$. For any morphism $f : (R,x) \to (S,y)$ in $\mathcal{C}_F$, a diagram

$$\begin{tikzcd}
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(F)(R,x),h_A) \arrow[r, "\theta_R(h_A)"] \arrow[d, "D(F)(f)^*"] & \mathcal{C}\left(A,\widehat{D}(F)(R,x)\right) \arrow[d, "\widehat{D}(F)(f)_*"] \\
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(F)(S,y),h_A) \arrow[r, "\theta_S(h_A)"] & \mathcal{C}\left(A,\widehat{D}(F)(S,y)\right)
\end{tikzcd}$$

commutes and it follows that

$$\left(\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_A) \xrightarrow{\theta_R(h_A)\varphi(F)^*_{(R,x)}} \mathcal{C}\left(A,\widehat{D}(F)(R,x)\right)\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$$

is a cone of the functor $(\widehat{D}(F))_A$ considered in (7.1.3).

On the other hand, since $\left(L(F) \xrightarrow{\pi_{(R,x)}} \widehat{D}(F)(R,x)\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$ is a limiting cone of $\widehat{D}(F)$,

$$\left(\mathcal{C}\left(A,L(F)\right) \xrightarrow{\pi_{(R,x)*}} \mathcal{C}\left(A,\widehat{D}(F)(R,x)\right)\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F} \quad \cdots (*)$$

is a limiting cone of $\Phi(\widehat{D}(F))_A : \mathcal{C}_F \to \mathcal{S}et$. Hence there exists a unique map $\Theta_{F,A} : \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_A) \to \mathcal{C}\left(A,L(F)\right)$ that makes the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_A) \arrow[r, "\Theta_{F,A}"] \arrow[d, "\varphi(F)^*_{(R,x)}"] & \mathcal{C}\left(A,L(F)\right) \arrow[d, "\pi_{(R,x)*}"] \\
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(F)(R,x),h_A) \arrow[r, "\theta_R(h_A)"] & \mathcal{C}\left(A,\widehat{D}(F)(R,x)\right)
\end{tikzcd}$$

If the condition (L) of (7.1.3) is satisfied for A and $\left(L(F) \xrightarrow{\pi_{(R,x)}} \widehat{D}(F)(R,x)\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$, the above limiting cone $(*)$ is the one in the category of topological spaces. In this case $\Theta_{F,A}$ is continuous.

By the naturality of $\theta_R(h_A) : \mathrm{Funct}(\mathcal{C},\mathcal{T}op)\left(D(F)(R,x),h_A\right) \to \mathcal{C}\left(A,\widehat{D}(F)(R,x)\right)$ in A, the following diagram commutes for a morphism $\psi : A \to B$ of $\mathcal{C}$.

$$\begin{tikzcd}
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_B) \arrow[r, "\Theta_{F,B}"] \arrow[d, "(h_\psi)_*"] & \mathcal{C}\left(B,L(F)\right) \arrow[d, "\psi^*"] \\
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_A) \arrow[r, "\Theta_{F,A}"] & \mathcal{C}\left(A,L(F)\right)
\end{tikzcd}$$

If $\psi : F \to G$ is a morphism of $\mathrm{Funct}(\mathcal{C},\mathcal{T}op)$ and a limit of $\widehat{D}(G)$ exist. Then, for any $(R,x) \in \mathrm{Ob}\,\mathcal{C}_F$, since

$$
\begin{tikzcd}
D(F)(R,x) \arrow[r, "\varphi(F)_{(R,x)}"] \arrow[d, equal] & F \arrow[d, "\psi"] \\
D(G)(R,\psi_R(x)) \arrow[r, "\varphi(G)_{(R,\psi_R(x))}"] & G
\end{tikzcd}
$$

commutes, the following diagram commutes.

$$
\begin{tikzcd}
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(G,h_A) \arrow[r, "\varphi(G)^*_{(R,\psi_R(x))}"] \arrow[d, "\psi^*"] & \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(G)(R,\psi_R(x)),h_A) \arrow[r, "\theta_R(h_A)"] \arrow[d, equal] & \mathcal{C}\left(A,\widehat{D}(G)(R,\varphi_R(x))\right) \arrow[d, equal] \\
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_A) \arrow[r, "\varphi(F)^*_{(R,x)}"] & \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(F)(R,x),h_A) \arrow[r, "\theta_R(h_A)"] & \mathcal{C}\left(A,\widehat{D}(F)(R,x)\right)
\end{tikzcd}
$$

Since $\pi'_{(R,\psi_R(x))*}\Theta_{G,A} = \varphi(G)^*_{(R,\psi_R(x))}\theta_R(h_A)$ and $\pi_{(R,x)*}\Theta_{F,A} = \varphi(F)^*_{(R,x)}\theta_R(h_A)$, the commutativity of the above diagram for any $(R,x) \in \operatorname{Ob}\mathcal{C}_F$ implies that the following diagram commutes.

$$
\begin{tikzcd}
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(G,h_A) \arrow[r, "\Theta_{G,A}"] \arrow[d, "\psi^*"] & \mathcal{C}(A,L(G)) \arrow[d, "L(\psi)_*"] \\
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_A) \arrow[r, "\Theta_{F,A}"] & \mathcal{C}(A,L(F))
\end{tikzcd}
$$

Proposition 7.3.4 *Let A be an object of a quasi-topological category $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{T}op$ a functor. Suppose that a limit $L(F)$ of the functor $\widehat{D}(F) : \mathcal{C}_F \to \mathcal{C}$ defined in (7.3.3) exists. If F is a colimit of representable functors, $\Theta_{F,A} : \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_A) \to \mathcal{C}(A,L(F)) = \mathcal{C}^{op}(L(F),A)$ is bijective. Moreover, if the condition (L) of (7.1.3) is satisfied for A and $\left(L(F) \xrightarrow{\pi_{(R,x)}} \widehat{D}(F)(R,x)\right)_{(R,x)\in\operatorname{Ob}\mathcal{C}_F}$, $\Theta_{F,A}$ is a homeomorphism.*

Proof. Since $\left(D(F)(R,x) \xrightarrow{\varphi(F)_{(R,x)}} F\right)_{(R,x)\in\operatorname{Ob}\mathcal{C}_F}$ is a colimiting cone of $D(F) : \mathcal{C}_F^{op} \to \mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$ by (7.3.1), it follows from (2) of (7.2.4) that

$$\left(\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_A) \xrightarrow{\varphi(F)^*_{(R,x)}} \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(F)(R,x),h_A)\right)_{(R,x)\in\operatorname{Ob}\mathcal{C}_F}$$

is a limiting cone of $(D(F))^{h_A} : \mathcal{C}_F \to \mathcal{T}op$ given by

$$(D(F))^{h_A}(R,x) = \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(F)(R,x),h_A), \quad (D(F))^{h_A}(\tau) = D(F)(\tau)^*.$$

Since $\theta_R(h_A) : \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(F)(R,x),h_A) \to \mathcal{C}\left(A,\widehat{D}(F)(R,x)\right)$ is an homeomorphism by (7.2.10) for any $R \in \operatorname{Ob}\mathcal{C}$, $\Theta_{F,A}$ is bijective. If $\mathcal{C}$ satisfies the condition (L) of (7.1.3), $(*)$ is the limiting cone in the category of topological spaces. Hence $\Theta_{F,A}$ is a homeomorphism. □

Corollary 7.3.5 *For objects F and G of $\mathrm{Funct}(\mathcal{C},\mathcal{T}op)$, suppose that limits of $\widehat{D}(F) : \mathcal{C}_F \to \mathcal{C}$ and $\widehat{D}(G) : \mathcal{C}_G \to \mathcal{C}$ exist and that F is a colimit of representable functors. If the condition (L) of (7.1.3) is satisfied for $L(G)$ and $\left(L(F) \xrightarrow{\pi_{(R,x)}} \widehat{D}(F)(R,x)\right)_{(R,x)\in\operatorname{Ob}\mathcal{C}_F}$, then $L : \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,G) \to \mathcal{C}(L(G),L(F))$ is continuous.*

Proof. It follows from (7.3.4) that $\Theta_{F,L(G)} : \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_{L(G)}) \to \mathcal{C}(L(G),L(F))$ is continuous and there exists a unique morphism $\eta_G : G \to h_{L(G)}$ that is mapped to the identity morphism of $L(G)$ by $\Theta_{G,L(G)} : \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(G,h_{L(G)}) \to \mathcal{C}(L(G),L(G))$. Since

$$
\begin{tikzcd}
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)\big(G,h_{L(G)}\big) \arrow[r, "\Theta_{G,L(G)}"] \arrow[d, "\psi^*"] & \mathcal{C}(L(G),L(G)) \arrow[d, "L(\psi)_*"] \\
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)\big(F,h_{L(G)}\big) \arrow[r, "\Theta_{F,L(G)}"] & \mathcal{C}(L(G),L(F))
\end{tikzcd}
$$

commutes, we have $L(\psi) = L(\psi)_*\left(id_{L(G)}\right) = L(\psi)_*\left(\Theta_{G,L(G)}(\eta_G)\right) = \Theta_{F,L(G)}(\eta_G\psi)$. In other words, L is the composition of continuous maps $(\eta_G)_* : \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,G) \to \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_{L(G)})$ and $\Theta_{F,L(G)} : \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(F,h_{L(G)}) \to \mathcal{C}(L(G),L(F))$. □

7.4 Colimit of representable functors

For a quasi-topological category $\mathcal{C}$, we denote by $\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$ the full subcategory of $\mathrm{Funct}(\mathcal{C}, \mathcal{T}op)$ consisting of functors which are colimit of representable functors. In other words, $\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$ is the image of the functor $\Psi : \mathrm{Funct}(\mathcal{C}, \mathcal{S}et) \to \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)$ given in the proof of (7.2.11). We consider the composition $\Psi\widetilde{\Phi}$: $\mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op) \to \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)$ of $\widetilde{\Phi} : \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op) \to \mathrm{Funct}(\mathcal{C}, \mathcal{S}et)$ and $\Psi : \mathrm{Funct}(\mathcal{C}, \mathcal{S}et) \to \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)$. We recall that the counit of the adjunction $\widetilde{\Phi} \dashv \Psi$ is denoted by $\rho : \Psi\widetilde{\Phi} \to id_{\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)}$ in the proof of (7.2.11).

Let us denote by $I_r : \mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op) \to \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)$ the inclusion functor.

Proposition 7.4.1 *Let F be an object of $\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$ and $f : I_r(F) \to G$ a morphism of $\mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)$. There exists a unique morphism $\tilde{f} : I_r(F) \to \Psi\widetilde{\Phi}(G)$ satisfying $\rho_G\tilde{f} = f$. In other words,*

$$(\rho_G)_* : \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)(I_r(F), \Psi\widetilde{\Phi}(G)) \to \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)(I_r(F), G)$$

is bijective.

Proof. Since $\widetilde{\Phi}\Psi = id_{\mathrm{Funct}(\mathcal{C},\mathcal{S}et)}$, we have $\Psi\widetilde{\Phi}\Psi\widetilde{\Phi} = \Psi\widetilde{\Phi}$. It follows from (7.3.1) that $\rho_{I_r(F)} : \Psi\widetilde{\Phi}I_r(F) \to I_r(F)$ is an isomorphism. We set $\tilde{f} = \Psi\widetilde{\Phi}(f)\rho^{-1}_{I_r(F)}$, then we have $\rho_G\tilde{f} = f$ by the naturality of ρ. Since $(\rho_G)_R$: $\Psi\widetilde{\Phi}(G)(R) \to G(R)$ is a continuous bijection for any $R \in \mathrm{Ob}\,\mathcal{C}$, the uniqueness of $\tilde{f}$ is clear. □

Corollary 7.4.2 *Let $\mathscr{R} : \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op) \to \mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$ be the functor that satisfies $I_r\mathscr{R} = \Psi\widetilde{\Phi}$. Then, $\mathscr{R}$ is a right adjoint of the inclusion functor $I_r : \mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op) \to \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)$.*

Proof. By (7.4.1), $\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)(F, \mathscr{R}(G)) \xrightarrow{I_r} \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)(I_r(F), \Psi\widetilde{\Phi}(G)) \xrightarrow{(\rho_G)_*} \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)(I_r(F), G)$ is a bijection. □

Remark 7.4.3 *The counit $\varepsilon : I_r\mathscr{R} \to id_{\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)}$ is given by $\varepsilon_G = \rho_G$ and the unit $\eta : id_{\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)} \to \mathscr{R}I_r$ is given by $I_r(\eta_F) = \rho^{-1}_{I_r(F)}$.*

Proposition 7.4.4 *For an object F of $\mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)$ and an object G of $\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$,*

$$\mathscr{R} : \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)(F, I_r(G)) \to \mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)(\mathscr{R}(F), \mathscr{R}I_r(G))$$

is continuous.

Proof. For $(R, x) \in \mathrm{Ob}\,\mathcal{C}_F$, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)(F, I(G)) \arrow[r, "\mathscr{R}"] \arrow[d, "E_R"] & \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)(\mathscr{R}(F), \mathscr{R}(I(G))) \arrow[d, "E_R"] \\
\mathcal{T}op(F(R), G(R)) \arrow[d, "ev_x"] & \mathcal{T}op(\mathscr{R}(F)(R), \mathscr{R}(I(G))(R)) \arrow[d, "ev_x"] \\
I(G)(R) & \mathscr{R}(I(G))(R) \arrow[l, "(\rho_{I(G)})_R"']
\end{tikzcd}$$

Since $(\rho_{I(G)})_R : \mathscr{R}(I(G))(R) \to I(G)(R)$ is an homeomorphism,

$$\mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)(F, I(G)) \xrightarrow{\mathscr{R}} \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)(\mathscr{R}(F), \mathscr{R}(G)) \xrightarrow{E_R} \mathcal{T}op(\mathscr{R}(F)(R), \mathscr{R}(I(G))(R)) \xrightarrow{ev_x} \mathscr{R}(I(G))(R)$$

is continuous. It follows from (2) of (7.2.2) that $\mathscr{R} : \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)(F, I(G)) \to \mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)(\mathscr{R}(F), \mathscr{R}I(G))$ is continuous. □

Let $\mathcal{C}$ be a quasi-topological category and $\mathcal{R}$ be a subcategory of $\mathcal{C}$. Let us denote by $I : \mathcal{R} \to \mathcal{C}$ the inclusion functor. Suppose that I has a right adjoint $\mathscr{R} : \mathcal{C} \to \mathcal{R}$. We denote by $\eta : id_{\mathcal{A}} \to \mathscr{R}I$ the unit of the adjunction, which is an equivalence if and only if $\mathcal{R}$ is a full subcategory of $\mathcal{C}$.

Proposition 7.4.5 *Suppose that $\mathcal{R}$ is a full subcategory of $\mathcal{C}$. Let $D:\mathcal{D}\to\mathcal{R}$ be a functor and*

$$\left(L \xrightarrow{\pi_i} ID(i)\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$$

a limiting cone of $ID:\mathcal{D}\to\mathcal{C}$.

(1) $\left(\mathscr{R}(L) \xrightarrow{\eta_{D(i)}^{-1}\mathscr{R}(\pi_i)} D(i)\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ *is a limiting cone of $D:\mathcal{D}\to\mathcal{R}$.*

(2) Assume that $\mathscr{R}:\mathcal{C}(I(F),L)\to\mathcal{R}(\mathscr{R}I(F),\mathscr{R}(L))$ and $\mathscr{R}:\mathcal{C}(I(F),ID(i))\to\mathcal{R}(\mathscr{R}I(F),\mathscr{R}ID(i))$ are continuous for any $i\in\mathrm{Ob}\,\mathcal{C}$. Let F be an object of $\mathcal{R}$. If the condition (L) of (7.1.3) is satisfied for ID and $I(F)$, then the condition (L) of (7.1.3) is satisfied for D and F.

Proof. (1) Let $\left(F \xrightarrow{\gamma_i} D(i)\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ be a cone of $D:\mathcal{D}\to\mathcal{R}$. Then $\left(I(F) \xrightarrow{I(\gamma_i)} ID(i)\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a cone of ID and there exists a unique morphism $\psi: I(F)\to L$ satisfying $\pi_i\psi = I(\gamma_i)$ for any $i\in\mathrm{Ob}\,\mathcal{D}$. Then, we have $\eta_{D(i)}^{-1}\mathscr{R}(\pi_i)\mathscr{R}(\psi) = \eta_{D(i)}^{-1}\mathscr{R}I(\gamma_i) = \gamma_i\eta_F^{-1}$. Hence $\eta_{D(i)}^{-1}\mathscr{R}(\pi_i)\mathscr{R}(\psi)\eta_F = \gamma_i$ for any $i\in\mathrm{Ob}\,\mathcal{D}$. Let us denote by $\mathrm{ad}_{F,G}:\mathcal{C}(I(F),G)\to\mathcal{R}(F,\mathscr{R}(G))$ the natural bijection. If $\xi: F\to\mathscr{R}(L)$ satisfies $\eta_{D(i)}^{-1}\mathscr{R}(\pi_i)\xi = \gamma_i$ for any $i\in\mathrm{Ob}\,\mathcal{D}$, then $\mathscr{R}(\pi_i)\xi = \mathscr{R}(\pi_i)\mathscr{R}(\psi)\eta_F\in\mathcal{R}(F,\mathscr{R}(D(i))$ which implies $\pi_i\mathrm{ad}_{F,ID(i)}^{-1}(\xi) = \pi_i\mathrm{ad}_{F,ID(i)}^{-1}(\mathscr{R}(\psi)\eta_F)$ by the commtativity of the following diagram.

$$\begin{array}{ccc} \mathcal{C}(I(F),L) & \xrightarrow{\mathrm{ad}_{F,L}} & \mathcal{R}(F,\mathscr{R}(L)) \\ \big\downarrow{\scriptstyle \pi_{i*}} & & \big\downarrow{\scriptstyle \mathscr{R}(\pi_i)_*} \\ \mathcal{C}(I(F),ID(i)) & \xrightarrow{\mathrm{ad}_{F,ID(i)}} & \mathcal{R}(F,\mathscr{R}ID(i)) \end{array}$$

Since $\left(L \xrightarrow{\pi_i} ID(i)\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of ID, it follows $\mathrm{ad}_{F,L}(\xi) = \mathrm{ad}_{F,L}(\mathscr{R}(\psi)\eta_F)$, therefore $\xi = \mathscr{R}(\psi)\eta_F$.

(2) By the assumption, $\left(\mathcal{C}(I(F),L) \xrightarrow{\pi_{i*}} \mathcal{C}(I(F),ID(i))\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of $ID_{I(F)}:\mathcal{D}\to\mathcal{T}op$. Since I is a continuous functor, $\mathrm{ad}_{F,G}:\mathcal{C}(I(F),G)\to\mathcal{R}(F,\mathscr{R}(G))$ is a homeomorphism for $G=L$ and $D(i)$ for any $i\in\mathrm{Ob}\mathcal{C}$ by (3) and (4) of (7.1.10). Hence $\left(\mathcal{R}(F,\mathscr{R}(L)) \xrightarrow{\mathscr{R}(\pi_i)_*} \mathcal{R}(F,\mathscr{R}ID(i))\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of $\mathscr{R}ID_{I(F)}:\mathcal{D}\to\mathcal{T}op$ by the commutative diagram of (1). Since $\eta_{D(i)}:D(i)\to\mathscr{R}I(i)$ is an isomorphism, $\left(\mathcal{R}(F,\mathscr{R}(L)) \xrightarrow{\left(\eta_{D(i)}^{-1}\mathscr{R}(\pi_i)\right)_*} \mathcal{R}(F,D(i))\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of $D_F:\mathcal{D}\to\mathcal{T}op$. □

Since $\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$ is a full subcategory of $\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$ and the inclusion functor $I_r:\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)\to\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$ has a right adjoint $\mathscr{R}:\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)\to\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$, the above result and (7.1.9) implies the following.

Corollary 7.4.6 *Let $D:\mathcal{D}\to\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$ be a functor. If $\left(L \xrightarrow{\pi_i} I_rD(i)\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of $I_rD:\mathcal{D}\to\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$, then*

$$\left(\mathscr{R}(L) \xrightarrow{\eta_{D(i)}^{-1}\mathscr{R}(\pi_i)} D(i)\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$$

is a limiting cone of $D:\mathcal{D}\to\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$. Hence $\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$ is complete.

Definition 7.4.7 *Let A and B be objects of a quasi-topological category $\mathcal{C}$. A topological coproduct of A and B is a coproduct $A \xrightarrow{\iota_1} A\coprod B \xleftarrow{\iota_2} B$ of A and B such that $\mathcal{C}(A,R) \xleftarrow{\iota_1^*} \mathcal{C}(A\coprod B,R) \xrightarrow{\iota_2^*} \mathcal{C}(B,R)$ is a product of $\mathcal{C}(A,R)$ and $\mathcal{C}(B,R)$ in $\mathcal{T}op$ for any $R\in\mathrm{Ob}\,\mathcal{C}$. If each pair of objects of $\mathcal{C}$ has a topological coproduct, we say that $\mathcal{C}$ is a category with finite topological coproducts.*

Remark 7.4.8 *We denote $\Psi\widetilde{\Phi}(F\times G)$ by $F\times_r G$ for $F,G\in\mathrm{Ob}\,\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$. We remark that, if a topological coproduct of objects A and B of $\mathcal{C}$ exists, then $\rho_{h_A\times h_B}: h_A\times_r h_B\to h_A\times h_B$ is an isomorphism.*

Lemma 7.4.9 *Let X,Y,Z,W be topolpgical spaces. A map $\mathrm{prod}:\mathcal{T}op(X,Y)\times\mathcal{T}op(Z,W)\to\mathcal{T}op(X\times Z,Y\times W)$ defined by $\mathrm{prod}(f,g) = f\times g$ is continuous.*

Proof. For $x \in X$ and $z \in Z$, since $E_{(x,z)}\mathrm{prod}(f,g) = (f(x), g(z)) = (E_x(f), E_z(g)) = (E_x \times E_z)(f,g)$, we have $E_{(x,z)}\mathrm{prod} = E_x \times E_z : \mathcal{T}\!op(X,Y) \times \mathcal{T}\!op(Z,W) \to Y \times W$ which is continuous. $\square$

Proposition 7.4.10 *Let $\mathcal{C}$ be a quasi-topological category. A functor $\times : \mathrm{Funct}(\mathcal{C}, \mathcal{T}\!op) \times \mathrm{Funct}(\mathcal{C}, \mathcal{T}\!op) \to \mathrm{Funct}(\mathcal{C}, \mathcal{T}\!op)$ defined by $\times(F,G) = F \times G$ and $\times(f,g) = f \times g$ is continuous.*

Proof. For $F, F'G, G' \in \mathrm{Ob}\mathrm{Funct}_c(\mathcal{C}, \mathcal{T}\!op)$ and $R \in \mathrm{Ob}\,\mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Funct}(\mathcal{C}, \mathcal{T}\!op)(F,F') \times \mathrm{Funct}(\mathcal{C}, \mathcal{T}\!op)(G,G') \arrow[r, "\times"] \arrow[d, "E_R \times E_R"] & \mathrm{Funct}(\mathcal{C}, \mathcal{T}\!op)(F \times G, F' \times G') \arrow[d, "E_R"] \\
\mathcal{T}\!op(F(R), F'(R)) \times \mathcal{T}\!op(G(R), G'(R)) \arrow[r, "\times"] & \mathcal{T}\!op(F(R) \times G(R), F'(R) \times G'(R))
\end{tikzcd}$$

Since the lower horizontal map is continuous by (7.4.9) and the left vertical map is also continuous, the assertion follows from (7.2.1). $\square$

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and $D : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ a functor. For each $j \in \mathrm{Ob}\,\mathcal{D}$, let $D_i : \mathcal{C} \to \mathcal{E}$ be the functor given by $D_i(j) = D(i,j)$ for $j \in \mathrm{Ob}\,\mathcal{D}$ and $D_i(\tau) = D(id_i, \tau)$ for $\tau \in \mathrm{Mor}, \mathcal{D}$. Suppose that there exists a colimiting cone $\left(D(i,j) \xrightarrow{\iota_{i,j}} X_i\right)_{j \in \mathrm{Ob}\,\mathcal{D}}$ of D_i for each $i \in \mathrm{Ob}\,\mathcal{C}$. Then, for a morphism $\sigma : i \to k$ in $\mathcal{C}$, there is a unique morphism $\bar{\sigma} : X_i \to X_k$ satisfying $\bar{\sigma}\iota_{i,j} = \iota_{k,j}D(\sigma, id_j)$ for any $j \in \mathrm{Ob}\,\mathcal{D}$. We define a functor $\overline{D} : \mathcal{C} \to \mathcal{E}$ by $\overline{D}(i) = X_i$ and $\overline{D}(\sigma) = \bar{\sigma}$.

Lemma 7.4.11 *Suppose that $\left(\overline{D}(i) \xrightarrow{\eta_i} Y\right)_{i \in \mathrm{Ob}\,\mathcal{C}}$ is a cone of $\overline{D}$. Then $\left(D(i,j) \xrightarrow{\eta_i\iota_{i,j}} Y\right)_{(i,j) \in \mathrm{Ob}\,\mathcal{C} \times \mathcal{D}}$ is a colimiting cone of D if and only if $\left(\overline{D}(i) \xrightarrow{\eta_i} Y\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $\overline{D}$.*

Proof. Assume that $\left(D(i,j) \xrightarrow{\eta_i\iota_{i,j}} Y\right)_{(i,j) \in \mathrm{Ob}\,\mathcal{C} \times \mathcal{D}}$ is a colimiting cone of D. Let $\left(\overline{D}(i) \xrightarrow{\lambda_i} Z\right)_{i \in \mathrm{Ob}\,\mathcal{C}}$ be a cone of $\overline{D}$. Since $\left(D(i,j) \xrightarrow{\lambda_i\iota_{i,j}} Z\right)_{(i,j) \in \mathrm{Ob}\,\mathcal{C} \times \mathcal{D}}$ is a cone of D, there is unique morphism $\varphi : Y \to Z$ satisfying $\varphi\eta_i\iota_{i,j} = \lambda_i\iota_{i,j}$. Since $\left(D(i,j) \xrightarrow{\iota_{i,j}} X_i\right)_{j \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of D_i for each $i \in \mathrm{Ob}\,\mathcal{C}$, we have $\varphi\eta_i = \lambda_i$. Hence $\left(\overline{D}(i) \xrightarrow{\eta_i} Y\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $\overline{D}$.

Conversely, assume that $\left(\overline{D}(i) \xrightarrow{\eta_i} Y\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $\overline{D}$. Let $\left(D(i,j) \xrightarrow{\mu_{i,j}} Z\right)_{(i,j) \in \mathrm{Ob}\,\mathcal{C} \times \mathcal{D}}$ be a cone of D. Then, $\left(D(i,j) \xrightarrow{\mu_{i,j}} Z\right)_{j \in \mathrm{Ob}\,\mathcal{D}}$ is a cone of D_i for each $i \in \mathrm{Ob}\,\mathcal{C}$. There exists unique morphism $\alpha_i : \overline{D}(i) = X_i \to Z$ satisfying $\alpha_i\iota_{i,j} = \mu_{i,j}$ for any $j \in \mathrm{Ob}\,\mathcal{D}$. Let $\sigma : i \to k$ be a morphism of $\mathcal{C}$. We have $\alpha_k\overline{D}(\sigma)\iota_{i,j} = \alpha_k\iota_{k,j}D(\sigma, id_j) = \mu_{k,j}D(\sigma, id_j) = \mu_{i,j} = \alpha_i\iota_{i,j}$ for any $j \in \mathrm{Ob}\,cd$. Since $\left(D(i,j) \xrightarrow{\iota_{i,j}} X_i\right)_{j \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of D_i, it follows that $\alpha_k\overline{D}(\sigma) = \alpha_i$. Thus $\left(\overline{D}(i) \xrightarrow{\alpha_i} Z\right)_{i \in \mathrm{Ob}\,\mathcal{C}}$ is a cone of $\overline{D}$ and there exists unique morphism $\beta : Y \to Z$ satisfying $\beta\eta_i = \alpha_i$ for any $i \in \mathrm{Ob}\,\mathcal{C}$. Hence $\left(D(i,j) \xrightarrow{\eta_i\iota_{i,j}} Y\right)_{(i,j) \in \mathrm{Ob}\,\mathcal{C} \times \mathcal{D}}$ is a colimiting cone of D. $\square$

Proposition 7.4.12 *Let $\mathcal{C}$ be a quasi-topological category with finite topological coproducts and $D : \mathcal{D}^{op} \to \mathcal{C}$ a functor. If $G \in \mathrm{Ob}\,\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}\!op)$ and $\left(h_{D(i)} \xrightarrow{\iota_i} F\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $hD : \mathcal{D} \to \mathrm{Funct}_r(\mathcal{C}, \mathcal{T}\!op)$, then $\left(h_{D(i)} \times_r G \xrightarrow{\iota_i \times_r id_G} F \times_r G\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of a functor $hD \times_r G : \mathcal{D} \to \mathrm{Funct}_r(\mathcal{C}, \mathcal{T}\!op)$ given by $i \mapsto h_{D(i)} \times_r G$ and $\tau \mapsto h_{D(\tau)} \times_r id_G$.*

Proof. First, we consider the case $G = h_R$ for some $R \in \mathrm{Ob}\,\mathcal{C}$. Since $\widetilde{\Phi}\Psi\widetilde{\Phi} = \widetilde{\Phi}$, it is easy to verify that $\left(\widetilde{\Phi}(h_{D(i)} \times_r h_R) \xrightarrow{\widetilde{\Phi}(\iota_i \times_r id_{h_R})} \widetilde{\Phi}(F \times_r h_R)\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of a functor $\widetilde{\Phi}(h_D \times_r h_R) : \mathcal{D} \to \mathrm{Funct}(\mathcal{C}, \mathcal{S}et)$. Since $h_{D(i)} \times h_R$ is naturally equivalent to $h_{D(i) \coprod R}$, $\left(h_D(i) \times_r h_R \xrightarrow{\iota_i \times_r id_{h_R}} F \times_r h_R\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of a functor $h_D \times_r h_R$ by (7.3.2).

Generally, since G is a colimit of representable functors, there exists a colimiting cone $\left(h_{E(j)} \xrightarrow{\eta_j} G\right)_{j\in \mathrm{Ob}\,\mathcal{E}}$ for some functor $E : \mathcal{E}^{op} \to \mathcal{C}$. We consider a functor $D : \mathcal{D}\times\mathcal{E} \to \mathrm{Funct}_r(\mathcal{C}, \mathcal{T}\!op)$ defined by $D(i, j) = h_{D(i)} \times h_{E(j)}$ and $D(\sigma, \tau) = h_{D(\sigma)} \times_r h_{E(\tau)}$. As we have seen above, for each $i \in \mathrm{Ob}\,\mathcal{D}$,

$$\left(D_i(j) = h_{D(i)} \times_r h_{E(j)} \xrightarrow{id_{h_{D(i)}}\times_r \eta_j} h_{D(i)} \times_r G\right)_{j\in \mathrm{Ob}\,\mathcal{E}}$$

is a colimiting cone of $D_i : \mathcal{E} \to \mathrm{Funct}_r(\mathcal{C}, \mathcal{T}\!op)$. Since $\left(h_{D(i)} \xrightarrow{\iota_i} F\right)_{i\in \mathrm{Ob}\,\mathcal{D}}$ and $\left(h_{E(j)} \xrightarrow{\eta_j} G\right)_{j\in \mathrm{Ob}\,\mathcal{E}}$ are colimiting cones, $\left(h_{D(i)}(R) \times h_{E(j)}(R) \xrightarrow{(\iota_i)_R\times(\eta_j)_R} F(R) \times G(R)\right)_{(i,j)\in \mathrm{Ob}\,\mathcal{C}\times\mathcal{D}}$ is a colimiting cone in $\mathcal{S}et$ for each $R \in \mathrm{Ob}\,\mathcal{C}$, namely, $\left(\widetilde{\Phi}(h_D(i) \times_r h_E(j)) \xrightarrow{\widetilde{\Phi}(\iota_i\times_r\eta_j)} \widetilde{\Phi}(F \times_r G)\right)_{(i,j)\in \mathrm{Ob}\,\mathcal{C}\times\mathcal{D}}$ is a colimiting cone of $\widetilde{\Phi}(h_D \times_r h_E) : \mathcal{D} \times \mathcal{E} \to \mathrm{Funct}(\mathcal{C}, \mathcal{S}et)$. Hence it follows from (7.3.2) that

$$\left(D(i, j) = h_D(i) \times_r h_E(j) \xrightarrow{\iota_i\times_r\eta_j} F \times_r G\right)_{(i,j)\in \mathrm{Ob}\,\mathcal{C}\times\mathcal{D}}$$

is a colimiting cone of D. Clearly, $\left(h_{D(i)} \times_r G \xrightarrow{\iota_i\times_r id_G} F \times_r G\right)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of $\overline{D} = h_D \times_r G$. Therefore the result follows from (7.4.11). $\square$

7.5 Exponential law

For a category $\mathcal{C}$, we consider the fibered category $p : \mathcal{C}^{(2)} \to \mathcal{C}$ given in (1) of (6.1.9). Note that, if $\mathcal{C}$ is a quasi-topological category, $\mathcal{C}^{(2)}$ is a quasi-topological category by (7.2.3) and p is continuous.

For a functor $F : \mathcal{C} \to \mathcal{T}$ and $R \in \mathrm{Ob}\,\mathcal{C}$, define a functor $F_R : \mathcal{C}^{(2)}_R \to \mathcal{T}$ by $F_R(\eta : R \to A) = F(A)$ and $F_R(id_R, \varphi) = F(\varphi)$. Moreover, if $f : R \to S$ is a morphism in $\mathcal{C}$, $G : \mathcal{C} \to \mathcal{T}$ is a functor and $\xi : F_R \to G_R$ is a natural transformation, let $f_*(\xi) : F_S \to G_S$ be the natural transformation such that $f_*(\xi)_\eta : F_S(\eta : S \to A) \to G_S(\eta : S \to A)$ is given by the following composition.

$$F_S(\eta : S \to A) = F(A) = F_R(\eta f : R \to A) \xrightarrow{\xi_{\eta f}} G_R(\eta f : R \to A) = G(A) = G_S(\eta : S \to A)$$

Suppose that $\mathcal{T}$ is a quasi-topological category. For functors $F, G : \mathcal{C} \to \mathcal{T}$, we define a functor $G^F : \mathcal{C} \to \mathcal{T}\!op$ as follows. For $R \in \mathrm{Ob}\,\mathcal{C}$, we set $G^F(R) = \mathrm{Funct}\big(\mathcal{C}^{(2)}_R, \mathcal{T}\!op\big)(F_R, G_R)$. For a morphism $f : R \to S$ in $\mathcal{C}$, $G^F(f) : G^F(R) \to G^F(S)$ is the map given by $G^F(f)(\xi) = f_*(\xi)$. Since the following diagram commutes for any $(\eta : S \to A) \in \mathrm{Ob}\,\mathcal{C}^{(2)}_S$, $G^F(f) : G^F(R) \to G^F(S)$ is continuous by (7.2.8).

$$\begin{tikzcd}
G^F(R) \arrow[r, "E_{\eta f}"] \arrow[d, "G^F(f)"] & \mathcal{T}(F_R(\eta f : R \to A), G_R(\eta f : R \to A)) \arrow[d, equal] \\
G^F(S) \arrow[r, "E_\eta"] & \mathcal{T}(F_S(\eta : S \to A), G_S(\eta : S \to A))
\end{tikzcd}$$

For a morphism $\zeta : F \to H$ in $\mathrm{Funct}(\mathcal{C}, \mathcal{T})$ and a functor $G : \mathcal{C} \to \mathcal{T}$, we define a morphism $G^\zeta : G^H \to G^F$ as follows. For $R \in \mathrm{Ob}\,\mathcal{C}$, we denote by $\zeta^R : F_R \to H_R$ the morphism in $\mathrm{Funct}\big(\mathcal{C}^{(2)}_R, \mathcal{T}\!op\big)$ defined by

$$\zeta^R_{(\eta:R\to A)} = \zeta_A : F_R(\eta : R \to A) = F(A) \to H(A) = H_R(\eta : R \to A).$$

Then, $G^\zeta_R : G^H(R) \to G^F(R)$ is the map $\zeta^{R*} : \mathrm{Funct}\big(\mathcal{C}^{(2)}_R, \mathcal{T}\!op\big)(H_R, G_R) \to \mathrm{Funct}\big(\mathcal{C}^{(2)}_R, \mathcal{T}\!op\big)(F_R, G_R)$ induced by ζ^R. It is easy to verify that G^ζ_R is natural in R. Similarly, for a morphism $\xi : G \to H$ in $\mathrm{Funct}(\mathcal{C}, \mathcal{T})$ and a functor $F : \mathcal{C} \to \mathcal{T}$, we define a morphism $\xi^F : G^F \to H^F$ as follows. For $R \in \mathrm{Ob}\,\mathcal{C}$, $\xi^F_R : G^F(R) \to H^F(R)$ is the map $\xi^R_* : \mathrm{Funct}\big(\mathcal{C}^{(2)}_R, \mathcal{T}\!op\big)(F_R, G_R) \to \mathrm{Funct}\big(\mathcal{C}^{(2)}_R, \mathcal{T}\!op\big)(F_R, H_R)$ induced by ξ^R. It is easy to verify that ξ^F_R is natural in R. The following is obvious from the definitions of the morphisms.

Proposition 7.5.1 *Let $\xi : G \to H$ and $\zeta : E \to F$ be morphisms in $\mathrm{Funct}(\mathcal{C}, \mathcal{T})$. Then, the following diagram commutes.*

$$\begin{array}{ccc} G^F & \xrightarrow{G^\zeta} & G^E \\ \downarrow{\scriptstyle \xi^F} & & \downarrow{\scriptstyle \xi^E} \\ H^F & \xrightarrow{H^\zeta} & H^E \end{array}$$

Lemma 7.5.2 *Let A and R be objects of a quasi-topological category $\mathcal{C}$ such that there exists a topological coproduct $R \xrightarrow{\iota_1} R \coprod A \xleftarrow{\iota_2} A$ of R and A. Then, $(h_A)_R : \mathcal{C}_R^{(2)} \to \mathcal{T}op$ is naturally equivalent to the functor represented by $(\iota_1 : R \to R \coprod A)$.*

Proof. For $(\eta : R \to B) \in \mathrm{Ob}\,\mathcal{C}_R^{(2)}$, define $\psi_\eta : h_{(\iota_1 : R \to R \coprod A)}(\eta : R \to B) \to \mathcal{C}(A, B) = (h_A)_R(\eta : R \to B)$ by $\psi_\eta(id_R, \varphi) = \varphi\iota_2$. It is clear that ψ_η is continuous and natural in η. Let us denote by $\mu : \mathcal{C}\left(R \coprod A, B\right) \to \mathcal{C}(R, B) \times \mathcal{C}(A, B)$ the homeomorphism induced by $\iota_1^* : \mathcal{C}\left(R \coprod A, B\right) \to \mathcal{C}(R, B)$ and $\iota_2^* : \mathcal{C}\left(R \coprod A, B\right) \to \mathcal{C}(R, A)$. Define a map $(h_A)_R(\eta : R \to B) \to h_{(\iota_1 : R \to R \coprod A)}(\eta : R \to B)$ by $\varphi \mapsto \left(id_R, \mu^{-1}(\eta, \varphi)\right)$. It is clear that this map is the continuous invrese of ψ_η. $\square$

Lemma 7.5.3 *Let A be an object of a quasi-topological category $\mathcal{C}$ such that, for any object R of $\mathcal{C}$, there exists a topological coproduct $R \xrightarrow{\iota_1} R \coprod A \xleftarrow{\iota_2} A$ of R and A. Define a functor $\Gamma_A : \mathcal{C} \to \mathcal{C}$ by $\Gamma_A(R) = R \coprod A$ and $\Gamma_A(f) = f \coprod id_A$. Then, Γ_A is continuous. Similarly, define a functor ${}_A\Gamma : \mathcal{C} \to \mathcal{C}$ by ${}_A\Gamma(R) = A \coprod R$ and ${}_A\Gamma(f) = id_A \coprod f$. Then, ${}_A\Gamma$ is also continuous.*

Proof. Let R and S be objects of $\mathcal{C}$. Since the following diagram commutes, $\iota_1^*\Gamma_A$ and $\iota_2^*\Gamma_A$ are continuous.

$$\begin{tikzcd} & \mathcal{C}(R, S) \arrow[dl, "\iota_{1*}"'] \arrow[d, "\Gamma_A"] \arrow[dr, "\iota_{2*}"] & \\ \mathcal{C}(R, S \coprod A) & \mathcal{C}\left(R \coprod A, S \coprod A\right) \arrow[l, "\iota_1^*"'] \arrow[r, "\iota_2^*"] & \mathcal{C}(A, S \coprod A) \end{tikzcd}$$

Hence Γ_A is continuous. $\square$

Proposition 7.5.4 *Let A be an object of a quasi-topological category $\mathcal{C}$ satisfying the conditions of (7.5.3). For a continuous functor $G : \mathcal{C} \to \mathcal{T}op$, $G^{h_A} : \mathcal{C} \to \mathcal{T}op$ is naturally equivalent to $G\Gamma_A$. Hence G^{h_A} is continuous.*

Proof. Define a natural transformation $\gamma : G^{h_A} \to G\Gamma_A$ as follows. For $R \in \mathrm{Ob}\,\mathcal{C}$, let us denote by $\psi : h_{(\iota_1 : R \to R \coprod A)} \to (h_A)_R$ the natural equivalence given in (7.5.2). Consider the composition of the map

$$\psi^* : G^{h_A}(R) = \mathrm{Funct}\big(\mathcal{C}_R^{(2)}, \mathcal{T}op\big)\left((h_A)_R, G_R\right) \to \mathrm{Funct}\big(\mathcal{C}_R^{(2)}, \mathcal{T}op\big)\left(h_{(\iota_1 : R \to R \coprod A)}, G_R\right)$$

induced by ψ and the map

$$\theta_{(\iota_1 : R \to R \coprod A)}(G_R) : \mathrm{Funct}\big(\mathcal{C}_R^{(2)}, \mathcal{T}op\big)\left(h_{(\iota_1 : R \to R \coprod A)}, G_R\right) \to G_R\left(\iota_1 : R \to R \coprod A\right) = G\Gamma_A(R)$$

introduced in (7.2.6) which is a homeomorphism by (7.2.10). This composition is natural in R, hence gives a natural equivalence $\gamma : G^{h_A} \to G\Gamma_A$. $\square$

For functors $F, H : \mathcal{C} \to \mathcal{T}op$, we define a natural transformation $\eta_H^F : H \to (H \times F)^F$ as follows. For $R \in \mathrm{Ob}\,\mathcal{C}$ and $x \in H(R)$, let $c_{R,x} : F_R \to H_R$ be the morphism given by $(c_{R,x})_{(\iota : R \to A)}(y) = H(\iota)(x)$ for $(\iota : R \to A) \in \mathrm{Ob}\,\mathcal{C}_R^{(2)}$ and $y \in F_R(\iota : R \to A) = F(A)$. Since $(c_{R,x})_{(\iota : R \to A)} : F_R(\iota : R \to A) \to H_R(\iota : R \to A)$ is a constant map, it is clearly continuous. Suppose that $\varphi : (\iota : R \to A) \to (\zeta : R \to B)$ is a morphimsm of $\mathrm{Funct}\Big(\mathcal{C}_R^{(2)}, \mathcal{T}op\Big)$. Since $\varphi\iota = \zeta$, we have $H(\varphi)H(\iota)(x) = H(\zeta)(x)$. It follows that $H_R(\varphi)(c_{R,x})_{(\iota : R \to A)}(y) = H_R(\zeta)(x) = H(\zeta)(x) = (c_{R,x})_{(\zeta : R \to B)}(F_R(y))$, namely $c_{R,x}$ is natural. Let us denote by $p_1 : H \times F \to H$ and $p_2 : H \times F \to H$ the projections. For $R \in \mathrm{Ob}\,\mathcal{C}$ and $x \in H(R)$, $\left(\eta_H^F\right)_R(x) \in (H \times F)^F(R) = \mathrm{Funct}\big(\mathcal{C}_R^{(2)}, \mathcal{T}op\big)(F_R, (H \times F)_R)$ is the unique morphism satisfying $p_1^R\left(\eta_H^F\right)_R(x) = c_{R,x}$ and $p_2^R\left(\eta_H^F\right)_R(x) = id_{F_R}$.

Let $f : R \to S$ be a morphism in $\mathcal{C}$. For $x \in H(R)$ and $(\iota : S \to A) \in \mathrm{Mor}\,\mathcal{C}_S^{(2)}$, we have
$\left(p_1^S(H \times F)^F(f)\left(\left(\eta_H^F\right)_R(x)\right)\right)_{(\iota : S \to A)} = (p_1^S)_{(\iota : S \to A)} f_* \left(\left(\eta_H^F\right)_R(x)\right)_{(\iota : S \to A)} = \left(p_1^R\left(\eta_H^F\right)_R(x)\right)_{(\iota f : R \to A)} = (c_{R,x})_{(\iota f : R \to A)} = (c_{S,H(f)(x)})_{(\iota : S \to A)} = \left(p_1^S\left(\eta_H^F\right)_S(H(f)(x))\right)_{(\iota : S \to A)}$ and
$\left(p_2^S(H \times F)^F(f)\left(\left(\eta_H^F\right)_R(x)\right)\right)_{(\iota : S \to A)} = (p_2^S)_{(\iota : S \to A)} f_*(\left(\eta_H^F\right)_R(x))_{(\iota : S \to A)} = \left(p_2^R\left(\eta_H^F\right)_R(x)\right)_{(\iota f : R \to A)} = (id_{F_R})_{(\iota f : R \to A)} = (id_{F_S})_{(\iota : S \to A)} = \left(p_2^S\left(\eta_H^F\right)_S(H(f)(x))\right)_{(\iota : S \to A)}$. Thus $\left(\eta_H^F\right)_R$ is natural in R.

Proposition 7.5.5 *$(\eta^F_H)_R : H(R) \to (H \times F)^F(R)$ is continuous for $R \in \operatorname{Ob}\mathcal{C}$. Hence $\eta^F_H : H \to (H \times F)^F$ is a morphism in $\operatorname{Funct}(\mathcal{C}, \mathcal{T}op)$.*

Proof. Define a map $i_y : H(R) \to H(A) \times F(A) = (H \times F)_R(\iota : R \to A)$ by $i_y(x) = (H(\iota)(x), y)$ for $(\iota : R \to A) \in \operatorname{Ob}\mathcal{C}^{(2)}_R$ and $y \in F_R(\iota : R \to A) = F(A)$. Then, i_y is continuous and the following diagram commutes.

$$\begin{tikzcd}
H(R) \arrow[r, "(\eta^F_H)_R"] \arrow[d, "i_y"] & \operatorname{Funct}(\mathcal{C}^{(2)}_R, \mathcal{T}op)(F_R, (H \times F)_R) \arrow[d, "E_{(\iota:R\to A)}"] \\
(H \times F)_R(\iota : R \to A) & \mathcal{T}op(F_R(\iota : R \to A), (H \times F)_R(\iota : R \to A)) \arrow[l, "ev_y"']
\end{tikzcd}$$

It follows from (7.2.2) that $(\eta^F_H)_R : H(R) \to (H \times F)^F(R)$ is continuous for $R \in \operatorname{Ob}\mathcal{C}$. □

Define a map $\operatorname{Ad}(H, F; G) : \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(H \times F, G) \to \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(H, G^F)$ by $\operatorname{Ad}(H, F; G)(\varphi) = \varphi^F \eta^F_H$.

Proposition 7.5.6 *Let A be an object of a quasi-topological category $\mathcal{C}$ satisfying the conditions of (7.5.3). For $R \in \operatorname{Ob}\mathcal{C}$ and a continuous functor G, $\operatorname{Ad}(h_R, h_A; G) : \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(h_R \times h_A, G) \to \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(h_R, G^{h_A})$ is a homeomorphism.*

Proof. Let $\kappa : h_{R \coprod A} \to h_R \times h_A$ be the natural equivalence induced by $\iota_1 : R \to R \coprod A$ and $\iota_2 : A \to R \coprod A$. Since both $\theta_{R \coprod A}(G)\kappa^*$ and $\gamma_R \theta_R(G^{h_A}) \operatorname{Ad}(h_R, h_A; G)$ map $\varphi \in \operatorname{Ob}\operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(h_R \times h_A, G)$ to $\varphi_{R \coprod A}(\iota_1, \iota_2) \in G(R \coprod A)$, the following diagram commutes. Hence the result follows from (7.2.10) and (7.5.4).

$$\begin{tikzcd}
\operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(h_{R \coprod A}, G) \arrow[d, "\theta_{R \coprod A}(G)"] & \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(h_R \times h_A, G) \arrow[l, "\kappa^*"'] \arrow[r, "\operatorname{Ad}(h_R, h_A; G)"] & \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(h_R, G^{h_A}) \arrow[d, "\theta_R(G^{h_A})"] \\
G(R \coprod A) \arrow[r, equal] & G\Gamma_A(R) & G^{h_A}(R) \arrow[l, "\gamma_R"']
\end{tikzcd}$$

□

For $R \in \operatorname{Ob}\mathcal{C}$, we define a functor $R_\sharp : \operatorname{Funct}(\mathcal{C}, \mathcal{T}) \to \operatorname{Funct}(\mathcal{C}^{(2)}_R, \mathcal{T}op)$ as follows. We set $R_\sharp(F) = F_R$ for $F \in \operatorname{Ob}\operatorname{Funct}(\mathcal{C}, \mathcal{T})$ and $R_\sharp(\zeta) = \zeta^R$.

Proposition 7.5.7 *For $R \in \operatorname{Ob}\mathcal{C}$, $R_\sharp : \operatorname{Funct}(\mathcal{C}, \mathcal{T}) \to \operatorname{Funct}(\mathcal{C}^{(2)}_R, \mathcal{T}op)$ preserves limits and colimits. Moreover, if $\mathcal{T}$ is a quasi-topological category, $R_\sharp$ is a continuous functor.*

Proof. For $(\eta : R \to A) \in \operatorname{Ob}\mathcal{C}^{(2)}_R$, we note that $E_\eta R_\sharp = E_A : \operatorname{Funct}(\mathcal{C}, \mathcal{T}) \to \mathcal{T}$. Let $D : \mathcal{D} \to \operatorname{Funct}(\mathcal{C}, \mathcal{T})$ a functor and $\left(L \xrightarrow{\pi_i} D(i)\right)_{i \in \operatorname{Ob}\mathcal{D}}$ a limiting cone of D. Then,

$$\left(L_R(\eta : R \to A) \xrightarrow{R_\sharp(\pi_i)_{(\eta:R\to A)}} D(i)_R(\eta : R \to A)\right)_{i \in \operatorname{Ob}\mathcal{D}}$$

is a limiting cone of $E_\eta R_\sharp D = E_A D : \mathcal{D} \to \mathcal{T}$. Hence $\left(L_R \xrightarrow{R_\sharp(\pi_i)} D(i)_R\right)_{i \in \operatorname{Ob}\mathcal{D}}$ is a limiting cone of $R_\sharp D$. Similarly, if $\left(D(i) \xrightarrow{\iota_i} C\right)_{i \in \operatorname{Ob}\mathcal{D}}$ a colimiting cone of D, then

$$\left(D(i)_R(\eta : R \to A) \xrightarrow{R_\sharp(\iota_i)_{(\eta:R\to A)}} C_R(\eta : R \to A)\right)_{i \in \operatorname{Ob}\mathcal{D}}$$

is a colimiting cone of $E_\eta R_\sharp D = E_A D : \mathcal{D} \to \mathcal{T}$. Hence $\left(D(i)_R \xrightarrow{R_\sharp(\iota_i)} C_R\right)_{i \in \operatorname{Ob}\mathcal{D}}$ is a limiting cone of $R_\sharp D$. Suppose that $\mathcal{T}$ is a quasi-topological category. For $(\eta : R \to S) \in \operatorname{Ob}\mathcal{C}^{(2)}_R$, since

$$\begin{tikzcd}
\operatorname{Funct}(\mathcal{C}, \mathcal{T})(X, Y) \arrow[r, "E_S"] \arrow[d, "R_\sharp"] & \mathcal{T}(X(S), Y(S)) \arrow[d, equal] \\
\operatorname{Funct}(\mathcal{C}^{(2)}_R, \mathcal{T}op)(X_R, Y_R) \arrow[r, "E_\eta"] & \mathcal{T}(X_R(\eta : R \to S), Y_R(\eta : R \to S))
\end{tikzcd}$$

commutes, $R_\sharp$ is a continuous functor. $\square$

The following result is straightforward from (7.5.2) and (7.5.7).

Corollary 7.5.8 *Let R be an object of a quasi-topological category $\mathcal{C}$ such that topological coproduct of R and A exists for any $A \in \mathrm{Ob}\,\mathcal{C}$. If $F \in \mathrm{Ob}\,\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$, then $F_R \in \mathrm{Ob}\,\mathrm{Funct}_r\left(\mathcal{C}_R^{(2)}, \mathcal{T}op\right)$.*

Corollary 7.5.9 *Let R be an object of a quasi-topological category $\mathcal{C}$ such that topological coproduct of R and A exists for any $A \in \mathrm{Ob}\,\mathcal{C}$. For $F, G \in \mathrm{Ob}\,\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$ and $R \in \mathrm{Ob}\,\mathcal{C}$, $(F \times_r G)_R$ is isomorphic to $F_R \times_r G_R$.*

Proof. Let us denote by $p_1 : F \times G \to F$ and $p_2 : F \times G \to G$ the projections. $(p_1\rho_{F\times G})^R : (F \times_r G)_R \to F_R$ and $(p_2\rho_{F\times G})^R : (F \times_r G)_R \to G_R$ induce a morphism $((p_1\rho_{F\times G})^R, (p_2\rho_{F\times G})^R) : (F \times_r G)_R \to F_R \times G_R$. By (7.4.1), there exists a unique morphism $\psi : (F \times_r G)_R \to F_R \times_r G_R$ satisfying $\psi\rho_{F_R\times G_R} = ((p_1\rho_{F\times G})^R, (p_2\rho_{F\times G})^R)$. We note that $\widetilde{\Phi}(\psi) : \widetilde{\Phi}((F \times_r G)_R) \to \widetilde{\Phi}(F_R \times_r G_R)$ is an isomorphism in $\mathrm{Funct}\left(\mathcal{C}_R^{(2)}, \mathcal{S}et\right)$. Consider the colimiting cone $\left(D(F \times G)(A, x) \xrightarrow{\varphi(F\times G)_{(A,x)}} F \times_r G\right)_{(A,x)\in \mathrm{Ob}\,\mathcal{C}_{F\times G}}$ of the functor $D(F \times G) : \mathcal{C}_{F\times G}^{op} \to \mathrm{Funct}_c(\mathcal{C}, \mathcal{T}op)$. It follows from (7.5.7) that $\left(D(F \times G)(A, x)_R \xrightarrow{\left(\varphi(F\times G)_{(A,x)}\right)^R} (F \times_r G)_R\right)_{(A,x)\in \mathrm{Ob}\,\mathcal{C}_{F\times G}}$ is a colimiting cone of $R_\sharp D(F \times G)$. Since $\widetilde{\Phi}(\psi)$ is an isomorphism,

$$\left(\widetilde{\Phi}(D(F \times G)(A, x)_R) \xrightarrow{\widetilde{\Phi}\left(\psi\left(\varphi(F\times G)_{(A,x)}\right)^R\right)} \widetilde{\Phi}(F_R \times_r G_R)\right)_{(A,x)\in \mathrm{Ob}\,\mathcal{C}_{F\times G}}$$

is a colimiting cone of $\widetilde{\Phi}(R_\sharp D(F \times G))$. For each $(A, x) \in \mathrm{Ob}\,\mathcal{C}_{F\times G}$, $D(F \times G)(A, x)_R$ is isomorphic to a representable functor by (7.5.2). Hence, by (7.3.2), $\left(D(F \times G)(A, x)_R \xrightarrow{\psi\left(\varphi(F\times G)_{(A,x)}\right)^R} F_R \times_r G_R\right)_{(A,x)\in \mathrm{Ob}\,\mathcal{C}_{F\times G}}$ is also a colimiting cone of the functor $R_\sharp D(F \times G)$. Therefore $\psi : (F \times_r G)_R \to F_R \times_r G_R$ is an isomorphism. $\square$

For functors $F, H \in \mathrm{Ob}\,\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$, we define a natural transformation $\tilde{\eta}_H^F : H \to (H \times_r F)^F$ as follows. Let us denote by $\tilde{p}_1 : H \times_r F \to H$ and $\tilde{p}_2 : H \times_r F \to F$ the projections. By (7.5.9), $H_R \xleftarrow{\tilde{p}_1^R} (H \times_r F)_R \xrightarrow{\tilde{p}_2^R} F_R$ is a product of H_R and F_R in $\mathrm{Funct}\left(\mathcal{C}_R^{(2)}, \mathcal{T}op\right)$. For $R \in \mathrm{Ob}\,\mathcal{C}$ and $x \in H(R)$, let $c_{R,x} : F_R \to H_R$ be the morphism given in the definition of η_H^F. $\left(\tilde{\eta}_H^F\right)_R(x) \in (H \times_r F)^F(R) = \mathrm{Funct}\left(\mathcal{C}_R^{(2)}, \mathcal{T}op\right)(F_R, (H \times_r F)_R)$ is the unique morphism $F_R \to (H \times_r F)_R$ satisfying $\tilde{p}_1^R \left(\tilde{\eta}_H^F\right)_R(x) = c_{R,x}$ and $\tilde{p}_2^R \left(\tilde{\eta}_H^F\right)_R(x) = id_{F_R}$. It can be verified that $\left(\tilde{\eta}_H^F\right)_R$ is natural in R.

Lemma 7.5.10 *(1) Let $\varphi : H \to G$ be a morphism of $\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$. Then, the following diagram commutes.*

$$\begin{tikzcd}
H \arrow[r, "\tilde{\eta}_H^F"] \arrow[d, "\varphi"] & (H \times_r F)^F \arrow[d, "(\varphi\times_r id_F)^F"] \\
G \arrow[r, "\tilde{\eta}_G^F"] & (G \times_r F)^F
\end{tikzcd}$$

(2) Let $\varphi : F \to G$ be a morphism of $\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$. Then the following diagram commutes.

$$\begin{tikzcd}
H \arrow[r, "\tilde{\eta}_H^F"] \arrow[d, "\tilde{\eta}_H^G"] & (H \times_r F)^F \arrow[d, "(id_H\times_r\varphi)^F"] \\
(H \times_r G)^G \arrow[r, "(H\times_r G)^\varphi"] & (H \times_r G)^F
\end{tikzcd}$$

Proof. (1) For $R \in \mathrm{Ob}\,\mathcal{C}$ and $x \in H(R)$, we have

$$\tilde{p}_1^R(\varphi \times_r id_F)_R^F\left(\tilde{\eta}_H^F\right)_R(x) = \tilde{p}_1^R(\varphi \times_r id_F)^R\left(\tilde{\eta}_H^F\right)_R(x) = \varphi^R\tilde{p}_1^R\left(\tilde{\eta}_H^F\right)_R(x) = \varphi^R c_{R,x} = c_{R,\varphi_R(x)} = \tilde{p}_1^R\left(\tilde{\eta}_G^F\right)_R\varphi_R(x)$$
$$\tilde{p}_2^R(\varphi \times_r id_F)_R^F\left(\tilde{\eta}_H^F\right)_R(x) = \tilde{p}_2^R(\varphi \times_r id_F)^R\left(\tilde{\eta}_H^F\right)_R(x) = \tilde{p}_2^R\left(\tilde{\eta}_H^F\right)_R(x) = id_{F_R} = \tilde{p}_2^R\left(\tilde{\eta}_G^F\right)_R\varphi_R(x).$$

Therefore $(\varphi\times_r id_F)^F\tilde{\eta}^F_H=\tilde{\eta}^F_G\varphi$.

(2) For $R\in\operatorname{Ob}\mathcal{C}$ and $x\in H(R)$, we have

$$\begin{aligned}\tilde{p}^R_1(H\times_r G)^\varphi\big(\tilde{\eta}^G_H\big)_R(x)&=\tilde{p}^R_1\big(\tilde{\eta}^G_H\big)_R(x)\varphi^R=c_{R,x}\varphi^R=c_{R,x}=\tilde{p}^R_1\big(\tilde{\eta}^F_H\big)_R(x)=\tilde{p}^R_1(id_H\times_r\varphi)^R\big(\tilde{\eta}^F_H\big)_R(x)\\&=\tilde{p}^R_1(id_H\times_r\varphi)^F_R\big(\tilde{\eta}^F_H\big)_R(x)\\ \tilde{p}^R_2(H\times_r G)^\varphi_R\big(\tilde{\eta}^G_H\big)_R(x)&=\tilde{p}^R_2\big(\tilde{\eta}^G_H\big)_R(x)\varphi^R=\varphi^R=\varphi^R\tilde{p}^R_2\big(\tilde{\eta}^F_H\big)_R(x)=\tilde{p}^R_2(id_H\times_r\varphi)^R\big(\tilde{\eta}^F_H\big)_R(x)\\&=\tilde{p}^R_2(id_H\times_r\varphi)^F_R\big(\tilde{\eta}^F_H\big)_R(x).\end{aligned}$$

Thus we have $(H\times_r G)^\varphi\tilde{\eta}^G_H=(id_H\times_r\varphi)^F\tilde{\eta}^F_H$. □

Proposition 7.5.11 *If $\mathcal{C}$ has finite topological coproducts, $\big(\tilde{\eta}^F_H\big)_R:H(R)\to(H\times_r F)^F(R)$ is continuous for $R\in\operatorname{Ob}\mathcal{C}$. Hence $\tilde{\eta}^F_H:H\to(H\times_r F)^F$ is a morphism in $\operatorname{Funct}(\mathcal{C},\mathcal{T}op)$. Moreover, $\tilde{\eta}^F_H$ satisfies $\rho^F_{H\times F}\tilde{\eta}^F_H=\eta^F_H$.*

Proof. By (7.2.2), it suffices to show that the following composition is continuous for any $(\iota:R\to A)\in\operatorname{Ob}\mathcal{C}^{(2)}_R$ and $y\in F_R(\iota:R\to A)=F(A)$.

$$H(R)\xrightarrow{(\tilde{\eta}^F_H)_R}\operatorname{Funct}\big(\mathcal{C}^{(2)}_R,\mathcal{T}op\big)(F_R,(H\times_r F)_R)\xrightarrow{E_{(\iota:R\to A)}}\mathcal{T}op(F(A),(H\times_r F)(A))\xrightarrow{ev_y}(H\times_r F)(A)$$

Since H is a colimit of representable functors, it suffices to show that the compisition of $(\varphi(H)_{(B,x)})_R:h_B(R)\to H(R)$ with the above composition is continuous for any $(B,x)\in\operatorname{Ob}\mathcal{C}_H$. It follows from (7.5.10) that the following diagram commutes.

$$\begin{tikzcd}
h_B(R) \arrow[r, "(\tilde{\eta}^F_{h_B})_R"] \arrow[d, "(\varphi(H)_{(B,x)})_R"] & \operatorname{Funct}\big(\mathcal{C}^{(2)}_R,\mathcal{T}op\big)(F_R,(h_B\times_r F)_R) \arrow[r, "E_{(\iota:R\to A)}"] \arrow[d, "(\varphi\times_r id_F)^R_*"] & \mathcal{T}op(F(A),(h_B\times_r F)(A)) \arrow[r, "ev_y"] \arrow[d, "(\varphi\times_r id_F)_{A*}"] & (h_B\times_r F)(A) \arrow[d, "(\varphi\times_r id_F)_A"'] \\
H(R) \arrow[r, "(\tilde{\eta}^F_H)_R"] & \operatorname{Funct}\big(\mathcal{C}^{(2)}_R,\mathcal{T}op\big)(F_R,(H\times_r F)_R) \arrow[r, "E_{(\iota:R\to A)}"] & \mathcal{T}op(F(A),(H\times_r F)(A)) \arrow[r, "ev_y"] & (H\times_r F)(A)
\end{tikzcd}$$

Define a map $\kappa:h_B(R)\to(h_B\times_r h_A)(A)=h_B(A)\times h_A(A)$ by $\kappa(\alpha)=(h_B(\iota)(\alpha),id_A)$. Since $h_B\times h_A$ is equivalent to a representable functor by the assumption, $(h_B\times_r h_A)(A)$ is identified with $h_B(A)\times h_A(A)$ as a topological space. Hence κ is continuous. Since $\alpha\in h_B(R)$ maps to $(h_B(\iota)(\alpha),y)\in(h_B\times_r F)(A)=h_B(A)\times F(A)$ by the composition of the upper horizontal maps of the above diagram, the following diagram commutes.

$$\begin{tikzcd}
h_B(R) \arrow[r, "\kappa"] \arrow[d, "(\tilde{\eta}^F_{h_B})_R"] & (h_B\times_r h_A)(A) \arrow[d, "\big(id_{h_B}\times_r\varphi(F)_{(A,y)}\big)_A"] \\
\operatorname{Funct}\big(\mathcal{C}^{(2)}_R,\mathcal{T}op\big)(F_R,(h_B\times_r F)_R) \arrow[r, "ev_yE_{(\iota:R\to A)}"] & (h_B\times_r F)(A)
\end{tikzcd}$$

Thus the composition of the upper horizontal maps of the above diagram is continuous, hence the continuity of $\big(\tilde{\eta}^F_H\big)_R$ follows. The second assertion is straightforward from the definitions of η^F_H and $\tilde{\eta}^F_H$. □

Proposition 7.5.12 *Let $F,G:\mathcal{C}\to\mathcal{T}$ and $D:\mathcal{D}\to\operatorname{Funct}(\mathcal{C},\mathcal{T})$ be functors.*

(1) Define a functor $D^F:\mathcal{D}\to\operatorname{Funct}(\mathcal{C},\mathcal{T}op)$ by $D^F(i)=D(i)^F$ and $D^F(\tau)=D(\tau)^F$. Suppose that, for any $R\in\operatorname{Ob}\mathcal{C}$ and $(\eta:R\to A)\in\operatorname{Ob}\mathcal{C}^{(2)}_R$, $E_\eta R_\sharp D:\mathcal{D}\to\mathcal{T}$ and $F_R(\eta)=F(A)\in\mathcal{T}$ satisfy the condition (L) of (7.1.3). If $\Big(L\xrightarrow{\pi_i}D(i)\Big)_{i\in\operatorname{Ob}\mathcal{D}}$ is a limiting cone of D, then $\Big(L^F\xrightarrow{\pi_i^F}D(i)^F\Big)_{i\in\operatorname{Ob}\mathcal{D}}$ is a limiting cone of D^F.

(2) Define a functor $G^D:\mathcal{D}^{op}\to\operatorname{Funct}(\mathcal{C},\mathcal{T}op)$ by $G^D(i)=G^{D(i)}$ and $G^D(\tau)=G^{D(\tau)}$. Suppose that, for any $R\in\operatorname{Ob}\mathcal{C}$ and $(\eta:R\to A)\in\operatorname{Ob}\mathcal{C}^{(2)}_R$, $E_\eta R_\sharp D:\mathcal{D}\to\mathcal{T}$ and $G_R(\eta)=G(A)\in\mathcal{T}$ satisfy the condition (C) of (7.1.3). If $\Big(D(i)\xrightarrow{\iota_i}C\Big)_{i\in\operatorname{Ob}\mathcal{D}}$ is a colimiting cone of D, then $\Big(G^C\xrightarrow{G^{\iota_i}}G^{D(i)}\Big)_{i\in\operatorname{Ob}\mathcal{D}}$ is a limiting cone of G^D.

Proof. (1) For each $R\in\operatorname{Ob}\mathcal{C}$, since $\Big(L_R\xrightarrow{R_\sharp(\pi_i)}D(i)_R\Big)_{i\in\operatorname{Ob}\mathcal{D}}$ is a limiting cone of $R_\sharp D$ by (7.5.7), it follows from (7.2.4) that

$$\Big(L^F(R)=\operatorname{Funct}\big(\mathcal{C}^{(2)}_R,\mathcal{T}op\big)(F_R,L_R)\xrightarrow{(R_\sharp(\pi_i))_*}\operatorname{Funct}\big(\mathcal{C}^{(2)}_R,\mathcal{T}op\big)(F_R,D(i)_R)=D(i)^F(R)\Big)_{i\in\operatorname{Ob}\mathcal{D}}$$

is a limiting cone of $E_R D^F$.

(2) For each $R \in \operatorname{Ob}\mathcal{C}$, since $\left(D(i)_R \xrightarrow{R_\sharp(\iota_i)} C_R\right)_{i\in \operatorname{Ob}\mathcal{D}}$ is a colimiting cone of $R_\sharp D$ by (7.5.7), it follows from (7.2.4) that

$$\left(G^C(R) = \operatorname{Funct}\bigl(\mathcal{C}_R^{(2)}, \mathcal{T}op\bigr)(C_R, G_R) \xrightarrow{(R_\sharp(\iota_i))^*} \operatorname{Funct}\bigl(\mathcal{C}_R^{(2)}, \mathcal{T}op\bigr)(D(i)_R, G_R) = G^{D(i)}(R)\right)_{i\in \operatorname{Ob}\mathcal{D}}$$

is a limiting cone of $E_R G^D$. $\square$

By (7.1.8), (7.5.4) and (7.5.12), we have the following result.

Proposition 7.5.13 *Let $\mathcal{C}$ be a quasi-topological category with finite topological coproducts. If $F : \mathcal{C} \to \mathcal{T}op$ is a colimit of representable functors and $G : \mathcal{C} \to \mathcal{T}op$ is a continuous functor, then G^F is a continuous functor.*

Let F and H be objects of $\operatorname{Funct}_r(\mathcal{C}, \mathcal{T}op)$ and G an object of $\operatorname{Funct}(\mathcal{C}, \mathcal{T}op)$. Define a map $\widetilde{\operatorname{Ad}}(H, F\,; G) : \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(H \times_r F, G) \to \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(H, G^F)$ by $\widetilde{\operatorname{Ad}}(H, F\,; G)(\varphi) = \varphi^F \tilde{\eta}_H^F$.

Proposition 7.5.14 *For morphisms $\lambda : H \to E$, $\mu : F \to E$ of $\operatorname{Funct}_r(\mathcal{C}, \mathcal{T}op)$ and $\nu : G \to T$ of $\operatorname{Funct}_c(\mathcal{C}, \mathcal{T}op)$, the following diagrams commute.*

$$\begin{array}{ccc}
\operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(E \times_r F, G) & \xrightarrow{\widetilde{\operatorname{Ad}}(E,\,F\,;G)} & \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)\bigl(E, G^F\bigr) \\
\Big\downarrow{\scriptstyle (\lambda\times_r id_F)^*} & & \Big\downarrow{\scriptstyle \lambda^*} \\
\operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(H \times_r F, G) & \xrightarrow{\widetilde{\operatorname{Ad}}(H,\,F\,;G)} & \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)\bigl(H, G^F\bigr)
\end{array}$$

$$\begin{array}{ccc}
\operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(H \times_r E, G) & \xrightarrow{\widetilde{\operatorname{Ad}}(H,\,E\,;G)} & \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)\bigl(H, G^E\bigr) \\
\Big\downarrow{\scriptstyle (id_H\times_r \mu)^*} & & \Big\downarrow{\scriptstyle G^\mu_*} \\
\operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(H \times_r F, G) & \xrightarrow{\widetilde{\operatorname{Ad}}(H,\,F\,;G)} & \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)\bigl(H, G^F\bigr)
\end{array}$$

$$\begin{array}{ccc}
\operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(H \times_r F, G) & \xrightarrow{\widetilde{\operatorname{Ad}}(H,\,F\,;G)} & \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)\bigl(H, G^F\bigr) \\
\Big\downarrow{\scriptstyle \nu_*} & & \Big\downarrow{\scriptstyle \nu^F_*} \\
\operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(H \times_r F, T) & \xrightarrow{\widetilde{\operatorname{Ad}}(H,\,F\,;T)} & \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)\bigl(H, T^F\bigr)
\end{array}$$

Proof. For $\varphi \in \operatorname{Ob}\operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(E \times_r F, G)$, by virtue of (7.5.10), $\lambda^*\widetilde{\operatorname{Ad}}(E, F\,; G)(\varphi) = \varphi^F \tilde{\eta}_E^F \lambda = \varphi^F(\lambda \times_r id_F)^F \tilde{\eta}_H^F = (\varphi(\lambda \times_r id_F))^F \tilde{\eta}_H^F = \widetilde{\operatorname{Ad}}(H, F\,; G)(\lambda \times_r id_F)^*(\varphi)$.

For $\varphi \in \operatorname{Ob}\operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(H \times_r E, G)$, the following diagram commutes by (7.5.10) and (7.5.1).

$$\begin{array}{ccccc}
H & \xrightarrow{\quad\tilde{\eta}_H^E\quad} & (H \times_r E)^E & \xrightarrow{\varphi^E} & G^E \\
\Big\downarrow{\scriptstyle \tilde{\eta}_H^F} & & \Big\downarrow{\scriptstyle (H\times_r E)^\mu} & & \Big\downarrow{\scriptstyle G^\mu} \\
(H \times_r F)^F & \xrightarrow{(id_H\times_r\varphi)^F} & (H \times_r E)^F & \xrightarrow{\varphi^E} & G^F
\end{array}$$

The commutativity of the third diagram is obvious. $\square$

Theorem 7.5.15 *Let $\mathcal{C}$ be a quasi-topological category with finite topological coproducts. If F and H are objects of $\operatorname{Funct}_r(\mathcal{C}, \mathcal{T}op)$ and G is an object of $\operatorname{Funct}_c(\mathcal{C}, \mathcal{T}op)$, then $\widetilde{\operatorname{Ad}}(H, F\,; G) : \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(H \times_r F, G) \to \operatorname{Funct}(\mathcal{C}, \mathcal{T}op)(H, G^F)$ is a homeomorphism.*

Proof. By the assumption, there are colimiting cones

$$\left(D(F)(R, x) \xrightarrow{\varphi(F)_{(R,x)}} F\right)_{(R,x)\in \operatorname{Ob}\mathcal{C}_F} \quad\text{and}\quad \left(D(H)(S, y) \xrightarrow{\varphi(H)_{(S,y)}} H\right)_{(S,y)\in \operatorname{Ob}\mathcal{C}_H}.$$

It follows from (7.5.12) that $\left(G^F \xrightarrow{G^{\varphi(F)(R,x)}} G^{D(F)(R,x)}\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$ is a limiting cone. Then, for $(S,y) \in \mathrm{Ob}\,\mathcal{C}_H$,

$$\left(\mathrm{Funct}(\mathcal{C},\mathcal{T}op)\big(D(H)(S,y),G^F\big) \xrightarrow{G_*^{\varphi(F)(R,x)}} \mathrm{Funct}(\mathcal{C},\mathcal{T}op)\Big(D(H)(S,y),G^{D(F)(R,x)}\Big)\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$$

is a limiting cone by (7.2.4). On the other hand, since

$$\left(D(H)(S,y)\times_r D(F)(R,x) \xrightarrow{id_{D(H)(S,y)}\times_r \varphi(F)_{(R,x)}} D(H)(S,y)\times_r F\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$$

is a colimiting cone by (7.4.12),

$$\left(\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(H)(S,y)\times_r F,G) \xrightarrow{\left(id_{D(H)(S,y)}\times_r \varphi(F)_{(R,x)}\right)^*} \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(H)(S,y)\times_r D(F)(R,x),G)\right)_{(R,x)\in \mathrm{Ob}\,\mathcal{C}_F}$$

is a limiting cone by (7.5.12). Since $\widetilde{\mathrm{Ad}}(D(H)(S,y),D(F)(R,x);G)$ is a homeomorphism by (7.5.6) and the following diagram commutes by (7.5.14), $\widetilde{\mathrm{Ad}}(D(H)(S,y),F\,;G)$ is also a homeomorphism.

$$\begin{tikzcd}
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(H)(S,y)\times_r F,G) \arrow[r, "\left(id_{D(H)(S,y)}\times_r \varphi(F)_{(R,x)}\right)^*"] \arrow[d, "\widetilde{\mathrm{Ad}}(D(H)(S,y),\,F\,;G)"] & \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(H)(S,y)\times_r D(F)(R,x),G) \arrow[d, "\widetilde{\mathrm{Ad}}(D(H)(S,y),\,D(F)(R,x);G)"] \\
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)\big(D(H)(S,y),G^F\big) \arrow[r, "G_*^{\varphi(F)(R,x)}"] & \mathrm{Funct}(\mathcal{C},\mathcal{T}op)\big(D(H)(S,y),G^{D(F)(R,x)}\big)
\end{tikzcd}$$

Moreover, since $\left(D(H)(S,y)\times_r F \xrightarrow{\varphi(H)_{(S,y)}\times_r id_F} H\times_r F\right)_{(S,y)\in \mathrm{Ob}\,\mathcal{C}_H}$ is a colimiting cone by (7.4.12), it follows from (7.5.12) that

$$\left(\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(H\times_r F,G) \xrightarrow{\left(\varphi(H)_{(S,y)}\times_r id_F\right)^*} \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(H)(S,y)\times_r F,G)\right)_{(S,y)\in \mathrm{Ob}\,\mathcal{C}_H}$$

is a limiting cone. Finally, $\left(\mathrm{Funct}(\mathcal{C},\mathcal{T}op)\big(H,G^F\big) \xrightarrow{\varphi(H)^*_{(S,y)}} \mathrm{Funct}(\mathcal{C},\mathcal{T}op)\big(D(H)(S,y),G^F\big)\right)_{(S,y)\in \mathrm{Ob}\,\mathcal{C}_F}$ is a limiting cone by (7.2.4) and the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)(H\times_r F,G) \arrow[r, "\left(\varphi(H)_{(S,y)}\times_r id_F\right)^*"] \arrow[d, "\widetilde{\mathrm{Ad}}(H,\,F\,;G)"] & \mathrm{Funct}(\mathcal{C},\mathcal{T}op)(D(H)(S,y)\times_r F,G) \arrow[d, "\widetilde{\mathrm{Ad}}(D(H)(S,y),\,F\,;G)"] \\
\mathrm{Funct}(\mathcal{C},\mathcal{T}op)\big(H,G^F\big) \arrow[r, "\varphi(H)^*_{(S,y)}"] & \mathrm{Funct}(\mathcal{C},\mathcal{T}op)\big(D(H)(S,y),G^F\big)
\end{tikzcd}$$

Since $\widetilde{\mathrm{Ad}}(D(H)(S,y),F\,;G)$ is a homeomorphism, $\widetilde{\mathrm{Ad}}(H,F\,;G)$ is also a homeomorphism. □

Remark 7.5.16 *Under the assumptions of (7.5.15), we denote by* $\tilde{\varepsilon}_G^F : \Psi\widetilde{\Phi}(G^F)\times_r F \to G$ *the unique morphism that is mapped to* $\rho_{G^F} : \Psi\widetilde{\Phi}(G^F) \to G^F$ *by*

$$\widetilde{\mathrm{Ad}}(\Psi\widetilde{\Phi}(G^F),F\,;G) : \mathrm{Funct}(\mathcal{C},\mathcal{T}op)\Big(\Psi\widetilde{\Phi}(G^F)\times_r F,G\Big) \to \mathrm{Funct}(\mathcal{C},\mathcal{T}op)\Big(\Psi\widetilde{\Phi}(G^F),G^F\Big).$$

Then, composition $\Psi\widetilde{\Phi}(G^F) \xrightarrow{\tilde{\eta}^F_{\Psi\widetilde{\Phi}(G^F)}} \Big(\Psi\widetilde{\Phi}(G^F)\times_r F\Big)^F \xrightarrow{(\tilde{\varepsilon}^F_G)^F} G^F$ *coincides with* ρ_{G^F}. *Hence the right rectangle of the following diagram commutes. It follows from (7.5.10) that the left and the center rectangles of the following diagram also commute.*

$$\begin{tikzcd}
H \arrow[r, "\rho_H^{-1}"] \arrow[d, "\tilde{\eta}^F_H"] & \Psi\widetilde{\Phi}(H) \arrow[r, "\Psi\widetilde{\Phi}(\tilde{\eta}^F_H)"] \arrow[d, "\tilde{\eta}^F_{\Psi\widetilde{\Phi}(H)}"] & \Psi\widetilde{\Phi}((H\times_r F)^F) \arrow[r, "\rho_{(H\times_r F)^F}"] \arrow[d, "\tilde{\eta}^F_{\Psi\widetilde{\Phi}((H\times_r F)^F)}"] & (H\times_r F)^F \\
(H\times_r F)^F \arrow[r, "(\rho_H^{-1}\times_r id_F)^F"] & (\Psi\widetilde{\Phi}(H)\times_r F)^F \arrow[r, "(\Psi\widetilde{\Phi}(\tilde{\eta}^F_H)\times_r id_F)^F"] & (\Psi\widetilde{\Phi}((H\times_r F)^F)\times_r F)^F \arrow[ur, "(\tilde{\varepsilon}^F_{H\times_r F})^F"']
\end{tikzcd}$$

Moreover, the composition of the upper horizontal maps coincides with $\tilde{\eta}^F_H$ by the naturality of $\rho : \Psi\widetilde{\Phi} \to id_{\mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)}$. On the other hand, the composition of the left vertical map and the lower horizontal maps is the image of composition $H \times_r F \xrightarrow{\rho_H^{-1} \times_r id_F} \Psi\widetilde{\Phi}(H) \times_r F \xrightarrow{\Psi\widetilde{\Phi}(\tilde{\eta}^F_H)\times_r id_F} \Psi\widetilde{\Phi}((H \times_r F)^F) \times_r F \xrightarrow{\tilde{\varepsilon}^F_{H\times_r F}} H \times_r F$ by $\widetilde{\mathrm{Ad}}(H, F\,; H \times_r F)$. Since the image of the identity morphism of $H \times_r F$ by $\widetilde{\mathrm{Ad}}(H, F\,; H \times_r F)$ is also $\tilde{\eta}^F_H$, it follows that composition $H \times_r F \xrightarrow{\rho_H^{-1} \times_r id_F} \Psi\widetilde{\Phi}(H) \times_r F \xrightarrow{\Psi\widetilde{\Phi}(\tilde{\eta}^F_H)\times_r id_F} \Psi\widetilde{\Phi}((H \times_r F)^F) \times_r F \xrightarrow{\tilde{\varepsilon}^F_{H\times_r F}} H \times_r F$ coincides with the identity morphism of $H \times_r F$.

It follows from (7.4.1) and (7.5.15) that a functor from $\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$ to $\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$ given by $G \mapsto \Psi\widetilde{\Phi}(G^F)$ is a right adjoint of a functor given by $G \mapsto G \times_r F$. Thus we have the following result.

Corollary 7.5.17 *If $\mathcal{C}$ is a quasi-topological category with finite topological coproducts, $\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$ is a cartesian closed category.*

The following assertion follows from (7.5.15) and (7.1.10).

Proposition 7.5.18 *Let $\mathcal{C}$ be a quasi-topological category with finite topological coproducts. For an object F of $\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$, define functors $(-) \times_r F, (-)^F : \mathrm{Funct}_r(\mathcal{C},\mathcal{T}op) \to \mathrm{Funct}_c(\mathcal{C},\mathcal{T}op)$ as follows. $(-) \times_r F$ maps $H \in \mathrm{Ob}\,\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$ to $H \times_r F$ and $f \in \mathrm{Mor}\,\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$ to $f \times_r id_F$. $(-)^F$ maps $H \in \mathrm{Ob}\,\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$ to H^F and $f \in \mathrm{Mor}\,\mathrm{Funct}_r(\mathcal{C},\mathcal{T}op)$ to f^F. Then, $(-) \times_r F$ and $(-)^F$ are continuous functors.*

For $F, G, H \in \mathrm{Ob}\,\mathrm{Funct}(\mathcal{C},\mathcal{T}op)$, we define a natural transformation $\mathrm{Prod}_H : G^F \to (G \times H)^{F\times H}$ as follows. We denote by $p_1 : F \times H \to F$, $p_2 : F \times H \to H$, $q_1 : G \times H \to G$ and $q_2 : G \times H \to H$ the projections. Then, for $R \in \mathrm{Ob}\,\mathcal{C}$, $(R_\sharp(p_1), R_\sharp(p_2)) : (F \times H)_R \to F_R \times H_R$ and $(R_\sharp(q_1), R_\sharp(q_2)) : (G \times H)_R \to G_R \times H_R$ are natural equivalences. For $R \in \mathrm{Ob}\,\mathcal{C}$,

$$(\mathrm{Prod}_H)_R : G^F(R) = \mathrm{Funct}\big(\mathcal{C}^{(2)}_R, \mathcal{T}op\big)(F_R, G_R) \to \mathrm{Funct}\big(\mathcal{C}^{(2)}_R, \mathcal{T}op\big)((F \times H)_R, (G \times H)_R) = (G \times H)^{F\times H}(R)$$

maps $\xi : F_R \to G_R$ to $(R_\sharp(q_1), R_\sharp(q_2))^{-1}(\xi \times id_{H_R})(R_\sharp(p_1), R_\sharp(p_2))$. Since the following diagram commutes for any $(\eta : R \to S) \in \mathrm{Ob}\,\mathcal{C}^{(2)}_R$ and $(x, y) \in F(S) \times H(S)$, $(\mathrm{Prod}_H)_R$ is continuous.

$$\begin{CD}
\mathrm{Funct}\big(\mathcal{C}^{(2)}_R, \mathcal{T}op\big)(F_R, G_R) @>{(\times H)_R}>> \mathrm{Funct}\big(\mathcal{C}^{(2)}_R, \mathcal{T}op\big)((F \times H)_R, (G \times H)_R)\\
@V{E_\eta}VV @V{E_\eta}VV\\
\mathcal{T}op(F(S), G(S)) @>{\times id_{H(S)}}>> \mathcal{T}op(F(S) \times H(S), G(S) \times H(S))\\
@V{E_x}VV @V{E_{(x,y)}}VV\\
G(S) @>{(i_1, y)}>> G(S) \times H(S)
\end{CD}$$

Suppose that $\mathcal{C}$ is a quasi-topological category with finite topological coproducts and that F, G and H are colimits of representable functors. We define a natural transformation $\mathrm{Prod}_H : \Psi\widetilde{\Phi}(G^F) \to (G \times_r H)^{F\times_r H}$ to be the image of $\tilde{\varepsilon}^F_G \times_r id_H : \Psi\widetilde{\Phi}(G^F) \times_r F \times_r H \to G \times_r H$ by

$$\begin{aligned}\widetilde{\mathrm{Ad}}\Big(\Psi\widetilde{\Phi}(G^F), F \times_r H\,; G \times_r H\Big) : \mathrm{Funct}(\mathcal{C},\mathcal{T}op)\Big(\Psi\widetilde{\Phi}(G^F) \times_r F \times_r H, G \times_r H\Big)&\\ \to \mathrm{Funct}(\mathcal{C},\mathcal{T}op)\Big(\Psi\widetilde{\Phi}(G^F), (G \times_r H)^{F\times_r H}\Big)&.\end{aligned}$$

7.6 Kan extensions

We first recall the definition of comma category.

Definition 7.6.1 *Let $T : \mathcal{A} \to \mathcal{C}$ and $S : \mathcal{B} \to \mathcal{C}$ be functors. We define the "comma category" $(T{\downarrow}S)$ as follows. Objects of $(T{\downarrow}S)$ are triples $\langle X, f, Y\rangle$ with $X \in \mathrm{Ob}\,\mathcal{A}$, $Y \in \mathrm{Ob}\,\mathcal{B}$ and $f \in \mathcal{C}(T(X), S(Y))$. Morphisms $\langle X, f, Y\rangle \to \langle Z, g, W\rangle$ are pairs $\langle\varphi, \psi\rangle$ of morphisms $\varphi : X \to Z$ in $\mathcal{A}$ and $\psi : Y \to W$ in $\mathcal{B}$ such that $gT(\varphi) = S(\psi)f$. The composite of $\langle\varphi, \psi\rangle : \langle X, f, Y\rangle \to \langle Z, g, W\rangle$ and $\langle\lambda, \mu\rangle : \langle Z, g, W\rangle \to \langle U, h, V\rangle$ is defined by $\langle\lambda\varphi, \mu\psi\rangle$.*

If $\mathcal{A}$ is a category consisting of a single object 1 and a single morphism id_1 and T is the functor given by $T(1) = X$, we denote $(T{\downarrow}S)$ by $(X{\downarrow}S)$. In this case, we denote by $\langle f, Y\rangle$ an object $\langle X, f, Y\rangle$ and by ψ a *morphism $\langle id_X, \psi\rangle$ in $(X{\downarrow}S)$. Similarly, if $\mathcal{B}$ is a category consisting of a single object 1 and a single morphism id_1 and S is the functor given by $S(1) = Y$, we denote $(T{\downarrow}S)$ by $(T{\downarrow}Y)$. In this case, we denote by $\langle X, f\rangle$ an object $\langle X, f, Y\rangle$ and by φ a morphism $\langle \varphi, id_Y\rangle$ of $(T{\downarrow}Y)$. Moreover, if $\mathcal{A} = \mathcal{C}$ and T is the identity functor of $\mathcal{C}$, $(id_{\mathcal{C}}{\downarrow}Y)$ is usually denoted by $\mathcal{C}/Y$.*

We have functors $P : (T{\downarrow}S) \to \mathcal{A}$, $Q : (T{\downarrow}S) \to \mathcal{B}$ and $R : (T{\downarrow}S) \to \mathcal{C}^{(2)}$ given by $P\langle X, f, Y\rangle = X$, $P\langle\varphi,\psi\rangle = \varphi$, $Q\langle X, f, Y\rangle = Y$, $Q\langle\varphi,\psi\rangle = \psi$ and $R\langle X, f, Y\rangle = f$, $R\langle\varphi,\psi\rangle = (T(\varphi), S(\psi))$.

Let $T, T' : \mathcal{A} \to \mathcal{C}$ and $S, S' : \mathcal{B} \to \mathcal{C}$ be functors and $\alpha : T' \to T$, $\beta : S \to S'$ natural transformations. Define a functor $(\alpha{\downarrow}\beta) : (T{\downarrow}S) \to (T'{\downarrow}S')$ by $(\alpha{\downarrow}\beta)(\langle X, f, Y\rangle) = \langle X, \beta_Y f \alpha_X, Y\rangle$, $(\alpha{\downarrow}\beta)(\langle\varphi,\psi\rangle) = \langle\varphi,\psi\rangle$. In particular, if $\alpha : X \to X'$ and $\beta : Y \to Y'$ are morphisms in $\mathcal{C}$, we have functors $(\alpha{\downarrow}id_S) : (X'{\downarrow}S) \to (X{\downarrow}S)$ and $(id_T{\downarrow}\beta) : (T{\downarrow}Y) \to (T{\downarrow}Y')$ which are given by $(\alpha{\downarrow}id_S)(\langle f, Y\rangle) = \langle f\alpha, Y\rangle$ and $(id_T{\downarrow}\beta)(\langle X, f\rangle) = \langle X, \beta f\rangle$, respectively.

Remark 7.6.2 *Suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are quasi-topological categories. For $\langle X, f, Y\rangle, \langle Z, g, W\rangle \in \mathrm{Ob}\,(T{\downarrow}S)$, since $(T{\downarrow}S)(\langle X, f, Y\rangle, \langle Z, g, W\rangle)$ is a subset of $\mathcal{A}(X, Z) \times \mathcal{B}(Y, W)$, we give $(T{\downarrow}S)(\langle X, f, Y\rangle, \langle Z, g, W\rangle)$ the topology induced by a product space $\mathcal{A}(X, Z)\times\mathcal{B}(Y, W)$. Then, it is clear that $(T{\downarrow}S)$ is a quasi-topological category and that $P : (T{\downarrow}S) \to \mathcal{A}$ and $Q : (T{\downarrow}S) \to \mathcal{B}$ are continuous functors. For $(f : X \to Y), (g : Z \to W) \in \mathrm{Ob}\,\mathcal{C}^{(2)}$, since $\mathcal{C}^{(2)}(f, g)$ is a subset of $\mathcal{C}(X, Z) \times \mathcal{C}(Y, W)$, we give $\mathcal{C}^{(2)}(f, g)$ the topology induced by the product space $\mathcal{C}(X, Z) \times \mathcal{C}(Y, W)$. It is straightforward that $\mathcal{C}^{(2)}$ is a quasi-topological category and that $R : (T{\downarrow}S) \to \mathcal{C}^{(2)}$ is a continuous functor if S and T are continuous functors.*

Definition 7.6.3 *Let $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{C}'$ a functor. We denote by $F^* : \mathrm{Funct}(\mathcal{C}', \mathcal{D}) \to \mathrm{Funct}(\mathcal{C}, \mathcal{D})$ a functor defined by $F^*(T) = TF$ and $F^*(\varphi : T \to U) = (\varphi_F : TF \to UF)$. If $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{D}$ are quasi-topological categories and F is a continuous functor, F^* defines a functor from $\mathrm{Funct}_c(\mathcal{C}', \mathcal{D})$ to $\mathrm{Funct}_c(\mathcal{C}, \mathcal{D})$ which is also denoted by F^*.*

Proposition 7.6.4 *If $\mathcal{D}$ is a quasi-topological category, then $F^* : \mathrm{Funct}(\mathcal{C}', \mathcal{D}) \to \mathrm{Funct}(\mathcal{C}, \mathcal{D})$ is a continuous functor.*

Proof. For an object R of $\mathcal{C}$, since a composition $\mathrm{Funct}(\mathcal{C}', \mathcal{D}) \xrightarrow{F^*} \mathrm{Funct}(\mathcal{C}, \mathcal{D}) \xrightarrow{E_R} \mathcal{T}op$ coincides with the evaluation functor $E_{F(R)} : \mathrm{Funct}(\mathcal{C}', \mathcal{D}) \to \mathcal{T}op$, F^* is a continuous functor by (1) of (7.2.2). □

Definition 7.6.5 *Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C} \to \mathcal{T}$ be functors.*

(1) A left Kan extension of G along F is a pair (L, η) of a functor $L : \mathcal{C}' \to \mathcal{T}$ and a natural transformation $\eta : G \to LF$ such that for any functor $H : \mathcal{C}' \to \mathcal{T}$, the composition of maps

$$\mathrm{Funct}(\mathcal{C}', \mathcal{T})(L, H) \xrightarrow{F^*} \mathrm{Funct}(\mathcal{C}, \mathcal{T})(F^*(L), F^*(H)) \xrightarrow{\eta^*} \mathrm{Funct}(\mathcal{C}, \mathcal{T})(G, F^*(H))$$

which maps σ to $\sigma_F\eta$ is bijective. We denote L by $F_!(G)$. It follows fram (7.2.3) and (7.6.4) that the above composition of maps is continuous if $\mathcal{T}$ is a quasi-topological category.

(2) A right Kan extension of G along F is a pair (R, ε) of a functor $R : \mathcal{C}' \to \mathcal{T}$ and a natural transformation $\varepsilon : RF \to G$ such that for any functor $H : \mathcal{C}' \to \mathcal{T}$, the composition of maps

$$\mathrm{Funct}(\mathcal{C}', \mathcal{T})(H, R) \xrightarrow{F^*} \mathrm{Funct}(\mathcal{C}, \mathcal{T})(F^*(H), F^*(R)) \xrightarrow{\varepsilon_*} \mathrm{Funct}(\mathcal{C}, \mathcal{T})(F^*(H), G)$$

which maps τ to $\varepsilon\tau_F$ is bijective. We denote R by $F_*(G)$. It follows fram (7.2.3) and (7.6.4) that the above composition of maps is continuous if $\mathcal{T}$ is a quasi-topological category.

Definition 7.6.6 *Let $\mathcal{C}'$ and $\mathcal{T}$ be quasi-topological categories and $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C} \to \mathcal{T}$ functors.*

A cone $\left(G(X) = GP\langle X, g\rangle \xrightarrow{\gamma_{\langle X,g\rangle}} C\right)_{\langle X,g\rangle\in\mathrm{Ob}\,(F{\downarrow}Z)}$ of $(F{\downarrow}Z) \xrightarrow{P} \mathcal{C} \xrightarrow{G} \mathcal{T}$ is called a continuous cone if a map $\Gamma_{X,Z} : \mathcal{C}'(F(X), Z) \to \mathcal{T}(G(X), C)$ defined by $\Gamma_{X,Z}(g) = \gamma_{\langle X,g\rangle}$ is continuous for any $X \in \mathrm{Ob}\,\mathcal{C}$.

A cone $\left(C \xrightarrow{\gamma_{\langle g,X\rangle}} GQ\langle g, X\rangle = G(X)\right)_{\langle g,X\rangle\in\mathrm{Ob}\,(Z{\downarrow}F)}$ of $(Z{\downarrow}F) \xrightarrow{Q} \mathcal{C} \xrightarrow{G} \mathcal{T}$ is called a continuous cone if a map $\Gamma_{X,Z} : \mathcal{C}'(Z, F(X)) \to \mathcal{T}(C, G(X))$ defined by $\Gamma_{X,Z}(g) = \gamma_{\langle g,X\rangle}$ is continuous for any $X \in \mathrm{Ob}\,\mathcal{C}$.

Proposition 7.6.7 *Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C} \to \mathcal{T}$ be functors. Assume that, for each object Y of $\mathcal{C}'$, the composite $(F{\downarrow}Y) \xrightarrow{P} \mathcal{C} \xrightarrow{G} \mathcal{T}$ has a colimit with a colimiting cone $\left(GP\langle X, f\rangle \xrightarrow{\lambda_{\langle X,f\rangle}} L(Y)\right)_{\langle X,f\rangle\in\mathrm{Ob}\,(F{\downarrow}Y)}$. Each morphism $g : Y \to Z$ in $\mathcal{C}'$ induces a unique morphism $L(g) : L(Y) \to L(Z)$ commuting with the colimiting* *cones. This defines a functor $L : \mathcal{C}' \to \mathcal{T}$. For each $X \in \mathrm{Ob}\,\mathcal{C}$, set $\eta_X = \lambda_{\langle X, id\rangle} : G(X) \to LF(X)$. Then, we have a natural transformation $\eta : G \to LF$ and (L, η) is a left Kan extension of G along F.*

Suppose that $\mathcal{C}'$ and $\mathcal{T}$ are quasi-topological categories and that $\left(GP\langle X, f\rangle \xrightarrow{\lambda_{\langle X,f\rangle}} L(Y)\right)_{\langle X,f\rangle\in \mathrm{Ob}\,(F\downarrow Y)}$ is a continuous cone. Then, L is a continuous functor.

Proof. Let $g : Y \to Z$ be a morphism in $\mathcal{C}'$. Consider the functor $(id_F{\downarrow}g) : (F{\downarrow}Y) \to (F{\downarrow}Z)$ defined in (7.6.1). Then, we have a cone

$$\left(GP(id_F{\downarrow}g)\langle X, f\rangle \xrightarrow{\lambda_{(id_F\downarrow g)\langle X,f\rangle}} L(Z)\right)_{\langle X,f\rangle\in \mathrm{Ob}\,(F\downarrow Y)}.$$

Since $GP(id_F{\downarrow}g)\langle X, f\rangle = G(X)$ for any $\langle X, f\rangle \in \mathrm{Ob}, (F{\downarrow}Y)$, there exists a unique morphism $L(g) : L(Y) \to L(Z)$ such that $L(g)\lambda_{\langle X,f\rangle} = \lambda_{(id_F\downarrow g)\langle X,f\rangle}$ for any $\langle X, f\rangle \in \mathrm{Ob}\,(F{\downarrow}Y)$. It is easy to verify that this choice of $L(g)$ makes L a functor.

Let $h : V \to W$ be a morphism in $\mathcal{C}$. It follows from the definition of $LF(h) : LF(V) \to LF(W)$ that $LF(h)\eta_V = LF(h)\lambda_{\langle V,id_{F(V)}\rangle} = \lambda_{(id_F\downarrow F(h))\langle V,id_{F(V)}\rangle} = \lambda_{\langle V,F(h)\rangle} = \lambda_{\langle W,id_{F(W)}\rangle}GP(h) = \eta_W G(h)$. Therefore $\eta : G \to LF$ is natural.

Let $H : \mathcal{C}' \to \mathcal{A}$ be a functor and $\alpha : G \to HF$ be a natural transformation. We construct a natural transformation $\sigma : L \to H$ as follows. For $Y \in \mathrm{Ob}\,\mathcal{C}'$, $(GP\langle X, f\rangle = G(X) \xrightarrow{H(f)\alpha_X} H(Y))_{\langle X,f\rangle\in \mathrm{Ob}\,(F\downarrow Y)}$ is a cone. In fact, if $\varphi : \langle X, f\rangle \to \langle W, k\rangle$ is a morphism in $(F{\downarrow}Y)$, $H(k)\alpha_W GP(\varphi) = H(k)\alpha_W G(\varphi) = H(k)HF(\varphi)\alpha_X = H(kF(\varphi))\alpha_X = H(f)\alpha_X$. Thus we have a unique morphism $\sigma_Y : L(Y) \to H(Y)$ such that $\sigma_Y\lambda_{\langle X,f\rangle} = H(f)\alpha_X$ for any $\langle X, f\rangle \in \mathrm{Ob}\,(F{\downarrow}Y)$.

To show the naturality of σ, take a morphism $g : Y \to Z$ in $\mathcal{C}'$. For each $\langle X, f\rangle \in \mathrm{Ob}\,(F{\downarrow}Y)$, since $H(g)\sigma_Y\lambda_{\langle X,f\rangle} = H(g)H(f)\alpha_X = H(gf)\alpha_X = \sigma_Z\lambda_{\langle X,gf\rangle} = \sigma_Z\lambda_{(id_F\downarrow g)\langle X,f\rangle} = \sigma_Z L(g)\lambda_{\langle X,f\rangle}$, we have $H(g)\sigma_Y = \sigma_Z L(g)$.

Finally, we show that the correspondence $\alpha \mapsto \sigma$ gives the inverse correspondence of the assignment $\sigma \mapsto \sigma_F\eta$. For given $\alpha \in \mathrm{Funct}(\mathcal{C}, \mathcal{A})(G, HF)$, construct $\sigma \in \mathrm{Funct}(\mathcal{C}', \mathcal{A})(L, H)$ as above, then for any $X \in \mathrm{Ob}\,\mathcal{C}$, $\sigma_{F(X)}\eta_X = \sigma_{F(X)}\lambda_{\langle X,id_{F(X)}\rangle} = \alpha_X$. Conversely, for given $\sigma \in \mathrm{Funct}(\mathcal{C}', \mathcal{A})(L, H)$, apply the above construction to $\sigma_F\eta$ to have a natural transformation $\sigma' : L \to H$. Since $\sigma'_Y\lambda_{\langle X,f\rangle} = H(f)\sigma_{F(X)}\eta_X = \sigma_Y L(f)\lambda_{\langle X,id_{F(X)}\rangle} = \sigma_Y\lambda_{(id_F\downarrow f)\langle X,id_Y\rangle} = \sigma_Y\lambda_{\langle X,f\rangle}$ for any $\langle X, f\rangle \in \mathrm{Ob}\,(F{\downarrow}Y)$, we have $\sigma'_Y = \sigma_Y$.

Suppose that $\mathcal{C}'$ and $\mathcal{T}$ are quasi-topological categories and that $\left(GP\langle X, f\rangle \xrightarrow{\lambda_{\langle X,f\rangle}} L(Y)\right)_{\langle X,f\rangle\in \mathrm{Ob}\,(F\downarrow Y)}$ is a continuous cone. By the assumption, since $\left(\mathcal{T}(L(Y), L(Z)) \xrightarrow{(\lambda_{\langle X,f\rangle})^*} \mathcal{T}(GP\langle X, f\rangle, L(Z))\right)_{\langle X,f\rangle\in \mathrm{Ob}\,(F\downarrow Y)}$ is a limiting cone for $Y, Z \in \mathrm{Ob}\,\mathcal{C}'$, it suffices to show that

$$\mathcal{C}'(Y, Z) \xrightarrow{L} \mathcal{T}(L(Y), L(Z)) \xrightarrow{(\lambda_{\langle X,f\rangle})^*} \mathcal{T}(GP\langle X, f\rangle, L(Z)) = \mathcal{T}(G(X), L(Z))$$

is continuous for any $\langle X, f\rangle \in \mathrm{Ob}\,(F{\downarrow}Y)$ to show that $L : \mathcal{C}'(Y, Z) \to \mathcal{T}(L(Y), L(Z))$ is continuous. It follows from the definition of L that the following diagram commutes and that the composition of the lower horizontal maps of the following diagram maps $g \in \mathcal{C}'(F(X), Z)$ to $\lambda_{\langle X,g\rangle}$ which is continuous by the assumption.

$$\begin{tikzcd}
\mathcal{C}'(Y, Z) \arrow[r, "L"] \arrow[d, "f^*"] & \mathcal{T}(L(Y), L(Z)) \arrow[r, "(\lambda_{\langle X,f\rangle})^*"] \arrow[d, "L(f)^*"] & \mathcal{T}(GP\langle X, f\rangle, L(Z)) \arrow[d, equal] \\
\mathcal{C}'(F(X), Z) \arrow[r, "L"] & \mathcal{T}(LF(X), L(Z)) \arrow[r, "(\lambda_{\langle X,id_{F(X)}\rangle})^*"] & \mathcal{T}(GP\langle X, id_{F(X)}\rangle, L(Z))
\end{tikzcd}$$

Hence the composition of the upper horizontal maps of the diagram is continuous. □

Proposition 7.6.8 *Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C} \to \mathcal{T}$ be functors. Assume that, for each object Y of $\mathcal{C}'$, the composite $(Y{\downarrow}F) \xrightarrow{Q} \mathcal{C} \xrightarrow{G} \mathcal{T}$ has a limit with a limiting cone $\left(R(Y) \xrightarrow{\lambda_{\langle f,X\rangle}} GQ\langle f, X\rangle\right)_{\langle f,X\rangle\in \mathrm{Ob}\,(Y\downarrow F)}$. Each morphism $g : Y \to Z$ in $\mathcal{C}'$ induces a unique morphism $R(g) : R(Y) \to R(Z)$ commuting with the limiting cones. This defines a functor $R : \mathcal{C}' \to \mathcal{T}$. For each $X \in \mathrm{Ob}\,\mathcal{C}$, set $\varepsilon_X = \lambda_{\langle id_{F(X)},X\rangle} : RF(X) \to G(X)$. Then, we have a natural transformation $\varepsilon : RF \to G$ and $(R, \varepsilon))$ is a right Kan extension of G along F.*

Suppose that $\mathcal{C}'$ and $\mathcal{T}$ are quasi-topological categories and that $\left(R(Y) \xrightarrow{\lambda_{\langle f,X\rangle}} GQ\langle f,X\rangle\right)_{\langle f,X\rangle\in \mathrm{Ob}\,(Y\downarrow F)}$ is a continuous cone. Then, R is a continuous functor.

Proof. Let $g : Y \to Z$ be a morphism in $\mathcal{C}'$. Consider the functor $(g\downarrow id_F) : (Z\downarrow F) \to (Y\downarrow F)$ defined in (7.6.1). Then, we have a cone

$$\left(R(Y) \xrightarrow{\lambda_{(g\downarrow id_F)\langle f,X\rangle}} GQ(g\downarrow id_F)\langle f,X\rangle\right)_{\langle f,X\rangle\in \mathrm{Ob}\,(Z\downarrow F)}.$$

Since $GQ(g\downarrow id_F)\langle f,X\rangle = G(X)$ for any $\langle f,X\rangle \in (Z\downarrow F)$, there exists a unique morphism $R(g) : R(Y) \to R(Z)$ such that $\lambda_{\langle f,X\rangle}R(g) = \lambda_{(g\downarrow id_F)\langle f,X\rangle}$ for any $\langle f,X\rangle \in \mathrm{Ob}\,(Z\downarrow F)$. It is easy to verify that this choice of $R(g)$ makes R a functor.

Let $h : V \to W$ be a morphism in $\mathcal{C}$. It follows from the definition of $RF(h) : RF(V) \to RF(W)$ that $\varepsilon_W RF(h) = \lambda_{\langle id_{F(W)},W\rangle}RF(h) = \lambda_{(F(h)\downarrow id_F)\langle id_{F(W)},W\rangle} = \lambda_{\langle F(h),W\rangle} = GQ(h)\lambda_{\langle id_{F(V)},V\rangle} = G(h)\varepsilon_V$. Therefore $\varepsilon : RF \to G$ is natural.

Let $H : \mathcal{C}' \to \mathcal{A}$ be a functor and $\beta : HF \to G$ be a natural transformation. We construct a natural transformation $\tau : H \to R$ as follows. For $Y \in \mathrm{Ob}\,\mathcal{C}'$, $(H(Y) \xrightarrow{\beta_X H(f)} G(X) = GQ\langle f,X\rangle)_{\langle f,X\rangle\in \mathrm{Ob}\,(Y\downarrow F)}$ is a cone. In fact, if $\varphi : \langle f,X\rangle \to \langle k,W\rangle$ is a morphism in $(Y\downarrow F)$, $GQ(\varphi)\beta_X H(f) = G(\varphi)\beta_X H(f) = \beta_W HF(\varphi)H(f) = \beta_W H(F(\varphi)f) = \beta_W H(k)$. Thus we have a unique morphism $\tau_Y : H(Y) \to R(Y)$ such that $\lambda_{\langle f,X\rangle}\tau_Y = \beta_X H(f)$ for any $\langle f,X\rangle \in \mathrm{Ob}\,(Y\downarrow F)$.

To show the naturality of τ, take a morphism $g : Y \to Z$ in $\mathcal{C}'$. For each $\langle f,X\rangle \in \mathrm{Ob}\,(Z\downarrow F)$, since $\lambda_{\langle f,X\rangle}\tau_Z H(g) = \beta_X H(f)\, H(g) = \beta_X H(fg) = \lambda_{(g\downarrow id_F)\langle f,X\rangle}\tau_Y = \lambda_{\langle f,X\rangle}R(g)\tau_Y$, we have $\tau_Z H(g) = R(g)\tau_Y$).

Finally, we show that the correspondence $\beta \mapsto \tau$ gives the inverse correspondence of the assignment $\tau \mapsto \varepsilon\tau_F$. For given $\beta \in \mathrm{Funct}(\mathcal{C},\mathcal{A})(HF,G)$, construct $\tau \in \mathrm{Funct}(\mathcal{C}',\mathcal{A})(H,R))$ as above, then for any $X \in \mathrm{Ob}\,\mathcal{C}$, $\varepsilon_X\tau_{F(X)} = \lambda_{\langle id_{F(X)},FX\rangle}\tau_{F(X)} = \beta_X$. Conversely, for given $\tau \in \mathrm{Funct}(\mathcal{C}',\mathcal{A})(H,R)$, apply the above construction to $\varepsilon\tau_F$ to have a natural transformation $\tau' : H \to R$. Since $\tau'_Y\lambda_{\langle X,f\rangle} = H(f)\tau_{F(X)}\eta_X = \tau_Y R(f)\lambda_{\langle X,id_{F(X)}\rangle} = \tau_Y\lambda_{(F\downarrow f)\langle X,id_Y\rangle} = \tau_Y\lambda_{\langle X,f\rangle}$ for any $\lambda_{\langle f,X\rangle}\tau'_Y = \varepsilon_X\tau_{F(X)}H(f) = \lambda_{\langle id_{F(X)},X\rangle}R(f)\tau_Y = \lambda_{(f\downarrow F)\langle id_Y,X\rangle}\tau_Y = \lambda_{\langle f,X\rangle}\tau_Y$ for any $\langle f,X\rangle \in \mathrm{Ob}\,(Y\downarrow F)$, we have $\tau'_Y = \tau_Y$.

Suppose that $\mathcal{C}'$ and $\mathcal{T}$ are quasi-topological categories and that $\left(R(Y) \xrightarrow{\lambda_{\langle f,X\rangle}} GQ\langle f,X\rangle\right)_{\langle f,X\rangle\in \mathrm{Ob}\,(Y\downarrow F)}$ is a continuous cone. By the assumption, since $\left(\mathcal{T}(R(Z),R(Y)) \xrightarrow{\left(\lambda_{\langle f,X\rangle}\right)_*} \mathcal{T}(R(Z),GQ\langle f,X\rangle)\right)_{\langle f,X\rangle\in \mathrm{Ob}\,(Y\downarrow F)}$ is a limiting cone for $Y,Z \in \mathrm{Ob}\,\mathcal{C}'$, it suffices to show that

$$\mathcal{C}'(Z,Y) \xrightarrow{R} \mathcal{T}(L(Z),L(Y)) \xrightarrow{\left(\lambda_{\langle f,X\rangle}\right)^*} \mathcal{T}(R(Z),GQ\langle f,X\rangle) = \mathcal{T}(R(Z),G(X))$$

is continuous for any $\langle f,X\rangle \in \mathrm{Ob}\,(Y\downarrow F)$ to show that $R : \mathcal{C}'(Z,Y) \to \mathcal{T}(R(Z),R(Y))$ is continuous. It follows from the definition of R that the following diagram commutes and that the composition of the lower horizontal maps of the following diagram maps $g \in \mathcal{C}'(Z,F(X))$ to $\lambda_{\langle g,X\rangle}$ which is continuous by the assumption.

$$\begin{tikzcd}
\mathcal{C}'(Z,Y) \arrow[r, "R"] \arrow[d, "f_*"] & \mathcal{T}(R(Z),R(Y)) \arrow[r, "\left(\lambda_{\langle f,X\rangle}\right)_*"] \arrow[d, "R(f)_*"] & \mathcal{T}(R(Z),GQ\langle f,X\rangle) \arrow[d, equal] \\
\mathcal{C}'(Z,F(X)) \arrow[r, "R"] & \mathcal{T}(R(Z),RF(X)) \arrow[r, "\left(\lambda_{\langle id_{F(X)},X\rangle}\right)_*"] & \mathcal{T}(R(Z),GQ\langle id_{F(X)},X\rangle)
\end{tikzcd}$$

Hence the composition of the upper horizontal maps of the diagram is continuous. □

Remark 7.6.9 *Let $\mathcal{T}$ be a quasi-topological category and $F : \mathcal{C} \to \mathcal{T}$ a functor. Suppose that, for any functor $G : \mathcal{C}' \to \mathcal{T}$, the right Kan extension of F along G exists. It follows from (2) of (7.1.10) that $F_* : \mathrm{Funct}(\mathcal{C},\mathcal{T}) \to \mathrm{Funct}(\mathcal{C}',\mathcal{T})$ is continuous.*

8 Topological affine group scheme

8.1 Definition and properties of topological affine schemes

For objects A^* and B^* of $\mathcal{T}op\mathcal{A}lg_{K^*}$, we define a topology on the set of morphisms $\mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*)$ by giving a uniform structure as follows. For $S \subset A^*$, $\mathfrak{b} \in \mathcal{I}_{B^*}$ and $p \in \mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*)$, we put

$$U(S, \mathfrak{b}) = \{(f, g) \in \mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*) \times \mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*) |\, f(x) - g(x) \in \mathfrak{b} \text{ for any } x \in S\}$$
$$U(p; S, \mathfrak{b}) = \{f \in \mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*) |\, (f, p) \in U(S, \mathfrak{b})\}.$$

We also put $\mathfrak{B} = \{U(S^*, \mathfrak{b}) |\, S^* \in \mathcal{F}_{A^*},\ \mathfrak{b} \in \mathcal{I}_{B^*}\}$, $\mathfrak{B}_p = \{U(p; S^*, \mathfrak{b}) |\, S^* \in \mathcal{F}_{A^*},\ \mathfrak{b} \in \mathcal{I}_{B^*}\}$. Here, $\mathcal{F}_{A^*}$ is the set of finitely generated K^*-submodules of A^*. Then, $\mathfrak{B}$ is a basis of a uniform structure of $\mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*)$ and $\mathfrak{B}_p$ is a basis of the neighborhood of p with respect to the topology defined by the uniform structure of $\mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*)$. If $\mathcal{C}$ is a subcategory of $\mathcal{T}op\mathcal{A}lg_{K^*}$, we give $\mathcal{C}(A^*, B^*)$ the topology induced by $\mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*)$ for $A^*, B^* \in \mathcal{C}$.

Remark 8.1.1 *(1) Suppose that $A^* \in \mathcal{T}op\mathcal{A}lg_{K^*}$ is finitely generated and B^* is discrete. If A^* is generated by $V^* \in \mathcal{F}_{A^*}$, we have $U(V^*, 0) = U(A^*, 0)$ which is just the diagonal subset of $\mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*) \times \mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*)$. Hence $\mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*)$ has the discrete topology in this case.*

(2) $\mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*)$ is a subspace of $\mathcal{H}om^0(A^*, B^*)$ if we regard A^* and B^* as left K^*-modules.

Proposition 8.1.2 *Let A^* and B^* be objects of $\mathcal{T}op\mathcal{A}lg_{K^*}$ and $S^*, T^* \in \mathcal{F}_{A^*}$, $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_{B^*}$.*

(1) $U(S^*, \mathfrak{a}) \subset U(T^*, \mathfrak{b})$ if $\mathfrak{a} \subset \mathfrak{b}$ and $T^* \subset S^*$.

(2) $U(S^*, \mathfrak{a}) \cap U(S^*, \mathfrak{b}) = U(S^*, \mathfrak{a} \cap \mathfrak{b})$.

(3) $U(S^*, \mathfrak{a}) \cap U(T^*, \mathfrak{a}) = U(S^* + T^*, \mathfrak{a})$. Hence $\left\{U(K^*x, \mathfrak{a}) \,\middle|\, x \in \bigcup_{n \in \boldsymbol{Z}} A^n,\ \mathfrak{a} \in \mathcal{I}_{B^*}\right\}$ is a subbase of the uniform structure of $\mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*)$.

Proposition 8.1.3 *Let $f : A^* \to B^*$ and $g : B^* \to C^*$ be morphisms in $\mathcal{T}op\mathcal{A}lg_{K^*}$ and consider maps $f^* : \mathcal{T}op\mathcal{A}lg_{K^*}(B^*, C^*) \to \mathcal{T}op\mathcal{A}lg_{K^*}(A^*, C^*)$ and $g_* : \mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*) \to \mathcal{T}op\mathcal{A}lg_{K^*}(A^*, C^*)$. Suppose $S^* \in \mathcal{F}_{A^*}$, $T^* \in \mathcal{F}_{B^*}$ and $\mathfrak{a} \in \mathcal{N}_{C^*}$.*

(1) $(f^* \times f^*)^{-1}(U(S^*, \mathfrak{a})) = U(f(S^*), \mathfrak{a})$ and $(g_* \times g_*)^{-1}(U(S^*, \mathfrak{a})) = U(S^*, g^{-1}(\mathfrak{a}))$ hold. Hence f^* and g_* are uniformly continuous.

(2) If f has a continuous left inverse $p : B^* \to A^*$, then $(f^* \times f^*)(U(T^*, \mathfrak{a})) \supset U(p(T^*), \mathfrak{a})$ holds and f^* is a surjective open map.

(3) If f is surjective, then $f^* : \mathcal{T}op\mathcal{A}lg_{K^*}(B^*, C^*) \to \mathcal{T}op\mathcal{A}lg_{K^*}(A^*, C^*)$ is a homeomorphism onto its image.

Proof. (1) is easy.

(2) For $(\varphi, \psi) \in U(p(T^*), \mathfrak{a})$, it is clear that $(\varphi p, \psi p) \in U(T^*, \mathfrak{a})$ and $f^*(\varphi p) = \varphi$, $f^*(\psi p) = \psi$ hold. Thus (φ, ψ) belongs to $(f^* \times f^*)(U(T^*, \mathfrak{a}))$.

(3) For $S^* \in \mathcal{F}_{B^*}$, take $T^* \in \mathcal{F}_{A^*}$ such that $f(T^*) = S^*$. It is clear that $(f^* \times f^*)(U(S^*, \mathfrak{a})) \subset U(T^*, \mathfrak{a})$ for $\mathfrak{a} \in \mathcal{N}_{C^*}$. Assume that $(gf, hf) \in U(T^*, \mathfrak{a})$ for $g, h \in \mathcal{T}op\mathcal{A}lg_{K^*}(B^*, C^*)$. Then, for any $y \in S^*$, take $x \in T^*$ satisfying $f(x) = y$, then we have $g(y) - h(y) = g(f(x)) - h(f(x)) \in \mathfrak{a}$, namely $(g, h) \in U(S^*, \mathfrak{a})$. It follows that $U(T^*, \mathfrak{a}) \cap \mathrm{Im}\,(f^* \times f^*) \subset (f^* \times f^*)(U(S^*, \mathfrak{a}))$. Hence we have $U(T^*, \mathfrak{a}) \cap \mathrm{Im}\,(f^* \times f^*) = (f^* \times f^*)(U(S^*, \mathfrak{a}))$ and f^* is an open map onto its image. □

Definition 8.1.4 *Let $\mathcal{T}op$ be the category of topological spaces and continuous maps. For an object A^* of $\mathcal{T}op\mathcal{A}lg_{K^*}$ and a subcategory $\mathcal{C}$ of $\mathcal{T}op\mathcal{A}lg_{K^*}$, we denote by $h_{A^*} : \mathcal{C} \to \mathcal{T}op$ the functor represented by A^*, that is, h_{A^*} maps $B^* \in \mathrm{Ob}\,\mathcal{C}$ to $\mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*)$. We call h_{A^*} a topological affine K^*-scheme. Thus we have a functor $h : \mathcal{C}^{op} \to \mathrm{Funct}(\mathcal{C}, \mathcal{T}op)$ given by $h(A^*) = h_{A^*}$ and $h(f) = f^*$. Genarally, we call a functor from a subcategory of $\mathcal{T}op\mathcal{A}lg_{K^*}$ to $\mathcal{T}op$ a topological K^*-functor.*

We note that $\mathcal{T}op\mathcal{A}lg_{K^*}$ is a quasi-topological category and its subcategory is also a quasi-topological category.

Definition 8.1.5 *An object A^* of $\mathcal{T}op\mathcal{A}lg_{K^*}$ is called profinite (resp. finite) if A^* is complete Hausdorff and $A^*/\mathfrak{a}$ is a finite K^*-module for any $\mathfrak{a} \in \mathcal{I}_{A^*}$ (resp. A^* is a discrete and finite K^*-module). We denote by $\mathcal{T}op\mathcal{A}lg_{pfK^*}$ (resp. $\mathcal{T}op\mathcal{A}lg_{fK^*}$) the full subcategory of $\mathcal{T}op\mathcal{A}lg_{K^*}$ consisting of objects which are profinite (resp. finite) topological K^*-algebras.*

Proposition 8.1.6 *For objects A_i^* and B^* of $\mathcal{TopAlg}_{K^*}$, the map*

$$(\iota_1^*, \iota_2^*) : \mathcal{TopAlg}_{K^*}(A_1^* \otimes_{K^*} A_2^*, B^*) \longrightarrow \mathcal{TopAlg}_{K^*}(A_1^*, B^*) \times \mathcal{TopAlg}_{K^*}(A_2^*, B^*).$$

induced by the canonical maps $\iota_i : A_i^* \to A_1^* \otimes_{K^*} A_2^*$ $(i = 1, 2)$ is an isomorphism.

Proof. Let $\mu : B^* \otimes_{K^*} B^* \to B^*$ be the product of B^*. $\mu_* \psi_{A_1^*, A_2^*, B^*, B^*} : \mathcal{TopAlg}_{K^*}(A_1^*, B^*) \times \mathcal{TopAlg}_{K^*}(A_2^*, B^*) \to \mathcal{TopAlg}_{K^*}(A_1^* \otimes_{K^*} A_2^*, B^*)$ is the inverse of (ι_1^*, ι_2^*). □

Proposition 8.1.7 *For objects A_i^*, B_i^* $(i = 1, 2)$ of $\mathcal{TopAlg}_{K^*}$, we define a map*

$$\psi = \psi_{A_1^*, A_2^*, B_1^*, B_2^*} : \mathcal{TopAlg}_{K^*}(A_1^*, B_1^*) \times \mathcal{TopAlg}_{K^*}(A_2^*, B_2^*) \longrightarrow \mathcal{TopAlg}_{K^*}(A_1^* \otimes_{K^*} A_2^*, B_1^* \otimes_{K^*} B_2^*)$$

by $\psi(f, g) = f \otimes_{K^*} g$. Then, ψ is uniformly continuous.

Proof. For $T^* \in \mathcal{F}_{A_1^* \otimes_{K^*} A_2^*}$ and $\mathfrak{c} \in \mathcal{I}_{B_1^* \otimes_{K^*} B_2^*}$, there exist $S_i^* \in \mathcal{F}_{A_i^*}$, $\mathfrak{b}_i \in \mathcal{I}_{B_i^*}$ $(i = 1, 2)$ such that $S_1^* \otimes_{K^*} S_2^* \supset T^*$ and $\mathfrak{b}_1 \otimes_{K^*} B_2^* + B_1^* \otimes_{K^*} \mathfrak{b}_2 \subset \mathfrak{c}$. If $(f_i, g_i) \in U(S_i^*, \mathfrak{b}_i)$ and $x_i \in S_i$ for $i = 1, 2$, we have $f_1(x_1) \otimes f_2(x_2) - g_1(x_1) \otimes g_2(x_2) = (f_1(x_1) - g_1(x_1)) \otimes f_2(x_2) + g_1(x_1) \otimes (f_2(x_2) - g_2(x_2)) \in \mathfrak{b}_1 \otimes_{K^*} B_2^* + B_1^* \otimes_{K^*} \mathfrak{b}_2 \subset \mathfrak{c}$. Hence $\psi(f_1, f_2) - \psi(g_1, g_2) \in U(T^*, \mathfrak{c})$. □

The following is an analog of (3.4.5)

Proposition 8.1.8 *Let $D : \mathcal{D} \to \mathcal{TopAlg}_{K^*}$ be a functor and A^* an object of $\mathcal{TopAlg}_{K^*}$. If $(L^* \xrightarrow{\pi_i} D(i))_{i \in \operatorname{Ob} \mathcal{D}}$ a limiting cone in $\mathcal{TopAlg}_{K^*}$, then $\left(\mathcal{TopAlg}_{K^*}(A^*, L^*) \xrightarrow{\pi_{i*}} \mathcal{TopAlg}_{K^*}(A^*, D(i))\right)_{i \in \operatorname{Ob} \mathcal{D}}$ is a limiting cone in the category of topological spaces.*

Proof. It is clear that $\left(\mathcal{TopAlg}_{K^*}(A^*, L^*) \xrightarrow{\pi_{i*}} \mathcal{TopAlg}_{K^*}(A^*, D(i))\right)_{i \in \operatorname{Ob} \mathcal{D}}$ is a limiting cone in the category of sets. For $\mathfrak{a} \in \mathcal{I}_{L^*}$, there exist $\mathfrak{a}_s \in \mathcal{I}_{D(i_s)}$ $(s = 1, 2, \ldots, l,\ i_s \in \operatorname{Ob} \mathcal{D})$ such that $\mathfrak{a} \supset \bigcap\limits_{s=1}^{n} \pi_{i_s}^{-1}(\mathfrak{a}_s)$. Then, we have $U(S^*, \mathfrak{a}) \supset \bigcap\limits_{s=1}^{n} U(S^*, \pi_{i_s}^{-1}(\mathfrak{a}_s)) = \bigcap\limits_{s=1}^{n} (\pi_{i_s} \times \pi_{i_s})^{-1}(U(S^*, \mathfrak{a}_s))$. Thus the topology on $\mathcal{TopAlg}_{K^*}(A^*, L^*)$ coincides with the one such that $\left(\mathcal{TopAlg}_{K^*}(A^*, L^*) \xrightarrow{\pi_{i*}} \mathcal{TopAlg}_{K^*}(A^*, D(i))\right)_{i \in \operatorname{Ob} \mathcal{D}}$ is a limiting cone in $\mathcal{Top}$. □

Corollary 8.1.9 *For $\mathfrak{b} \in \mathcal{I}_{B^*}$, let $\pi_{\mathfrak{b}} : \widehat{B}^* \to B^*/\mathfrak{b}$ be the map induced by the quotient map $p_{\mathfrak{b}} : B^* \to B^*/\mathfrak{b}$, then $\left(\mathcal{TopAlg}_{K^*}(A^*, \widehat{B}^*) \xrightarrow{\pi_{\mathfrak{b}*}} \mathcal{TopAlg}_{K^*}(A^*, B^*/\mathfrak{b})\right)_{\mathfrak{b} \in \mathcal{I}_{B^*}}$ is a limiting cone in the category of topological spaces.*

We can show the following as (3.4.2).

Proposition 8.1.10 *If B^* is an object of $\mathcal{TopAlg}_{K^*}$ which is complete Hausdorff, then for $A^* \in \operatorname{Ob} \mathcal{TopAlg}_{K^*}$, $\eta_{A^*}^* : \mathcal{TopAlg}_{K^*}(\widehat{A}^*, B^*) \to \mathcal{TopAlg}_{K^*}(A^*, B^*)$ is a homeomorphism.*

Proof. By (1.3.17) and (1.3.4), $\eta_{A^*}^*$ is a continuous bijection. For $S^* \in \mathcal{F}_{A^*}$ and $\mathfrak{b} \in \mathcal{I}_{B^*}$, it follows from 1) of (8.1.3) that $(\eta_{A^*}^* \times \eta_{A^*}^*)(U(\eta_{A^*}(S^*), \mathfrak{b})) \subset U(S^*, \mathfrak{b})$. For $(f, g) \in U(S^*, \mathfrak{b})$, let $f', g' \in \mathcal{TopAlg}_{K^*}(\widehat{A}^*, B^*)$ be the unique morphisms such that $f'\eta_{A^*} = f$, $g'\eta_{A^*} = g$. Then, $f' - g'$ maps $\eta_{A^*}(S^*)$ into $\mathfrak{b}$. In other words, $(f', g') \in U(\eta_{A^*}(S^*), \mathfrak{b})$. Thus we have $(\eta_{A^*}^* \times \eta_{A^*}^*)(U(\eta_{A^*}(S^*), \mathfrak{b})) = U(S^*, \mathfrak{b})$ and η_{A^*} is an open map. □

The following is an analog of (3.4.17).

Proposition 8.1.11 *For $A^*, B^* \in \operatorname{Ob} \mathcal{TopAlg}_{K^*}$, let $c_{A^*, B^*} : \mathcal{TopAlg}_{K^*}(A^*, B^*) \to \mathcal{TopAlg}_{K^*}(\widehat{A}^*, \widehat{B}^*)$ be the map defined by $c_{A^*, B^*}(f) = \hat{f}$. Then c_{A^*, B^*} is continuous and the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{TopAlg}_{K^*}(\widehat{A}^*, B^*) \arrow[r, "\eta_{A^*}^*"] \arrow[dr, "\eta_{B^*{*}}"'] & \mathcal{TopAlg}_{K^*}(A^*, B^*) \arrow[r, "\eta_{B^*{*}}"] \arrow[d, "c_{A^*, B^*}"] & \mathcal{TopAlg}_{K^*}(A^*, \widehat{B}^*) \\
& \mathcal{TopAlg}_{K^*}(\widehat{A}^*, \widehat{B}^*) \arrow[ur, "\eta_{A^*}^*"'] &
\end{tikzcd}$$

Proposition 8.1.12 *Let $D : \mathcal{D} \to \mathcal{T}op\mathcal{A}lg_{K^*}$ be a functor and $\left(D(i) \xrightarrow{\iota_i} A^*\right)_{i\in \mathrm{Ob}\,\mathcal{D}}$ a colimiting cone of D. Then, $\left(\mathcal{T}op\mathcal{A}lg_{K^*}(A^*, R^*) \xrightarrow{\iota_i^*} \mathcal{T}op\mathcal{A}lg_{K^*}(D(i), R^*)\right)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of the functor $\mathcal{D}^{op} \to \mathcal{T}op$ given by $i \mapsto \mathcal{T}op\mathcal{A}lg_{K^*}(D(i), R^*)$.*

Proof. It is clear that $\left(\mathcal{T}op\mathcal{A}lg_{K^*}(A^*, R^*) \xrightarrow{\iota_i} \mathcal{T}op\mathcal{A}lg_{K^*}(D(i), R^*)\right)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone in the category of sets. Take $S^* \in \mathcal{F}_{A^*}$ and $\mathfrak{b} \in \mathcal{I}_{R^*}$. There exist $i_1, i_2, \ldots, i_n \in \mathrm{Ob}\,\mathcal{D}$ and $S_k^* \in \mathcal{F}_{D(i_k)}$ $(k = 1, 2, \ldots, n)$ such that $S^* \subset \iota_{i_1}(S_1^*) + \iota_{i_2}(S_2^*) + \cdots + \iota_{i_n}(S_n^*)$. Then, $U(S^*, \mathfrak{b}) \supset \bigcap_{k=1}^{n} (\iota_{i_k} \times \iota_{i_k})^{-1} U(S_k, \mathfrak{b})$ and the assertion follows. □

Definition 8.1.13 *Let $\mathcal{C}$ be a full subcategory of $\mathcal{T}op\mathcal{A}lg_{K^*}$ which is complete. For a topological K^*-functor $X : \mathcal{C} \to \mathcal{T}op$, we consider the category $\mathcal{C}_X$ of X-models and define a functor $\widehat{D}_X : \mathcal{C}_X \to \mathcal{C}$ by $\widehat{D}_X(R^*, x) = R^*$. We denote by $K^*[X]$ the limit of $\widehat{D}_X$ and call this the ring of functions on X.*

We note that if X is a topological affine scheme represented by $A^* \in \mathrm{Ob}\,\mathcal{C}$, $K^*[X]$ is isomorphic to A^* The following is a special case of (7.3.4).

Proposition 8.1.14 *Let A^* be an object of $\mathcal{C}$. If a topological K^*-functor $X : \mathcal{C} \to \mathcal{T}op$ is a colimit of representable functors, there is a natural equivalence*

$$\mathrm{Funct}\,(\mathcal{C}, \mathcal{T}op)\,(X, h_{A^*}) \to \mathcal{C}\,(A^*, K^*[X])\,.$$

Proposition 8.1.15 *Let $H : \mathcal{S} \to \mathcal{T}$ be a functor. Suppose that for any $X \in \mathrm{Ob}\,\mathcal{T}$ and $(f : X \to Y) \in \mathrm{Mor}\,\mathcal{T}$, there exist a filtered category $\mathcal{F}_X$, a functor $D_X : \mathcal{F}_X \to \mathcal{S}$, a limiting cone $\left(X \xrightarrow{p(X)_i} HD_X(i)\right)_{i\in \mathrm{Ob}\,\mathcal{F}_X}$ of HD_X, a functor $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ and a natural transformation $\rho_f : D_X f^* \to D_Y$ which satisfy the following conditions.*

(i) For any $S \in \mathcal{S}$ and $i \in \mathcal{F}_{H(S)}$, there exists a morphism $\pi(S)_i : S \to D_{H(S)}(i)$ in $\mathcal{C}$ which satisfies $p(H(S))_i = H(\pi(S)_i)$ (for example, H is full).
(ii) The following diagram commutes for any $j \in \mathrm{Ob}\,\mathcal{F}_Y$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p(X)_{f^*(j)}"] & Y \arrow[d, "p(Y)_j"] \\
HD_X f^*(j) \arrow[r, "H((\rho_f)_j)"] & HD_Y(j)
\end{tikzcd}$$

Let $F, G : \mathcal{T} \to \mathcal{C}$ be functors and $\lambda : FH \to GH$ a natural transformation. If G preserves filtered limits, there exists a unique natural transfomation $\bar{\lambda} : F \to G$ satisfying $\bar{\lambda}_{H(S)} = \lambda_S : FH(S) \to GH(S)$ for any $S \in \mathrm{Ob}\,\mathcal{S}$. Hence if F also preserves filtered limits and λ is a natural equivalence, so is $\bar{\lambda}$.

Proof. Let X be an object of $\mathcal{T}$. Since G preserves filtered limits, $\left(G(X) \xrightarrow{G(p(X)_i)} GHD_X(i)\right)_{i\in \mathrm{Ob}\,\mathcal{F}_X}$ is a limiting cone of $GHD_X : \mathcal{F}_X \to \mathcal{C}$. On the other hand, $\left(F(X) \xrightarrow{\lambda_{D_X(i)}F(p(X)_i)} GHD_X(i)\right)_{i\in \mathrm{Ob}\,\mathcal{F}_X}$ is a cone of $GHD_X : \mathcal{F}_X \to \mathcal{C}$. Hence we have a unique morphism $\bar{\lambda}_X : F(X) \to G(X)$ making the following left diagram commutes for any $i \in \mathrm{Ob}\,\mathcal{F}_X$. Since $p(H(S))_i = H(\pi(S)_i)$ for any $S \in \mathrm{Ob}\,\mathcal{S}$ and $i \in \mathrm{Ob}\,\mathcal{F}_{H(S)}$, the naturality of λ implies that the following right diagram also commutes.

$$\begin{tikzcd}
F(X) \arrow[r, "\bar{\lambda}_X"] \arrow[d, "F(p(X)_i)"] & G(X) \arrow[d, "G(p(X)_i)"] \\
FHD_X(i) \arrow[r, "\lambda_{D_X(i)}"] & GHD_X(i)
\end{tikzcd}
\qquad
\begin{tikzcd}
FH(S) \arrow[r, "\lambda_S"] \arrow[d, "F(p(H(S))_i)"] & GH(S) \arrow[d, "G(p(H(S))_i)"] \\
FHD_{H(S)}(i) \arrow[r, "\lambda_{D_{H(S)}(i)}"] & GHD_{H(S)}(i)
\end{tikzcd}$$

Thus we have $\bar{\lambda}_{H(S)} = \lambda_S$ by the uniqueness of $\bar{\lambda}_{H(S)}$. Let $f : X \to Y$ be a morphism in $\mathcal{T}$. By the assumption, for $j \in \operatorname{Ob}\mathcal{F}_Y$, we have

$$\begin{aligned} G(p(Y)_j)\bar{\lambda}_Y F(f) &= \lambda_{D_Y(j)}F(p(Y)_j)F(f) = \lambda_{D_Y(j)}F(p(Y)_j f) = \lambda_{D_Y(j)}F(H((\rho_f)_j)p(X)_{f^*(j)}) \\ &= \lambda_{D_Y(j)}FH((\rho_f)_j)F(p(X)_{f^*(j)}) = GH((\rho_f)_j)\lambda_{D_X(f^*(j))}F(p(X)_{f^*(j)}) \\ &= G(H((\rho_f)_j))G(p(X)_{f^*(j)})\bar{\lambda}_X = G(H((\rho_f)_j)p(X)_{f^*(j)})\bar{\lambda}_X = G(p(Y)_j f)\bar{\lambda}_X \\ &= G(p(Y)_j)G(f)\bar{\lambda}_X. \end{aligned}$$

Since $\left(G(Y) \xrightarrow{G(p(Y)_j)} GHD_Y(j)\right)_{j\in\operatorname{Ob}\mathcal{F}_Y}$ is a limiting cone of GHD_Y, it follows that $\bar{\lambda}_Y F(f) = G(f)\bar{\lambda}_X$ which shows the naturality of $\bar{\lambda}$. $\square$

Corollary 8.1.16 *Let $F, G : \mathcal{T}op\mathcal{A}lg_{pfK^*} \to \mathcal{C}$ be functors. Let us denote by $H : \mathcal{T}op\mathcal{A}lg_{fK^*} \to \mathcal{T}op\mathcal{A}lg_{pfK^*}$ the inclusion functor. Suppose that G preserves filtered limits. For a natural transformation $\lambda : FH \to GH$, λ extends uniquely to a natural transformation $\bar{\lambda} : F \to G$. Hence if F also preserves filtered limits and λ is a natural equivalence, so is $\bar{\lambda}$.*

Proof. For $R^* \in \operatorname{Ob}\mathcal{T}op\mathcal{A}lg_{pfK^*}$, let $D_{R^*} : \mathcal{I}_{R^*} \to \mathcal{T}op\mathcal{A}lg_{fK^*}$ be a functor defined by $D_{R^*}(\mathfrak{a}) = R^*/\mathfrak{a}$ and $D_{R^*}(i : \mathfrak{a} \to \mathfrak{b}) = (\bar{i} : R^*/\mathfrak{a} \to R^*/\mathfrak{b})$, where i is the inclusion map and $\bar{i}$ is the map induced by the identity map of R^*. We denote by $p(R^*)_\mathfrak{a} : R^* \to R^*/\mathfrak{a}$ the quotient map. Since R^* is profinite, $\left(R^* \xrightarrow{p(R^*)_\mathfrak{a}} HD_{R^*}(\mathfrak{a})\right)_{\mathfrak{a}\in\operatorname{Ob}\mathcal{I}_{R^*}}$ is a limiting cone of D_{R^*}. For a morphism $f : R^* \to S^*$ in $\mathcal{T}op\mathcal{A}lg_{pfK^*}$, define a functor $f^* : \mathcal{I}_{S^*} \to \mathcal{I}_{R^*}$ and a natural transformation $\rho_f : D_{R^*}f^* \to D_{S^*}$ as follows. For $\mathfrak{a} \in \operatorname{Ob}\mathcal{I}_{S^*}$, we set $f^*(\mathfrak{a}) = f^{-1}(\mathfrak{a})$ and let $(\rho_f)_\mathfrak{a} : D_{R^*}f^*(\mathfrak{a}) = R^*/f^{-1}(\mathfrak{a}) \to S^*/\mathfrak{a} = D_{S^*}(\mathfrak{a})$ be the map induced by f^*. It is clear that the conditions of (8.1.15) are all satisfied. $\square$

Proposition 8.1.17 *Let $\mathcal{C}$ be a category and $H : \mathcal{S} \to \mathcal{C}$ a functor. Suppose that objects X, Y of $\mathcal{C}$ and a natural transformation $T : h_X H \to h_Y H$ are given. If there exist a functor $D : \mathcal{D} \to \mathcal{S}$ and a limiting cone $\left(X \xrightarrow{p_i} HD(i)\right)_{i\in\operatorname{Ob}\mathcal{D}}$ of HD which satisfy the following condition $(*)$. Then, there exists a unique morphism $f : Y \to X$ in $\mathcal{C}$ such that $T_S(g) = gf$ for any $S \in \operatorname{Ob}\mathcal{S}$ and $g \in h_X(H(S))$.*

$(*)$ For any object S of $\mathcal{S}$ and morphism $g : X \to H(S)$, there exists an object i_0 of $\mathcal{D}$ and a morphism $g' : D(i_0) \to S$ in $\mathcal{S}$ satisfying $H(g')p_{i_0} = g$.

Proof. Since $\left(Y \xrightarrow{T_{D(i)}(p_i)} HD(i)\right)_{i\in\operatorname{Ob}\mathcal{D}}$ is a cone of HD, there exists a unique morphism $f : Y \to X$ satisfying $p_i f = T_{D(i)}(p_i)$ for any $i \in \operatorname{Ob}\mathcal{D}$. For $S \in \operatorname{Ob}\mathcal{S}$ and $g \in h_X(H(S))$, there exists an object i_0 of $\mathcal{D}$ and a morphism $g' : D(i_0) \to S$ in $\mathcal{S}$ satisfying $H(g')p_{i_0} = g$. By the naturarity of T, we have

$$gf = H(g')p_{i_0}f = (h_Y H)(g')T_{D(i_0)}(p_{i_0}) = T_S(h_X H)(g')(p_{i_0}) = T_S(H(g')p_{i_0}) = T_S(g).$$

If a morphism $f' : Y \to X$ satisfies $T_{D(i)}(p_i) = p_i f'$ for any $i \in \operatorname{Ob}\mathcal{D}$, then, $p_i f' = p_i f$ which implies $f' = f$. $\square$

Corollary 8.1.18 *Let A^*, B^* be objects of $\mathcal{T}op\mathcal{A}lg_{K^*}$, $H : \mathcal{T}op\mathcal{A}lg_{fK^*} \to \mathcal{T}op\mathcal{A}lg_{K^*}$ the inclusion functor and $T : h_{A^*}H \to h_{B^*}H$ a natural transformation. If A^* is profinite, there exists a unique morphism $f : B^* \to A^*$ in $\mathcal{T}op\mathcal{A}lg_{K^*}$ inducing T.*

Proof. Let $D : \mathcal{I}_{A^*} \to \mathcal{T}op\mathcal{A}lg_{fK^*}$ be a functor defined by $D(\mathfrak{a}) = A^*/\mathfrak{a}$ and $D(i : \mathfrak{a} \to \mathfrak{b}) = (\bar{i} : A^*/\mathfrak{a} \to A^*/\mathfrak{b})$, where i is the inclusion map and $\bar{i}$ is the map induced by the identity map of A^*. Since A^* is profinte, $\left(A^* \xrightarrow{p_\mathfrak{a}} A^*/\mathfrak{a}\right)_{\mathfrak{a}\in\mathcal{I}_{A^*}}$ is a limiting cone of HD, where $p_\mathfrak{a} : A^* \to A^*/\mathfrak{a}$ is the quotient map. For any object R^* of $\mathcal{T}op\mathcal{A}lg_{fK^*}$ and morphism $g : A^* \to R^*$, since R^* is discrete, $\operatorname{Ker} g \in \mathcal{I}_{A^*}$ and g induces $g' : A^*/\operatorname{Ker} g \to R^*$ such that $g' p_{\operatorname{Ker} g} = g$. Hence the condition $(*)$ of (8.1.17) is satisfied. $\square$

Definition 8.1.19 *(1) We say that a topological graded K^*-algebra A^* has the cofinite topology if the set of all ideals $\mathfrak{a}$ of A^* such that $A^*/\mathfrak{a}$ is a finite K^*-module is a fundamental system of the neighborhood of 0. We denote by $\mathcal{T}op\mathcal{A}lg_{cfK^*}$ the full subcategory of $\mathcal{T}op\mathcal{A}lg_{K^*}$ consisting of objects of $\mathcal{T}op\mathcal{A}lg_{K^*}$ which have the cofinite topology.*

(2) If the topology of A^* is coarser (resp. finer) than the cofinite topology, we say that A^* is subcofinite (supercofinite). Hence A^* is subcofinite (resp. supercofinite) if and only if every graded open ideal $\mathfrak{a}$ satisfies the condition that $A^*/\mathfrak{a}$ is a finite K^*-module (resp. every graded ideal $\mathfrak{a}$ such that $A^*/\mathfrak{a}$ is a finite K^*-module is open).

For a graded commutative K^*-algebra A^*, we set

$$I_{cf}(A^*) = \{\mathfrak{a}|\, \mathfrak{a} \text{ is a graded ideal of } A^* \text{ such that } A^*/\mathfrak{a} \text{ is a finte } K^*\text{-module}\}.$$

We give A^* a topology such that $I_{cf}(A^*)$ is a fundamental system of the neighborhood of 0. We call this topology on A^* the cofinite topology. Let us denote by A^*_{cf} the topological K^*-module A^* with the cofinite topology. Let $\mu : A^* \otimes_{K^*} A^* \to A^*$ be the multiplication of A^*. For $\mathfrak{a} \in I_{cf}(A^*)$, since $\mathfrak{a} \otimes_{K^*} A^* + A^* \otimes_{K^*} \mathfrak{a} \subset \mu^{-1}(\mathfrak{a})$ and $\mathfrak{a} \otimes_{K^*} A^* + A^* \otimes_{K^*} \mathfrak{a}$ is an open ideal in $A^*_{cf} \otimes_{K^*} A^*_{cf}$, $\mu^{-1}(\mathfrak{a})$ is also open and μ is continuous. Thus A^*_{cf} is a topological K^*-algebra.

Let us denote by $\mathcal{A}lg^*_{K^*}$ the category of graded commutative K^*-algebras. By assigning $A^* \in \operatorname{Ob} \mathcal{A}lg^*_{K^*}$ to A^*_{cf}, we have an isomorphism $\mathcal{A}lg^*_{K^*} \to \mathcal{T}op\mathcal{A}lg_{cfK^*}$ of categories.

Remark 8.1.20 *(1) A topological K^*-algebra is profinite if and only if it is complete Hausdorff and a subcofinite.*

(2) For a morphism $f : A^* \to B^*$ of graded K^*-algebras and a cofinite ideal $\mathfrak{b}$ of B^*, since f induces a monomorphism $A^*/f^{-1}(\mathfrak{b}) \to B^*/\mathfrak{b}$, $f^{-1}(\mathfrak{b})$ is also cofinite. Hence if A^* is supercofinite and B^* is subcofinite, every K^*-algebra homomorphism from A^* to B^* is continuous.

Example 8.1.21 *Let K^* be a linearly topologized graded commutative topological ring. For $k_1, k_2, \dots, k_n \in \boldsymbol{Z}$, we assign degree $-2k_i$ to a variable X_i and regard the polynomial ring $K^*[X_1, X_2, \dots, X_n]$ as a graded K^*-algebra. Let I_n be the ideal of $K^*[X_1, X_2, \dots, X_n]$ generated by $X_1, X_2, \dots, X_n$. We give $K^*[X_1, X_2, \dots, X_n]$ a topology such that $\{\mathfrak{a} + I_n^l|\, \mathfrak{a} \in \mathcal{I}_{K^*},\ l = 1, 2, \dots\}$ is a basis of the neighborhood of 0. Then, it is easy to verify that $K^*[X_1, X_2, \dots, X_n]^\wedge$ is isomorphic to $\hat{K}^*[[X_1, X_2, \dots, X_n]]$. Define a functor $F : \mathcal{T}op\mathcal{A}lg_{K^*} \to \mathcal{T}op$ by*

$$F(R^*) = (R^*[[X_1, X_2, \dots, X_n]])^0 .$$

We denote by Seq_n the set of sequences $(j_1, j_2, \dots, j_n)$ of non-negative integers of length n. For $J \in \operatorname{Seq}_n$, let x_J be a variable of degree $\sum\limits_{i=1}^{n} 2j_ik_i$ if $J = (j_1, j_2, \dots, j_n)$ and consider a graded polynomial algebra $K^*[x_J|\, J \in \operatorname{Seq}_n]$ with the cofinite topology. We set $X^J = X_1^{j_1}X_2^{j_2}\cdots X_n^{j_n}$ and $|J| = \sum\limits_{i=1}^{n} j_i$ if $J = (j_1, j_2, \dots, j_n) \in \operatorname{Seq}_n$. Define a map $T_{R^*} : h_{K^*[x_J|\, J\in\operatorname{Seq}_n]}(R^*) \to F(R^*)$ by $T_{R^*}(\varphi) = \sum\limits_{J\in\operatorname{Seq}_n} \varphi(x_J)X^J$. Let V_k^* be the submodule of $K^*[x_J|\, J \in \operatorname{Seq}_n]$ generated by $\{x_J|\, J \in \operatorname{Seq}_n,\ |J| \le k\}$. For $k \in \boldsymbol{N}$ and $\mathfrak{a} \in \mathcal{I}_{R^*}$, $(\varphi, \psi) \in U(V_k^*, \mathfrak{a})$ if and only if $\varphi(x_J) - \psi(x_J) \in \mathfrak{a}$ for $J \in \operatorname{Seq}_n$ satisfying $|J| \le k$. Moreover, $X^J \in I_n^k$ if and only if $|J| \ge k$. Hence $(T_{R^*} \times T_{R^*})(U(V_k^*, \mathfrak{a}))$ is contained in

$$\left\{ (\alpha, \beta) \in (R^*[[X_1, X_2, \dots, X_n]])^0 \times (R^*[[X_1, X_2, \dots, X_n]])^0 \middle|\, \alpha - \beta \in \mathfrak{a} + I_n^{k+1} \right\}$$

and it follows that T_{R^*} is uniformly continuous. Clearly, T_{R^*} is injective and natural in R^*. If the topology of R^* is coarser than the cofinite topology (R^* is profinite, for example), T_{R^*} is bijective and $T_{R^*}^{-1} \times T_{R^*}^{-1}$ maps above set onto $U(V_k^*, \mathfrak{a})$. Let $\mathcal{T}op\mathcal{A}lg_{scfK^*}$ be a full subcatgory of $\mathcal{T}op\mathcal{A}lg_{K^*}$ consisting of objects which are subcofinite. We denote by $\iota : \mathcal{T}op\mathcal{A}lg_{scfK^*} \to \mathcal{T}op\mathcal{A}lg_{K^*}$ the inclusion functor. Then, $T : h_{K^*[x_0, x_1, \dots]} \to F$ induces a natutal equivalence $T : h_{K^*[x_0, x_1, \dots]}\iota \to F\iota$.

Proposition 8.1.22 *For morphisms $f : A^* \to R^*$ and $g : A^* \to S^*$ in $\mathcal{T}op\mathcal{A}lg_{K^*}$, let $i_1 : R^* \to R^* \otimes_{A^*} S^*$ and $i_2 : S^* \to R^* \otimes_{A^*} S^*$ be maps defined by $i_1(x) = x \otimes 1$, $i_2(y) = 1 \otimes y$. Then $R^* \xrightarrow{i_1} R^* \otimes_{A^*} S^* \xleftarrow{i_2} S^*$ is a push-out of a diagram $R^* \xleftarrow{f} A^* \xrightarrow{g} S^*$ in $\mathcal{T}op\mathcal{A}lg_{K^*}$.*

Proof. Let $\varphi : R^* \to B^*$ and $\psi : S^* \to B^*$ be morphisms in $\mathcal{T}op\mathcal{A}lg_{K^*}$ satisfying $\varphi f = \psi g$. Define $\xi : R^* \otimes_{K^*} S^* \to B^*$ by $\xi(x \otimes_{K^*} y) = \varphi(x)\psi(y)$. For $\mathfrak{b} \in \mathcal{I}_{B^*}$, since $\xi(\varphi^{-1}(\mathfrak{b}) \otimes_{K^*} S^* + R^* \otimes_{K^*} \psi^{-1}(\mathfrak{b})) \subset \mathfrak{b}$, ξ is continuous. For $a \in A^*$, $x \in R^*$ and $y \in S^*$, we have $\xi(xa \otimes y) = \varphi(xf(a))\psi(y) = \varphi(x)\varphi(f(a))\psi(y) = \varphi(x)\psi(g(a))\psi(y) = \varphi(x)\psi(g(a)y) = \xi(x \otimes ay)$. Hence ξ induces a unique homomorphism $\zeta : R^* \otimes_{A^*} S^* \to B^*$ satisfying $\zeta i_1 = \varphi$ and $\zeta i_2 = \psi$. □

8.2 Topological modules

Let M^* and N^* be linearly topologized right K^*-modules. We identify $\mathrm{Hom}^c_{K^*}(M^*, N^*)$ with $\mathcal{H}om^0(M^*, N^*)$ and give $\mathrm{Hom}^c_{K^*}(M^*, N^*)$ the topology induced by $\mathcal{H}om^*(M^*, N^*)$. If we denote by M^*_{top} the underlying topological space of M^*, then $\{W(x, y+U^*) \mid x \in M^*,\, y \in N^*,\, U^* \in \mathcal{V}_{N^*}\}$ is a subbasis of the topology of $\mathcal{T}op(M^*_{top}, N^*_{top})$. It follows that $\mathrm{Hom}^c_{K^*}(M^*, N^*)$ is a subspace of $\mathcal{T}op(M^*_{top}, N^*_{top})$. It is easy to verify that the category of topological K^*-modules is a quasi-topological category.

Proposition 8.2.1 *Let M^* be a linearly topologized right K^*-module. Define a functor $F_{M^*} : \mathcal{T}op\mathcal{A}lg_{K^*} \to \mathcal{T}op\mathcal{M}od_{K^*}$ by $F_{M^*}(A^*) = M^* \otimes_{K^*} A^*$ and $F_{M^*}(f) = id_{M^*} \otimes f$. Then, F_{M^*} is a continuous topological K^*-functor.*

Proof. Let A^* and B^* be objects of $\mathcal{T}op\mathcal{A}lg_{K^*}$. Take $x \in F_{M^*}(A^*) = M^* \otimes_{K^*} A^*$ and an open ideal $\mathfrak{b}$ of B^*. Suppose $x = \sum_{i=1}^{n} m_i \otimes a_i$ and let us denote by S^* the submodule generated by $a_1, a_2, \ldots, a_n$. If $(g, f) \in U(S^*, \mathfrak{b})$, then $g(a_i) - f(a_i) \in \mathfrak{b}$ for $i = 1, 2, \ldots, n$ and it follows that $F_{M^*}(g)(x) - F_{M^*}(f)(x) = \sum_{i=1}^{n} m_i \otimes (g(a_i) - f(a_i)) \in M^* \otimes_{K^*} \mathfrak{b}$. Hence if $g \in U(f; S^*, \mathfrak{b})$, we have $F_{M^*}(g) \in W(x, F_{M^*}(f)(x) + M^* \otimes_{K^*} \mathfrak{b})$. Thus $F_{M^*} : \mathcal{T}op\mathcal{A}lg_{K^*}(A^*, B^*) \to \mathcal{T}op\mathcal{M}od_{K^*}(F_{M^*}(A^*), F_{M^*}(B^*))$ is continuous at f. □

Lemma 8.2.2 *(1) If N^* is complete Hausdorff, the map $\eta^*_{M^*} : \mathrm{Hom}^c_{K^*}(\widehat{M}^*, N^*) \to \mathrm{Hom}^c_{K^*}(M^*, N^*)$ induced by $\eta_{M^*} : M^* \to \widehat{M}^*$ is a homeomorphism.*

(2) The map $\gamma : \mathrm{Hom}^c_{K^*}(M^*, N^*) \to \mathrm{Hom}^c_{K^*}(\widehat{M}^*, \widehat{N}^*)$ given by $\gamma(f) = \hat{f}$ is continuous.

Proof. (1) It is clear that $\eta^*_{M^*}$ is a continuous bijection. For $x \in M^*$ and $U^* \in \mathcal{V}_{N^*}$, we note that $\eta^*_{M^*}$ maps $W(\eta_{M^*}(x), U^*)$ into $W(x, U^*)$. For $f \in W(x, U^*) \subset \mathrm{Hom}^c_{K^*}(M^*, N^*)$, let $g \in \mathrm{Hom}^c_{K^*}(\widehat{M}^*, N^*)$ be the unique morphism such that $g\eta_{M^*} = f$. Then, g maps $\eta_{M^*}(x)$ into U^*. In other words, $g \in W(\eta_{M^*}(x), U^*) \subset \mathcal{H}om^c_{M^*}(\widehat{M}^*, N^*)$. Thus we have $\eta^*_{M^*}(W(\eta_{M^*}(x), U^*)) = W(x, U^*)$ and η_{M^*} is an open map.

(2) We note that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}^c_{K^*}(\widehat{M}^*, N^*) \arrow[r, "\eta^*_{M^*}"] \arrow[dr, "\eta_{N^**}"'] & \mathrm{Hom}^c_{K^*}(M^*, N^*) \arrow[d, "\gamma"] \\
& \mathrm{Hom}^c_{K^*}(\widehat{M}^*, \widehat{N}^*)
\end{tikzcd}$$

Since $\eta_{N^**} : \mathrm{Hom}^c_{K^*}(\widehat{M}^*, N^*) \to \mathrm{Hom}^c_{K^*}(\widehat{M}^*, \widehat{N}^*)$ is continuous and $\eta^*_{M^*} : \mathrm{Hom}^c_{K^*}(\widehat{M}^*, N^*) \to \mathrm{Hom}^c_{K^*}(M^*, N^*)$ is a homeomorphism by 1), γ is continuous. □

Proposition 8.2.3 *Let M^* be a linearly topologized right K^*-module. Define a functor $\widehat{F}_{M^*} : \mathcal{T}op\mathcal{A}lg_{K^*} \to \mathcal{T}op\mathcal{M}od_{K^*}$ by $\widehat{F}_{M^*}(A^*) = M^* \widehat{\otimes}_{K^*} A^*$ and $\widehat{F}_{M^*}(f) = id_{M^*} \widehat{\otimes}_{K^*} f$. Then, $\widehat{F}_{M^*}$ is continuous.*

Definition 8.2.4 *Let A^* be an object of $\mathcal{T}op\mathcal{A}lg_{K^*}$. For a right A^*-module M^* with structure map $\alpha : M^* \otimes_{K^*} A^* \to M^*$ and a left A^*-module N^* with structure map $\beta : A^* \otimes_{K^*} N^* \to N^*$, we define $M^* \otimes_{A^*} N^*$ to be the cokernel of $\alpha \otimes_{K^*} id_{N^*} - id_{M^*} \otimes_{K^*} \beta : M^* \otimes_{K^*} A^* \otimes_{K^*} N^* \to M^* \otimes_{K^*} N^*$.*

Remark 8.2.5 *If M^* is a right (resp. left) A^*-module with structure map $\alpha : M^* \otimes_{K^*} A^* \to M^*$ (resp. $\alpha : A^* \otimes_{K^*} M^* \to M^*$), we regard M^* as a left (resp. right) A^*-module with structure map $\alpha T_{A^*, M^*} : A^* \otimes_{K^*} M^* \to M^*$ (resp. $\alpha T_{M^*, A^*} : M^* \otimes_{K^*} A^* \to M^*$).*

It is easy to verify the following result.

Proposition 8.2.6 *For right (resp. left) R^*-modules L^*, M^*, N^* there are natural isomorphisms $(L^* \otimes_{R^*} M^*) \otimes_{R^*} N^* \cong L^* \otimes_{R^*} (M^* \otimes_{R^*} N^*)$ and $M^* \otimes_{R^*} N^* \cong N^* \otimes_{R^*} M^*$.*

8.3 Topological group functors

Let $\mathcal{C}$ be a quasi-topological category. Recall that $\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$ is category with finite products. We denote by $1 : \mathcal{C} \to \mathcal{T}op$ the terminal object of $\mathrm{Funct}(\mathcal{C}, \mathcal{T}op)$ that maps each object R of $\mathcal{C}$ to the terminal object $\{*\}$ of $\mathcal{T}op$ consisting of a single point $*$. It is clear that 1 is an object of $\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$. If $\mathcal{C}$ has an initial object $\emptyset$, 1 is represented by $\emptyset$.

Definition 8.3.1 *Let $\mathcal{C}$ be a quasi-topological category. We call a group object in* $\mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$ *a topological $\mathcal{C}$-group functor.*

If $\mathcal{C}$ is a subcategory of $\mathcal{T}op\mathcal{A}lg_{K^*}$, we call a topological K^*-group functor instead of a topological $\mathcal{C}$-group functor.

Lemma 8.3.2 *Let R be an object of $\mathcal{C}$.*

(1) For $x \in X(R)$, $\mathrm{Ad}_r(\alpha)(x) \in X^G(R) = \mathrm{Funct}\left(\mathcal{C}^{(2)}_R, \mathcal{T}op\right)(G_R, X_R)$ is given as follows.

For $(\eta : R \to S) \in \mathrm{Ob}\,\mathcal{C}^{(2)}_R$, $\mathrm{Ad}_r(\alpha)(x)_{(\eta:R\to S)} : G_R(\eta : R \to S) \to X_R(\eta : R \to S)$ maps $g \in G_R(\eta : R \to S) = G(S)$ to $\alpha_S(X(\eta)(x), g)$.

(2) For $g \in G(R)$, $\mathrm{Ad}_l(\alpha)(g) \in X^X(R) = \mathrm{Funct}\left(\mathcal{C}^{(2)}_R, \mathcal{T}op\right)(X_R, X_R)$ is given as follows.

For $(\eta : R \to S) \in \mathrm{Ob}\,\mathcal{C}^{(2)}_R$, $\mathrm{Ad}_l(\alpha)(g)_{(\eta:R\to S)} : X_R(\eta : R \to S) \to X_R(\eta : R \to S)$ maps $x \in X_R(\eta : R \to S) = X(S)$ to $\alpha_S(x, G(\eta)(g))$.

The following fact is straightforward from the definition.

Proposition 8.3.3 *Let $\alpha, \beta : X \times_r G \to X$ be right actions of G on X, For a subfunctor H of G and subfunctors Y, Z of X, the following equality holds for $R \in \mathrm{Ob}\,\mathcal{C}$.*

$$X^{\alpha,\beta}_H(R) = \left\{x \in X(R) \,\middle|\, \alpha_S(X(\eta)(x), g) = \beta_S(X(\eta)(x), g) \text{ for all } (\eta : R \to S) \in \mathrm{Ob}\,\mathcal{C}^{(2)}_R,\ g \in H(S)\right\}$$
$$G^Y_{\alpha,\beta}(R) = \left\{g \in G(R) \,\middle|\, \alpha_S(x, G(\eta)(g)) = \beta_S(x, G(\eta)(g)) \text{ for all } (\eta : R \to S) \in \mathrm{Ob}\,\mathcal{C}^{(2)}_R,\ x \in Y(S)\right\}$$
$$\mathrm{Transp}_\alpha(Y, Z)(R) = \left\{g \in G(R) \,\middle|\, \alpha_S(x, G(\eta)(g)) \in Z(S) \text{ for all } (\eta : R \to S) \in \mathrm{Ob}\,\mathcal{C}^{(2)}_R,\ x \in Y(S)\right\}$$
$$\mathrm{Norm}_\alpha(Y)(R) = \{g \in G(R) \,|\, g, \iota_R(g) \in \mathrm{Stab}_\alpha(Y)(R)\}$$

Remark 8.3.4 *(1) If X takes values in the full subcategory of $\mathcal{T}op$ consisting of Hausdorff spaces, it follows from the above result that $X^{\alpha,\beta}_H(R)$ is a closed subset of $X(R)$ and that $G^Y_{\alpha,\beta}(R)$ is a closed subset of $G(R)$. If $Z(S)$ is a closed set of $X(S)$ for any $(\eta : R \to S) \in \mathrm{Ob}\,\mathcal{C}^{(2)}_R$, $\mathrm{Transp}_\alpha(Y, Z)(R)$ is a closed subset of $G(R)$.*

(2) It is easy to verify that $G^Y_{\alpha,\beta}$ and $\mathrm{Norm}_\alpha(Y)$ are subgroup functors of G and that $\mathrm{Stab}_\alpha(Y)$ is a submonoid functor of G.

By the above definition and (8.3.3), we have the following.

Proposition 8.3.5 *Let G be a topological $\mathcal{C}$-group functor with multiplication $\mu : G \times G \to G$ and inverse $\iota : G \to G$. The following equality holds for a subfunctor H of G and $R \in \mathrm{Ob}\,\mathcal{C}$.*

$$Z_G(H)(R) = \left\{g \in G(R) \,\middle|\, \mu_S(x, G(\eta)(g))) = \mu_S(G(\eta)(g), x) \text{ for all } (\eta : R \to S) \in \mathrm{Ob}\,\mathcal{C}^{(2)}_R,\ x \in H(S)\right\}$$
$$N_G(H)(R) = \left\{g \in G(R) \,\middle|\, \mu_S(H(S) \times \{G(\eta)(g)\}) = \mu_S(\{G(\eta)(g)\} \times H(S)) \text{ for all } (\eta : R \to S) \in \mathrm{Ob}\,\mathcal{C}^{(2)}_R\right\}$$

Suppose that G, X and Y are colimits of representable functors and that $\mathcal{C}$ is a quasi-topological category with finite topological coproducts. Let $\alpha : X \times G \to X$ and $\beta : Y \times G \to Y$ be right actions of G on X and Y, respectively. We put $\tilde{\alpha} = \alpha\rho_{X\times_r G} : X \times_r G \to X$ and $\tilde{\beta} = \beta\rho_{Y\times_r G} : Y \times_r G \to Y$. Let $e : (Y, \beta)^{(X,\alpha)} \to \Psi\widetilde{\Phi}(Y^X)$ the equalizer of $\Psi\widetilde{\Phi}(Y^X) \xrightarrow{\times_r G} (Y \times_r G)^{X\times_r G} \xrightarrow{\tilde{\beta}^{X\times_r G}} Y^{X\times_r G}$ and $Y^X \xrightarrow{Y^{\tilde{\alpha}}} Y^{X\times_r G}$.

Let $\tilde{p}_1 : F \times_r H \to F$, $\tilde{p}_2 : F \times_r H \to H$, $\tilde{q}_1 : G \times_r H \to G$ and $\tilde{q}_2 : G \times_r H \to H$ the projections. It follows from (7.5.9) that $(R_\sharp(\tilde{p}_1), R_\sharp(\tilde{p}_2)) : (F \times H)_R \to F_R \times_r H_R$ and $(R_\sharp(\tilde{q}_1), R_\sharp(\tilde{q}_2)) : (G \times H)_R \to G_R \times H_R$ are natural equivalences for any $R \in \mathrm{Ob}\,\mathcal{C}$. We denote by $\alpha(R) : X_R \times G_R \to X_R$ and $\beta(R) : Y_R \times G_R \to Y_R$ the compositions $X_R \times G_R \xrightarrow{(R_\sharp(p_1), R_\sharp(p_2))^{-1}} (X \times G)_R \xrightarrow{R_\sharp(\alpha)} X_R$, $Y_R \times G_R \xrightarrow{(R_\sharp(q_1), R_\sharp(q_2))^{-1}} (Y \times G)_R \xrightarrow{R_\sharp(\beta)} Y_R$, respectively. Then, $\alpha(R)$ is a right action of G_R on X_R and $\beta(R)$ is a right action of G_R on Y_R. Since $\Psi\widetilde{\Phi}(Y^X)(R)$ is $Y^X(R) = \mathrm{Funct}\big(\mathcal{C}^{(2)}, \mathcal{T}op\big)(X_R, Y_R)$ as a set, it is easy to verify the following.

Proposition 8.3.6 *$(Y,\beta)^{(X,\alpha)}$ is a subfunctor of $\Psi\widetilde{\Phi}(Y^X)$ given by*

$$(Y,\beta)^{(X,\alpha)}(R) = \left\{\theta \in \Psi\widetilde{\Phi}(Y^X)(R) \,\middle|\, \theta\alpha(R) = \beta(R)(\theta \times id_{G_R})\right\}.$$

8.4 Hopf algebra and topological affine group scheme

We denote by $\mathcal{TopAlg}_{cK^*}$ the full subcategory of $\mathcal{TopAlg}_{K^*}$ consisting of objects which are complete Hausdorff topological K^*-algebras.

Let A^* be a complete Hausdorff Hopf algebra with product $m_{A^*} : A^* \otimes_{K^*} A^* \to A^*$, unit $u_{A^*} : K^* \to A^*$, coproduct $\mu : A^* \to A^* \widehat{\otimes} A^*$ and counit $\varepsilon : A^* \to K^*$. We denote by $\hat{m} : A^* \widehat{\otimes} A^* \to A^*$ the morphism induced by m_{A^*}. For $R^* \in \operatorname{Ob}\mathcal{TopAlg}_{cK^*}$, let us denote by $\hat{m}_{R^*} : R^* \widehat{\otimes} R^* \to R^*$ the map induced by the product of R^* and by $u_{R^*} : K^* \to R^*$ the unit of R^*. Then, the following composition makes $\mathcal{TopAlg}_{K^*}(A^*, R^*)$ a topological monoid with unit $u_{R^*}\varepsilon$ by (8.1.7).

$$\mathcal{TopAlg}_{K^*}(A^*,R^*) \times \mathcal{TopAlg}_{K^*}(A^*,R^*) \xrightarrow{\psi} \mathcal{TopAlg}_{K^*}(A^* \otimes_{K^*} A^*, R^* \otimes_{K^*} R^*) \xrightarrow{c_{A^* \otimes_{K^*} A^*, R^* \otimes_{K^*} R^*}}$$
$$\mathcal{TopAlg}_{K^*}(A^* \widehat{\otimes}_{K^*} A^*, R^* \widehat{\otimes}_{K^*} R^*) \xrightarrow{\mu^*} \mathcal{TopAlg}_{K^*}(A^*, R^* \widehat{\otimes}_{K^*} R^*) \xrightarrow{\hat{m}_{R^*}{}_*} \mathcal{TopAlg}_{K^*}(A^*, R^*)$$

Let $\iota : A^* \to A^*$ be the conjugation of A^*, namely, a mophism of $\mathcal{TopAlg}_{K^*}$ which makes the following diagram commute.

$$\begin{tikzcd}
K^* \arrow[d, "u_{A^*}"] & A^* \arrow[l, "\varepsilon"'] \arrow[r, "\varepsilon"] \arrow[d, "\mu"] & K^* \arrow[d, "u_{A^*}"] \\
A^* & A^* \widehat{\otimes}_{K^*} A^* \arrow[l, "\hat{m}_{A^*}(id_{A^*} \widehat{\otimes}_{K^*} \iota)"'] \arrow[r, "\hat{m}_{A^*}(\iota \widehat{\otimes}_{K^*} id_{A^*})"] & A^*
\end{tikzcd}$$

Then, $\mathcal{TopAlg}_{K^*}(A^*, R^*)$ has a structure of topological group. Hence the functor $h_{A^*} : \mathcal{TopAlg}_{K^*} \to \mathcal{Top}$ represented by A^* induces a functor from $\mathcal{TopAlg}_{cK^*}$ to $\mathcal{TopGr}$. If $\mathcal{C}$ is a subcategory of $\mathcal{TopAlg}_{cK^*}$ ($\mathcal{C} = \mathcal{TopAlg}_{pfK^*}$ or $\mathcal{TopAlg}_{fK^*}$ for example), we also denote by h_{A^*} the restriction of the functor represented by A^* to $\mathcal{C}$. We call this functor $h_{A^*} : \mathcal{C} \to \mathcal{TopGr}$ the topological affine group scheme represented by A^*.

Remark 8.4.1 *If A^* is connective or coconnective and the coproduct $\mu : A^* \to A^* \widehat{\otimes}_{K^*} A^*$ lifts to $A^* \to A^* \otimes_{K^*} A^*$, it is not necessary to assume the completeness of A^* and the domain of h_{A^*} extends to $\mathcal{TopAlg}_{K^*}$. For a subcategory $\mathcal{C}$ of $\mathcal{TopAlg}_{K^*}$, we call a continuous functor from $\mathcal{C}$ to $\mathcal{TopGr}$ a topological K^*-group functor instead of a topological $\mathcal{C}$-group functor.*

For a graded K^*-module V^*, let $T(V^*)$ the tensor algebra generated by V^* and $I(V^*)$ the two-sided ideal of $T(V^*)$ generated by the set

$$\{x \otimes y - (-1)^{mn} y \otimes x | \, x \in V^m, \; y \in V^n, \; m, n \in \boldsymbol{Z}\}.$$

Put $S(V^*) = T(V^*)/I(V^*)$. We denote by $T_n(V^*)$ the n-fold tensor product of V^* and by $S_n(V^*)$ the image of $T_n(V^*)$ by the quotient map $S_n(V^*)$. Let $\mathcal{S}(V^*)$ be the topological graded K-algebra whose underlying algebra is $S(V^*)$ with the cofinite topology. Let us denote by $i_{V^*} : V^* \to S(V^*)$ the composition map $V^* = T_1(V^*) \to S_1(V^*) \subset S(V^*)$.

Proposition 8.4.2 *We denote by $\mathcal{Mod}^*_{K^*}$ the category of graded topological K^*-modules. Let $F : \mathcal{TopAlg}_{K^*} \to \mathcal{Mod}^*_{K^*}$ be the forgetful functor. The map $\Phi : \mathcal{TopAlg}_{K^*}(\mathcal{S}(V^*), R^*) \to \mathcal{Mod}^*_{K^*}(V^*, F(R^*))$ defined by $f \mapsto f i_{V^*}$ is bijective if R^* is subcofinite.*

Proof. Since $\mathcal{S}(V^*)$ generated by $S_1(V^*)$, Φ is injective. For a linear map $g : V^* \to R^*$, there exists a homomorphism $f : S(V^*) \to R^*$ of K-algebras satisfying $f i_{V^*} = g$ by the construction of $\mathcal{S}(V^*)$. The continuity of f follows from 2) of (8.1.20). Thus Φ is bijective. □

Put $\mathfrak{a}_{V^*} = \sum\limits_{n \geq 1} S_n(V^*)$. Then $\mathfrak{a}_{V^*}$ is an ideal of $S(V^*)$ and $\mathfrak{a}^k_{V^*} = \sum\limits_{n \geq k} S_n(V^*)$. If V^* is finite dimensional, $\mathcal{S}(V^*)$ is noetharian and the $\mathfrak{a}_{V^*}$-adic topology on $S(V^*)$ is subcofinite.

Proposition 8.4.3 *If $V^i = \{0\}$ for all $i \leqq 0$ or $V^i = \{0\}$ for all $i \geqq 0$, the $\mathfrak{a}_{V^*}$-adic topology on $S(V^*)$ is finer than the cofinite topology. Hence, if moreover V^* is finite dimensional, $\mathfrak{a}_{V^*}$-adic topology on $S(V^*)$ coincides with the cofinite topology.*

Proof. Assume that V^* is finite dimensional and $V^i = \{0\}$ for all $i \leqq 0$. Let $\mathfrak{b}$ be a graded ideal of $S(V^*)$ of finite codimension. Then, $(S(V^*)/\mathfrak{b})^*[N] = \{0\}$, namely $\mathfrak{b} \supset S(V^*)^{[N]*}$ for some $N > 0$. Since $\mathfrak{a}_{V^*} \subset S(V^*)^*[1]$, we have $\mathfrak{a}_{V^*}^N \subset S(V^*)^*[N] \subset \mathfrak{b}$. $\square$

Example 8.4.4 *For $n \in \boldsymbol{Z}$, let $G_{a,n} : \mathcal{T}op\mathcal{A}lg_{pfK^*} \to \mathcal{T}op\mathcal{G}r$ be the functor defined by $G_{a,n}(R^*) = R^n$. The group operation of $G_{a,n}(R^*)$ is the addition of R^n. Let V_n^* be a one dimensional graded vector space generated by a single element x_n of degree n. Define $\varepsilon : \mathcal{S}(V_n^*) \to K^*$ and $\mu : \mathcal{S}(V_n^*) \to \mathcal{S}(V_n^*) \otimes_{K^*} \mathcal{S}(V_n^*)$ by $\varepsilon(x_n) = 0$ and $\mu(x_n) = 1 \otimes x_n + x_n \otimes 1$, respectively. Then, $\mathcal{S}(V_n^*)$ is a graded topological Hopf algebra. Define a natural transformation $\theta_n : h_{\mathcal{S}(V_n^*)} \to G_{a,n}$ by $\theta_{nR^*}(f) = f(x_n)$ for $R^* \in \operatorname{Ob}\mathcal{T}op\mathcal{A}lg_{pfK^*}$ and $f \in h_{\mathcal{S}(V_n^*)}(R^*)$. Since an element f of $h_{\mathcal{S}(V_n^*)}(R^*)$ is uniquely determined by the image of x_n which belongs to R^n, $(\theta_n)_{R^*}$ is injective. For any $x \in G_{a,n}(R^*) = R^n$, let $f : \mathcal{S}(V_n^*) \to R^*$ be the unique K-algebra homomorphism that maps x_n to x. Since $\mathcal{S}(V_n^*)$ is cofinite and R^* is subcofinite, f is continuous. Hence θ_{nR^*} is surjective, thus θ_n is a natural equivalence.*

For $m, n \in \boldsymbol{Z}$, define $\rho_{m,n} : G_{a,m} \times G_{a,n} \to G_{a,m+n}$ by $(\rho_{m,n})_{R^*}(x,y) = xy$ for $R^* \in \operatorname{Ob}\mathcal{T}op\mathcal{A}lg_{pfK^*}$ and $x \in G_{a,m}(R^*)$, $y \in G_{a,n}(R^*)$. . We also define $\tilde{\rho}_{m,n} : \mathcal{S}(V_{m+n}^*) \to \mathcal{S}(V_m^*) \otimes_{K^*} \mathcal{S}(V_n^*)$ by $\tilde{\rho}_{m,n}(x_{m+n}) = x_m \otimes x_n$. Then $\tilde{\rho}_{m,n}$ induces $h_{\tilde{\rho}_{m,n}} : h_{\mathcal{S}(V_m^*)} \times h_{\mathcal{S}(V_n^*)} \cong h_{\mathcal{S}(V_m^*) \otimes_{K^*} \mathcal{S}(V_n^*)} \to h_{\mathcal{S}(V_{m+n}^*)}$ and the following diagram commutes.

$$\begin{tikzcd}
h_{\mathcal{S}(V_m^*)} \times h_{\mathcal{S}(V_n^*)} \arrow[r, "h_{\tilde{\rho}_{m,n}}"] \arrow[d, "\theta_m \times \theta_n"] & h_{\mathcal{S}(V_{m+n}^*)} \arrow[d, "\theta_{m+n}"] \\
G_{a,m} \times G_{a,n} \arrow[r, "\rho_{m,n}"] & G_{a,m+n}
\end{tikzcd}$$

Suppose that X is a topological K^*-functor which is a colimit of representable functors. It follows from (8.1.13) that there is a natural homeomorphism

$$\operatorname{Funct}\left(\mathcal{T}op\mathcal{A}lg_{pfK^*}, \mathcal{T}op\right)(X, G_{a,n}) \to \mathcal{T}op\mathcal{A}lg_{pfK^*}\left(\mathcal{S}(V_n^*), K^*[X]\right)$$

which is a homomorphism of groups by the naturality. Moreover, $\mathcal{T}op\mathcal{A}lg_{pfK^*}\left(\mathcal{S}(V_n^*), K^*[X]\right) = h_{\mathcal{S}(V_n^*)}(K^*[X])$ is identified with $G_{a,n}(K^*[X]) = (K^*[X])^n$, hence we have an isomorphism

$$\xi_n : \operatorname{Funct}\left(\mathcal{T}op\mathcal{A}lg_{pfK^*}, \mathcal{T}op\right)(X, G_{a,n}) \to (K^*[X])^n.$$

Let us denote by $\rho(X)_{m,n} : (K^*[X])^m \times (K^*[X])^n \to (K^*[X])^{m+n}$ the product of $K^*[X]$. Then, the following diagram commutes.

$$\begin{tikzcd}
\operatorname{Funct}\left(\mathcal{T}op\mathcal{A}lg_{pfK^*}, \mathcal{T}op\right)(X, G_{a,m}) \times \operatorname{Funct}\left(\mathcal{T}op\mathcal{A}lg_{pfK^*}, \mathcal{T}op\right)(X, G_{a,n}) \arrow[r, "\xi_m \times \xi_n"] & (K^*[X])^m \times (K^*[X])^n \arrow[dd, "\rho(X)_{m,n}"] \\
\operatorname{Funct}\left(\mathcal{T}op\mathcal{A}lg_{pfK^*}, \mathcal{T}op\right)(X, G_{a,m} \times G_{a,n}) \arrow[u, "\cong"] \arrow[d, "(\rho_{m,n})_*"] & \\
\operatorname{Funct}\left(\mathcal{T}op\mathcal{A}lg_{pfK^*}, \mathcal{T}op\right)(X, G_{a,m+n}) \arrow[r, "\xi_{m+n}"] & (K^*[X])^{m+n}
\end{tikzcd}$$

Example 8.4.5 *Let $T : \mathcal{T}op\mathcal{A}lg_{pfK^*} \to \mathcal{T}op\mathcal{G}r$ be the functor defined by*

$$T(R^*) = \left\{ x \in R^0 \middle| \, x \text{ is topologically nilpotent} \right\}.$$

The group operation of $T(R^*)$ is the addition of R^0. We give $S(V_0^*)$ the (x_0)-adic topology. Define $\varepsilon : S(V_0^*) \to K^*$, $\mu : S(V_0^*) \to S(V_0^*) \otimes_{K^*} S(V_0^*)$ and $\theta : h_{S(V_0^*)} \to T$ as the above example. Since a K^*-algebra homomorphism $f : S(V_0^*) \to R^*$ is continuous if and only if $f(x_0)$ is topologically nilpotent. Therefore θ is a natural equivalence.

Remark 8.4.6 *Suppose that an object R^* of $\mathcal{T}op\mathcal{A}lg_{K^*}$ has a topology coarser than the skeletal topology. For non-zero integer n, each element x of R^n is topologically nilpotent, in particular, R^n does not contain a unit.*

Example 8.4.7 *Let G be a functor $\mathcal{T}op\mathcal{A}lg_{K^*} \to \mathcal{T}op\mathcal{G}r$ defined as follows. For $R^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{A}lg_{K^*}$, give $R^*[[X]]$ the topology as in (8.1.21), set*

$$G(R^*) = \left\{ f(X) \in (R^*[[X]])^{-2} \middle|\, f(0) = 0,\ f'(0) \in (R^0)^\times \right\}$$

and regard $G(R^*)$ as a subspace of $R^*[[X]]$. Then, $G(R^*)$ is a group with respect to the composition of power series. Define $\mu_{R^*} : G(R^*) \times G(R^*) \to G(R^*)$ by $\mu_{R^*}(f(X), g(X)) = g(f(X))$. If $f(X) = \sum_{i=1}^{\infty} a_i X^i$ and $g(X) = \sum_{j=1}^{\infty} b_j X^j$, we have $g(f(X)) = \sum_{k=1}^{\infty} c_k X^k$ where

$$c_k = \sum_{s_1+2s_2+\cdots+ls_l+\cdots=k} \frac{(s_1+s_2+\cdots+s_l+\cdots)!}{s_1!s_2!\cdots s_l!\cdots} a_1^{s_1}a_2^{s_2}\cdots a_l^{s_l}\cdots b_{s_1+s_2+\cdots+s_l+\cdots}.$$

It can be verified that $g_1(f_1(X)) - g_2(f_2(X)) \in \mathfrak{a} + (X)^n$ if $f_1(X) - f_2(X), g_1(X) - g_2(X) \in \mathfrak{a} + (X)^n$ for $\mathfrak{a} \in \mathcal{I}_{R^*}$ and $n \in \boldsymbol{N}$. Hence μ_{R^*} is continuous. We also define $\iota_{R^*} : G(R^*) \to G(R^*)$ by $\iota_{R^*}(f(X)) = \sum_{k=1}^{\infty} d_k X^k$ where d_k is inductively defined by $d_1 = a_1^{-1}$ and

$$\sum_{n=1}^{k} \left(\sum_{\substack{s_1+2s_2+\cdots+ls_l+\cdots=k \\ s_1+s_2+\cdots+s_l+\cdots=n}} a_1^{s_1}a_2^{s_2}\cdots a_l^{s_l}\cdots \right) d_n = 0$$

for $k \geqq 2$ so that $\mu_{R^*}(f(X), \iota_{R^*}(f(X))) = X$ holds. It is easily seen that $a_1^k d_k$ is a polynomial of $a_1, a_2, \ldots, a_k$ and that ι_{R^*} is continuous. We can define a map $\iota'_{R^*} : G(R^*) \to G(R^*)$ satisfying $\mu_{R^*}(\iota'_{R^*}(f(X)), f(X)) = X$, similarly. Then, we have $\iota'_{R^*} = \iota_{R^*}$ and it follows that $G(R^*)$ is a topological group.

Put $S^* = K^*[x_1, x_1^{-1}, x_2, x_3, \ldots]$ $(\deg x_i = 2i-2)$ and let $\mathfrak{a}$ be the ideal generated by $x_1 - 1$ and x_i for $i \geqq 2$. We give S^* the cofinite topology. Define $\mu : S^* \to S^* \otimes_{K^*} S^*$, $\varepsilon : S^* \to K^*$ as follows.

$$\mu(x_k) = \sum_{s_1+2s_2+\cdots+ls_l+\cdots=k} \frac{(s_1+s_2+\cdots+s_l+\cdots)!}{s_1!s_2!\cdots s_l!\cdots} x_0^{s_1}x_1^{s_2}\cdots x_{l-1}^{s_l} \otimes x_{s_1+s_2+\cdots+s_l+\cdots}, \qquad \varepsilon(x_i) = \begin{cases} 1 & i = 1 \\ 0 & i \geqq 2 \end{cases}$$

Also define $\iota : S^* \to S^*$ inductively by $\iota(x_1) = x_1^{-1}$,

$$\sum_{n=1}^{k} \left(\sum_{\substack{s_1+2s_2+\cdots+ls_l+\cdots=k \\ s_1+s_2+\cdots+s_l+\cdots=n}} \frac{n!}{s_1!s_2!\cdots s_l!\cdots} x_1^{s_1}x_2^{s_2}\cdots x_l^{s_l}\cdots \right) \iota(x_n) = 0.$$

Thus we have a Hopf algebra S^* in $\mathcal{T}op\mathcal{M}od_{K^*}$. The natural transformation $T : h_{S^*} \to G$ defined by $T_{R^*}(\varphi) = \sum_{i=1}^{\infty} \varphi(x_i) X^i$ is a natural equivalence if we restrict the domains of h_{S^*} and G to $\mathcal{T}op\mathcal{A}lg_{fK^*}$. By (8.1.16), T is a natural equivalence if we restrict the domains of h_{S^*} and G to $\mathcal{T}op\mathcal{A}lg_{pfK^*}$.

Definition 8.4.8 *Let A^* be a topological Hopf algebra and G_{A^*} the topological affine group scheme represented by A^*. For an object B^* of $\mathcal{T}op\mathcal{A}lg_{K^*}$, let $\beta : h_{B^*} \times G_{A^*} \to h_{B^*}$ be a natural transformation such that $\beta_{R^*} : h_{B^*}(R^*) \times G_{A^*}(R^*) \to h_{B^*}(R^*)$ is a continuous right $G_{A^*}(R^*)$-action on $h_{B^*}(R^*)$. We call a pair (h_{B^*}, β) a right G_{A^*}-scheme.*

Recall from (8.1.6) that $(\iota_1^*, \iota_2^*) : h_{B^* \widehat{\otimes}_{K^*} A^*}(R^*) \to h_{B^*}(R^*) \times h_{A^*}(R^*)$ is a homeomorphism. For a right G_{A^*}-scheme (h_{B^*}, β), we put $\psi_\beta = \left(\beta(\iota_1^*, \iota_2^*)^{-1}\right)_{B^* \widehat{\otimes}_{K^*} A^*}(id_{B^* \widehat{\otimes}_{K^*} A^*}) : B^* \to B^* \widehat{\otimes}_{K^*} A^* \in h_{B^*}(B^* \widehat{\otimes}_{K^*} A^*)$. It can be seen that ψ_β is a structure map of right A^*-comodule algebra. Conversely, for a right A^*-comodule algebra $(B^*, \psi : B^* \to B^* \widehat{\otimes}_{K^*} A^*)$, we put $\beta_\psi = h_\psi(\iota_1^*, \iota_2^*)^{-1} : h_{B^*} \times G_{A^*} \to h_{B^*}$.

Proposition 8.4.9 *The correspondences $\beta \mapsto \psi_\beta$ and $\psi \mapsto \beta_\psi$ give an isomorphism between the category of right G_{A^*}-schemes and the category of right A^*-comodule algebras.*

8.5 Distributions of affine group schemes

For a fileld K^*, let A^* be a Hopf algebra in $\mathcal{T}op\mathcal{A}lg_{pfK^*}$ and G_{A^*} an affine group scheme represented by A^*. We assume that the coproduct μ of A^* takes values in $A^* \otimes_{K^*} A^*$. We denote by I the kernel of the counit $\varepsilon : A^* \to K^*$ of A^*. We also denote by $\rho_n : A^* \to A^*/I^{n+1}$ and $u_{A^*} : K^* \to A^*$ the quotient map and the unit of A^*, respectively.

Definition 8.5.1 *For a non-negative integer n, we define graded K^*-submodule $\mathrm{Dist}_n(G_{A^*})$ of A^{**} to be the image of $\rho_n^* : \mathcal{H}om^*(A^*/I^{n+1}, K^*) \to \mathcal{H}om^*(A^*, K^*) = A^{**}$. We call $\mathrm{Dist}_n(G_{A^*})$ the distribution of order $\leqq n$ on G_{A^*}. Define $\mathrm{Dist}_n^+(G_{A^*})$ to be the kernel of the composition $\mathrm{Dist}_n(G_{A^*}) \xrightarrow{inclusion} \mathcal{H}om^*(A^*, K^*) \xrightarrow{u_{A^*}^*} \mathcal{H}om^*(K^*, K^*)$. We put $\mathrm{Dist}(G_{A^*}) = \bigcup\limits_{n\geqq 0} \mathrm{Dist}_n(G_{A^*})$ and $\mathrm{Dist}^+(G_{A^*}) = \bigcup\limits_{n\geqq 0} \mathrm{Dist}_n^+(G_{A^*})$.*

Remark 8.5.2 *Let $\tau_n : A^*/I^{n+2} \to A^*/I^{n+1}$ be the quotient map. We note that ρ_n^* and τ_n^* are injective and since $\tau_n\rho_{n+1} = \rho_n$, we have $\rho_{n+1}^*\tau_n^* = \rho_n^*$ which implies that $\mathrm{Dist}_n(G_{A^*})$ is a submodule of $\mathrm{Dist}_{n+1}(G_{A^*})$. Since ε induces an isomorphism $A/I \to K^*$, $\mathrm{Dist}_0(G_{A^*}) \cong \mathcal{H}om^*(A^*/I, K^*)$ is isomorphic to K^*. Hence there is the following filtration of A^{**}.*

$$K^* \cong \mathrm{Dist}_0(G_{A^*}) \subset \mathrm{Dist}_1(G_{A^*}) \subset \cdots \subset \mathrm{Dist}_n(G_{A^*}) \subset \mathrm{Dist}_{n+1}(G_{A^*}) \subset \cdots \subset \mathrm{Dist}(G_{A^*}) \subset A^{**}$$

Proposition 8.5.3 *Let us denote by $i_I : I \to A^*$ the inclusion map and by $\bar{\rho}_n : I \to I/I^{n+1}$ the quotient map. Then, $i_I^* : \mathcal{H}om^*(A^*, K^*) \to \mathcal{H}om^*(I, K^*)$ maps $\mathrm{Dist}_n^+(G_{A^*})$ isomorphically onto the image of $\bar{\rho}_n^* : \mathcal{H}om^*(I/I^{n+1}, K^*) \to \mathcal{H}om^*(I, K^*)$ and there is a split short exact sequence*

$$0 \to \mathrm{Dist}_n^+(G_{A^*}) \xrightarrow{inclusion} \mathrm{Dist}_n(G_{A^*}) \to K^* \to 0.$$

Proof. We denote by $\bar{i}_I : I/I^{n+1} \to A^*/I^{n+1}$ and $\varepsilon_n : A^*/I^{n+1} \to K^*$ the map induced by i_I and ε, respectively. Then, the following diagram is commutative and both raws are split exact. In fact, $K^* \xrightarrow{u_{A^*}} A^* \xrightarrow{\rho_n} A^*/I^{n+1}$ is a right inverse of $\varepsilon_n : A^*/I^{n+1} \to K^*$ and $A^*/I \xrightarrow{\varepsilon_0^{-1}} K^* \xrightarrow{u_{A^*}} A^*$ is a right inverse of $\rho_0 : A^* \to A^*/I$

$$\begin{array}{ccccccccc}
0 \to & \mathcal{H}om^*(K^*, K^*) & \xrightarrow{\varepsilon_n^*} & \mathcal{H}om^*(A^*/I^{n+1}, K^*) & \xrightarrow{\bar{i}_I^*} & \mathcal{H}om^*(I/I^{n+1}, K^*) & \to 0 \\
& \cong\downarrow \varepsilon_0^* & & \downarrow \rho_n^* & & \downarrow \bar{\rho}_n^* & \\
0 \to & \mathcal{H}om^*(A^*/I, K^*) & \xrightarrow{\rho_0^*} & \mathcal{H}om^*(A^*, K^*) & \xrightarrow{i_I^*} & \mathcal{H}om^*(I, K^*) & \to 0
\end{array}$$

We note that ε_0^* is an isomorphism. If $x \in \mathrm{Ker}\, i_I^*$, there exists unique $y \in \mathcal{H}om^*(K^*, K^*)$ such that $\rho_0^*\varepsilon_0^*(y) = x$. On the other hand, since $\varepsilon_0\rho_0 u_{A^*} = \varepsilon u_{A^*} = id_{K^*}$, we have $u_{A^*}^*\rho_0^*\varepsilon_0^* = id_{\mathcal{H}om^*(K^*,K^*)}$, Hence if $x \in \mathrm{Ker}\, u_{A^*}^*$, then $y = u_{A^*}^*\rho_0^*\varepsilon_0^*(y) = u_{A^*}^*(x) = 0$ which shows $x = 0$. If $z \in \mathrm{Im}\,\bar{\rho}_n^*$, there exists $w \in \mathcal{H}om^*(A^*/I^{n+1}, K^*)$ such that $i_I^*\rho_n^*(w) = \bar{\rho}_n^*\bar{i}_I^*(w) = z$. Put $v = u_{A^*}^*\rho_n^*(w)$. Since $v = u_{A^*}^*\rho_0^*\varepsilon_0^*(v) = u_{A^*}^*\rho_n^*\varepsilon_n^*(v)$, it follows that $\rho_n^*(w - \varepsilon_n^*(v)) \in \mathrm{Ker}\, u_{A^*}^*$. We also have $i_I^*\rho_n^*(w - \varepsilon_n^*(v)) = i_I^*\rho_n^*(w) - i_I^*\rho_n^*\varepsilon_n^*(v) = z - \bar{\rho}_n^*\bar{i}_I^*\varepsilon_n^*(v) = z$. Therefore z is in the image of the kernel of $u_{A^*}^*$ by i_I^*. The second assertion is a direct □

Since the coproduct $\mu : A^* \to A^* \otimes_{K^*} A^*$ of A^* maps I into $I \otimes_{K^*} A^* + A^* \otimes_{K^*} I$, we have

$$\mu(I^{n+1}) \subset I^{n+1} \otimes_{K^*} A^* + I^n \otimes_{K^*} I + \cdots + I^{n-m+1} \otimes_{K^*} I^m + \cdots + A^* \otimes_{K^*} I^{n+1}.$$

We denote by $m_{K^*} : K^* \otimes_{K^*} K^* \to K^*$ the isomorphism induced by the product of K^*. For $\alpha \in \mathrm{Dist}_m(G_{A^*})$ and $\beta \in \mathrm{Dist}_n(G_{A^*})$, since $\alpha(\Sigma^a I^{m+1}) = \beta(\Sigma^b I^{n+1}) = \{0\}$ if $\deg\alpha = a$, $\deg\beta = b$ and $\alpha\beta \in A^{**}$ is a composition

$$\Sigma^{a+b}A^* \xrightarrow{\Sigma^{a+b}\mu} \Sigma^{a+b}(A^* \otimes_{K^*} A^*) \xrightarrow{\left(\tau_{A^*,A^*}^{a,b}\right)^{-1}} (\Sigma^a A^*) \otimes_{K^*} (\Sigma^b A^*) \xrightarrow{\alpha\otimes_{K^*}\beta} K^* \otimes_{K^*} K^* \xrightarrow{m_{K^*}} K^*,$$

$\alpha\beta$ maps $\Sigma^{a+b}I^{m+n+1}$ to $\{0\}$, that is, $\alpha\beta \in \mathrm{Dist}_{m+n}(G_{A^*})$. Hence $\mathrm{Dist}(G_{A^*})$ is a filtered algebra. For $\alpha \in \mathrm{Dist}_m^+(G_{A^*})$ and $\beta \in \mathrm{Dist}_n^+(G_{A^*})$, $\alpha\Sigma^a u_{A^*} = 0$ and $\beta\Sigma^b u_{A^*} = 0$ if $\deg\alpha = a$, $\deg\beta = b$. We denote by $\mu_{K^*} : K^* \to K^* \otimes_{K^*} K^*$ the map defined by $\mu_{K^*}(1) = 1 \otimes 1$. Then, the commutativity of the following diagram implies $\alpha\beta \in \mathrm{Dist}_{m+n}^+(G_{A^*})$.

$$\begin{array}{ccccccc}
\Sigma^{a+b}K^* & \xrightarrow{\Sigma^{a+b}\mu_{K^*}} & \Sigma^{a+b}(K^*\otimes_{K^*}K^*) & \xrightarrow{\left(\tau^{a,b}_{K^*,K^*}\right)^{-1}} & (\Sigma^a K^*)\otimes_{K^*}(\Sigma^b K^*) & & \\
\Sigma^{a+b}u_{A^*}\downarrow & & \Sigma^{a+b}(u_{A^*}\otimes_{K^*}u_{A^*})\downarrow & & \downarrow\Sigma^{a+b}u_{A^*}\otimes_{K^*}\Sigma^{a+b}u_{A^*} & & \\
\Sigma^{a+b}A^* & \xrightarrow{\Sigma^{a+b}\mu} & \Sigma^{a+b}(A^*\otimes_{K^*}A^*) & \xrightarrow{\left(\tau^{a,b}_{A^*,A^*}\right)^{-1}} & (\Sigma^a A^*)\otimes_{K^*}(\Sigma^b A^*) & \xrightarrow{\alpha\otimes_{K^*}\beta} & K^*\otimes_{K^*}K^*
\end{array}$$

Lemma 8.5.4 *For $x \in I$, $\mu(x) - (1\otimes x + x\otimes 1) \in I\otimes_{K^*} I$ holds.*

Proof. We put $\mu(x) = 1\otimes x + x\otimes 1 + \sum_{j=1}^{k} y_j\otimes z_j$ for $y_j, z_j \in A^*$. Then, $\sum_{j=1}^{k}\varepsilon(y_j)z_j = \sum_{j=1}^{k}\varepsilon(z_j)y_j = 0$, hence we have $\sum_{j=1}^{k}\varepsilon(y_jz_j) = 0$. Put $y'_j = y_j - \varepsilon(y_j)1$ and $z'_j = z_j - \varepsilon(z_j)1$. Then, we have $y'_j, z'_j \in I$ and the above equalities imply $\sum_{j=1}^{k}\varepsilon(y_j)z'_j = \sum_{j=1}^{k}\varepsilon(y_j)(z_j - \varepsilon(z_j)1) = 0$, $\sum_{j=1}^{k}\varepsilon(z_j)y'_j = \sum_{j=1}^{k}\varepsilon(z_j)(y_j - \varepsilon(y_j)1) = 0$. It follows $\sum_{j=1}^{k} y_j\otimes z_j = \sum_{j=1}^{k}(y'_j + \varepsilon(y_j)1)\otimes(z'_j + \varepsilon(z_j)1) = \sum_{j=1}^{k} y'_j\otimes z'_j \in I\otimes_{K^*} I$. □

Proposition 8.5.5 *For $\alpha \in \mathrm{Dist}_m(G_{A^*})$ and $\beta \in \mathrm{Dist}_n(G_{A^*})$, $\alpha\beta - (-1)^{\deg\alpha\deg\beta}\beta\alpha \in \mathrm{Dist}_{m+n-1}(G_{A^*})$.*

Proof. For $x_i \in I$ $(i = 1,2,\dots,m+n)$, we have $\mu(x_i) = 1\otimes x_i + x_i\otimes 1 + Y_i$ for some $Y_i \in I\otimes_{K^*} I$ by (8.5.4). For a sequence of integers $1 \leqq i_1 < i_2 < \cdots < i_k \leqq m+n$, let $\bar{i}_1 < \bar{i}_2 < \cdots < \bar{i}_{m+n-k}$ be the sequence of integers which satisfies $\{\bar{i}_1, \bar{i}_2, \dots, \bar{i}_{m+n-k}\} = \{1,2,\dots,m+n\} - \{i_1, i_2, \dots, i_k\}$. Then, we have

$$\prod_{s=1}^{k}(1\otimes x_{i_s} + x_{i_s}\otimes 1)\prod_{t=1}^{m+n-k} Y_{\bar{i}_t} \in I^{m+n-k}\otimes_{K^*} I^{m+n} + I^{m+n-k+1}\otimes_{K^*} I^{m+n-1} + \cdots + I^{m+n}\otimes_{K^*} I^{m+n-k}$$

which implies the following.

$$\mu(x_1x_2\cdots x_{m+n}) - \prod_{i=1}^{m+n}(1\otimes x_i + x_i\otimes 1) \in I\otimes_{K^*} I^{m+n} + I^2\otimes_{K^*} I^{m+n-1} + \cdots + I^{m+n}\otimes_{K^*} I \ \cdots(*)$$

Put $x = x_1x_2\cdots x_{m+n}$ and $\mu(x) = \prod_{i=1}^{m+n}(1\otimes x_i + x_i\otimes 1) + \sum_{j=1}^{l} y_j\otimes z_j$. Then, $y_j\otimes z_j \in I^{i_j}\otimes_{K^*} I^{m+n+1-i_j}$ for some $1 \leqq i_j \leqq m+n$ by $(*)$. Put $\deg\alpha = a$ and $\deg\beta = b$. Then $\alpha : \Sigma^a A^* \to K^*$ maps $\Sigma^a I^{m+1}$ maps to zero and $\beta : \Sigma^b A^* \to K^*$ maps $\Sigma^b I^{n+1}$ maps to zero. Hence $\alpha([a], y_j)\beta([b], z_j) = \beta([b], y_j)\alpha([a], z_j) = 0$. Therefore we have

$$\begin{aligned}
(\alpha\beta)([a+b], x) &= m_{K^*}(\alpha\otimes_{K^*}\beta)\left(\tau^{a,b}_{A^*,A^*}\right)^{-1}([a+b], \mu(x)) \\
&= m_{K^*}(\alpha\otimes_{K^*}\beta)\left(\tau^{a,b}_{A^*,A^*}\right)^{-1}\Big([a+b], \prod_{i=1}^{m+n}(1\otimes x_i + x_i\otimes 1)\Big) \ \cdots(i) \\
(\beta\alpha)([a+b], x) &= m_{K^*}(\beta\otimes_{K^*}\alpha)\left(\tau^{b,a}_{A^*,A^*}\right)^{-1}([a+b], \mu(x)) \\
&= m_{K^*}(\beta\otimes_{K^*}\alpha)\left(\tau^{b,a}_{A^*,A^*}\right)^{-1}\Big([a+b], \prod_{i=1}^{m+n}(1\otimes x_i + x_i\otimes 1)\Big) \ \cdots(ii)
\end{aligned}$$

We put $\deg x_i = \nu(i)$. It follows from

$$\begin{aligned}
\prod_{i=1}^{m+n}(1\otimes x_i + x_i\otimes 1) &= \sum_{1\leqq i_1<i_2<\cdots<i_k\leqq m+n}(-1)^{\sum_{s=1}^{k}\nu(i_s)\left(\sum_{\bar{i}_t<i_s}\nu(\bar{i}_t)\right)} x_{i_1}x_{i_2}\cdots x_{i_k}\otimes x_{\bar{i}_1}x_{\bar{i}_2}\cdots x_{\bar{i}_{m+n-k}} \\
&= \sum_{1\leqq i_1<i_2<\cdots<i_k\leqq m+n}(-1)^{\sum_{t=1}^{m+n-k}\nu(\bar{i}_t)\left(\sum_{i_s<\bar{i}_t}\nu(i_s)\right)} x_{\bar{i}_1}x_{\bar{i}_2}\cdots x_{\bar{i}_{m+n-k}}\otimes x_{i_1}x_{i_2}\cdots x_{i_k},
\end{aligned}$$

and $\sum_{s=1}^{k}\nu(i_s)\Big(\sum_{\bar{i}_t<i_s}\nu(\bar{i}_t)\Big)+\sum_{t=1}^{m+n-k}\nu(\bar{i}_t)\Big(\sum_{i_s<\bar{i}_t}\nu(i_s)\Big)=\Big(\sum_{s=1}^{k}\nu(i_s)\Big)\Big(\sum_{t=1}^{m+n-k}\nu(\bar{i}_t)\Big)$ that we have the following equalities.

$$(i)=\sum_{1\leqq i_1<i_2<\cdots<i_k\leqq m+n}(-1)^{b\sum\limits_{s=1}^{k}\nu(i_s)+\sum\limits_{s=1}^{k}\nu(i_s)\Big(\sum\limits_{\bar{i}_t<i_s}\nu(\bar{i}_t)\Big)}\alpha([a],x_{i_1}x_{i_2}\cdots x_{i_k})\beta([b],x_{\bar{i}_1}x_{\bar{i}_2}\cdots x_{\bar{i}_{m+n-k}})$$

$$(ii)=\sum_{1\leqq i_1<i_2<\cdots<i_k\leqq m+n}(-1)^{a\sum\limits_{t=1}^{m+n-k}\nu(\bar{i}_t)+\sum\limits_{t=1}^{m+n-k}\nu(\bar{i}_t)\Big(\sum\limits_{i_s<\bar{i}_t}\nu(i_s)\Big)}\beta([b],x_{\bar{i}_1}x_{\bar{i}_2}\cdots x_{\bar{i}_{m+n-k}})\alpha([a],x_{i_1}x_{i_2}\cdots x_{i_k})$$

$$=\sum_{1\leqq i_1<i_2<\cdots<i_k\leqq m+n}(-1)^{ab+b\sum\limits_{s=1}^{k}\nu(i_s)+\sum\limits_{s=1}^{k}\nu(i_s)\Big(\sum\limits_{\bar{i}_t<i_s}\nu(\bar{i}_t)\Big)}\alpha([a],x_{i_1}x_{i_2}\cdots x_{i_k})\beta([b],x_{\bar{i}_1}x_{\bar{i}_2}\cdots x_{\bar{i}_{m+n-k}})$$

Therefore $(\alpha\beta)([a+b],x)=(-1)^{ab}(\beta\alpha)([a+b],x)$ for any $x\in I^{m+n}$ and this shows $\alpha\beta-(-1)^{ab}\beta\alpha$ belongs to $\mathcal{H}om^*(A^*/I^{m+n},K^*)=\mathrm{Dist}_{m+n-1}(G_{A^*})$. $\square$

Definition 8.5.6 *We define the Lie algebra* $\mathrm{Lie}(G_{A^*})$ *of* G_{A^*} *by* $\mathrm{Lie}(G_{A^*})=\mathrm{Dist}_1^+(G_{A^*})$. *The bracket operation* $\mathrm{Lie}(G_{A^*})\times\mathrm{Lie}(G_{A^*})\to\mathrm{Lie}(G_{A^*})$ *is given by* $(\alpha,\beta)\mapsto\alpha\beta-(-1)^{\deg\alpha\deg\beta}\beta\alpha$ *for* $\alpha,\beta\in\mathrm{Lie}(G_{A^*})$.

Let $m_{A^*}:A^*\otimes_{K^*}A^*\to A^*$ be the product of A^*. Then, m_{A^*} induces a coproduct of $A^{**}=\mathcal{H}om^*(A^*,K^*)$ which is the following composition.

$$\mathcal{H}om^*(A^*,K^*)\xrightarrow{m_{A^*}^*m_{K^*}^{-1}}\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*\otimes_{K^*}K^*)\xrightarrow{\phi^{-1}}\mathcal{H}om^*(A^*,K^*)\otimes_{K^*}\mathcal{H}om^*(A^*,K^*)$$

we denote this coproduct by $\psi_{A^{**}}$, which is a morphism of graded K^*-algebras by (5.1.6).

Proposition 8.5.7 $\alpha\in\mathrm{Ker}\,u_{A^*}^*$ *belongs to* $\mathrm{Lie}(G_{A^*})$ *if and only if* $\psi_{A^{**}}(\alpha)=1\otimes\alpha+\alpha\otimes 1$.

Proof. Since the counit $\varepsilon\in\mathcal{H}om^0(A^*,K^*)$ corresponds to the unit of A^{**}, $m_{K^*}\phi(1\otimes\alpha)$ and $m_{K^*}\phi(\alpha\otimes 1)$ are the following compositions if $\deg\alpha=a$, respectively.

$$\Sigma^a(A^*\otimes_{K^*}A^*)\xrightarrow{\left(\tau^{0,a}_{A^*,A^*}\right)^{-1}}A^*\otimes_{K^*}\Sigma^aA^*\xrightarrow{\varepsilon\otimes_{K^*}\alpha}K^*\otimes_{K^*}K^*\xrightarrow{m_{K^*}}K^*$$

$$\Sigma^a(A^*\otimes_{K^*}A^*)\xrightarrow{\left(\tau^{a,0}_{A^*,A^*}\right)^{-1}}\Sigma^aA^*\otimes_{K^*}A^*\xrightarrow{\alpha\otimes_{K^*}\varepsilon}K^*\otimes_{K^*}K^*\xrightarrow{m_{K^*}}K^*$$

Hence $\psi_{A^{**}}(\alpha)=1\otimes\alpha+\alpha\otimes 1$ if and only if $\alpha\Sigma^am_{A^*}=m_{K^*}(\varepsilon\otimes_{K^*}\alpha)\big(\tau^{0,a}_{A^*,A^*}\big)^{-1}+m_{K^*}(\alpha\otimes_{K^*}\varepsilon)\big(\tau^{a,0}_{A^*,A^*}\big)^{-1}$. Suppose $\psi_{A^{**}}(\alpha)=1\otimes\alpha+\alpha\otimes 1$. For $x,y\in I$, the following chain of equalities shows that $\alpha\in\mathrm{Lie}(G_{A^*})$.

$$\begin{aligned}\alpha([a],xy)&=\alpha\Sigma^am_{A^*}([a],x\otimes y)\\&=m_{K^*}(\varepsilon\otimes_{K^*}\alpha)\big(\tau^{0,a}_{A^*,A^*}\big)^{-1}([a],x\otimes y)+m_{K^*}(\alpha\otimes_{K^*}\varepsilon)\big(\tau^{a,0}_{A^*,A^*}\big)^{-1}([a],x\otimes y)\\&=m_{K^*}(\varepsilon\otimes_{K^*}\alpha)((-1)^{a\deg x}x\otimes([a],y))+m_{K^*}(\alpha\otimes_{K^*}\varepsilon)(([a],x)\otimes y)\\&=(-1)^{a\deg x}\varepsilon(x)\alpha([a],y)+\alpha([a],x)\varepsilon(y)=0.\end{aligned}$$

Conversely, assume that $\alpha\in\mathrm{Lie}(G_{A^*})$. For $x,y\in I$, we have $\alpha\Sigma^am_{A^*}([a],x\otimes y)=\alpha([a],xy)=0$ and

$$\Big(m_{K^*}(\varepsilon\otimes_{K^*}\alpha)\big(\tau^{0,a}_{A^*,A^*}\big)^{-1}+m_{K^*}(\alpha\otimes_{K^*}\varepsilon)\big(\tau^{a,0}_{A^*,A^*}\big)^{-1}\Big)([a],x\otimes y)=(-1)^{ad}\varepsilon(x)\alpha([a],y)+\alpha([a],x)\varepsilon(y)=0,$$

where we put $\deg x=d$. We also have $\alpha\Sigma^am_{A^*}([a],1\otimes x)=\alpha\Sigma^am_{A^*}([a],x\otimes 1)=\alpha([a],x)$ and the following.

$$\begin{aligned}\Big(m_{K^*}(\varepsilon\otimes_{K^*}\alpha)\big(\tau^{0,a}_{A^*,A^*}\big)^{-1}+m_{K^*}(\alpha\otimes_{K^*}\varepsilon)\big(\tau^{a,0}_{A^*,A^*}\big)^{-1}\Big)([a],1\otimes x)&=\varepsilon(1)\alpha([a],x)+\alpha([a],1)\varepsilon(x)=\alpha([a],x)\\\Big(m_{K^*}(\varepsilon\otimes_{K^*}\alpha)\big(\tau^{0,a}_{A^*,A^*}\big)^{-1}+m_{K^*}(\alpha\otimes_{K^*}\varepsilon)\big(\tau^{a,0}_{A^*,A^*}\big)^{-1}\Big)([a],x\otimes 1)&=(-1)^{ad}\varepsilon(x)\alpha([a],1)+\alpha([a],x)\varepsilon(1)\\&=\alpha([a],x)\end{aligned}$$

Finally, $\alpha\Sigma^am_{A^*}([a],1\otimes 1)=\alpha([a],1)=\alpha\Sigma^au_{A^*}([a],1)=0$ and

$$\Big(m_{K^*}(\varepsilon\otimes_{K^*}\alpha)\big(\tau^{0,a}_{A^*,A^*}\big)^{-1}+m_{K^*}(\alpha\otimes_{K^*}\varepsilon)\big(\tau^{a,0}_{A^*,A^*}\big)^{-1}\Big)([a],1\otimes 1)=\varepsilon(1)\alpha([a],1)+\alpha([a],1)\varepsilon(1)=0.$$

Thus we have $\alpha\Sigma^am_{A^*}=m_{K^*}(\varepsilon\otimes_{K^*}\alpha)\big(\tau^{0,a}_{A^*,A^*}\big)^{-1}+m_{K^*}(\alpha\otimes_{K^*}\varepsilon)\big(\tau^{a,0}_{A^*,A^*}\big)^{-1}$. $\square$

Proposition 8.5.8 *Suppose that the characteristic of K^* is a prime number p. If $\alpha \in \mathrm{Lie}(G_{A^*})$ and the degree of α is even or $p = 2$, then $\alpha^p \in \mathrm{Lie}(G_{A^*})$.*

Proof. Since $\psi_{A^{**}}(\alpha) = 1 \otimes \alpha + \alpha \otimes 1$ by (8.5.7), we have $\psi_{A^{**}}(\alpha^p) = (1 \otimes \alpha + \alpha \otimes 1)^p$. $1 \otimes \alpha$ and $\alpha \otimes 1$ are commutative in $A^{**} \otimes_{K^*} A^{**}$ since the degree of α is even or $p = 2$. It follows $\psi_{A^{**}}(\alpha^p) = (1 \otimes \alpha + \alpha \otimes 1)^p = 1 \otimes \alpha^p + \alpha^p \otimes 1$ which implies $\alpha^p \in \mathrm{Lie}(G_{A^*})$. □

8.6 General linear group

Let R^* be an object of $\mathcal{T}op\mathcal{A}lg_{K^*}$ and M^*, N^* right R^*-modules. We denote by $\mathcal{H}om^*_{R^*}(M^*, N^*)$ the subspace of $\mathcal{H}om^*(M^*, N^*)$ consisting of homomorphisms of right R^*-modules (3.1.9).

We assume that $K^i = \{0\}$ if $i \neq 0$ in this subsection. It follows from (3.4.19) that the composition

$$\mu_{L^*,M^*,N^*} : \mathcal{H}om^s_{R^*}(L^*, M^*) \times \mathcal{H}om^t_{R^*}(M^*, N^*) \to \mathcal{H}om^{s+t}_{R^*}(L^*, N^*)$$

of right R^*-module homomorphisms is continuous for $s, t \in \boldsymbol{Z}$ if L^* is supercofinite, M^* is superskeletal and N^* is profinite.

Definition 8.6.1 *Let M^* be a right R^*-module such that M^* is finite type as a K^*-module and has cofinite topology (1.4.2). We put $\mathcal{E}nd^*_{R^*}(M^*) = \mathcal{H}om^*_{R^*}(M^*, M^*)$ and let $\mathcal{GL}_{R^*}(M^*)$ be the set of invertible homomorphisms in $\mathcal{E}nd^0_{R^*}(M^*)$. Note that $\mathcal{E}nd^0_{R^*}(M^*)$ is a topological monoid by the composition of morphisms if M^* is finite type, profinite and has cofinite topology.*

For an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, let $\mathcal{GL}(M^*)$ be a group functor which assigns an object R^* of $\mathcal{T}op\mathcal{A}lg_{K^*}$ to $\mathcal{GL}_{R^*}(M^* \widehat{\otimes}_{K^*} R^*)$. Suppose that K^* is a field such that $K^i = \{0\}$ if $i \neq 0$ and that M^* is finite dimensional. Then $M^* \widehat{\otimes}_{K^*} R^*$ is supercofinite, superskeletal and profinite if R^* is finite type, profinite and has cofinite topology.

Definition 8.6.2 *(1) Suppose that a set B and a map $d : B \to \boldsymbol{Z}$ are given. We denote by $V^*(B, d)$ the graded vector space over a field K^* spanned by B such that the degree of $x \in B$ is $d(x)$. We give $V^*(B, d)$ the skeletal topology and regard this as an object of $\mathcal{T}op\mathcal{M}od_{K^*}$.*

(2) For a non-increasing sequence $\boldsymbol{v} = (s_1, s_2, \ldots, s_n)$ of integers, let $B_{\boldsymbol{v}}$ be a set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ of n-elements and $d_{\boldsymbol{v}} : B_{\boldsymbol{v}} \to \boldsymbol{Z}$ a map given by $d_{\boldsymbol{v}}(\boldsymbol{v}_i) = s_i$ for $i = 1, 2, \ldots, n$. We denote $V^*(B_{\boldsymbol{v}}, d_{\boldsymbol{v}})$ by $V^*_{\boldsymbol{v}}$ for short.

(3) Let $\boldsymbol{w} = (t_1, t_2, \ldots, t_m)$ be another non-increasing sequence of integers and R^* an object of $\mathcal{T}op\mathcal{A}lg_{K^*}$. We denote by $M(\boldsymbol{v}, \boldsymbol{w}\,; R^*)$ the set of $m \times n$ matrices whose (i, j)-entry belongs to $R^{s_j - t_i}$. We regard $M(\boldsymbol{v}, \boldsymbol{w}\,; R^*)$ as a subspace of mn-fold product space of R^*. Let us denote by $\mathrm{GL}_{\boldsymbol{v}}(R^*)$ the subspace of $M(\boldsymbol{v}, \boldsymbol{v}\,; R^*)$ consisting of invertible matrices.

Remark 8.6.3 *For $n \in \boldsymbol{Z}$ and a map $d : B \to \boldsymbol{Z}$, let $\Sigma^n d : B \to \boldsymbol{Z}$ be the map defined by $\Sigma^n d(x) = d(x) + n$. Since $(\Sigma^n V^*(B, d))^k = V^*(B, d)^{k-n}$ is spanned by $d^{-1}(k-n) = (\Sigma^n d)^{-1}(k)$, we have $\Sigma^n V^*(B, d) = V^*(B, \Sigma^n d)$.*

Let $f : V^*_{\boldsymbol{v}} \otimes_{K^*} R^* \to V^*_{\boldsymbol{w}} \otimes_{K^*} R^*$ be a homomorphism of right R^*-modules. For each $j = 1, 2, \ldots, n$, put $f(\boldsymbol{v}_j \otimes 1) = \sum_{i=1}^{m} \boldsymbol{w}_i \otimes a_{ij}$. Let A_f be the matrix whose (i, j)-entry is a_{ij}. Then, A_f is an elemeqt of $M(\boldsymbol{v}, \boldsymbol{w}\,; R^*)$. We define a map $\Phi_{\boldsymbol{v},\boldsymbol{w}}(R^*) : \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{w}} \otimes_{K^*} R^*) \to M(\boldsymbol{v}, \boldsymbol{w}\,; R^*)$ by $(\Phi_{\boldsymbol{v},\boldsymbol{w}}(R^*))(f) = A_f$.

Lemma 8.6.4 *For $\mathfrak{a} \in \mathcal{I}_{R^*}$, put $O(\mathfrak{a}) = \{f \in \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{w}} \otimes_{K^*} R^*) |\, \mathrm{Im}\, f \subset V^*_{\boldsymbol{w}} \otimes_{K^*} \mathfrak{a}\}$. Then, $\{O(\mathfrak{a}) |\, \mathfrak{a} \in \mathcal{I}_{R^*}\}$ is a fundamental system of the neighborhoods of zero map.*

Proof. Since $V^*_{\boldsymbol{v}} \otimes_{K^*} R^*$ is generated by a finite dimensional subspace $V^*_{\boldsymbol{v}} \otimes_{K^*} K^*$ over R^*, we have $O(V^*_{\boldsymbol{v}} \otimes_{K^*} K^*, V^*_{\boldsymbol{w}} \otimes_{K^*} \mathfrak{a})^0 = O(\mathfrak{a})$. □

For $\mathfrak{a} \in \mathcal{I}_{R^*}$, we put $N(\mathfrak{a}) = \{(a_{ij}) \in M(\boldsymbol{v}, \boldsymbol{w}\,; R^*) |\, a_{ij} \in \mathfrak{a}\}$. Then, $\{N(\mathfrak{a}) |\, \mathfrak{a} \in \mathcal{I}_{R^*}\}$ is a fundamental system of the neighborhoods of zero matrix.

Proposition 8.6.5 *For $\mathfrak{a} \in \mathcal{I}_{R^*}$, $(\Phi_{\boldsymbol{v},\boldsymbol{w}}(R^*))(O(\mathfrak{a})) = N(\mathfrak{a})$ holds and $\Phi_{\boldsymbol{v},\boldsymbol{w}}(R^*)$ is an isomorphism of topological vector spaces.*

Proof. Clearly, $\Phi_{\boldsymbol{v},\boldsymbol{w}}(R^*)$ is injective. Since the topologies of $V^*_{\boldsymbol{v}} \otimes_{K^*} R^*$ and $V^*_{\boldsymbol{w}} \otimes_{K^*} R^*$ are induced by R^*, it follows from (1.1.11) that every R^*-module homomorphism is continuous. Thus $\Phi_{\boldsymbol{v},\boldsymbol{w}}(R^*)$ is surjective. We can easily verify $(\Phi_{\boldsymbol{v},\boldsymbol{w}}(R^*))(O(\mathfrak{a})) = N(\mathfrak{a})$ from the definitions. Hence $\Phi_{\boldsymbol{v},\boldsymbol{w}}(R^*)$ is continuous and open map. $\square$

Proposition 8.6.6 *Let $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{z}$ be non-increasing sequences of integers. Then, the composition*

$$\mu : \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{w}} \otimes_{K^*} R^*) \times \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{w}} \otimes_{K^*} R^*, V^*_{\boldsymbol{z}} \otimes_{K^*} R^*) \to \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{z}} \otimes_{K^*} R^*)$$

is continuous.

Proof. Define $\nu : M(\boldsymbol{v},\boldsymbol{w}\,; R^*) \times M(\boldsymbol{w},\boldsymbol{z}\,; R^*) \to M(\boldsymbol{v},\boldsymbol{z}\,; R^*)$ by $\nu(A,B) = BA$. Clearly, ν is continuous and the following diagram commutes.

$$\begin{tikzcd}
\mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{w}} \otimes_{K^*} R^*) \times \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{w}} \otimes_{K^*} R^*, V^*_{\boldsymbol{z}} \otimes_{K^*} R^*) \arrow[r, "\mu"] \arrow[d, "\Phi_{\boldsymbol{v},\boldsymbol{w}}(R^*) \times \Phi_{\boldsymbol{w},\boldsymbol{z}}(R^*)"] & \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{z}} \otimes_{K^*} R^*) \arrow[d, "\Phi_{\boldsymbol{v},\boldsymbol{z}}(R^*)"] \\
M(\boldsymbol{v},\boldsymbol{w}\,; R^*) \times M(\boldsymbol{w},\boldsymbol{z}\,; R^*) \arrow[r, "\nu"] & M(\boldsymbol{v},\boldsymbol{z}\,; R^*)
\end{tikzcd}$$

Hence the result follows from (8.6.5). $\square$

Corollary 8.6.7 $\Phi_{\boldsymbol{v},\boldsymbol{v}}(R^*) : \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{v}} \otimes_{K^*} R^*) \to M(\boldsymbol{v},\boldsymbol{v}\,; R^*)$ *induces a homeomorphism*

$$\Phi_{\boldsymbol{v}}(R^*) : \mathcal{GL}_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*) \to \mathrm{GL}_{\boldsymbol{v}}(R^*)$$

which satisfies $(\Phi_{\boldsymbol{v}}(R^*))(gf) = (\Phi_{\boldsymbol{v}}(R^*))(g)(\Phi_{\boldsymbol{v}}(R^*))(f)$ for $f, g \in \mathcal{GL}_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*)$.

Let $\varphi : R^* \to S^*$ be a morphism in $\mathcal{T}op\mathcal{A}lg_{K^*}$ and regard S^* as a left R^*-module by φ. Consider the isomorphism $\chi_\varphi : R^* \otimes_{R^*} S^* \to S^*$ given by $\chi_\varphi(x \otimes y) = \varphi(x)y$. For $f \in \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{w}} \otimes_{K^*} R^*)$, let $f_\varphi : V^*_{\boldsymbol{v}} \otimes_{K^*} S^* \to V^*_{\boldsymbol{w}} \otimes_{K^*} S^*$ be the following composition.

$$V^*_{\boldsymbol{v}} \otimes_{K^*} S^* \xrightarrow{id_{V^*_{\boldsymbol{v}}} \otimes_{K^*} \chi_\varphi^{-1}} V^*_{\boldsymbol{v}} \otimes_{K^*} R^* \otimes_{R^*} S^* \xrightarrow{f \otimes_{K^*} id_{S^*}} V^*_{\boldsymbol{w}} \otimes_{K^*} R^* \otimes_{R^*} S^* \xrightarrow{id_{V^*_{\boldsymbol{w}}} \otimes_{K^*} \chi_\varphi} V^*_{\boldsymbol{w}} \otimes_{K^*} S^*$$

Then, f_φ is a homomorphism of S^*-modules and the following diagram commutes.

$$\begin{tikzcd}
V^*_{\boldsymbol{v}} \otimes_{K^*} R^* \arrow[r, "f"] \arrow[d, "id_{V^*_{\boldsymbol{v}}} \otimes_{K^*} \varphi"] & V^*_{\boldsymbol{w}} \otimes_{K^*} R^* \arrow[d, "id_{V^*_{\boldsymbol{w}}} \otimes_{K^*} \varphi"] \\
V^*_{\boldsymbol{v}} \otimes_{K^*} S^* \arrow[r, "f_\varphi"] & V^*_{\boldsymbol{w}} \otimes_{K^*} S^*
\end{tikzcd}$$

Define a map $T_\varphi : \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{w}} \otimes_{K^*} R^*) \to \mathcal{H}om^0_{S^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} S^*, V^*_{\boldsymbol{w}} \otimes_{K^*} S^*)$ by $T_\varphi(f) = f_\varphi$.

It is straightforward to show the following fact.

Lemma 8.6.8 *Let $\varphi : R^* \to S^*$ and $\psi : S^* \to T^*$ be morphisms of $\mathcal{T}op\mathcal{A}lg_{K^*}$. Then $T_\psi T_\varphi = T_{\psi\varphi}$.*

For a morphism $\varphi : R^* \to S^*$ in $\mathcal{T}op\mathcal{A}lg_{K^*}$, let $M_\varphi : M(\boldsymbol{v},\boldsymbol{w}\,; R^*) \to M(\boldsymbol{v},\boldsymbol{w}\,; S^*)$ be the map defined by $M_\varphi((a_{ij})) = (\varphi(a_{ij}))$. The following fact is also straightforward.

Lemma 8.6.9 *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{w}} \otimes_{K^*} R^*) \arrow[r, "\Phi_{\boldsymbol{v},\boldsymbol{w}}(R^*)"] \arrow[d, "T_\varphi"] & M(\boldsymbol{v},\boldsymbol{w}\,; R^*) \arrow[d, "M_\varphi"] \\
\mathcal{H}om^0_{S^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} S^*, V^*_{\boldsymbol{w}} \otimes_{K^*} S^*) \arrow[r, "\Phi_{\boldsymbol{v},\boldsymbol{w}}(S^*)"] & M(\boldsymbol{v},\boldsymbol{w}\,; S^*)
\end{tikzcd}$$

Proposition 8.6.10 *T_φ is continuous.*

Proof. Since M_φ is continuous and both $\Phi_{\boldsymbol{v},\boldsymbol{w}}(R^*)$ and $\Phi_{\boldsymbol{v},\boldsymbol{w}}(S^*)$ are isomorphisms, the continuity follows from (8.6.9). $\square$

The following fact is also straightforward.

Proposition 8.6.11 *Let $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{z}$ be non-increasing sequence of integers. For a morphism $\varphi : R^* \to S^*$ in $\mathcal{T}op\mathcal{A}lg_{K^*}$, the following diagram commutes.*

$$\begin{CD}
\mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{w}} \otimes_{K^*} R^*) \times \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{w}} \otimes_{K^*} R^*, V^*_{\boldsymbol{z}} \otimes_{K^*} R^*) @>{\mu}>> \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{z}} \otimes_{K^*} R^*) \\
@V{T_\varphi \times T_\varphi}VV @VV{T_\varphi}V \\
\mathcal{H}om^0_{S^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} S^*, V^*_{\boldsymbol{w}} \otimes_{K^*} S^*) \times \mathcal{H}om^0_{S^*}(V^*_{\boldsymbol{w}} \otimes_{K^*} S^*, V^*_{\boldsymbol{z}} \otimes_{K^*} S^*) @>{\mu}>> \mathcal{H}om^0_{S^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} S^*, V^*_{\boldsymbol{z}} \otimes_{K^*} S^*)
\end{CD}$$

By the above result, $T_\varphi : \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{v}} \otimes_{K^*} R^*) \to \mathcal{H}om^0_{S^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} S^*, V^*_{\boldsymbol{v}} \otimes_{K^*} S^*)$ maps $\mathcal{GL}_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*)$ into $\mathcal{GL}_{S^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} S^*)$. We also denote by $T_\varphi : \mathcal{GL}_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*) \to \mathcal{GL}_{S^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} S^*)$ the map induced by $T_\varphi : \mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{v}} \otimes_{K^*} R^*) \to \mathcal{H}om^0_{S^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} S^*, V^*_{\boldsymbol{v}} \otimes_{K^*} S^*)$.

Proposition 8.6.12 *Let $\left(R^* \xrightarrow{p_i} D(i)\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ be a limiting cone of a functor $D : \mathcal{D} \to \mathcal{T}op\mathcal{A}lg_{K^*}$. Define a functor $E : \mathcal{D} \to \mathcal{T}op\mathcal{M}od_{K^*}$ by $E(i) = \mathcal{H}om^0_{D(i)}(V^*_{\boldsymbol{v}} \otimes_{K^*} D(i), V^*_{\boldsymbol{w}} \otimes_{K^*} D(i))$ and $E(\varphi) = T_{D(\varphi)}$ for $i \in \mathrm{Ob}\,\mathcal{D}$ and $\varphi \in \mathrm{Mor}\,\mathcal{D}$. Then*

$$\left(\mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{w}} \otimes_{K^*} R^*) \xrightarrow{T_{p_i}} \mathcal{H}om^0_{D(i)}(V^*_{\boldsymbol{v}} \otimes_{K^*} D(i), V^*_{\boldsymbol{w}} \otimes_{K^*} D(i))\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$$

is a limiting cone of E.

Proof. Suppose that $(f_i)_{i \in \mathrm{Ob}\,\mathcal{D}} \in \prod_{\mathfrak{a} \in \mathrm{Ob}\,\mathcal{D}} \mathcal{H}om^0_{D(i)}(V^*_{\boldsymbol{v}} \otimes_{K^*} D(i), V^*_{\boldsymbol{w}} \otimes_{K^*} D(i))$ satisfies $T_{D(\varphi)}(f_i) = f_j$ for $\varphi \in \mathcal{D}(i,j)$. Then, $\left(V^*_{\boldsymbol{v}} \otimes_{K^*} R^* \xrightarrow{f_i(id_{V^*_{\boldsymbol{v}}} \otimes_{K^*} p_i)} V^*_{\boldsymbol{w}} \otimes_{K^*} D(i)\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a cone of a functor $D' : \mathrm{Ob}\,\mathcal{D} \to \mathcal{T}op\mathcal{M}od_{K^*}$ given by $D'(i) = V^*_{\boldsymbol{w}} \otimes_{K^*} D(i)$. Since $\left(V^*_{\boldsymbol{w}} \otimes_{K^*} R^* \xrightarrow{id_{V^*_{\boldsymbol{w}}} \otimes_{K^*} p_i} V^*_{\boldsymbol{w}} \otimes_{K^*} D(i)\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of D' by (2.3.9), there is a unique map $f : V^*_{\boldsymbol{v}} \otimes_{K^*} R^* \to V^*_{\boldsymbol{w}} \otimes_{K^*} R^*$ satisfying $(id_{V^*_{\boldsymbol{w}}} \otimes_{K^*} p_i)f = f_i(id_{V^*_{\boldsymbol{v}}} \otimes_{K^*} p_i)$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. It can be verified that f is a homomorphism of R^*-modules and that $T_{p_i}(f) = f_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$.

For any $i \in \mathrm{Ob}\,\mathcal{D}$ and $\mathfrak{a} \in \mathcal{I}_{D(i)}$, we claim that $T^{-1}_{p_i}(O(\mathfrak{a})) = O(p_i^{-1}(\mathfrak{a}))$ holds. In fact, if $f \in T^{-1}_{p_i}(O(\mathfrak{a}))$ then $f \in O(p_i^{-1}(\mathfrak{a}))$ by the commutativity of the following diagram.

$$\begin{CD}
V^*_{\boldsymbol{v}} \otimes_{K^*} R^* @>{f}>> V^*_{\boldsymbol{w}} \otimes_{K^*} R^* \\
@V{id_{V^*_{\boldsymbol{v}}} \otimes_{K^*} p_i}VV @VV{id_{V^*_{\boldsymbol{w}}} \otimes_{K^*} p_i}V \\
V^*_{\boldsymbol{v}} \otimes_{K^*} D(i) @>{T_{p_i}(f)}>> V^*_{\boldsymbol{w}} \otimes_{K^*} D(i)
\end{CD}$$

Suppose $f \in O(p_i^{-1}(\mathfrak{a}))$ and put $A_f = (a_{kl})$, then $a_{kl} \in p_i^{-1}(\mathfrak{a})$. By (8.6.9), $(T_{p_i}(f))(\boldsymbol{v}_l) = \sum_{k=1}^{m} \boldsymbol{v}_k \otimes_{K^*} p_i(a_{kl}) \in V^*_{\boldsymbol{w}} \otimes_{K^*} \mathfrak{a}$ for $l = 1, 2, \dots, n$. Thus we have $f \in T^{-1}_{p_i}(O(\mathfrak{a}))$. Since $\{p_i^{-1}(\mathfrak{a}) | \, i \in \mathrm{Ob}\,\mathcal{D},\ \mathfrak{a} \in \mathcal{I}_{D(i)}\}$ is a fundamental system of neighborhood of 0 of R^* by the assumption, the equality we have just shown implies the result. □

In the above proof, if $f_i \in \mathcal{H}om^0_{D(i)}(V^*_{\boldsymbol{v}} \otimes_{K^*} D(i), V^*_{\boldsymbol{w}} \otimes_{K^*} D(i))$ is an isomorphism for every $i \in \mathrm{Ob}\,\mathcal{D}$, the map $f : V^*_{\boldsymbol{v}} \otimes_{K^*} R^* \to V^*_{\boldsymbol{w}} \otimes_{K^*} R^*$ induced by f_i's is also an isomorphism. Thus we have the following result.

Corollary 8.6.13 *Let $\left(R^* \xrightarrow{p_i} D(i)\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$ be a limiting cone of a functor $D : \mathcal{D} \to \mathcal{T}op\mathcal{A}lg_{K^*}$. Define a functor $G : \mathcal{D} \to \mathcal{T}op$ by $G(i) = \mathcal{GL}_{D(i)}(V^*_{\boldsymbol{v}} \otimes_{K^*} D(i))$ and $G(\varphi) = T_{D(\varphi)}$ for $i \in \mathrm{Ob}\,\mathcal{D}$ and $\varphi \in \mathrm{Mor}\,\mathcal{D}$. Then*

$$\left(\mathcal{GL}_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*) \xrightarrow{T_{p_i}} \mathcal{GL}_{D(i)}(V^*_{\boldsymbol{v}} \otimes_{K^*} D(i))\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$$

is a limiting cone of G.

Remark 8.6.14 *Suppose that R^* is profinite and consider the limiting cone $\left(R^* \xrightarrow{p_{\mathfrak{a}}} R^*/\mathfrak{a}\right)_{\mathfrak{a} \in \mathcal{I}_{R^*}}$ of the functor $d_{R^*} : \mathcal{I}_{R^*} \to \mathcal{T}op\mathcal{A}lg_{K^*}$ given by $d_{R^*}(\mathfrak{a}) = R^*/\mathfrak{a}$. Since $V^*_{\boldsymbol{v}} \otimes_{K^*} R^*/\mathfrak{a}$ is discrete, $\mathcal{GL}_{R^*/\mathfrak{a}}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*/\mathfrak{a})$ is a discrete group. It follows from (8.6.13) that $\mathcal{GL}_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*)$ is a topological group. If K^* is a finite field, $\mathcal{GL}_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*)$ is a profinite group.*

For a non-increasing sequence $\boldsymbol{v} = (s_1, s_2, \dots, s_n)$ of integers, we define a functor $\mathcal{GL}_{\boldsymbol{v}} : \mathcal{TopAlg}_{pfK^*} \to \mathcal{TopGr}$ by $\mathcal{GL}_{\boldsymbol{v}}(R^*) = \mathcal{GL}_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*)$ and $\mathcal{GL}_{\boldsymbol{v}}(\varphi) = T_\varphi$. The product $\boldsymbol{\mu}_{\boldsymbol{v}} : \mathcal{GL}_{\boldsymbol{v}} \times \mathcal{GL}_{\boldsymbol{v}} \to \mathcal{GL}_{\boldsymbol{v}}$, the unit $\boldsymbol{\varepsilon}_{\boldsymbol{v}} : h_{K^*} \to \mathcal{GL}_{\boldsymbol{v}}$ and the inverse $\boldsymbol{\iota}_{\boldsymbol{v}} : \mathcal{GL}_{\boldsymbol{v}} \to \mathcal{GL}_{\boldsymbol{v}}$ are given by $\boldsymbol{\mu}_{\boldsymbol{v}R^*}(f,g) = gf$, $\boldsymbol{\varepsilon}_{\boldsymbol{v}R^*}(u_{R^*}) = id_{V^*_{\boldsymbol{v}} \otimes_{K^*} R^*}$ and $\boldsymbol{\iota}_{\boldsymbol{v}R^*}(f) = f^{-1}$, respectively for $R^* \in \operatorname{Ob}\mathcal{C}$ and $f, g \in \mathcal{GL}_{\boldsymbol{v}}(R^*)$.

Proposition 8.6.15 *$\mathcal{GL}_{\boldsymbol{v}}$ is representable.*

Proof. Let $E_{\boldsymbol{v}}$ be a set of variables $\{x_{ij}, y_{ij} | i, j = 1, 2, \dots, n\}$ and $D_{\boldsymbol{v}} : E_{\boldsymbol{v}} \to \boldsymbol{Z}$ a map defined by $D_{\boldsymbol{v}}(x_{ij}) = D_{\boldsymbol{v}}(y_{ij}) = s_j - s_i$. Let $J_{\boldsymbol{v}}$ be an ideal of $S(V^*(E_{\boldsymbol{v}}, D_{\boldsymbol{v}}))$ generated by the union of the following sets.

$$\left\{ \sum_{k=1}^{n} x_{ik} y_{ki} - 1 \,\middle|\, i = 1, 2, \dots n \right\}, \qquad \left\{ \sum_{k=1}^{n} x_{ik} y_{kj} \,\middle|\, i, j = 1, 2, \dots n,\ i \neq j \right\},$$

$$\left\{ \sum_{k=1}^{n} y_{ik} x_{ki} - 1 \,\middle|\, i = 1, 2, \dots n \right\}, \qquad \left\{ \sum_{k=1}^{n} y_{ik} x_{kj} \,\middle|\, i, j = 1, 2, \dots n,\ i \neq j \right\}$$

Put $A^*_{\boldsymbol{v}} = S(V^*(E_{\boldsymbol{v}}, D_{\boldsymbol{v}}))/J_{\boldsymbol{v}}$ and we also denote by x_{ij}, y_{ij} the classes of $x_{ij}, y_{ij} \in S(V^*(E_{\boldsymbol{v}}, D_{\boldsymbol{v}}))$ in $A^*_{\boldsymbol{v}}$. We give $A^*_{\boldsymbol{v}}$ the cofinite topology. Consider $n \times n$ matrices X and Y whose (i,j)-entries are x_{ij} and y_{ij} of $A^*_{\boldsymbol{v}}$, respectively. Then, both XY and YX are the unit matrix and X is invertible, namely $X^{-1} = Y$.

Define maps $\mu_{\boldsymbol{v}} : A^*_{\boldsymbol{v}} \to A^*_{\boldsymbol{v}} \otimes_{K^*} A^*_{\boldsymbol{v}}$, $\varepsilon_{\boldsymbol{v}} : A^*_{\boldsymbol{v}} \to K^*$ and $\iota_{\boldsymbol{v}} : A^*_{\boldsymbol{v}} \to A^*_{\boldsymbol{v}}$ by

$$\mu_{\boldsymbol{v}}(x_{ij}) = \sum_{k=1}^{n} (-1)^{(s_j - s_k)(s_k - s_i)} x_{kj} \otimes x_{ik}, \quad \mu_{\boldsymbol{v}}(y_{ij}) = \sum_{k=1}^{n} y_{ik} \otimes y_{kj},$$

$$\varepsilon_{\boldsymbol{v}}(x_{ij}) = \varepsilon_{\boldsymbol{v}}(y_{ij}) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \quad \iota_{\boldsymbol{v}}(x_{ij}) = y_{ij}, \quad \iota_{\boldsymbol{v}}(y_{ij}) = x_{ij}.$$

It is easy to verify that $A^*_{\boldsymbol{v}}$ is a Hopf algebra.

For an object R^* of $\mathcal{TopAlg}_{pfK^*}$, we define a map $\varphi_{\boldsymbol{v}R^*} : h_{A^*_{\boldsymbol{v}}}(R^*) \to \mathcal{GL}_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*)$ as follows. For $f \in h_{A^*_{\boldsymbol{v}}}(R^*)$, let $\varphi_{\boldsymbol{v}R^*}(f) : V^*_{\boldsymbol{v}} \otimes_{K^*} R^* \to V^*_{\boldsymbol{v}} \otimes_{K^*} R^*$ be the unique homomorphism of right R^*-modules satisfying $(\varphi_{\boldsymbol{v}R^*}(f))(\boldsymbol{v}_j \otimes 1) = \sum\limits_{i=1}^{n} \boldsymbol{v}_i \otimes f(x_{ij})$. Then, $\varphi_{\boldsymbol{v}R^*}(f)$ is continuous by (1.1.11) and $\varphi_{\boldsymbol{v}R^*}$ is a homomorphism of groups. In fact, we have the following equalities. Here $m_{R^*} : R^* \otimes_{K^*} R^* \to R^*$ denotes the multiplication of R^*.

$$\begin{aligned}
(\varphi_{\boldsymbol{v}R^*}(g)\varphi_{\boldsymbol{v}R^*}(f))(\boldsymbol{v}_j \otimes 1) &= \sum_{k=1}^{n} \varphi_{\boldsymbol{v}R^*}(g)(\boldsymbol{v}_k \otimes f(x_{kj})) = \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{v}_i \otimes g(x_{ik}) f(x_{kj}) \\
&= \sum_{i=1}^{n} \boldsymbol{v}_i \otimes \left(\sum_{k=1}^{n} (-1)^{(s_j - s_k)(s_k - s_i)} f(x_{kj}) g(x_{ik}) \right) \\
&= \sum_{i=1}^{n} \boldsymbol{v}_i \otimes m_{R^*}(f \otimes g)\mu_{R^*}(x_{ij}) = (\varphi_{\boldsymbol{v}R^*}(m_{R^*}(f \otimes g)\mu_{R^*}))(\boldsymbol{v}_j \otimes 1) \\
(\varphi_{\boldsymbol{v}R^*}(u_{R^*}\varepsilon_{\boldsymbol{v}}))(\boldsymbol{v}_j \otimes 1) &= \sum_{i=1}^{n} \boldsymbol{v}_i \otimes u_{R^*}\varepsilon_{\boldsymbol{v}}(x_{ij}) = \boldsymbol{v}_j \otimes 1 \\
(\varphi_{\boldsymbol{v}R^*}(f\iota_{\boldsymbol{v}})\varphi_{\boldsymbol{v}R^*}(f))(\boldsymbol{v}_j \otimes 1) &= \sum_{k=1}^{n} \varphi_{\boldsymbol{v}R^*}(f\iota_{\boldsymbol{v}})(\boldsymbol{v}_k \otimes f(x_{kj})) = \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{v}_i \otimes f\iota_{\boldsymbol{v}}(x_{ik}) f(x_{kj}) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} \boldsymbol{v}_i \otimes f(y_{ik} x_{kj}) = \boldsymbol{v}_j \otimes 1 \\
(\varphi_{\boldsymbol{v}R^*}(f)\varphi_{\boldsymbol{v}R^*}(f\iota_{\boldsymbol{v}}))(\boldsymbol{v}_j \otimes 1) &= \sum_{k=1}^{n} \varphi_{\boldsymbol{v}R^*}(f)(\boldsymbol{v}_k \otimes f\iota_{\boldsymbol{v}}(x_{kj})) = \sum_{k=1}^{n} \sum_{i=1}^{n} \boldsymbol{v}_i \otimes f(x_{ik}) f\iota_{\boldsymbol{v}}(x_{kj}) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{n} \boldsymbol{v}_i \otimes f(x_{ik} y_{kj}) = \boldsymbol{v}_j \otimes 1
\end{aligned}$$

Clearly, $\varphi_{\boldsymbol{v}R^*}$ is natural in R^* hence we have a natural transformation $\varphi_{\boldsymbol{v}} : h_{A^*_{\boldsymbol{v}}} \to \mathcal{GL}_{\boldsymbol{v}}$.

Suppose that R^* is finite. Since $A^*_{\boldsymbol{v}}$ is finitely generated, it follows from (8.1.1) that $h_{A^*_{\boldsymbol{v}}}(R^*)$ is discrete. On the other hand, since $V^*_{\boldsymbol{v}} \otimes_{K^*} R^*$ is finite, $\mathcal{H}om^0_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*, V^*_{\boldsymbol{v}} \otimes_{K^*} R^*)$ is discrete. Hence $\mathcal{GL}_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*)$ is also discrete. Let L be an element of $\mathcal{GL}_{R^*}(V^*_{\boldsymbol{v}} \otimes_{K^*} R^*)$ and let a_{ij} be the (i,j)-entry of $\Phi_{\boldsymbol{v}}(R^*)(L)$. Since $\Phi_{\boldsymbol{v}}(R^*)(L)$ is invertible, let b_{ij} be the (i,j)-entry of the inverse of $\Phi_{\boldsymbol{v}}(R^*)(L)$. Define a K^*-algebra homomorphism $f : A^*_{\boldsymbol{v}} \to R^*$ by $f(x_{ij}) = a_{ij}$ and $f(y_{ij}) = b_{ij}$. Since $A^*_{\boldsymbol{v}}$ is cofinite, f is continuous, namely f is an element of $h_{A^*_{\boldsymbol{v}}}(R^*)$. It is clear that $\varphi_{\boldsymbol{v}R^*}(f) = L$. Thus $\varphi_{\boldsymbol{v}R^*}(f) : V^*_{\boldsymbol{v}} \otimes_{K^*} R^* \to V^*_{\boldsymbol{v}} \otimes_{K^*} R^*$ is an isomorphism if R^* is finite.

It follows from the naturality of $\varphi_{\boldsymbol{v}}$ and (8.6.13) that $\varphi_{\boldsymbol{v}R^*}(f) : V^*_{\boldsymbol{v}} \otimes_{K^*} R^* \to V^*_{\boldsymbol{v}} \otimes_{K^*} R^*$ is an isomorphism for any profinite R^*. Let $\hat{A}^*_{\boldsymbol{v}}$ be the completion of $A^*_{\boldsymbol{v}}$. Then $\hat{A}^*_{\boldsymbol{v}}$ is profinite and $\eta_{A^*_{\boldsymbol{v}}} : A^*_{\boldsymbol{v}} \to \hat{A}^*_{\boldsymbol{v}}$ induces an isomorphism $h_{\eta_{A^*_{\boldsymbol{v}}}} : h_{\hat{A}^*_{\boldsymbol{v}}}(R^*) \to h_{A^*_{\boldsymbol{v}}}(R^*)$ if R^* is profinite. Therefore $\mathcal{GL}_{\boldsymbol{v}} : \mathcal{T}op\mathcal{A}lg_{pfK^*} \to \mathcal{T}op\mathcal{G}r$ is represented by $\hat{A}^*_{\boldsymbol{v}}$, namely a composition $\varphi_{\boldsymbol{v}} h_{\eta_{A^*_{\boldsymbol{v}}}} : h_{\hat{A}^*_{\boldsymbol{v}}} \to \mathcal{GL}_{\boldsymbol{v}}$ gives a natural equvalence. □

8.7 The Steenrod group

Definition 8.7.1 *We define $\boldsymbol{F}_p$-group functors G_p, G^{ev}_p and G^{od}_p as follows. Let A^* be a graded commutative algebra over $\boldsymbol{F}_p$. If $p=2$, we assign degree -1 to a variable X and define $G_2(A^*)$ to be the following subset of $A^*[[X]]$.*

$$\left\{\alpha(X) \in A^*[[X]] \,\middle|\, \alpha(X) = \sum_{i=0}^{\infty} \alpha_i X^{2^i},\ \deg \alpha_i = 2^i - 1\ (i \geqq 0),\ \alpha_0 = 1\right\}$$

If p is an odd prime, we assign degree -2 to a variable X and consider a graded exterior algebra $\boldsymbol{F}_p[\epsilon]/(\epsilon^2)$ with $\deg \epsilon = -1$. Define $G_p(A^*)$ to be the following subset of $A^* \otimes_{\boldsymbol{F}_p} \boldsymbol{F}_p[\epsilon]/(\epsilon^2)[[X]] = A^*[\epsilon]/(\epsilon^2)[[X]]$.

$$\left\{\alpha(X) \in A^*[\epsilon]/(\epsilon^2)[[X]] \,\middle|\, \alpha(X) = \sum_{i=0}^{\infty} \alpha_i X^{p^i}, \deg \alpha_i = 2(p^i - 1)\ (i \geqq 0),\ \alpha_0 - 1 \in (\epsilon)\right\}$$

We give a group structure to $G_p(A^*)$ by the composition of formal power series. Namely, the product $\alpha(X)\cdot\beta(X)$ of $\alpha(X)$ and $\beta(X)$ is defined to be

$$\beta(\alpha(X)) = \sum_{i=0}^{\infty} \beta_i \alpha(X)^{p^i} = \sum_{i=0}^{\infty} \beta_i \left(\sum_{j=0}^{\infty} \alpha_j X^{p^j}\right)^{p^i} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_j^{p^i} \beta_i X^{p^{i+j}} = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} \alpha_{i-j}^{p^j} \beta_j\right) X^{p^i}.$$

We call G_p the mod p Steenrod group. For an odd prime p, $G^{ev}_p(A^*)$ is defined to be the following subset of $A^*[[X]]$.

$$\left\{\alpha(X) \in A^*[[X]] \,\middle|\, \alpha(X) = \sum_{i=0}^{\infty} \alpha_i X^{p^i}, \deg \alpha_i = 2(p^i - 1)\ (i \geqq 0),\ \alpha_0 = 1\right\}$$

Since A^* is a subalgebra of $A^*[\epsilon]/(\epsilon^2)$, we regard $G^{ev}_p(A^*)$ as a subgroup of $G_p(A^*)$. We also define $G^{od}_p(A^*)$ to be the following subset of $A^*[\epsilon]/(\epsilon^2)[[X]]$.

$$\left\{\alpha(X) \in A^*[\epsilon]/(\epsilon^2)[[X]] \,\middle|\, \alpha(X) = \sum_{i=0}^{\infty} \alpha_i X^{p^i}, \deg \alpha_i = 2(p^i - 1)\ (i \geqq 0),\ \alpha_0 - 1, \alpha_i \in (\epsilon)\ (i \geqq 1)\right\}$$

Remark 8.7.2 *The quotient map $A^*[\epsilon]/(\epsilon^2) \to A^*[\epsilon]/(\epsilon) = A^*$ defines a homomorphism $\pi^{ev}_{A^*} : G_p(A^*) \to G^{ev}_p(A^*)$ of groups which is a left inverse of the inclusion map and that $G^{od}_p(A^*)$ is the kernel of $\pi^{ev}_{A^*}$. Hence $G_p(A^*)$ is a semi-direct product of $G^{od}_p(A^*)$ and $G^{ev}_p(A^*)$.*

We denote by $\mathcal{A}^{ev}_{p*}$ the polynomial part $\boldsymbol{F}_p[\xi_1, \xi_2, \dots]$ of $\mathcal{A}_{p*}$ which is a Hopf subalgebra of $\mathcal{A}_{p*}$.

Proposition 8.7.3 *(G. Nishida,[22]) The mod p dual Steenrod algebra $\mathcal{A}_{p*}$ represents G_p. If p is an odd prime, $\mathcal{A}^{ev}_{p*}$ represents G^{ev}_p.*

Remark 8.7.4 *We denote by $\mathcal{A}^{od}_{p*}$ the quotient of $\mathcal{A}_{p*}$ by the ideal generated by $\xi_1, \xi_2, \dots$. Then, $\mathcal{A}^{od}_{p*} = E(\tau_0, \tau_1, \tau_2, \dots)$ and each τ_i is primitive. $\mathcal{A}^{od}_{p*}$ represents G^{od}_p and the quotient map $\mathcal{A}_{p*} \to \mathcal{A}^{od}_{p*}$ induces the inclusion morphism $G^{od}_p \to G_p$. In fact, for a graded $\boldsymbol{F}_p$-algebra A^*, the natural bijection is given by assigning* *a morphism* $f : \mathcal{A}_{p*}^{od} \to A^*$ *to an element* $(1+f(\tau_0)\epsilon)X + \sum_{i=1}^{\infty} f(\tau_i)\epsilon X^{p^i}$ *of* $G_p^{od}(A^*)$*. Since* $\alpha(X)^p = X^p$ *if* $\alpha(X) \in G_p^{od}(A^*)$*,* $G_p^{od}(A^*)$ *is an abelian normal subgroup of* $G_p(A^*)$*. In fact,* $G_p^{od}(A^*)$ *is isomorphic to* $\prod_{i=0}^{\infty} A^{2p^i-1}$ *as an additive group.*

For a positive integer n, a partition of n is a sequence $(\nu(1), \nu(2), \ldots, \nu(l))$ of positive integers which satisfies $\nu(1)+\nu(2)+\cdots+\nu(l) = n$. We denote by $\mathrm{Part}(n)$ the set of all partitions of n. For a partition $\nu = (\nu(1), \nu(2), \ldots, \nu(l))$ of n, we put $\ell(\nu) = l$ and $\sigma(\nu)(i) = \sum_{s=1}^{i-1} \nu(s)$ $(1 \leqq i \leqq l)$. We call $\ell(\nu)$ the length of ν and denote by $\mathrm{Part}_l(n)$ the subset of $\mathrm{Part}(n)$ consisting of partitions of length l.

Lemma 8.7.5 *For integers* $1 \leqq l \leqq k < m$*, we define a map* $F_{l,k} : \mathrm{Part}_l(k) \to \mathrm{Part}_{l+1}(m)$ *by*

$$F_{l,k}((\nu(1), \nu(2), \ldots, \nu(l))) = (\nu(1), \nu(2), \ldots, \nu(l), m-k).$$

Let $F : \bigcup_{k=1}^{m-1} \mathrm{Part}(k) \to \mathrm{Part}(m)$ *be the map induced by* $F_{l,k}$*'s. Then,* F *is an injection whose image is partitions of* m *of length greater than one.*

Proof. Since each $F_{l,k}$ is injective and the images of $F_{l,k}$'s are disjoint each other, F is injective. For each $\nu = (\nu(1), \nu(2), \ldots, \nu(l), \nu(l+1)) \in \mathrm{Part}_{l+1}(m)$, $F_{l,\sigma(\nu)(l+1)}$ maps $(\nu(1), \nu(2), \ldots, \nu(l)) \in \mathrm{Part}_l(\sigma(\nu)(l+1))$ to ν. □

Proposition 8.7.6 *([16]) Let* A^* *be a graded commutative algebra over* $\boldsymbol{F}_p$ *and* c *a fixed integer which is even if* p *is odd. Suppose that sequences of elements* $(\alpha_i)_{i \geqq 0}$ *and* $(\beta_i)_{i \geqq 0}$ *of* A^* *satisfy the following conditions.*

(i) $\alpha_0 = 1$ *if* $p = 2$*,* $(\alpha_0 - 1)^2 = 0$ *if* p *is odd.*
(ii) $\deg \alpha_0 = \deg \beta_0 = 0$*,* $\deg \alpha_i = \deg \beta_i = c(1+p+\cdots+p^{i-1})$*.*
(iii) $\alpha_0\beta_0 = 1$ *and* $\sum_{k=0}^{i} \alpha_{i-k}^{p^k}\beta_k = 0$ *for any positive integer* i*.*

Then, $\beta_0 = \alpha_0^{-1} = 2 - \alpha_0$ *and the following equality holds for each positive integer* n*.*

$$\beta_n = \beta_0 \sum_{\nu \in \mathrm{Part}(n)} (-1)^{\ell(\nu)} \prod_{j=1}^{\ell(\nu)} \alpha_{\nu(j)}^{p^{\sigma(\nu)(j)}}$$

Proof. We have $\alpha_0(2-\alpha_0) = 1$ by (i). Thus it follows from $\alpha_0\beta_0 = 1$ that $\beta_0 = \alpha_0^{-1} = 2 - \alpha_0$. We put $\tilde{\beta}_i = \beta_i\alpha_0$. Then $\tilde{\beta}_0 = 1$ and $\sum_{k=0}^{i} \alpha_{i-k}^{p^k}\tilde{\beta}_k = 0$ for any positive integer i by (iii). It suffices to show the following.

$$\tilde{\beta}_n = \sum_{\nu \in \mathrm{Part}(n)} (-1)^{\ell(\nu)} \prod_{j=1}^{\ell(\nu)} \alpha_{\nu(j)}^{p^{\sigma(\nu)(j)}}$$

Since $\alpha_0 = 1 + (\alpha_0 - 1)$ and $(\alpha_0-1)^2 = 0$, we have $\alpha_0^p = 1$. Then, $\alpha_1\tilde{\beta}_0 + \alpha_0^p\tilde{\beta}_1 = 0$ implies $\tilde{\beta}_1 = -\alpha_1$ and the assertion holds for $n = 1$. Suppose that assertion holds for $1 \leqq n \leqq m-1$. We consider the map $F : \bigcup_{k=1}^{m-1} \mathrm{Part}(k) \to \mathrm{Part}(m)$ in (8.7.5). For $\nu \in \mathrm{Part}(k)$, we have $\sigma(F(\nu))(j) = \sigma(\nu)(j)$ if $1 \leqq j \leqq \ell(\nu)$ and $\sigma(F(\nu))(\ell(\nu)+1) = k$. Hence it follows from the inductive hypothesis and (8.7.5) that

$$\begin{aligned}
\tilde{\beta}_m &= -\sum_{k=0}^{m-1} \alpha_{m-k}^{p^k}\tilde{\beta}_k = -\alpha_m\tilde{\beta}_0 - \sum_{k=1}^{m-1} \alpha_{m-k}^{p^k} \sum_{\nu \in \mathrm{Part}(k)} (-1)^{\ell(\nu)} \prod_{j=1}^{\ell(\nu)} \alpha_{\nu(j)}^{p^{\sigma(\nu)(j)}} \\
&= -\alpha_m + \sum_{k=1}^{m-1} \sum_{\nu \in \mathrm{Part}(k)} (-1)^{\ell(F(\nu))} \alpha_{F(\nu)(\ell(\nu)+1)}^{p^{\sigma(F(\nu))(\ell(\nu)+1)}} \prod_{j=1}^{\ell(\nu)} \alpha_{F(\nu)(j)}^{p^{\sigma(F(\nu))(j)}} \\
&= -\alpha_m + \sum_{k=1}^{m-1} \sum_{\nu \in \mathrm{Part}(k)} (-1)^{\ell(F(\nu))} \prod_{j=1}^{\ell(\nu)+1} \alpha_{F(\nu)(j)}^{p^{\sigma(F(\nu))(j)}} = \sum_{\nu \in \mathrm{Part}(m)} (-1)^{\ell(\nu)} \prod_{j=1}^{\ell(\nu)} \alpha_{\nu(j)}^{p^{\sigma(\nu)(j)}}.
\end{aligned}$$

□

The next result is a direct consequence of (8.7.6) and the above equality.

Proposition 8.7.7 *The inverse of* $\alpha(X) = \sum_{i=0}^{\infty} \alpha_i X^{p^i} \in G_p(A^*)$ *is given as follows.*

$$\alpha(X)^{-1} = \alpha_0^{-1}X + \sum_{i=1}^{\infty} \alpha_0^{-1}\left(\sum_{\nu\in\mathrm{Part}(i)} (-1)^{\ell(\nu)} \prod_{j=1}^{\ell(\nu)} \alpha_{\nu(j)}^{p^{\sigma(\nu)(j)}}\right) X^{p^i}$$

Remark 8.7.8 *Suppose that* p *is an odd prime and that* $\alpha_i = \alpha_{0i} + \alpha_{1i}\epsilon$ *for* $\alpha_{0i} \in A_{2(p^i-1)}$*,* $\alpha_{1i} \in A_{2p^i-1}$*. For* $\nu \in \mathrm{Part}(i)$*, we have*

$$\prod_{j=1}^{\ell(\nu)} \alpha_{\nu(j)}^{p^{\sigma(\nu)(j)}} = \left(\alpha_{0\nu(1)} + \alpha_{1\nu(1)}\epsilon\right) \prod_{j=2}^{\ell(\nu)} \alpha_{0\nu(j)}^{p^{\sigma(\nu)(j)}} = \prod_{j=1}^{\ell(\nu)} \alpha_{0\nu(j)}^{p^{\sigma(\nu)(j)}} + \left(\alpha_{1\nu(1)} \prod_{j=2}^{\ell(\nu)} \alpha_{0\nu(j)}^{p^{\sigma(\nu)(j)}}\right)\epsilon.$$

Hence we have the following formula.

$$\alpha(X)^{-1} = (1-\alpha_{10}\epsilon)X + \sum_{i=1}^{\infty}(1-\alpha_{10}\epsilon)\left(\sum_{\nu\in\mathrm{Part}(i)} (-1)^{\ell(\nu)}\left(\prod_{j=1}^{\ell(\nu)} \alpha_{0\nu(j)}^{p^{\sigma(\nu)(j)}} + \left(\alpha_{1\nu(1)} \prod_{j=2}^{\ell(\nu)} \alpha_{0\nu(j)}^{p^{\sigma(\nu)(j)}}\right)\epsilon\right)\right) X^{p^i}$$

We define "quotient groups" G_p^k of G_p as follows. If $p=2$, we assign degree -1 to a variable X and define $G_2^k(A^*)$ to be the following subset of $A^*[X]/(X^{2^{k+1}})$.

$$\left\{\alpha(X) \in A^*[X]/(X^{2^{k+1}}) \,\middle|\, \alpha(X) = \sum_{i=0}^{k} \alpha_i X^{2^i},\ \deg\alpha_i = 2^i - 1\,(0 \leqq i \leqq k),\ \alpha_0 = 1\right\}$$

If p is an odd prime, we assign degree -2 to a variable X and consider a graded exterior algebra $\boldsymbol{F}_p[\epsilon]/(\epsilon^2)$ with $\deg\epsilon = -1$. Define $G_p^k(A^*)$ to be the following subset of $A^*\otimes_{\boldsymbol{F}_p}\boldsymbol{F}_p[\epsilon]/(\epsilon^2)[X]/(X^{p^{k+1}}) = A^*[\epsilon]/(\epsilon^2)[X]/(X^{p^{k+1}})$.

$$\left\{\alpha(X) \in A^*[\epsilon]/(\epsilon^2)[X]/(X^{p^{k+1}}) \,\middle|\, \alpha(X) = \sum_{i=0}^{k} \alpha_i X^{p^i}, \deg\alpha_i = 2(p^i-1)\,(0 \leqq i \leqq k),\ \alpha_0 - 1 \in (\epsilon)\right\}$$

We give a group structure to $G_p^k(A^*)$ by the composition of stunted polynomials. Namely,

$$\alpha(X)\cdot\beta(X) = \beta(\alpha(X)) = \sum_{i=0}^{k} \beta_i\alpha(X)^{p^i} = \sum_{i=0}^{k}\beta_i\left(\sum_{j=0}^{k}\alpha_j X^{p^j}\right)^{p^i} = \sum_{i=0}^{k}\sum_{j=0}^{k}\alpha_j^{p^i}\beta_i X^{p^{i+j}} = \sum_{i=0}^{k}\left(\sum_{l=0}^{i}\alpha_{i-l}^{p^l}\beta_l\right)X^{p^i}.$$

Define maps $\pi_{A^*}^k : G_p(A^*) \to G_p^k(A^*)$ to be the restrictions of the quotient maps $A^*[[X]] \to A^*[X]/(X^{2^{k+1}})$ if $p=2$ and $A^*[\epsilon]/(\epsilon^2)[[X]] \to (A^*[\epsilon]/(\epsilon^2)[X])/(X^{p^{k+1}})$ if p is an odd prime. It is clear that $\pi_{A^*}^k$ is a homomorphism of groups and natural in A^*. We denote by $G_p^{(k)}(A^*)$ the kernel of $\pi_{A^*}^k$, that is,

$$G_p^{(k)}(A^*) = \left\{\alpha(X) \in G_p(A^*) \,\middle|\, \alpha(X) = X + \sum_{i=k+1}^{\infty}\alpha_i X^{p^i}\right\}.$$

We regard A^* as a subalgebra of $A^*[\epsilon]/(\epsilon^2)$ and define a subset $G_p^{k+0.5}(A^*)$ of $G_p^{k+1}(A^*)$ by

$$G_p^{k+0.5}(A^*) = \left\{\alpha(X) \in G_p^{k+1}(A^*) \,\middle|\, \alpha(X) = \sum_{i=0}^{k+1}\alpha_i X^{p^i},\ \alpha_{k+1} \in A^{2p^{k+1}-2}\right\}.$$

Let $\rho_{A^*} : A^*[\epsilon]/(\epsilon^2) \to A^*[\epsilon]/(\epsilon) = A^*$ be the quotient map and define a map $\rho_{A^*}^k : G_p^{k+1}(A^*) \to G_p^{k+0.5}(A^*)$ by $\rho_{A^*}^k(\alpha(X)) = \sum_{i=0}^{k}\alpha_i X^{p^i} + \rho_{A^*}(\alpha_{k+1})X^{p^{k+1}}$ if $\alpha(X) = \sum_{i=0}^{k+1}\alpha_i X^{p^i}$. For $\alpha(X), \beta(X) \in G_p^{k+0.5}(A^*)$, we set

$\alpha(X){*}\beta(X) = \rho^{k+1}_{A^*}(\alpha(X)\cdot\beta(X))$. Then, the correspondence $(\alpha(X),\beta(X)) \mapsto \alpha(X){*}\beta(X)$ defines a group structure on $G^{k+0.5}_p(A^*)$. In fact, the inverse of $\alpha(X)$ is $\rho^{k+1}_{A^*}(\alpha(X)^{-1})$. We note that $\rho^k_{A^*}$ is a homomorphism of groups. Let us denote by $G^{(k+0.5)}_p(A^*)$ the kernel of a composition $G_p(A^*) \xrightarrow{\pi^{k+1}_{A^*}} G^{k+1}_p(A^*) \xrightarrow{\rho^k_{A^*}} G^{k+0.5}_p(A^*)$. Then, we have $G^{(k+0.5)}_p(A^*) = \left\{\alpha(X)\in G_p(A^*)\,\middle|\,\alpha(X) = X + \sum\limits_{i=k+1}^{\infty}\alpha_i X^{p^i}, \alpha_{k+1}\in(\epsilon)\right\}$. Here we put $\epsilon = 0$ if $p=2$. Then, $G_2(A^*) = G^{(0)}_2(A^*)$, $G^{(k+0.5)}_2(A^*) = G^{(k+1)}_2(A^*)$ and we have a decreasing filtration of $G_p(A^*)$.

$$G_p(A^*) \supset G^{(0)}_p(A^*) \supset G^{(0.5)}_p(A^*) \supset G^{(1)}_p(A^*) \supset \cdots \supset G^{(k)}_p(A^*) \supset G^{(k+0.5)}_p(A^*) \supset G^{(k+1)}_p(A^*) \supset \cdots$$

Lemma 8.7.9 *Suppose that* $\alpha(X) = \sum\limits_{i=0}^{\infty}\alpha_i X^{p^i}, \beta(X) = \sum\limits_{i=0}^{\infty}\beta_i X^{p^i} \in G_p(A^*)$ *satisfy* $\alpha_i = 0$ *for* $i = 1,2,\dots,k$ *and* $\beta_i = 0$ *for* $i=1,2,\dots,l$, *respectively. We put* $\beta(X)^{-1} = \sum\limits_{i=0}^{\infty}\bar\beta_i X^{p^i}$.

(1) *If* $k = l = 0$, *the following equality holds.*

$$\begin{aligned}[\alpha(X),\beta(X)] = X &+ (\alpha_1(\beta_0-1) + (1-\alpha_0)\beta_1)X^p \\ &+ ((\alpha_2-\alpha_1^{p+1})(\beta_0-1) + (1-\alpha_0)(\beta_2-\beta_1^{p+1}) + \alpha_0\alpha_1^p\beta_1 - \alpha_1\beta_0\beta_1^p)X^{p^2} + (\textit{higher terms})\end{aligned}$$

(2) *If* $k \geqq 1$ *and* $l = 0$, *the following equality holds.*

$$\begin{aligned}[\alpha(X),\beta(X)] = X &+ (\alpha_{k+1}(\beta_0-1) - (1-\alpha_0)\beta_0\bar\beta_{k+1})X^{p^{k+1}} \\ &+ \left(\alpha_{k+2}(\beta_0-1) - (1-\alpha_0)\beta_0\bar\beta_{k+2} + \alpha_0\alpha_{k+1}^p\beta_1 - \alpha_{k+1}\beta_0\beta_1^{p^{k+1}}\right)X^{p^{k+2}} + (\textit{higher terms})\end{aligned}$$

(3) *If* $k \geqq l \geqq 1$, *the following equality holds.*

$$\begin{aligned}[\alpha(X),\beta(X)] = X &+ (\alpha_{k+1}(\beta_0-1) - (1-\alpha_0)\beta_0\bar\beta_{k+1})X^{p^{k+1}} + (\alpha_{k+2}(\beta_0-1) - (1-\alpha_0)\beta_0\bar\beta_{k+2})X^{p^{k+2}} \\ &+ (\textit{higher terms})\end{aligned}$$

Proof. Since $\alpha_0^p = \beta_0^p = 1$ and $\alpha(X)\cdot\beta(X) = \sum\limits_{i=0}^{\infty}\left(\sum\limits_{j=0}^{i}\alpha_{i-j}^{p^j}\beta_j\right)X^{p^i}$, we have the following equality.

$$\alpha(X)\cdot\beta(X) = \alpha_0\beta_0 X + \sum_{i=1}^{k+1}(\alpha_i\beta_0+\beta_i)X^{p^i} + (\alpha_{k+2}\beta_0 + \alpha_{k+1}^p\beta_1 + \beta_{k+2})X^{p^{k+2}} + (\text{higher terms})$$

Hence if we put $\alpha(X)\cdot\beta(X) = \sum\limits_{i=0}^{\infty}\gamma_i X^{p^i}$, γ_i's are given by $\gamma_0 = \alpha_0\beta_0$, $\gamma_i = \beta_i$ for $1 \leqq i \leqq k$, $\gamma_{k+1} = \alpha_{k+1}\beta_0+\beta_{k+1}$ and $\gamma_{k+2} = \alpha_{k+2}\beta_0 + \alpha_{k+1}^p\beta_1 + \beta_{k+2}$.

Put $\alpha(X)^{-1} = \sum\limits_{i=0}^{\infty}\bar\alpha_i X^{p^i}$. If $1 \leqq i \leqq k$ and $\nu \in \mathrm{Part}(i)$, then $\nu(j) \leqq k$ for any $1 \leqq j \leqq \ell(\nu)$. Since $\alpha_i = 0$ for $1 \leqq i \leqq k$, we have $\bar\alpha_i = 0$ for $1 \leqq i \leqq k$ by (8.7.7). If $\nu \in \mathrm{Part}(k+1)$ satisfies $\nu(j) \geqq k+1$ for $1 \leqq j \leqq \ell(\nu)$, then $\nu = (k+1)$, hence (8.7.7) implies $\bar\alpha_{k+1} = -\alpha_0^{-1}\alpha_{k+1}$. If $\nu \in \mathrm{Part}(k+2)$ satisfies $\nu(j) \geqq k+1$ for $1 \leqq j \leqq \ell(\nu)$, then $\nu = (k+2)$ or "$\nu = (1,1)$ and $k=0$". It follows from (8.7.7) that $\bar\alpha_{k+2} = \begin{cases}\alpha_0^{-1}(\alpha_1^{p+1}-\alpha_2) & k=0\\ -\alpha_0^{-1}\alpha_{k+2} & k \geqq 1\end{cases}$ holds. Similarly, we have $\bar\beta_1 = -\beta_0^{-1}\beta_1$ and $\bar\beta_2 = \beta_0^{-1}(\beta_1^{p+1}-\beta_2)$ if $l=0$. Hence, if we put $\alpha(X)^{-1}\cdot\beta(X)^{-1} = \sum\limits_{i=0}^{\infty}\bar\gamma_i X^{p^i}$, then $\bar\gamma_i$'s are given by $\bar\gamma_0 = \alpha_0^{-1}\beta_0^{-1}$, $\bar\gamma_i = \bar\beta_i$ for $1 \leqq i \leqq k$ and

$$\begin{aligned}\bar\gamma_{k+1} &= \bar\alpha_{k+1}\beta_0^{-1} + \bar\beta_{k+1} = \bar\beta_{k+1} - \alpha_0^{-1}\alpha_{k+1}\beta_0^{-1}\\ \bar\gamma_{k+2} &= \bar\alpha_{k+2}\beta_0^{-1} + \bar\alpha_{k+1}^p\bar\beta_1 + \bar\beta_{k+2} = \begin{cases}\beta_0^{-1}(\alpha_0^{-1}(\alpha_1^{p+1}-\alpha_2) + \alpha_1^p\beta_1 + \beta_1^{p+1} - \beta_2) & k=l=0\\ -\alpha_0^{-1}\alpha_{k+2}\beta_0^{-1} + \alpha_{k+1}^p\beta_0^{-1}\beta_1 + \bar\beta_{k+2} & k \geqq 1,\, l=0\\ -\alpha_0^{-1}\alpha_{k+2}\beta_0^{-1} + \bar\beta_{k+2} & k \geqq l \geqq 1\end{cases}.\end{aligned}$$

It follows that $\bar{\gamma}_0\gamma_0 = \alpha_0^{-1}\beta_0^{-1}\alpha_0\beta_0 = 1$ and $\sum_{j=0}^{i} \bar{\gamma}_{i-j}^{p^j}\gamma_j = \sum_{j=0}^{i} \bar{\beta}_{i-j}^{p^j}\beta_j = 0$ if $1 \leqq i \leqq k$. We also have

$$\begin{aligned}\sum_{j=0}^{k+1} \bar{\gamma}_{k+1-j}^{p^j}\gamma_j &= \bar{\gamma}_{k+1}\gamma_0 + \sum_{j=1}^{k} \bar{\gamma}_{k+1-j}^{p^j}\gamma_j + \bar{\gamma}_0^{p^{k+1}}\gamma_{k+1} = \alpha_{k+1}(\beta_0 - 1) - (1-\alpha_0)\beta_0\bar{\beta}_{k+1} + \sum_{j=0}^{k+1} \bar{\beta}_{k+1-j}^{p^j}\beta_j \\ &= \alpha_{k+1}(\beta_0 - 1) - (1-\alpha_0)\beta_0\bar{\beta}_{k+1}\end{aligned}$$

If $k = l = 0$, we have

$$\sum_{j=0}^{2} \bar{\gamma}_{2-j}^{p^j}\gamma_j = \bar{\gamma}_2\gamma_0 + \bar{\gamma}_1^p\gamma_1 + \bar{\gamma}_0^{p^2}\gamma_2 = (\alpha_2 - \alpha_1^{p+1})(\beta_0 - 1) + (1-\alpha_0)(\beta_2 - \beta_1^{p+1}) + \alpha_0\alpha_1^p\beta_1 - \alpha_1\beta_0\beta_1^p.$$

If $k \geqq 1$ and $l = 0$, the following equalities hold.

$$\begin{aligned}\sum_{j=0}^{k+2} \bar{\gamma}_{k+2-j}^{p^j}\gamma_j &= \bar{\gamma}_{k+2}\gamma_0 + \bar{\gamma}_{k+1}^p\gamma_1 + \sum_{j=2}^{k} \bar{\gamma}_{k+2-j}^{p^j}\gamma_j + \bar{\gamma}_1^{p^{k+1}}\gamma_{k+1} + \bar{\gamma}_0^{p^{k+2}}\gamma_{k+2} \\ &= \alpha_{k+2}(\beta_0 - 1) - (1-\alpha_0)\beta_0\bar{\beta}_{k+2} + \alpha_0\alpha_{k+1}^p\beta_1 - \alpha_{k+1}\beta_0\beta_1^{p^{k+1}} + \sum_{j=0}^{k+2} \bar{\beta}_{k+2-j}^{p^j}\beta_j \\ &= \alpha_{k+2}(\beta_0 - 1) - (1-\alpha_0)\beta_0\bar{\beta}_{k+2} + \alpha_0\alpha_{k+1}^p\beta_1 - \alpha_{k+1}\beta_0\beta_1^{p^{k+1}}\end{aligned}$$

If $k \geqq l \geqq 1$, the following equalities hold.

$$\begin{aligned}\sum_{j=0}^{k+2} \bar{\gamma}_{k+2-j}^{p^j}\gamma_j &= \bar{\gamma}_{k+2}\gamma_0 + \bar{\gamma}_{k+1}^p\gamma_1 + \sum_{j=2}^{k} \bar{\gamma}_{k+2-j}^{p^j}\gamma_j + \bar{\gamma}_1^{p^{k+1}}\gamma_{k+1} + \bar{\gamma}_0^{p^{k+2}}\gamma_{k+2} \\ &= \alpha_{k+2}(\beta_0 - 1) - (1-\alpha_0)\beta_0\bar{\beta}_{k+2} + \sum_{j=0}^{k+2} \bar{\beta}_{k+2-j}^{p^j}\beta_j = \alpha_{k+2}(\beta_0 - 1) - (1-\alpha_0)\beta_0\bar{\beta}_{k+2}\end{aligned}$$

□

Proposition 8.7.10 *The following relations hold.*

$$\begin{aligned}[\,G_p(A^*), G_p(A^*)\,] &\subset G_p^{(0.5)}(A^*) \\ [\,G_p^{(0.5)}(A^*), G_p^{(0.5)}(A^*)\,] &\subset G_p^{(2)}(A^*) \\ [\,G_p^{(k)}(A^*), G_p^{(k)}(A^*)\,] &\subset G_p^{(k+2)}(A^*) \quad \textit{if } k \textit{ is a positive integer.} \\ [\,G_p^{(k)}(A^*), G_p(A^*)\,] &\subset G_p^{(k+0.5)}(A^*) \quad \textit{if } k \textit{ is a non-negative integer.} \\ [\,G_p^{(k+0.5)}(A^*), G_p(A^*)\,] &\subset G_p^{(k+1.5)}(A^*) \quad \textit{if } k \textit{ is a non-negative integer.}\end{aligned}$$

Proof. The first and second relations are direct consequence of (1) of (8.7.9). The third relation follows from (3) of (8.7.9). For $\alpha(X) = X + \sum_{i=k+1}^{\infty} \alpha_i X^{p^i} \in G_p^{(k)}(A^*)$ and $\beta(X) = \sum_{i=0}^{\infty} \beta_i X^{p^i} \in G_p(A^*)$, since $\beta_0 - 1 \in (\epsilon)$, the fourth relation follows from (2) of (8.7.9). If $\alpha(X) \in G_p^{(k+0.5)}(A^*)$, then $\alpha_{k+1} \in (\epsilon)$ which implies that $\alpha_{k+1}(\beta_0 - 1) = 0$ and $\alpha_{k+2}(\beta_0 - 1) + \alpha_{k+1}^p\beta_1 - \beta_0\alpha_{k+1}\beta_1^{p^{k+1}} \in (\epsilon)$. Hence the fifth relation also follows from (2) of (8.7.9). □

For a group G and a non-negative integer k, we define subgroups $D_k(G)$ and $\Gamma_i(G)$ of G inductively by $D_0(G) = \Gamma_0(G) = G$ and $D_{k+1}(G) = [\,D_k(G), D_k(G)\,]$, $\Gamma_{k+1}(G) = [\,\Gamma_k(G), G\,]$. The following result is a direct consequence of (8.7.10)

Corollary 8.7.11 *We have* $D_1(G_p(A^*)) = \Gamma_1(G_p(A^*)) \subset G_p^{(0.5)}(A^*)$. *For positive integer k, the following relations hold.*

$$D_{k+1}(G_p(A^*)) \subset G_p^{(2k)}(A^*), \qquad \Gamma_{k+1}(G_p(A^*)) \subset G_p^{(k+0.5)}(A^*)$$

Remark 8.7.12 *If H is a subgroup of $G_p(A^*)$, we have $D_1(H) = \Gamma_1(H) \subset G_p^{(0.5)}(A^*) \cap H$ and $D_{k+1}(H) \subset G_p^{(2k)}(A^*) \cap H$, $\Gamma_{k+1}(H) \subset G_p^{(k+0.5)}(A^*) \cap H$. Since $G_p^{(k+0.5)}(A^*) \cap G_p^{ev}(A^*) \subset G_p^{(k+1)}(A^*)$, we have $\Gamma_{k+1}(H) \subset G_p^{(k+1)}(A^*) \cap H$ if H is a subgroup of $G_p^{ev}(A^*)$.*

We define another filtration of G_p as follows. For a non-negative integer n and a graded commutative algebra A^* over $\boldsymbol{F}_p$, let $G_{p,n}(A^*)$ be a subset of $G_p(A^*)$ consisting of elements $\alpha(X) = \sum_{i=0}^{\infty} \alpha_i X^{p^i}$ which satisfy $\alpha_i^{p^{n-i+1}} = 0$ for $i = 1, 2, \dots, n$ and $\alpha_i = 0$ for $i \geqq n+1$.

Proposition 8.7.13 *$G_{p,n}(A^*)$ is a subgroup of $G_p(A^*)$.*

Proof. Suppose $\alpha(X) = \sum_{i=0}^{\infty} \alpha_i X^{p^i}, \beta(X) = \sum_{i=0}^{\infty} \beta_i X^{p^i} \in G_{p,n}(A^*)$. Put $\alpha(X) \cdot \beta(X) = \sum_{i=0}^{\infty} \gamma_i X^{p^i}$, then we have $\gamma_i = \sum_{j=0}^{i} \alpha_{i-j}^{p^j} \beta_j$. Since $\alpha_k^{p^{n-k+1}} = \beta_k^{p^{n-k+1}} = 0$ for $k = 1, 2, \dots, n$, it follows $\gamma_i^{p^{n-i+1}} = \sum_{j=0}^{i} \alpha_{i-j}^{p^{n-(i-j)+1}} \beta_j^{p^{n-i+1}} = 0$ if $1 \leqq i \leqq n$. Assume that $i \geqq n+1$. Since $\alpha_{i-j}^{p^j} = \alpha_{i-j}^{p^{n-(i-j)+1+(i-n-1)}} = 0$ for $i - n \leqq j \leqq n$, we have $\gamma_i = \sum_{j=i-n}^{n} \alpha_{i-j}^{p^j} \beta_j = 0$. Thus $\alpha(X) \cdot \beta(X) \in G_{p,n}(A^*)$.

We put $\alpha(X)^{-1} = \alpha_0^{-1} X + \sum_{i=1}^{\infty} \delta_i X^{p^i}$. Then, $\delta_i = \alpha_0^{-1} \left(\sum_{\nu \in \mathrm{Part}(i)} (-1)^{\ell(\nu)} \prod_{j=1}^{\ell(\nu)} \alpha_{\nu(j)}^{p^{\sigma(\nu)(j)}} \right)$ by (8.7.7). Suppose that $\prod_{j=1}^{\ell(\nu)} \alpha_{\nu(j)}^{p^{\sigma(\nu)(j)+n-i+1}} \neq 0$ for some $1 \leqq i \leqq n$ and $\nu \in \mathrm{Part}(i)$. Then, $\sigma(\nu)(j) + n - i + 1 \leqq n - \nu(j)$ for $1 \leqq j \leqq \ell(\nu)$, which implies a contradiction $n + 1 \leqq n$ if $j = \ell(\nu)$. Hence we have $\delta_i^{n-i+1} = 0$ for $1 \leqq i \leqq n$. Suppose that $\prod_{j=1}^{\ell(\nu)} \alpha_{\nu(j)}^{p^{\sigma(\nu)(j)}} \neq 0$ for some $i \geqq n+1$ and $\nu \in \mathrm{Part}(i)$. Then, $\nu(j) \leqq n$ and $\sigma(\nu)(j) \leqq n - \nu(j)$ for $1 \leqq j \leqq \ell(\nu)$. The latter inequality implies $i \leqq n$ if $j = \ell(\nu)$, which contradicts the assumption. Hence we have $\delta_i = 0$ for $i \geqq n+1$ and $\alpha(X)^{-1} \in G_{p,n}(A^*)$. □

Thus we have the following increasing filtration of subgroups of $G_p(A^*)$.

$$G_{p,0}(A^*) \subset G_{p,1}(A^*) \subset G_{p,2}(A^*) \subset \cdots \subset G_{p,n}(A^*) \subset G_{p,n+1}(A^*) \subset \cdots \subset G_p(A^*)$$

If p is an odd prime, we define $G_{p,n}^{ev}(A^*)$ by $G_{p,n}^{ev}(A^*) = G_{p,n}(A^*) \cap G_p^{ev}(A^*)$, we have the following increasing filtration of subgroups of $G_p^{ev}(A^*)$.

$$G_{p,0}^{ev}(A^*) \subset G_{p,1}^{ev}(A^*) \subset G_{p,2}^{ev}(A^*) \subset \cdots \subset G_{p,n}^{ev}(A^*) \subset G_{p,n+1}^{ev}(A^*) \subset \cdots \subset G_p^{ev}(A^*)$$

We note that $G_{2,0}(A^*)$ and $G_{p,0}^{ev}(A^*)$ are the trivial groups and that $G_{p,0}(A^*)$ is isomorphic to the additive group A^1. Since $G_p^{(n)}(A^*) \cap G_{p,n}(A^*)$ is the trivial group, (8.7.12) implies the following fact.

Proposition 8.7.14 *We have the following lower central series.*

$$G_{p,n}(A^*) \supset \Gamma_1(G_{p,n}(A^*)) \supset \cdots \supset \Gamma_i(G_{p,n}(A^*)) \supset \Gamma_{i+1}(G_{p,n}(A^*)) \supset \cdots \supset \Gamma_{n+1}(G_{p,n}(A^*)) = \{X\}$$
$$G_{p,n}^{ev}(A^*) \supset \Gamma_1(G_{p,n}^{ev}(A^*)) \supset \cdots \supset \Gamma_i(G_{p,n}^{ev}(A^*)) \supset \Gamma_{i+1}^{ev}(G_{p,n}(A^*)) \supset \cdots \supset \Gamma_n(G_{p,n}^{ev}(A^*)) = \{X\}$$

Let $I_{2,n}$ be an ideal of $\mathcal{A}_{2*}$ generated by $\zeta_1^{2^n}, \zeta_2^{2^{n-1}}, \dots, \zeta_n^2$ and ζ_i for $i \geqq n+1$. For an odd prime p, let $I_{p,n}$ be an ideal of $\mathcal{A}_{p*}$ generated by $\xi_1^{p^n}, \xi_2^{p^{n-1}}, \dots, \xi_n^p$ and τ_i, ξ_i for $i \geqq n+1$ and $I_{p,n}^{ev}$ an ideal of $\mathcal{A}_{p*}^{ev}$ generated by $\xi_1^{p^n}, \xi_2^{p^{n-1}}, \dots, \xi_n^p$ and ξ_i for $i \geqq n+1$. We put

$$\begin{aligned}
\mathcal{A}_2(n)_* &= \mathcal{A}_{2*}/I_{2,n} = \boldsymbol{F}_2[\zeta_1, \zeta_2, \dots, \zeta_n]/(\zeta_1^{2^n}, \zeta_2^{2^{n-1}}, \dots, \zeta_n^2) \\
\mathcal{A}_p(n)_* &= \mathcal{A}_{p*}/I_{p,n} = E(\tau_0, \tau_1, \dots, \tau_n) \otimes_{\boldsymbol{F}_p} \boldsymbol{F}_p[\xi_1, \xi_2, \dots, \xi_n]/(\xi_1^{p^n}, \xi_2^{p^{n-1}}, \dots, \xi_n^p) \\
\mathcal{A}_p^{ev}(n)_* &= \mathcal{A}_{p*}^{ev}/I_{p,n}^{ev} = \boldsymbol{F}_p[\xi_1, \xi_2, \dots, \xi_n]/(\xi_1^{p^n}, \xi_2^{p^{n-1}}, \dots, \xi_n^p).
\end{aligned}$$

We have the following fact from (8.7.3).

Proposition 8.7.15 *$G_{p,n}$ is represented by $\mathcal{A}_p(n)_*$ and $G^{ev}_{p,n}$ is represented by $\mathcal{A}^{ev}_p(n)_*$.*

Let $\mathcal{H}_2(n)_*$ be the subalgebra of $\mathcal{A}_2(n)_*$ generated by $\zeta_1^{2^{n-1}}, \zeta_2^{2^{n-2}}, \dots, \zeta_n$. For an odd prime p, let $\mathcal{H}_p(n)_*$ be the subalgebra of $\mathcal{A}_p(n)_*$ generated by $\xi_1^{p^{n-1}}, \xi_2^{p^{n-2}}, \dots, \xi_n, \tau_n$ and $\mathcal{H}^{ev}_p(n)_*$ the subalgebra of $\mathcal{A}^{ev}_p(n)_*$ generated by $\xi_1^{p^{n-1}}, \xi_2^{p^{n-2}}, \dots, \xi_n$. We denote by $\pi_{p,n} : \mathcal{A}_p(n)_* \to \mathcal{A}_p(n-1)_*$ and $\pi^{ev}_n : \mathcal{A}^{ev}_p(n)_* \to \mathcal{A}^{ev}_p(n-1)_*$ the quotient maps. Then, the kernel of $\pi_{2,n}$ is the ideal of $\mathcal{A}_2(n)_*$ generated by $\zeta_1^{2^{n-1}}, \zeta_2^{2^{n-2}}, \dots, \zeta_n$. If p is an odd prime, the kernels of $\pi_{p,n}$ and π^{ev}_n are the ideals of $\mathcal{A}_p(n)_*$ and $\mathcal{A}^{ev}_p(n)_*$ generated by $\xi_1^{p^{n-1}}, \xi_2^{p^{n-2}}, \dots, \xi_n, \tau_n$ and $\xi_1^{p^{n-1}}, \xi_2^{p^{n-2}}, \dots, \xi_n$, respectively. We put $\zeta_{n,i} = \zeta_i^{2^{n-i}}$ and $\xi_{n,i} = \xi_i^{p^{n-i}}$. Then we have

$$\begin{aligned}
\mathcal{H}_2(n)_* &= \boldsymbol{F}_2[\zeta_{n,1}, \zeta_{n,2}, \cdots, \zeta_{n,n}]/(\zeta_{n,1}^2, \zeta_{n,2}^2, \cdots, \zeta_{n,n}^2) \\
\mathcal{H}_p(n)_* &= E(\tau_n) \otimes_{\boldsymbol{F}_p} \boldsymbol{F}_p[\xi_{n,1}, \xi_{n,2}, \cdots, \xi_{n,n}]/(\xi_{n,1}^p, \xi_{n,2}^p, \cdots, \xi_{n,n}^p) \\
\mathcal{H}^{ev}_p(n)_* &= \boldsymbol{F}_p[\xi_{n,1}, \xi_{n,2}, \cdots, \xi_{n,n}]/(\xi_{n,1}^p, \xi_{n,2}^p, \cdots, \xi_{n,n}^p).
\end{aligned}$$

Proposition 8.7.16 *The inclusion maps $\iota_{p,n} : \mathcal{H}_p(n)_* \to \mathcal{A}_p(n)_*$ and $\iota^{ev}_{p,n} : \mathcal{H}^{ev}_p(n)_* \to \mathcal{A}^{ev}_p(n)_*$ are faithfully flat.*

Proof. Let $M^*_{2,i}$ be a 2^{n-i} dimensional subspace of $\mathcal{A}_2(n)_*$ spanned by $1, \zeta_i, \zeta_i^2, \dots, \zeta_i^{2^{n-i}-1}$ and $M^*_{p,i}$ a p^{n-i} dimensional subspace of $\mathcal{A}_p(n)_*$ spanned by $1, \xi_i, \xi_i^2, \dots, \xi_i^{p^{n-i}-1}$ if p is an odd prime. Then, the following equalities hold as $\boldsymbol{F}_2\big[\zeta_i^{2^{n-i}}\big]/\big((\zeta_i^{2^{n-i}})^2\big)$-module and $\boldsymbol{F}_p\big[\xi_i^{p^{n-i}}\big]/\big((\xi_i^{p^{n-i}})^p\big)$-module, reapectively.

$$\boldsymbol{F}_2[\zeta_i]/\big(\zeta_i^{2^{n-i+1}}\big) = M^*_{2,i} \otimes_{\boldsymbol{F}_2} \boldsymbol{F}_2\big[\zeta_i^{2^{n-i}}\big]/\big((\zeta_i^{2^{n-i}})^2\big) \qquad \boldsymbol{F}_p[\xi_i]/\big(\xi_i^{p^{n-i+1}}\big) = M^*_{p,i} \otimes_{\boldsymbol{F}_p} \boldsymbol{F}_p\big[\xi_i^{p^{n-i}}\big]/\big((\xi_i^{p^{n-i}})^p\big)$$

Hence we have $\mathcal{A}_2(n)_* = \Big(\bigotimes_{i=1}^n M^*_{2,i}\Big) \otimes_{\boldsymbol{F}_p} \mathcal{H}_2(n)_*$ and $\mathcal{A}_p(n)_* = E(\tau_0, \tau_1, \dots, \tau_{n-1}) \otimes_{\boldsymbol{F}_p} \Big(\bigotimes_{i=1}^n M^*_{p,i}\Big) \otimes_{\boldsymbol{F}_p} \mathcal{H}_p(n)_*$ as $\mathcal{H}_p(n)_*$-modules, which implies that the inclusion map $\iota_{p,n} : \mathcal{H}_p(n)_* \to \mathcal{A}_p(n)_*$ is flat. Since ζ_i's, τ_i's and ξ_i's are all nilpotent in $\mathcal{A}_p(n)_*$, the nilradicals of $\mathcal{A}_p(n)_*$ and $\mathcal{H}_p(n)_*$ are maximal ideals and which are unique prime ideals of $\mathcal{A}_p(n)_*$ and $\mathcal{H}_p(n)_*$, hence the underlying spaces of $\operatorname{Spec}\mathcal{A}_p(n)_*$ and $\operatorname{Spec}\mathcal{H}_p(n)_*$ consist of single point. Therefore $\operatorname{Spec}(\iota_{p,n}) : \operatorname{Spec}\mathcal{A}_p(n)_* \to \operatorname{Spec}\mathcal{H}_p(n)_*$ is surjective and $\iota_{p,n}$ is faithfully flat. It can be shown similarly that $\iota^{ev}_{p,n}$ is faithfully flat. □

Let us denote by $H_{p,n}$ and $H^{ev}_{p,n}$ the affine schemes represented by $\mathcal{H}_p(n)_*$ and $\mathcal{H}^{ev}_p(n)_*$, respectively. Since $\mathcal{H}^{ev}_p(n)_*$ is regarded as a quotient algebra of $\mathcal{H}_p(n)_*$ by the ideal generated by τ_n, $\mathcal{H}^{ev}_p(n)_*$ is regarded as a subscheme of $\mathcal{H}_p(n)_*$. We also denote by $\iota^*_{p,n} : G_{p,n} \to H_{p,n}$ and $\iota^{ev*}_{p,n} : G^{ev}_{p,n} \to H^{ev}_{p,n}$ the morphisms induced by $\iota_{p,n}$ and $\iota^{ev}_{p,n}$.

Proposition 8.7.17 *Let A^* be a graded $\boldsymbol{F}_p$-algebra. For $\alpha(X) = \sum_{i=0}^{\infty} \alpha_i X^{p^i}, \beta(X) = \sum_{i=0}^{\infty} \beta_i X^{p^i} \in G_{p,n}(A^*)$, the following conditions are equivalent.*
(i) $\iota^*_{nA^*}(\alpha(X)) = \iota^*_{nA^*}(\beta(X))$
(ii) $\alpha_i^{p^{n-i}} = \beta_i^{p^{n-i}}$ for $i = 1, 2, \dots, n$.
(iii) $\alpha(X) = \beta(\gamma(X))$ for some $\gamma(X) \in G_{p,n-1}(A^*)$.

Proof. (*i*) $\Leftrightarrow$ (*ii*): Let $f, g : \mathcal{A}_p(n)_* \to A^*$ the morphisms of graded $\boldsymbol{F}_p$-algebras which corresponds $\alpha(X)$ and $\beta(X)$, respectively. Then, $f(\zeta_i) = \alpha_i$, $g(\zeta_i) = \beta_i$ for $i = 1, 2, \dots, n$ if $p = 2$ and $1 + \epsilon f(\tau_0) = \alpha_0$, $1 + \epsilon g(\tau_0) = \beta_0$, $f(\xi_i) + \epsilon f(\tau_i) = \alpha_i$, $g(\xi_i) + \epsilon g(\tau_i) = \beta_i$ for $i = 1, 2, \dots, n$ if p is an odd prime. Hence $f\big(\zeta_i^{2^{n-i}}\big) = \alpha_i^{2^{n-i}}$, $g\big(\zeta_i^{2^{n-i}}\big) = \beta_i^{2^{n-i}}$ for $i = 1, 2, \dots, n$ if $p = 2$ and $f\big(\xi_i^{p^{n-i}}\big) = \alpha_i^{p^{n-i}}$, $g\big(\xi_i^{p^{n-i}}\big) = \beta_i^{p^{n-i}}$ for $i = 1, 2, \dots, n-1$, $f(\xi_n) + \epsilon f(\tau_n) = \alpha_n$, $g(\xi_n) + \epsilon g(\tau_n) = \beta_n$ if p is an odd prime. Hence (*i*) and (*ii*) are equivalent from the definition of $\mathcal{H}_p(n)_*$.

(*iii*) $\Rightarrow$ (*ii*): We put $\gamma(X) = \sum_{i=0}^{\infty} \gamma_i X^{p^i}$. Then $\gamma_i^{p^{n-i}} = 0$ for $i = 1, 2, \dots, n-1$ and $\gamma_i = 0$ for $i \geqq n$. It follows from the assumption that $\alpha_i = \sum_{j=0}^{i} \gamma_{i-j}^{p^j} \beta_j$. Hence $\alpha_i^{p^{n-i}} = \sum_{j=0}^{i} \gamma_{i-j}^{p^{n-(i-j)}} \beta_j^{p^{n-i}} = \gamma_0^{p^n} \beta_i^{p^{n-i}} = \beta_i^{p^{n-i}}$ for $i = 1, 2, \dots, n$.

$(ii) \Rightarrow (iii)$: Put $\gamma(X) = \sum_{i=0}^{\infty} \gamma_i X^{p^i} = \alpha(X) \cdot \beta(X)^{-1}$. Then, $\alpha(X) = \beta(\gamma(X))$ which implies $\alpha_i = \sum_{j=0}^{i} \gamma_{i-j}^{p^j} \beta_j$. Hence $\beta_i^{p^{n-i}} = \alpha_i^{p^{n-i}} = \sum_{j=0}^{i} \gamma_{i-j}^{p^{n-(i-j)}} \beta_j^{p^{n-i}}$ for $i = 1, 2, \ldots, n$. It follows $\gamma_i^{p^{n-i}} \beta_0^{p^{n-i}} + \sum_{j=1}^{i-1} \gamma_{i-j}^{p^{n-(i-j)}} \beta_j^{p^{n-i}} = 0$. For $i = 1$, we have $\gamma_1^{p^{n-1}} \beta_0^{p^{n-1}} = 0$, thus $\gamma_1^{p^{n-1}} = 0$ since β_0 is a unit. Assume inductively that $\gamma_j^{p^{n-j}} = 0$ for $j = 1, 2, \ldots, i-1$. Then we have $\gamma_i^{p^{n-i}} \beta_0^{p^{n-i}} = 0$ which implies $\gamma_i^{p^{n-i}} = 0$. Thus $\gamma(X) \in G_{p,n-1}(A^*)$. □

Remark 8.7.18 *(1) For an odd prime p and $\alpha(X) = \sum_{i=0}^{\infty} \alpha_i X^{p^i}, \beta(X) = \sum_{i=0}^{\infty} \beta_i X^{p^i} \in G_{p,n}^{ev}(A^*)$, It follows from (8.7.17) that three conditions "$\iota_{nA^*}^{ev*}(\alpha(X)) = \iota_{nA^*}^{ev*}(\beta(X))$" and "$\alpha_i^{p^{n-i}} = \beta_i^{p^{n-i}}$ for $i = 1, 2, \ldots, n$." and "$\alpha(X) = \beta(\gamma(X))$ for some $\gamma(X) \in G_{p,n-1}^{ev}(A^*)$." are equivalent.*

(2) The above result shows that there exist unique injections

$$j_{nA^*} : G_{p,n-1}(A^*)\backslash G_{p,n}(A^*) \to H_{p,n}(A^*) \quad \textit{and} \quad j_{nA^*}^{ev} : G_{p,n-1}^{ev}(A^*)\backslash G_{p,n}^{ev}(A^*) \to H_{p,n}^{ev}(A^*)$$

that make the following diagrams commute, where

$$\pi_{nA^*} : G_{p,n}(A^*) \to G_{p,n-1}(A^*)\backslash G_{p,n}(A^*) \quad \textit{and} \quad \pi_{nA^*}^{ev} : G_{p,n}^{ev}(A^*) \to G_{p,n-1}^{ev}(A^*)\backslash G_{p,n}^{ev}(A^*)$$

denote the quotient maps.

$$\begin{tikzcd} G_{p,n}(A^*) \arrow[r, "\iota_{nA^*}^*"] \arrow[d, "\pi_{nA^*}"] & H_{p,n}(A^*) \\ G_{p,n-1}(A^*)\backslash G_{p,n}(A^*) \arrow[ur, "j_{nA^*}"'] & \end{tikzcd} \qquad \begin{tikzcd} G_{p,n}^{ev}(A^*) \arrow[r, "\iota_{nA^*}^{ev*}"] \arrow[d, "\pi_{nA^*}^{ev}"] & H_{p,n}^{ev}(A^*) \\ G_{p,n-1}^{ev}(A^*)\backslash G_{p,n}^{ev}(A^*) \arrow[ur, "j_{nA^*}^{ev}"'] & \end{tikzcd}$$

(3) Since $\mu(\zeta_i) = \sum_{k=0}^{i} \zeta_{i-k}^{2^k} \otimes \zeta_k$, we have $\mu(\zeta_i^{2^{n-i}}) = \sum_{k=0}^{i} \zeta_{i-k}^{2^{n-(i-k)}} \otimes \zeta_k^{2^{n-i}}$. Similarly, since $\mu(\xi_i) = \sum_{k=0}^{i} \xi_{i-k}^{p^k} \otimes \xi_k$, we have $\mu(\xi_i^{p^{n-i}}) = \sum_{k=0}^{i} \xi_{i-k}^{p^{n-(i-k)}} \otimes \xi_k^{p^{n-i}}$. Moreover, we have $\mu(\tau_n) = \sum_{k=0}^{n} \xi_k^{p^{n-k}} \otimes \tau_{n-k} + \tau_n \otimes 1$. Thus the coproduct $\mu : \mathcal{A}_p(n)_* \to \mathcal{A}_p(n)_* \otimes_{\boldsymbol{F}_p} \mathcal{A}_p(n)_*$ defines a right coaction $\tilde{\mu} : \mathcal{H}_p(n)_* \to \mathcal{H}_p(n)_* \otimes_{\boldsymbol{F}_p} \mathcal{A}_p(n)_*$ of $\mathcal{A}_p(n)_*$ on $\mathcal{H}_p(n)_*$. Hence we have a right action $\tilde{\mu}^* : H_{p,n} \times G_{p,n} \to H_{p,n}$ of $G_{p,n}$ on $H_{p,n}$.

For a commutative graded algebra A^* and $k, n \in \boldsymbol{Z}$ $(k > 0)$, we put $J_k(A^*) = \{x \in A^* \,|\, x^k = 0\}$ and $J_k^n(A^*) = A^n \cap J_k(A^*)$. We define maps

$$\begin{aligned} &\Phi_{2,n} : G_{2,n}(A^*) \longrightarrow \prod_{i=1}^{n} J_{2^{n-i+1}}^{2^i-1}(A^*) && \Psi_{2,n} : H_{2,n}(A^*) \longrightarrow \prod_{i=1}^{n} J_2^{2^n-2^{n-i}}(A^*) \\ &\Phi_{p,n} : G_{p,n}(A^*) \longrightarrow \prod_{i=0}^{n} A^{2p^i-1} \times \prod_{i=1}^{n} J_{p^{n-i+1}}^{2(p^i-1)}(A^*) && \Psi_{p,n} : H_{p,n}(A^*) \longrightarrow A^{2p^n-1} \times \prod_{i=1}^{n} J_p^{2(p^n-p^{n-i})}(A^*) \\ &\Phi_{p,n}^{ev} : G_{p,n}^{ev}(A^*) \longrightarrow \prod_{i=1}^{n} J_{p^{n-i+1}}^{2(p^i-1)}(A^*) && \Psi_{p,n}^{ev} : H_{p,n}^{ev}(A^*) \longrightarrow \prod_{i=1}^{n} J_p^{2(p^n-p^{n-i})}(A^*) \end{aligned}$$

as follows.

$$\begin{aligned} &\Phi_{2,n}(f) = (f(\zeta_1), f(\zeta_2), \cdots, f(\zeta_n)) && \Psi_{2,n}(f) = \left(f(\zeta_1^{2^{n-1}}), f(\zeta_2^{2^{n-2}}), \cdots, f(\zeta_n)\right) \\ &\Phi_{p,n}(f) = (f(\tau_0), f(\tau_1), \ldots, f(\tau_n), f(\xi_1), f(\xi_2), \cdots, f(\xi_n)) && \Psi_{p,n}(f) = \left(f(\tau_n), f(\xi_1^{p^{n-1}}), f(\xi_2^{p^{n-2}}), \cdots, f(\xi_n)\right) \\ &\Phi_{p,n}^{ev}(f) = (f(\xi_1), f(\xi_2), \cdots, f(\xi_n)) && \Psi_{p,n}^{ev}(f) = \left(f(\xi_1^{p^{n-1}}), f(\xi_2^{p^{n-2}}), \cdots, f(\xi_n)\right) \end{aligned}$$

Then, $\Phi_{p,n}$ and $\Psi_{p,n}$ are natural in A^*. The following result is clear from the $\boldsymbol{F}_p$-algebra structures of $\mathcal{A}_p(n)_*$, $\mathcal{A}_p^{ev}(n)_*$ and $\mathcal{H}_p(n)_*$.

Proposition 8.7.19 *$\Phi_{p,n}$, $\Psi_{p,n}$, $\Phi_{p,n}^{ev}$, $\Psi_{p,n}^{ev}$ are bijective.*

Remark 8.7.20 *For an $\boldsymbol{F}_p$-algebra A^*, define a map $\rho_p^k : A^* \to A^*$ by $\rho_p^k(x) = x^{p^k}$. Then, ρ_p^k maps $J_{p^m}^n(A^*)$ into $J_{p^{m-k}}^{p^k n}(A^*)$ and the following diagrams are commutative.*

$$\begin{tikzcd}
G_{2,n}(A^*) \arrow[r, "\iota^*_{nA^*}"] \arrow[d, "\Phi_{2,n}"] & H_{2,n}(A^*) \arrow[d, "\Psi_{2,n}"] \\
\prod_{i=1}^{n} J^{2^i-1}_{2^{n-i+1}}(A^*) \arrow[r, "\prod_{i=1}^{n}\rho_2^{n-i}"] & \prod_{i=1}^{n} J_2^{2^n-2^{n-i}}(A^*)
\end{tikzcd}
\qquad
\begin{tikzcd}
G^{ev}_{p,n}(A^*) \arrow[r, "\iota^{ev*}_{nA^*}"] \arrow[d, "\Phi^{ev}_{p,n}"] & H^{ev}_{p,n}(A^*) \arrow[d, "\Psi^{ev}_{p,n}"] \\
\prod_{i=1}^{n} J^{2(p^i-1)}_{p^{n-i+1}}(A^*) \arrow[r, "\prod_{i=1}^{n}\rho_p^{n-i}"] & \prod_{i=1}^{n} J_p^{2(p^n-p^{n-i})}(A^*)
\end{tikzcd}$$

We put $\delta^m_k(A^*) = \dim J^m_k(A^*)$ for a graded $\boldsymbol{F}_p$-algebra A^* which is finite type. For a finite set S, we denote by $\sharp(S)$ the number of elements of S. (8.7.19) implies the following fact.

Corollary 8.7.21 *Let A^* be a graded $\boldsymbol{F}_p$-algebra which is finite type.*

(1) If $p=2$, we have $\log_2(\sharp(G_{2,n}(A^*))) = \sum_{i=1}^{n}\delta^{2^i-1}_{n-i+1}(A^*)$ and $\log_2(\sharp(H_{2,n}(A^*)) = \sum_{i=1}^{n}\delta_1^{2^n-2^{n-i}}(A^*)$.

(2) If p is an odd prime, the following equalities holds.

$$\log_p(\sharp(G_{p,n}(A^*))) = \sum_{i=0}^{n}\dim\ A^{2p^i-1} + \sum_{i=1}^{n}\delta^{2(p^i-1)}_{n-i+1}(A^*)$$

$$\log_p(\sharp(H_{p,n}(A^*))) = \dim\ A^{2p^n-1} + \sum_{i=1}^{n}\delta_1^{2(p^n-p^{n-i})}(A^*)$$

$$\log_p(\sharp(G^{ev}_{p,n}(A^*))) = \sum_{i=1}^{n}\delta^{2(p^i-1)}_{n-i+1}(A^*)$$

$$\log_p(\sharp(H^{ev}_{p,n}(A^*))) = \sum_{i=1}^{n}\delta_1^{2(p^n-p^{n-i})}(A^*)$$

Proposition 8.7.22 *If $l < 2^i$ and $m < 2p^i(p-1)$, then $J^l_{2^{n-i}}(\mathcal{A}_2(n)_*) = J^m_{p^{n-i}}(\mathcal{A}^{ev}_p(n)_*) = \{0\}$ holds for $i = 1, 2, \cdots, n-1$. Hence $G_{2,n-1}(\mathcal{A}_2(n)_*)$ and $G^{ev}_{p,n-1}(\mathcal{A}^{ev}_p(n)_*)$ are trivial groups.*

Proof. It follows from the structure of $\mathcal{A}_2(n)_*$ that $\zeta_1^{j_1}\zeta_2^{j_2}\cdots\zeta_n^{j_n} \in J_{2^{n-i}}(\mathcal{A}_2(n)_*)$ if and only if $j_k \geqq 2^{i-k+1}$ for some $k = 1, 2, \ldots, n$. Hence we have

$$\deg\zeta_1^{j_1}\zeta_2^{j_2}\cdots\zeta_n^{j_n} \geqq \deg\zeta_k^{j_k} = j_k(2^k-1) \geqq 2^{i-k+1}(2^k-1) = 2^{i+1} - 2^{i-k+1} \geqq 2^i.$$

Similarly, $\xi_1^{j_1}\xi_2^{j_2}\cdots\xi_n^{j_n} \in J_{p^{n-i}}(\mathcal{A}^{ev}_p(n)_*)$ if and only if $j_k \geqq p^{i-k+1}$ for some $k = 1, 2, \ldots, n$. Hence we have

$$\deg\xi_1^{j_1}\xi_2^{j_2}\cdots\xi_n^{j_n} \geqq \deg\xi_k^{j_k} = 2j_k(p^k-1) \geqq 2p^{i-k+1}(p^k-1) = 2(p^{i+1}-p^{i-k+1}) \geqq 2p^i(p-1).$$

Thus the first assertion follows. The second assertion follows from the first assertion and (8.7.19). □

We have the following result by (8.7.17) and (8.7.19).

Corollary 8.7.23 *$\iota^*_{n\mathcal{A}_2(n)_*} : G_{2,n}(\mathcal{A}_2(n)_*) \to H_{2,n}(\mathcal{A}_2(n)_*)$ and $\iota^*_{n\mathcal{A}^{ev}_p(n)_*} : G^{ev}_{2,n}(\mathcal{A}^{ev}_p(n)_*) \to H^{ev}_{2,n}(\mathcal{A}^{ev}_p(n)_*)$ are injective.*

Proposition 8.7.24 *The following equalities hold.*

$$\delta^m_k(\mathcal{A}_2(n)_*) = \dim\mathcal{A}_2(n)_m - \dim\mathcal{A}_2(n-k)_m$$
$$\delta^m_k(\mathcal{A}^{ev}_p(n)_*) = \dim\mathcal{A}^{ev}_p(n)_m - \dim\mathcal{A}^{ev}_p(n-k)_m$$
$$\delta^m_k(\mathcal{A}_p(n)_*) = \dim\mathcal{A}_p(n)_m - \dim\mathcal{A}^{ev}_p(n-k)_m$$

Proof. $J_k(\mathcal{A}_2(n)_*)$ and $J_k(\mathcal{A}^{ev}_p(n)_*)$ are ideals of $\mathcal{A}_2(n)_*$ and $\mathcal{A}^{ev}_p(n)_*$ generated by the following sequence of elemements, respectively.

$$\zeta_1^{2^{n-k}}, \zeta_2^{2^{n-k-1}}, \ldots, \zeta_i^{2^{n-i-k+1}}, \ldots, \zeta_{n-k}^2, \zeta_{n-k+1}, \zeta_{n-k+2}, \ldots, \zeta_n$$

$$\xi_1^{p^{n-k}}, \xi_2^{p^{n-k-1}}, \ldots, \xi_i^{p^{n-i-k+1}}, \ldots, \xi_{n-k}^p, \xi_{n-k+1}, \xi_{n-k+2}, \ldots, \xi_n$$

Hence $\mathcal{A}_2(n)_*/J_k(\mathcal{A}_2(n)_*)$ is isomorphic to $\mathcal{A}_2(n-k)_* = \boldsymbol{F}_2[\zeta_1,\zeta_2,\ldots,\zeta_{n-k}]/(\zeta_1^{2^{n-k}},\zeta_2^{2^{n-k-1}},\ldots,\zeta_{n-k}^2)$ and $\mathcal{A}_p^{ev}(n)_*/J_k(\mathcal{A}_p^{ev}(n)_*)$ is isomorphic to $\mathcal{A}_p^{ev}(n-k)_* = \boldsymbol{F}_p[\xi_1,\xi_2,\ldots,\xi_{n-k}]/(\xi_1^{p^{n-k}},\xi_2^{p^{n-k-1}},\ldots,\xi_{n-k}^p)$. It follows that $\dim\mathcal{A}_2(n)_m/J_k^m(\mathcal{A}_2(n)_*) = \dim\mathcal{A}_2(n-k)_m$ and $\dim\mathcal{A}_p^{ev}(n)_m/J_k^m(\mathcal{A}_p^{ev}(n)_*) = \dim\mathcal{A}_p^{ev}(n-k)_m$, which imply $\delta_k^m(\mathcal{A}_2(n)_*) = \dim\mathcal{A}_2(n)_m - \dim\mathcal{A}_2(n-k)_m$ and $\delta_k^m(\mathcal{A}_p^{ev}(n)_*) = \dim\mathcal{A}_p^{ev}(n)_m - \dim\mathcal{A}_p^{ev}(n-k)_m$, respectively. $J_k(\mathcal{A}_p(n)_*)$ is an ideal of $\mathcal{A}_p(n)_*$ generated by

$$\xi_1^{p^{n-k}},\xi_2^{p^{n-k-1}},\ldots,\xi_i^{p^{n-i-k+1}},\ldots,\xi_{n-k}^p,\xi_{n-k+1},\xi_{n-k+2},\ldots,\xi_n,\tau_0,\tau_1,\ldots,\tau_n$$

Hence $\mathcal{A}_p(n)_*/J_k(\mathcal{A}_p(n)_*)$ is isomorphic to $\mathcal{A}_p^{ev}(n-k)_* = \boldsymbol{F}_p[\xi_1,\xi_2,\ldots,\xi_{n-k}]/(\xi_1^{p^{n-k}},\xi_2^{p^{n-k-1}},\ldots,\xi_{n-k}^p)$. It follows that $\dim\mathcal{A}_p(n)_m/J_k^m(\mathcal{A}_p(n)_*) = \dim\mathcal{A}_p^{ev}(n-k)_m$ and this implies the required equality. $\square$

Let $P_{p,n}(t)$ be the Poincaré series of $\mathcal{A}_p(n)_*$ and $P_{p,n}^{ev}(t)$ the Poincaré series of $\mathcal{A}_p^{ev}(n)_*$. Then, $P_{p,n}(t)$ and $P_{p,n}^{ev}(t)$ are given as follows.

$$P_{2,n}(t) = \prod_{k=1}^{n}\left(\sum_{i=1}^{2^{n-k+1}} t^{(i-1)(2^k-1)}\right) = \prod_{k=1}^{n}\left(1+t^{2^k-1}+t^{2(2^k-1)}+\cdots+t^{(2^{n-k+1}-1)(2^k-1)}\right)$$
$$P_{p,n}^{ev}(t) = \prod_{k=1}^{n}\left(\sum_{i=1}^{p^{n-k+1}} t^{2(i-1)(p^k-1)}\right) = \prod_{k=1}^{n}\left(1+t^{2(p^k-1)}+t^{4(p^k-1)}+\cdots+t^{2(p^{n-k+1}-1)(p^k-1)}\right)$$
$$P_{p,n}(t) = P_{p,n}^{ev}(t)\prod_{k=0}^{n}(1+t^{2p^k-1}) = \prod_{k=0}^{n}(1+t^{2p^k-1})\prod_{k=1}^{n}\left(1+t^{2(p^k-1)}+t^{4(p^k-1)}+\cdots+t^{2(p^{n-k+1}-1)(p^k-1)}\right)$$

Since $\dim\mathcal{A}_p(n)_m = \dfrac{1}{m!}\dfrac{d^mP_{p,n}}{dt^m}(0)$ and $\dim\mathcal{A}_p^{ev}(n)_m = \dfrac{1}{m!}\dfrac{d^mP_{p,n}^{ev}}{dt^m}(0)$, we have the following equalities by (8.7.24). Here we put $P_{2,l}(t)=1$ if $l \leqq 0$.

$$\delta_k^m(\mathcal{A}_2(n)_*) = \frac{1}{m!}\left(\frac{d^mP_{2,n}}{dt^m}(0) - \frac{d^mP_{2,n-k}}{dt^m}(0)\right)$$
$$\delta_k^m(\mathcal{A}_p^{ev}(n)_*) = \frac{1}{m!}\left(\frac{d^mP_{p,n}^{ev}}{dt^m}(0) - \frac{d^mP_{p,n-k}^{ev}}{dt^m}(0)\right)$$
$$\delta_k^m(\mathcal{A}_p(n)_*) = \frac{1}{m!}\left(\frac{d^mP_{p,n}}{dt^m}(0) - \frac{d^mP_{p,n-k}^{ev}}{dt^m}(0)\right)$$

We have the following results by (8.7.21).

Proposition 8.7.25 *The following formulas hold.*

$$\log_2(\sharp(G_{2,n}(\mathcal{A}_2(n)_*))) = \sum_{i=1}^{n}\frac{1}{(2^i-1)!}\left(\frac{d^{2^i-1}P_{2,n}}{dt^{2^i-1}}(0) - \frac{d^{2^i-1}P_{2,i-1}}{dt^{2^i-1}}(0)\right)$$
$$\log_2(\sharp(H_{2,n}(\mathcal{A}_2(n)_*))) = \sum_{i=0}^{n-1}\frac{1}{(2^n-2^i)!}\left(\frac{d^{2^n-2^i}P_{2,n}}{dt^{2^n-2^i}}(0) - \frac{d^{2^n-2^i}P_{2,n-1}}{dt^{2^n-2^i}}(0)\right)$$
$$\log_p(\sharp(G_{p,n}(\mathcal{A}_p(n)_*))) = \sum_{i=0}^{n}\frac{1}{(2p^i-1)!}\frac{d^{2p^i-1}P_{p,n}}{dt^{2p^i-1}}(0) + \sum_{i=1}^{n}\frac{1}{2(p^i-1)!}\left(\frac{d^{2(p^i-1)}P_{p,n}}{dt^{2(p^i-1)}}(0) - \frac{d^{2(p^i-1)}P_{p,i-1}^{ev}}{dt^{2(p^i-1)}}(0)\right)$$
$$\log_p(\sharp(H_{p,n}(\mathcal{A}_p(n)_*))) = \frac{1}{(2p^n-1)!}\frac{d^{2p^n-1}P_{p,n}}{dt^{2p^n-1}}(0) + \sum_{i=0}^{n-1}\frac{1}{2(p^n-p^i)!}\left(\frac{d^{2(p^n-p^i)}P_{p,n}}{dt^{2(p^n-p^i)}}(0) - \frac{d^{2(p^n-p^i)}P_{p,n-1}^{ev}}{dt^{2(p^n-p^i)}}(0)\right)$$
$$\log_p(\sharp(G_{p,n}^{ev}(\mathcal{A}_p^{ev}(n)_*))) = \sum_{i=1}^{n}\frac{1}{2(p^i-1)!}\left(\frac{d^{2(p^i-1)}P_{p,n}^{ev}}{dt^{2(p^i-1)}}(0) - \frac{d^{2(p^i-1)}P_{p,i-1}^{ev}}{dt^{2(p^i-1)}}(0)\right)$$
$$\log_p(\sharp(H_{p,n}^{ev}(\mathcal{A}_p^{ev}(n)_*))) = \sum_{i=0}^{n-1}\frac{1}{2(p^n-p^i)!}\left(\frac{d^{2(p^n-p^i)}P_{p,n}^{ev}}{dt^{2(p^n-p^i)}}(0) - \frac{d^{2(p^n-p^i)}P_{p,n-1}^{ev}}{dt^{2(p^n-p^i)}}(0)\right)$$

Remark 8.7.26 *By using (8.7.25), the orders of $G_{2,n}(\mathcal{A}_2(n)_*)$ for $n = 1,2,3,4,5,6$ are 2, 2^3, 2^7, 2^{15}, 2^{34}, 2^{98}, respectively. Similarly, we see that the number of elements of $H_{2,n}(\mathcal{A}_2(n)_*)$ for $n = 1,2,3,4,5,6$ are 2, 2^3, 2^7, 2^{17}, 2^{49}, 2^{209}, respectively. Therefore $\iota^*_{n,\mathcal{A}_2(n)_*} : G_{2,n}(\mathcal{A}_2(n)_*) \to H_{2,n}(\mathcal{A}_2(n)_*)$ is bijective if $n = 1,2,3$ but not surjective if $n = 4,5,6$.*

For a non-negative integer k, let us denote by $\mathcal{A}_2[k]_*$ a subalgebra of $\mathcal{A}_{2*}$ generated by $\zeta_1^{2^k}, \zeta_2^{2^k}, \dots, \zeta_n^{2^k}, \dots$ and by $\mathcal{A}_p[k]_*$ a subalgebra of $\mathcal{A}_{p*}$ generated by $\xi_1^{p^k}, \xi_2^{p^k}, \dots, \xi_n^{p^k}, \dots$ if p is an odd prime. Thus we have the following decreasing sequence of subalgebras of $\mathcal{A}^{ev}_{p*}$. Here we put $\mathcal{A}^{ev}_{2*} = \mathcal{A}_{2*}$.

$$\mathcal{A}^{ev}_{p*} = \mathcal{A}_p[0]_* \supset \mathcal{A}_p[1]_* \supset \cdots \supset \mathcal{A}_p[k]_* \supset \mathcal{A}_p[k+1]_* \supset \cdots$$

Definition 8.7.27 *Let Seq be the set of all infinite sequences $(r_1, r_2, \dots, r_i, \dots)$ of non-negative integers such that $r_i = 0$ for all but finite number of i's. We regard Seq as an abelian monoid with unit $\mathbf{0} = (0,0,\dots)$ by componentwise addition. We denote by Seq^b a subset of Seq consisting of all sequences $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_i, \dots)$ such that $\varepsilon_i = 0, 1$ for all $i = 0, 1, \dots$. If $r_i = 0$ for $i > n$, we denote $(r_1, r_2, \dots, r_i, \dots)$ by $(r_1, r_2, \dots, r_n)$. For $E = (e_0, e_1, \dots, e_m) \in \mathrm{Seq}^b$ and $R = (r_1, r_2, \dots, r_n) \in \mathrm{Seq}$, we put $\tau(E) = \tau_0^{e_0}\tau_1^{e_0}\cdots\tau_m^{e_m}$, $\xi(R) = \xi_1^{r_1}\xi_2^{r_2}\cdots\xi_n^{r_n}$, $\zeta(R) = \zeta_1^{r_1}\zeta_2^{r_2}\cdots\zeta_i^{r_n}$ and $|R| = \sum\limits_{i=1}^{n} r_i$.*

We consider the monomial basis $B_p = \{\tau(E)\xi(R) \,|\, E \in \mathrm{Seq}^b,\ R \in \mathrm{Seq}\}$ of $\mathcal{A}_{p*}$ if p is an odd prime and $B_2 = \{\zeta(R) \,|\, R \in \mathrm{Seq}\}$ of $\mathcal{A}_{2*}$. Let $\wp(R)$ be the dual of $\xi(R)$ with respect to B_p if p is an odd prime and $Sq(R)$ the dual of $\zeta(R)$ with respect to B_2 if $p = 2$.

Then, the Milnor basis is defined as follows.

Definition 8.7.28 *([16]) For $R \in \mathrm{Seq}$ and $E \in \mathrm{Seq}^b$, let us denote by $\rho(E,R)$ the dual of $\tau(E)\xi(R)$ with respect to B_p and by $Sq(R)$ the dual of $\zeta(R)$ with respect to B_2. If p is odd, let Q_n be the dual of τ_n with respect to B_p. We put $\wp(R) = \rho(\mathbf{0}, R)$ and $Q(E) = Q_0^{\varepsilon_0}Q_1^{\varepsilon_1}\cdots Q_n^{\varepsilon_n}$ for $E = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) \in \mathrm{Seq}^b$.*

Proposition 8.7.29 *$\{Q(E)\wp(R) \,|\, |E| + |R| \leqq n,\ E \in \mathrm{Seq}^b,\ R \in \mathrm{Seq}\}$ is a basis of $\mathrm{Dist}_n(G_p)$ if p is an odd prime and $\{Sq(R) \,|\, |R| \leqq n,\ R \in \mathrm{Seq}\}$ is a basis of $\mathrm{Dist}_n(G_2)$.*

Proof. $\{\tau(E)\xi(R) \,|\, |E| + |R| \geqq n+1\}$ is a basis of I^{n+1} if p is an odd prime and $\{\zeta(R) \,|\, |E| + |R| \geqq n+1\}$ is a basis of I^{n+1} if $p = 2$. Let $\rho(E,R)$ be the dual of $\tau(E)\xi_(R)$ with respect to B_p if p is an odd prime. Since $\rho(E,R)(\tau(E')\xi(R')) = \begin{cases} 0 & |E'| + |R'| \geqq |E| + |R| + 1 \\ 1 & (E', R') = (E, R) \end{cases}$, $\rho(E,R)$ maps $I^{|E|+|R|+1}$ to $\{0\}$ but $\rho(E,R)$ does not map $I^{|E|+|R|}$ to $\{0\}$, hence $\rho(E,R) \in \mathrm{Dist}_n(G_p) - \mathrm{Dist}_{n-1}(G_p)$. Since $Q(E)\wp(R) = \pm\rho(E,R)$ by [16], the assertion follows. We can show the assertion for $p = 2$, similarly. □

We put $E_i = (\overbrace{0, 0, \dots, 0}^{i-1}, 1) \in \mathrm{Seq}^b$.

Proposition 8.7.30 *$\{Q_0, Q_2, \dots, Q_i \dots, \wp(E_1), \wp(E_2), \dots, \wp(E_i), \dots\}$ is a basis of $\mathrm{Lie}(G_p)$ if p is an odd prime and $\{Sq(E_1), Sq(E_2), \dots, Sq(E_i), \dots\}$ is a basis of $\mathrm{Lie}(G_2)$. The brackets of $\mathrm{Lie}(G_p)$ are given by $[Q_i, Q_j] = 0$ for $i, j \geqq 0$, $[\wp(E_i), Q_0] = Q_i$ for $i \geqq 1$ and $[\wp(E_i), Q_j] = [\wp(E_i), \wp(E_j)] = 0$ for $i, j \geqq 1$ if p is an odd prime, $[Sq(E_i), Sq(E_j)] = 0$ for $i, j \geqq 1$ if $p = 2$. The p-th power map of $\mathrm{Lie}(G_p)$ is trivial.*

9 Actions of group objects in a cartesian closed category

9.1 Group objects

Let $\mathcal{T}$ be a category with finite products and 1 a fixed terminal object of $\mathcal{T}$.

Definition 9.1.1 *A group object $(G, \mu, \varepsilon, \iota)$ of $\mathcal{T}$ consists of an object G of $\mathcal{T}$ and morphisms $\mu : G \times G \to G$, $\varepsilon : 1 \to G$, $\iota : G \to G$ of $\mathcal{T}$ which make the following diagrams commute.*

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "\mu \times id_G"] \arrow[d, "id_G \times \mu"'] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G
\end{tikzcd}
\quad
\begin{tikzcd}
G \times 1 \arrow[r, "id_G \times \varepsilon"] \arrow[dr, "\mathrm{pr}_1"'] & G \times G \arrow[d, "\mu"] & 1 \times G \arrow[l, "\varepsilon \times id_G"'] \arrow[dl, "\mathrm{pr}_2"] \\
& G &
\end{tikzcd}
\quad
\begin{tikzcd}
G \times G \arrow[r, "id_G \times \iota"] & G \times G \arrow[d, "\mu"] & G \times G \arrow[l, "\iota \times id_G"'] \\
G \arrow[u, "\Delta_G"] \arrow[r, "\varepsilon o_G"] & G & G \arrow[l, "\varepsilon o_G"'] \arrow[u, "\Delta_G"']
\end{tikzcd}
$$

Here, we denote by $\Delta_G : G \to G \times G$ the diagonal morphism, by $o_G : G \to 1$ the unique morphism to the terminal object.

Let $(G, \mu, \varepsilon, \iota)$ and $(G', \mu', \varepsilon', \iota')$ be group objects of $\mathcal{T}$. A morphism $f : G \to G'$ of $\mathcal{T}$ is called a homomorphism of group objects if the following diagram commute.

$$
\begin{tikzcd}
G \times G \arrow[r, "\mu"] \arrow[d, "f \times f"'] & G \arrow[d, "f"] \\
G' \times G' \arrow[r, "\mu'"] & G'
\end{tikzcd}
$$

We denote by $\mathrm{Hom}(G, G')$ the set of homomorphisms of group objects from G to G'. If there is a homomorphism $G \to G'$ which is a monomorphism, we say that G is a subgroup object of G'.

For $f, g \in \mathrm{Hom}(G, G')$, we say that f and g are conjugate if there exists a morphism $\varphi : 1 \to G'$ which makes the following diagram commute.

$$
\begin{tikzcd}
G \arrow[r, "(f, \varphi o_G)"] \arrow[d, "(\varphi o_G, g)"'] & G' \times G' \arrow[d, "\mu'"] \\
G' \times G' \arrow[r, "\mu'"] & G'
\end{tikzcd}
$$

Thus we have a relation $\equiv$ in $\mathrm{Hom}(G, G')$ defined by "$f \equiv g$ if and only if f and g are conjugate.".

Remark 9.1.2 *The projection $\mathrm{pr}_1 : X \times 1 \to X$ is an isomorphism whose inverse is $(id_X, o_X) : X \to X \times 1 \to X$. It is clear that $(1, o_{1\times 1}, id_1, id_1)$ is a group object in $\mathcal{T}$ which is called a trivial group.*

Lemma 9.1.3 *Let $(G, \mu, \varepsilon, \iota)$ be a group object of $\mathcal{T}$. If morphisms $\alpha, \beta, \gamma : X \to G$ make the following diagram commute, then $\alpha = \gamma$*

$$
\begin{tikzcd}
X \arrow[r, "(\alpha, \beta)"] \arrow[d, "o_X"'] & G \times G \arrow[d, "\mu"] & X \arrow[l, "(\beta, \gamma)"'] \arrow[d, "o_X"] \\
1 \arrow[r, "\varepsilon"] & G & 1 \arrow[l, "\varepsilon"']
\end{tikzcd}
$$

Proof. By the commutativity of the middle diagram of (9.1.1), we have $\mu(id_G, \varepsilon o_G) = \mu(\varepsilon o_G, id_G) = id_G$. Hence, by the assumption and the commutativity of the left diagram of (9.1.1), we have $\alpha = \mu(id_G, \varepsilon o_G)\alpha = \mu(\alpha, \varepsilon o_G \alpha) = \mu(\alpha, \varepsilon o_X) = \mu(\alpha, \mu(\beta, \gamma)) = \mu(\mu(\alpha, \beta), \gamma) = \mu(\varepsilon o_X, \gamma) = \mu(\varepsilon o_G \gamma, \gamma) = \mu(\varepsilon o_G, id_G)\gamma = \gamma$. □

Proposition 9.1.4 *Let $(G, \mu, \varepsilon, \iota)$ and $(G', \mu', \varepsilon', \iota')$ be group objects of $\mathcal{T}$. If $f : G \to G'$ is a homomorphism of group objects, the following diagrams commute.*

$$
\begin{tikzcd}
1 \arrow[r, "\varepsilon"] \arrow[dr, "\varepsilon'"'] & G \arrow[d, "f"] \\
& G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\iota"] \arrow[d, "f"'] & G \arrow[d, "f"] \\
G' \arrow[r, "\iota'"] & G'
\end{tikzcd}
$$

If $G = G'$ and $\mu = \mu'$, since the identity morphism of G is a homomorphism of group objects, we have $\varepsilon = \varepsilon'$ and $\iota = \iota'$.

Proof. By the commutativity of the middle diagram of (9.1.1), we have $\mu(\varepsilon,\varepsilon)=\mu(id_G\times\varepsilon)(\varepsilon,id_1)=\mathrm{pr}_1(\varepsilon,id_1)=\varepsilon$. Therefore $\mu'(f\varepsilon,f\varepsilon)=\mu'(f\times f)(\varepsilon,\varepsilon)=f\mu(\varepsilon,\varepsilon)=f\varepsilon$. On the other hand, by the commutativity of the right diagram of (9.1.1), we have $\mu'(\iota' f\varepsilon,f\varepsilon)=\mu'(\iota',id_{G'})f\varepsilon=\varepsilon' o_G f\varepsilon=\varepsilon'$. The above equalities and the commutativity of the left and the middle diagram of (9.1.1) imply $\mu'(\iota' f\varepsilon,f\varepsilon)=\mu'(\iota' f\varepsilon,\mu'(f\varepsilon,f\varepsilon))=\mu'(\mu'(\iota' f\varepsilon,f\varepsilon),f\varepsilon)=\mu'(\varepsilon',f\varepsilon)=\mu'(\varepsilon'\times id_{G'})(id_1,f\varepsilon)=\mathrm{pr}_2(id_1,f\varepsilon)=f\varepsilon$. Thus we have $f\varepsilon=\varepsilon'$.

By the commutativity of the right diagram of (9.1.1) and $f\varepsilon=\varepsilon'$, we have $\mu'(f\iota,f)=\mu'(f\times f)(\iota,id_G)=f\mu(\iota,id_G)=f\varepsilon o_G=\varepsilon' o_G$ and $\mu'(f,\iota' f)=\mu'(id_{G'},\iota')f=\varepsilon' o_{G'}f=\varepsilon' o_G$. Hence $f\iota=\iota' f$ by (9.1.3). □

Proposition 9.1.5 *The relation $\equiv$ is an equivalence relation in* $\mathrm{Hom}(G,G')$.

Proof. For $f\in\mathrm{Hom}(G,G')$, the commutativity of the left diagram of (9.1.1) implies

$$\mu'(f,\varepsilon' o_G)=\mu'(id_{G'}\times\varepsilon')(f,o_G)=\mathrm{pr}_1(f,o_G)=f,\quad \mu'(\varepsilon' o_G,f)=\mu'(\varepsilon'\times id_{G'})(o_G,f)=\mathrm{pr}_2(o_G,f)=f.$$

Hence $f\equiv f$.

Suppose $f\equiv g$ for $f,g\in\mathrm{Hom}(G,G')$. Then, there exists a morphism $\varphi:1\to G'$ satisfying $\mu'(f,\varphi o_G)=\mu'(\varphi o_G,g)$ and we have

$$\begin{aligned}\mu'(\mu'(\iota'\varphi o_G,f),\varphi o_G)&=\mu'(\iota'\varphi o_G,\mu'(f,\varphi o_G))=\mu'(\iota'\varphi o_G,\mu'(\varphi o_G,g))=\mu'(\mu'(\iota'\varphi o_G,\varphi o_G),g)\\&=\mu'(\mu'(\iota',id_{G'})\varphi o_G,g)=\mu'(\varepsilon' o_{G'}\varphi o_G,g)=\mu'(\varepsilon' o_G,g)\\&=\mu'(\varepsilon'\times id_{G'})(o_G,g)=\mathrm{pr}_2(o_G,g)=g.\end{aligned}$$

Therefore,

$$\begin{aligned}\mu'(g,\iota'\varphi o_G)&=\mu'(\mu'(\mu'(\iota'\varphi o_G,f),\varphi o_G),\iota'\varphi o_G)=\mu'(\mu'(\iota'\varphi o_G,f),\mu'(\varphi o_G,\iota'\varphi o_G))\\&=\mu'(\mu'(\iota'\varphi o_G,f),\mu'(id_{G'}\times\iota')\varphi o_G)=\mu'(\mu'(\iota'\varphi o_G,f),\varepsilon' o_{G'}\varphi o_G)=\mu'(\mu'(\iota'\varphi o_G,f),\varepsilon' o_G)\\&=\mu'(\iota'\varphi o_G,\mu'(f,\varepsilon' o_G))=\mu'(\iota'\varphi o_G,\mu'(id_{G'}\times\varepsilon')(f,o_G))=\mu'(\iota'\varphi o_G,\mathrm{pr}_1(f,o_G))=\mu'(\iota'\varphi o_G,f).\end{aligned}$$

Hence $g\equiv f$.

Suppose $f\equiv g$ and $g\equiv h$ for $f,g,h\in\mathrm{Hom}(G,G')$. Then, there exists morphisms $\varphi,\psi:1\to G'$ satisfying $\mu'(f,\varphi o_G)=\mu'(\varphi o_G,g)$ and $\mu'(g,\psi o_G)=\mu'(\psi o_G,h)$. We have

$$\begin{aligned}\mu'(f,\mu'(\varphi,\psi)o_G)&=\mu'(f,\mu'(\varphi o_G,\psi o_G))=\mu'(\mu'(f,\varphi o_G),\psi o_G)=\mu'(\mu'(\varphi o_G,g),\psi o_G)=\mu'(\varphi o_G,\mu'(g,\psi o_G))\\&=\mu'(\varphi o_G,\mu'(\psi o_G,h))=\mu'(\mu'(\varphi o_G,\psi o_G),h)=\mu'(\mu'(\varphi,\psi)o_G,h)\end{aligned}$$

Thus we see $f\equiv h$. □

Definition 9.1.6 *Let $(G,\mu,\varepsilon,\iota)$ be a group object of $\mathcal{T}$ and X an object of $\mathcal{T}$. We call a morphism $\alpha:X\times G\to X$ (resp. $\alpha:G\times X\to X$) in $\mathcal{T}$ a right (resp. left) action on X if it satisfies the following conditions.*

(i)
$$\begin{tikzcd} X\times G\times G \arrow[r, "\alpha\times id_G"] \arrow[d, "id_X\times\mu"] & X\times G \arrow[d, "\alpha"] \\ X\times G \arrow[r, "\alpha"] & X \end{tikzcd}\quad\left(\text{resp. }\begin{tikzcd} G\times G\times X \arrow[r, "id_G\times\alpha"] \arrow[d, "\mu\times id_X"] & G\times X \arrow[d, "\alpha"] \\ G\times X \arrow[r, "\alpha"] & X \end{tikzcd}\right)\text{ commutes.}$$

(ii)
$$\begin{tikzcd} X\times 1 \arrow[r, "id_X\times\varepsilon"] \arrow[dr, "\mathrm{pr}_1"'] & X\times G \arrow[d, "\alpha"] \\ & X \end{tikzcd}\quad\left(\text{resp. }\begin{tikzcd} 1\times X \arrow[r, "\varepsilon\times id_X"] \arrow[dr, "\mathrm{pr}_2"'] & G\times X \arrow[d, "\alpha"] \\ & X \end{tikzcd}\right)\text{ comutes}$$

We call an object with right (resp. left) G-action a right (resp. left) G-object. Let $\beta:Y\times G\to Y$ (resp. $\beta:G\times Y\to Y$) be a right (resp. left) G-action on Y and $f:X\to Y$ a morphism of $\mathcal{T}$. If f makes the following diagram commute,

$$\begin{tikzcd} X\times G \arrow[r, "\alpha"] \arrow[d, "f\times id_G"] & X \arrow[d, "f"] \\ Y\times G \arrow[r, "\beta"] & Y \end{tikzcd}\quad\left(\text{resp. }\begin{tikzcd} G\times X \arrow[r, "\alpha"] \arrow[d, "id_G\times f"] & X \arrow[d, "f"] \\ G\times Y \arrow[r, "\beta"] & Y \end{tikzcd}\right)$$

we call f a morphism of G-functors. We denote by $\mathrm{Act}_r(G)$ (resp. $\mathrm{Act}_l(G)$) the category of right G-objects and morphisms of right G-objects in $\mathcal{T}$ (resp. the category of left G-objects and morphisms of left G-objects).

Remark 9.1.7 *For an object (X, α) of $\mathrm{Act}_r(G)$, let $\check{\alpha} : G \times X \to X$ be the following composition.*

$$G \times X \xrightarrow{T_{G,X}} X \times G \xrightarrow{id_X \times \iota} X \times G \xrightarrow{\alpha} X$$

Here, $T_{G,X}$ denotes the switching morphism. Define a functor $\mathfrak{I}_G : \mathrm{Act}_r(G) \to \mathrm{Act}_l(G)$ by $\mathfrak{I}_G(X, \alpha) = (X, \check{\alpha})$ and $\mathfrak{I}_G(f) = f$. Then, $\mathfrak{I}_G$ is an isomorphism of categories.

Example 9.1.8 *(1) It is clear that the projection $X \times G \to X$ (resp. $G \times X \to X$) to the first (resp. second) factor is a right (resp. left) action of G on X. We call this the trivial right (resp. left) action of G.*

(2) The multiplication $\mu : G \times G \to G$ of G is regarded as both a right action and a left action of G on itself.

(3) Let us denote by $\Delta_G : G \to G \times G$ the diagonal morphism and by $T_{G,G} : G \times G \to G \times G$ the switching morphism, respectively. Define morphisms $\gamma_r, \gamma_l : G \times G \to G$ to be the following compositions.

$$G \times G \xrightarrow{id_G \times \Delta_G} G \times G \times G \xrightarrow{T_{G,G} \times id_G} G \times G \times G \xrightarrow{\iota \times \mu} G \times G \xrightarrow{\mu} G$$

$$G \times G \xrightarrow{\Delta_G \times id_G} G \times G \times G \xrightarrow{id_G \times T_{G,G}} G \times G \times G \xrightarrow{\mu \times \iota} G \times G \xrightarrow{\mu} G$$

Then, γ_r is a right action of G on G and γ_l is a left action of G on G. γ_r and γ_l are called the adjoint actions of G. We note that $\mathfrak{I}_G(\gamma_r) = \gamma_l$ and that $\mu(\gamma_l, \mathrm{pr}_1) = \mu(\mathrm{pr}_2, \gamma_r) = \mu$, where $\mathrm{pr}_i : G \times G \to G$ ($i = 1, 2$) denotes the projection onto the i-th component.

Proposition 9.1.9 $\mathrm{Act}_r(G)$ *is a category with finite products.*

Proof. For objects (X, α) and (Y, β) of $\mathrm{Act}_r(G)$, define a right G-action on $X \times Y$ to be the following composition

$$X \times Y \times G \xrightarrow{id_X \times id_Y \times \Delta_G} X \times Y \times G \times G \xrightarrow{id_X \times T_{Y,G} \times id_G} X \times G \times Y \times G \xrightarrow{\alpha \times \beta} X \times Y$$

Then, projections $\mathrm{pr}_X : X \times Y \to X$, $\mathrm{pr}_Y : X \times Y \to Y$ gives morphisms

$$\mathrm{pr}_X : (X \times Y, (\alpha \times \beta)(id_X \times T_{Y,G} \times id_G)(id_X \times id_Y \times \Delta_G)) \to (X, \alpha),$$

$$\mathrm{pr}_Y : (X \times Y, (\alpha \times \beta)(id_X \times T_{Y,G} \times id_G)(id_X \times id_Y \times \Delta_G)) \to (Y, \beta)$$

of $\mathrm{Act}_r(G)$. In fact the following diagram commutes.

$$\begin{tikzcd}
 & X \times G \arrow[rr, "\alpha"] & & X \\
X \times Y \times G \arrow[ur, "\mathrm{pr}_X \times id_G"] \arrow[dr, "\mathrm{pr}_Y \times id_G"'] \arrow[r, "id_X \times id_Y \times \Delta_G"] & X \times Y \times G \times G \arrow[u, "\mathrm{pr}_X \times \mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_Y \times \mathrm{pr}_2"] \arrow[r, "id_X \times T_{Y,G} \times id_G"] & X \times G \times Y \times G \arrow[ul, "(\mathrm{pr}_1, \mathrm{pr}_2)"'] \arrow[dl, "(\mathrm{pr}_3, \mathrm{pr}_4)"] \arrow[r, "\alpha \times \beta"] & X \times Y \arrow[u, "\mathrm{pr}_X"'] \arrow[d, "\mathrm{pr}_Y"] \\
 & Y \times G \arrow[rr, "\beta"] & & Y
\end{tikzcd}$$

Let $f : (Z, \gamma) \to (X, \alpha)$ and $g : (Z, \gamma) \to (Y, \beta)$ be morphisms of $\mathrm{Act}_r(G)$. Since

$$\begin{tikzcd}
 & Z \times G \arrow[dl, "(f,g) \times id_G"'] \arrow[d, "(f,g) \times \Delta_G"] \arrow[dr, "(f, id_G, g, id_G)"] \arrow[rr, "\gamma"] & & Z \arrow[d, "(f,g)"] \\
X \times Y \times G \arrow[r, "id_X \times id_Y \times \Delta_G"] & X \times Y \times G \times G \arrow[r, "id_X \times T_{Y,G} \times id_G"] & X \times G \times Y \times G \arrow[r, "\alpha \times \beta"] & X \times Y
\end{tikzcd}$$

commutes, $(f, g) : Z \to X \times Y$ is a morphism of $\mathrm{Act}_r(G)$ which satisfies $\mathrm{pr}_X(f, g) = f$ and $\mathrm{pr}_Y(f, g) = g$. It is clear that $(1, o_{1 \times G})$ is a terminal object of $\mathrm{Act}_r(G)$. □

Proposition 9.1.10 *If $\mathcal{T}$ is a category with finite limits, so is* $\mathrm{Act}_r(G)$*.*

Proof. Let $f, g : (X, \alpha) \to (Y, \beta)$ be morphisms of $\mathrm{Act}_r(G)$ and $h : Z \to X$ an equalizer of $f, g : X \to Y$ in $\mathcal{T}$. Then, $f\alpha(h \times id_G) = \beta(f \times id_G)(h \times id_G) = \beta(fh \times id_G) = \beta(gh \times id_G) = \beta(g \times id_G)(h \times id_G) = g\alpha(h \times id_G)$. It follows that there exists unique morphism $\gamma : Z \times G \to Z$ that satisfies $h\gamma = \alpha(h \times id_G)$. Since h is a monomorphism and α is a right G-action on X, γ is a right G-action on X. Thus we have a morphism $h : (Z, \gamma) \to (X, \alpha)$ of $\mathrm{Act}_r(G)$. Let $k : (W, \delta) \to (X, \alpha)$ be a morphism of $\mathrm{Act}_r(G)$ satisfying $kf = kg$. There exists unique morphism $p : W \to Z$ in $\mathcal{T}$ satisfying $hp = k$. Then, we have $h\gamma(p \times id_G) = \alpha(h \times id_G)(p \times id_G) = \alpha(k \times id_G) = k\delta = hp\delta$. Since h is a monomorphism, it follows $\gamma(p \times id_G) = p\delta$, that is, p is a morphism $(W, \delta) \to (Z, \gamma)$ of $\mathrm{Act}_r(G)$. □

Definition 9.1.11 *Let $f : H \to G$ be a morphism of group objects.*

(1) Let (X, α) be an object of $\mathrm{Act}_r(G)$ (resp. $\mathrm{Act}_l(G)$). A subobject $i : Y \to X$ of X is said to be H-invarinat if there exists a morphism $\alpha' : Y \times H \to Y$ (resp. $\alpha' : H \times Y \to Y$) which satisfies $\alpha(i \times f) = i\alpha'$ (resp. $\alpha(f \times i)) = i\alpha'$). We note that, since i is a monomorphism, such α' is unique if exists and (Y, α') is an object of $\mathrm{Act}_r(H)$ (resp $\mathrm{Act}_l(H)$).

(2) Consider the adjoint action (G, γ_r) on G. If a subobject N of G is H-invariant subobject of G, we say that $f : H \to G$ normalizes N.

Proposition 9.1.12 *Let $f : H \to G$ be a morphism of group objects and N a subobject of G. N is an H-invariant subobject of G with respect to γ_l if and only if $f : H \to G$ normalizes N.*

Proof. Let $i : N \to G$ be the inclusion morphism. Suppose that there is a morphism $\alpha : H \times N \to N$ satisfying $\gamma_l(f \times i) = i\alpha$. Then, (N, α) is an object of $\mathrm{Act}_l(H)$ and put $(N, \beta) = \mathfrak{I}_H^{-1}(N, \alpha)$. $\beta : N \times H \to N$ satisfies $\gamma_r(i \times f) = i\beta$. Conversely, if there is a morphism $\beta : N \times H \to N$ satisfying $\gamma_r(f \times i) = i\beta$. Then, (N, β) is an object of $\mathrm{Act}_r(H)$ and put $(N, \alpha) = \mathfrak{I}_H(N, \beta)$. Then, $\alpha : H \times N \to N$ satisfies $\gamma_l(i \times f) = i\alpha$. □

Lemma 9.1.13 *Suppose that a morphism $f : H \to G$ of group objects normalizes a subobject $i : N \to G$ of G. Let $\alpha : H \times N \to N$, $\beta : N \times H \to N$ be morphisms which satisfy $\gamma_l(f \times i) = i\alpha$ and $\gamma_r(i \times f) = i\beta$, respectively. Then we have*

$$\mu(f \times i) = \mu(i \times id_G)(\alpha \times f)(id_H \times T_{H,N})(\Delta_H \times id_N)$$
$$\mu(i \times f) = \mu(id_G \times i)(f \times \beta)(T_{N,H} \times id_H)(id_N \times \Delta_H).$$

Proof. Since $\mu(id_G \times \varepsilon o_G) = \mathrm{pr}_1$, $\mu(\varepsilon o_G \times id_G) = \mathrm{pr}_2$, $\mu(\iota \times id_G)\Delta_G = \mu(id_G \times \iota)\Delta_G = \varepsilon o_G$ by the definition of group objects and $\gamma_l = \mu(\mu \times \iota)(id_G \times T_{G,G})(\Delta_G \times id_G)$, $\gamma_r = \mu(\iota \times \mu)(T_{G,G} \times id_G)(id_G \times \Delta_G)$,

$$\begin{aligned}
\mu(\gamma_l \times id_G)(id_G \times T_{G,G})(\Delta_G \times id_G) &= \mu(\mu(\mu \times \iota)(id_G \times T_{G,G})(\Delta_G \times id_G) \times id_G)(id_G \times T_{G,G})(\Delta_G \times id_G) \\
&= \mu(\mu \times id_G)(\mu \times \iota \times id_G)(id_G \times T_{G,G} \times id_G)(\Delta_G \times id_G \times id_G)(id_G \times T_{G,G})(\Delta_G \times id_G) \\
&= \mu(id_G \times \mu)(\mu \times \iota \times id_G)(id_G \times T_{G,G} \times id_G)(id_G \times id_G \times T_{G,G})(\Delta_G \times id_G \times id_G)(\Delta_G \times id_G) \\
&= \mu(\mu \times \mu(\iota \times id_G))(id_G \times T_{G,G} \times id_G)(id_G \times id_G \times T_{G,G})(id_G \times \Delta_G \times id_G)(\Delta_G \times id_G) \\
&= \mu(\mu \times \mu(\iota \times id_G))(id_G \times (T_{G,G} \times id_G)(id_G \times T_{G,G})(\Delta_G \times id_G))(\Delta_G \times id_G) \\
&= \mu(\mu \times \mu(\iota \times id_G))(id_G \times (id_G \times \Delta_G)T_{G,G})(\Delta_G \times id_G) \\
&= \mu(\mu \times \mu(\iota \times id_G))(id_G \times id_G \times \Delta_G)(id_G \times T_{G,G})(\Delta_G \times id_G) \\
&= \mu(\mu \times \mu(\iota \times id_G)\Delta_G)(id_G \times T_{G,G})(\Delta_G \times id_G) = \mu(\mu \times \varepsilon o_G)(id_G \times T_{G,G})(\Delta_G \times id_G) \\
&= \mu(id_G \times \varepsilon o_G)(\mu \times id_G)(id_G \times T_{G,G})(\Delta_G \times id_G) = \mathrm{pr}_1(\mu \times id_G)(id_G \times T_{G,G})(\Delta_G \times id_G) = \mu \\
\mu(id_G \times \gamma_r)(T_{G,G} \times id_G)(id_G \times \Delta_G) &= \mu(id_G \times \mu(\iota \times \mu)(T_{G,G} \times id_G)(id_G \times \Delta_G))(T_{G,G} \times id_G)(id_G \times \Delta_G) \\
&= \mu(id_G \times \mu)(id_G \times \iota \times \mu)(id_G \times T_{G,G} \times id_G)(id_G \times id_G \times \Delta_G)(T_{G,G} \times id_G)(id_G \times \Delta_G) \\
&= \mu(\mu \times id_G)(id_G \times \iota \times \mu)(id_G \times T_{G,G} \times id_G)(T_{G,G} \times id_G \times id_G)(id_G \times id_G \times \Delta_G)(id_G \times \Delta_G) \\
&= \mu(\mu(id_G \times \iota) \times \mu)(id_G \times T_{G,G} \times id_G)(T_{G,G} \times id_G \times id_G)(id_G \times \Delta_G \times id_G)(id_G \times \Delta_G) \\
&= \mu(\mu(id_G \times \iota) \times \mu)((id_G \times T_{G,G})(T_{G,G} \times id_G)(id_G \times \Delta_G) \times id_G)(id_G \times \Delta_G) \\
&= \mu(\mu(id_G \times \iota) \times \mu)((\Delta_G \times id_G)T_{G,G} \times id_G)(id_G \times \Delta_G) \\
&= \mu(\mu(id_G \times \iota) \times \mu)(\Delta_G \times id_G \times id_G)(T_{G,G} \times id_G)(id_G \times \Delta_G) \\
&= \mu(\mu(id_G \times \iota)\Delta_G \times \mu)(T_{G,G} \times id_G)(id_G \times \Delta_G) = \mu(\varepsilon o_G \times \mu)(T_{G,G} \times id_G)(id_G \times \Delta_G) \\
&= \mu(\varepsilon o_G \times id_G)(id_G \times \mu)(T_{G,G} \times id_G)(id_G \times \Delta_G) = \mathrm{pr}_2(id_G \times \mu)(T_{G,G} \times id_G)(id_G \times \Delta_G) = \mu.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mu(f \times i) &= \mu(\gamma_l \times id_G)(id_G \times T_{G,G})(\Delta_G \times id_G)(f \times i) = \mu(\gamma_l \times id_G)(id_G \times T_{G,G})(f \times f \times i)(\Delta_H \times id_N) \\
&= \mu(\gamma_l \times id_G)(f \times i \times f)(id_H \times T_{H,N})(\Delta_H \times id_N) = \mu(\gamma_l(f \times i) \times f)(id_H \times T_{H,N})(\Delta_H \times id_N) \\
&= \mu(i\alpha \times f)(id_H \times T_{H,N})(\Delta_H \times id_N) = \mu(i \times id_G)(\alpha \times f)(id_H \times T_{H,N})(\Delta_H \times id_N) \\
\mu(i \times f) &= \mu(id_G \times \gamma_r)(T_{G,G} \times id_G)(id_G \times \Delta_G)(i \times f) = \mu(id_G \times \gamma_r)(T_{G,G} \times id_G)(i \times f \times f)(id_N \times \Delta_H) \\
&= \mu(id_G \times \gamma_r)(f \times i \times f)(T_{N,H} \times id_H)(id_N \times \Delta_H) = \mu(f \times \gamma_r(i \times f))(T_{N,H} \times id_H)(id_N \times \Delta_H) \\
&= \mu(f \times i\beta)(T_{N,H} \times id_H)(id_N \times \Delta_H) = \mu(id_G \times i)(f \times \beta)(T_{N,H} \times id_H)(id_N \times \Delta_H).
\end{aligned}$$

□

Proposition 9.1.14 *Let $F : \mathrm{Act}_r(G) \to \mathcal{T}$ be the forgetful functor given by $F(X, \alpha) = X$. Then, F has a left adjoint.*

Proof. For $X, Y \in \mathrm{Ob}\,\mathcal{T}$ and $f \in \mathcal{T}(X, Y)$, it is clear that $id_X \times \mu : X \times G \times G \to X \times G$ is a right action of G on $X \times G$ and that $f \times id_G : X \times G \to Y \times G$ is a morphism in $\mathrm{Act}_r(G)$. We define a functor $L : \mathcal{T} \to \mathrm{Act}_r(G)$ by $L(X) = (X \times G, id_X \times \mu)$ and $L(f) = f \times id_G$. For $(Y, \beta) \in \mathrm{Ob}\,\mathrm{Act}_r(G)$, define a map $\Phi : \mathcal{T}(X, F(Y, \beta)) \to \mathrm{Act}_r(G)(L(X), (Y, \beta))$ by $\Phi(f) = \beta(f \times id_G)$. Then, Φ is natural and the inverse of Φ is given by $\Phi^{-1}(f) = f(id_X, \varepsilon o_X)$. Hence L is a left adjoint of F. $\square$

Definition 9.1.15 *Let $\varphi : H \to G$ be a morphism of group objects in $\mathcal{T}$ and $\alpha : X \times G \to X$ a right action of G on X. Then, $\alpha(id_X \times \varphi) : X \times H \to X$ is a right action of H on X. We denote this action by $\varphi^*(\alpha)$. In particular, if $\varphi : H \to G$ is a subgroup object of G, we denote $\varphi^*(\alpha)$ by $\mathrm{Res}^G_H(\alpha)$. If $\alpha' : Y \times G \to Y$ is a right action of G on Y and $f : X \to Y$ is a morphism of $\mathrm{Act}_r(G)$, the following diagram commutes.*

$$\begin{tikzcd} X \times H \arrow[r, "\varphi^*(\alpha)"] \arrow[d, "f \times id_H"] & X \arrow[d, "f"] \\ Y \times H \arrow[r, "\varphi^*(\alpha')"] & Y \end{tikzcd}$$

We denote $(X, \alpha(id_X \times \varphi))$ by $\varphi^*(X, \alpha)$. Thus we have a functor $\varphi^* : \mathrm{Act}_r(G) \to \mathrm{Act}_r(H)$.

Theorem 9.1.16 *For an object (X, α) of $\mathrm{Act}_r(H)$, we assume that a coequalizer of $X \times H \times G \xrightarrow{\alpha \times id_G} X \times G$ and $X \times H \times G \xrightarrow{id_X \times \mu(\varphi \times id_G)} X \times G$ exists. Let us denote by $Q^\alpha_\varphi : X \times G \to X^\alpha_\varphi$ a coequalizer of the above morphisms. We also assume that $Q^\alpha_\varphi \times id_G : X \times G \times G \to X^\alpha_\varphi \times G$ is a coequalizer of $X \times H \times G \times G \xrightarrow{\alpha \times id_G \times id_G} X \times G \times G$ and $X \times H \times G \times G \xrightarrow{id_X \times \mu(\varphi \times id_G) \times id_G} X \times G \times G$ and that $Q^\alpha_\varphi \times id_G \times id_G : X \times G \times G \to X^\alpha_\varphi \times G$ is an epimorphism. Then, a functor $\mathrm{Act}_r(G) \to \mathcal{S}et$ given by $(Y, \beta) \mapsto \mathrm{Act}_r(H)((X, \alpha), \varphi^*(Y, \beta))$ is representable.*

Proof. Since diagrams

$$\begin{tikzcd} X \times H \times G \times G \arrow[r, "\alpha \times id_G \times id_G"] \arrow[d, "id_X \times id_H \times \mu"] & X \times G \times G \arrow[d, "id_X \times \mu"] \\ X \times H \times G \arrow[r, "\alpha \times id_G"] & X \times G \end{tikzcd} \qquad \begin{tikzcd} X \times H \times G \times G \arrow[r, "id_X \times \mu(\varphi \times id_G) \times id_G"] \arrow[d, "id_X \times id_H \times \mu"] & X \times G \times G \arrow[d, "id_X \times \mu"] \\ X \times H \times G \arrow[r, "id_X \times \mu(\varphi \times id_G)"] & X \times G \end{tikzcd}$$

commute, there exists unique morphism $\alpha_\varphi : X^\alpha_\varphi \times G \to X^\alpha_\varphi$ that makes the following diagram commute.

$$\begin{tikzcd} X \times G \times G \arrow[r, "Q^\alpha_\varphi \times id_G"] \arrow[d, "id_X \times \mu"] & X^\alpha_\varphi \times G \arrow[d, "\alpha_\varphi"] \\ X \times G \arrow[r, "Q^\alpha_\varphi"] & X^\alpha_\varphi \end{tikzcd}$$

By the definition of α_φ, we have the following equalities.

$$\begin{aligned} \alpha_\varphi(\alpha_\varphi \times id_G)(Q^\alpha_\varphi \times id_G \times id_G) &= \alpha_\varphi(\alpha_\varphi(Q^\alpha_\varphi \times id_G) \times id_G) = \alpha_\varphi(Q^\alpha_\varphi(id_X \times \mu) \times id_G) \\ &= \alpha_\varphi(Q^\alpha_\varphi \times id_G)(id_X \times \mu \times id_G) = Q^\alpha_\varphi(id_X \times \mu)(id_X \times \mu \times id_G) \\ &= Q^\alpha_\varphi(id_X \times \mu(\mu \times id_G)) \\ \alpha_\varphi(id_{X^\alpha_\varphi} \times \mu)(Q^\alpha_\varphi \times id_G \times id_G) &= \alpha_\varphi(Q^\alpha_\varphi \times id_G)(id_X \times id_G \times \mu) = Q^\alpha_\varphi(id_X \times \mu)(id_X \times id_G \times \mu) \\ &= Q^\alpha_\varphi(id_X \times \mu(id_G \times \mu)) \\ \alpha_\varphi(id_{X^\alpha_\varphi} \times \varepsilon)(Q^\alpha_\varphi \times id_1) &= \alpha_\varphi(Q^\alpha_\varphi \times \varepsilon) = \alpha_\varphi(Q^\alpha_\varphi \times id_G)(id_X \times id_G \times \varepsilon) \\ &= Q^\alpha_\varphi(id_X \times \mu)(id_X \times id_G \times \varepsilon) = Q^\alpha_\varphi(id_X \times \mu(id_G \times \varepsilon)) \\ &= Q^\alpha_\varphi(id_X \times \mathrm{pr}_1) = \mathrm{pr}_1(Q^\alpha_\varphi \times id_1) \end{aligned}$$

Thus $(X^\alpha_\varphi, \alpha_\varphi)$ is an object of $\mathrm{Act}_r(G)$ and we denote this by $\varphi_!(X, \alpha)$. $Q^\alpha_\varphi : L(X) = (X \times G, id_X \times \mu) \to \varphi_!(X, \alpha)$ is a morphism of $\mathrm{Act}_r(G)$.

Define a morphism $\eta_{(X,\alpha)} : X \to X^\alpha_\varphi$ to be a composition $X \xrightarrow{(id_X, \varepsilon o_X)} X \times G \xrightarrow{Q^\alpha_\varphi} X^\alpha_\varphi$. Since

$$\begin{aligned}\eta_{(X,\alpha)}\alpha &= Q^\alpha_\varphi(id_X, \varepsilon o_X)\alpha = Q^\alpha_\varphi(\alpha, \varepsilon o_{X\times H}) = Q^\alpha_\varphi(\alpha \times id_G)(id_{X\times H}, \varepsilon o_{X\times H})\\ &= Q^\alpha_\varphi(id_X \times \mu(\varphi \times id_G))(id_X \times (id_H, \varepsilon o_H)) = Q^\alpha_\varphi(id_X \times \mu(\varphi, \varepsilon o_H))\\ &= Q^\alpha_\varphi(id_X \times \varphi)\\ \varphi^*(\alpha_\varphi)(\eta_{(X,\alpha)} \times id_H) &= \alpha_\varphi(id_{X^\alpha_\varphi} \times \varphi)(Q^\alpha_\varphi(id_X, \varepsilon o_X) \times id_H) = \alpha_\varphi(Q^\alpha_\varphi(id_X, \varepsilon o_X) \times \varphi)\\ &= \alpha_\varphi(Q^\alpha_\varphi \times id_G)((id_X, \varepsilon o_X) \times \varphi) = Q^\alpha_\varphi(id_X \times \mu)((id_X, \varepsilon o_X) \times \varphi)\\ &= Q^\alpha_\varphi(id_X \times \mu(\varepsilon o_X, \varphi)) = Q^\alpha_\varphi(id_X \times \varphi),\end{aligned}$$

$\eta_{(X,\alpha)} : (X,\alpha) \to \varphi^*\varphi_!(X,\alpha)$ is a morphism in $\mathrm{Act}_r(H)$.

Define a map $\Phi^{(X,\alpha)}_{(Y,\beta)} : \mathrm{Act}_r(G)(\varphi_!(X,\alpha),(Y,\beta)) \to \mathrm{Act}_r(H)((X,\alpha),\varphi^*(Y,\beta))$ by $\Phi^{(X,\alpha)}_{(Y,\beta)}(f) = \varphi^*(f)\eta_{(X,\alpha)}$. Suppose $\Phi^{(X,\alpha)}_{(Y,\beta)}(f) = \Phi^{(X,\alpha)}_{(Y,\beta)}(g)$ for $f, g \in \mathrm{Act}_r(G)(\varphi_!(X,\alpha),(Y,\beta))$. Then, $fQ^\alpha_\varphi(id_X, \varepsilon o_X) = gQ^\alpha_\varphi(id_X, \varepsilon o_X)$. Since $fQ^\alpha_\varphi, gQ^\alpha_\varphi \in \mathrm{Act}_r(L(X),(Y,\beta))$, it follows from the proof of (9.1.14) that $fQ^\alpha_\varphi = gQ^\alpha_\varphi$. Thus we see that $f = g$ and $\Phi^{(X,\alpha)}_{(Y,\beta)}$ is injective. For $h \in \mathrm{Act}_r(H)((X,\alpha),\varphi^*(Y,\beta))$, since

$$\begin{aligned}\beta(h \times id_G)(\alpha \times id_G) &= \beta(h\alpha \times id_G) = \beta(\varphi^*(\beta)(h \times id_H) \times id_G) = \beta(\beta(id_Y \times \varphi)(h \times id_H) \times id_G)\\ &= \beta(\beta \times id_G)(h \times \varphi \times id_G) = \beta(id_Y \times \mu)(h \times \varphi \times id_G) = \beta(h \times \mu(\varphi \times id_G))\\ &= \beta(h \times id_G)(id_X \times \mu(\varphi \times id_G)),\end{aligned}$$

there exists unique morphism $f : X^\alpha_\varphi \to Y$ that satisfies $fQ^\alpha_\varphi = \beta(h \times id_G)$. Moreover, since we have

$$\begin{aligned}f\alpha_\varphi(Q^\alpha_\varphi \times id_G) &= fQ^\alpha_\varphi(id_X \times \mu) = \beta(h \times id_G)(id_X \times \mu) = \beta(id_X \times \mu)(h \times id_G \times id_G)\\ &= \beta(\beta \times id_G)(h \times id_G \times id_G) = \beta(\beta(h \times id_G) \times id_G) = \beta(fQ^\alpha_\varphi \times id_G)\\ &= \beta(f \times id_G)(Q^\alpha_\varphi \times id_G),\end{aligned}$$

it follows $f\alpha_\varphi = \beta(f \times id_G)$, namely, $f : \varphi_!(X,\alpha) \to (Y,\beta)$ is a morphism of $\mathrm{Act}_r(G)$. We also have

$$\Phi^{(X,\alpha)}_{(Y,\beta)}(f) = \varphi^*(f)\eta_{(X,\alpha)} = fQ^\alpha_\varphi(id_X, \varepsilon o_X) = \beta(h \times id_G)(id_X, \varepsilon o_X) = \beta(h, \varepsilon o_X) = h.$$

Hence $\Phi^{(X,\alpha)}_{(Y,\beta)}$ is surjective. It is clear that $\Phi^{(X,\alpha)}_{(Y,\beta)}$ is natural in (Y,β). □

Remark 9.1.17 *If $\mathcal{T}$ is a cartesian closed category and the first assumption of (9.1.16) is satisfied, the second and the third assumptions are satisfied by (4) of (9.2.9) below. If the assumptions of (9.1.16) are all satisfied for arbitrary object (X,α) of $\mathrm{Act}_r(H)$, $\varphi^* : \mathrm{Act}_r(G) \to \mathrm{Act}_r(H)$ has a left adjoint $\varphi_! : \mathrm{Act}_r(H) \to \mathrm{Act}_r(G)$.*

9.2 Group objects in cartesian closed categories

Let $\mathcal{T}$ be a cartesian closed category, namely, $\mathcal{T}$ has finite products and, for any $Y, Z \in \mathrm{Ob}\,\mathcal{T}$, the functor $P_{Y,Z} : \mathcal{T}^{op} \to \mathcal{S}et$ given by $P_{Y,Z}(X) = \mathcal{T}(X \times Y, Z)$ is representable. We denote by Z^Y an object of $\mathcal{T}$ which represents $P_{Y,Z}$ and by $\exp_{X,Y,Z} : \mathcal{T}(X \times Y, Z) \to \mathcal{T}(X, Z^Y)$ the natural equivalence. We also assume that $\mathcal{T}$ has finite limits below.

We put $\eta^Y_X = \exp_{X,Y,X\times Y}(id_{X\times Y}) : X \to (X \times Y)^Y$ and $\varepsilon^Y_Z = \exp^{-1}_{Z^Y,Y,Z}(id_{Z^Y}) : Z^Y \times Y \to Z$. For a morphism $f : Z \to W$ of $\mathcal{T}$, define a morphism $f^Y : Z^Y \to W^Y$ to be the image of $f\varepsilon^Y_Z$ by $\exp_{Z^Y,Y,W} : \mathcal{T}(Z^Y \times Y, W) \to \mathcal{T}(Z^Y, W^Y)$.

Proposition 9.2.1 *For objects X, Y, Z and W of $\mathcal{T}$ and a morphism $f : X \to Y$, the following diagrams commute.*

$$\begin{tikzcd}\mathcal{T}(Y \times W, Z) \arrow[r, "\exp_{Y,W,Z}"] \arrow[d, "(f\times id_W)^*"] & \mathcal{T}(Y, Z^W) \arrow[d, "f^*"] \\ \mathcal{T}(X \times W, Z) \arrow[r, "\exp_{X,W,Z}"] & \mathcal{T}(X, Z^W)\end{tikzcd} \qquad \begin{tikzcd}\mathcal{T}(W \times Z, X) \arrow[r, "\exp_{W,Z,X}"] \arrow[d, "f_*"] & \mathcal{T}(W, X^Z) \arrow[d, "f^Z_*"] \\ \mathcal{T}(W \times Z, Y) \arrow[r, "\exp_{W,Z,Y}"] & \mathcal{T}(W, Y^Z)\end{tikzcd}$$

Proof. The commutativity of the left diagram is the naturality of exp. For a morphism $f : X \to Y$, consider a natural transformation $P_{Z,f} : P_{Z,X} \to P_{Z,Y}$ given by $(P_{Z,f})_W(g) = fg$ for $g \in \mathcal{T}(W \times Z, X)$. Then, a composition of maps $\mathcal{T}(W, X^Z) \xrightarrow{\exp_{W,Z,X}^{-1}} \mathcal{T}(W \times Z, X) \xrightarrow{(P_{Z,f})_W} \mathcal{T}(W \times Z, Y) \xrightarrow{\exp_{W,Y,Z}} \mathcal{T}(W, Y^Z)$ defines a natural transformation from the functor represented by X^Z to the functor represented by Y^Z. Hence it follows from Yoneda's lemma that this natural transformation is induced by

$$\exp_{X^Z,Y,Z}((P_{Z,f})_{X^Z}(\exp^{-1}_{X^Z,Z,X}(id_{X^Z}))) = \exp_{X^Z,Y,Z}((P_{Z,f})_{X^Z}(\varepsilon^Z_X)) = \exp_{X^Z,Y,Z}(f\varepsilon^Z_X) = f^Z$$

and the right diagram is commutative. □

Proposition 9.2.2 *For a morphism $f : X \to Y$, the following diagrams commute.*

$$\begin{tikzcd} X^Z \times Z \arrow[r, "f^Z \times id_Z"] \arrow[d, "\varepsilon^Z_X"] & Y^Z \times Z \arrow[d, "\varepsilon^Z_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta^Z_X"] & Y \arrow[d, "\eta^Z_Y"] \\ (X \times Z)^Z \arrow[r, "(f \times id_Z)^Z"] & (Y \times Z)^Z \end{tikzcd}$$

Proof. The commutativity of the left (resp. right) diagram follows from the commutativity of the left (resp. right) diagram below.

$$\begin{tikzcd} \mathcal{T}(Y^Z \times Z, Y) \arrow[r, "\exp_{Y^Z,Z,Y}"] \arrow[d, "(f^Z \times id_Z)^*"] & \mathcal{T}(Y^Z, Y^Z) \arrow[d, "(f^Z)^*"] \\ \mathcal{T}(X^Z \times Z, Y) \arrow[r, "\exp_{X^Z,Z,Y}"] & \mathcal{T}(X^Z, Y^Z) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}(X \times Z, X \times Z) \arrow[r, "\exp_{X,Z,X \times Z}"] \arrow[d, "(f \times id_Z)_*"] & \mathcal{T}(X, (X \times Z)^Z) \arrow[d, "(f \times id_Z)^Z_*"] \\ \mathcal{T}(X \times Z, Y \times Z) \arrow[r, "\exp_{X,Z,Y \times Z}"] & \mathcal{T}(X, (Y \times Z)^Z) \\ \mathcal{T}(Y \times Z, Y \times Z) \arrow[r, "\exp_{Y,Z,Y \times Z}"] \arrow[u, "(f \times id_Z)^*"] & \mathcal{T}(Y, (Y \times Z)^Z) \arrow[u, "f^*"] \end{tikzcd}$$

□

Proposition 9.2.3 *For objects X and Y of $\mathcal{T}$, $\varepsilon^Y_{X \times Y}(\eta^Y_X \times id_Y) = id_{X \times Y}$ and $\left(\varepsilon^Y_X\right)^Y \eta^Y_{X^Y} = id_{X^Y}$.*

Proof. It follows from the definition of $\varepsilon^Y_{X \times Y}$ and the commutativity of the following diagram, we have

$$\exp_{X,Y,X \times Y}(\varepsilon^Y_{X \times Y}(\eta^Y_X \times id_Y)) = \exp_{(X \times Y)^Y,Y,X \times Y}(\varepsilon^Y_{X \times Y})\eta^Y_X = \eta^Y_X = \exp_{X,Y,X \times Y}(id_{X \times Y}).$$

$$\begin{tikzcd} \mathcal{T}((X \times Y)^Y \times Y, X \times Y) \arrow[r, "\exp_{(X \times Y)^Y,Y,X \times Y}"] \arrow[d, "(\eta^Y_X \times id_Y)^*"] & \mathcal{T}((X \times Y)^Y, (X \times Y)^Y) \arrow[d, "(\eta^Y_X)^*"] \\ \mathcal{T}(X \times Y, X \times Y) \arrow[r, "\exp_{X,Y,X \times Y}"] & \mathcal{T}(X, (X \times Y)^Y) \end{tikzcd}$$

Since $\exp_{X,Y,X \times Y}$ is bijective, the first equality follows.

It follows from the definition of $\eta^Y_{X^Y}$ and the commutativity of the following diagram, we have

$$\exp^{-1}_{X^Y,Y,X}\left(\left(\varepsilon^Y_X\right)^Y \eta^Y_{X^Y}\right) = \varepsilon^Y_X \exp^{-1}_{X^Y,Y,X^Y \times Y}(\eta^Y_{X^Y}) = \varepsilon^Y_X = \exp^{-1}_{X^Y,Y,X}(id_{X^Y}).$$

$$\begin{tikzcd} \mathcal{T}(X^Y \times Y, X^Y \times Y) \arrow[r, "\exp_{X^Y,Y,X^Y \times Y}"] \arrow[d, "(\varepsilon^Y_X)_*"] & \mathcal{T}(X^Y, (X^Y \times Y)^Y) \arrow[d, "(\varepsilon^Y_X)^Y_*"] \\ \mathcal{T}(X^Y \times Y, X) \arrow[r, "\exp_{X^Y,Y,X}"] & \mathcal{T}(X^Y, X^Y) \end{tikzcd}$$

Since $\exp_{X^Y,Y,X}$ is bijective, the second equality follows. □

Lemma 9.2.4 *Let X be an object of $\mathcal{T}$ and $\mathrm{pr}_X : X \times 1 \to X$ the projection. Then $\exp_{X,1,X}(\mathrm{pr}_X) = \mathrm{pr}^1_X \eta^1_X : X \to X^1$ is an isomorphism whose inverse is composition $X^1 \xrightarrow{(id_{X^1}, o_{X^1})} X^1 \times 1 \xrightarrow{\varepsilon^1_X} X$.*

Proof. For a morphism $f : Y \to X$, $\exp^{-1}_{Y,1,X}(\mathrm{pr}^1_X \eta^1_X f) = \mathrm{pr}_1(f \times id_1) = f\mathrm{pr}_Y$, where $\mathrm{pr}_Y : Y \times 1 \to Y$ is the projection. Hence the composite $\mathcal{C}(Y, X) \xrightarrow{(\mathrm{pr}^1_X \eta^1_X)_*} \mathcal{C}(Y, X^1) \xrightarrow{\exp^{-1}_{Y,1,X}} \mathcal{C}(Y \times 1, X)$ coincides with pr^*_Y. Since pr_Y is an isomorphism by (9.1.2), it follows that $(\mathrm{pr}^1_X \eta^1_X)_*$ is a bijection. Therefore $\mathrm{pr}^1_X \eta^1_X$ is an isomorphism. Consider the case $Y = X^1$ and $f = \varepsilon^1_X(id_{X^1}, o_{X^1})$. Since $(id_{X^1}, o_{X^1})\mathrm{pr}_{X^1}$ is the identity morphism of $X^1 \times 1$, the image of f by the above composition is $\varepsilon^1_X = \exp^{-1}_{X^1,1,X}(id_{X^1})$. Hence we have $\mathrm{pr}^1_X \eta^1_X f = id_{X^1}$ and f is the inverse of $\mathrm{pr}^1_X \eta^1_X$. □

For a morphism $f : Z \to Y$ of $\mathcal{T}$, define a morphism $X^f : X^Y \to X^Z$ and to be the image of $\varepsilon^Y_X(id_{X^Y} \times f)$ by $\exp_{X^Y,Z,X} : \mathcal{T}(X^Y \times Z, X) \to \mathcal{T}(X^Y, X^Z)$.

Proposition 9.2.5 *For a morphism $f : X \to Y$ of $\mathcal{T}$, the following diagram is commutative for any object Z and W of $\mathcal{T}$.*

$$\begin{array}{ccc}
\mathcal{T}(W \times Y, Z) & \xrightarrow{\exp_{W,Y,Z}} & \mathcal{T}(W, Z^Y) \\
\Big\downarrow {\scriptstyle (id_W \times f)^*} & & \Big\downarrow {\scriptstyle Z^f_*} \\
\mathcal{T}(W \times X, Z) & \xrightarrow{\exp_{W,X,Z}} & \mathcal{T}(W, Z^X)
\end{array}$$

Proof. For any morphism $\alpha : W \times Y \to Z$, the following is commutative by (9.2.3) and (9.2.2).

$$\begin{tikzcd}
W \times X \arrow[r, "\eta^Y_W \times id_X"] \arrow[d, "id_W \times f"] & (W \times Y)^Y \times X \arrow[r, "\alpha^Y \times id_X"] \arrow[d, "id_{(W\times Y)^Y} \times f"] & Z^Y \times X \arrow[d, "id_{Z^Y} \times f"] \\
W \times Y \arrow[r, "\eta^Y_W \times id_Y"] \arrow[dr, "id_{W\times Y}"'] & (W \times Y)^Y \times Y \arrow[r, "\alpha^Y \times id_Y"] \arrow[d, "\varepsilon^Y_{W\times Y}"] & Z^Y \times Y \arrow[d, "\varepsilon^Y_Z"] \\
 & W \times Y \arrow[r, "\alpha"] & Z
\end{tikzcd}$$

Hence we have

$$\begin{aligned}
\exp_{W,X,Z}((id_W \times f)^*(\alpha)) &= \exp_{W,X,Z}(\alpha(id_W \times f)) = \exp_{W,X,Z}(\varepsilon^Y_Z(id_{Z^Y} \times f)(\alpha^Y \eta^Y_W \times id_X)) \\
&= \exp_{W,X,Z}((\alpha^Y \eta^Y_W \times id_X)^*(\varepsilon^Y_Z(id_{Z^Y} \times f)) = (\alpha^Y \eta^Y_W)^*(\exp_{Z^Y,X,Z}(\varepsilon^Y_Z(id_{Z^Y} \times f))) \\
&= Z^f \alpha^Y \eta^Y_W = Z^f_*(\exp_{W,Y,Z}(\alpha)).
\end{aligned}$$

□

Proposition 9.2.6 *For morphisms $f : Y \to Z$, $g : Z \to W$ and an object X of $\mathcal{T}$, $X^{gf} = X^f X^g$.*

Proof. The following diagram commutes by (9.2.5).

$$\begin{array}{ccccc}
\mathcal{T}(X^W \times W, X) & \xrightarrow{(id_{X^W} \times g)^*} & \mathcal{T}(X^W \times Z, X) & \xrightarrow{(id_{X^W} \times f)^*} & \mathcal{T}(X^W \times Y, X) \\
\Big\downarrow {\scriptstyle \exp_{X^W,Y,X}} & & \Big\downarrow {\scriptstyle \exp_{X^W,Y,Z}} & & \Big\downarrow {\scriptstyle \exp_{X^W,X,Z}} \\
\mathcal{T}(X^W, X^W) & \xrightarrow{X^g_*} & \mathcal{T}(X^W, X^Z) & \xrightarrow{X^f_*} & \mathcal{T}(X^W, X^Y)
\end{array}$$

Hence $X^f X^g = \exp_{X^W,X,Z}(id_{X^W} \times f)^*(id_{X^W} \times g)^*(\varepsilon^W_X) = \exp_{X^W,X,Z}(\varepsilon^W_X(id_{X^W} \times gf)) = X^{gf}$. □

Proposition 9.2.7 *For morphisms $f : X \to Z$ and $g : Y \to W$ in $\mathcal{T}$, the following diagram commutes.*

$$\begin{array}{ccc}
X^W & \xrightarrow{X^g} & X^Y \\
\Big\downarrow {\scriptstyle f^W} & & \Big\downarrow {\scriptstyle f^Y} \\
Z^W & \xrightarrow{Z^g} & Z^Y
\end{array}$$

Proof. It follows from (9.2.1) that $f^Y X^g = f^Y_*(X^g) = \exp_{X^W,Y,Z}(f\varepsilon^W_X(id_{X^W} \times g))$ and that $Z^g f^W = (f^W)^*(Z^g) = \exp_{X^W,Y,Z}(\varepsilon^W_Z(id_{Z^W} \times g)(f^W \times id_Y))$. By (9.2.2), the following diagram commutes.

$$\begin{tikzcd}
X^W \times Y \arrow[r, "id_{X^W}\times g"] \arrow[d, "f^W\times id_Y"] & X^W \times W \arrow[r, "\varepsilon^W_X"] \arrow[d, "f^W\times id_W"] & X \arrow[d, "f"] \\
Z^W \times Y \arrow[r, "id_{Z^W}\times g"] & Z^W \times W \arrow[r, "\varepsilon^W_Z"] & Z
\end{tikzcd}$$

Thus we have $f\varepsilon^W_X(id_{X^W} \times g) = \varepsilon^W_Z(id_{Z^W} \times g)(f^W \times id_Y)$ and the result follows. □

Proposition 9.2.8 *For a morphism $f : X \to Y$ of $\mathcal{T}$ and $Z \in \mathrm{Ob}\,\mathcal{T}$, the following diagrams commute.*

$$\begin{tikzcd}
Z^Y \times X \arrow[r, "id_{Z^Y}\times f"] \arrow[d, "Z^f\times id_X"] & Z^Y \times Y \arrow[d, "\varepsilon^Y_Z"] \\
Z^X \times X \arrow[r, "\varepsilon^X_Z"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "\eta^Y_Z"] \arrow[d, "\eta^X_Z"] & (Z\times Y)^Y \arrow[d, "(Z\times Y)^f"] \\
(Z\times X)^X \arrow[r, "(id_Z\times f)^X"] & (Z\times Y)^X
\end{tikzcd}$$

Proof. By (9.2.5) and (9.2.1), we have

$$\exp_{Z^Y,X,Z}(\varepsilon^Y_Z(id_{Z^Y} \times f)) = Z^f \exp_{X^Y,X,Y}(\varepsilon^Y_X) = Z^f = \exp_{Z^X,X,Z}(\varepsilon^X_Z)Z^f = \exp_{Z^Y,X,Z}(\varepsilon^X_Z(Z^f \times id_X))$$
$$\begin{aligned}\exp^{-1}_{Z,X,Z\times Y}(\eta^Y_Z(Z\times Y)^f) &= \exp^{-1}_{Z,Y,Z\times Y}(\eta^Y_Z)(id_Z \times f) = id_Z \times f = (id_Z \times f)\exp^{-1}_{Z,X,Z\times X}(\eta^X_Z)\\ &= \exp^{-1}_{Z,X,Z\times Y}((id_Z \times f)^X\eta^X_Z)\end{aligned}$$

Hence $\varepsilon^Y_Z(id_{Z^Y} \times f) = \varepsilon^X_Z(Z^f \times id_X)$ and $\eta^Y_Z(Z \times Y)^f = (id_Z \times f)^X\eta^X_Z$. □

Proposition 9.2.9 *Let $f : X \to Z$ be a morphism in a cartesian closed category $\mathcal{T}$.*
(1) If f is a monomorphism, so is $f^Y : X^Y \to Z^Y$.
(2) If f is an epimorphism, so is $f \times g : X \times Y \to Z \times W$.
(3) If f is an epimorphism, $Y^f : Y^Z \to Y^X$ is a monomorphism.
(4) If f is a coequalizer of $g : Y \to X$ and $h : Y \to X$, then $f \times id_V : X \times V \to Z \times V$ is a coequalizer of $g \times id_V : Y \times V \to X \times V$ and $h \times id_V : Y \times V \to X \times V$.

Proof. Let W be an object of $\mathcal{T}$.
(1) Since the following diagram commutes by (9.2.1) and f_* is injective, f^Y_* is injective.

$$\begin{tikzcd}
\mathcal{T}(W\times Y, X) \arrow[r, "\exp_{W,Y,X}"] \arrow[d, "f_*"] & \mathcal{T}(W, X^Y) \arrow[d, "f^Y_*"] \\
\mathcal{T}(W\times Y, Z) \arrow[r, "\exp_{W,Y,Z}"] & \mathcal{T}(W, Z^Y)
\end{tikzcd}$$

Hence f^Y is a monomorphism.
(2) Since the following diagram commutes by (9.2.1) and f^* is injective, $(f \times id_Y)^*$ is injective.

$$\begin{tikzcd}
\mathcal{T}(Z\times Y, W) \arrow[r, "\exp_{Z,Y,W}"] \arrow[d, "(f\times id_Y)^*"] & \mathcal{T}(Z, W^Y) \arrow[d, "f^*"] \\
\mathcal{T}(X\times Y, W) \arrow[r, "\exp_{X,Y,W}"] & \mathcal{T}(X, W^Y)
\end{tikzcd}$$

Hence $f \times id_Y$ is an epimorphism.
(3) Since the following diagram commutes by (9.2.5) and f^* is injective, $(id_W \times f)^*$ is injective by (2).

$$\begin{tikzcd}
\mathcal{T}(W\times Z, Y) \arrow[r, "\exp_{W,Z,Y}"] \arrow[d, "(id_W\times f)^*"] & \mathcal{T}(W, Y^Z) \arrow[d, "Y^f_*"] \\
\mathcal{T}(W\times X, Y) \arrow[r, "\exp_{W,X,Y}"] & \mathcal{T}(W, Y^X)
\end{tikzcd}$$

Hence Y^f is a monomorphism.
(4) The following diagrams commutes by (9.2.1).

$$\begin{tikzcd}
\mathcal{T}(Z\times V,W) \arrow[r, "(f\times id_V)^*"] \arrow[d, "\exp_{Z,V,W}"] & \mathcal{T}(X\times V,W) \arrow[r, "(g\times id_V)^*"] \arrow[d, "\exp_{X,V,W}"] & \mathcal{T}(Y\times V,W) \arrow[d, "\exp_{Y,V,W}"] \\
\mathcal{T}(Z,W^V) \arrow[r, "f^*"] & \mathcal{T}(X,W^V) \arrow[r, "g^*"] & \mathcal{T}(Y,W^V)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T}(X\times V,W) \arrow[r, "(h\times id_V)^*"] \arrow[d, "\exp_{X,V,W}"] & \mathcal{T}(Y\times V,W) \arrow[d, "\exp_{Y,V,W}"] \\
\mathcal{T}(X,W^V) \arrow[r, "h^*"] & \mathcal{T}(Y,W^V)
\end{tikzcd}$$

Since $f^* : \mathcal{T}(Z,W^V) \to \mathcal{T}(X,W^V)$ is an equalizer of $g^* : \mathcal{T}(X,W^V) \to \mathcal{T}(Y,W^V)$ and $h^* : \mathcal{T}(X,W^V) \to \mathcal{T}(Y,W^V)$ by the assumption, $(f \times id_V)^* : \mathcal{T}(Z \times V,W) \to \mathcal{T}(X \times V,W)$ is an equalizer of $(g \times id_V)^* : \mathcal{T}(X \times V,W) \to \mathcal{T}(Y \times V,W)$ and $(h \times id_V)^* : \mathcal{T}(X \times V,W) \to \mathcal{T}(Y \times V,W)$. □

Remark 9.2.10 *If $f : X \to Z$ and $g : Y \to W$ are epimorphisms, so is $f \times g : X \times Y \to Z \times W$. In fact, $id_Z \times g = T_{W,Z}(g \times id_Z)T_{Z,Y}$ is an epimorphism by (2) of (9.2.9), where $T_{Z,Y} : Z \times Y \to Y \times Z$ and $T_{W,Z} : W \times Z \to Z \times W$ are the switching maps. Thus $f \times g = (id_Z \times g)(f \times id_Y)$ is an epimorphism.*

For objects X, Y, Z of $\mathcal{T}$, we define a morphism $\gamma_{X,Y,Z} : Z^Y \times Y^X \to Z^X$ to be the image of the following composition of morphisms by $\exp_{Z^Y\times Y^X,X,Z} : \mathcal{T}(Z^Y \times Y^X \times X, Z) \to \mathcal{T}(Z^Y \times Y^X, Z^X)$.

$$Z^Y \times Y^X \times X \xrightarrow{id_{Z^Y}\times\varepsilon_Y^X} Z^Y \times Y \xrightarrow{\varepsilon_Z^Y} Z$$

We also define a morphism $\epsilon_X : 1 \to X^X$ to be the image of $\mathrm{pr}_2 : 1 \times X \to X$ by $\exp_{1,X,X} : \mathcal{T}(1 \times X, X) \to \mathcal{T}(1, X^X)$.

Proposition 9.2.11 *Let X, Y, Z and W be objects of $\mathcal{T}$.*

(1) The following diagram commutes.

$$\begin{tikzcd}
Z^Y \times Y^X \times X \arrow[r, "id_{Z^Y}\times\varepsilon_Y^X"] \arrow[d, "\gamma_{X,Y,Z}\times id_X"] & Z^Y \times Y \arrow[d, "\varepsilon_Z^Y"] \\
Z^X \times X \arrow[r, "\varepsilon_Z^X"] & Z
\end{tikzcd}$$

(2) The following diagram commutes.

$$\begin{tikzcd}
W^Z \times Z^Y \times Y^X \arrow[r, "\gamma_{Y,Z,W}\times id_{Y^X}"] \arrow[d, "id_{W^Z}\times\gamma_{X,Y,Z}"] & W^Y \times Y^X \arrow[d, "\gamma_{X,Y,W}"] \\
W^Z \times Z^X \arrow[r, "\gamma_{X,Z,W}"] & W^X
\end{tikzcd}$$

(3) The following diagrams commute.

$$\begin{tikzcd}
Y^X \times 1 \arrow[r, "id_{Y^X}\times\epsilon_X"] \arrow[dr, "\mathrm{pr}_1"'] & Y^X \times X^X \arrow[d, "\gamma_{X,X,Y}"] \\
& Y^X
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \times Y^X \arrow[r, "\epsilon_Y\times id_{Y^X}"] \arrow[dr, "\mathrm{pr}_2"'] & Y^Y \times Y^X \arrow[d, "\gamma_{X,Y,Y}"] \\
& Y^X
\end{tikzcd}$$

Proof. (1) By (9.2.1), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}(Z^X \times X, Z) \arrow[r, "\exp_{Z^X,X,Z}"] \arrow[d, "(\gamma_{X,Y,Z}\times id_X)^*"] & \mathcal{T}(Z^X, Z^X) \arrow[d, "\gamma_{X,Y,Z}^*"] \\
\mathcal{T}(Z^Y \times Y^X \times X, Z) \arrow[r, "\exp_{Z^Y\times Y^X,X,Z}"] & \mathcal{T}(Z^Y \times Y^X, Z^Y)
\end{tikzcd} \quad \cdots (i)$$

Since $\varepsilon_Z^X = \exp_{Z^X,X,Z}^{-1}(id_{Z^X})$, we have $\gamma_{X,Y,Z} = \exp_{Z^Y\times Y^X,X,Z}(\varepsilon_Z^X(\gamma_{X,Y,Z} \times id_X))$ by the commutativity of (i). On the other hand, since $\gamma_{X,Y,Z} = \exp_{Z^Y\times Y^X,X,Z}(\varepsilon_Z^Y(id_{Z^Y} \times \varepsilon_Y^X))$ by the definition of $\gamma_{X,Y,Z}$, it follows that $\varepsilon_Z^X(\gamma_{X,Y,Z} \times id_X) = \varepsilon_Z^Y(id_{Z^Y} \times \varepsilon_Y^X)$.

(2) By (9.2.1), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}(W^Z\times Z^X\times X,W) \arrow[r, "(id_{W^Z}\times\gamma_{X,Y,Z}\times id_X)^*"] \arrow[d, "\exp_{W^Z\times Z^X,X,W}"] & \mathcal{T}(W^Z\times Z^Y\times Y^X\times X,W) \arrow[d, "\exp_{W^Z\times Z^Y\times Y^X,X,W}"] & \mathcal{T}(W^Y\times Y^X\times X,W) \arrow[l, "(\gamma_{Y,Z,W}\times id_{Y^X}\times id_X)^*"'] \arrow[d, "\exp_{W^Y\times Y^X,X,W}"] \\
\mathcal{T}(W^Z\times Z^X,W^X) \arrow[r, "(id_{W^Z}\times\gamma_{X,Y,Z})^*"] & \mathcal{T}(W^Z\times Z^Y\times Y^X,W^X) & \mathcal{T}(W^Y\times Y^X,W^X) \arrow[l, "(\gamma_{Y,Z,W}\times id_{Y^X})^*"']
\end{tikzcd}$$

Since $\gamma_{X,Z,W} = \exp_{W^Z\times Z^X,X,W}(\varepsilon_W^Z(id_{W^Z}\times\varepsilon_Z^X))$ and $\gamma_{X,Y,W} = \exp_{W^Y\times Y^X,X,W}(\varepsilon_W^Y(id_{W^Y}\times\varepsilon_Y^X))$, it follows from the commutativity of the diagram of (1) and the above diagram that

$$\begin{aligned}
\exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon_W^Z(id_{W^Z}\times\varepsilon_Z^Y(id_{Z^Y}\times\varepsilon_Y^X)) &= \exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon_W^Z(id_{W^Z}\times\varepsilon_Z^X(\gamma_{X,Y,Z}\times id_X))\\
&= \exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon_W^Z(id_{W^Z}\times\varepsilon_Z^X)(id_{W^Z}\times\gamma_{X,Y,Z}\times id_X))\\
&= \gamma_{X,Z,W}(id_{W^Z}\times\gamma_{X,Y,Z})\\
\exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon_W^Z(id_{W^Z}\times\varepsilon_Z^Y(id_{Z^Y}\times\varepsilon_Y^X)) &= \exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon_W^Z(id_{W^Z}\times\varepsilon_Z^Y)(id_{W^Z}\times id_{Z^Y}\times\varepsilon_Y^X))\\
&= \exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon_W^Y(\gamma_{Y,Z,W}\times id_Y)(id_{W^Z\times Z^Y}\times\varepsilon_Y^X))\\
&= \exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon_W^Y(\gamma_{Y,Z,W}\times\varepsilon_Y^X))\\
&= \exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon_W^Y(id_{W^Y}\times\varepsilon_Y^X)(\gamma_{Y,Z,W}\times id_{Y^X}\times id_X))\\
&= \gamma_{X,Y,W}(\gamma_{Y,Z,W}\times id_{Y^X}).
\end{aligned}$$

(3) We first claim that the following diagram commutes.

$$\begin{tikzcd}
1\times X \arrow[r, "\epsilon_X\times id_X"] \arrow[dr, "\mathrm{pr}_2"'] & X^X\times X \arrow[d, "\varepsilon_X^X"] \\
& X
\end{tikzcd} \qquad \cdots\ (ii)$$

By (9.2.1), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}(X^X\times X,X) \arrow[r, "\exp_{X^X,X,X}"] \arrow[d, "(\epsilon_X\times id_X)^*"] & \mathcal{T}(X^X,X^X) \arrow[d, "\epsilon_X^*"] \\
\mathcal{T}(1\times X,X) \arrow[r, "\exp_{1,X,X}"] & \mathcal{T}(1,X^X)
\end{tikzcd} \qquad \cdots\ (iii)$$

Since $\epsilon_X = \exp_{1,X,X}(\mathrm{pr}_2)$ and $\exp_{X^X,X,X}(\varepsilon_X^X) = id_{X^X}$, it follows from the commutativity of (iii) that

$$\exp_{1,X,X}(\varepsilon_X^X(\epsilon_X\times id_X)) = \exp_{X^X,X,X}(\varepsilon_X^X)\epsilon_X = \exp_{1,X,X}(\mathrm{pr}_2).$$

Thus we have $\varepsilon_X^X(\epsilon_X\times id_X) = \mathrm{pr}_2$.

By (9.2.1), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}(Y^X\times X^X\times X,Y) \arrow[r, "(id_{Y^X}\times\epsilon_X\times id_X)^*"] \arrow[d, "\exp_{Y^X\times X^X,X,Y}"] & \mathcal{T}(Y^X\times 1\times X,Y) \arrow[d, "\exp_{Y^X\times 1,X,Y}"] & \mathcal{T}(Y^X\times X,Y) \arrow[l, "(\mathrm{pr}_1\times id_X)^*"'] \arrow[d, "\exp_{Y^X,X,Y}"] \\
\mathcal{T}(Y^X\times X^X,Y^X) \arrow[r, "(id_{Y^X}\times\epsilon_X)^*"] & \mathcal{T}(Y^X\times 1,X^Y) & \mathcal{T}(Y^X,Y^X) \arrow[l, "\mathrm{pr}_1^*"']
\end{tikzcd}$$

Since $\exp_{Y^X\times X^X,X,Y}(\varepsilon_Y^X(id_{Y^X}\times\varepsilon_X^X)) = \gamma_{X,X,Y}$ and $\exp_{Y^X,X,Y}(\varepsilon_Y^X) = id_{Y^X}$, it follows from the commutativity of (ii) and the above diagram that

$$\begin{aligned}
\exp_{Y^X\times 1,X,Y}(\varepsilon_Y^X(id_{Y^X}\times\mathrm{pr}_2)) &= \exp_{Y^X\times 1,X,Y}(\varepsilon_Y^X(id_{Y^X}\times\varepsilon_X^X(\epsilon_X\times id_X)))\\
&= \exp_{Y^X\times 1,X,Y}(\varepsilon_Y^X(id_{Y^X}\times\varepsilon_X^X)(id_{Y^X}\times\epsilon_X\times id_X))\\
&= \exp_{Y^X\times X^X,X,Y}(\varepsilon_Y^X(id_{Y^X}\times\varepsilon_X^X))(id_{Y^X}\times\epsilon_X) = \gamma_{X,X,Y}(id_{Y^X}\times\epsilon_X)\\
\exp_{Y^X\times 1,X,Y}(\varepsilon_Y^X(id_{Y^X}\times\mathrm{pr}_2)) &= \exp_{Y^X\times 1,X,Y}(\varepsilon_Y^X(\mathrm{pr}_1\times id_X)) = \exp_{Y^X,X,Y}(\varepsilon_Y^X)\mathrm{pr}_1 = \mathrm{pr}_1.
\end{aligned}$$

By (9.2.1), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}(Y^Y\times Y^X\times X,Y) \arrow[r, "(\epsilon_Y\times id_{Y^X}\times id_X)^*"] \arrow[d, "\exp_{Y^Y\times Y^X,X,Y}"] & \mathcal{T}(1\times Y^X\times X,Y) \arrow[d, "\exp_{1\times Y^X,X,Y}"] & \mathcal{T}(Y^X\times X,Y) \arrow[l, "(\mathrm{pr}_2\times id_X)^*"'] \arrow[d, "\exp_{Y^X,X,Y}"] \\
\mathcal{T}(Y^Y\times Y^X,Y^X) \arrow[r, "(\epsilon_Y\times id_{Y^X})^*"] & \mathcal{T}(1\times Y^X,X^Y) & \mathcal{T}(Y^X,Y^X) \arrow[l, "\mathrm{pr}_2^*"']
\end{tikzcd}$$

Since $\exp_{Y^Y\times Y^X,X,Y}(\varepsilon_Y^Y(id_{Y^Y}\times\varepsilon_Y^X)) = \gamma_{X,Y,Y}$ and $\varepsilon_Y^X = \exp_{Y^X,X,Y}(id_{Y^X})$, it follows from the commutativity of (iii) and the above diagram that

$$\exp_{1\times Y^X,X,Y}(\varepsilon_Y^X(\mathrm{pr}_2\times id_X)) = \exp_{1\times Y^X,X,Y}(\mathrm{pr}_2(id_1\times\varepsilon_Y^X)) = \exp_{1\times Y^X,X,Y}(\varepsilon_Y^Y(\epsilon_Y\times id_Y)(id_1\times\varepsilon_Y^X))$$
$$= \exp_{1\times Y^X,X,Y}(\varepsilon_Y^Y(id_{Y^Y}\times\varepsilon_Y^X)(\epsilon_Y\times id_{Y^X}\times id_X))$$
$$= \exp_{Y^Y\times Y^X,X,Y}(\varepsilon_Y^Y(id_{Y^Y}\times\varepsilon_Y^X))(\epsilon_Y\times id_{Y^X}) = \gamma_{X,Y,Y}(\epsilon_Y\times id_{Y^X})$$
$$\exp_{1\times Y^X,X,Y}(\varepsilon_Y^X(\mathrm{pr}_2\times id_X)) = \exp_{Y^X,X,Y}(\varepsilon_Y^X)\mathrm{pr}_2 = \mathrm{pr}_2.$$

□

Remark 9.2.12 *By the above result, $(X^X,\gamma_{X,X,X},\epsilon_X)$ is a monoid object of $\mathcal{T}$.*

Proposition 9.2.13 *Let X, Y, Z and W be objects of $\mathcal{T}$ and $f: W\to X$, $g: Z\to W$ morphisms of $\mathcal{T}$. The following diagram commutes.*

$$\begin{tikzcd}
Z^Y\times Y^W \arrow[d, "\gamma_{W,Y,Z}"] & Z^Y\times Y^X \arrow[l, "id_{Z^Y}\times Y^f"'] \arrow[r, "g^Y\times id_{Y^X}"] \arrow[d, "\gamma_{X,Y,Z}"] & W^Y\times Y^X \arrow[d, "\gamma_{X,Y,W}"] \\
Z^W & Z^X \arrow[l, "Z^f"'] \arrow[r, "g^X"] & W^X
\end{tikzcd}$$

Proof. By (9.2.5), (9.2.8), (1) of (9.2.11), (9.2.1) and (9.2.2), we have

$$\begin{aligned}
Z^f\gamma_{X,Y,Z} &= Z^f\exp_{Z^Y\times Y^X,X,Z}(\varepsilon_Z^Y(id_{Z^Y}\times\varepsilon_Y^X)) = \exp_{Z^Y\times Y^X,W,Z}(\varepsilon_Z^Y(id_{Z^Y}\times\varepsilon_Y^X(id_{Y^X}\times f)))\\
&= \exp_{Z^Y\times Y^X,W,Z}(\varepsilon_Z^Y(id_{Z^Y}\times\varepsilon_Y^W(Y^f\times id_W))) = \exp_{Z^Y\times Y^X,W,Z}(\varepsilon_Z^Y(id_{Z^Y}\times\varepsilon_Y^W)(id_{Z^Y}\times Y^f\times id_W))\\
&= \exp_{Z^Y\times Y^X,W,Z}(\varepsilon_Z^W(\gamma_{W,Y,Z}\times id_W)(id_{Z^Y}\times Y^f\times id_W))\\
&= \exp_{Z^Y\times Y^X,W,Z}(\varepsilon_Z^W(\gamma_{W,Y,Z}(id_{Z^Y}\times Y^f)\times id_W))\\
&= \exp_{Z^W,W,Z}(\varepsilon_Z^W)\gamma_{W,Y,Z}(id_{Z^Y}\times Y^f) = \gamma_{W,Y,Z}(id_{Z^Y}\times Y^f)\\
g^X\gamma_{X,Y,Z} &= g^X\exp_{Z^Y\times Y^X,X,Z}(\varepsilon_Z^Y(id_{Z^Y}\times\varepsilon_Y^X)) = \exp_{Z^Y\times Y^X,X,W}(g\varepsilon_Z^Y(id_{Z^Y}\times\varepsilon_Y^X))\\
&= \exp_{Z^Y\times Y^X,X,W}(\varepsilon_W^Y(g^Y\times id_Y)(id_{Z^Y}\times\varepsilon_Y^X)) = \exp_{X^Y\times Y^X,X,W}(\varepsilon_W^Y(g^Y\times\varepsilon_Y^X))\\
&= \exp_{X^Y\times Y^X,X,W}(\varepsilon_W^Y(id_{W^Y}\times\varepsilon_Y^X)(g^Y\times id_{Y^X}\times id_X))\\
&= \exp_{W^Y\times Y^X,X,W}(\varepsilon_W^Y(id_{W^Y}\times\varepsilon_Y^X))(g^Y\times id_{Y^X}) = \gamma_{X,Y,W}(g^Y\times id_{Y^X}).
\end{aligned}$$

□

Let G be a group object in $\mathcal{T}$ and $\alpha: X\times G\to X$ an action of G on $X\in\mathrm{Ob}\,\mathcal{T}$. We define $\mathrm{Ad}_r(\alpha): X\to X^G$ and $\mathrm{Ad}_l(\alpha): G\to X^X$ by $\mathrm{Ad}_r(\alpha) = \exp_{X,G,X}(\alpha)$ and $\mathrm{Ad}_l(\alpha) = \exp_{G,X,X}(\alpha T_{G,X})$, where $T_{Y,X}: Y\times X\to X\times Y$ is the switching map.

Proposition 9.2.14 *Let (X,α) be an object of $\mathrm{Act}_r(G)$.*

(1) The following diagram commutes.

$$\begin{tikzcd}
X\times G \arrow[r, "\alpha"] \arrow[d, "T_{X,G}"] & X \\
G\times X \arrow[r, "\mathrm{Ad}_l(\alpha)\times id_X"] & X^X\times X \arrow[u, "\varepsilon_X^X"']
\end{tikzcd}$$

(2) Let us denote by $\mathrm{pr}_X: X\times 1\to X$ the projection. The following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{Ad}_r(\alpha)"] \arrow[dr, "\exp_{X,1,X}(\mathrm{pr}_X)"'] & X^G \arrow[d, "X^\varepsilon"] \\
& X^1
\end{tikzcd}$$

(3) $(id_{X^G},\varepsilon o_{X^G})\mathrm{Ad}_r(\alpha) = (\mathrm{Ad}_r(\alpha)\times id_G)(id_X,\varepsilon o_X): X\to X^G\times G$ is a right inverse of $\varepsilon_X^G: X^G\times G\to X$.

(4) Let $f: (X,\alpha)\to(Y,\beta)$ be a morphism of $\mathrm{Act}_r(G)$. The following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{Ad}_r(\alpha)"] \arrow[d, "f"] & X^G \arrow[d, "f^G"] \\
Y \arrow[r, "\mathrm{Ad}_r(\beta)"] & Y^G
\end{tikzcd}$$

Proof. (1) By (9.2.1), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}(X^X \times X, X) \arrow[r, "\exp_{X^X,X,X}"] \arrow[d, "(\mathrm{Ad}_l(\alpha)\times id_X)^*"] & \mathcal{T}(X^X, X^X) \arrow[d, "\mathrm{Ad}_l(\alpha)^*"] \\
\mathcal{T}(G \times X, X) \arrow[r, "\exp_{G,X,X}"] & \mathcal{T}(G \times G \times X, X)
\end{tikzcd}$$

Therefore, $\exp_{G,X,X}(\varepsilon^X_X(\mathrm{Ad}_l(\alpha) \times id_X)) = \exp_{X^X,X,X}(\varepsilon^X_X)\mathrm{Ad}_l(\alpha) = \mathrm{Ad}_l(\alpha) = \exp_{G,X,X}(\alpha T_{G,X})$, which implies $\alpha T_{G,X} = \varepsilon^X_X(\mathrm{Ad}_l(\alpha) \times id_X)$. Hence we have $\alpha = \varepsilon^X_X(\mathrm{Ad}_l(\alpha) \times id_X)T_{X,G}$.

(2) By (9.2.5), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}(X \times G, X) \arrow[r, "(id_X\times\varepsilon)^*"] \arrow[d, "\exp_{X,G,X}"] & \mathcal{T}(X \times 1, x) \arrow[d, "\exp_{X,1,X}"] \\
\mathcal{T}(X, X^G) \arrow[r, "X^{\varepsilon}_*"] & \mathcal{T}(X, X^1)
\end{tikzcd}$$

Since $\alpha(id_X \times \varepsilon) = \mathrm{pr}_X$, the assertion follows.

(3) Since $\mathrm{Ad}_r(\alpha) = \alpha^G\eta^G_X$, it follows from (9.2.2) and (9.2.3) that

$$\begin{aligned}
\varepsilon^G_X(id_{X^G}, \varepsilon o_{X^G})\mathrm{Ad}_r(\alpha) &= \varepsilon^G_X(id_{X^G}, \varepsilon o_{X^G})\alpha^G\eta^G_X = \varepsilon^G_X(\alpha^G\eta^G_X, \varepsilon o_X) = \varepsilon^G_X(\alpha^G \times id_G)(\eta^G_X \times id_G)(id_X, \varepsilon o_X) \\
&= \alpha\varepsilon^G_{X\times G}(\eta^G_X \times id_G)(id_X, \varepsilon o_X) = \alpha(id_X, \varepsilon o_X) = id_X.
\end{aligned}$$

(4) By (9.2.1), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}(X \times G, X) \arrow[r, "f_*"] \arrow[d, "\exp_{X,G,X}"] & \mathcal{T}(X \times G, Y) \arrow[d, "\exp_{X,G,Y}"] & \mathcal{T}(Y \times G, Y) \arrow[l, "(f\times id_G)^*"'] \arrow[d, "\exp_{Y,G,Y}"] \\
\mathcal{T}(X, X^G) \arrow[r, "f^G_*"] & \mathcal{T}(X, Y^G) & \mathcal{T}(Y, Y^G) \arrow[l, "f^*"']
\end{tikzcd}$$

Since $f\alpha = \beta(f \times id_G)$, the right diagram commutes. $\square$

Proposition 9.2.15 *The following diagrams commute.*

$$\begin{tikzcd}
G \times G \arrow[r, "\mu"] \arrow[d, "\mathrm{Ad}_l(\alpha)\times\mathrm{Ad}_l(\alpha)"] & G \arrow[d, "\mathrm{Ad}_l(\alpha)"] \\
X^X \times X^X \arrow[r, "\gamma_{X,X,X}T_{X^X,X^X}"] & X^X
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\varepsilon"] \arrow[dr, "\epsilon_X"'] & G \arrow[d, "\mathrm{Ad}_l(\alpha)"] \\
 & X^X
\end{tikzcd}$$

Proof. It follows from (1) of (9.2.14) that

$$\begin{aligned}
\alpha(\alpha \times id_G)T_{G\times G,X} &= \varepsilon^X_X(\mathrm{Ad}_l(\alpha) \times id_X)T_{X,G}(\varepsilon^X_X(\mathrm{Ad}_l(\alpha) \times id_X)T_{X,G} \times id_G)T_{G\times G,X} \\
&= \varepsilon^X_X(\mathrm{Ad}_l(\alpha) \times id_X)(id_G \times \varepsilon^X_X(\mathrm{Ad}_l(\alpha) \times id_X)T_{X,G})T_{X\times G,G}T_{G\times G,X} \\
&= \varepsilon^X_X(\mathrm{Ad}_l(\alpha) \times \varepsilon^X_X(\mathrm{Ad}_l(\alpha) \times id_X))(id_G \times T_{X,G})T_{X\times G,G}T_{G\times G,X} \\
&= \varepsilon^X_X(id_{X^X} \times \varepsilon^X_X)(\mathrm{Ad}_l(\alpha) \times \mathrm{Ad}_l(\alpha) \times id_X)(T_{G,G} \times id_X) \\
&= \varepsilon^X_X(id_{X^X} \times \varepsilon^X_X)(T_{X^X,X^X}(\mathrm{Ad}_l(\alpha) \times \mathrm{Ad}_l(\alpha)) \times id_X).
\end{aligned}$$

By (9.2.1), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}(G \times X, X) \arrow[r, "(\mu\times id_X)^*"] \arrow[d, "\exp_{G,X,X}"] & \mathcal{T}(G \times G \times X, X) \arrow[d, "\exp_{G\times G,X,X}"] & \mathcal{T}(X^X \times X^X \times X, X) \arrow[l, "(T_{X^X,X^X}(\mathrm{Ad}_l(\alpha)\times\mathrm{Ad}_l(\alpha))\times id_X)^*"'] \arrow[d, "\exp_{X^X\times X^X,X,X}"] \\
\mathcal{T}(G, X^X) \arrow[r, "\mu^*"] & \mathcal{T}(G \times G, X^X) & \mathcal{T}(X^X \times X^X, X^X) \arrow[l, "(T_{X^X,X^X}(\mathrm{Ad}_l(\alpha)\times\mathrm{Ad}_l(\alpha)))^*"']
\end{tikzcd}$$

Hence we have

$$\begin{aligned}\mathrm{Ad}_l(\alpha)\mu &= \exp_{G,X,X}(\alpha T_{G,X})\mu = \exp_{G\times G,X,X}(\alpha T_{G,X}(\mu\times id_X))\\ &= \exp_{G\times G,X,X}(\alpha(id_X\times\mu)T_{G\times G,X}) = \exp_{G\times G,X,X}(\alpha(\alpha\times id_G)T_{G\times G,X})\\ &= \exp_{G\times G,X,X}(\varepsilon^X_X(id_{X^X}\times\varepsilon^X_X)(T_{X^X,X^X}(\mathrm{Ad}_l(\alpha)\times\mathrm{Ad}_l(\alpha))\times id_X))\\ &= \exp_{X^X\times X^X,X,X}(\varepsilon^X_X(id_{X^X}\times\varepsilon^X_X))T_{X^X,X^X}(\mathrm{Ad}_l(\alpha)\times\mathrm{Ad}_l(\alpha))\\ &= \gamma_{X,X,X}T_{X^X,X^X}(\mathrm{Ad}_l(\alpha)\times\mathrm{Ad}_l(\alpha)).\end{aligned}$$

By (9.2.1), we have

$$\begin{aligned}\mathrm{Ad}_l(\alpha)\varepsilon &= \exp_{G,X,X}(\alpha T_{G,X})\varepsilon = \exp_{1,X,X}(\alpha T_{G,X}(\varepsilon\times id_X)) = \exp_{1,X,X}(\alpha(id_X\times\varepsilon)T_{1,X})\\ &= \exp_{1,X,X}(\mathrm{pr}_1T_{1,X}) = \exp_{1,X,X}(\mathrm{pr}_2) = \epsilon_X.\end{aligned}$$

□

Definition 9.2.16 *Let $\alpha,\beta : X\times G\to X$ be right actions of a group object G in $\mathcal{T}$ on $X\in\mathrm{Ob}\,\mathcal{T}$.*

(1) For a morphism $f : H\to G$, we denote by $e^{\alpha,\beta}_f : X^{\alpha,\beta}_f\to X$ the equalizer of $X\xrightarrow{\mathrm{Ad}_r(\alpha)}X^G\xrightarrow{X^f}X^H$ and $X\xrightarrow{\mathrm{Ad}_r(\beta)}X^G\xrightarrow{X^f}X^H$. We denote $X^{\alpha,\beta}_{id_G}$ by $X^{\alpha,\beta}$ and if β is the trivial action, we denote $X^{\alpha,\beta}_{id_G}$ by X^α.

(2) For a morphism $f : Y\to X$, we denote by $e^f_{\alpha,\beta} : G^f_{\alpha,\beta}\to G$ the equalizer of $G\xrightarrow{\mathrm{Ad}_l(\alpha)}X^X\xrightarrow{X^f}X^Y$ and $G\xrightarrow{\mathrm{Ad}_l(\beta)}X^X\xrightarrow{X^f}X^Y$. If β is the trivial action and $f : Y\to X$ is a subobject of X, we denote $G^f_{\alpha,\beta}$ by $\mathrm{Cent}_\alpha(Y)$.

(3) For morphisms $f : Y\to X$, $g : Z\to X$ of X, we denote by $\tau^{f,g}_\alpha : \mathrm{Transp}_\alpha(f,g)\to G$ the pull-back of $g^Y : Z^Y\to X^Y$ along $G\xrightarrow{\mathrm{Ad}_l(\alpha)}X^X\xrightarrow{X^f}X^Y$. We denote $\tau^{f,f}_\alpha : \mathrm{Transp}_\alpha(f,f)\to G$ by $\tau^Y_\alpha : \mathrm{Stab}_\alpha(Y)\to G$ if $Y=Z$, $f=g$ and f is a subobject of X.

(4) For a subobject $i : Y\to X$ of X, we denote by $\nu^Y_\alpha : \mathrm{Norm}_\alpha(Y)\to\mathrm{Stab}_\alpha(Y)$ the pull-back of $\mathrm{Stab}_\alpha(Y)\xrightarrow{\tau^Y_\alpha}G\xrightarrow{\iota}G$ along $\tau^Y_\alpha : \mathrm{Stab}_\alpha(Y)\to G$.

Remark 9.2.17 *(1) By (9.2.5) and (9.2.1), a morphism $\varphi : Z\to X$ satisfies $X^f\mathrm{Ad}_r(\alpha)\varphi = X^f\mathrm{Ad}_r(\beta)\varphi$ if and only if $\alpha(\varphi\times f)=\beta(\varphi\times f)$. In particular, $\alpha(e^{\alpha,\beta}_f\times f)=\beta(e^{\alpha,\beta}_f\times f)$.*

(2) It follows from (9.2.9) that $\tau^{f,g}_\alpha : \mathrm{Transp}_\alpha(f,g)\to G$ is a monomorphism if g is a monomorphism. Hence $\nu^Y_\alpha : \mathrm{Norm}_\alpha(Y)\to\mathrm{Stab}_\alpha(Y)$ is also a monomorphism. We can regard $\mathrm{Stab}_\alpha(Y)$ as a subobject of G and regard $\mathrm{Norm}_\alpha(Y)$ as a subobject of $\mathrm{Stab}_\alpha(Y)$.

Proposition 9.2.18 *Let $\alpha : X\times G\to X$ be a right action of a group object $(G,\mu,\varepsilon,\iota)$ in $\mathcal{T}$ on $X\in\mathrm{Ob}\,\mathcal{T}$ and $\tau : X\times G\to X$ the trivial action. If $i : H\to G$ is a morphism of group objects and $j : N\to G$ is a subgroup object of G such that H normalizes N, then $e^{\alpha,\tau}_j : X^{\alpha,\tau}_j\to X$ is an H-invariant subobject of X.*

Proof. Since H normalizes N, there exists a morphism $\lambda : H\times N\to N$ satisfying $j\lambda=\gamma_l(i\times j)$. We have $X^j\mathrm{Ad}_r(\alpha)e^{\alpha,\tau}_j = X^j\mathrm{Ad}_r(\tau)e^{\alpha,\tau}_j$ by the definition of $e^{\alpha,\tau}_j$. It follows from the naturality of the adjunctions that $\alpha(e^{\alpha,\tau}_j\times j)=e^{\alpha,\tau}_j\mathrm{pr}_1 : X^{\alpha,\tau}_j\times N\to X$. Since $\mu=\mu(\gamma_l,\mathrm{pr}_1)$ by the definition of γ_l, we have

$$\begin{aligned}\alpha(e^{\alpha,\tau}_j\times\mu(i\times j)) &= \alpha(e^{\alpha,\tau}_j\times\mu(\gamma_l,\mathrm{pr}_1)(i\times j)) = \alpha(e^{\alpha,\tau}_j\times\mu(\gamma_l(i\times j),\mathrm{pr}_1(i\times j)))\\ &= \alpha(e^{\alpha,\tau}_j\times\mu(j\lambda,i\mathrm{pr}_1)) = \alpha(\alpha(e^{\alpha,\tau}_j\mathrm{pr}_1,j\lambda(\mathrm{pr}_2,\mathrm{pr}_3)),i\mathrm{pr}_2)\\ &= \alpha(\alpha(e^{\alpha,\tau}_j,j\lambda),i\mathrm{pr}_2) = \alpha(e^{\alpha,\tau}_j\mathrm{pr}_1,i\mathrm{pr}_2) = \alpha(e^{\alpha,\tau}_j\times\mathrm{pr}_1).\end{aligned}$$

The following diagram commutes by the naturality of adjunctions.

$$\begin{tikzcd}
\mathcal{T}(X \times G, X) \arrow[r, "\exp_{X,G,X}"] \arrow[d, "(\alpha\times id_G)^*"] & \mathcal{T}(X, X^G) \arrow[d, "\alpha^*"] \\
\mathcal{T}(X \times G \times G, X) \arrow[r, "\exp_{X\times G,G,X}"] \arrow[d, "(e_j^{\alpha,\tau}\times i\times id_G)^*"] & \mathcal{T}(X \times G, X^G) \arrow[d, "(e_j^{\alpha,\tau}\times i)^*"] \\
\mathcal{T}(X_j^{\alpha,\tau} \times H \times G, X) \arrow[r, "\exp_{X_j^{\alpha,\tau}\times H,G,X}"] \arrow[d, "(id_{X_j^{\alpha,\tau}}\times id_H\times j)^*"] & \mathcal{T}(X_j^{\alpha,\tau} \times H, X^G) \arrow[d, "X^j_*"] \\
\mathcal{T}(X_j^{\alpha,\tau} \times H \times N, X) \arrow[r, "\exp_{X_j^{\alpha,\tau}\times H,N,X}"] & \mathcal{T}(X_j^{\alpha,\tau} \times H, X^N)
\end{tikzcd}$$

α and τ are mapped to $\alpha(e_j^{\alpha,\tau} \times \mu(i \times j))$ and $\alpha(e_j^{\alpha,\tau} \times \mathrm{pr}_1)$, respectively by the composition of $(\alpha \times id_G)^*$, $(e_j^{\alpha,\tau} \times i \times id_G)^*$ and $(id_{X_j^{\alpha,\tau}} \times id_H \times j)^*$. Here we denote by $\mathrm{pr}_1 : H \times N \to N$ the projection onto the first component. On the other hand, α and τ in $\mathcal{T}(X \times G, X)$ are mapped to $X^j \mathrm{Ad}_r(\alpha)\alpha(e_j^{\alpha,\tau} \times i)$ and $X^j \mathrm{Ad}_r(\tau)\alpha(e_j^{\alpha,\tau} \times i)$, respectively by the composition of $\exp_{X,G,X}$, α^*, $(e_j^{\alpha,\tau} \times i)^*$ and X^j_*. Therefore $\alpha(e_j^{\alpha,\tau} \times i) : X_j^{\alpha,\tau} \times H \to X$ satisfies $X^j \mathrm{Ad}_r(\alpha)\alpha(e_j^{\alpha,\tau} \times i) = X^j \mathrm{Ad}_r(\tau)\alpha(e_j^{\alpha,\tau} \times i)$, which implies there exists unique morphism $\beta : X_j^{\alpha,\tau} \times H \to X_j^{\alpha,\tau}$ that satisfies $e_j^{\alpha,\tau}\beta = \alpha(e_j^{\alpha,\tau} \times i)$. □

Proposition 9.2.19 *Let $\alpha, \beta : X \times G \to X$ be right actions of a group object $(G, \mu, \varepsilon, \iota)$ in $\mathcal{T}$ on $X \in \mathrm{Ob}\,\mathcal{T}$ and $f : Y \to X$ a morphism of $\mathcal{T}$. Suppose that there exists a morphism $\gamma : Y \times G \to Y$ satisfying $\alpha(f \times id_G) = f\gamma$. Then, $e^f_{\alpha,\beta} : G^f_{\alpha,\beta} \to G$ is a subgroup object of G, that is, there exist morphisms $\lambda : G^f_{\alpha,\beta} \times G^f_{\alpha,\beta} \to G^f_{\alpha,\beta}$ and $\kappa : G^f_{\alpha,\beta} \to G^f_{\alpha,\beta}$ which make the following diagrams commute.*

$$\begin{tikzcd}
G^f_{\alpha,\beta} \times G^f_{\alpha,\beta} \arrow[r, "e^f_{\alpha,\beta}\times e^f_{\alpha,\beta}"] \arrow[d, "\lambda"] & G \times G \arrow[d, "\mu"] \\
G^f_{\alpha,\beta} \arrow[r, "e^f_{\alpha,\beta}"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G^f_{\alpha,\beta} \arrow[r, "e^f_{\alpha,\beta}"] \arrow[d, "\kappa"] & G \arrow[d, "\iota"] \\
G^f_{\alpha,\beta} \arrow[r, "e^f_{\alpha,\beta}"] & G
\end{tikzcd}$$

Proof. For $Z \in \mathrm{Ob}\,\mathcal{T}$ and $g \in \mathcal{T}(Z, G)$, we have

$$\begin{aligned}
\exp^{-1}_{Z,Y,X} X^f_* \mathrm{Ad}_l(\alpha)_*(g) &= (id_Z \times f)^* \exp^{-1}_{Z,X,X} \mathrm{Ad}_l(\alpha)_*(g) = (id_Z \times f)^* \exp^{-1}_{Z,X,X}(\exp_{G,X,X}(\alpha T_{G,X})g) \\
&= (id_Z \times f)^* \exp^{-1}_{Z,X,X}(\exp_{Z,X,X}(\alpha T_{G,X}(g \times id_X))) = (id_Z \times f)^*(\alpha T_{G,X}(g \times id_X)) \\
&= \alpha(id_X \times g)T_{Z,X}(id_Z \times f) = \alpha(f \times g)T_{Z,Y}.
\end{aligned}$$

Hence $X^f_* \mathrm{Ad}_l(\alpha)_*(g) = X^f_* \mathrm{Ad}_l(\beta)_*(g)$ if and only if $\alpha(f \times g) = \beta(f \times g)$. If $X^f_* \mathrm{Ad}_l(\alpha)_*(g) = X^f_* \mathrm{Ad}_l(\beta)_*(g)$ and $X^f_* \mathrm{Ad}_l(\alpha)_*(h) = X^f_* \mathrm{Ad}_l(\beta)_*(h)$ for $g, h \in \mathcal{T}(Z, G)$, then $\alpha(f \times g) = \beta(f \times g)$ and $\alpha(f \times h) = \beta(f \times h)$. We note that $\alpha(f \times g) = \alpha(f \times id_G)(id_Y \times g) = f\gamma(id_Y \times g)$. Put $k = \mu(g \times h)\Delta_Z$, where $\Delta_Z : Z \to Z \times Z$ is the diagonal morphism, then the following shows $X^f_* \mathrm{Ad}_l(\alpha)_*(k) = X^f_* \mathrm{Ad}_l(\beta)_*(k)$.

$$\begin{aligned}
\beta(f \times k) &= \beta(f \times \mu(g \times h)\Delta_Z) = \beta(id_X \times \mu(g \times h)\Delta_Z)(f \times id_Z) = \beta(id_X \times \mu)(f \times (g \times h)\Delta_Z) \\
&= \beta(\beta \times id_G)(f \times g \times h)(id_Y \times \Delta_Z) = \beta(\beta(f \times g) \times h)(id_Y \times \Delta_Z) = \beta(\alpha(f \times g) \times h)(id_Y \times \Delta_Z) \\
&= \beta(f\gamma(id_Y \times g) \times h)(id_Y \times \Delta_Z) = \beta(f \times h)(\gamma(id_Y \times g) \times id_Z)(id_Y \times \Delta_Z) \\
&= \alpha(f \times h)(\gamma(id_Y \times g) \times id_Z)(id_Y \times \Delta_Z) = \alpha(f\gamma(id_Y \times g) \times h)(id_Y \times \Delta_Z) \\
&= \alpha(\alpha(f \times g) \times h)(id_Y \times \Delta_Z) = \alpha(\alpha \times id_G)(f \times g \times h)(id_Y \times \Delta_Z) = \alpha(id_X \times \mu)(f \times (g \times h)\Delta_Z) \\
&= \alpha(f \times \mu(g \times h)\Delta_Z) = \alpha(f \times k).
\end{aligned}$$

Consider the case $Z = G^f_{\alpha,\beta} \times G^f_{\alpha,\beta}$ and $g = e^f_{\alpha,\beta}\mathrm{pr}_1$, $h = e^f_{\alpha,\beta}\mathrm{pr}_2$. Then, $k = \mu(g \times h)\Delta_Z = \mu(e^f_{\alpha,\beta} \times e^f_{\alpha,\beta})$ and since k satisfies $X^f_* \mathrm{Ad}_l(\alpha)_*(k) = X^f_* \mathrm{Ad}_l(\beta)_*(k)$, there exists unique morphism $\lambda : G^f_{\alpha,\beta} \times G^f_{\alpha,\beta} \to G^f_{\alpha,\beta}$ that satisfies $e^f_{\alpha,\beta}\lambda = k$.

For $Z \in \mathrm{Ob}\,\mathcal{T}$ and $g \in \mathcal{T}(Z, G)$, we assume $X^f_* \mathrm{Ad}_l(\alpha)_*(g) = X^f_* \mathrm{Ad}_l(\beta)_*(g)$. Then, $\alpha(f \times g) = \beta(f \times g)$ and

$$\begin{aligned}
\beta(f \times g)(\gamma \times id_Z)(id_Y \times (\iota g \times id_Z)\Delta_Z) &= \beta(f\gamma \times g)(id_Y \times (\iota g \times id_Z)\Delta_Z) = \beta(f\gamma \times id_G)(id_Y \times (\iota g \times g)\Delta_Z) \\
&= \beta(f\gamma \times id_G)(id_Y \times (\iota \times id_G)\Delta_G g) \\
&= \beta(f\gamma \times id_G)(id_Y \times (\iota \times id_G)\Delta_G)(id_Y \times g) \\
\alpha(f \times g)(\gamma \times id_Z)(id_Y \times (\iota g \times id_Z)\Delta_Z) &= \alpha(f\gamma \times g)(id_Y \times (\iota g \times id_Z)\Delta_Z) \\
&= \alpha(\alpha(f \times id_G) \times id_G)(id_Y \times (\iota g \times g)\Delta_Z) \\
&= \alpha(\alpha \times id_G)(f \times (\iota \times id_G)\Delta_G g) = \alpha(id_X \times \mu)(f \times (\iota \times id_G)\Delta_G g) \\
&= \alpha(f \times \mu(\iota \times id_G)\Delta_G g) = \alpha(f \times \varepsilon o_G g) \\
&= \alpha(id_X \times \varepsilon)(f \times o_X) = \mathrm{pr}_1(f \times o_X) = f\mathrm{pr}_1
\end{aligned}$$

Thus $f\mathrm{pr}_1 = \beta(f\gamma \times id_G)(id_Y \times (\iota \times id_G)\Delta_G)(id_Y \times g)$ and the following shows $X^f_* \mathrm{Ad}_l(\alpha)_*(\iota g) = X^f_* \mathrm{Ad}_l(\beta)_*(\iota g)$.

$$\begin{aligned}
\beta(f \times \iota g) &= \beta(f \times \iota g)(\mathrm{pr}_1 \times id_Z)(id_Y \times \Delta_Z) = \beta(f\mathrm{pr}_1 \times \iota g)(id_Y \times \Delta_Z) \\
&= \beta(\beta(f\gamma \times id_G)(id_Y \times (\iota \times id_G)\Delta_G g) \times \iota g)(id_Y \times \Delta_Z) \\
&= \beta(\beta(f\gamma \times id_G) \times id_G)(id_Y \times (\iota \times id_G)\Delta_G \times id_G)(id_Y \times g \times \iota g)(id_Y \times \Delta_Z) \\
&= \beta(\beta \times id_G)(f\gamma \times id_G \times id_G)(id_Y \times (\iota \times id_G)\Delta_G \times id_G)(id_Y \times (id_G \times \iota)\Delta_G g) \\
&= \beta(id_X \times \mu)(f\gamma \times id_G \times id_G)(id_Y \times ((\iota \times id_G)\Delta_G \times id_G)(id_G \times \iota)\Delta_G g) \\
&= \beta(f\gamma \times \mu)(id_Y \times ((\iota \times id_G)\Delta_G \times \iota)\Delta_G g) = \beta(f\gamma \times \mu)(id_Y \times (\iota \times id_G \times \iota)(\Delta_G \times id_G)\Delta_G g) \\
&= \beta(f\gamma \times \mu)(id_Y \times (\iota \times id_G \times \iota)(id_G \times \Delta_G)\Delta_G g) = \beta(f\gamma \times id_G)(id_Y \times (\iota \times \mu(id_G \times \iota)\Delta_G)\Delta_G g) \\
&= \beta(f\gamma \times id_G)(id_Y \times (\iota \times \varepsilon o_G)\Delta_G g) = \beta(f\gamma(id_Y \times \iota) \times \varepsilon o_G)(id_Y \times \Delta_G g) \\
&= \beta(id_X \times \varepsilon)(f\gamma(id_Y \times \iota) \times o_G)(id_Y \times \Delta_G g) = \mathrm{pr}_1(f\gamma(id_Y \times \iota) \times o_G)(id_Y \times \Delta_G g) \\
&= f\gamma(id_Y \times \iota g) = \alpha(f \times \iota g)
\end{aligned}$$

Consider the case $Z = G^f_{\alpha,\beta}$ and $g = e^f_{\alpha,\beta}$. Then, there exists unique morphism $\kappa : G^f_{\alpha,\beta} \to G^f_{\alpha,\beta}$ that satisfies $e^f_{\alpha,\beta}\kappa = \iota g$. □

Corollary 9.2.20 *Let $\alpha, \beta : X \times G \to X$ be right actions of a group object G in $\mathcal{T}$ on $X \in \mathrm{Ob}\,\mathcal{T}$ and $f : Y \to X$ a morphism of $\mathcal{T}$. If β is a trivial action, then $e^f_{\alpha,\beta} : G^f_{\alpha,\beta} \to G$ is a subgroup object of G. In particular, $\mathrm{Cent}_\alpha(Y)$ is a subgroup object of G.*

Proposition 9.2.21 *Let $\alpha : X \times G \to X$ be a right action of a group object $(G, \mu, \varepsilon, \iota)$ in $\mathcal{T}$ on $X \in \mathrm{Ob}\,\mathcal{T}$ and $i : Y \to X$ a subobject of X. Then, $\tau^Y_\alpha : \mathrm{Stab}_\alpha(Y) \to G$ is a submonoid object of G, that is, there exists a morphism $\lambda : \mathrm{Stab}_\alpha(Y) \times \mathrm{Stab}_\alpha(Y) \to \mathrm{Stab}_\alpha(Y)$ which makes the following diagram commute.*

$$\begin{CD}
\mathrm{Stab}_\alpha(Y) \times \mathrm{Stab}_\alpha(Y) @>{\tau^Y_\alpha \times \tau^Y_\alpha}>> G \times G \\
@V{\lambda}VV @VV{\mu}V \\
\mathrm{Stab}_\alpha(Y) @>{\tau^Y_\alpha}>> G
\end{CD}$$

Proof. By the definition of $\mathrm{Stab}_\alpha(Y)$, there is a cartesian square

$$\begin{CD}
\mathrm{Stab}_\alpha(Y) @>{\sigma}>> Y^Y \\
@V{\tau^Y_\alpha}VV @VV{i^Y}V \\
G @>{X^i\mathrm{Ad}_l(\alpha)}>> X^Y
\end{CD} .$$

Put $\tilde{\sigma} = \exp^{-1}_{\mathrm{Stab}_\alpha(Y),Y,Y}(\sigma)$. It follows from (9.2.1), (9.2.5) and the commutativity of the above diagram that $\alpha T_{G,X}(\tau^Y_\alpha \times i) = i\tilde{\sigma}$. For $Z \in \mathrm{Ob}\,\mathcal{T}$, we consider a subset $S_\alpha(Z; Y)$ of $\mathcal{T}(Z, G)$ defined by

$$S_\alpha(Z; Y) = \{g \in \mathcal{T}(Z, G) \mid \alpha(i \times g) \in i_*(\mathcal{T}(Y \times Z, Y))\}.$$

For $\varphi \in \mathcal{T}(Z, \mathrm{Stab}_\alpha(Y))$, since $\alpha(i \times \tau_\alpha^Y \varphi) = \alpha T_{G,X}(\tau_\alpha^Y \times i)(\varphi \times id_Y)T_{Y,Z} = i\tilde{\sigma}(\varphi \times id_Y)T_{Y,Z} \in i_*(\mathcal{T}(Y \times Z, Y))$, we define a map $\Phi : \mathcal{T}(Z, \mathrm{Stab}_\alpha(Y)) \to S_\alpha(Z;Y)$ by $\Phi(\varphi) = \tau_\alpha^Y \varphi$. Since τ_α^Y is a monomorphism, Φ is injective. For $g \in S_\alpha(Z;Y)$, there exists $f \in \mathcal{T}(Y \times Z, Y)$ satisfying $\alpha(i \times g) = if$. Then, we have

$$\begin{aligned} X^i\mathrm{Ad}_l(\alpha)g &= \exp_{G,Y,X}(\alpha T_{G,X}(id_G \times i))g = \exp_{Z,Y,X}(\alpha T_{G,X}(g \times i)) = \exp_{Z,Y,X}(\alpha(i \times g)T_{Z,Y}) \\ &= \exp_{Z,Y,X}(ifT_{Z,Y}) = i^Y \exp_{Z,Y,Y}(fT_{Z,Y}). \end{aligned}$$

Hence there exists $\varphi \in \mathcal{T}(Z, \mathrm{Stab}_\alpha(Y))$ satisfying $\Phi(\varphi) = \tau_\alpha^Y \varphi = g$ and $\sigma\varphi = \exp_{Z,Y,Y}(fT_{Z,Y})$. It follows

$$\alpha T_{G,X}(g \times i) = \alpha T_{G,X}(\tau_\alpha^Y \times i)(\varphi \times id_Y) = i\tilde{\sigma}(\varphi \times id_Y) \in i_*(\mathcal{T}(Y \times Z, Y)).$$

Therefore $g \in S_\alpha(Z;Y)$ and Φ is surjective.

For $g, h \in S_\alpha(Z;Y)$, we take $f, e \in \mathcal{T}(Y \times Z, Y)$ satisfying $\alpha(i \times g) = if$, $\alpha(i \times h) = ie$. Put $k = \mu(g \times h)\Delta_Z$, then the following shows $k \in S_\alpha(Z;Y)$.

$$\begin{aligned} \alpha(i \times k) &= \alpha(i \times \mu(g \times h)\Delta_Z) = \alpha(id_X \times \mu)(i \times (g \times h)\Delta_Z) = \alpha(\alpha \times id_G)(i \times g \times h)(id_Y \times \Delta_Z) \\ &= \alpha(\alpha(i \times g) \times h)(id_Y \times \Delta_Z) = \alpha(if \times h)(id_Y \times \Delta_Z) = \alpha(i \times h)(f \times \Delta_Z) = ie(f \times \Delta_Z). \end{aligned}$$

Consider the case $Z = \mathrm{Stab}_\alpha(Y) \times \mathrm{Stab}_\alpha(Y)$ and $g = \tau_\alpha^Y \mathrm{pr}_1 = \Phi(\mathrm{pr}_1)$, $h = \tau_\alpha^Y \mathrm{pr}_2 = \Phi(\mathrm{pr}_2)$. Then, $g, h \in S_\alpha(Z;Y)$, thus k belongs to $S_\alpha(Z;Y)$. This implies that there exists a morphism $\lambda : \mathrm{Stab}_\alpha(Y) \times \mathrm{Stab}_\alpha(Y) \to \mathrm{Stab}_\alpha(Y)$ which satisfies $\tau_\alpha^Y \lambda = \Phi(\lambda) = k = \mu(g \times h)\Delta_Z = \mu(\tau_\alpha^Y \times \tau_\alpha^Y)$. □

Proposition 9.2.22 *Let $\alpha : X \times G \to X$ be a right action of a group object $(G, \mu, \varepsilon, \iota)$ in $\mathcal{T}$ on $X \in \mathrm{Ob}\,\mathcal{T}$ and $i : Y \to X$ a subobject of X. Then, $\tau_\alpha^Y \nu_\alpha^Y : \mathrm{Norm}_\alpha(Y) \to G$ is a subgroup object of G, that is, there exist morphisms $\lambda : \mathrm{Norm}_\alpha(Y) \times \mathrm{Norm}_\alpha(Y) \to \mathrm{Norm}_\alpha(Y)$ and $\kappa : \mathrm{Norm}_\alpha(Y) \to \mathrm{Norm}_\alpha(Y)$ which makes the following diagrams comnute.*

$$\begin{tikzcd} \mathrm{Norm}_\alpha(Y) \times \mathrm{Norm}_\alpha(Y) \arrow[r, "\tau_\alpha^Y \nu_\alpha^Y \times \tau_\alpha^Y \nu_\alpha^Y"] \arrow[d, "\lambda"] & G \times G \arrow[d, "\mu"] \\ \mathrm{Norm}_\alpha(Y) \arrow[r, "\tau_\alpha^Y \nu_\alpha^Y"] & G \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Norm}_\alpha(Y) \arrow[r, "\tau_\alpha^Y \nu_\alpha^Y"] \arrow[d, "\kappa"] & G \arrow[d, "\iota"] \\ \mathrm{Norm}_\alpha(Y) \arrow[r, "\tau_\alpha^Y \nu_\alpha^Y"] & G \end{tikzcd}$$

Proof. By the definition of $\mathrm{Stab}_\alpha(Y)$, there is a cartesian square

$$\begin{tikzcd} \mathrm{Norm}_\alpha(Y) \arrow[r, "\xi"] \arrow[d, "\nu_\alpha^Y"] & \mathrm{Stab}_\alpha(Y) \arrow[d, "\iota\tau_\alpha^Y"] \\ \mathrm{Stab}_\alpha(Y) \arrow[r, "\tau_\alpha^Y"] & G \end{tikzcd} .$$

For $Z \in \mathrm{Ob}\,\mathcal{T}$, we define a subset $T_\alpha(Z;Y)$ of $S_\alpha(Z;Y)$ in the proof of (9.2.21) by $T_\alpha(Z;Y) = \{g \in S_\alpha(Z;Y) \,|\, \iota g \in S_\alpha(Z;Y)\}$. For $\psi \in \mathcal{T}(Z, \mathrm{Norm}_\alpha(Y))$, since $\iota\tau_\alpha^Y \xi\psi = \tau_\alpha^Y \nu_\alpha^Y \psi = \Phi(\nu_\alpha^Y \psi)$, we have $\iota\Phi(\nu_\alpha^Y \psi) = \tau_\alpha^Y \xi\psi = \Phi(\xi\psi) \in S_\alpha(Z;Y)$. It follows $\tau_\alpha^Y \nu_\alpha^Y \psi = \Phi(\nu_\alpha^Y \psi) \in T_\alpha(Z;Y)$. Thus we can define a map $\Psi : \mathcal{T}(Z, \mathrm{Norm}_\alpha(Y)) \to T_\alpha(Z;Y)$ by $\Psi(\psi) = \tau_\alpha^Y \nu_\alpha^Y \psi$. Since ν_α^Y and τ_α^Y are monomorphisms, Ψ is injective. For $g \in T_\alpha(Z;Y)$, there exists $\varphi, \varphi' \in \mathcal{T}(Z, \mathrm{Stab}_\alpha(Y))$ satisfying $\Phi(\varphi) = g, \Phi(\varphi') = \iota g$. Then, there exists a morphism $\psi \in \mathcal{T}(Z, \mathrm{Norm}_\alpha(Y))$ which satisfies $\nu_\alpha^Y \psi = \varphi$ and $\xi\psi = \varphi'$. It follows that $\Psi(\psi) = \tau_\alpha^Y \nu_\alpha^Y \psi = \Phi(\varphi) = g$, which implies that Ψ is surjective.

For $g, h \in T_\alpha(Z;Y)$, put $k = \mu(g \times h)\Delta_Z$, then $\iota k = \mu(\iota h \times \iota g)\Delta_Z$. Since $g, \iota g, h, \iota h \in S_\alpha(Z;Y)$, we have $k, \iota k \in S_\alpha(Z;Y)$ by the proof of (9.2.21). Therefore $k \in T_\alpha(Z;Y)$. Consider the case $Z = \mathrm{Norm}_\alpha(Y) \times \mathrm{Norm}_\alpha(Y)$ and $g = \tau_\alpha^Y \nu_\alpha^Y \mathrm{pr}_1 = \Psi(\mathrm{pr}_1)$, $h = \tau_\alpha^Y \nu_\alpha^Y \mathrm{pr}_2 = \Psi(\mathrm{pr}_2)$. Then, $g, h \in T_\alpha(Z;Y)$, thus k belongs to $T_\alpha(Z;Y)$. This implies that there exists a morphism $\lambda : \mathrm{Norm}_\alpha(Y) \times \mathrm{Norm}_\alpha(Y) \to \mathrm{Norm}_\alpha(Y)$ which satisfies $\tau_\alpha^Y \nu_\alpha^Y \lambda = \Psi(\lambda) = k = \mu(g \times h)\Delta_Z = \mu(\tau_\alpha^Y \nu_\alpha^Y \times \tau_\alpha^Y \nu_\alpha^Y)$.

Consider the case $Z = \mathrm{Norm}_\alpha(Y)$ and $g = \tau_\alpha^Y \nu_\alpha^Y$. Then, $g, \iota g \in T_\alpha(Z;Y)$ and there exist a morphism $\kappa : \mathrm{Norm}_\alpha(Y) \to \mathrm{Norm}_\alpha(Y)$ which satisfies $\tau_\alpha^Y \nu_\alpha^Y \kappa = \Psi(\kappa) = \iota g = \iota\tau_\alpha^Y \nu_\alpha^Y$. □

Definition 9.2.23 *Let G be a group object in $\mathcal{T}$ with multiplication $\mu : G \times G \to G$ and inverse $\iota : G \to G$. Define a morphism $\mathrm{ad} : G \times G \to G$ to be the following composition. ad is a right action on G and call this the adjoint action on G.*

$$G \times G \xrightarrow{id_G \times \Delta_G} G \times G \times G \xrightarrow{T_{G,G} \times id_G} G \times G \times G \xrightarrow{\iota \times \mu} G \times G \xrightarrow{\mu} G$$

Definition 9.2.24 *Let G be a group object in $\mathcal{T}$ and $i : H \to G$ a subgroup object of G. We denote* $\mathrm{Cent}_{\mathrm{ad}}(H)$ *(resp.* $\mathrm{Norm}_{\mathrm{ad}}(H)$*) by $Z_G(H)$ (resp. $N_G(H)$) and call this a centralizer (resp. normalizer) of H. If N is a subgroup object of $N_G(H)$, we say that N normalizes H.*

Proposition 9.2.25 *Let $i : H \to G$ be a subgroup object of G and $\alpha : X \times G \to X$ a right action of G on X. We denote by $\tau : X \times G \to X$ the trivial action. Then, $X_i^{\alpha,\tau}$ is closed under the action α of $N_G(H)$, namely, there exists a morphism $\alpha' : X_i^{\alpha,\tau} \times N_G(H) \to X_i^{\alpha,\tau}$ such that the following diagram commute.*

$$\begin{tikzcd} X_i^{\alpha,\tau} \times N_G(H) \arrow[r, "\alpha'"] \arrow[d, "e_i^{\alpha,\tau} \times \tau_{\mathrm{ad}}^H \nu_{\mathrm{ad}}^H"'] & X_i^{\alpha,\tau} \arrow[d, "e_i^{\alpha,\tau}"] \\ X \times G \arrow[r, "\alpha"] & X \end{tikzcd}$$

Proof. Since $e_i^{\alpha,\tau} : X_i^{\alpha,\tau} \to X$ is the equalizer of $X \xrightarrow{\mathrm{Ad}_r(\alpha)} X^G \xrightarrow{X^i} X^H$ and $X \xrightarrow{\mathrm{Ad}_r(\tau)} X^G \xrightarrow{X^i} X^H$, it suffices to show that $X^i \mathrm{Ad}_r(\alpha)\alpha\big(e_i^{\alpha,\tau} \times \tau_{\mathrm{ad}}^H \nu_{\mathrm{ad}}^H\big) = X^i \mathrm{Ad}_r(\tau)\alpha\big(e_i^{\alpha,\tau} \times \tau_{\mathrm{ad}}^H \nu_{\mathrm{ad}}^H\big)$, which is equivalent to

$$\alpha(\alpha \times id_G)\big(e_i^{\alpha,\tau} \times \tau_{\mathrm{ad}}^H \nu_{\mathrm{ad}}^H \times i\big) = \tau(\alpha \times id_G)\big(e_i^{\alpha,\tau} \times \tau_{\mathrm{ad}}^H \nu_{\mathrm{ad}}^H \times i\big) : X_i^{\alpha,\tau} \times N_G(H) \times H \to X$$

by (1) of (9.2.17). Put $Z = X_i^{\alpha,\tau} \times N_G(H) \times H$ and let us denote by $\mathrm{pr}_{X_i^{\alpha,\tau}} : Z \to X_i^{\alpha,\tau}$, $\mathrm{pr}_{N_G(H)} : Z \to N_G(H)$ and $\mathrm{pr}_H : Z \to H$ the projections. We put $x = e_i^{\alpha,\tau} \mathrm{pr}_{X_i^{\alpha,\tau}} : Z \to X$, $g = \tau_{\mathrm{ad}}^H \nu_{\mathrm{ad}}^H \mathrm{pr}_{N_G(H)}$, $h = i\,\mathrm{pr}_H : Z \to G$. Since $\mathrm{ad} = \mu(\iota \times \mu)(T_{G,G} \times id_G)(id_G \times \Delta_G)$, we have

$$\begin{aligned} \mathrm{ad}(h, \iota g) &= \mu(\iota \times \mu)(T_{G,G} \times id_G)(id_G \times \Delta_G)(h, \iota g) = \mu(\iota \times \mu)(T_{G,G} \times id_G)(h, \iota g, \iota g) \\ &= \mu(\iota \times \mu)(\iota g, h, \iota g) = \mu(g, \mu(h, \iota g)) = \mu(\mu(g, h), \iota g). \end{aligned}$$

By the proof of (9.2.21), there exists a morphism $\bar{\sigma} : H \times \mathrm{Stab}_{\mathrm{ad}}(H) \to H$ satisfying $\mathrm{ad}\big(i \times \tau_{\mathrm{ad}}^H\big) = i\bar{\sigma}$. There also exists a morphism $\kappa : N_G(H) \to N_G(H)$ satisfying $\iota \tau_{\mathrm{ad}}^H \nu_{\mathrm{ad}}^H = \tau_{\mathrm{ad}}^H \nu_{\mathrm{ad}}^H \kappa$. Hence

$$\mathrm{ad}(h, \iota g) = \mathrm{ad}\big(i \times \iota\tau_{\mathrm{ad}}^H \nu_{\mathrm{ad}}^H\big)\big(\mathrm{pr}_H, \mathrm{pr}_{N_G(H)}\big) = \mathrm{ad}\big(i \times \tau_{\mathrm{ad}}^H \nu_{\mathrm{ad}}^H\big)\big(\mathrm{pr}_H, \kappa\mathrm{pr}_{N_G(H)}\big) = i\bar{\sigma}\big(\mathrm{pr}_H, \kappa\mathrm{pr}_{N_G(H)}\big).$$

We note that $\alpha(e_i^{\alpha,\tau} \times i) = \tau(e_i^{\alpha,\tau} \times i) = e_i^{\alpha,\tau}\mathrm{pr}_1 : X_i^{\alpha,\tau} \times H \to X$ by the definition of $X_i^{\alpha,\tau}$, we have

$$\begin{aligned} \alpha(\alpha \times id_G)\big(e_i^{\alpha,\tau} \times \tau_{\mathrm{ad}}^H \nu_{\mathrm{ad}}^H \times i\big) &= \alpha(id_X \times \mu)(x, g, h) = \alpha(x, \mu(g, h)) = \alpha(\alpha(x, \mu(g, h)), \varepsilon o_Z) \\ &= \alpha(\alpha(x, \mu(g, h)), \mu(\iota g, g)) = \alpha(x, \mu(\mu(g, h), \mu(\iota g, g))) \\ &= \alpha(x, \mu(\mu(\mu(g, h), \iota g), g)) = \alpha(x, \mu(\mathrm{ad}(h, \iota g), g)) \\ &= \alpha\big(x, \mu\big(i\bar{\sigma}\big(\mathrm{pr}_H, \kappa\mathrm{pr}_{N_G(H)}\big), g\big)\big) = \alpha\big(\alpha\big(x, i\bar{\sigma}\big(\mathrm{pr}_H, \kappa\mathrm{pr}_{N_G(H)}\big)\big), g\big) \\ &= \alpha\big(\alpha\big(e_i^{\alpha,\tau} \times i\big)\big(\mathrm{pr}_{X_i^{\alpha,\tau}}, \bar{\sigma}\big(\mathrm{pr}_H, \kappa\mathrm{pr}_{N_G(H)}\big)\big), g\big) = \alpha\big(e_i^{\alpha,\tau}\mathrm{pr}_{X_i^{\alpha,\tau}}, g\big) \\ &= \alpha(x, g) = \tau(\alpha \times id_G)(x, g, h) = \tau(\alpha \times id_G)\big(e_i^{\alpha,\tau} \times \tau_{\mathrm{ad}}^H \nu_{\mathrm{ad}}^H \times i\big) \end{aligned}$$

This completes the proof. □

Let $\alpha : G \times X \to X$ be a left G-action on X. For an object Y of $\mathcal{T}$, define a morphism $\rho_Y^\alpha : Y^X \times G \to Y^X$ to the image of $Y^X \times G \times X \xrightarrow{id_{Y^X} \times \alpha} Y^X \times X \xrightarrow{\varepsilon_Y^X} Y$ by $\exp_{Y^X \times G, X, Y} : \mathcal{T}(Y^X \times G \times X, Y) \to \mathcal{T}(Y^X \times G, Y^X)$.

For objects X, Y, Z of $\mathcal{T}$, we define a map $\times Z : \mathcal{T}(X, Y) \to \mathcal{T}(X \times Z, Y \times Z)$ by $(\times Z)(f) = f \times id_Z$.

Lemma 9.2.26 *(1) The following diagram commutes.*

$$\begin{tikzcd} Y^X \times G \times X \arrow[r, "\rho_Y^\alpha \times id_X"] \arrow[d, "id_{Y^X} \times \alpha"'] & Y^X \times X \arrow[d, "\varepsilon_Y^X"] \\ Y^X \times X \arrow[r, "\varepsilon_Y^X"] & Y \end{tikzcd}$$

(2) $\rho_Y^\alpha : Y^X \times G \to Y^X$ is a right G-action on Y^X.
(3) For a morphism $f : Y \to Z$ of $\mathcal{T}$, the following diagram commutes.

$$\begin{array}{ccc} Y^X\times G & \xrightarrow{\rho_Y^\alpha} & Y^X \\ \downarrow{\scriptstyle f^X\times id_G} & & \downarrow{\scriptstyle f^X} \\ Z^X\times G & \xrightarrow{\rho_Z^\alpha} & Z^X \end{array}$$

(4) Let $\beta : G\times Z\to Z$ be a left G-action on Z and $f : Z\to X$ a morphism of left G-objects. The following diagram commutes.

$$\begin{array}{ccc} Y^X\times G & \xrightarrow{\rho_Y^\alpha} & Y^X \\ \downarrow{\scriptstyle Y^f\times id_G} & & \downarrow{\scriptstyle Y^f} \\ Y^Z\times G & \xrightarrow{\rho_Y^\beta} & Y^Z \end{array}$$

(5) For an object Z of $\mathcal{T}$, the following diagram commutes.

$$\begin{array}{ccccc} \mathcal{T}(W\times X,Y) & & \xrightarrow{(id_W\times\alpha)^*} & & \mathcal{T}(W\times G\times X,Y) \\ \downarrow{\scriptstyle \exp_{W,X,Y}} & & & & \downarrow{\scriptstyle \exp_{W\times G,X,Y}} \\ \mathcal{T}(W,Y^X) & \xrightarrow{\times G} & \mathcal{T}(W\times G,Y^X\times G) & \xrightarrow{(\rho_Y^\alpha)_*} & \mathcal{T}(W\times G,Y^X) \end{array}$$

(6) We regard $\mu : G\times G\to G$ as a left G-action on G. Let us denote by $\mathrm{pr}_Y : Y\times 1\to Y$ the projection. Then, the following diagram commutes.

$$\begin{array}{ccc} Y^G\times G & \xrightarrow{\varepsilon_Y^G} & Y \\ \downarrow{\scriptstyle \rho_Y^\mu} & & \downarrow{\scriptstyle \exp_{Y,1,Y}(\mathrm{pr}_Y)} \\ Y^G & \xrightarrow{Y^\varepsilon} & Y^1 \end{array}$$

(7) For an object (X,α) of $\mathrm{Act}_r(G)$, the following diagram commutes.

$$\begin{array}{ccc} X\times G & \xrightarrow{\alpha} & X \\ \downarrow{\scriptstyle \mathrm{Ad}_r(\alpha)\times id_G} & & \downarrow{\scriptstyle \mathrm{Ad}_r(\alpha)} \\ X^G\times G & \xrightarrow{\rho_X^\mu} & X^G \end{array}$$

Proof. (1) By the commutativity of the following diagram,

$$\begin{array}{ccc} \mathcal{T}(Y^X\times X,Y) & \xrightarrow{\exp_{Y^X,X,Y}} & \mathcal{T}(Y^X,Y^X) \\ \downarrow{\scriptstyle (\rho_Y^\alpha\times id_X)^*} & & \downarrow{\scriptstyle (\rho_Y^\alpha)^*} \\ \mathcal{T}(Y^X\times G\times X,Y) & \xrightarrow{\exp_{Y^X\times G,X,Y}} & \mathcal{T}(Y^X\times G,Y^X) \end{array}$$

we have $\exp_{Y^X\times X,G,Y}(\varepsilon_Y^X(id_{Y^X}\times\alpha)) = \rho_Y^\alpha = \exp_{Y^X\times G,X,Y}(\varepsilon_Y^X(\rho_Y^\alpha\times id_X))$, which implies $\varepsilon_Y^X(id_{Y^X}\times\alpha) = \varepsilon_Y^X(\rho_Y^\alpha\times id_X)$.

(2) By the result of (1),

$$\begin{aligned} \varepsilon_Y^X(id_{Y^X}\times\alpha)(\rho_Y^\alpha\times id_G\times id_X) &= \varepsilon_Y^X(\rho_Y^\alpha\times\alpha) = \varepsilon_Y^X(\rho_Y^\alpha\times id_X)(id_{Y^X}\times id_G\times\alpha) \\ &= \varepsilon_Y^X(id_{Y^X}\times\alpha)(id_{Y^X}\times id_G\times\alpha) = \varepsilon_Y^X(id_{Y^X}\times\alpha)(id_{Y^X}\times\mu\times id_X) \end{aligned}$$

holds in $\mathcal{T}(Y^X\times G\times G\times X,Y)$. It follows from a commutative diagram

$$\begin{array}{ccccc} \mathcal{T}(Y^X\times G\times X,Y) & \xrightarrow{(id_{Y^X}\times\mu\times id_X)^*} & \mathcal{T}(Y^X\times G\times G\times X,Y) & \xleftarrow{(\rho_Y^\alpha\times id_G\times id_X)^*} & \mathcal{T}(Y^X\times G\times X,Y) \\ \downarrow{\scriptstyle \exp_{Y^X\times G,X,Y}} & & \downarrow{\scriptstyle \exp_{Y^X\times G\times G,X,Y}} & & \downarrow{\scriptstyle \exp_{Y^X\times G,X,Y}} \\ \mathcal{T}(Y^X\times G,Y^X) & \xrightarrow{(id_{Y^X}\times\mu)^*} & \mathcal{T}(Y^X\times G\times G,Y^X) & \xleftarrow{(\rho_Y^\alpha\times id_G)^*} & \mathcal{T}(Y^X\times G,Y^X) \end{array}$$

that the following diagram commutes.

$$\begin{tikzcd}
Y^X \times G \times G \arrow[r, "\rho_Y^\alpha \times id_G"] \arrow[d, "id_{Y^X} \times \mu"] & Y^X \times G \arrow[d, "\rho_Y^\alpha"] \\
Y^X \times G \arrow[r, "\rho_Y^\alpha"] & Y^X
\end{tikzcd}$$

Since

$$\varepsilon_Y^X(id_{Y^X} \times \alpha)(id_{Y^X} \times \varepsilon \times id_X) = \varepsilon_Y^X(id_{Y^X} \times \alpha(\varepsilon \times id_X)) = \varepsilon_Y^X(id_{Y^X} \times \mathrm{pr}_2) = \varepsilon_Y^X(\mathrm{pr}_1 \times id_X)$$

in $\mathcal{T}(Y^X \times 1 \times X, Y)$, and

$$\begin{tikzcd}
\mathcal{T}(Y^X \times X, Y) \arrow[r, "(\mathrm{pr}_1 \times id_X)^*"] \arrow[d, "\exp_{Y^X, X, Y}"] & \mathcal{T}(Y^X \times 1 \times X, Y) \arrow[d, "\exp_{Y^X \times 1, X, Y}"] & \mathcal{T}(Y^X \times G \times X, Y) \arrow[l, "(id_{Y^X} \times \varepsilon \times id_X)^*"'] \arrow[d, "\exp_{Y^X \times X, X, Y}"] \\
\mathcal{T}(Y^X, Y^X) \arrow[r, "\mathrm{pr}_1^*"] & \mathcal{T}(Y^X \times 1, Y^X) & \mathcal{T}(Y^X \times G, Y^X) \arrow[l, "(id_{Y^X} \times \varepsilon)^*"']
\end{tikzcd}$$

commutes, the following diagram is commutative.

$$\begin{tikzcd}
Y^X \times 1 \arrow[r, "id_{Y^X} \times \varepsilon"] \arrow[dr, "\mathrm{pr}_1"] & Y^X \times G \arrow[d, "\rho_Y^\alpha"] \\
& Y^X
\end{tikzcd}$$

Thus (Y^X, ρ_Y^α) is an object of $\mathrm{Act}_r(X)$.

(3) It follows from (9.2.2) that

$$\begin{tikzcd}
Y^X \times G \times X \arrow[r, "id_{Y^X} \times \alpha"] \arrow[d, "f^X \times id_G \times id_X"] & Y^X \times X \arrow[r, "\varepsilon_Y^X"] \arrow[d, "f^X \times id_X"] & Y \arrow[d, "f"] \\
Z^X \times G \times X \arrow[r, "id_{Z^X} \times \alpha"] & Z^X \times X \arrow[r, "\varepsilon_Z^X"] & Z
\end{tikzcd}$$

is commutative. Then, by (9.2.1), we have

$$\begin{aligned}
f^X \rho_Y^\alpha &= f^X \exp_{Y^X \times G, X, Y}(\varepsilon_Y^X(id_{Y^X} \times \alpha)) = \exp_{Y^X \times G, X, Y}(f \varepsilon_Y^X(id_{Y^X} \times \alpha)) \\
&= \exp_{Y^X \times G, X, Y}(\varepsilon_Z^X(id_{Z^X} \times \alpha)(f^X \times id_G \times id_X)) \\
&= \exp_{Z^X \times G, X, Y}(\varepsilon_Z^X(id_{Z^X} \times \alpha))(f^X \times id_G) = \rho_Z^\alpha(f^X \times id_G).
\end{aligned}$$

(4) The following diagram commutes by (9.2.8).

$$\begin{tikzcd}
& Y^X \times G \times Z \arrow[r, "id_{Y^X} \times id_G \times f"] \arrow[d, "id_{Y^X} \times \beta"] & Y^X \times G \times X \arrow[d, "id_{Y^X} \times \alpha"] \\
Y^X \times G \times Z \arrow[r, "id_{Y^X} \times \beta"] \arrow[d, "Y^f \times id_G \times id_Z"] & Y^X \times Z \arrow[r, "id_{Z^X} \times f"] \arrow[d, "Y^f \times id_Z"] & Y^X \times X \arrow[d, "\varepsilon_Y^X"] \\
Y^Z \times G \times Z \arrow[r, "id_{Y^Z} \times \beta"] & Y^Z \times Z \arrow[r, "\varepsilon_Y^Z"] & Y
\end{tikzcd}$$

Then, by (9.2.5), we have

$$\begin{aligned}
Y^f \rho_Y^\alpha &= Y^f \exp_{Y^X \times G, X, Y}(\varepsilon_Y^X(id_{Y^X} \times \alpha)) = \exp_{Y^X \times G, Z, Y}(\varepsilon_Y^X(id_{Y^X} \times \alpha)(id_{Y^X} \times id_G \times f)) \\
&= \exp_{Y^X \times G, Z, Y}(\varepsilon_Y^Z(id_{Y^Z} \times \beta)(Y^f \times id_G \times id_Z)) = \exp_{Y^Z \times G, Z, Y}(\varepsilon_Y^Z(id_{Y^Z} \times \beta))(Y^f \times id_G) \\
&= \rho_Y^\beta(Y^f \times id_G).
\end{aligned}$$

(5) For a morphism $f : W \times X \to Y$, we put $\bar{f} = \exp_{W,X,Y}(f)$. Then, $f = \exp_{W,X,Y}^{-1}(\bar{f}) = \varepsilon_Y^X(\bar{f} \times id_X)$. Thus we have

$$\begin{aligned}
(\rho_Y^\alpha)_*(\times G) \exp_{W,X,Y}(f) &= \exp_{Y^X \times G, X, Y}(\varepsilon_Y^X(id_{Y^X} \times \alpha))(\bar{f} \times id_G) \\
&= \exp_{W \times G, X, Y}(\varepsilon_Y^X(id_{Y^X} \times \alpha)(\bar{f} \times id_G \times id_X)) \\
&= \exp_{W \times G, X, Y}(\varepsilon_Y^X(\bar{f} \times id_X)(id_W \times \alpha)) \\
&= \exp_{W \times G, X, Y}(f(id_W \times \alpha)) = \exp_{W \times G, X, Y}(id_W \times \alpha)^*(f)
\end{aligned}$$

by (9.2.2).

(6) Let $\mathrm{pr}_G : G \times G \to G$ be the projection. Since

$$\varepsilon^G_Y(id_{Y^G} \times \mu)(id_{Y^G} \times id_G \times \varepsilon) = \varepsilon^G_Y(id_{Y^G} \times \mu(id_G \times \varepsilon)) = \varepsilon^G_Y(id_{Y^G} \times \mathrm{pr}_G) = \mathrm{pr}_Y(\varepsilon^G_Y \times id_1),$$

the assertion follows from the following diagram which is commutative by (9.2.1) and (9.2.5).

$$\begin{tikzcd}
\mathcal{T}(Y^G \times G \times G, Y) \arrow[r, "(id_{Y^G}\times id_G \times \varepsilon)^*"] \arrow[d, "\exp_{Y^G\times G,G,Y}"] & \mathcal{T}(Y^G \times G \times 1, Y) \arrow[d, "\exp_{Y^G\times G,1,Y}"] & \mathcal{T}(Y \times 1, Y) \arrow[l, "(\varepsilon^G_Y \times id_1)^*"'] \arrow[d, "\exp_{Y,1,Y}"] \\
\mathcal{T}(Y^G \times G, Y^G) \arrow[r, "Y^{\varepsilon}_*"] & \mathcal{T}(Y^G \times G, Y^1) & \mathcal{T}(Y, Y^1) \arrow[l, "(\varepsilon^G_Y)^*"']
\end{tikzcd}$$

(7) By the commutativity of a diagram

$$\begin{tikzcd}
\mathcal{T}(X^G \times G \times G, X) \arrow[r, "(\mathrm{Ad}_r(\alpha)\times id_G \times id_G)^*"] \arrow[d, "\exp_{X^G\times G,G,X}"] & \mathcal{T}(X \times G \times G, X) \arrow[d, "\exp_{X\times G,G,X}"] & \mathcal{T}(X \times G, X) \arrow[l, "(\alpha \times id_G)^*"'] \arrow[d, "\exp_{X,G,X}"] \\
\mathcal{T}(X^G \times G, X^G) \arrow[r, "(\mathrm{Ad}_r(\alpha)\times id_G)^*"] & \mathcal{T}(X \times G, X^G) & \mathcal{T}(X, X^G) \arrow[l, "\alpha^*"']
\end{tikzcd}$$

and an equality $\varepsilon^G_X(\mathrm{Ad}_r(\alpha) \times id_G) = \alpha$, we have

$$\begin{aligned}
\rho^\mu_X(\mathrm{Ad}_r(\alpha) \times id_G) &= \exp_{X\times G,G,X}(\varepsilon^G_X(id_{X^G} \times \mu)(\mathrm{Ad}_r(\alpha) \times id_G \times id_G)) \\
&= \exp_{X\times G,G,X}(\varepsilon^G_X(\mathrm{Ad}_r(\alpha) \times id_G)(id_X \times \mu)) \\
&= \exp_{X\times G,G,X}(\alpha(id_X \times \mu)) \\
&= \exp_{X\times G,G,X}(\alpha(\alpha \times id_G)) = \mathrm{Ad}_r(\alpha)\alpha.
\end{aligned}$$

□

Proposition 9.2.27 *Let $F : \mathrm{Act}_r(G) \to \mathcal{T}$ be the forgetful functor given by $F(X, \alpha) = X$. Then, F has a right adjoint.*

Proof. We regard the multiplication $\mu : G\times G \to G$ as a left G-action on G and define a functor $R : \mathcal{T} \to \mathrm{Act}_r(R)$ by $R(X) = (X^G, \rho^\mu_X)$ and $R(f) = f^G$. For $(X, \alpha) \in \mathrm{Ob}\,\mathrm{Act}_r(G)$ and $f \in \mathcal{T}(F(X, \alpha), Y)$, since the following diagram commutes by (3) and (7) of (9.2.26), $f^G\mathrm{Ad}_r(\alpha) : X \to Y^G$ is a morphism of $\mathrm{Act}_r(G)$.

$$\begin{tikzcd}
X \times G \arrow[r, "\mathrm{Ad}_r(\alpha)\times id_G"] \arrow[d, "\alpha"] & X^G \times G \arrow[r, "f^G \times id_G"] \arrow[d, "\rho^\mu_X"] & Y^G \times G \arrow[d, "\rho^\mu_Y"] \\
X \arrow[r, "\mathrm{Ad}_r(\alpha)"] & X^G \arrow[r, "f^G"] & Y^G
\end{tikzcd}$$

We define a map $\Psi : \mathcal{T}(F(X, \alpha), Y) \to \mathrm{Act}_r((X, \alpha), R(Y))$ by $\Psi(f) = f^G\mathrm{Ad}_r(\alpha)$. It is easy to verify that Ψ is natural. The inverse of Ψ is given by $\Psi^{-1}(g) = \varepsilon^G_Y(g, \varepsilon o_X)$ for $g \in \mathrm{Act}_r((X, \alpha), R(Y))$. In fact,

$$\begin{aligned}
\Psi^{-1}(\Psi(f)) = \varepsilon^G_Y(f^G\mathrm{Ad}_r(\alpha), \varepsilon o_X) &= \varepsilon^G_Y(f^G \times id_G)(\mathrm{Ad}_r(\alpha) \times id_G)(id_X, \varepsilon o_X) \\
&= f\varepsilon^G_X(\mathrm{Ad}_r(\alpha) \times id_G)(id_X, \varepsilon o_X) = f\alpha(id_X, \varepsilon o_X) = f
\end{aligned}$$

by (9.2.2). On the other hand, for $g \in \mathrm{Act}_r(G)\mathrm{Act}_r((X, \alpha), R(Y))$, since $g\alpha = \rho^\mu_Y(g\times id_G)$ and $\mathrm{Ad}_r(\alpha) = \alpha^G\eta^G_X$,

$$\begin{aligned}
\Psi(\Psi^{-1}(g)) = \left(\varepsilon^G_Y(g, \varepsilon o_X)\right)^G\mathrm{Ad}_r(\alpha) &= \left(\varepsilon^G_Y(g \times \varepsilon o_X)\Delta_X\alpha\right)^G\eta^G_X = \left(\varepsilon^G_Y(g\alpha \times \varepsilon o_X\alpha)\Delta_{X\times G}\right)^G\eta^G_X \\
&= \left(\varepsilon^G_Y(\rho^\mu_Y(g \times id_G) \times \varepsilon o_{X\times G})\Delta_{X\times G}\right)^G\eta^G_X = \left(\varepsilon^G_Y(\rho^\mu_Y \times id_G)(g \times id_G \times \varepsilon o_{X\times G})\Delta_{X\times G}\right)^G\eta^G_X \\
&= \left(\varepsilon^G_Y(id_{Y^G} \times \mu)(g\,\mathrm{pr}_1, \mathrm{pr}_2, \varepsilon o_G\mathrm{pr}_2)\right)^G\eta^G_X = \left(\varepsilon^G_Y(g \times \mu(id_G \times \varepsilon o_G)\Delta_G)\right)^G\eta^G_X \\
&= \left(\varepsilon^G_Y(g \times id_G)\right)^G\eta^G_X = \left(\varepsilon^G_Y\right)^G(g \times id_G)^G\eta^G_X = \left(\varepsilon^G_Y\right)^G\eta^G_{Y^G}g = g
\end{aligned}$$

follows from (9.2.2) and (9.2.3). Hence R is a right adjoint of F. □

9.3 Right induction

Let $\mathcal{T}$ be a cartesian closed category with finite limits.

We define a morphism $\mathrm{Prod}_Z : Y^X \to (Y \times Z)^{X \times Z}$ to be the image of $\varepsilon_Y^X \times id_Z : Y^X \times X \times Z \to Y \times Z$ by

$$\exp_{Y^X, X\times Z, Y\times Z} : \mathcal{T}(Y^X \times X \times Z, Y \times Z) \to \mathcal{T}(Y^X, (Y \times Z)^{X\times Z}).$$

Proposition 9.3.1 *For objects X, Y, Z of $\mathcal{T}$, the following diagrams are commutative.*

$$\begin{tikzcd} X \arrow[d, "\eta_X^Y"'] \arrow[dr, "\eta_X^{Y\times Z}"] & \\ (X\times Y)^Y \arrow[r, "\mathrm{Prod}_Z"] & (X\times Y\times Z)^{Y\times Z} \end{tikzcd} \qquad \begin{tikzcd} Y^X \times X \times Z \arrow[r, "\mathrm{Prod}_Z \times id_{X\times Z}"] \arrow[dr, "\varepsilon_X^Y \times id_Z"'] & (Y\times Z)^{X\times Z} \times X \times Z \arrow[d, "\varepsilon_{X\times Z}^{Y\times Z}"] \\ & Y \times Z \end{tikzcd}$$

Proof. The commutativity of the left diagram follows from the commutativity of the following diagram.

$$\begin{tikzcd} \mathcal{T}((X\times Y)^Y \times Y \times Z, X\times Y\times Z) \arrow[r, "(\eta_X^Y \times id_{Y\times Z})^*"] \arrow[d, "\exp_{(X\times Y)^Y, Y\times Z, X\times Y\times Z}"] & \mathcal{T}(X\times Y\times Z, X\times Y\times Z) \arrow[d, "\exp_{X, Y\times Z, X\times Y\times Z}"] \\ \mathcal{T}((X\times Y)^Y, (X\times Y\times Z)^{Y\times Z}) \arrow[r, "(\eta_X^Y)^*"] & \mathcal{T}(X, (X\times Y\times Z)^{Y\times Z}) \end{tikzcd}$$

The commutativity of the right diagram follows from the commutativity of the following diagram.

$$\begin{tikzcd} \mathcal{T}((Y\times Z)^{X\times Z} \times X \times Z, Y\times Z) \arrow[r, "(\mathrm{Prod}_Z \times id_{X\times Z})^*"] \arrow[d, "\exp_{(Y\times Z)^{X\times Z}, X\times Z, Y\times Z}"] & \mathcal{T}(Y^X \times X\times Z, Y\times Z) \arrow[d, "\exp_{Y^X, X\times Z, Y\times Z}"] \\ \mathcal{T}((Y\times Z)^{X\times Z}, (Y\times Z)^{Y\times Z}) \arrow[r, "\mathrm{Prod}_Z^*"] & \mathcal{T}(Y^X, (Y\times Z)^{X\times Z}) \end{tikzcd}$$

□

Proposition 9.3.2 *For morphisms $f : Y \to W$, $g : W \to X$, $h : W \to Z$ of $\mathcal{T}$, the following diagrams are commutative.*

$$\begin{tikzcd} Y^X \arrow[r, "\mathrm{Prod}_Z"] \arrow[d, "f^X"] & (Y\times Z)^{X\times Z} \arrow[d, "(f\times id_Z)^{X\times Z}"] \\ W^X \arrow[r, "\mathrm{Prod}_Z"] & (W\times Z)^{X\times Z} \end{tikzcd} \quad \begin{tikzcd} Y^X \arrow[r, "\mathrm{Prod}_Z"] \arrow[d, "Y^g"] & Y\times Z)^{X\times Z} \arrow[d, "(Y\times Z)^{g\times id_Z}"] \\ Y^W \arrow[r, "\mathrm{Prod}_Z"] & (Y\times Z)^{W\times Z} \end{tikzcd} \quad \begin{tikzcd} Y^X \arrow[r, "\mathrm{Prod}_Z"] \arrow[d, "\mathrm{Prod}_W"] & (Y\times Z)^{X\times Z} \arrow[d, "(Y\times Z)^{id_X\times h}"] \\ (Y\times W)^{X\times W} \arrow[r, "(id_Y\times h)^{X\times W}"] & (Y\times Z)^{X\times W} \end{tikzcd}$$

Proof. The commutativity of the left diagram follows from the commutativity of the following diagram and the equality $f\varepsilon_Y^X = \varepsilon_W^X(f^X \times id_X)$ given in (9.2.2).

$$\begin{tikzcd} \mathcal{T}(Y^X \times X\times Z, Y\times Z) \arrow[r, "(f\times id_Z)_*"] \arrow[d, "\exp_{Y^X, X\times Z, Y\times Z}"] & \mathcal{T}(Y^X \times X\times Z, W\times Z) \arrow[d, "\exp_{Y^X, X\times Z, W\times Z}"] & \mathcal{T}(W^X \times X\times Z, W\times Z) \arrow[l, "(f^X\times id_{X\times Z})^*"'] \arrow[d, "\exp_{W^X, X\times Z, W\times Z}"] \\ \mathcal{T}(Y^X, (Y\times Z)^{X\times Z}) \arrow[r, "(f\times id_Z)_*^{X\times Z}"] & \mathcal{T}(Y^X, (W\times Z)^{X\times Z}) & \mathcal{T}(W^X, (W\times Z)^{X\times Z}) \arrow[l, "(f^X)^*"'] \end{tikzcd}$$

The commutativity of the center diagram follows from the commutativity of the following diagram and the equality $\varepsilon_Y^X(id_{Y^X} \times g) = \varepsilon_X^W(Y^g \times id_W)$ given in (9.2.8).

$$\begin{tikzcd} \mathcal{T}(Y^X\times X\times Z, Y\times Z) \arrow[r, "(id_{Y^X}\times g\times id_Z)^*"] \arrow[d, "\exp_{Y^X, X\times Z, Y\times Z}"] & \mathcal{T}(Y^X\times W\times Z, Y\times Z) \arrow[d, "\exp_{Y^X, W\times Z, Y\times Z}"] & \mathcal{T}(Y^W\times W\times Z, Y\times Z) \arrow[l, "(Y^g\times id_{W\times Z})^*"'] \arrow[d, "\exp_{Y^W, W\times Z, Y\times Z}"] \\ \mathcal{T}(Y^X, (Y\times Z)^{X\times Z}) \arrow[r, "(Y\times Z)_*^{g\times id_Z}"] & \mathcal{T}(Y^X, (Y\times Z)^{W\times Z}) & \mathcal{T}(Y^W, (Y\times Z)^{W\times Z}) \arrow[l, "(Y^g)^*"'] \end{tikzcd}$$

The commutativity of the center diagram follows from the commutativity of the following diagram and the definition of Prod_Z, Prod_W.

$$\begin{tikzcd} \mathcal{T}(Y^X\times X\times Z, Y\times Z) \arrow[r, "(id_{Y^X}\times id_X\times h)^*"] \arrow[d, "\exp_{Y^X, X\times Z, Y\times Z}"] & \mathcal{T}(Y^X\times X\times W, Y\times Z) \arrow[d, "\exp_{Y^X, X\times W, Y\times Z}"] & \mathcal{T}(Y^X\times X\times W, Y\times W) \arrow[l, "(id_Y\times h)_*"'] \arrow[d, "\exp_{Y^X, X\times W, Y\times W}"] \\ \mathcal{T}(Y^X, (Y\times Z)^{X\times Z}) \arrow[r, "(Y\times Z)_*^{id_X\times h}"] & \mathcal{T}(Y^X, (Y\times Z)^{X\times W}) & \mathcal{T}(Y^X, (Y\times W)^{X\times W}) \arrow[l, "(id_Y\times h)_*^{X\times W}"'] \end{tikzcd}$$

$\square$

Proposition 9.3.3 *For objects X, Y, Z and W of $\mathcal{T}$, the following diagram commutes.*

$$\begin{tikzcd} \mathcal{T}(W\times X, Y) \arrow[r, "\times Z"] \arrow[d, "\exp_{W,X,Y}"] & \mathcal{T}(W\times X\times Z, Y\times Z) \arrow[d, "\exp_{W,X\times Z,Y\times Z}"] \\ \mathcal{T}(W, Y^X) \arrow[r, "\mathrm{Prod}_Z"] & \mathcal{T}(W, (Y\times Z)^{X\times Z}) \end{tikzcd}$$

Proof. For $f\in\mathcal{T}(W\times X, Y)$, we have

$$\exp_{W,X\times Z,Y\times Z}(f\times id_Z) = (f\times id_Z)^{Y\times Z}\eta_W^{Y\times Z} = (f\times id_Z)^{Y\times Z}\mathrm{Prod}_Z\eta_W^Y = \mathrm{Prod}_Z f^Y\eta_W^Y = \mathrm{Prod}_Z \exp_{W,X,Y}(f)$$

by (9.3.1) and (9.3.2). $\square$

For objects X, Y, Z of $\mathcal{T}$, the image of $\varepsilon_Z^{X\times Y}\in\mathcal{T}(Z^{X\times Y}\times X\times Y, Z)$ by a composition

$$\mathcal{T}(Z^{X\times Y}\times X\times Y, Z)\xrightarrow{\exp_{Z^{X\times Y}\times X,Y,Z}}\mathcal{T}(Z^{X\times Y}\times X, Z^Y)\xrightarrow{\exp_{Z^{X\times Y},X,Z^Y}}\mathcal{T}(Z^{X\times Y},(Z^Y)^X)$$

is denoted by $\omega_Z^{X,Y}: Z^{X\times Y}\to(Z^Y)^X$.

Proposition 9.3.4 *The following diagram $(*)$ commutes for any object W of $\mathcal{T}$.*

$$\begin{tikzcd} \mathcal{T}(W\times X\times Y, Z) \arrow[r, "\exp_{W\times X,Y,Z}"] \arrow[d, "\exp_{W,X\times Y,Z}"] & \mathcal{T}(W\times X, Z^Y) \arrow[d, "\exp_{W,X,Z^Y}"] & \cdots(*) \\ \mathcal{T}(W, Z^{X\times Y}) \arrow[r, "(\omega_Z^{X,Y})_*"] & \mathcal{T}(W,(Z^Y)^X) & \end{tikzcd}$$

Thus $\omega_Z^{X,Y}$ is an isomorphism and the inverse of $\omega_Z^{X,Y}$ is the image of a composition

$$(Z^Y)^X\times X\times Y\xrightarrow{\varepsilon_{Z^Y}^X\times id_Y}Z^Y\times Y\xrightarrow{\varepsilon_Z^Y}Z$$

by $\exp_{(Z^Y)^X,X\times Y,Z}:\mathcal{T}((Z^Y)^X\times X\times Y, Z)\to\mathcal{T}((Z^Y)^X, Z^{X\times Y})$.

Proof. For $f\in\mathcal{T}(W\times X\times Y, Z)$, put $g=\exp_{W,X\times Y,Z}(f)\in\mathcal{T}(W, Z^{X\times Y})$. It follows from (9.2.1) that the following diagram commutes.

$$\begin{tikzcd} \mathcal{T}(Z^{X\times Y}, Z^{X\times Y}) \arrow[r, "g^*"] & \mathcal{T}(W, Z^{X\times Y}) \\ \mathcal{T}(Z^{X\times Y}\times X\times Y, Z) \arrow[r, "(g\times id_{X\times Y})^*"] \arrow[u, "\exp_{Z^{X\times Y},X\times Y,Z}"] \arrow[d, "\exp_{Z^{X\times Y}\times X,Y,Z}"] & \mathcal{T}(W\times X\times Y, Z) \arrow[u, "\exp_{W,X\times Y,Z}"] \arrow[d, "\exp_{W\times X,Y,Z}"] \\ \mathcal{T}(Z^{X\times Y}\times X, Z^Y) \arrow[r, "(g\times id_X)^*"] \arrow[d, "\exp_{Z^{X\times Y},X,Z^Y}"] & \mathcal{T}(W\times X, Z^Y) \arrow[d, "\exp_{W,X,Z^Y}"] \\ \mathcal{T}(Z^{X\times Y},(Z^Y)^X) \arrow[r, "g^*"] & \mathcal{T}(W,(Z^Y)^X) \end{tikzcd}$$

Hence $f=\varepsilon_Z^{X\times Y}(g\times id_Z)$ and $\exp_{W,X,Z^Y}\exp_{W\times X,Y,Z}(f)=g^*(\omega_Z^{X,Y})=(\omega_Z^{X,Y})^*\exp_{W,X\times Y,Z}(f)$. Thus we have $\exp_{W,X,Z^Y}\exp_{W\times X,Y,Z}=(\omega_Z^{X,Y})^*\exp_{W,X\times Y,Z}$, that is, the diagram $(*)$ commutes.

Therefore $(\omega_Z^{X,Y})_*:\mathcal{T}(W,Z^{X\times Y})\to\mathcal{T}(W,(Z^Y)^X)$ is bijective for any $W\in\mathrm{Ob}\,\mathcal{T}$ and $\omega_Z^{X,Y}$ is an isomorphism. Consider the case $W=(Z^Y)^X$ and $f=\varepsilon_Z^Y(\varepsilon_{Z^Y}^X\times id_Y)\in\mathcal{T}((Z^Y)^X\times X\times Y, Z)$. Since

$$\begin{tikzcd} \mathcal{T}(Z^Y\times Y, Z) \arrow[r, "(\varepsilon_{Z^Y}^X\times id_Y)^*"] \arrow[d, "\exp_{Z^Y,Y,Z}"] & \mathcal{T}((Z^Y)^X\times X\times Y, Z) \arrow[d, "\exp_{(Z^Y)\times X,Y,Z}"] \\ \mathcal{T}(Z^Y, Z^Y) \arrow[r, "(\varepsilon_{Z^Y}^X)^*"] & \mathcal{T}((Z^Y)^X\times X, Z^Y) \end{tikzcd}$$

is commutative by (9.2.1), we have $\exp_{(Z^Y)^X\times X,Y,Z}(f)=\varepsilon^X_{Z^Y}$. It follows from the commutativity of $(*)$ that $\exp_{(Z^Y)^X,X\times Y,Z}(f)\omega^{X,Y}_Z$ is the identity morphism of $(Z^Y)^X$. Hence $\exp_{(Z^Y)^X,X\times Y,Z}(f)$ is the inverse of $\omega^{X,Y}_Z$. $\square$

For $(X,\alpha),(Y,\beta)\in\operatorname{Ob}\operatorname{Act}_r(G)$, we denote by $E^{(X,\alpha)}_{(Y,\beta)}:(Y,\beta)^{(X,\alpha)}\to Y^X$ the equalizer of $Y^X\xrightarrow{Y^\alpha}Y^{X\times G}$ and $Y^X\xrightarrow{\mathrm{Prod}_G}(Y\times G)^{X\times G}\xrightarrow{\beta^{X\times G}}Y^{X\times G}$.

Proposition 9.3.5 *Let (X,α), (Y,β) and (Z,γ) be objects of $\operatorname{Act}_r(G)$ and $f:(X,\alpha)\to(Y,\beta)$ a morphism of $\operatorname{Act}_r(G)$. There exist morphisms $f^{(Z,\gamma)}:(X,\alpha)^{(Z,\gamma)}\to(Y,\beta)^{(Z,\gamma)}$, $(Z,\gamma)^f:(Z,\gamma)^{(Y,\beta)}\to(Z,\gamma)^{(X,\alpha)}$ which make the following diagrams commute.*

$$\begin{tikzcd}
(X,\alpha)^{(Z,\gamma)} \arrow[r, "E^{(Z,\gamma)}_{(X,\alpha)}"] \arrow[d, "f^{(Z,\gamma)}"] & X^Z \arrow[d, "f^Z"] \\
(Y,\beta)^{(Z,\gamma)} \arrow[r, "E^{(Z,\gamma)}_{(Y,\beta)}"] & Y^Z
\end{tikzcd}
\qquad
\begin{tikzcd}
(Z,\gamma)^{(Y,\beta)} \arrow[r, "E^{(Y,\beta)}_{(Z,\gamma)}"] \arrow[d, "(Z,\gamma)^f"] & Z^Y \arrow[d, "Z^f"] \\
(Z,\gamma)^{(X,\alpha)} \arrow[r, "E^{(X,\alpha)}_{(Z,\gamma)}"] & Z^X
\end{tikzcd}$$

Proof. The following diagrams commutes by (9.2.7) and (9.3.2).

$$\begin{tikzcd}
X^Z \arrow[r, "X^\gamma"] \arrow[d, "f^Z"] & X^{Z\times G} \arrow[d, "f^{Z\times G}"] \\
Y^Z \arrow[r, "Y^\gamma"] & Y^{Z\times G}
\end{tikzcd}
\qquad
\begin{tikzcd}
X^Z \arrow[r, "\mathrm{Prod}_G"] \arrow[d, "f^Z"] & (X\times G)^{Z\times G} \arrow[r, "\alpha^{Z\times G}"] \arrow[d, "(f\times id_G)^{Z\times G}"] & X^{Z\times G} \arrow[d, "f^{Z\times G}"] \\
Y^Z \arrow[r, "\mathrm{Prod}_G"] & (Y\times G)^{Z\times G} \arrow[r, "\beta^{Z\times G}"] & Y^{Z\times G}
\end{tikzcd}$$

Hence we have $Y^\gamma f^Z E^{(Z,\gamma)}_{(X,\alpha)}=f^{Z\times G}X^\gamma E^{(Z,\gamma)}_{(X,\alpha)}=f^{Z\times G}\alpha^{Z\times G}\mathrm{Prod}_G E^{(Z,\gamma)}_{(X,\alpha)}=\beta^{Z\times G}\mathrm{Prod}_G f^Z E^{(Z,\gamma)}_{(X,\alpha)}$ and this implies that there is a unique morphism $f^{(Z,\gamma)}:(X,\alpha)^{(Z,\gamma)}\to(Y,\beta)^{(Z,\gamma)}$ that satisfies $E^{(Z,\gamma)}_{(Y,\beta)}f^{(Z,\gamma)}=f^Z E^{(Z,\gamma)}_{(X,\alpha)}$.

The following diagrams commutes by (9.2.6), (9.2.7) and (9.3.2).

$$\begin{tikzcd}
Z^Y \arrow[r, "Z^\beta"] \arrow[d, "Z^f"] & Z^{Y\times G} \arrow[d, "Z^{f\times id_G}"] \\
Z^X \arrow[r, "Z^\alpha"] & Z^{X\times G}
\end{tikzcd}
\qquad
\begin{tikzcd}
Z^Y \arrow[r, "\mathrm{Prod}_G"] \arrow[d, "Z^f"] & (Z\times G)^{Y\times G} \arrow[r, "\gamma^{Y\times G}"] \arrow[d, "(Z\times G)^{f\times id_G}"] & Z^{Y\times G} \arrow[d, "Z^{f\times id_G}"] \\
Z^X \arrow[r, "\mathrm{Prod}_G"] & (Z\times G)^{X\times G} \arrow[r, "\gamma^{X\times G}"] & Z^{X\times G}
\end{tikzcd}$$

Hence we have $Z^\alpha Z^f E^{(Y,\beta)}_{(Z,\gamma)}=Z^{f\times id_G}Z^\beta E^{(Y,\beta)}_{(Z,\gamma)}=Z^{f\times id_G}\gamma^{Y\times G}\mathrm{Prod}_G E^{(Y,\beta)}_{(Z,\gamma)}=\gamma^{X\times G}\mathrm{Prod}_G Z^f E^{(Y,\beta)}_{(Z,\gamma)}$, which implies that there is a unique morphism $(Z,\gamma)^f:(Z,\gamma)^{(Y,\beta)}\to(Z,\gamma)^{(X,\alpha)}$ that satisfies $E^{(X,\alpha)}_{(Z,\gamma)}f^*=Z^f E^{(Y,\beta)}_{(Z,\gamma)}$. $\square$

Lemma 9.3.6 *Let (X,α) and (Y,β) be objects of $\operatorname{Act}_r(G)$. For a morphism $f:Z\to Y^X$ of $\mathcal{T}$, we put $\bar f=\exp^{-1}_{Z,X,Y}(f)$. Then, f satisfies $Y^\alpha f=\beta^{X\times G}\mathrm{Prod}_G f$ if and only if $\bar f$ satisfies $\bar f(id_Z\times\alpha)=\beta(\bar f\times id_G)$.*

Proof. The assertion follows from the commutativity of the following diagram.

$$\begin{tikzcd}
\mathcal{T}(Z\times X,Y) \arrow[r, "(id_Z\times\alpha)^*"] \arrow[d, "\exp_{Z,X,Y}"] & \mathcal{T}(Z\times X\times G,Y) \arrow[d, "\exp_{Z,X\times G,Y}"] & \mathcal{T}(Z\times X\times G,Y\times G) \arrow[l, "\beta_*"'] \arrow[d, "\exp_{Z,X\times G,Y\times G}"] & \mathcal{T}(Z\times X,Y) \arrow[l, "\times G"'] \arrow[d, "\exp_{Z,X,Y}"] \\
\mathcal{T}(Z,Y^X) \arrow[r, "Y^\alpha_*"] & \mathcal{T}(Z,Y^{X\times G}) & \mathcal{T}(Z,(Y\times G)^{X\times G}) \arrow[l, "\beta^{X\times G}_*"'] & \mathcal{T}(Z,Y^X) \arrow[l, "\mathrm{Prod}_{G*}"']
\end{tikzcd}$$

$\square$

Proposition 9.3.7 *(1) Let X be an object of $\mathcal{T}$ and (Y,β) an object of $\operatorname{Act}_r(G)$. Then, a composition*

$$Y^X\xrightarrow{\mathrm{Prod}_G}(Y\times G)^{X\times G}\xrightarrow{\beta^{X\times G}}Y^{X\times G}$$

is an equalizer of $Y^{id_X\times\mu}:Y^{X\times G}\to Y^{X\times G\times G}$ and $\beta^{X\times G\times G}\mathrm{Prod}_G:Y^{X\times G}\to Y^{X\times G\times G}$. Hence $(Y,\beta)^{L(X)}$ is isomorphic to Y^X.

(2) Let (X,α) be an object of $\operatorname{Act}_r(G)$ and Y an object of $\mathcal{T}$. Then, a composition

$$Y^X\xrightarrow{Y^\alpha}Y^{X\times G}\xrightarrow{\omega^{X,G}_Y}(Y^G)^X$$

is an equalizer of $(Y^G)^\alpha:(Y^G)^X\to(Y^G)^{X\times G}$ and $(\rho^\mu_Y)^{X\times G}\mathrm{Prod}_G:(Y^G)^X\to(Y^G)^{X\times G}$. Hence $R(Y)^{(X,\alpha)}$ is isomorphic to Y^X.

Proof. (1) By (9.3.2) and (9.2.7), the following diagrams commute.

$$\begin{tikzcd}
(Y\times G)^{X\times G} \arrow[r, "\mathrm{Prod}_G"] \arrow[d, "\beta^{X\times G}"] & (Y\times G\times G)^{X\times G\times G} \arrow[r, "(id_Y\times\mu)^{X\times G\times G}"] \arrow[d, "(\beta\times id_G)^{X\times G\times G}"] & (Y\times G)^{X\times G\times G} \arrow[d, "\beta^{X\times G\times G}"] \\
Y^{X\times G} \arrow[r, "\mathrm{Prod}_G"] & (Y\times G)^{X\times G\times G} \arrow[r, "\beta^{X\times G\times G}"] & Y^{X\times G\times G}
\end{tikzcd}$$

$$\begin{tikzcd}
Y^X \arrow[r, "\mathrm{Prod}_G"] \arrow[d, "\mathrm{Prod}_{G\times G}"] & (Y\times G)^{X\times G} \arrow[r, "\beta^{X\times G}"] \arrow[d, "(Y\times G)^{id_X\times\mu}"] & Y^{X\times G} \arrow[d, "Y^{id_X\times\mu}"] \\
(Y\times G\times G)^{X\times G\times G} \arrow[r, "(id_Y\times\mu)^{X\times G\times G}"] & (Y\times G)^{X\times G\times G} \arrow[r, "\beta^{X\times G\times G}"] & Y^{X\times G\times G}
\end{tikzcd}$$

Thus we have

$$\begin{aligned}\beta^{X\times G\times G}\mathrm{Prod}_G\beta^{X\times G}\mathrm{Prod}_G &= \beta^{X\times G\times G}(id_Y\times\mu)^{X\times G\times G}\mathrm{Prod}_G\mathrm{Prod}_G = \beta^{X\times G\times G}(id_Y\times\mu)^{X\times G\times G}\mathrm{Prod}_{G\times G}\\ &= Y^{id_X\times\mu}\beta^{X\times G}\mathrm{Prod}_G\end{aligned}$$

For morphisms $f : Z\to Y^{X\times G}$ and $g : Z\to Y^X$, we put $\bar f = \exp^{-1}_{Z,X\times G,Y}(f)$ and $\bar g = \exp^{-1}_{Z,X,Y}(g)$. Since the following diagram commutes, g satisfies $\beta^{X\times G}\mathrm{Prod}_G g = f$ if and only if $\bar g$ satisfies $\beta(\bar g\times id_G) = \bar f$.

$$\begin{tikzcd}
\mathcal{T}(Z\times X, Y) \arrow[r, "\times G"] \arrow[d, "\exp_{Z,X,Y}"] & \mathcal{T}(Z\times X\times G, Y\times G) \arrow[r, "\beta_*"] \arrow[d, "\exp_{Z,X\times G,Y\times G}"] & \mathcal{T}(Z\times X\times G, Y) \arrow[d, "\exp_{Z,X\times G,Y}"] \\
\mathcal{T}(Z, Y^X) \arrow[r, "\mathrm{Prod}_{G*}"] & \mathcal{T}(Z, (Y\times G)^{X\times G}) \arrow[r, "\beta^{X\times G}_*"] & \mathcal{T}(Z, Y^{X\times G})
\end{tikzcd}$$

Assume that f satisfies $Y^{id_X\times\mu}f = \beta^{X\times G\times G}\mathrm{Prod}_G f$ and put $\bar g = \bar f(id_Z\times id_X, \varepsilon o_{Z\times X})$. Then, since $\bar f$ satisfies $\bar f(id_Z\times id_X\times\mu) = \beta(\bar f\times id_G)$ by (9.3.6), we have

$$\begin{aligned}\bar f &= \bar f(id_Z\times id_X\times\mu(\varepsilon o_G\times id_G)\Delta_G) = \bar f(id_Z\times id_X\times\mu)(id_Z\times id_X\times(\varepsilon o_G\times id_G)\Delta_G)\\ &= \beta(\bar f\times id_G)(id_Z\times id_X\times(\varepsilon o_G\times id_G)\Delta_G) = \beta(\bar f(id_Z\times id_X, \varepsilon o_{Z\times X})\times id_G) = \beta(\bar g\times id_G).\end{aligned}$$

Thus $g = \exp_{Z,X,Y}(\bar g)$ satisfies $\beta^{X\times G}\mathrm{Prod}_G g = f$. If $g_1, g_2\in\mathcal{T}(Z, Y^X)$ satisfy $\beta^{X\times G}\mathrm{Prod}_G g_1 = \beta^{X\times G}\mathrm{Prod}_G g_2$, then $\bar g_1 = \exp^{-1}_{Z,X,Y}(g_1)$ and $\bar g_2 = \exp^{-1}_{Z,X,Y}(g)$ satisfy $\beta(\bar g_1\times id_G) = \beta(\bar g_2\times id_G)$, which implies

$$\bar g_1 = \beta(\bar g_1\times\varepsilon o_G) = \beta(\bar g_1\times id_G)(id_Z\times id_X\times\varepsilon o_G) = \beta(\bar g_2\times id_G)(id_Z\times id_X\times\varepsilon o_G) = \beta(\bar g_2\times\varepsilon o_G) = \bar g_2.$$

Therefore, we have $g_1 = g_2$ and $\beta^{X\times G}\mathrm{Prod}_G$ is a monomorphism.

(2) Put $\varphi = \omega^{X,G}_Y Y^\alpha$ and $\bar\varphi = \exp^{-1}_{Y^X,G,Y}(\varphi)$, then $\bar\varphi = \exp_{Y^X\times X,G,Y}(id_{Y^X}\times\alpha)$ by (9.3.4). Hence

$$id_{Y^X}\times\alpha = \exp^{-1}_{Y^X\times X,G,Y}(\bar\varphi) = \varepsilon^G_Y(\bar\varphi\times id_G)$$

and we have

$$\begin{aligned}\varepsilon^G_Y(id_{Y^G}\times\mu)(\bar\varphi\times id_{G\times G}) &= \varepsilon^G_Y(\bar\varphi\times id_G)(id_{Y^G}\times id_X\times\mu) = (id_{Y^X}\times\alpha)(id_{Y^G}\times id_X\times\mu)\\ &= (id_{Y^X}\times\alpha)(id_{Y^G}\times\alpha\times id_G).\end{aligned}$$

By the commutativity of

$$\begin{tikzcd}
\mathcal{T}(Y^X\times X\times G, Y) \arrow[r, "(id_{Y^G}\times\alpha\times id_G)^*"] \arrow[d, "\exp_{Y^X\times X,G,Y}"] & \mathcal{T}(Y^X\times X\times G\times G, Y) \arrow[d, "\exp_{Y^X\times X\times G,G,Y}"] & \mathcal{T}(Y^G\times G\times G, Y) \arrow[l, "(\bar\varphi\times id_{G\times G})^*"'] \arrow[d, "\exp_{Y^G\times G,G,Y}"] \\
\mathcal{T}(Y^X\times X, Y^G) \arrow[r, "(id_{Y^G}\times\alpha)^*"] & \mathcal{T}(Y^X\times X\times G, Y^G) & \mathcal{T}(Y^G\times G, Y^G) \arrow[l, "(\bar\varphi\times id_G)^*"']
\end{tikzcd},$$

we see $\bar\varphi(id_{Y^X}\times\alpha) = \rho^\mu_Y(\bar\varphi\times id_G)$, which implies $(Y^G)^\alpha\varphi = (\rho^\mu_Y)^{X\times G}\mathrm{Prod}_G\varphi$ by (9.3.6).

Suppose that a morphism $f : Z\to (Y^G)^X$ satisfies $(Y^G)^\alpha f = (\rho^\mu_Y)^{X\times G}\mathrm{Prod}_G f$. Put $\bar f = \exp^{-1}_{Z,X,Y^G}(f)$, then we have $\bar f(id_Z\times\alpha) = \rho^\mu_Y(\bar f\times id_G)$ by (9.3.6). We define a morphism $\bar g : Z\times X\to Y$ by

$$\bar g = \exp^{-1}_{Z\times X,G,Y}(\bar f)(id_Z\times(id_X, \varepsilon o_X)) = \varepsilon^G_Y(\bar f\times id_G)(id_Z\times(id_X, \varepsilon o_X)).$$

It follows from (6) of (9.2.26) and (9.2.4) that $\varepsilon_Y^G = \exp_{Y,1,Y}(\mathrm{pr}_Y)^{-1}Y^{\varepsilon}\rho_Y^{\mu}$. Thus we have

$$\begin{aligned}
\bar{g}(id_Z \times \alpha) &= \varepsilon_Y^G(\bar{f} \times id_G)(id_Z \times (id_X, \varepsilon o_X))(id_Z \times \alpha) \\
&= \exp_{Y,1,Y}(\mathrm{pr}_Y)^{-1}Y^{\varepsilon}\rho_Y^{\mu}(\bar{f} \times id_G)(id_Z \times (id_X, \varepsilon o_X))(id_Z \times \alpha) \\
&= \exp_{Y,1,Y}(\mathrm{pr}_Y)^{-1}Y^{\varepsilon}\bar{f}(id_Z \times \alpha)(id_Z \times (id_X, \varepsilon o_X))(id_Z \times \alpha) \\
&= \exp_{Y,1,Y}(\mathrm{pr}_Y)^{-1}Y^{\varepsilon}\bar{f}(id_Z \times \alpha(id_X, \varepsilon o_X))(id_Z \times \alpha) \\
&= \exp_{Y,1,Y}(\mathrm{pr}_Y)^{-1}Y^{\varepsilon}\bar{f}(id_Z \times \alpha) = \exp_{Y,1,Y}(\mathrm{pr}_Y)^{-1}Y^{\varepsilon}\rho_Y^{\mu}(\bar{f} \times id_G) \\
&= \varepsilon_Y^G(\bar{f} \times id_G) = \exp_{Z\times X,G,Y}^{-1}(\bar{f}).
\end{aligned}$$

Therefore $\exp_{Z\times X,G,Y}(\bar{g}(id_Z \times \alpha)) = \bar{f}$. We put $g = \exp_{Z,X,Y}(\bar{g})$. By the commutativity of

$$\begin{tikzcd}
\mathcal{T}(Z \times X, Y) \arrow[r, "(id_Z\times\alpha)^*"] \arrow[d, "\exp_{Z,X,Y}"] & \mathcal{T}(Z \times X \times G, Y) \arrow[r, "\exp_{Z\times X,G,Y}"] \arrow[d, "\exp_{Z,X\times G,Y}"] & \mathcal{T}(Z \times X, Y^G) \arrow[d, "\exp_{Z,X,Y^G}"] \\
\mathcal{T}(Z, Y^X) \arrow[r, "Y_*^{\alpha}"] & \mathcal{T}(Z, Y^{X\times G}) \arrow[r, "(\omega_Y^{X,G})_*"] & \mathcal{T}(Z, (Y^G)^X)
\end{tikzcd},$$

$\exp_{Z\times X,G,Y}(\bar{g}(id_Z \times \alpha)) = \bar{f}$ implies $\omega_Y^{X,G}Y^{\alpha}g = f$.

Since α has a right inverse $(id_X, \varepsilon o_X) : X \to X \times G$, α is an epimorphism. Hence Y^{α} is a monomorphism by (9.2.9) and so is $\omega_Y^{X,G}Y^{\alpha}$ by (9.3.4). □

For a group object $(G, \mu, \varepsilon, \iota)$ in $\mathcal{T}$, we regard (G, μ) as an object of $\mathrm{Act}_r(G)$.

Lemma 9.3.8 *Suppose that a morphism $\varphi : H \to G$ of group objects and a right H-action $\alpha : X \times H \to X$ on X are given. There exists a morphism $\tilde{\alpha} : (X, \alpha)^{\varphi^*(G,\mu)} \times G \to (X, \alpha)^{\varphi^*(G,\mu)}$ which makes the following diagram commute.*

$$\begin{tikzcd}
(X, \alpha)^{\varphi^*(G,\mu)} \times G \arrow[r, "\tilde{\alpha}"] \arrow[d, "E_{(X,\alpha)}^{\varphi^*(G,\mu)} \times id_G"] & (X, \alpha)^{\varphi^*(G,\mu)} \arrow[d, "E_{(X,\alpha)}^{\varphi^*(G,\mu)}"] \\
X^G \times G \arrow[r, "\rho_X^{\mu}"] & X^G
\end{tikzcd}$$

Proof. By the commutativity of diagrams

$$\begin{tikzcd}
\mathcal{T}(X^G \times G \times G, X) \arrow[r, "\exp_{X^G\times G,G,X}"] \arrow[d, "\left(E_{(X,\alpha)}^{\varphi^*(G,\mu)} \times id_G \times id_G\right)^*"] & \mathcal{T}(X^G \times G, X^G) \arrow[d, "\left(E_{(X,\alpha)}^{\varphi^*(G,\mu)} \times id_G\right)^*"] \\
\mathcal{T}((X, \alpha)^{\varphi^*(G,\mu)} \times G \times G, X) \arrow[r, "\exp_{(X,\alpha)^{\varphi^*(G,\mu)}\times G,G,X}"] \arrow[d, "(id_{(X,\alpha)^{\varphi^*(G,\mu)}} \times id_G \times \mu)^*"] & \mathcal{T}((X, \alpha)^{\varphi^*(G,\mu)} \times G, X^G) \arrow[d, "X_*^{\mu}"] \\
\mathcal{T}((X, \alpha)^{\varphi^*(G,\mu)} \times G \times G \times G, X) \arrow[r, "\exp_{(X,\alpha)^{\varphi^*(G,\mu)}\times G,G\times G,X}"] \arrow[d, "(id_{(X,\alpha)^{\varphi^*(G,\mu)}} \times id_G \times id_G \times \varphi)^*"] & \mathcal{T}((X, \alpha)^{\varphi^*(G,\mu)} \times G, X^{G\times G}) \arrow[d, "X_*^{id_G\times\varphi}"] \\
\mathcal{T}((X, \alpha)^{\varphi^*(G,\mu)} \times G \times G \times H, X) \arrow[r, "\exp_{(X,\alpha)^{\varphi^*(G,\mu)}\times G,G\times H,X}"] & \mathcal{T}((X, \alpha)^{\varphi^*(G,\mu)} \times G, X^{G\times H})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{T}(X^G \times G \times G, X) \arrow[r, "\exp_{X^G\times G,G,X}"] \arrow[d, "\left(E_{(X,\alpha)}^{\varphi^*(G,\mu)} \times id_G \times id_G\right)^*"] & \mathcal{T}(X^G \times G, X^G) \arrow[d, "\left(E_{(X,\alpha)}^{\varphi^*(G,\mu)} \times id_G\right)^*"] \\
\mathcal{T}((X, \alpha)^{\varphi^*(G,\mu)} \times G \times G, X) \arrow[r, "\exp_{(X,\alpha)^{\varphi^*(G,\mu)}\times G,G,X}"] \arrow[d, "\times H"] & \mathcal{T}((X, \alpha)^{\varphi^*(G,\mu)} \times G, X^G) \arrow[d, "(\mathrm{Prod}_H)_*"] \\
\mathcal{T}((X, \alpha)^{\varphi^*(G,\mu)} \times G \times G \times H, X \times H) \arrow[r, "\exp_{(X,\alpha)^{\varphi^*(G,\mu)}\times G,G\times H,X\times H}"] \arrow[d, "\alpha_*"] & \mathcal{T}((X, \alpha)^{\varphi^*(G,\mu)} \times G, (X \times H)^{G\times H}) \arrow[d, "\alpha_*^{G\times H}"] \\
\mathcal{T}((X, \alpha)^{\varphi^*(G,\mu)} \times G \times G \times H, X) \arrow[r, "\exp_{(X,\alpha)^{\varphi^*(G,\mu)}\times G,G\times H,X}"] & \mathcal{T}((X, \alpha)^{\varphi^*(G,\mu)} \times G, X^{G\times H})
\end{tikzcd}$$

$X^{\mu(id_G\times\varphi)}\rho^\mu_X\big(E^{\varphi^*(G,\mu)}_{(X,\alpha)}\times id_G\big)=\alpha^{G\times H}\mathrm{Prod}_H\rho^\mu_X\big(E^{\varphi^*(G,\mu)}_{(X,\alpha)}\times id_G\big)$ is equivalent to the following equality.

$$\varepsilon^G_X\big(E^{\varphi^*(G,\mu)}_{(X,\alpha)}\times\mu(\mu\times\varphi)\big)=\alpha\big(\varepsilon^G_X\big(E^{\varphi^*(G,\mu)}_{(X,\alpha)}\times\mu\big)\times id_H\big)\ \cdots(*)$$

Since $\exp^{-1}_{(X,\alpha)^{\varphi^*(G,\mu)},G\times H,X}\big(E^{\varphi^*(G,\mu)}_{(X,\alpha)}\big)=\varepsilon^G_X\big(E^{\varphi^*(G,\mu)}_{(X,\alpha)}\times id_G\big)$ and

$$\begin{tikzcd}
\mathcal{T}((X,\alpha)^{\varphi^*(G,\mu)}\times G,X) \arrow[r, "\exp_{(X,\alpha)^{\varphi^*(G,\mu)},G,X}"] \arrow[d, "(id_{(X,\alpha)^{\varphi^*(G,\mu)}}\times\mu(id_G\times\varphi))^*"] & \mathcal{T}((X,\alpha)^{\varphi^*(G,\mu)},X^G) \arrow[d, "X^{\mu(id_G\times\varphi)}_*"] \\
\mathcal{T}((X,\alpha)^{\varphi^*(G,\mu)}\times G\times H,X) \arrow[r, "\exp_{(X,\alpha)^{\varphi^*(G,\mu)},G\times H,X}"] & \mathcal{T}((X,\alpha)^{\varphi^*(G,\mu)},X^{G\times H}) \\
\mathcal{T}((X,\alpha)^{\varphi^*(G,\mu)}\times G\times H,X\times H) \arrow[r, "\exp_{(X,\alpha)^{\varphi^*(G,\mu)},G\times H,Y\times H}"] \arrow[u, "\alpha_*"] & \mathcal{T}((X,\alpha)^{\varphi^*(G,\mu)},(X\times H)^{G\times H}) \arrow[u, "\alpha^{G\times H}_*"'] \\
\mathcal{T}((X,\alpha)^{\varphi^*(G,\mu)}\times G,X) \arrow[r, "\exp_{(X,\alpha)^{\varphi^*(G,\mu)},G,X}"] \arrow[u, "\times H"] & \mathcal{T}((X,\alpha)^{\varphi^*(G,\mu)},X^G) \arrow[u, "\mathrm{Prod}_{H*}"']
\end{tikzcd}$$

commutes, we have the following equality.

$$\varepsilon^G_X\big(E^{\varphi^*(G,\mu)}_{(X,\alpha)}\times\mu(id_G\times\varphi)\big)=\alpha\big(\varepsilon^G_X\big(E^{\varphi^*(G,\mu)}_{(X,\alpha)}\times id_G\big)\times id_H\big)\ \cdots(**)$$

The left hand side of $(*)$ is equal to $\varepsilon^G_X\big(E^{\varphi^*(G,\mu)}_{(X,\alpha)}\times\mu(id_G\times\varphi)\big)\big(id_{(X,\alpha)^{\varphi^*(G,\mu)}}\times\mu\times id_H\big)$ and the right hand side of $(*)$ is equal to $\alpha\big(\varepsilon^G_X\big(E^{\varphi^*(G,\mu)}_{(X,\alpha)}\times id_G\big)\times id_H\big)\big(id_{(X,\alpha)^{\varphi^*(G,\mu)}}\times\mu\times id_H\big)$. Thus $(*)$ follows from $(**)$. Since $E^{\varphi^*(G,\mu)}_{(X,\alpha)}$ is an equalizer of $X^{\mu(id_G\times\varphi)}$ and $\alpha^{G\times H}\mathrm{Prod}_H$, the assertion follows. □

Since $E^{\varphi^*(G,\mu)}_{(X,\alpha)}$ is a monomorphism, the morphism $\tilde\alpha$ in (9.3.8) is unique and it defines a right G-action on $(X,\alpha)^{\varphi^*(G,\mu)}$. Let us denote this $\tilde\alpha$ by α^φ, $(X,\alpha)^{\varphi^*(G,\mu)}$ by X^φ_α and $E^{\varphi^*(G,\mu)}_{(X,\alpha)}$ by E^φ_α.

Lemma 9.3.9 *Let $f:(X,\alpha)\to(Y,\beta)$ be a morphism in $\mathrm{Act}_r(H)$ and put $f^\varphi=f^{\varphi^*(G,\mu)}$. The following diagram commutes.*

$$\begin{tikzcd}
X^\varphi_\alpha\times G \arrow[r, "\alpha^\varphi"] \arrow[d, "f^\varphi\times id_G"] & X^\varphi_\alpha \arrow[d, "f^\varphi"] \\
Y^\varphi_\beta\times G \arrow[r, "\beta^\varphi"] & Y^\varphi_\beta
\end{tikzcd}$$

Proof. By (9.3.5), (9.3.8) and (3) of (9.2.26), the following diagram commutes.

$$\begin{tikzcd}
 & X^\varphi_\alpha\times G \arrow[r, "\alpha^\varphi"] \arrow[d, "E^\varphi_\alpha\times id_G"] & X^\varphi_\alpha \arrow[d, "E^\varphi_\alpha"] & \\
X^\varphi_\alpha\times G \arrow[r, "E^\varphi_\alpha\times id_G"] \arrow[d, "f^\varphi\times id_G"] & X^G\times G \arrow[r, "\rho^\mu_X"] \arrow[d, "f^G\times id_G"] & X^G \arrow[d, "f^G"] & X^\varphi_\alpha \arrow[l, "E^\varphi_\alpha"'] \arrow[d, "f^\varphi"] \\
Y^\varphi_\beta\times G \arrow[r, "E^\varphi_\beta\times id_G"] & Y^G\times G \arrow[r, "\rho^\mu_Y"] & Y^G & Y^\varphi_\beta \arrow[l, "E^\varphi_\beta"'] \\
 & Y^\varphi_\beta\times G \arrow[r, "\beta^\varphi"] \arrow[u, "E^\varphi_\beta\times id_G"] & Y^\varphi_\beta. \arrow[u, "E^\varphi_\beta"] &
\end{tikzcd}$$

Hence $E^\varphi_\beta f^\varphi\alpha^\varphi=f^GE^\varphi_\alpha\alpha^\varphi=\rho^\mu_Y(f^G\times id_G)(E^\varphi_\alpha\times id_G)=\rho^\mu_Y(E^\varphi_\beta\times id_G)(f^\varphi\times id_G)=E^\varphi_\beta\beta^\varphi(f^\varphi\times id_G)$. Since E^φ_β is a monomorphism, the assertion follows. □

We define a functor $\varphi_*:\mathrm{Act}_r(H)\to\mathrm{Act}_r(G)$ by $\varphi_*(X,\alpha)=(X^\varphi_\alpha,\alpha^\varphi)$ and $\varphi_*(f)=f^\varphi$.

Theorem 9.3.10 *$\varphi_*:\mathrm{Act}_r(H)\to\mathrm{Act}_r(G)$ is a right adjoint of $\varphi^*:\mathrm{Act}_r(G)\to\mathrm{Act}_r(H)$.*

Proof. Let (X,α) be an object of $\mathrm{Act}_r(G)$. The following diagram commutes by (9.2.1), (9.2.5) and (9.3.3).

$$\begin{tikzcd}
\mathcal{T}(X\times G,X) \arrow[r, "(id_X\times\mu(id_G\times\varphi))^*"] \arrow[d, "\exp_{X,G,X}"] & \mathcal{T}(X\times G\times H,X) \arrow[d, "\exp_{X,G\times H,X}"] & \mathcal{T}(X\times G\times H,X\times H) \arrow[l, "(\alpha(id_X\times\varphi))_*"'] \arrow[d, "\exp_{X,G\times H,X\times H}"] & \mathcal{T}(X\times G,X) \arrow[l, "\times H"'] \arrow[d, "\exp_{X,G,X}"] \\
\mathcal{T}(X,X^G) \arrow[r, "X_*^{\mu(id_G\times\varphi)}"] & \mathcal{T}(X,X^{G\times H}) & \mathcal{T}(X,(X\times H)^{G\times H}) \arrow[l, "(\alpha(id_X\times\varphi))_*^{G\times H}"'] & \mathcal{T}(X,X^G) \arrow[l, "\mathrm{Prod}_{H*}"']
\end{tikzcd}$$

Since $\alpha(id_X\times\mu(id_G\times\varphi))=\alpha(\alpha\times\varphi)=\alpha(id_X\times\varphi)(\alpha\times id_H)$, it follows from the above diagram that $\mathrm{Ad}_r(\alpha)$ satisfies $X^{\mu(id_G\times\varphi)}\mathrm{Ad}_r(\alpha)=(\alpha(id_X\times\varphi))^{G\times H}\mathrm{Prod}_H\mathrm{Ad}_r(\alpha)$. Hence there exists unique morphism $\eta_{(X,\alpha)}:X\to X^{\varphi}_{\varphi^*(\alpha)}$ that satisfies $E^{\varphi}_{\varphi^*(\alpha)}\eta_{(X,\alpha)}=\mathrm{Ad}_r(\alpha)$. It follows from (4) of (9.2.14) and (9.3.5) that $\eta_{(X,\alpha)}$ is natural in (X,α). Since $\mathrm{Ad}_r(\alpha):(X,\alpha)\to(X^G,\rho^{\mu}_X)$ and $E^{\varphi}_{\varphi^*(\alpha)}:\big(X^{\varphi}_{\varphi^*(\alpha)},\varphi^*(\alpha)^{\varphi}\big)\to(X^G,\rho^{\mu}_X)$ are morphisms of $\mathrm{Act}_r(G)$ by (7) of (9.2.26) and the definition of $\varphi^*(\alpha)^{\varphi}$, we have

$$E^{\varphi}_{\varphi^*(\alpha)}\varphi^*(\alpha)^{\varphi}(\eta_{(X,\alpha)}\times id_G)=\rho^{\mu}_X\big(E^{\varphi}_{\varphi^*(\alpha)}\times id_G\big)(\eta_{(X,\alpha)}\times id_G)=\rho^{\mu}_X(\mathrm{Ad}_r(\alpha)\times id_G)=\mathrm{Ad}_r(\alpha)\alpha=E^{\varphi}_{\varphi^*(\alpha)}\eta_{(X,\alpha)}\alpha.$$

Since $E^{\varphi}_{\varphi^*(\alpha)}$ is a monomorphism, $\eta_{(X,\alpha)}:(X,\alpha)\to\big(X^{\varphi}_{\varphi^*(\alpha)},\varphi^*(\alpha)^{\varphi}\big)=\varphi_*\varphi^*(X,\alpha)$ is a morphism of $\mathrm{Act}_r(G)$ by the above equality.

For an object (Y,β) of $\mathrm{Act}_r(H)$, define a morphism $\varepsilon_{(Y,\beta)}:Y^{\varphi}_{\beta}\to Y$ to be the following composition.

$$Y^{\varphi}_{\beta}\xrightarrow{E^{\varphi}_{\beta}}Y^G\xrightarrow{(id_{Y^G},\varepsilon o_{Y^G})}Y^G\times G\xrightarrow{\varepsilon^G_Y}Y$$

It follows from (9.3.5) and (9.2.2) that $\varepsilon_{(Y,\beta)}$ is natural in (Y,β). By $(**)$ of the proof of (9.3.8), we have $\varepsilon^G_Y(E^{\varphi}_{\beta}\times id_G)(id_{Y^{\varphi}_{\beta}}\times\mu(id_G\times\varphi))=\beta(\varepsilon^G_Y(E^{\varphi}_{\beta}\times id_G)\times id_H)$. Hence

$$\begin{aligned}
\beta(\varepsilon_{(Y,\beta)}\times id_H)&=\beta(\varepsilon^G_Y(id_{Y^G},\varepsilon o_{Y^G})E^{\varphi}_{\beta}\times id_H)=\beta(\varepsilon^G_Y(E^{\varphi}_{\beta},\varepsilon o_{Y^{\varphi}_{\beta}})\times id_H)\\
&=\beta(\varepsilon^G_Y(E^{\varphi}_{\beta}\times id_G)\times id_H)(id_{Y^{\varphi}_{\beta}}\times(\varepsilon o_H,id_H))\\
&=\varepsilon^G_Y(E^{\varphi}_{\beta}\times id_G)(id_{Y^{\varphi}_{\beta}}\times\mu(id_G\times\varphi))(id_{Y^{\varphi}_{\beta}}\times(\varepsilon o_H,id_H))\\
&=\varepsilon^G_Y(E^{\varphi}_{\beta}\times\mu(\varepsilon o_H,\varphi))=\varepsilon^G_Y(E^{\varphi}_{\beta}\times\varphi).
\end{aligned}$$

On the other hand, by (1) of (9.2.26), we have

$$\begin{aligned}
\varepsilon_{(Y,\beta)}\beta^{\varphi}(id_{Y^{\varphi}_{\beta}}\times\varphi)&=\varepsilon^G_Y(id_{Y^G},\varepsilon o_{Y^G})E^{\varphi}_{\beta}\beta^{\varphi}(id_{Y^{\varphi}_{\beta}}\times\varphi)\\
&=\varepsilon^G_Y(id_{Y^G},\varepsilon o_{Y^G})\rho^{\mu}_Y(E^{\varphi}_{\beta}\times id_G)(id_{Y^{\varphi}_{\beta}}\times\varphi)\\
&=\varepsilon^G_Y(\rho^{\mu}_Y(E^{\varphi}_{\beta}\times\varphi),\varepsilon o_{Y^{\varphi}_{\beta}\times H})\\
&=\varepsilon^G_Y(\rho^{\mu}_Y\times id_G)(E^{\varphi}_{\beta}\times id_G\times id_G)(id_{Y^{\varphi}_{\beta}}\times(\varphi,\varepsilon o_H))\\
&=\varepsilon^G_Y(id_{Y^G}\times\mu)(E^{\varphi}_{\beta}\times id_G\times id_G)(id_{Y^{\varphi}_{\beta}}\times(\varphi,\varepsilon o_H))\\
&=\varepsilon^G_Y(E^{\varphi}_{\beta}\times\mu(\varphi,\varepsilon o_H))=\varepsilon^G_Y(E^{\varphi}_{\beta}\times\varphi).
\end{aligned}$$

Therefore $\varepsilon_{(Y,\beta)}:\varphi^*\varphi_*(Y,\beta)=(Y^{\varphi}_{\beta},\beta^{\varphi}(id_{Y^{\varphi}_{\beta}}\times\varphi))\to(Y,\beta)$ is a morphism of $\mathrm{Act}_r(H)$.

Let (X,α) be an object of $\mathrm{Act}_r(G)$. It follows from $\mathrm{Ad}_r(\alpha)=E^{\varphi}_{\varphi^*(\alpha)}\eta_{(X,\alpha)}$ and (3) of (9.2.14) that we have

$$\begin{aligned}
\varepsilon_{\varphi^*(X,\alpha)}\varphi^*(\eta_{(X,\alpha)})&=\varepsilon^G_X(id_{X^G},\varepsilon o_{X^G})E^{\varphi}_{\varphi^*(\alpha)}\eta_{(X,\alpha)}=\varepsilon^G_X(id_{X^G},\varepsilon o_{X^G})\mathrm{Ad}_r(\alpha)\\
&=\varepsilon^G_X(\mathrm{Ad}_r(\alpha),\varepsilon o_X)=\varepsilon^G_X(\mathrm{Ad}_r(\alpha)\times id_G)(id_X,\varepsilon o_X)=id_X.
\end{aligned}$$

Let (Y,β) be an object of $\mathrm{Act}_r(G)$. It follows from the definition of β^{φ} and (4) of (9.2.14) that the lower left rectangle of the following diagram commutes. Other rectangles of the following diagram commutes by the definitions of $\eta_{\varphi_*(Y,\beta)}$, $\varepsilon_{(Y,\beta)}$ and $\varphi_*(\varepsilon_{(Y,\beta)})$.

$$\begin{tikzcd}
Y^{\varphi}_{\beta} \arrow[r, "\eta_{\varphi_*(Y,\beta)}"] \arrow[dr, "\mathrm{Ad}_r(\beta^{\varphi})"] \arrow[dd, "E^{\varphi}_{\beta}"'] & (Y^{\varphi}_{\beta})^{\varphi}_{\varphi^*(\beta^{\varphi})} \arrow[r, "\varphi_*(\varepsilon_{(Y,\beta)})"] \arrow[d, "E^{\varphi}_{\varphi^*(\beta^{\varphi})}"] & Y^{\varphi}_{\beta} \arrow[d, "E^{\varphi}_{\beta}"] \\
 & (Y^{\varphi}_{\beta})^G \arrow[r, "\varepsilon^G_{(Y,\beta)}"] \arrow[d, "(E^{\varphi}_{\beta})^G"] & Y^G \\
Y^G \arrow[r, "\mathrm{Ad}_r(\rho^{\mu}_Y)^G"] & (Y^G)^G \arrow[r, "(id_{Y^G},\varepsilon o_{Y^G})^G"] & (Y^G\times G)^G \arrow[u, "(\varepsilon^G_Y)^G"']
\end{tikzcd}$$

Thus we have $E^{\varphi}_{\beta}\varphi_*(\varepsilon_{(Y,\beta)})\eta_{\varphi_*(Y,\beta)} = (\varepsilon^G_Y)^G(id_{Y^G}, \varepsilon o_{Y^G})^G \mathrm{Ad}_r(\rho^{\mu}_Y)^G E^{\varphi}_{\beta} = E^{\varphi}_{\beta}$ by (3) of (9.2.14). Since E^{φ}_{β} is a monomorphism, the above equality implies that $\varphi_*(\varepsilon_{(Y,\beta)})\eta_{\varphi_*(Y,\beta)}$ is the identity morphism of Y^{φ}_{β}.

Therefore

$$\varphi^*(X,\alpha) \xrightarrow{\varphi^*(\eta_{(X,\alpha)})} \varphi^*\varphi_*\varphi^*(X,\alpha) \xrightarrow{\varepsilon_{\varphi^*(X,\alpha)}} \varphi^*(X,\alpha)$$

is the identity morphism of $\varphi^*(X,\alpha)$ and

$$\varphi_*(Y,\beta) \xrightarrow{\eta_{\varphi_*(Y,\beta)}} \varphi_*\varphi^*\varphi_*(Y,\beta) \xrightarrow{\varphi_*(\varepsilon_{(Y,\beta)})} \varphi_*(Y,\beta)$$

is the identity morphism of $\varphi_*(Y,\beta)$. □

10 Fibered category of modules

10.1 Fibered category of affine modules

Let K^* be a linearly topologized graded commutative algebra and $\mathcal{C}$ a subcategory of $\mathcal{TopAlg}_{K^*}$, $\mathcal{M}$ a subcategory of $\mathcal{TopMod}_{K^*}$.

Condition 10.1.1 *We assume one of the following conditions.*

(*i*) *If a morphism $S^* \to R^*$ of $\mathcal{C}$ and a right S^* module structure on $N^* \in \operatorname{Ob}\mathcal{M}$ are given, then $N^* \otimes_{S^*} R^*$ is an object of $\mathcal{M}$.*

(*ii*) *If a morphism $S^* \to R^*$ of $\mathcal{C}$ and a right S^*-module structure on $N^* \in \operatorname{Ob}\mathcal{M}$ are given, then $N^* \widehat{\otimes}_{S^*} R^*$ is an object of $\mathcal{M}$ and every object of $\mathcal{C}$ and $\mathcal{M}$ is complete Hausdorff.*

Definition 10.1.2 *We define a category $\mathcal{Mod}(\mathcal{C},\mathcal{M})$ as follows. $\operatorname{Ob}\mathcal{Mod}(\mathcal{C},\mathcal{M})$ consists of triples (R^*, M^*, α) where $R^* \in \operatorname{Ob}\mathcal{C}$, $M^* \in \operatorname{Ob}\mathcal{M}$ and $\alpha : M^* \otimes_{K^*} R^* \to M^*$ is a right R^*-module structure of M^*. Since $\alpha\beta_{M^*,R^*} : M^* \times R^* \to M^*$ is a strongly continuous bilinear map by (2.1.6), it follows from (2.1.9) that M^* has a fundamental system of neighborhoods of 0 which consists of open R^*-submodules and the topology of M^* is coarser than the topology induced by R^*. A morphism from (R^*, M^*, α) to (S^*, N^*, β) is a pair (λ, φ) of morphisms $\lambda \in \mathcal{C}(R^*, S^*)$ and $\varphi \in \mathcal{M}(M^*, N^*)$ such that the following diagram commutes.*

$$\begin{tikzcd}
M^* \otimes_{K^*} R^* \arrow[r, "\alpha"] \arrow[d, "\varphi\otimes_{K^*}\lambda"] & M^* \arrow[d, "\varphi"] \\
N^* \otimes_{K^*} S^* \arrow[r, "\beta"] & N^*
\end{tikzcd}$$

Composition of $(\lambda, \varphi) : (R^*, M^*, \alpha) \to (S^*, N^*, \beta)$ and $(\nu, \psi) : (S^*, N^*, \beta) \to (T^*, L^*, \gamma)$ is defined to be $(\nu\lambda, \psi\varphi)$. Hence if $\boldsymbol{M} = (R^*, M^*, \alpha)$ and $\boldsymbol{N} = (S^*, N^*, \beta)$ are objects of $\mathcal{Mod}(\mathcal{C},\mathcal{M})$, $\mathcal{Mod}(\mathcal{C},\mathcal{M})(\boldsymbol{M},\boldsymbol{N})$ is a subset of $\mathcal{C}(R^*, S^*) \times \mathcal{M}(M^*, N^*)$. We give $\mathcal{C}(R^*, S^*) \times \mathcal{M}(M^*, N^*)$ the topology of product spaces and give $\mathcal{Mod}(\mathcal{C},\mathcal{M})(\boldsymbol{M},\boldsymbol{N})$ the induced topology. Thus $\mathcal{Mod}(\mathcal{C},\mathcal{M})$ is a quasi-topological category.

Define functors $p_{\mathcal{C}} : \mathcal{Mod}(\mathcal{C},\mathcal{M}) \to \mathcal{C}$ and $p_{\mathcal{M}} : \mathcal{Mod}(\mathcal{C},\mathcal{M}) \to \mathcal{M}$ by $p_{\mathcal{C}}(R^*, M^*, \alpha) = R^*$, $p_{\mathcal{C}}(\lambda, \varphi) = \lambda$ and $p_{\mathcal{M}}(R^*, M^*, \alpha) = M^*$, $p_{\mathcal{M}}(\lambda, \varphi) = \varphi$. Then, $p_{\mathcal{C}}$ and $p_{\mathcal{M}}$ are continuous functors.

For $R^* \in \operatorname{Ob}\mathcal{C}$, we denote by $\mathcal{Mod}(\mathcal{C},\mathcal{M})_{R^*}$ a subcategory of $\mathcal{Mod}(\mathcal{C},\mathcal{M})$ consisting of objects which map to R^* by $p_{\mathcal{C}}$ and morphisms which map the identity morphism of R^* by $p_{\mathcal{C}}$. Hence $\mathcal{Mod}(\mathcal{C},\mathcal{M})_{R^*}$ is a subcategory of the category of right R^*-modules. We remark that, for objects $\boldsymbol{M} = (K^*, M^*, \alpha)$ and $\boldsymbol{N} = (K^*, N^*, \beta)$ of $\mathcal{Mod}(\mathcal{C},\mathcal{M})_{R^*}$, a map $\mathcal{Mod}(\mathcal{C},\mathcal{M})_{R^*}(\boldsymbol{M},\boldsymbol{N}) \to \operatorname{Hom}^c_{K^*}(M^*, N^*)$ which maps (id_{K^*}, φ) to φ is bijective. Thus, $\mathcal{Mod}(\mathcal{C},\mathcal{M})_{R^*}(\boldsymbol{M},\boldsymbol{N})$ is identified with $\operatorname{Hom}^c_{K^*}(M^*, N^*)$.

Proposition 10.1.3 *If $\mathcal{C}$ and $\mathcal{M}$ are complete, so is $\mathcal{Mod}(\mathcal{C},\mathcal{M})$.*

Proof. For a functor $D : \mathcal{I} \to \mathcal{Mod}(\mathcal{C},\mathcal{M})$, we assume that limits of $p_{\mathcal{C}}D : \mathcal{I} \to \mathcal{C}$ and $p_{\mathcal{M}}D : \mathcal{I} \to \mathcal{M}$ exist. Let $\left(A^* \xrightarrow{\rho_i} p_{\mathcal{C}}D(i)\right)_{i\in\operatorname{Ob}\mathcal{I}}$ be a limiting cone of $p_{\mathcal{C}}D : \mathcal{I} \to \mathcal{C}$ and $\left(L^* \xrightarrow{\pi_i} p_{\mathcal{M}}D(i)\right)_{i\in\operatorname{Ob}\mathcal{I}}$ a limiting cone of $p_{\mathcal{M}}D : \mathcal{I} \to \mathcal{M}$. For $i \in \operatorname{Ob}\mathcal{I}$ and $(\tau : i \to j) \in \operatorname{Mor}\mathcal{I}$, we put $D(i) = (R^*_i, M^*_i, \alpha_i)$ and $D(\tau) = (\lambda_\tau, \varphi_\tau)$. Since the following diagram commutes for any $(\tau : i \to j) \in \operatorname{Mor}\mathcal{I}$, there exists unique morphism $\lambda : L^* \otimes_{K^*} A^* \to L^*$ satisfying $\pi_i\lambda = \alpha_i(\pi_i \otimes_{K^*} \rho_i)$ for any $i \in \operatorname{Ob}\mathcal{I}$.

$$\begin{tikzcd}
L^* \otimes_{K^*} A^* \arrow[r, "\pi_i\otimes_{K^*}\rho_i"] \arrow[dr, "\pi_j\otimes_{K^*}\rho_j"'] & M^*_i \otimes_{K^*} R^*_i \arrow[r, "\alpha_i"] \arrow[d, "\varphi_\tau\otimes_{K^*}\lambda_\tau"] & M^*_i \arrow[d, "\varphi_\tau"] \\
& M^*_j \otimes_{K^*} R^*_j \arrow[r, "\alpha_j"] & M_j
\end{tikzcd}$$

It can be verified that (A^*, L^*, λ) is an object of $\mathcal{Mod}(\mathcal{C},\mathcal{M})$ and that $\left((A^*, L^*, \lambda) \xrightarrow{(\rho_i,\pi_i)} D(i)\right)_{i\in\operatorname{Ob}\mathcal{I}}$ is a limiting cone of D. □

Proposition 10.1.4 $p^{op}_{\mathcal{C}} : \mathcal{Mod}(\mathcal{C},\mathcal{M})^{op} \to \mathcal{C}^{op}$ *is a fibered category.*

Proof. We assume that $\mathcal{M}$ satisfies the condition (i) of (10.1.1). For a morphism $\lambda : S^* \to R^*$ of $\mathcal{C}$ and $(S^*, N^*, \beta) \in \mathrm{Ob}\,\mathcal{M}od(\mathcal{C}, \mathcal{M})$, let $i_{N^*} : N^* \to N^* \otimes_{S^*} R^*$ be a map defined by $i_{N^*}(x) = x \otimes 1$ and $\beta_\lambda : (N^* \otimes_{S^*} R^*) \otimes_{K^*} R^* \to N^* \otimes_{S^*} R^*$ the map induced by the product $\mu : R^* \otimes_{K^*} R^* \to R^*$ of R^*. Since the following diagram commutes, $(\lambda, i_{N^*}) : (R^*, N^* \otimes_{S^*} R^*, \beta_\lambda) \to (S^*, N^*, \beta)$ is a morphism of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}$.

$$\begin{tikzcd}
N^* \otimes_{K^*} S^* \arrow[r, "\beta"] \arrow[d, "i_{N^*} \otimes_{K^*} \lambda"] & N^* \arrow[d, "i_{N^*}"] \\
(N^* \otimes_{S^*} R^*) \otimes_{K^*} R^* \arrow[r, "\beta_\lambda"] & N^* \otimes_{S^*} R^*
\end{tikzcd}$$

A map $(\lambda, i_{N^*})_* : \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{R^*}((R^*, M^*, \alpha), (R^*, N^* \otimes_{S^*} R^*, \beta_\lambda)) \to \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_\lambda((R^*, M^*, \alpha), (S^*, N^*, \beta))$ given by $(\lambda, i_{N^*})_*((id_{R^*}, \varphi)) = (\lambda, \varphi i_{N^*})$ is bijective. In fact, for $(\lambda, \psi) \in \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_\lambda((R^*, M^*, \alpha), (S^*, N^*, \beta))$, since $\psi\beta = \alpha(\psi \otimes_{K^*} \lambda) : N^* \otimes_{K^*} S^* \to M^*$, we have

$$\begin{aligned}
\alpha(\psi \otimes_{K^*} id_{R^*})(z \otimes \lambda(y)x) &= \alpha(\psi(z) \otimes \lambda(y)x) = \alpha(\alpha(\psi(z) \otimes \lambda(y)) \otimes x) \\
&= \alpha(\psi\beta(z \otimes y) \otimes x) = \alpha(\psi \otimes_{K^*} id_{R^*})(\beta(z \otimes y) \otimes x)
\end{aligned}$$

for $x \in R^*$, $y \in S^*$ and $z \in N^*$. Hence there exists unique morphism $\tilde{\psi} : N^* \otimes_{S^*} R^* \to M^*$ that makes the following diagram commute. Here, $\pi_\lambda : N^* \otimes_{K^*} R^* \to N^* \otimes_{S^*} R^*$ denotes the quotient map.

$$\begin{tikzcd}
N^* \otimes_{K^*} R^* \arrow[r, "\psi \otimes_{K^*} id_{R^*}"] \arrow[d, "\pi_\lambda"] & M^* \otimes_{K^*} R^* \arrow[d, "\alpha"] \\
N^* \otimes_{S^*} R^* \arrow[r, "\tilde{\psi}"] & M^*
\end{tikzcd}$$

Then, a correspondence $(\lambda, \psi) \mapsto (id_{R^*}, \tilde{\psi})$ gives the inverse of $(\lambda, i_{N^*})_*$. In fact, since

$$\begin{tikzcd}
N^* \otimes_{K^*} R^* \arrow[r, "i_{N^*} \otimes_{K^*} id_{R^*}"] \arrow[dr, "\pi_\lambda"'] & N^* \otimes_{S^*} R^* \otimes_{K^*} R^* \arrow[r, "\varphi \otimes_{K^*} id_{R^*}"] \arrow[d, "\beta_\lambda"] & M^* \otimes_{K^*} R^* \arrow[d, "\alpha"] \\
& N^* \otimes_{S^*} R^* \arrow[r, "\varphi"] & M^*
\end{tikzcd}$$

commutes for $(id_{R^*}, \varphi) \in \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{R^*}((R^*, M^*, \alpha), (R^*, N^* \otimes_{S^*} R^*, \beta_\lambda))$, the correspondence $(\lambda, \psi) \mapsto (id_{R^*}, \tilde{\psi})$ is a left inverse of $(\lambda, i_{N^*})_*$. For $(\lambda, \psi) \in \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_\lambda((R^*, M^*, \alpha), (S^*, N^*, \beta))$ and $x \in N^*$, since

$$\tilde{\psi} i_{N^*}(x) = \tilde{\psi}(x \otimes_{S^*} 1) = \tilde{\psi}\pi_\lambda(x \otimes_{K^*} 1) = \alpha(\psi \otimes_{K^*} id_{R^*})(x \otimes_{K^*} 1) = \psi(x),$$

it follows that the correspondence $(\lambda, \psi) \mapsto (id_{R^*}, \tilde{\psi})$ is a right inverse of $(\lambda, i_{N^*})_*$. Thus (λ, i_{N^*}) is a cartesian morphism and $p^{op}_{\mathcal{C}} : \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op} \to \mathcal{C}^{op}$ is a prefibered category. We set $\lambda^*(S^*, N^*, \beta) = (R^*, N^* \otimes_{S^*} R^*, \beta_\lambda)$ and $\alpha_\lambda(S^*, N^*, \beta) = (\lambda, i_{N^*}) : \lambda^*(S^*, N^*, \beta) \to (S^*, N^*, \beta)$ in $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}$.

For morphisms $\lambda : S^* \to R^*$, $\nu : T^* \to S^*$ of $\mathcal{C}$ and $(T^*, L^*, \gamma) \in \mathrm{Ob}\,\mathcal{M}od(\mathcal{C}, \mathcal{M})$, there is an isomorphism $\bar{c}_{\lambda,\nu,L^*} : L^* \otimes_{T^*} R^* \to (L^* \otimes_{T^*} S^*) \otimes_{S^*} R^*$ given by $\bar{c}_{\lambda,\nu,L^*}(w \otimes x) = w \otimes 1 \otimes x$. We put $\boldsymbol{c}_{\nu,\lambda}(T^*, L^*, \gamma) = (id_{R^*}, \bar{c}_{\lambda,\nu,L^*})$. Then, $\boldsymbol{c}_{\nu,\lambda}(T^*, L^*, \gamma) : \lambda^*\nu^*(T^*, L^*, \gamma) \to (\lambda\nu)^*(T^*, L^*, \gamma)$ is an isomorphism of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{R^*}$ and the following diagram commutes.

$$\begin{tikzcd}
\lambda^*\nu^*(T^*, L^*, \gamma) \arrow[r, "\alpha_\lambda(\nu^*(T^*, L^*, \gamma))"] \arrow[d, "\boldsymbol{c}_{\nu,\lambda}(T^*, L^*, \gamma)"] & \nu^*(T^*, L^*, \gamma) \arrow[d, "\alpha_\nu(T^*, L^*, \gamma)"] \\
(\lambda\nu)^*(T^*, L^*, \gamma) \arrow[r, "\alpha_{\lambda\nu}(T^*, L^*, \gamma)"] & (T^*, L^*, \gamma)
\end{tikzcd}$$

Therefore $p^{op}_{\mathcal{C}} : \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op} \to \mathcal{C}^{op}$ is a fibered category.

Next, we assume that $\mathcal{M}$ satisfies the condition (ii) of (10.1.1). For a morphism $\lambda : S^* \to R^*$ of $\mathcal{C}$ and $(S^*, N^*, \beta) \in \mathrm{Ob}\,\mathcal{M}od(\mathcal{C}, \mathcal{M})$, let $\hat{i}_{N^*} : N^* \to N^* \widehat{\otimes}_{S^*} R^*$ be the composition of $i_{N^*} : N^* \to N^* \otimes_{S^*} R^*$ and $\eta_{N^* \otimes_{S^*} R^*} : N^* \otimes_{S^*} R^* \to N^* \widehat{\otimes}_{S^*} R^*$. Define $\hat{\beta}_\lambda : (N^* \widehat{\otimes}_{S^*} R^*) \otimes_{K^*} R^* \to N^* \widehat{\otimes}_{S^*} R^*$ to be the following composition.

$$(N^* \widehat{\otimes}_{S^*} R^*) \otimes_{K^*} R^* \xrightarrow{\eta_{(N^* \widehat{\otimes}_{S^*} R^*) \otimes_{K^*} R^*}} (N^* \widehat{\otimes}_{S^*} R^*) \widehat{\otimes}_{K^*} R^* \xrightarrow{\cong} N^* \widehat{\otimes}_{S^*} (R^* \widehat{\otimes}_{K^*} R^*) \xrightarrow{id_{N^*} \widehat{\otimes}_{S^*} \hat{\mu}} N^* \widehat{\otimes}_{S^*} \hat{R}^*$$
$$\xrightarrow{(id_{N^*} \widehat{\otimes}_{S^*} \eta_{R^*})^{-1}} N^* \widehat{\otimes}_{S^*} R^*$$

Here $\hat{\mu} : R^* \widehat{\otimes}_{K^*} R^* \to \hat{R}^*$ is the map induced by the product $\mu : R^* \otimes_{K^*} R^* \to R^*$ of R^*. Since the following diagram commutes, $(\lambda, \hat{i}_{N^*}) : (R^*, N^* \widehat{\otimes}_{S^*} R^*, \hat{\beta}_\lambda) \to (S^*, N^*, \beta)$ is a morphism of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}$.

$$\begin{tikzcd}
N^* \otimes_{K^*} S^* \arrow[r, "\beta"] \arrow[d, "i_{N^*} \otimes_{K^*} \lambda"] & N^* \arrow[d, "i_{N^*}"] \\
(N^* \otimes_{S^*} R^*) \otimes_{K^*} R^* \arrow[r, "\beta_\lambda"] \arrow[d, "\eta_{N^* \otimes_{S^*} R^*} \otimes_{K^*} id_{R^*}"] & N^* \otimes_{S^*} R^* \arrow[d, "\eta_{N^* \otimes_{S^*} R^*}"] \\
(N^* \widehat{\otimes}_{S^*} R^*) \otimes_{K^*} R^* \arrow[r, "\hat{\beta}_\lambda"] & N^* \widehat{\otimes}_{S^*} R^*
\end{tikzcd}$$

A map $(\lambda, \hat{i}_{N^*})_* : \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{R^*}((R^*, M^*, \alpha), (R^*, N^* \widehat{\otimes}_{S^*} R^*, \hat{\beta}_\lambda)) \to \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{\lambda}((R^*, M^*, \alpha), (S^*, N^*, \beta))$ given by $(\lambda, \hat{i}_{N^*})_*((id_{R^*}, \varphi)) = (\lambda, \varphi \hat{i}_{N^*})$ is bijective. In fact, for $(\lambda, \psi) \in \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{\lambda}((R^*, M^*, \alpha), (S^*, N^*, \beta))$, since $\psi\beta = \alpha(\psi \otimes_{K^*} \lambda) : N^* \otimes_{K^*} S^* \to M^*$, we have

$$\begin{aligned}
\alpha(\psi \otimes_{K^*} id_{R^*})(z \otimes \lambda(y)x) &= \alpha(\psi(z) \otimes \lambda(y)x) = \alpha(\alpha(\psi(z) \otimes \lambda(y)) \otimes x) \\
&= \alpha(\psi\beta(z \otimes y) \otimes x) = \alpha(\psi \otimes_{K^*} id_{R^*})(\beta(z \otimes y) \otimes x)
\end{aligned}$$

for $x \in R^*$, $y \in S^*$ and $z \in N^*$. Hence there exists unique map $\tilde{\psi} : N^* \otimes_{S^*} R^* \to M^*$ that makes the following diagram commute. Here, $\pi_\lambda : N^* \otimes_{K^*} R^* \to N^* \otimes_{S^*} R^*$ denotes the quotient map.

$$\begin{tikzcd}
N^* \otimes_{K^*} R^* \arrow[r, "\psi \otimes_{K^*} id_{R^*}"] \arrow[d, "\pi_\lambda"] & M^* \otimes_{K^*} R^* \arrow[d, "\alpha"] \\
N^* \otimes_{S^*} R^* \arrow[r, "\tilde{\psi}"] & M^*
\end{tikzcd}$$

Since M^* is complete Hausdorff, there exists unique map $\hat{\psi} : N^* \widehat{\otimes}_{S^*} R^* \to M^*$ satisfying $\hat{\psi}\eta_{N^* \otimes_{S^*} R^*} = \tilde{\psi}$. Then, a correspondence $(\lambda, \psi) \mapsto (id_{R^*}, \hat{\psi})$ gives the inverse of $(\lambda, \hat{i}_{N^*})_*$. In fact, since

$$\begin{tikzcd}
N^* \otimes_{K^*} R^* \arrow[r, "\hat{i}_{N^*} \otimes_{K^*} id_{R^*}"] \arrow[d, "\pi_\lambda"] & (N^* \widehat{\otimes}_{S^*} R^*) \otimes_{K^*} R^* \arrow[r, "\varphi \otimes_{K^*} id_{R^*}"] \arrow[d, "\hat{\beta}_\lambda"] & M^* \otimes_{K^*} R^* \arrow[d, "\alpha"] \\
N^* \otimes_{S^*} R^* \arrow[r, "\eta_{N^* \otimes_{S^*} R^*}"] & N^* \widehat{\otimes}_{S^*} R^* \arrow[r, "\varphi"] & M^*
\end{tikzcd}$$

commutes for $(id_{R^*}, \varphi) \in \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{R^*}((R^*, M^*, \alpha), (R^*, N^* \widehat{\otimes}_{S^*} R^*, \hat{\beta}_\lambda))$, the correspondence $(\lambda, \psi) \mapsto (id_{R^*}, \hat{\psi})$ is a left inverse of $(\lambda, \hat{i}_{N^*})_*$. For $(\lambda, \psi) \in \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{\lambda}((R^*, M^*, \alpha), (S^*, N^*, \beta))$ and $x \in N^*$, since

$$\hat{\psi}\hat{i}_{N^*}(x) = \hat{\psi}\eta_{N^* \otimes_{S^*} R^*} i_{N^*}(x) = \tilde{\psi}(x \otimes_{S^*} 1) = \tilde{\psi}\pi_\lambda(x \otimes_{K^*} 1) = \alpha(\psi \otimes_{K^*} id_{R^*})(x \otimes_{K^*} 1) = \psi(x),$$

it follows that the correspondence $(\lambda, \psi) \mapsto (id_{R^*}, \hat{\psi})$ is a right inverse of $(\lambda, i_{N^*})_*$. Thus (λ, i_{N^*}) is a cartesian morphism and $p^{op}_{\mathcal{C}} : \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op} \to \mathcal{C}^{op}$ is a prefibered category. We set $\lambda^*(S^*, N^*, \beta) = (R^*, N^* \widehat{\otimes}_{S^*} R^*, \hat{\beta}_\lambda)$ and $\alpha_\lambda(S^*, N^*, \beta) = (\lambda, \hat{i}_{N^*}) : \lambda^*(S^*, N^*, \beta) \to (S^*, N^*, \beta)$ in $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}$.

For morphisms $\lambda : S^* \to R^*$, $\nu : T^* \to S^*$ of $\mathcal{C}$ and $(T^*, L^*, \gamma) \in \mathrm{Ob}\,\mathcal{M}od(\mathcal{C}, \mathcal{M})$, there is an isomorphism $\bar{c}_{\lambda,\nu,L^*} : L^* \otimes_{T^*} R^* \to (L^* \otimes_{T^*} S^*) \otimes_{S^*} R^*$ given by $\bar{c}_{\lambda,\nu,L^*}(w \otimes x) = w \otimes 1 \otimes x$. Let $\hat{c}_{\lambda,\nu,L^*} : L^* \widehat{\otimes}_{T^*} R^* \to (L^* \widehat{\otimes}_{T^*} S^*) \widehat{\otimes}_{S^*} R^*$ be the map induced by $\bar{c}_{\lambda,\nu,L^*}$, which is also an isomorphism. We put

$$\boldsymbol{c}_{\nu,\lambda}(T^*, L^*, \gamma) = (id_{R^*}, \hat{c}_{\lambda,\nu,L^*}) : \left(R^*, (L^* \widehat{\otimes}_{T^*} S^*) \widehat{\otimes}_{S^*} R^*, \widehat{(\hat{\gamma}_\nu)}_\lambda\right) \to (R^*, L^* \widehat{\otimes}_{T^*} R^*, \gamma_{\lambda\nu}).$$

Then, $\boldsymbol{c}_{\nu,\lambda}(T^*, L^*, \gamma) : \lambda^*\nu^*(T^*, L^*, \gamma) \to (\lambda\nu)^*(T^*, L^*, \gamma)$ is an isomorphism of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{R^*}$ and the following diagram commutes.

$$\begin{tikzcd}
\lambda^*\nu^*(T^*, L^*, \gamma) \arrow[r, "\alpha_\lambda(\nu^*(T^*, L^*, \gamma))"] \arrow[d, "\boldsymbol{c}_{\nu,\lambda}(T^*, L^*, \gamma)"] & \nu^*(T^*, L^*, \gamma) \arrow[d, "\alpha_\nu(T^*, L^*, \gamma)"] \\
(\lambda\nu)^*(T^*, L^*, \gamma) \arrow[r, "\alpha_{\lambda\nu}(T^*, L^*, \gamma)"] & (T^*, L^*, \gamma)
\end{tikzcd}$$

Therefore $p^{op}_{\mathcal{C}} : \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op} \to \mathcal{C}^{op}$ is a fibered category. $\square$

Remark 10.1.5 *For a morphism $\lambda : A^* \to R^*$ of $\mathcal{C}$ and an object $\boldsymbol{M} = (A^*, M^*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{A^*}$, we denote by $\hat{i}_{M^*,\lambda} : M^* \to M^* \widehat{\otimes}_{A^*} R^*$ the composition of a map $M^* \to M^* \otimes_{A^*} R^*$ given by $x \mapsto x \otimes 1$ and the completion map $M^* \otimes_{A^*} R^* \to M^* \widehat{\otimes}_{A^*} R^*$. For a morphism $\gamma : R^* \to S^*$ of $\mathcal{C}$, if we regard S^* as an A^*-algebra by $\gamma\lambda$, the isomorphism*

$$\hat{c}_{\gamma,\lambda,M^*} : M^* \widehat{\otimes}_{A^*} S^* \to (M^* \widehat{\otimes}_{A^*} R^*) \widehat{\otimes}_{R^*} S^*$$

given in the proof of (10.1.4) coincides with the following composition.

$$M^* \widehat{\otimes}_{A^*} S^* \xrightarrow{\hat{i}_{M^*,\lambda} \widehat{\otimes}_{A^*} id_{S^*}} (M^* \widehat{\otimes}_{A^*} R^*) \widehat{\otimes}_{A^*} S^* \xrightarrow{\widehat{\otimes}_\lambda} (M^* \widehat{\otimes}_{A^*} R^*) \widehat{\otimes}_{R^*} S^*$$

Proposition 10.1.6 *For a morphism $\lambda : S^* \to R^*$ of $\mathcal{C}$, $\lambda^* : \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{S^*} \to \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{R^*}$ has a left adjoint.*

Proof. Define a functor $\lambda_* : \mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*} \to \mathcal{M}od(\mathcal{C}, \mathcal{M})_{S^*}$ as follows. For $(R^*, M^*, \alpha) \in \mathrm{Ob}\, \mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}$, set $\lambda_*(R^*, M^*, \alpha) = (S^*, M^*, \alpha(id_{M^*} \otimes_{K^*} \lambda))$. For $(id_{R^*}, \psi) \in \mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}((R^*, L^*, \gamma), (R^*, M^*, \alpha))$, we set $\lambda_*(id_{R^*}, \psi) = (id_{S^*}, \psi)$. It is clear that $(id_{S^*}, \varphi) \in \mathcal{M}od(\mathcal{C}, \mathcal{M})_{S^*}((S^*, N^*, \beta), \lambda_*(R^*, M^*, \alpha))$ if and only if $(\lambda, \varphi) \in \mathcal{M}od(\mathcal{C}, \mathcal{M})_\lambda((S^*, N^*, \beta), (R^*, M^*, \alpha))$. It follows from the proof of (10.1.4) that we have a natural bijection

$$(\lambda, i_{N^*})^* : \mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}(\lambda^*(S^*, N^*, \beta), (R^*, M^*, \alpha)) \to \mathcal{M}od(\mathcal{C}, \mathcal{M})_\lambda((S^*, N^*, \beta), (R^*, M^*, \alpha))$$

if $\mathcal{M}$ satisfies the condition (i) of (10.1.1) and that we have a natural bijection

$$(\lambda, \hat{i}_{N^*})^* : \mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}(\lambda^*(S^*, N^*, \beta), (R^*, M^*, \alpha)) \to \mathcal{M}od(\mathcal{C}, \mathcal{M})_\lambda((S^*, N^*, \beta), (R^*, M^*, \alpha))$$

if $\mathcal{M}$ satisfies the condition (ii) of (10.1.1). Thus a correspondence $(id_{R^*}, \varphi) \mapsto (id_{S^*}, \varphi\, i_{N^*})$ or $(id_{R^*}, \varphi) \mapsto (id_{S^*}, \varphi\, \hat{i}_{N^*})$ gives a bijection

$$\mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}(\lambda^*(S^*, N^*, \beta), (R^*, M^*, \alpha)) \to \mathcal{M}od(\mathcal{C}, \mathcal{M})_{S^*}((S^*, N^*, \beta), \lambda_*(R^*, M^*, \alpha))$$

which is natural. Hence λ_* is a right adjoint of $\lambda^* : \mathcal{M}od(\mathcal{C}, \mathcal{M})_{S^*} \to \mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}$. □

Remark 10.1.7 *Let $\lambda : S^* \to R^*$ be a morphism of $\mathcal{C}$.*

(1) *The unit $\eta(\lambda) : id_{\mathcal{M}od(\mathcal{C},\mathcal{M})_{S^*}} \to \lambda_*\lambda^*$ is given as follows. For an object $\boldsymbol{N} = (S^*, N^*, \beta)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{S^*}$, $\eta(\lambda)_{\boldsymbol{N}} : \boldsymbol{N} \to \lambda_*\lambda^*(\boldsymbol{N})$ is defined to be*

$$(id_{S^*}, i_{N^*}) : (S^*, N^*, \beta) \to (S^*, N^* \otimes_{S^*} R^*, \beta_\lambda(id_{N^* \otimes_{S^*} R^*} \otimes_{K^*} \lambda))$$

if $\mathcal{M}$ satisfies the condition (i) of (10.1.1). If $\mathcal{M}$ satisfies the condition (ii) of (10.1.1), $\eta(\lambda)_{\boldsymbol{N}}$ is defined to be

$$(id_{S^*}, \hat{i}_{N^*}) : (S^*, N^*, \beta) \to (S^*, N^* \widehat{\otimes}_{S^*} R^*, \hat{\beta}_\lambda(id_{N^* \widehat{\otimes}_{S^*} R^*} \widehat{\otimes}_{K^*} \lambda)).$$

(2) *The counit $\varepsilon(\lambda) : \lambda^*\lambda_* \to id_{\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}}$ is given as follows. For an object $\boldsymbol{M} = (R^*, M^*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}$, we put $\beta = \alpha(id_{M^*} \otimes_{K^*} \lambda)$. Suppose that $\mathcal{M}$ satisfies the condition (i) of (10.1.1). Then, $\lambda^*\lambda_*(\boldsymbol{M}) = (R^*, M^* \otimes_{S^*} R^*, \beta_\lambda)$, where $\beta_\lambda : (M^* \otimes_{S^*} R^*) \otimes_{K^*} R^* \to M^* \otimes_{S^*} R^*$ is the following composition.*

$$(M^* \otimes_{S^*} R^*) \otimes_{K^*} R^* \cong M^* \otimes_{S^*} (R^* \otimes_{K^*} R^*) \xrightarrow{id_{M^*} \otimes_{S^*} \mu} M^* \otimes_{S^*} R^*$$

Here, we denote by $\mu : R^* \otimes_{K^*} R^* \to R^*$ is the product of R^*. Let us denote by $\bar{\alpha} : M^* \otimes_{R^*} R^* \to M^*$ the isomorphism induced by α and by $\alpha' : (M^* \otimes_{R^*} R^*) \otimes_{K^*} R^* \to M^* \otimes_{R^*} R^*$ the following composition.

$$(M^* \otimes_{R^*} R^*) \otimes_{K^*} R^* \cong M^* \otimes_{R^*} (R^* \otimes_{K^*} R^*) \xrightarrow{id_{M^*} \otimes_{R^*} \mu} M^* \otimes_{R^*} R^*$$

Then, $\varepsilon(\lambda)_{\boldsymbol{M}} : \lambda^*\lambda_*(\boldsymbol{M}) \to \boldsymbol{M}$ is defined to be the following composition.

$$(R^*, M^* \otimes_{S^*} R^*, \beta_\lambda) \xrightarrow{(id_{R^*}, \otimes_\lambda)} (R^*, M^* \otimes_{R^*} R^*, \alpha') \xrightarrow{(id_{R^*}, \bar{\alpha})} (R^*, M^*, \alpha).$$

Suppose that $\mathcal{M}$ satisfies the condition (ii) of (10.1.1). Then, we have $\lambda^*\lambda_*(\boldsymbol{M}) = (R^*, M^* \widehat{\otimes}_{S^*} R^*, \hat{\beta}_\lambda)$, where $\hat{\beta}_\lambda : (M^* \widehat{\otimes}_{S^*} R^*) \otimes_{K^*} R^* \to M^* \widehat{\otimes}_{S^*} R^*$ is the following composition.

$$(M^* \widehat{\otimes}_{S^*} R^*) \otimes_{K^*} R^* \xrightarrow{completion} (M^* \widehat{\otimes}_{S^*} R^*) \widehat{\otimes}_{K^*} R^* \cong M^* \widehat{\otimes}_{S^*} (R^* \widehat{\otimes}_{K^*} R^*) \xrightarrow{id_{M^*} \widehat{\otimes}_{S^*} \hat{\mu}} M^* \widehat{\otimes}_{S^*} R^*$$

Here, we denote by $\hat{\mu} : R^* \widehat{\otimes}_{K^*} R^* \to R^*$ is the map induced by μ. Let $\hat{\alpha} : M^* \widehat{\otimes}_{K^*} R^* \to M^*$ be the map induced by $\alpha : M^* \otimes_{K^*} R^* \to M^*$. $\hat{\alpha}$ induces an isomorphism $\tilde{\alpha} : M^* \widehat{\otimes}_{R^*} R^* \to M^*$ which satisfies $\tilde{\alpha}\,\widehat{\otimes}_u = \hat{\alpha}$, where $u : K^* \to R^*$ is the unit of R^*. We also denote by $\hat{\alpha}' : (M^* \widehat{\otimes}_{R^*} R^*) \otimes_{K^*} R^* \to M^* \widehat{\otimes}_{R^*} R^*$ the following composition.

$$(M^* \widehat{\otimes}_{R^*} R^*) \otimes_{K^*} R^* \xrightarrow{completion} (M^* \widehat{\otimes}_{R^*} R^*) \widehat{\otimes}_{K^*} R^* \cong M^* \widehat{\otimes}_{R^*} (R^* \widehat{\otimes}_{K^*} R^*) \xrightarrow{id_{M^*} \widehat{\otimes}_{R^*} \hat{\mu}} M^* \widehat{\otimes}_{R^*} R^*$$

Then, $\varepsilon(\lambda)_{\boldsymbol{M}} : \lambda^*\lambda_*(\boldsymbol{M}) \to \boldsymbol{M}$ is defined to be a composition

$$(R^*, M^* \widehat{\otimes}_{S^*} R^*, \hat{\beta}_\lambda) \xrightarrow{(id_{R^*}, \widehat{\otimes}_\lambda)} (R^*, M^* \widehat{\otimes}_{R^*} R^*, \hat{\alpha}') \xrightarrow{(id_{R^*}, \tilde{\alpha})} (R^*, M^*, \alpha).$$

By the definition of $\gamma^\sharp_{\boldsymbol{M},\boldsymbol{N}}$, we have the following result.

Proposition 10.1.8 *Let $\lambda : A^* \to R^*$, $\nu : B^* \to R^*$, $\gamma : R^* \to S^*$ be morphisms of $\mathcal{C}$ and $\boldsymbol{M} = (A^*, M^*, \alpha)$, $\boldsymbol{M} = (B^*, N^*, \beta)$ objects of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{A^*}$, $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{B^*}$, respectively. Then,*

$$\gamma^\sharp_{\boldsymbol{M},\boldsymbol{N}} : \mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}(\nu^*(\boldsymbol{N}), \lambda^*(\boldsymbol{M})) \to \mathcal{M}od(\mathcal{C}, \mathcal{M})_{S^*}((\gamma\nu)^*(\boldsymbol{N}), (\gamma\lambda)^*(\boldsymbol{M}))$$

maps (id_{R^*}, φ) to $(id_{S^*}, \hat{c}^{-1}_{\gamma,\lambda,M^*}(\varphi \widehat{\otimes}_{R^*} id_{S^*})\hat{c}_{\gamma,\nu,N^*})$.

For an object R^* of $\mathcal{C}$ and an object $\boldsymbol{M} = (K^*, M^*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$, let $u_{R^*} : K^* \to R^*$ the unit of R^* and we put $\alpha_{R^*} = \hat{\alpha}_{u_{R^*}}(id_{M^* \widehat{\otimes}_{K^*} R^*} \otimes_{K^*} u_{R^*}) : (M^* \widehat{\otimes}_{K^*} R^*) \otimes_{K^*} K^* \to M^* \widehat{\otimes}_{K^*} R^*$.

Proposition 10.1.9 *Let R^* be an object of $\mathcal{C}$ and $\boldsymbol{M} = (K^*, M^*, \alpha)$ an object of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$.*

(1) *We have $R^* \times \boldsymbol{M} = u_{R^**}(u^*_{R^*}(\boldsymbol{M})) = (K^*, M^* \widehat{\otimes}_{K^*} R^*, \alpha_{R^*})$.*

(2) *We denote by $\hat{m} : R^* \widehat{\otimes}_{K^*} R^* \to R^*$ the map induced by the product $m : R^* \otimes_{K^*} R^* \to R^*$ of R^* and define $i_{R^*}(\boldsymbol{M}) : (M^* \widehat{\otimes}_{K^*} R^*) \widehat{\otimes}_{K^*} R^* \to M^* \widehat{\otimes}_{K^*} R^*$ to be the following composition.*

$$(M^* \widehat{\otimes}_{K^*} R^*) \widehat{\otimes}_{K^*} R^* \xrightarrow{\cong} M^* \widehat{\otimes}_{K^*} (R^* \widehat{\otimes}_{K^*} R^*) \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} \hat{m}} M^* \widehat{\otimes}_{K^*} R^*$$

Then, $\iota_{R^*}(\boldsymbol{M}) : u^*_{R^*}(R^* \times \boldsymbol{M}) = (R^*, (M^* \widehat{\otimes}_{K^*} R^*) \widehat{\otimes}_{K^*} R^*, (\alpha_{R^*})_{u_{R^*}}) \to (R^*, M^* \widehat{\otimes}_{K^*} R^*, \hat{\alpha}_{u_{R^*}}) = u^*_{R^*}(\boldsymbol{M})$ is given by $\iota_{R^*}(\boldsymbol{M}) = (id_{R^*}, i_{R^*}(\boldsymbol{M}))$.

(3) *For objects $\boldsymbol{M}$ and $\boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$,*

$$P_{R^*}(\boldsymbol{M})_{\boldsymbol{N}} : \mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}(u^*_{R^*}(\boldsymbol{N}), u^*_{R^*}(\boldsymbol{M})) \to \mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}(\boldsymbol{N}, R^* \times \boldsymbol{M})$$

maps (id_{R^*}, φ) to $(id_{K^*}, \varphi\hat{i}_{M^*,u_{R^*}})$.

(4) *For a morphism $\boldsymbol{\varphi} = (id_{K^*}, \varphi) : \boldsymbol{M} \to \boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$, $R^* \times \boldsymbol{\varphi} : R^* \times \boldsymbol{M} \to R^* \times \boldsymbol{N}$ is given by $u_*(u^*(\boldsymbol{\varphi})) = (id_{K^*}, \varphi \widehat{\otimes}_{K^*} id_{R^*})$.*

(5) *For a morphism $\gamma : R^* \to S^*$ of $\mathcal{C}$, $\gamma \times \boldsymbol{M} : R^* \times \boldsymbol{M} \to S^* \times \boldsymbol{M}$ is given by $\gamma \times \boldsymbol{M} = (id_{K^*}, id_{M^*} \widehat{\otimes}_{K^*} \gamma)$.*

Proof. (1) The assertion follows from (10.1.4), (10.1.6) and (6.3.1).

(2) Since $\iota_{R^*}(\boldsymbol{M}) = \eta(u_{R^*})_{u^*_{R^*}(\boldsymbol{M})}$ by (6.3.1), the assertion follows from and (10.1.7).

(3) The assertion follows from (6.3.1) and (10.1.6).

(4) This is a direct consequence of (6.3.4).

(5) The assertion can be verified from (6.3.7) and (10.1.7). □

Proposition 10.1.10 *For an object R^* of $\mathcal{C}$ and an object $\boldsymbol{M} = (K^*, M^*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$, define a map $\bar{\delta}_{R^*,\boldsymbol{M}} : (M^* \otimes_{K^*} R^*) \otimes_{K^*} R^* \to M^* \otimes_{K^*} R^*$ by $\bar{\delta}_{R^*,\boldsymbol{M}}(x \otimes r \otimes r) = x \otimes rs$. Let*

$$\tilde{\delta}_{R^*,\boldsymbol{M}} : (M^* \widehat{\otimes}_{K^*} R^*) \widehat{\otimes}_{K^*} R^* \to M^* \widehat{\otimes}_{K^*} R^*$$

be the map induced by $\bar{\delta}_{R^*,\boldsymbol{M}}$. Then, $\delta_{R^*,\boldsymbol{M}} : R^* \times (R^* \times \boldsymbol{M}) \to R^* \times \boldsymbol{M}$ is given by $\delta_{R^*,\boldsymbol{M}} = (id_{K^*}, \tilde{\delta}_{R^*,\boldsymbol{M}})$.

Proof. First we note that it follows from (1) of (10.1.9) that $R^* \times (R^* \times \boldsymbol{M})$ is given as follows.

$$R^* \times (R^* \times \boldsymbol{M}) = R^* \times (K^*, M^* \widehat{\otimes}_{K^*} R^*, \alpha_{R^*}) = (K^*, (M^* \widehat{\otimes}_{K^*} R^*) \widehat{\otimes}_{K^*} R^*, (\alpha_{R^*})_{R^*})$$

Since $\delta_{R^*,\boldsymbol{M}} = u^*_{R^*}\left(\eta(u)_{u^*_{R^*}(\boldsymbol{M})}\right)$ by (6.3.12), the assertion follows from (2) of (10.1.7). □

Proposition 10.1.11 *For objects R^*, S^* of $\mathcal{C}$ and an object $\boldsymbol{M} = (K^*, M^*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$, define a map $\bar{\theta}_{R^*,S^*}(\boldsymbol{M}) : (M^* \otimes_{K^*} S^*) \otimes_{K^*} R^* \to M^* \otimes_{K^*} (R^* \otimes_{K^*} S^*)$ by $\bar{\theta}_{R^*,S^*}(\boldsymbol{M})((x \otimes s) \otimes r) = (-1)^{\deg r \deg s} x \otimes (r \otimes s)$. Let*

$$\tilde{\theta}_{R^*,S^*}(\boldsymbol{M}) : (M^* \widehat{\otimes}_{K^*} S^*) \widehat{\otimes}_{K^*} R^* \to M^* \widehat{\otimes}_{K^*} (R^* \widehat{\otimes}_{K^*} S^*)$$

be the map induced by $\bar{\theta}_{R^*,S^*}(\boldsymbol{M})$. Then, $\theta_{R^*,S^*}(\boldsymbol{M}) : R^* \times (S^* \times \boldsymbol{M}) \to (R^* \times S^*) \times \boldsymbol{M}$ is given by $\theta_{R^*,S^*}(\boldsymbol{M}) = (id_{K^*}, \tilde{\theta}_{R^*,S^*}(\boldsymbol{M}))$. Hence $\theta_{R^*,S^*}(\boldsymbol{M})$ is an isomorphism of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$.

Proof. We have the following equalities by (1) of (10.1.9).

$$\begin{aligned} R^* \times (S^* \times \boldsymbol{M}) &= (K^*, (M^* \widehat{\otimes}_{K^*} S^*) \widehat{\otimes}_{K^*} R^*, (\alpha_{S^*})_{R^*}) \\ (R^* \times S^*) \times ((R^* \times S^*) \times \boldsymbol{M}) &= (K^*, (M^* \widehat{\otimes}_{K^*} (R^* \widehat{\otimes}_{K^*} S^*)) \widehat{\otimes}_{K^*} (R^* \widehat{\otimes}_{K^*} S^*), (\alpha_{R^* \widehat{\otimes}_{K^*} S^*})_{R^* \widehat{\otimes}_{K^*} S^*}) \\ (R^* \times S^*) \times \boldsymbol{M} &= (K^*, M^* \widehat{\otimes}_{K^*} (R^* \widehat{\otimes}_{K^*} S^*), \alpha_{R^* \widehat{\otimes}_{K^*} S^*}) \end{aligned}$$

We denote by $i_1 : R^* \to R^* \otimes_{K^*} S^*$, $i_2 : S^* \to R^* \otimes_{K^*} S^*$ maps defined by $i_1(r) = r \otimes 1$, $i_2(s) = 1 \otimes s$ and by $\hat{i}_1 : R^* \to R^* \widehat{\otimes}_{K^*} S^*$, $\hat{i}_2 : S^* \to R^* \widehat{\otimes}_{K^*} S^*$ the following compositions.

$$R^* \xrightarrow{i_1} R^* \otimes_{K^*} S^* \xrightarrow{\text{completion}} R^* \widehat{\otimes}_{K^*} S^*, \qquad S^* \xrightarrow{i_2} R^* \otimes_{K^*} S^* \xrightarrow{\text{completion}} R^* \widehat{\otimes}_{K^*} S^*$$

Since $\theta_{R^*,S^*}(\boldsymbol{M})$ is defined to be a composition

$$R^* \times (S^* \times \boldsymbol{M}) \xrightarrow{\hat{i}_1 \times (\hat{i}_2 \times \boldsymbol{M})} (R^* \times S^*) \times ((R^* \times S^*) \times \boldsymbol{M}) \xrightarrow{\delta_{R^* \times S^*, \boldsymbol{M}}} (R^* \times S^*) \times \boldsymbol{M},$$

the assertion follows from (3) of (10.1.6) and (10.1.10). □

Recall that $\mathrm{prod}_{R^*} : \mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M}od_{cK^*})_{K^*} \to \mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M}od_{cK^*})_{K^*}$ is a functor which assigns an object $\boldsymbol{M}$ of $\mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M}od_{cK^*})_{K^*}$ to $R^* \times \boldsymbol{M}$ and a morphism $\boldsymbol{\varphi}$ of $\mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M}od_{cK^*})_{K^*}$ to $R^* \times \boldsymbol{\varphi}$.

Proposition 10.1.12 *For an object R^* of $\mathcal{T}op\mathcal{A}lg_{K^*}$,*

$$\mathrm{prod}_{R^*} : \mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M}od_{cK^*})_{K^*} \to \mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M}od_{cK^*})_{K^*}$$

preserves epimorphims and coequalizers. It preserves monomorphisms and equalizers if K^* is a field.

Proof. The first assertion is a direct consequence of (2.3.14), (2.3.15) and (10.1.9). The second assertion follows from (2.1.5), (1.3.12) and (1.3.14). □

Proposition 10.1.13 *Let K^* be a field such that $K^i = \{0\}$ for $i \neq 0$. For an object R^* of $\mathcal{C}$ and an object $\boldsymbol{N} = (K^*, N^*, \beta)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$, we define a functor $F_{\boldsymbol{N}}^{R^*} : \mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*} \to \mathcal{T}op$ by $F_{\boldsymbol{N}}^{R^*}(\boldsymbol{M}) = \mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}(u_{R^*}^*(\boldsymbol{N}), u_{R^*}^*(\boldsymbol{M}))$ and $F_{\boldsymbol{N}}^{R^*}(\boldsymbol{\varphi}) = u_{R^*}^*(\boldsymbol{\varphi})^*$. $F_{\boldsymbol{N}}^{R^*}$ is representable if the following conditions are satisfied.*

(i) *R^* is finite type, connective and has skeletal topology.*
(ii) *N^* is finite type, coconnective and has skeletal topology.*
(iii) *Every object of $\mathcal{M}$ is profinite.*

Proof. For $\boldsymbol{M} = (K^*, M^*, \alpha) \in \mathrm{Ob}\,\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$, since $u_{R^*}^* : \mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*} \to \mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}$ has a right adjoint u_{R^**} by (10.1.6), $F_{\boldsymbol{N}}^{R^*}(\boldsymbol{M})$ is naturally isomorphic to

$$\begin{aligned} \mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}(\boldsymbol{N}, u_{R^**}u_{R^*}^*(\boldsymbol{M})) &= \mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}((K^*, N^*, \beta), (K^*, M^* \widehat{\otimes}_{K^*} R^*, \hat{\alpha}_{u_{R^*}}(id_{M^* \widehat{\otimes}_{K^*} R^*} \otimes_{K^*} u_{R^*}))) \\ &= \{(id_{K^*}, \varphi) \,|\, \varphi \in \mathrm{Hom}_{K^*}^c(N^*, M^* \widehat{\otimes}_{K^*} R^*)\}. \end{aligned}$$

Since R^* is finite type, $R^{**} = \mathcal{H}om^*(R^*, K^*)$ is also finite type and has skeletal topology by (3.1.36). Thus R^{**} is supercofinite by (1.4.6) and $\hat{\varphi}_{M^*}^{R^{**}} : \mathcal{H}om^*(R^{**}, K^*) \widehat{\otimes}_{K^*} M^* \to \mathcal{H}om^*(R^{**}, M^*)$ is an isomorphism by (4.1.14). On the other hand, since $\chi_{R^*,K^*} : R^* \to \mathcal{H}om^*(R^{**}, K^*)$ is an isomorphism by (3.3.6), we have the following chain of isomorphisms.

$$M^* \widehat{\otimes}_{K^*} R^* \xrightarrow{\widehat{T}_{M^*,R^*}} R^* \widehat{\otimes}_{K^*} M^* \xrightarrow{\chi_{R^*,K^*} \widehat{\otimes}_{K^*} id_{M^*}} \mathcal{H}om^*(R^{**}, K^*) \widehat{\otimes}_{K^*} M^* \xrightarrow{\hat{\varphi}_{M^*}^{R^{**}}} \mathcal{H}om^*(R^{**}, M^*)$$

Since R^{**} is coconnective, it follows from (2.1.20) that $N^*\otimes_{K^*}R^{**}$ has skeletal topology, hence it is supercofinite by (1.4.6). Hence (3.2.6) implies that $\Phi_{N^*,R^{**},M^*}:\mathrm{Hom}^c_{K^*}(N^*\otimes_{K^*}R^{**},M^*)\to\mathrm{Hom}^c_{K^*}(N^*,\mathcal{H}om^*(R^{**},M^*))$ is an isomorphism. We note that $N^*\otimes_{K^*}R^{**}$ is complete Hausdorff by (2.3.3). Therefore, if we denote by $\beta':N^*\otimes_{K^*}R^{**}\otimes_{K^*}K^*\to N^*\otimes_{K^*}R^{**}$ the right K^*-module structure of $N^*\otimes_{K^*}R^{**}$, $F^{R^*}_{\boldsymbol{N}}$ is represented by $(K^*,N^*\otimes_{K^*}R^{**},\beta')$. □

Remark 10.1.14 *Under the assumptions of (10.1.13), we put $\boldsymbol{N}^{R^*}=(K^*,R^{**}\otimes_{K^*}N^*,\beta^{R^*})$ and the natural equivalence*

$$E_{R^*}(\boldsymbol{N})_{\boldsymbol{M}}:\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}(u^*_{R^*}(\boldsymbol{N}),u^*_{R^*}(\boldsymbol{M}))\to\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}(\boldsymbol{N}^{R^*},\boldsymbol{M})$$

is given by the following composition of isomorphisms.

$$\begin{aligned}\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}(u^*_{R^*}(\boldsymbol{N}),u^*_{R^*}(\boldsymbol{M}))&\xrightarrow{(id,\hat{\imath}_{N^*,u_{R^*}})^*}\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}(\boldsymbol{N},u_{R^**}u^*_{R^*}(\boldsymbol{M}))\xrightarrow{\cong}\mathrm{Hom}^c_{K^*}(N^*,M^*\,\widehat{\otimes}_{K^*}R^*)\\&\xrightarrow{(\widehat{T}_{M^*,R^*})_*}\mathrm{Hom}^c_{K^*}(N^*,R^*\,\widehat{\otimes}_{K^*}M^*)\xrightarrow{(\chi_{R^*,K^*}\widehat{\otimes}_{K^*}id_{M^*})_*}\mathrm{Hom}^c_{K^*}(N^*,\mathcal{H}om^*(R^{**},K^*)\,\widehat{\otimes}_{K^*}M^*)\\&\xrightarrow{\left(\hat{\varphi}^{R^{**}}_{M^*}\right)_*}\mathrm{Hom}^c_{K^*}(N^*,\mathcal{H}om^*(R^{**},M^*))\xrightarrow{\Phi^{-1}_{N^*,R^{**},M^*}}\mathrm{Hom}^c_{K^*}(N^*\otimes_{K^*}R^{**},M^*)\\&\xrightarrow{T^*_{R^{**},N^*}}\mathrm{Hom}^c_{K^*}(R^{**}\otimes_{K^*}N^*,M^*)\xrightarrow{\cong}\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}(\boldsymbol{N}^{R^*},\boldsymbol{M})\end{aligned}$$

It follows from (3.2.5) that $\Phi^{-1}_{N^*,R^{**},M^*}:\mathrm{Hom}^c_{K^*}(N^*,\mathcal{H}om^*(R^{**},M^*))\to\mathrm{Hom}^c_{K^*}(N^*\otimes_{K^*}R^{**},M^*)$ is given by $\Phi^{-1}_{N^*,R^{**},M^*}(g)=ev^{R^{**}}_{M^*}(g\otimes_{K^*}id_{R^{**}})$. Hence if we put $\boldsymbol{\xi}=(id_{R^*},\xi)$ and $E_{R^*}(\boldsymbol{N})_{\boldsymbol{M}}(\boldsymbol{\xi})=(id_{K^*},\check{\xi})$ for $\boldsymbol{\xi}\in\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}(u^*_{R^*}(\boldsymbol{N}),u^*_{R^*}(\boldsymbol{M}))$, $\check{\xi}:R^{**}\otimes_{K^*}N^*\to M^*$ is the following composition.

$$\begin{aligned}R^{**}\otimes_{K^*}N^*&\xrightarrow{T_{R^{**},N^*}}N^*\otimes_{K^*}R^{**}\xrightarrow{\hat{\imath}_{N^*,u_{R^*}}\otimes_{K^*}id_{R^{**}}}(N^*\,\widehat{\otimes}_{K^*}R^*)\otimes_{K^*}R^{**}\xrightarrow{\xi\otimes_{K^*}id_{R^{**}}}(M^*\,\widehat{\otimes}_{K^*}R^*)\otimes_{K^*}R^{**}\\&\xrightarrow{\widehat{T}_{M^*,R^*}\otimes_{K^*}id_{R^{**}}}(R^*\widehat{\otimes}_{K^*}M^*)\otimes_{K^*}R^{**}\xrightarrow{(\chi_{R^*,K^*}\widehat{\otimes}_{K^*}id_{M^*})\otimes_{K^*}id_{R^{**}}}(\mathcal{H}om^*(R^{**},K^*)\widehat{\otimes}_{K^*}M^*)\otimes_{K^*}R^{**}\\&\xrightarrow{\hat{\varphi}^{R^{**}}_{M^*}\otimes_{K^*}id_{R^{**}}}\mathcal{H}om^*(R^{**},M^*)\otimes_{K^*}R^{**}\xrightarrow{ev^{R^{**}}_{M^*}}M^*\end{aligned}$$

For the rest of this subsection, we assume that K^* is a field such that $K^i=\{0\}$ for $i\neq 0$.

Assume that every object of $\mathcal{M}$ is profinite. Moreover, when we consider $\boldsymbol{N}^{R^*}$ for an object R^* of $\mathcal{C}$ and an object $\boldsymbol{N}=(K^*,N^*,\beta)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$, we always assume that R^* and N^* satisfy the conditions (i) and (ii) of (10.1.13), respectively. Let us denote by $\lambda^{R^*}_{\boldsymbol{N}}:N^*\to\mathcal{H}om^*(R^{**},R^{**}\otimes_{K^*}N^*)$ be the image of the switching map of $T_{N^*,R^{**}}:N^*\otimes_{K^*}R^{**}\to R^{**}\otimes_{K^*}N^*$ by

$$\Phi_{N^*,R^{**},R^{**}\otimes_{K^*}N^*}:\mathrm{Hom}^c_{K^*}(N^*\otimes_{K^*}R^{**},R^{**}\otimes_{K^*}N^*)\to\mathrm{Hom}^c_{K^*}(N^*,\mathcal{H}om^*(R^{**},R^{**}\otimes_{K^*}N^*)).$$

Proposition 10.1.15 *Let R^* be an object of $\mathcal{C}$ which is finite type, connective and has skeletal topology and $\boldsymbol{N}=(K^*,N^*,\beta)$ an object of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$ such that N^* is finite type, coconnective and has skeletal topology.*

(1) *Let $j_{R^*}(\boldsymbol{N}):N^*\,\widehat{\otimes}_{K^*}R^*\to(R^{**}\otimes_{K^*}N^*)\,\widehat{\otimes}_{K^*}R^*$ be the homomorphism of right R^*-modules induced by*

$$\begin{aligned}N^*&\xrightarrow{\lambda^{R^*}_{\boldsymbol{N}}}\mathcal{H}om^*(R^{**},R^{**}\otimes_{K^*}N^*)\xrightarrow{\left(\hat{\varphi}^{R^{**}}_{R^{**}\otimes_{K^*}N^*}\right)^{-1}}\mathcal{H}om^*(R^{**},K^*)\,\widehat{\otimes}_{K^*}(R^{**}\otimes_{K^*}N^*)\\&\xrightarrow{\chi^{-1}_{R^*,K^*}\widehat{\otimes}_{K^*}id_{R^{**}\otimes_{K^*}N^*}}R^*\,\widehat{\otimes}_{K^*}(R^{**}\otimes_{K^*}N^*)\xrightarrow{\widehat{T}_{R^*,R^{**}\otimes_{K^*}N^*}}(R^{**}\otimes_{K^*}N^*)\,\widehat{\otimes}_{K^*}R^*.\end{aligned}$$

Then, $\pi_{R^*}(\boldsymbol{N}):u^*_{R^*}(\boldsymbol{N})=(R^*,N^*\,\widehat{\otimes}_{K^*}R^*,\hat{\alpha}_{u_{R^*}})\to(R^*,(R^{**}\otimes_{K^*}N^*)\,\widehat{\otimes}_{K^*}R^*,(\beta^{R^*})_{u_{R^*}})=u^*_{R^*}(\boldsymbol{N}^{R^*})$ is given by $\pi_{R^*}(\boldsymbol{N})=(id_{R^*},j_{R^*}(\boldsymbol{N}))$.

(2) *Let $\boldsymbol{M}=(K^*,M^*,\alpha)$ be an object of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$ such that M^* is finite type, coconnective and has skeletal topology. For a morphism $\boldsymbol{\varphi}=(id_{K^*},\varphi):\boldsymbol{M}\to\boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$, $\boldsymbol{\varphi}^{R^*}:\boldsymbol{M}^{R^*}\to\boldsymbol{N}^{R^*}$ is given by $\boldsymbol{\varphi}^{R^*}=(id_{K^*},id_{R^{**}}\otimes_{K^*}\varphi)$.*

(3) *Let S^* be an object of $\mathcal{C}$ which is finite type, connective and has skeletal topology. For a morphism $\gamma:R^*\to S^*$ of $\mathcal{C}$, $\boldsymbol{N}^\gamma:\boldsymbol{N}^{S^*}\to\boldsymbol{N}^{R^*}$ is given by $\boldsymbol{N}^\gamma=(id_{K^*},\gamma^*\otimes_{K^*}id_{N^*})$.*

Proof. (1) The assertion is a direct consequence of (10.1.14).

(2) We note that the the following diagram is commutative by (3.2.1).

$$\begin{tikzcd}
\mathrm{Hom}^c_{K^*}(M^* \otimes_{K^*} R^{**}, R^{**} \otimes_{K^*} M^*) \arrow[r, "\Phi_{M^*,R^{**},R^{**}\otimes_{K^*}M^*}"] \arrow[d, "(id_{R^{**}}\otimes_{K^*}\varphi)_*"] & \mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(R^{**}, R^{**} \otimes_{K^*} M^*)) \arrow[d, "((id_{R^{**}}\otimes_{K^*}\varphi)_*)_*"] \\
\mathrm{Hom}^c_{K^*}(M^* \otimes_{K^*} R^{**}, R^{**} \otimes_{K^*} N^*) \arrow[r, "\Phi_{N^*,R^{**},R^{**}\otimes_{K^*}N^*}"] & \mathrm{Hom}^c_{K^*}(M^*, \mathcal{H}om^*(R^{**}, R^{**} \otimes_{K^*} N^*)) \\
\mathrm{Hom}^c_{K^*}(N^* \otimes_{K^*} R^{**}, R^{**} \otimes_{K^*} N^*) \arrow[r, "\Phi_{N^*,R^{**},R^{**}\otimes_{K^*}N^*}"] \arrow[u, "(\varphi\otimes_{K^*}id_{R^{**}})^*"] & \mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(R^{**}, R^{**} \otimes_{K^*} N^*)) \arrow[u, "\varphi^*"]
\end{tikzcd}$$

Since $(id_{R^{**}} \otimes_{K^*} \varphi)T_{M^*,R^{**}} = T_{N^*,R^{**}}(\varphi \otimes_{K^*} id_{R^{**}})$, it follows from the above diagram that the upper triangle of the following diagram is commutative. The lower left rectangle of the following diagram is commutative by the definition of $j_{R^*}(\boldsymbol{N})$.

$$\begin{tikzcd}
 & & \mathcal{H}om^*(R^{**}, R^{**} \otimes_{K^*} M^*) \arrow[d, "(id_{R^{**}}\otimes_{K^*}\varphi)_*"] \\
M^* \arrow[urr, "\lambda^{R^*}_{\boldsymbol{M}}"] \arrow[r, "\varphi"] \arrow[d, "\hat{\imath}_{M^*,u_{R^*}}"] & N^* \arrow[r, "\lambda^{R^*}_{\boldsymbol{N}}"] \arrow[d, "\hat{\imath}_{N^*,u_{R^*}}"] & \mathcal{H}om^*(R^{**}, R^{**} \otimes_{K^*} N^*) \arrow[d, "\left(\hat{\varphi}^{R^{**}}_{R^{**}\otimes_{K^*}N^*}\right)^{-1}"] \\
M^* \mathbin{\widehat{\otimes}}_{K^*} R^* \arrow[r, "\varphi \mathbin{\widehat{\otimes}}_{K^*} id_{R^*}"] & N^* \mathbin{\widehat{\otimes}}_{K^*} R^* \arrow[d, "j_{R^*}(\boldsymbol{N})"] & \mathcal{H}om^*(R^{**}, K^*) \mathbin{\widehat{\otimes}}_{K^*} (R^{**} \otimes_{K^*} N^*) \arrow[d, "\chi^{-1}_{R^*,K^*} \mathbin{\widehat{\otimes}}_{K^*} id_{R^{**}\otimes_{K^*}N^*}"] \\
 & (R^{**} \otimes_{K^*} N^*) \mathbin{\widehat{\otimes}}_{K^*} R^* & R^* \mathbin{\widehat{\otimes}}_{K^*} (R^{**} \otimes_{K^*} N^*) \arrow[l, "\widehat{T}_{R^*,R^{**}\otimes_{K^*}N^*}"']
\end{tikzcd}$$

Then, it follows from (10.1.14) that we have $\boldsymbol{\varphi}^{R^*} = E_{R^*}(\boldsymbol{M})_{\boldsymbol{N}^{R^*}}(\pi_{R^*}(\boldsymbol{N})u^*_{R^*}(\boldsymbol{\varphi})) = (id_{K^*}, id_{R^{**}} \otimes_{K^*} \varphi)$.

(3) We first note that the following diagram is commutative.

$$\begin{tikzcd}
N^* \arrow[rr, "\hat{\imath}_{N^*,u_{S^*}}"] \arrow[d, "\hat{\imath}_{N^*,u_{R^*}}"] & & N^* \mathbin{\widehat{\otimes}}_{K^*} S^* \arrow[d, "\hat{c}_{\gamma,u_{R^*},N^*}"] \\
N^* \mathbin{\widehat{\otimes}}_{K^*} R^* \arrow[rr, "\hat{\imath}_{N^* \mathbin{\widehat{\otimes}}_{K^*} R^*,\gamma}"] \arrow[d, "j_{R^*}(\boldsymbol{N})"] & & (N^* \mathbin{\widehat{\otimes}}_{K^*} R^*) \mathbin{\widehat{\otimes}}_{R^*} S^* \arrow[d, "j_{R^*}(\boldsymbol{N}) \mathbin{\widehat{\otimes}}_{R^*} id_{S^*}"] \\
(R^{**} \otimes_{K^*} N^*) \mathbin{\widehat{\otimes}}_{K^*} R^* \arrow[rr, "\hat{\imath}_{(R^{**}\otimes_{K^*}N^*) \mathbin{\widehat{\otimes}}_{K^*} R^*,\gamma}"] \arrow[dr, "id_{R^{**}\otimes_{K^*}N^*} \mathbin{\widehat{\otimes}}_{K^*} \gamma"'] & & ((R^{**} \otimes_{K^*} N^*) \mathbin{\widehat{\otimes}}_{K^*} R^*) \mathbin{\widehat{\otimes}}_{R^*} S^* \arrow[dl, "\hat{c}^{-1}_{\gamma,u_{R^*},R^{**}\otimes_{K^*}N^*}"] \\
 & (R^{**} \otimes_{K^*} N^*) \mathbin{\widehat{\otimes}}_{K^*} S^* &
\end{tikzcd}$$

It follows from the definition of $\boldsymbol{N}^\gamma$ and (1) that we have the following equality.

$$\boldsymbol{N}^\gamma = E_{S^*}(\boldsymbol{N})_{\boldsymbol{N}^{R^*}}(\gamma^\sharp_{\boldsymbol{N}^{R^*},\boldsymbol{N}}(\pi_{R^*}(\boldsymbol{N}))) = E_{S^*}(\boldsymbol{N})_{\boldsymbol{N}^{R^*}}(id_{S^*}, \hat{c}^{-1}_{\gamma,u_{R^*},N^*}(j_{R^*}(\boldsymbol{N}) \mathbin{\widehat{\otimes}}_{R^*} id_{S^*})\hat{c}_{\gamma,u_{R^*},N^*})$$

We also note that the following equality holds in $\mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(S^{**}, R^{**} \otimes_{K^*} N^*))$ by (3.2.1).

$$\begin{aligned}
\Phi_{N^*,S^{**},R^{**}\otimes_{K^*}N^*}((\gamma^* \otimes_{K^*} id_{N^*})T_{N^*,S^{**}}) &= \Phi_{N^*,S^{**},R^{**}\otimes_{K^*}N^*}(T_{N^*,R^{**}}(id_{N^*} \otimes_{K^*} \gamma^*)) \\
&= (\gamma^*)^*\Phi_{N^*,R^{**},R^{**}\otimes_{K^*}N^*}(T_{N^*,R^{**}}) = (\gamma^*)^*\lambda^{R^*}_{\boldsymbol{N}}
\end{aligned}$$

Hence, by (10.1.14) it suffices to show that

$$\hat{c}^{-1}_{\gamma,u_{R^*},N^*}(j_{R^*}(\boldsymbol{N}) \mathbin{\widehat{\otimes}}_{R^*} id_{S^*})\hat{c}_{\gamma,u_{R^*},N^*}\hat{\imath}_{N^*,u_{S^*}} = (id_{R^{**}\otimes_{K^*}N^*} \mathbin{\widehat{\otimes}}_{K^*} \gamma)j_{R^*}(\boldsymbol{N})\hat{\imath}_{N^*,u_{R^*}}$$

which belongs to $\mathrm{Hom}^c_{K^*}(N^*, (R^{**} \otimes_{K^*} N^*) \mathbin{\widehat{\otimes}}_{K^*} S^*)$ maps to $(\gamma^*)^*\lambda^{R^*}_{\boldsymbol{N}}$ by the following composition.

$$\begin{aligned}
&\mathrm{Hom}^c_{K^*}(N^*, (R^{**} \otimes_{K^*} N^*) \mathbin{\widehat{\otimes}}_{K^*} S^*) \xrightarrow{(\widehat{T}_{R^{**}\otimes_{K^*}N^*,S^*})_*} \mathrm{Hom}^c_{K^*}(N^*, S^* \mathbin{\widehat{\otimes}}_{K^*} (R^{**} \otimes_{K^*} N^*)) \xrightarrow{(\chi_{S^*,K^*} \mathbin{\widehat{\otimes}}_{K^*} id_{R^{**}\otimes_{K^*}N^*})_*} \\
&\qquad \mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(S^{**}, K^*) \mathbin{\widehat{\otimes}}_{K^*} (R^{**} \otimes_{K^*} N^*)) \xrightarrow{\left(\hat{\varphi}^{S^{**}}_{R^{**}\otimes_{K^*}N^*}\right)_*} \mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(S^{**}, R^{**} \otimes_{K^*} N^*)) \cdots (*)
\end{aligned}$$

Since the following diagram is commutative by the naturality of $\hat{\varphi}^{M^*}_{N^*}$ and χ_{M^*,N^*},

$$\begin{tikzcd}
\mathrm{Hom}^c_{K^*}(N^*,(R^{**}\otimes_{K^*}N^*)\,\widehat{\otimes}_{K^*}R^*) \arrow[r, "(id_{R^{**}\otimes_{K^*}N^*}\widehat{\otimes}_{K^*}\gamma)_*"] \arrow[d, "(\widehat{T}_{R^{**}\otimes_{K^*}N^*,R^*})_*"] & \mathrm{Hom}^c_{K^*}(N^*,(R^{**}\otimes_{K^*}N^*)\,\widehat{\otimes}_{K^*}S^*) \arrow[d, "(\widehat{T}_{R^{**}\otimes_{K^*}N^*,S^*})_*"] \\
\mathrm{Hom}^c_{K^*}(N^*,R^*\,\widehat{\otimes}_{K^*}(R^{**}\otimes_{K^*}N^*)) \arrow[r, "(\gamma\widehat{\otimes}_{K^*}id_{R^{**}\otimes_{K^*}N^*})_*"] \arrow[d, "(\chi_{R^*,K^*}\widehat{\otimes}_{K^*}id_{R^{**}\otimes_{K^*}N^*})_*"] & \mathrm{Hom}^c_{K^*}(N^*,S^*\,\widehat{\otimes}_{K^*}(R^{**}\otimes_{K^*}N^*)) \arrow[d, "(\chi_{S^*,K^*}\widehat{\otimes}_{K^*}id_{R^{**}\otimes_{K^*}N^*})_*"] \\
\mathrm{Hom}^c_{K^*}(N^*,\mathcal{H}om^*(R^{**},K^*)\,\widehat{\otimes}_{K^*}(R^{**}\otimes_{K^*}N^*)) \arrow[r, "((\gamma^*)^*\widehat{\otimes}_{K^*}id_{R^{**}\otimes_{K^*}N^*})_*"] \arrow[d, "(\hat{\varphi}^{R^{**}}_{R^{**}\otimes_{K^*}N^*})_*"] & \mathrm{Hom}^c_{K^*}(N^*,\mathcal{H}om^*(S^{**},K^*)\,\widehat{\otimes}_{K^*}(R^{**}\otimes_{K^*}N^*)) \arrow[d, "(\hat{\varphi}^{S^{**}}_{R^{**}\otimes_{K^*}N^*})_*"] \\
\mathrm{Hom}^c_{K^*}(N^*,\mathcal{H}om^*(R^{**},R^{**}\otimes_{K^*}N^*)) \arrow[r, "((\gamma^*)^*)_*"] & \mathrm{Hom}^c_{K^*}(N^*,\mathcal{H}om^*(S^{**},R^{**}\otimes_{K^*}N^*))
\end{tikzcd}$$

$(id_{R^{**}\otimes_{K^*}N^*}\widehat{\otimes}_{K^*}\gamma)\widehat{T}_{R^*,R^{**}\otimes_{K^*}N^*}(\chi^{-1}_{R^*,K^*}\widehat{\otimes}_{K^*}id_{R^{**}\otimes_{K^*}N^*})(\hat{\varphi}^{R^{**}}_{R^{**}\otimes_{K^*}N^*})^{-1}\lambda^{R^*}_{\boldsymbol{N}} = (id_{R^{**}\otimes_{K^*}N^*}\widehat{\otimes}_{K^*}\gamma)j_{R^*}(\boldsymbol{N})\hat{i}_{N^*,u_{R^*}}$ maps to $(\gamma^*)^*\lambda^{R^*}_{\boldsymbol{N}}$ by the above composition $(*)$. $\square$

We choose a basis $b_{i1},b_{i2},\dots,b_{id_i}$ of R^i and let $b^*_{i1},b^*_{i2},\dots,b^*_{id_i}$ ($b^*_{ij}\in\mathcal{H}om^{-i}(R^*,K^*)$) be the dual basis of $b_{i1},b_{i2},\dots,b_{id_i}$.

Lemma 10.1.16 *Suppose that $\boldsymbol{N}=(K^*,N^*,\beta)$ is an object of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$ such that N^* is finite type, coconnective and has the skeletal topology. Let $\tilde{j}_{R^*}(\boldsymbol{N}):N^*\to(R^{**}\otimes_{K^*}N^*)\,\widehat{\otimes}_{K^*}R^*$ be the following composition.*

$$N^*\xrightarrow{\lambda^{R^*}_{\boldsymbol{N}}}\mathcal{H}om^*(R^{**},R^{**}\otimes_{K^*}N^*)\xrightarrow{\left(\hat{\varphi}^{R^{**}}_{R^{**}\otimes_{K^*}N^*}\right)^{-1}}\mathcal{H}om^*(R^{**},K^*)\,\widehat{\otimes}_{K^*}(R^{**}\otimes_{K^*}N^*)$$
$$\xrightarrow{\chi^{-1}_{R^*,K^*}\widehat{\otimes}_{K^*}id_{R^{**}\otimes_{K^*}N^*}}R^*\,\widehat{\otimes}_{K^*}(R^{**}\otimes_{K^*}N^*)\xrightarrow{\widehat{T}_{R^*,R^{**}\otimes_{K^*}N^*}}(R^{**}\otimes_{K^*}N^*)\,\widehat{\otimes}_{K^*}R^*$$

Then, $\tilde{j}_{R^*}(\boldsymbol{N})(y)=\sum\limits_{k\in\boldsymbol{Z}}\sum\limits_{l=1}^{d_k}(-1)^{k(k+n)}b^*_{kl}\otimes y\otimes b_{kl}$ for $y\in N^{-n}$.

Proof. Since $\chi_{R^*,K^*}(b_{ij}):\Sigma^iR^{**}\to K^*$ maps $([i],b^*_{kl})$ to $(-1)^{ik}b^*_{kl}([k],b_{ij})$ which is $(-1)^i$ if $(k,l)=(i,j)$, otherwise 0, $(-1)^i\chi_{R^*,K^*}(b_{i1}),(-1)^i\chi_{R^*,K^*}(b_{i2}),\dots,(-1)^i\chi_{R^*,K^*}(b_{id_i})$ is the dual basis of $b^*_{i1},b^*_{i2},\dots,b^*_{id_i}$. For $y\in N^{-n}$, $\lambda^{R^*}_{\boldsymbol{N}}(y):\Sigma^{-n}R^{**}\to R^{**}\otimes_{K^*}N^*$ maps $([-n],f)$ to $(-1)^{kn}f\otimes y$ if $f\in(R^{**})^{-k}$. By (4.1.15), we have

$$\left(\hat{\varphi}^{R^{**}}_{R^{**}\otimes_{K^*}N^*}\right)^{-1}\lambda^{R^*}_{\boldsymbol{N}}(y)=\sum_{k\in\boldsymbol{Z}}\sum_{l=1}^{d_k}(-1)^{k(k-n+1)}\chi_{R^*,K^*}(b_{kl})\otimes(\lambda^{R^*}_{\boldsymbol{N}}(y))([-n],b^*_{kl})=\sum_{k\in\boldsymbol{Z}}\sum_{l=1}^{d_k}\chi_{R^*,K^*}(b_{kl})\otimes b^*_{kl}\otimes y$$

Thus we see $\tilde{j}_{R^*}(\boldsymbol{N})(y)=\sum\limits_{k\in\boldsymbol{Z}}\sum\limits_{l=1}^{d_k}(-1)^{k(k+n)}b^*_{kl}\otimes y\otimes b_{kl}$. $\square$

By the assumptions on R^*, $R^{**}=\mathcal{H}om^*(R^*,K^*)$ is finite type, coconective and has skeletal topology. Hence $R^{**}\otimes_{K^*}R^{**}$ has skeletal topology and is finite type and complete. It follows from (4.1.10) that

$$\phi:R^{**}\otimes_{K^*}R^{**}=\mathcal{H}om^*(R^*,K^*)\otimes_{K^*}\mathcal{H}om^*(R^*,K^*)\to\mathcal{H}om^*(R^*\otimes_{K^*}R^*,K^*)$$

is an isomorphism. Let $m:R^*\otimes_{K^*}R^*\to R^*$ be the multiplication of R^*. We put

$$m(b_{ij}\otimes b_{kl})=\sum_{u=1}^{d_{i+k}}a_u(i,j:k,l)b_{i+k\,u}.$$

Lemma 10.1.17 *For an object $\boldsymbol{N}=(K^*,N^*,\beta)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$, let*

$$\tilde{\epsilon}^{R^*}_{\boldsymbol{N}}:R^{**}\otimes_{K^*}N^*\to R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*)$$

be the following composition.

$$R^{**}\otimes_{K^*}N^*=\mathcal{H}om^*(R^*,K^*)\otimes_{K^*}N^*\xrightarrow{m^*\otimes_{K^*}id_{N^*}}\mathcal{H}om^*(R^*\otimes_{K^*}R^*,K^*)\otimes_{K^*}N^*\xrightarrow{\phi^{-1}\otimes_{K^*}id_{N^*}}$$
$$(\mathcal{H}om^*(R^*,K^*)\otimes_{K^*}\mathcal{H}om^*(R^*,K^*))\otimes_{K^*}N^*=(R^{**}\otimes_{K^*}R^{**})\otimes_{K^*}N^*\xrightarrow{\cong}R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*)$$

Then, for $y\in N^{-n}$, $\tilde{\epsilon}^{R^*}_{\boldsymbol{N}}(b^*_{st}\otimes y)=\sum\limits_{i+k=s}\sum\limits_{l=1}^{e_k}\sum\limits_{j=1}^{d_i}(-1)^{ik}a_t(i,j:k,l)b^*_{ij}\otimes(b^*_{kl}\otimes y)$.

Proof. It follows from (4.1.15) that we have

$$\begin{aligned}\phi^{-1}(b^*_{st}\Sigma^{-s}m) &= \sum_{i+k=s}\sum_{l=1}^{e_k}\sum_{j=1}^{d_i}(-1)^{ik}b^*_{st}\Sigma^{-s}m([-s],b_{ij}\otimes b_{kl})b^*_{ij}\otimes b^*_{kl}\\ &= \sum_{i+k=s}\sum_{l=1}^{e_k}\sum_{j=1}^{d_i}\sum_{u=1}^{d_{i+k}}(-1)^{ik}a_u(i,j:k,l)b^*_{st}([-s],b_{su})b^*_{ij}\otimes b^*_{kl}\\ &= \sum_{i+k=s}\sum_{l=1}^{e_k}\sum_{j=1}^{d_i}(-1)^{ik}a_t(i,j:k,l)b^*_{ij}\otimes b^*_{kl}.\end{aligned}$$

Hence the assertion follows. □

Proposition 10.1.18 *$\epsilon^{R^*}_{\boldsymbol{N}} : \boldsymbol{N}^{R^*} \to (\boldsymbol{N}^{R^*})^{R^*}$ is given by $\epsilon_{R^*,\boldsymbol{N}} = (id_{K^*}, \tilde{\epsilon}^{R^*}_{\boldsymbol{N}})$.*

Proof. First we note that it follows from (10.1.14) that $(\boldsymbol{N}^{R^*})^{R^*}$ is given as follows.

$$(\boldsymbol{N}^{R^*})^{R^*} = (K^*, R^{**}\otimes_{K^*}N^*, \beta^{R^*})^{R^*} = (K^*, R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*), (\beta^{R^*})^{R^*})$$

We have $\epsilon^{R^*}_{\boldsymbol{N}} = E_{R^*}(\boldsymbol{N})_{(\boldsymbol{N}^{R^*})^{R^*}}(\pi_{R^*}(\boldsymbol{N}^{R^*})\pi_X(\boldsymbol{N}))$ and $\pi_{R^*}(\boldsymbol{N}^{R^*})\pi_X(\boldsymbol{N}) = (id_{K^*}, j_{R^*}(\boldsymbol{N}^{R^*})j_{R^*}(\boldsymbol{N}))$ by the definition of $\epsilon^{R^*}_{\boldsymbol{N}}$ and (10.1.15). Since

$$\begin{tikzcd}
\mathrm{Hom}^c_{K^*}(N^*\otimes_{K^*}R^{**}, R^{**}\otimes_{K^*}N^*) \arrow[r, "\tilde{\epsilon}^{R^*}_{\boldsymbol{N}*}"] \arrow[d, "\Phi_{N^*,R^{**},R^{**}\otimes_{K^*}N^*}"] & \mathrm{Hom}^c_{K^*}(N^*\otimes_{K^*}R^{**}, R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*)) \arrow[d, "\Phi_{N^*,R^{**},R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*)}"] \\
\mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(R^{**}, R^{**}\otimes_{K^*}N^*)) \arrow[r, "(\tilde{\epsilon}^{R^*}_{\boldsymbol{N}*})_*"] & \mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(R^{**}, R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*)))
\end{tikzcd}$$

is commutative by (3.2.1), $\Phi_{N^*,R^{**},R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*)}(\tilde{\epsilon}^{R^*}_{\boldsymbol{N}}T_{N^*,R^{**}})$ coincides with the following composition.

$$N^* \xrightarrow{\lambda^{R^*}_{\boldsymbol{N}}} \mathcal{H}om^*(R^{**}, R^{**}\otimes_{K^*}N^*) \xrightarrow{\tilde{\epsilon}^{R^*}_{\boldsymbol{N}*}} \mathcal{H}om^*(R^{**}, R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*))$$

Hence it suffices to show that the following composition coincides with $\tilde{\epsilon}^{R^*}_{\boldsymbol{N}*}\lambda^{R^*}_{\boldsymbol{N}}$ by (10.1.14).

$$\begin{aligned}
&N^* \xrightarrow{j_{R^*}(\boldsymbol{N}^{R^*})j_{R^*}(\boldsymbol{N})\hat{\imath}_{N,u_{R^*}}} (R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*))\,\widehat{\otimes}_{K^*}R^* \xrightarrow{\widehat{T}_{R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*),R^*}}\\
(*)\qquad &R^*\,\widehat{\otimes}_{K^*}(R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*)) \xrightarrow{\chi_{R^*,K^*}\widehat{\otimes}_{K^*}id_{R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*)}}\\
&R^{***}\,\widehat{\otimes}_{K^*}(R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*)) \xrightarrow{\hat{\varphi}^{R^{**}}_{R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*)}} \mathcal{H}om^*(R^{**}, R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*))
\end{aligned}$$

Here, we put $R^{***} = \mathcal{H}om^*(R^{**}, K^*)$. It follows from (10.1.15) that

$$j_{R^*}(\boldsymbol{N}^{R^*})j_{R^*}(\boldsymbol{N})\hat{\imath}_{N,u_{R^*}} : N^* \to (R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*))\,\widehat{\otimes}_{K^*}R^*$$

is the following composition.

$$\begin{aligned}
&N^* \xrightarrow{\tilde{\jmath}_{R^*}(\boldsymbol{N})} (R^{**}\otimes_{K^*}N^*)\,\widehat{\otimes}_{K^*}R^* \xrightarrow{\tilde{\jmath}_{R^*}(\boldsymbol{N}^{R^*})\,\widehat{\otimes}_{K^*}id_{R^*}} ((R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*))\,\widehat{\otimes}_{K^*}R^*)\,\widehat{\otimes}_{K^*}R^* \xrightarrow{\cong}\\
&(R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*))\,\widehat{\otimes}_{K^*}(R^*\,\widehat{\otimes}_{K^*}R^*) \xrightarrow{id_{R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*)}\widehat{\otimes}_{K^*}\hat{m}} (R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*))\,\widehat{\otimes}_{K^*}R^*
\end{aligned}$$

It follows from (10.1.17) that, for $y \in N^{-n}$, $\tilde{\epsilon}^{R^*}_{\boldsymbol{N}*}\lambda^{R^*}_{\boldsymbol{N}}(y)$ maps $([-n], b^*_{st}) \in \Sigma^{-n}R^{**}$ to

$$(-1)^{ns}\tilde{\epsilon}^{R^*}_{\boldsymbol{N}}(b^*_{st}\otimes y) = (-1)^{ns}\sum_{i+k=s}\sum_{l=1}^{e_k}\sum_{j=1}^{d_i}(-1)^{ik}a_t(i,j:k,l)b^*_{ij}\otimes(b^*_{kl}\otimes y).$$

On the other hand, by (10.1.16), composition $(*)$ maps $y \in N^{-n}$ to a map $\Sigma^{-n}R^{**} \to R^{**}\otimes_{K^*}(R^{**}\otimes_{K^*}N^*)$ which maps $([-n], b^*_{st})$ to $\sum\limits_{i+k=s}\sum\limits_{l=1}^{d_k}\sum\limits_{j=1}^{d_i}(-1)^{ik+ns}a_t(i,j:k,l)b^*_{ij}\otimes b^*_{kl}\otimes y$. □

We note that if R^* and S^* satisfy the condition (i) of (10.1.13), so does $R^*\otimes_{K^*}S^*$ by (2) of (2.1.20).

Proposition 10.1.19 *For objects R^*, S^* of $\mathcal{C}$ and an object $\boldsymbol{N} = (K^*, N^*, \beta)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$, define a map*

$$\tilde{\theta}^{R^*,S^*}(\boldsymbol{N}) : \mathcal{H}om^*(R^* \otimes_{K^*} S^*, K^*) \otimes_{K^*} N^* \to S^{**} \otimes_{K^*} (R^{**} \otimes_{K^*} N^*)$$

by to be the following composition.

$$\mathcal{H}om^*(R^* \otimes_{K^*} S^*, K^*) \otimes_{K^*} N^* \xrightarrow{\phi^{-1} \otimes_{K^*} id_{N^*}} (R^{**} \otimes_{K^*} S^{**}) \otimes_{K^*} N^* \xrightarrow{T_{R^{**},S^{**}} \otimes_{K^*} id_{N^*}} (S^{**} \otimes_{K^*} R^{**}) \otimes_{K^*} N^* \xrightarrow{\cong} S^{**} \otimes_{K^*} (R^{**} \otimes_{K^*} N^*)$$

Then, $\theta^{R^*,S^*}(\boldsymbol{N}) : \boldsymbol{N}^{R^* \otimes_{K^*} S^*} \to (\boldsymbol{N}^{R^*})^{S^*}$ is given by $\theta^{R^*,S^*}(\boldsymbol{N}) = (id_{K^*}, \tilde{\theta}^{R^*,S^*}(\boldsymbol{N}))$. Hence $\theta^{R^*,S^*}(\boldsymbol{N})$ is an isomorphism of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$.

Proof. Let $i_1 : R^* \to R^* \otimes_{K^*} S^*$ and $i_2 : S^* \to R^* \otimes_{K^*} S^*$ be maps defined by $i_1(r) = r \otimes 1$, $i_2(s) = 1 \otimes s$. We denote by $m_{R^*} : R^* \otimes_{K^*} R^* \to R^*$ and $m_{S^*} : S^* \otimes_{K^*} S^* \to S^*$ the products of R^* and S^*, respectively. Then, the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{H}om^*(R^* \otimes_{K^*} S^*, K^*) \arrow[rr, "(m_{R^*} \otimes_{K^*} m_{S^*})^*"] \arrow[d, "\phi^{-1}"] \arrow[dr, "T^*_{S^*,R^*}"] & & \mathcal{H}om^*((R^* \otimes_{K^*} R^*) \otimes_{K^*} (S^* \otimes_{K^*} S^*), K^*) \arrow[d, "(id_{R^*} \otimes_{K^*} T_{S^*,R^*} \otimes_{K^*} id_{S^*})^*"] \\
R^{**} \otimes_{K^*} S^{**} \arrow[dr, "T_{R^{**},S^{**}}"] & \mathcal{H}om^*(S^* \otimes_{K^*} R^*, K^*) \arrow[d, "\phi^{-1}"] & \mathcal{H}om^*((R^* \otimes_{K^*} S^*) \otimes_{K^*} (R^* \otimes_{K^*} S^*), K^*) \arrow[l, "(i_2 \otimes_{K^*} i_1)^*"'] \arrow[d, "\phi^{-1}"] \\
& S^{**} \otimes_{K^*} R^{**} & \mathcal{H}om^*(R^* \otimes_{K^*} S^*, K^*) \otimes_{K^*} \mathcal{H}om^*(R^* \otimes_{K^*} S^*, K^*) \arrow[l, "i_2^* \otimes_{K^*} i_1^*"']
\end{tikzcd}$$

Since $\theta^{R^*,S^*}(\boldsymbol{N})$ is a composition $\boldsymbol{N}^{R^* \otimes_{K^*} S^*} \xrightarrow{\epsilon_{\boldsymbol{N}}^{R^* \otimes_{K^*} S^*}} (\boldsymbol{N}^{R^* \otimes_{K^*} S^*})^{R^* \otimes_{K^*} S^*} \xrightarrow{(\boldsymbol{N}^{i_1})^{i_2}} (\boldsymbol{N}^{R^*})^{S^*}$, it follows from (10.1.15) and (10.1.18) that the commutativity of the above diagram implies the result. □

Since we assumed that K^* is a field, we have the following result by (2.1.3) and (2.1.5).

Proposition 10.1.20 *Let $\mathcal{M}$ be a full subcategory of $\mathcal{M}od_{cK^*}$ consisting of objects which satisfy the condition (ii) of (10.1.13). For an object R^* of $\mathcal{T}op\mathcal{A}lg_{K^*}$,*

$$\exp_{R^*} : \mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M})_{K^*} \to \mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M})_{K^*}$$

preserves monomorphisms, equalizers, epimorphims and coequalizers.

10.2 Fibered category of functorial modules

Definition 10.2.1 *For a functor $F : \mathcal{C} \to \mathcal{T}op$, we define a functor $U_F : \mathcal{C}_F \to \mathcal{C}$ by $U_F(R^*, \rho) = R^*$ and $U_F(\lambda : (R^*, \rho) \to (S^*, \sigma)) = (\lambda : S^* \to R^*)$. A functor $M : \mathcal{C}_F \to \mathcal{M}od(\mathcal{C}, \mathcal{M})$ is called an F-module if M satisfies $p_{\mathcal{C}} M = U_F$. A natural transformation $\varphi : M \to N$ of F-modules is called a morphism of F-modules if φ satisfies $p_{\mathcal{C}}\big(\varphi_{(R^*,\rho)}\big) = id_{R^*}$ for $(R^*, \rho) \in \operatorname{Ob}\mathcal{C}_F$. We denote by $\mathcal{M}od(F)$ the category of F-modules and morphisms of F-modules. We say that an F-module M is continuous if M is a continuous functor. The full subcategory of $\mathcal{M}od(F)$ which consists of continuous F-modules is denoted by $\mathcal{M}od_c(F)$. Since $\mathcal{M}od(F)$ is a subcategory of a quasi-topological category* $\mathrm{Funct}(\mathcal{C}_F, \mathcal{M}od(\mathcal{C}, \mathcal{M}))$*, $\mathcal{M}od(F)$ is a quasi-topological category.*

We put $\mathcal{T} = \mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$. For a morphism $f : G \to F$ of $\mathcal{T}$, define a functor $\tilde{f} : \mathcal{C}_G \to \mathcal{C}_F$ by $\tilde{f}(R^*, \rho) = (R^*, f_{R^*}(\rho))$ for $(R^*, \rho) \in \operatorname{Ob}\mathcal{C}_G$ and $\tilde{f}(\lambda : (R^*, \rho) \to (S^*, \sigma)) = (\lambda : (R^*, f_{R^*}(\rho)) \to (S^*, f_{S^*}(\sigma)))$. Define a functor $f^* : \mathcal{M}od(F) \to \mathcal{M}od(G)$ by $f^*(M) = M\tilde{f}$ and $f^*(\varphi)_{(R^*,\rho)} = \varphi_{\tilde{f}(R^*,\rho)} = \varphi_{(R^*, f_{R^*}(\rho))}$ for $(R^*, \rho) \in \operatorname{Ob}\mathcal{C}_G$. Since $\tilde{f}$ is continuous, $f^*(M)$ is continuous if M is so. It follows from (7.6.4) that f^* is a continuous functor. Note that $(gf)^* = f^* g^* : \mathcal{M}od(H) \to \mathcal{M}od(G)$ holds for morphisms $f : G \to F$ and $g : F \to H$ of $\mathrm{Funct}(\mathcal{C}, \mathcal{T}op)$ and that id_F^* is the identity functor of $\mathcal{M}od(F)$.

We define a category $\mathcal{MOD}$ as follows. Objects of $\mathcal{MOD}$ are pairs (F, M) of $F \in \operatorname{Ob}\mathcal{T}$ and an F-module M. A morphism $(G, N) \to (F, M)$ is a pair (f, φ) of a morphism $f : G \to F$ of $\mathcal{T}$ and a morphism of G-modules $\varphi : f^*(M) \to N$. Composition of morphisms $(f, \varphi) : (G, N) \to (F, M)$ and $(g, \psi) : (F, M) \to (H, L)$ is defined to be $(gf, \varphi f^*(\psi))$.

Define a functor $p_{\mathcal{T}} : \mathcal{MOD} \to \mathcal{T}$ by $p_{\mathcal{T}}(F, M) = F$ and $p_{\mathcal{T}}(f, \varphi) = f$. Then, for each $F \in \operatorname{Ob}\mathcal{T}$, the subcategory $\mathcal{MOD}_F$ of $\mathcal{MOD}$ consisting of objects of the form (F, M) and morphisms of the form (id_F, φ) is identified with the opposite category $\mathcal{M}od(F)^{op}$ of F-modules.

Proposition 10.2.2 *$p_{\mathcal{T}} : \mathcal{MOD} \to \mathcal{T}$ is a fibered category.*

Proof. For a morphism $f : G \to F$ of $\mathcal{T}$ and $(F, M) \in \mathrm{Ob}\,\mathcal{MOD}_F$, it is clear that a map

$$(f, id_{f^*(M)})_* : \mathcal{MOD}_G((G, N), (G, f^*(M))) \to \mathcal{MOD}_f((G, N), (F, M))$$

which maps $(id_G, \boldsymbol{\varphi})$ to $(f, \boldsymbol{\varphi})$ is bijective. Thus $(f, id_{f^*(M)}) : (G, f^*(M)) \to (F, M)$ is a cartesian morphism and $p_{\mathcal{T}} : \mathcal{MOD} \to \mathcal{T}$ is a prefibered category. We set $f^*(F, M) = (G, f^*(M))$ and $\alpha_f(F, M) = (f, id_{f^*(M)})$.

For morphisms $f : G \to F$, $g : F \to H$ of $\mathcal{T}$ and $(H, L) \in \mathrm{Ob}\,\mathcal{MOD}_H$, we note that $f^*g^*(H, L) = f^*(F, g^*(L)) = (G, f^*(g^*(L))) = (G, (gf)^*(L)) = (gf)^*(H, L)$. Define $c_{g,f}(H, L)$ to be the identity morphism of $f^*g^*(H, L) = (gf)^*(H, L)$. Then, the following diagram commutes.

$$\begin{tikzcd}
f^*g^*(H, L) \arrow[r, "\alpha_f(g^*(H, L))"] \arrow[d, "c_{g,f}(H, L)"] & g^*(H, L) \arrow[d, "\alpha_g(H, L)"] \\
(fg)^*(H, L) \arrow[r, "\alpha_{fg}(H, L)"] & (H, L)
\end{tikzcd}$$

Therefore $p_{\mathcal{T}} : \mathcal{MOD} \to \mathcal{T}$ is a fibered category. □

Remark 10.2.3 *(1) For a morphism $f : G \to F$ of $\mathcal{T}$, the functor $f^* : \mathcal{MOD}_F \to \mathcal{MOD}_G$ is given by $f^*(F, M) = (G, f^*(M))$ and $f^*(id_F, \boldsymbol{\varphi}) = (id_G, f^*(\boldsymbol{\varphi}))$ for $M \in \mathcal{M}od(F)$ and $\boldsymbol{\varphi} \in \mathcal{M}od(F)(M, N)$.*

(2) A morphism $(f, \boldsymbol{\varphi}) : (G, N) \to (F, M)$ of $\mathcal{MOD}$ is cartesian if and only if $\boldsymbol{\varphi} : f^*(M) \to N$ is an isomorphism of F-modules.

Proposition 10.2.4 *$\mathcal{MOD}$ has coproducts.*

Proof. Let $((F_i, M_i))_{i\in I}$ be a family of objects of $\mathcal{MOD}$. Put $F = \coprod_{i\in I} F_i$ and we denote by $\iota_i : F_i \to F$ be the canonical morphism. Define an F-module $M : \mathcal{C}_F \to \mathcal{M}od(\mathcal{C}, \mathcal{M})$ as follows. For $(R^*, \rho) \in \mathrm{Ob}\,\mathcal{C}_F$, we set $M(R^*, \rho) = M_i(R^*, \rho)$ if $\rho \in F_i(R^*)$. If $\lambda : (R^*, \rho) \to (S^*, \sigma)$ is a morphism of $\mathcal{C}_F$ such that $\rho \in F_i(R^*)$, then $\sigma = F(\lambda)(\rho) = F_i(\lambda)(\rho) \in F_i(S^*)$. We define $M(\lambda) : M(R^*, \rho) \to M(S^*, \sigma)$ by $M(\lambda) = M_i(\lambda)$ if $\rho \in F_i(R^*)$. We note that, if (R^*, ρ) is an F_i-model, then $\iota_i^*(M)(R^*, \rho) = M(R^*, (\iota_i)_{R^*}(\rho)) = M_i(R^*, \rho)$. Define a morphism $\boldsymbol{\iota}_i : \iota_i^*(M) \to M_i$ of F_i-modules by $(\boldsymbol{\iota}_i)_{(R^*,\rho)} = id_{M_i(R^*,\rho)} : \iota_i^*(M)(R^*, \rho) \to M_i(R^*, \rho)$.

Let $((g_i, \boldsymbol{\gamma}_i) : (F_i, M_i) \to (G, N))_{i\in I}$ be a family of morphism of $\mathcal{MOD}$. There exists unique morphism $g : F \to G$ satisfying $g\iota_i = g_i$ for any $i \in I$. Since $g^*(N)(R^*, \rho) = N(R^*, g_{R^*}(\iota_i)_{R^*}(\rho)) = N(R^*, (g_i)_{R^*}(\rho)) = g_i^*(N)(R^*, \rho)$ for $(R^*, \rho) \in \mathrm{Ob}\,\mathcal{C}_F$ if $\rho \in F_i(R^*)$, we define a morphism $\boldsymbol{\gamma} : g^*(N) \to M$ of F-modules by $\boldsymbol{\gamma}_{(R^*,\rho)} = (\boldsymbol{\gamma}_i)_{(R^*,\rho)}$. Since $\iota_i^* g^*(N)(R^*, \rho) = N(R^*, g_{R^*}(\iota_i)_{R^*}(\rho)) = N(R^*, (g_i)_{R^*}(\rho))$ if $\rho \in F_i(R^*)$, it follows $(\boldsymbol{\iota}_i\iota_i^*(\boldsymbol{\gamma}))_{(R^*,\rho)} = (\boldsymbol{\iota}_i)_{(R^*,\rho)}\iota_i^*(\boldsymbol{\gamma})_{(R^*,\rho)} = \boldsymbol{\gamma}_{(R^*,(\iota_i)_{R^*}(\rho))} = (\boldsymbol{\gamma}_i)_{(R^*,\rho)}$, thai is, $\boldsymbol{\iota}_i\iota_i^*(\boldsymbol{\gamma}) = \boldsymbol{\gamma}_i$. Hence we have $(g, \boldsymbol{\gamma})(\iota_i, \boldsymbol{\iota}_i) = (g_i, \boldsymbol{\gamma}_i)$. Suppose that a morphism $(g', \boldsymbol{\gamma}') : (F, M) \to (G, N)$ also satisfies $(g', \boldsymbol{\gamma}')(\iota_i, \boldsymbol{\iota}_i) = (g_i, \boldsymbol{\gamma}_i)$ for any $i \in I$. Since $g'\iota_i = g\iota_i$ for all $i \in I$, it follows $g' = g$. Then, we have

$$\boldsymbol{\gamma}'_{(R^*,(\iota_i)_{R^*}(\rho))} = \iota_i^*(\boldsymbol{\gamma}')_{(R^*,\rho)} = (\boldsymbol{\iota}_i)_{(R^*,\rho)}\iota_i^*(\boldsymbol{\gamma}')_{(R^*,\rho)} = (\boldsymbol{\gamma}_i)_{(R^*,\rho)} = (\boldsymbol{\iota}_i)_{(R^*,\rho)}\iota_i^*(\boldsymbol{\gamma})_{(R^*,\rho)} = \boldsymbol{\gamma}_{(R^*,(\iota_i)_{R^*}(\rho))}$$

for any $i \in I$ and $(R^*, \rho) \in \mathcal{C}_{F_i}$. Therefore $\boldsymbol{\gamma}' = \boldsymbol{\gamma}$. □

The following assertion is straightforward.

Lemma 10.2.5 *For $R^* \in \mathrm{Ob}\,\mathcal{C}$, let $(\boldsymbol{M}_i)_{i\in I}$ be a family of objects of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}$ and put $\boldsymbol{M}_i = (R^*, M_i^*, \alpha_i)$. Assume that a coproduct $\coprod_{i\in I} M_i^*$ in $\mathcal{M}$ exists and we denote by $\iota_j : M_j^* \to \coprod_{i\in I} M_i^*$ the inclusion map to j-summand for $j \in I$. Let $\alpha : \left(\coprod_{i\in I} M_i^*\right) \otimes_{K^*} R^* \to \coprod_{i\in I} M_i^*$ be the unique map that makes the following diagram commute for any $j \in I$.*

$$\begin{tikzcd}
M_j^* \otimes_{K^*} R^* \arrow[r, "\alpha_j"] \arrow[d, "\iota_j \otimes_{K^*} id_{R^*}"] & M_j^* \arrow[d, "\iota_j"] \\
\left(\coprod_{i\in I} M_i^*\right) \otimes_{K^*} R^* \arrow[r, "\alpha"] & \coprod_{i\in I} M_i^*
\end{tikzcd}$$

Then $\left(R^*, \coprod_{i\in I} M_i^*, \alpha\right)$ *is a coproduct of* $(\boldsymbol{M}_i)_{i\in I}$ *in* $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$*. Hence if* $\mathcal{M}$ *has coproducts,* $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$ *has coproducts for any* $R^* \in \operatorname{Ob}\mathcal{C}$*.*

Proposition 10.2.6 *If* $\mathcal{M}$ *has coproducts,* $f^* : \mathcal{M}od(F) \to \mathcal{M}od(G)$ *has a left adjoint for any morphism* $f : G \to F$ *of* $\mathcal{T}$*.*

Proof. Let $N : \mathcal{C}_G \to \mathcal{M}od(\mathcal{C},\mathcal{M})$ be a G-module. For $(R^*,\rho) \in \operatorname{Ob}\mathcal{C}_F$, we put

$$f_!(N)(R^*,\rho) = \coprod_{\kappa\in f_{R^*}^{-1}(\rho)} N(R^*,\kappa).$$

Here, $\coprod_{\kappa\in f_{R^*}^{-1}(\rho)} N(R^*,\kappa)$ denotes a coproduct in $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$. We also denote by

$$i(N)_{(R^*,\kappa)} : N(R^*,\kappa) \to \coprod_{\kappa\in f_{R^*}^{-1}(\rho)} N(R^*,\kappa)$$

the inclusion morphism into κ-component below. If $\lambda \in \mathcal{C}_F((R^*,\rho),(S^*,\sigma))$, then $F(U_F(\lambda))(\rho) = \sigma$ and it follows that $\kappa \in f_{R^*}^{-1}(\rho)$ implies $G(U_F(\lambda))(\kappa) \in f_{S^*}^{-1}(\sigma)$. For $\kappa \in G(R^*)$, let $\lambda_\kappa \in \mathcal{C}_G((R^*,\kappa),(S^*,G(U_F(\lambda))(\kappa)))$ be the morphism satisfying $U_G(\lambda_\kappa) = U_F(\lambda)$. Let

$$f_!(N)(\lambda) : f_!(N)(R^*,\rho) = \coprod_{\kappa\in f_{R^*}^{-1}(\rho)} N(R^*,\kappa) \to \coprod_{\nu\in f_{S^*}^{-1}(\sigma)} N(S^*,\nu) = f_!(N)(S^*,\sigma)$$

be the unique morphism that make the following diagram commute for any $\kappa \in f_{R^*}^{-1}(\rho)$.

$$\begin{tikzcd}
N(R^*,\kappa) \arrow[r, "N(\lambda_\kappa)"] \arrow[d, "i(N)_{(R^*,\kappa)}"] & N(S^*,G(\lambda)(\kappa)) \arrow[d, "i(N)_{(S^*,G(\lambda)(\kappa))}"] \\
\coprod_{\kappa\in f_{R^*}^{-1}(\rho)} N(R^*,\kappa) \arrow[r, "f_!(N)(\lambda)"] & \coprod_{\nu\in f_{S^*}^{-1}(\sigma)} N(S^*,\nu)
\end{tikzcd}$$

For a morphism $\boldsymbol{\varphi} : M \to N$ of G-modules, we define a morphism $f_!(\boldsymbol{\varphi}) : f_!(M) \to f_!(N)$ of F-modules as follows. For $(R^*,\rho) \in \operatorname{Ob}\mathcal{C}_F$, let

$$f_!(\boldsymbol{\varphi})_{(R^*,\rho)} : f_!(M)(R^*,\rho) = \coprod_{\kappa\in f_{R^*}^{-1}(\rho)} M(R^*,\kappa) \to \coprod_{\kappa\in f_{R^*}^{-1}(\rho)} N(R^*,\kappa) = f_!(N)(R^*,\rho)$$

be the unique morphism that makes the following diagram commute.

$$\begin{tikzcd}
M(R^*,\kappa) \arrow[r, "\boldsymbol{\varphi}_{(R^*,\kappa)}"] \arrow[d, "i(M)_{(R^*,\kappa)}"] & N(R^*,\kappa) \arrow[d, "i(N)_{(R^*,\kappa)}"] \\
\coprod_{\kappa\in f_{R^*}^{-1}(\rho)} M(R^*,\kappa) \arrow[r, "f_!(\boldsymbol{\varphi})_{(R^*,\rho)}"] & \coprod_{\kappa\in f_{R^*}^{-1}(\rho)} N(R^*,\kappa)
\end{tikzcd}$$

We define a map $\operatorname{Ad} : \mathcal{M}od(G)(N, f^*(M)) \to \mathcal{M}od(F)(f_!(N), M)$ as follows. For $\boldsymbol{\varphi} \in \mathcal{M}od(G)(N, f^*(M))$ and $(R^*,\rho) \in \operatorname{Ob}\mathcal{C}_F$, let

$${}^t\boldsymbol{\varphi}_{(R^*,\rho)} : f_!(N)(R^*,\rho) = \coprod_{\kappa\in f_{R^*}^{-1}(\rho)} N(R^*,\kappa) \to M(R^*,\rho)$$

be the unique morphism that makes the following diagram commute for every $\kappa \in f_{R^*}^{-1}(\rho)$.

$$\begin{tikzcd}
N(R^*,\kappa) \arrow[r, "\boldsymbol{\varphi}_{(R^*,\kappa)}"] \arrow[d, "i(N)_{(R^*,\kappa)}"] & M(R^*,f_{R^*}(\kappa)) \arrow[d, equal] \\
\coprod_{\kappa\in f_{R^*}^{-1}(\rho)} N(R^*,\kappa) \arrow[r, "{}^t\boldsymbol{\varphi}_{(R^*,\rho)}"] & M(R^*,\rho)
\end{tikzcd}$$

Then, the naturality of φ implies the naturality of ${}^t\varphi$. Put $\mathrm{Ad}(\varphi) = {}^t\varphi$. The inverse of Ad is given as follows. For $\psi \in \mathcal{M}od(F)(f_!(N), M)$ and $(T^*, \tau) \in \mathrm{Ob}\,\mathcal{C}_G$, let $\tilde{\psi}_{(T^*,\tau)} : N(T^*, \tau) \to M(T^*, f_{T^*}(\tau)) = f^*(M)(T^*, \tau)$ be the following composition.

$$N(T^*, \tau) \xrightarrow{i(N)_{(T^*,\tau)}} \coprod_{\kappa \in f_{T^*}^{-1}(f_{T^*}(\tau))} N(T^*, \kappa) = f_!(N)(T^*, f_{T^*}(\tau)) \xrightarrow{\psi_{(T^*, f_{T^*}(\tau))}} M(T^*, f_{T^*}(\tau))$$

Then, the naturality of ψ implies the naturality of $\tilde{\psi}$. Put $\mathrm{Ad}^{-1}(\psi) = \tilde{\psi}$. □

Remark 10.2.7 *The unit $\bar{\eta} : id_{\mathcal{M}od(G)} \to f^* f_!$ and the counit $\bar{\varepsilon} : f_! f^* \to id_{\mathcal{M}od(F)}$ are given as follows. For $N \in \mathrm{Ob}\,\mathcal{M}od(G)$ and $(T^*, \tau) \in \mathrm{Ob}\,\mathcal{C}_G$,*

$$(\bar{\eta}_N)_{(T^*,\tau)} : N(T^*, \tau) \to \coprod_{\kappa \in f_{T^*}^{-1}(f_{T^*}(\tau))} N(T^*, \kappa) = f^* f_!(N)(T^*, \tau)$$

is the inclusion morphism into τ-component. For $M \in \mathrm{Ob}\,\mathcal{M}od(F)$ and $(R^*, \rho) \in \mathrm{Ob}\,\mathcal{C}_F$,

$$(\bar{\varepsilon}_M)_{(R^*,\rho)} : f_! f^*(M)(R^*, \rho) = \coprod_{\kappa \in f_{R^*}^{-1}(\rho)} M(R^*, f_{R^*}(\kappa)) \to M(R^*, \rho)$$

is the morphism induced by the identity morphism of $M(R^*, \rho)$.

Since $\mathcal{MOD}_F$ is identified with $\mathcal{M}od(F)^{op}$ and the inverse image functor $f^* : \mathcal{MOD}_F \to \mathcal{MOD}_G$ is identified with the functor $(f^*)^{op} : \mathcal{M}od(F)^{op} \to \mathcal{M}od(G)^{op}$, (10.2.6) implies the following result.

Corollary 10.2.8 *If $\mathcal{M}$ has coproducts, the inverse image functor $f^* : \mathcal{MOD}_F \to \mathcal{MOD}_G$ has a right adjoint for any morphism $f : G \to F$ of $\mathcal{T}$.*

Remark 10.2.9 *The unit $\eta_f : id_{\mathcal{M}od_F} \to f_! f^*$ and the counit $\varepsilon_f : f^* f_! \to id_{\mathcal{M}od_G}$ of the adjunction $f^* \dashv f_!$ are given as follows. For $X \in \mathrm{Ob}\,\mathcal{M}od(F)$, $(\eta_f)_{(F,X)} = (id_F, \bar{\varepsilon}_X) : (F, X) \to (F, f^* f_!(X)) = f^* f_!(F, X)$. For $N \in \mathrm{Ob}\,\mathcal{M}od(G)$, $(\varepsilon_f)_{(G,N)} : f_! f^*(G, N) = (id_G, \bar{\eta}_N) : (G, f_! f^*(N)) \to (G, N)$.*

Proposition 10.2.10 *Suppose that $\mathcal{M}$ is complete. For any morphism $f : G \to F$ of $\mathcal{T}$, $f^* : \mathcal{M}od(F) \to \mathcal{M}od(G)$ has a right adjoint.*

Proof. Let N be a G-module. For $(T^*, t) \in \mathrm{Ob}\,\mathcal{C}_G$, we put $N(T^*, t) = (T^*, N^*_{(T^*,t)}, \mu_{(T^*,t)})$. Then, we have $p_{\mathcal{M}} NQ\langle\alpha, (T^*, t)\rangle = p_{\mathcal{M}} N(T^*, t) = N^*_{(T^*,t)}$ for $(R^*, x) \in \mathrm{Ob}\,\mathcal{C}_F$ and $\langle\alpha, (T^*, t)\rangle \in \mathrm{Ob}\,((R^*, x)\downarrow\tilde{f})$. Let

$$\left(N^*_f(R^*, x) \xrightarrow{\pi_{\langle\alpha,(T^*,t)\rangle}} p_{\mathcal{M}} NQ\langle\alpha, (T^*, t)\rangle\right)_{\langle\alpha,(T^*,t)\rangle \in \mathrm{Ob}\,((R^*,x)\downarrow\tilde{f})}$$

be a limiting cone of composition $((R^*, x)\downarrow\tilde{f}) \xrightarrow{Q} \mathcal{C}_G \xrightarrow{N} \mathcal{M}od(\mathcal{C}, \mathcal{M}) \xrightarrow{p_{\mathcal{M}}} \mathcal{M}$. Let $\tau : \langle\alpha, (T^*, t)\rangle \to \langle\beta, (S^*, s)\rangle$ be a morphism of $((R^*, x)\downarrow\tilde{f})$ and put $NQ(\tau) = (\tau, \tilde{\tau})$. Then, we have $p_{\mathcal{M}} NQ(\tau)\pi_{\langle\alpha,(T^*,t)\rangle} = \pi_{\langle\beta,(S^*,s)\rangle}$, $\tau U_F(\alpha) = U_F(\beta)$ and the following diagram commutes.

$$\begin{tikzcd}
N^*_{(T^*,t)} \otimes_{K^*} T^* \arrow[r, "\mu_{(T^*,t)}"] \arrow[d, "\tilde{\tau} \otimes_{K^*} \tau"] & N^*_{(T^*,t)} \arrow[d, "\tilde{\tau}"] \\
N^*_{(S^*,s)} \otimes_{K^*} S^* \arrow[r, "\mu_{(S^*,s)}"] & N^*_{(S^*,s)}
\end{tikzcd}$$

Thus we have

$$\begin{aligned}
p_{\mathcal{M}} NQ(\tau)\mu_{(T^*,t)}(\pi_{\langle\alpha,(T^*,t)\rangle} \otimes_{K^*} U_F(\alpha)) &= \tilde{\tau}\mu_{(T^*,t)}(\pi_{\langle\alpha,(T^*,t)\rangle} \otimes_{K^*} U_F(\alpha)) \\
&= \mu_{(S^*,s)}(\tilde{\tau} \otimes_{K^*} \tau)(\pi_{\langle\alpha,(T^*,t)\rangle} \otimes_{K^*} U_F(\alpha)) \\
&= \mu_{(S^*,s)}(p_{\mathcal{M}} NQ(\tau)\pi_{\langle\alpha,(T^*,t)\rangle} \otimes_{K^*} \tau U_F(\alpha)) \\
&= \mu_{(S^*,s)}(\pi_{\langle\beta,(S^*,s)\rangle} \otimes_{K^*} U_F(\beta)).
\end{aligned}$$

Hence $\left(N_f^*(R^*,x)\otimes_{K^*}R^* \xrightarrow{\mu_{(T^*,t)}(\pi_{\langle\alpha,(T^*,t)\rangle}\otimes_{K^*}U_F(\alpha))} N^*_{(T^*,t)}\right)_{\langle\alpha,(T^*,t)\rangle\in \mathrm{Ob}\,((R^*,x)\downarrow\tilde{f})}$ is a cone of $p_{\mathcal{M}}NQ$ and there exists unique map $\rho_{(R^*,x)} : N_f^*(R^*,x)\otimes_{K^*}R^* \to N_f^*(R^*,x)$ satisfying

$$\pi_{\langle\alpha,(T^*,t)\rangle}\rho_{(R^*,x)} = \mu_{(T^*,t)}(\pi_{\langle\alpha,(T^*,t)\rangle}\otimes_{K^*}U_F(\alpha))$$

for any object $\langle\alpha,(T^*,t)\rangle$ of $((R^*,x)\downarrow\tilde{f})$. Let $\nu_{T^*} : T^*\otimes_{K^*}T^*\to T^*$ be the multiplication of T^*. Then

$$\begin{aligned}
\pi_{\langle\alpha,(T^*,t)\rangle}\rho_{(R^*,x)}(\rho_{(R^*,x)}\otimes_{K^*} id_{R^*}) &= \mu_{(T^*,t)}(\pi_{\langle\alpha,(T^*,t)\rangle}\otimes_{K^*}U_F(\alpha))(\rho_{(R^*,x)}\otimes_{K^*} id_{R^*})\\
&= \mu_{(T^*,t)}(\pi_{\langle\alpha,(T^*,t)\rangle}\rho_{(R^*,x)}\otimes_{K^*}U_F(\alpha))\\
&= \mu_{(T^*,t)}(\mu_{(T^*,t)}(\pi_{\langle\alpha,(T^*,t)\rangle}\otimes_{K^*}U_F(\alpha))\otimes_{K^*}U_F(\alpha))\\
&= \mu_{(T^*,t)}(\mu_{(T^*,t)}\otimes_{K^*} id_{T^*})(\pi_{\langle\alpha,(T^*,t)\rangle}\otimes_{K^*}U_F(\alpha)\otimes_{K^*}U_F(\alpha))\\
&= \mu_{(T^*,t)}(id_{N^*_{(T^*,t)}}\otimes_{K^*}\nu_{T^*})(\pi_{\langle\alpha,(T^*,t)\rangle}\otimes_{K^*}U_F(\alpha)\otimes_{K^*}U_F(\alpha))\\
&= \mu_{(T^*,t)}(\pi_{\langle\alpha,(T^*,t)\rangle}\otimes_{K^*}U_F(\alpha))(id_{N_f^*(R^*,x)}\otimes_{K^*}\nu_{R^*})\\
&= \pi_{\langle\alpha,(T^*,t)\rangle}\rho_{(R^*,x)}(id_{N_f^*(R^*,x)}\otimes_{K^*}\nu_{R^*})
\end{aligned}$$

for any $\langle\alpha,(T^*,t)\rangle\in\mathrm{Ob}\,((R^*,x)\downarrow\tilde{f})$. Therefore $\rho_{(R^*,x)}(\rho_{(R^*,x)}\otimes_{K^*} id_{R^*}) = \rho_{(R^*,x)}(id_{N_f^*(R^*,x)}\otimes_{K^*}\nu_{R^*})$. For a K^*-module N^* and a K^*-algebra R^*, let $i_{N^*,R^*} : N^*\to N^*\otimes_{K^*}R^*$ be a map defined by $i_{N^*,R^*}(x) = x\otimes_{K^*}1$. Then, for any $\langle\alpha,(T^*,t)\rangle\in\mathrm{Ob}\,((R^*,x)\downarrow\tilde{f})$, we have

$$\begin{aligned}
\pi_{\langle\alpha,(T^*,t)\rangle}\rho_{(R^*,x)}i_{N_f^*(R^*,x),R^*} &= \mu_{(T^*,t)}(\pi_{\langle\alpha,(T^*,t)\rangle}\otimes_{K^*}U_F(\alpha))i_{N_f^*(R^*,x),R^*}\\
&= \mu_{(T^*,t)}i_{N^*_{(T^*,t)},T^*}\pi_{\langle\alpha,(T^*,t)\rangle} = \pi_{\langle\alpha,(T^*,t)\rangle}.
\end{aligned}$$

Thus $\rho_{(R^*,x)}i_{N_f^*(R^*,x),R^*} = id_{N_f^*(R^*,x)}$ and $\rho_{(R^*,x)} : N_f^*(R^*,x)\otimes_{K^*}R^*\to N_f^*(R^*,x)$ is a right R^*-module structure of $N_f^*(R^*,x)$. We note that $(U_F(\alpha),\pi_{\langle\alpha,(T^*,t)\rangle}) : (R^*,N_f^*(R^*,x),\rho_{(R^*,x)})\to(T^*,N^*_{(T^*,t)},\mu_{(T^*,t)})$ is a morphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})$.

Recall that a morphism $\gamma : (S^*,y)\to(R^*,x)$ of $\mathcal{C}_F$ defines a functor $(\gamma\downarrow id_{\tilde{f}}) : ((R^*,x)\downarrow\tilde{f})\to((S^*,y)\downarrow\tilde{f})$ by $(\gamma\downarrow id_{\tilde{f}})\langle\alpha,(T^*,t)\rangle = \langle\alpha\gamma,(T^*,t)\rangle$. Hence we have a cone

$$\left(N_f^*(R^*,x)\xrightarrow{\pi_{(\gamma\downarrow id_{\tilde{f}})\langle\alpha,(T^*,t)\rangle}} p_{\mathcal{M}}NQ(\gamma\downarrow id_{\tilde{f}})\langle\alpha,(T^*,t)\rangle\right)_{\langle\alpha,(T^*,t)\rangle\in\mathrm{Ob}\,((R^*,x)\downarrow\tilde{f})}.$$

Since $p_{\mathcal{M}}NQ(\gamma\downarrow id_{\tilde{f}})\langle\alpha,(T^*,t)\rangle = p_{\mathcal{M}}N(T^*,t)$ for any $\langle\alpha,(T^*,t)\rangle\in((R^*,x)\downarrow\tilde{f})$, there exists unique morphism $N_f^*(\gamma) : N_f^*(S^*,y)\to N_f^*(R^*,x)$ such that $\pi_{\langle\alpha,(T^*,t)\rangle}N_f^*(\gamma) = \pi_{(\gamma\downarrow id_{\tilde{f}})\langle\alpha,(T^*,t)\rangle}$ for any $\langle\alpha,(T^*,t)\rangle\in\mathrm{Ob}\,((R^*,x)\downarrow\tilde{f})$. It is easy to verify that this choice of $N_f^*(\gamma)$ makes N_f^* a functor. Since

$$\begin{aligned}
\pi_{\langle\alpha,(T^*,t)\rangle}\rho_{(R^*,x)}(N_f^*(\gamma)\otimes_{K^*}U_F(\gamma)) &= \mu_{(T^*,t)}(\pi_{\langle\alpha,(T^*,t)\rangle}\otimes_{K^*}U_F(\alpha))(N_f^*(\gamma)\otimes_{K^*}U_F(\gamma))\\
&= \mu_{(T^*,t)}(\pi_{\langle\alpha,(T^*,t)\rangle}N_f^*(\gamma)\otimes_{K^*}U_F(\alpha\gamma))\\
&= \mu_{(T^*,t)}(\pi_{(\gamma\downarrow id_{\tilde{f}})\langle\alpha,(T^*,t)\rangle}\otimes_{K^*}U_F(\alpha\gamma))\\
&= \pi_{(\gamma\downarrow id_{\tilde{f}})\langle\alpha,(T^*,t)\rangle}\rho_{(S^*,y)} = \pi_{\langle\alpha,(T^*,t)\rangle}N_f^*(\gamma)\rho_{(S^*,y)}
\end{aligned}$$

for any $\langle\alpha,(T^*,t)\rangle\in\mathrm{Ob}\,((R^*,x)\downarrow\tilde{f})$, we have $\rho_{(R^*,x)}(N_f^*(\gamma)\otimes_{K^*}U_F(\gamma)) = N_f^*(\gamma)\rho_{(S^*,y)}$, in other words, $(U_F(\gamma),N_f^*(\gamma)) : (S^*,N_f^*(S^*,y),\rho_{(S^*,y)})\to(R^*,N_f^*(R^*,x),\rho_{(R^*,x)})$ is a morphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})$. We define an F-module $f_*(N)$ by $f_*(N)(R^*,x) = (R^*,N_f^*(R^*,x),\rho_{(R^*,x)})$ and $f_*(N)(\gamma) = (U_F(\gamma),N_f^*(\gamma))$.

For each $(T^*,t)\in\mathrm{Ob}\,\mathcal{C}_G$, we define a morphism $\varepsilon_{(T^*,t)} : f_*(N)\tilde{f}(T^*,t)\to N(T^*,t)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})$ by $\varepsilon_{(T^*,t)} = (id_{T^*},\pi_{\langle id_{\tilde{f}(T^*,t)},(T^*,t)\rangle})$. We note that a morphism $\lambda : (T^*,t)\to(S^*,s)$ of $\mathcal{C}_G$ defines a morphism $\lambda : \langle id_{\tilde{f}(T^*,t)},(T^*,t)\rangle\to\langle\tilde{f}(\lambda),(S^*,s)\rangle$ of $(\tilde{f}(T^*,t)\downarrow\tilde{f})$. It follows from the definition of $f_*(N)\tilde{f}(\lambda) : f_*(N)\tilde{f}(T^*,t)\to f_*(N)\tilde{f}(S^*,s)$ that

$$\begin{aligned}
\varepsilon_{(S^*,s)}f_*(N)\tilde{f}(\lambda) &= (id_{S^*},\pi_{\langle id_{\tilde{f}(S^*,s)},(S^*,s)\rangle})(U_F(\tilde{f}(\lambda)),N_f^*(\tilde{f}(\lambda))) = (U_G(\lambda),\pi_{\langle id_{\tilde{f}(S^*,s)},(S^*,s)\rangle}N_f^*(\tilde{f}(\lambda)))\\
&= (U_G(\lambda),\pi_{\langle id_{\tilde{f}(S^*,s)},(S^*,s)\rangle}N_f^*(\tilde{f}(\lambda))) = (U_G(\lambda),\pi_{(\tilde{f}(\lambda)\downarrow id_{\tilde{f}})\langle id_{\tilde{f}(S^*,s)},(S^*,s)\rangle})\\
&= (U_G(\lambda),\pi_{\langle\tilde{f}(\lambda),(S^*,s)\rangle}) = (U_G(\lambda),p_{\mathcal{M}}NQ(\lambda)\pi_{\langle id_{\tilde{f}(T^*,t)},(T^*,t)\rangle}) = N(\lambda)\varepsilon_{(T^*,t)}.
\end{aligned}$$

Therefore we have a morphism $\varepsilon : f_*(N)\tilde{f} \to N$ of F-modules.

Let $M : \mathcal{C}_F \to \mathcal{M}od(\mathcal{C}, \mathcal{M})$ be an F-module and $\zeta : M\tilde{f} \to N$ a morphism of G-modules. For $(R^*, x) \in \mathrm{Ob}\, \mathcal{C}_F$, we put $M(R^*, x) = (R^*, M^*_{(R^*,x)}, \chi_{(R^*,x)})$. If $\varphi : \langle \alpha, (T^*, t)\rangle \to \langle \beta, (S^*, s)\rangle$ is a morphism of $((R^*, x)\downarrow\tilde{f})$, since

$$NQ(\varphi)\zeta_{(T^*,t)}M(\alpha) = \zeta_{(S^*,s)}M\tilde{f}Q(\varphi)M(\alpha) = \zeta_{(S^*,s)}M(\tilde{f}(Q(\varphi))\alpha) = \zeta_{(S^*,s)}M(\beta),$$

$\left(M(R^*, x) \xrightarrow{\zeta_{(T^*,t)}M(\alpha)} NQ\langle \alpha, (T^*, t)\rangle\right)_{\langle \alpha,(T^*,t)\rangle \in \mathrm{Ob}\,((R^*,x)\downarrow\tilde{f})}$ is a cone of $NQ : ((R^*, x)\downarrow\tilde{f}) \to \mathcal{M}od(\mathcal{C}, \mathcal{M})$. We have unique morphism $\bar{\zeta}_{(R^*,x)} : M^*_{(R^*,x)} \to N^*_f(R^*, x)$ such that $\pi_{\langle \alpha,(T^*,t)\rangle}\bar{\zeta}_{(R^*,x)} = p_{\mathcal{M}}(\zeta_{(T^*,t)}M(\alpha))$ for any $\langle \alpha, (T^*, t)\rangle \in \mathrm{Ob}\,((R^*, x)\downarrow\tilde{f})$. Define $\check{\zeta}_{(R^*,x)} : M(R^*, x) \to f_*(N)(R^*, x)$ by $\check{\zeta}_{(R^*,x)} = (id_{R^*}, \bar{\zeta}_{(R^*,x)})$. Let $\gamma : (L^*, y) \to (R^*, x)$ be a morphism of $\mathcal{C}_F$. For each $\langle \alpha, (T^*, t)\rangle \in \mathrm{Ob}\,((R^*, x)\downarrow\tilde{f})$, since

$$\begin{aligned}\pi_{\langle \alpha,(T^*,t)\rangle}\bar{\zeta}_{(R^*,x)}p_{\mathcal{M}}(M(\gamma)) &= p_{\mathcal{M}}(\zeta_{(T^*,t)}M(\alpha))p_{\mathcal{M}}(M(\gamma)) = p_{\mathcal{M}}(\zeta_{(T^*,t)}M(\alpha\gamma))\\ &= \pi_{(\gamma\downarrow id_{\tilde{f}})\langle \alpha,(T^*,t)\rangle}\bar{\zeta}_{(L^*,y)} = \pi_{\langle \alpha,(T^*,t)\rangle}N^*_f(\gamma)\bar{\zeta}_{(L^*,y)},\end{aligned}$$

we have $\bar{\zeta}_{(R^*,x)}p_{\mathcal{M}}(M(\gamma)) = N^*_f(\gamma)\bar{\zeta}_{(L^*,y)}$, which implies the naturality of $\check{\zeta}$. Since diagrams

$$\begin{array}{ccc} N^*_f(R^*, x) \otimes_{K^*} R^* & \xrightarrow{\pi_{\langle \alpha,(T^*,t)\rangle}\otimes_{K^*}U_F(\alpha)} & N^*_{(T^*,t)} \otimes_{K^*} T^* \\ \Big\downarrow{\scriptstyle \rho_{(R^*,x)}} & & \Big\downarrow{\scriptstyle \mu_{(T^*,t)}} \\ N^*_f(R^*, x) & \xrightarrow{\pi_{\langle \alpha,(T^*,t)\rangle}} & N^*_{(T^*,t)} \end{array}$$

$$\begin{array}{ccccc} M^*_{(R^*,x)} \otimes_{K^*} R^* & \xrightarrow{p_{\mathcal{M}}(M(\alpha))\otimes_{K^*}U_F(\alpha)} & M^*_{\tilde{f}(T^*,t)} \otimes_{K^*} T^* & \xrightarrow{p_{\mathcal{M}}(\zeta_{(T^*,t)})\otimes_{K^*}id_{T^*}} & N^*_{(T^*,t)} \otimes_{K^*} T^* \\ \Big\downarrow{\scriptstyle \chi_{(R^*,x)}} & & \Big\downarrow{\scriptstyle \chi_{\tilde{f}(T^*,t)}} & & \Big\downarrow{\scriptstyle \mu_{(T^*,t)}} \\ M^*_{(R^*,x)} & \xrightarrow{p_{\mathcal{M}}(M(\alpha))} & M^*_{\tilde{f}(T^*,t)} & \xrightarrow{p_{\mathcal{M}}(\zeta_{(T^*,t)})} & N^*_{(T^*,t)} \end{array}$$

commute for any $(R^*, x) \in \mathrm{Ob}\, \mathcal{C}_F$ and $\langle \alpha, (T^*, t)\rangle \in \mathrm{Ob}\,((R^*, x)\downarrow\tilde{f})$, we have

$$\begin{aligned}\pi_{\langle \alpha,(T^*,t)\rangle}\rho_{(R^*,x)}(\bar{\zeta}_{(R^*,x)} \otimes_{K^*} id_{R^*}) &= \mu_{(T^*,t)}(\pi_{\langle \alpha,(T^*,t)\rangle} \otimes_{K^*} U_F(\alpha))(\bar{\zeta}_{(R^*,x)} \otimes_{K^*} id_{R^*})\\ &= \mu_{(T^*,t)}(\pi_{\langle \alpha,(T^*,t)\rangle}\bar{\zeta}_{(R^*,x)} \otimes_{K^*} U_F(\alpha))\\ &= \mu_{(T^*,t)}(p_{\mathcal{M}}(\zeta_{(T^*,t)})p_{\mathcal{M}}(M(\alpha)) \otimes_{K^*} U_F(\alpha))\\ &= \mu_{(T^*,t)}(p_{\mathcal{M}}(\zeta_{(T^*,t)}) \otimes_{K^*} id_{T^*})(p_{\mathcal{M}}(M(\alpha)) \otimes_{K^*} U_F(\alpha))\\ &= p_{\mathcal{M}}(\zeta_{(T^*,t)}M(\alpha))\chi_{(R^*,x)} = \pi_{\langle \alpha,(T^*,t)\rangle}\bar{\zeta}_{(R^*,x)}\chi_{(R^*,x)}.\end{aligned}$$

It follows $\rho_{(R^*,x)}(\bar{\zeta}_{(R^*,x)} \otimes_{K^*} id_{R^*}) = \bar{\zeta}_{(R^*,x)}\chi_{(R^*,x)}$, that is, $\check{\zeta} : M \to f_*(N)$ is a morphism of F-modules. Thus we have a map

$$ad_{M,N} : \mathcal{M}od(G)(M\tilde{f}, N) \to \mathcal{M}od(F)(M, f_*(N))$$

which maps ζ to $\check{\zeta}$.

Finally, we show that $ad_{M,N}$ is the inverse of the map $\mathcal{M}od(F)(M, f_*(N)) \to \mathcal{M}od(G)(M\tilde{f}, N)$ given by $\xi \mapsto \varepsilon\xi_{\tilde{f}}$. For $\zeta \in \mathcal{M}od(G)(M\tilde{f}, N)$ and $(T^*, t) \in \mathrm{Ob}\, \mathcal{C}_G$, we have

$$\varepsilon_{(T^*,t)}ad_{N,M}(\zeta)_{\tilde{f}(T^*,t)} = (id_{T^*}, \pi_{\langle id_{\tilde{f}(T^*,t)},(T^*,t)\rangle}\bar{\zeta}_{\tilde{f}(T^*,t)}) = (id_{T^*}, p_{\mathcal{M}}(\zeta_{(T^*,t)}M(id_{\tilde{f}(T^*,t)}))) = \zeta_{(T^*,t)}.$$

For $\xi \in \mathcal{M}od(F)(M, f_*(N))$ and $(R^*, x) \in \mathrm{Ob}\, \mathcal{C}_F$, we put $\bar{\xi}_{(R^*,x)} = p_{\mathcal{M}}(\xi_{(R^*,x)}) : p_{\mathcal{M}}(M(R^*, x)) \to N^*_f(R^*, x)$ and $\bar{\zeta}_{(R^*,x)} = p_{\mathcal{M}}(ad_{M,N}(\varepsilon\xi_{\tilde{f}})_{(R^*,x)}) : p_{\mathcal{M}}(M(R^*, x)) \to N^*_f(R^*, x)$. For each $\langle \alpha, (T^*, t)\rangle \in \mathrm{Ob}\,((R^*, x)\downarrow\tilde{f})$, by the naturality of ξ, it follows that

$$\begin{aligned}\pi_{\langle \alpha,(T^*,t)\rangle}\bar{\zeta}_{(R^*,x)} &= p_{\mathcal{M}}(\varepsilon_{(T^*,t)}\xi_{\tilde{f}(T^*,t)}M(\alpha)) = p_{\mathcal{M}}(\varepsilon_{(T^*,t)}f_*(N)(\alpha)\xi_{(R^*,x)}) = \pi_{\langle id_{\tilde{f}(T^*,t)},(T^*,t)\rangle}N^*_f(\alpha)\bar{\xi}_{(R^*,x)}\\ &= \pi_{(\alpha\downarrow id_{\tilde{f}})\langle id_{\tilde{f}(T^*,t)},(T^*,t)\rangle}\bar{\xi}_{(R^*,x)} = \pi_{\langle \alpha,(T^*,t)\rangle}p_{\mathcal{M}}(\xi_{(R^*,x)})\end{aligned}$$

and this implies $p_{\mathcal{M}}(ad_{M,N}(\varepsilon\xi_{\tilde{f}})_{(R^*,x)}) = p_{\mathcal{M}}(\xi_{(R^*,x)})$ for any $(R^*, x) \in \mathrm{Ob}\, \mathcal{C}_F$. Therefore $ad_{M,N}(\varepsilon\xi_{\tilde{f}}) = \xi$. $\square$

Corollary 10.2.11 *$p_{\mathcal{T}} : \mathcal{MOD} \to \mathcal{T}$ is a bifibered category if $\mathcal{M}$ is complete.*

10.3 Cartesian closedness of the fibered category of functorial modules

Suppose that $\mathcal{M}$ has coproducts. It follows from (10.2.8) and (6.4.1) that the presheaf $F^X_{(h_{K^*},N)} : \mathcal{MOD}_1^{op} \to \mathcal{S}et$ on $\mathcal{MOD}_1$ is representable for any $X \in \mathrm{Ob}\,\mathcal{T}$ and $(h_{K^*},N) \in \mathrm{Ob}\,\mathcal{MOD}_1$.

For an object X of $\mathcal{T}$ and an object (h_{K^*},N) of $\mathcal{MOD}_1$, it follows from (6.4.1) that $(h_{K^*},N)^X$ is given by $o_{X!}o_X^*(h_{K^*},N) = (h_{K^*}, o_{X!}(N\tilde{o}_X))$. Let us denote by $u_{R^*} : K^* \to R^*$ the unit of $R^*\mathrm{Ob}\,\mathcal{C}$. Since $\tilde{o}_X(R^*,\kappa) = (R^*,u_{R^*})$ for $(R^*,\kappa) \in \mathrm{Ob}\,\mathcal{C}_X$, $o_{X!}(N\tilde{o}_X)$ is given by

$$o_{X!}(N\tilde{o}_X)(R^*,u_{R^*}) = \coprod_{\kappa\in X(R^*)} N\tilde{o}_X(R^*,\kappa) = \coprod_{\kappa\in X(R^*)} N(R^*,u_{R^*})$$

for $(R^*,u_{R^*}) \in \mathrm{Ob}\,\mathcal{C}_{h_{K^*}}$.

Let $\varphi : N \to M$ be a morphism of h_{K^*}-modules. It follows from (6.4.4) that

$$(id_{h_{K^*}},\varphi)^X : (h_{K^*},M)^X = (h_{K^*}, o_{X!}(M\tilde{o}_X)) \to (h_{K^*}, o_{X!}(N\tilde{o}_X)) = (h_{K^*},N)^X$$

is given by $(id_{h_{K^*}},\varphi)^X = (id_{h_{K^*}}, o_{X!}o_X^*(\varphi))$. If we denote by

$$i_X(N)_{(R^*,\rho)} : N(R^*,u_{R^*}) = N\tilde{o}_X(R^*,\rho) \longrightarrow \coprod_{\kappa\in X(R^*)} N\tilde{o}_X(R^*,\kappa) = o_{X!}(N\tilde{o}_X)(R^*,u_{R^*})$$

the inclusion morphism to ρ-summand for $\rho \in X(R^*)$, the following diagram commutes for $(R^*,\rho) \in \mathrm{Ob}\,\mathcal{C}_X$.

$$\begin{tikzcd}
N(R^*,u_{R^*}) \arrow[r, "\varphi_{(R^*,u_{R^*})}"] \arrow[d, "i_X(N)_{(R^*,\rho)}"] & M(R^*,u_{R^*}) \arrow[d, "i_X(M)_{(R^*,\rho)}"] \\
o_{X!}(N\tilde{o}_X)(R^*,u_{R^*}) \arrow[r, "o_{X!}o_X^*(\varphi)_{(R^*,u_{R^*})}"] & o_{X!}(M\tilde{o}_X)(R^*,u_{R^*})
\end{tikzcd}$$

Let $f : X \to Y$ be a morphism of $\mathcal{T}$ and N an h_{K^*}-module. For $(R^*,u_{R^*}) \in \mathrm{Ob}\,\mathcal{C}_{h_{K^*}}$, we define a morphism

$$N^f_{(R^*,u_{R^*})} : o_{X!}(N\tilde{o}_X)(R^*,u_{R^*}) = \coprod_{\kappa\in X(R^*)} N(R^*,u_{R^*}) \longrightarrow \coprod_{\kappa\in Y(R^*)} N(R^*,u_{R^*}) = o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*})$$

of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$ to be the unique homomorphism that makes the following diagram commute for any $\rho \in X(R^*)$.

$$\begin{tikzcd}
N(R^*,u_{R^*}) \arrow[r, "id_{N(R^*,u_{R^*})}"] \arrow[d, "i_X(N)_{(R^*,\rho)}"] & N(R^*,u_{R^*}) \arrow[d, "i_Y(N)_{(R^*,f_{R^*}(\rho))}"] \\
o_{X!}(N\tilde{o}_X)(R^*,u_{R^*}) \arrow[r, "N^f_{(R^*,u_{R^*})}"] & o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*})
\end{tikzcd}$$

Let $\lambda : (R^*,u_{R^*}) \to (S^*,u_{S^*})$ a morphism of $\mathcal{C}_{h_{K^*}}$ and ρ an element of $X(R^*)$. The left rectangle of the following diagram commutes by the definition of $o_{X!}(N\tilde{o}_X)(\lambda)$. Since $N^f_{(R^*,u_{R^*})}i_X(N)_{(R^*,\rho)} = i_Y(N)_{(R^*,f_{R^*}(\rho))}$ and $N^f_{(S^*,u_{S^*})}i_X(N)_{(S^*,X(U_X(\lambda))(\rho))} = i_Y(N)_{(S^*,f_{S^*}(X(U_X(\lambda))(\rho)))}$ by the definition of $N^f_{(R^*,u_{R^*})}$ and $N^f_{(S^*,u_{S^*})}$, the outer rectangle of the following diagram commutes by the definition of $o_{Y!}(N\tilde{o}_Y)(\lambda)$. Thus the right rectangle of the following diagram is commutative.

$$\begin{tikzcd}
N(R^*,u_{R^*}) \arrow[r, "i_X(N)_{(R^*,\rho)}"] \arrow[d, "N(\lambda)"] & o_{X!}(N\tilde{o}_X)(R^*,u_{R^*}) \arrow[r, "N^f_{(R^*,u_{R^*})}"] \arrow[d, "o_{X!}(N\tilde{o}_X)(\lambda)"] & o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*}) \arrow[d, "o_{Y!}(N\tilde{o}_Y)(\lambda)"] \\
N(S^*,u_{S^*}) \arrow[r, "i_X(N)_{(S^*,X(U_X(\lambda))(\rho))}"] & o_{X!}(N\tilde{o}_X)(S^*,u_{S^*}) \arrow[r, "N^f_{(S^*,u_{S^*})}"] & o_{Y!}(N\tilde{o}_Y)(S^*,u_{S^*})
\end{tikzcd}$$

Hence we have a morphism $N^f : o_{X!}(N\tilde{o}_X) \to o_{Y!}(N\tilde{o}_Y)$ of h_{K^*}-modules.

Proposition 10.3.1 *Let $f : X \to Y$ be a morphism of $\mathcal{T}$ and N an h_{K^*}-module. The morphism*

$$(h_{K^*},N)^f : (h_{K^*},N)^Y = (h_{K^*}, o_{Y!}(N\tilde{o}_Y)) \to (h_{K^*}, o_{X!}(N\tilde{o}_X)) = (h_{K^*},N)^X$$

of $\mathcal{MOD}_1$ is given by $(h_{K^*},N)^f = (id_{h_{K^*}}, N^f)$.

Proof. It follows from (1) of (6.4.7) that $(h_{K^*},N)^f$ is the following composition.

$$(h_{K^*},N)^Y = o_{Y!}o_Y^*(h_{K^*},N) \xrightarrow{\eta^X_{o_{Y!}o_Y^*(h_{K^*},N)}} o_{X!}o_X^*o_{Y!}o_Y^*(h_{K^*},N) = o_{X!}f^*o_Y^*o_{Y!}o_Y^*(h_{K^*},N)$$
$$\xrightarrow{o_{X!}f^*\left(\varepsilon^Y_{o_Y^*(h_{K^*},N)}\right)} o_{X!}f^*o_Y^*(h_{K^*},N) = o_{X!}o_X^*(h_{K^*},N) = (h_{K^*},N)^X$$

We recall from (10.2.9) that

$$\eta^X_{o_{Y!}o_Y^*(h_{K^*},N)} = (id_{h_{K^*}}, \bar{\varepsilon}_{o_{Y!}(N\tilde{o}_Y)}) : (h_{K^*}, o_{Y!}(N\tilde{o}_Y)) \longrightarrow (h_{K^*}, o_{X!}(o_{Y!}(N\tilde{o}_Y)\tilde{o}_X))$$
$$o_{X!}f^*(\varepsilon^Y_{o_Y^*(h_{K^*},N)}) = (id_{h_{K^*}}, o_{X!}f^*(\bar{\eta}_{o_Y^*(N)})) : (h_{K^*}, o_{X!}(o_{Y!}(N\tilde{o}_Y)\tilde{o}_X)) \longrightarrow (h_{K^*}, o_{X!}(N\tilde{o}_X)).$$

It follows from (10.2.7) that

$$\left(\bar{\varepsilon}_{o_{Y!}(N\tilde{o}_Y)}\right)_{(R^*,u_{R^*})} : o_{X!}(o_{Y!}(N\tilde{o}_Y)\tilde{o}_X)(R^*,u_{R^*}) = \coprod_{\tau\in X(R^*)} o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*}) \longrightarrow o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*})$$

is the morphism induced by the identity morphism of $o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*})$ for $(R^*,u_{R^*})\in \operatorname{Ob}\mathcal{C}_{h_{K^*}}$ and that

$$f^*(\bar{\eta}_{o_Y^*(N)})_{(R^*,\rho)} : (N\tilde{o}_X)(R^*,\rho) = N(R^*,u_{R^*}) \longrightarrow \coprod_{\kappa\in Y(R^*)} N(R^*,u_{R^*}) = o_{Y!}(N\tilde{o}_Y)\tilde{o}_X(R^*,\rho)$$

is the inclusion morphism to $f_{R^*}(\rho)$-summand for $(R^*,\rho)\in\operatorname{Ob}\mathcal{C}_X$. Hence, for $(R^*,u_{R^*})\in\operatorname{Ob}\mathcal{C}_{h_{K^*}}$, we have

$$o_{X!}(N\tilde{o}_X)(R^*,u_{R^*}) = \coprod_{\tau\in X(R^*)} N(R^*,u_{R^*}), \qquad o_{X!}(o_{Y!}(N\tilde{o}_Y)\tilde{o}_X)(R^*,u_{R^*}) = \coprod_{\tau\in X(R^*)} o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*})$$

and the following commutative diagram.

$$\begin{tikzcd}
N(R^*,u_{R^*}) \arrow[r, "f^*(\bar{\eta}_{o_Y^*(N)})_{(R^*,\rho)}"] \arrow[d, "i_X(N)_{(R^*,\rho)}"] & o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*}) \arrow[r, "id_{o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*})}"] \arrow[d, "i_X(o_{Y!}(N\tilde{o}_Y))_{(R^*,\rho)}"] & o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*}) \\
o_{X!}(N\tilde{o}_X)(R^*,u_{R^*}) \arrow[r, "\left(o_{X!}f^*\left(\bar{\eta}_{o_Y^*(N)}\right)\right)_{(R^*,u_{R^*})}"] & o_{X!}(o_{Y!}(N\tilde{o}_Y)\tilde{o}_X)(R^*,u_{R^*}) \arrow[ur, bend right, "\left(\bar{\varepsilon}_{o_{Y!}(N\tilde{o}_Y)}\right)_{(R^*,u_{R^*})}"'] &
\end{tikzcd}$$

Thus a composition

$$o_{X!}(N\tilde{o}_X)(R^*,u_{R^*}) \xrightarrow{\left(o_{X!}f^*\left(\bar{\eta}_{o_Y^*(N)}\right)\right)_{(R^*,u_{R^*})}} o_{X!}(o_{Y!}(N\tilde{o}_Y)\tilde{o}_X) \xrightarrow{\left(\bar{\varepsilon}_{o_{Y!}(N\tilde{o}_Y)}\right)_{(R^*,u_{R^*})}} o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*})$$

maps ρ-summand of $o_{X!}(N\tilde{o}_X)(R^*,u_{R^*})$ to $f_{R^*}(\rho)$-summand of $o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*})$ and this implies the assertion. □

Lemma 10.3.2 *Let $f : X\to Z$ and $g : Y\to W$ be morphisms of $\mathcal{T}$. For an h_{K^*}-module N, a morphism*

$$((h_{K^*},N)^f)^g : ((h_{K^*},N)^Z)^W = (h_{K^*}, o_{W!}(o_{Z!}(N\tilde{o}_Z)\tilde{o}_W)) \to (h_{K^*}, o_{Y!}(o_{X!}(N\tilde{o}_X)\tilde{o}_Y)) = ((h_{K^*},N)^X)^Y$$

is given by $((h_{K^*},N)^f)^g = (id_{h_{K^*}}, o_{W!}o_W^*(N^f)o_{X!}(N\tilde{o}_X)^g)$.

Proof. Since $(h_{K^*},N)^X = (h_{K^*}, o_{X!}(N\tilde{o}_X))$, we have $((h_{K^*},N)^X)^g = (id_{h_{K^*}}, o_{X!}(N\tilde{o}_X)^g)$ by (10.3.1). We also have $((h_{K^*},N)^f)^W = (id_{h_{K^*}}, N^f)^W = (id_{h_{K^*}}, o_{W!}o_W^*(N^f))$. Hence $((h_{K^*},N)^f)^g = ((h_{K^*},N)^X)^g((h_{K^*},N)^f)^W$ implies the assertion. □

We investigate the morphism $o_{W!}o_W^*(N^f)o_{X!}(N\tilde{o}_X)^g : o_{Y!}(o_{X!}(N\tilde{o}_X)\tilde{o}_Y) \to o_{W!}(o_{Z!}(N\tilde{o}_Z)\tilde{o}_W)$ below. Put $M = o_{X!}(N\tilde{o}_X)$ and $L = o_{Z!}(N\tilde{o}_Z)$, then the following diagram is commutative.

$$\begin{tikzcd}
N(R^*,u_{R^*}) \arrow[r, "id_{N(R^*,u_{R^*})}"] \arrow[d, "i_X(N)_{(R^*,\kappa)}"] & N(R^*,u_{R^*}) \arrow[r, "id_{N(R^*,u_{R^*})}"] \arrow[d, "i_X(N)_{(R^*,\kappa)}"] & N(R^*,u_{R^*}) \arrow[d, "i_Z(N)_{(R^*,f_{R^*}(\kappa))}"] \\
M(R^*,u_{R^*}) \arrow[r, "id_{M(R^*,u_{R^*})}"] \arrow[d, "i_Y(M)_{(R^*,\rho)}"] & M(R^*,u_{R^*}) \arrow[r, "N^f_{(R^*,u_{R^*})}"] \arrow[d, "i_W(M)_{(R^*,g_{R^*}(\rho))}"] & L(R^*,u_{R^*}) \arrow[d, "i_W(L)_{(R^*,g_{R^*}(\rho))}"] \\
o_{Y!}(M\tilde{o}_Y)(R^*,u_{R^*}) \arrow[r, "M^g_{(R^*,u_{R^*})}"] & o_{W!}(M\tilde{o}_W)(R^*,u_{R^*}) \arrow[r, "o_{W!}o_W^*(N^f)_{(R^*,u_{R^*})}"] & o_{W!}(L\tilde{o}_W)(R^*,u_{R^*})
\end{tikzcd}$$

Note that

$$o_{Y!}(o_{X!}(N\tilde{o}_X)\tilde{o}_Y)(R^*,u_{R^*}) = o_{Y!}(M\tilde{o}_Y)(R^*,u_{R^*}) = \coprod_{\tau\in Y(R^*)}\coprod_{\sigma\in X(R^*)} N(R^*,u_{R^*})$$
$$o_{W!}(o_{Z!}(N\tilde{o}_Z)\tilde{o}_W)(R^*,u_{R^*}) = o_{W!}(L\tilde{o}_W)(R^*,u_{R^*}) = \coprod_{\tau\in W(R^*)}\coprod_{\sigma\in Z(R^*)} N(R^*,u_{R^*}).$$

Since $i_Y(M)_{(R^*,\rho)}i_X(N)_{(R^*,\kappa)}$ is the inclusion morphism to "ρ-κ-summand" of $o_{Y!}(o_{X!}(N\tilde{o}_X)\tilde{o}_Y)(R^*,u_{R^*})$ and $i_W(L)_{(R^*,g_{R^*}(\rho))}i_Z(N)_{(R^*,f_{R^*}(\kappa))}$ is the inclusion morphism to "$g_{R^*}(\rho)$-$f_{R^*}(\kappa)$-summand" of $o_{W!}(o_{Z!}(N\tilde{o}_Z)\tilde{o}_W)$, $(o_{W!}o_W^*(N^f)_{(R^*,u_{R^*})}o_{X!}(N\tilde{o}_X)^g_{(R^*,u_{R^*})}$ maps "ρ-κ-summand" of $o_{Y!}(o_{X!}(N\tilde{o}_X)\tilde{o}_Y)(R^*,u_{R^*})$ to "$g_{R^*}(\rho)$-$f_{R^*}(\kappa)$-summand" of $o_{W!}(o_{Z!}(N\tilde{o}_Z)\tilde{o}_W)$.

For $(h_{K^*},N)\in \mathrm{Ob}\,\mathcal{MOD}_1$ and $X\in\mathrm{Ob}\,\mathcal{T}$, $\epsilon^X_{(h_{R^*},N)}:((h_{K^*},N)^X)^X\to(h_{K^*},N)^X$ is described as follows. First of all, recall that

$$(h_{K^*},N)^X = o_{X!}o_X^*(h_{K^*},N) = (h_{K^*},o_{X!}(N\tilde{o}_X))$$
$$((h_{K^*},N)^X)^X = o_{X!}o_X^*o_{X!}o_X^*(h_{K^*},N) = (h_{K^*},o_{X!}(o_{X!}(N\tilde{o}_X)\tilde{o}_X)).$$

It follows from (6.4.12) and (10.2.9) that $\epsilon^X_{(h_{R^*},N)} = o_{X!}\big(\varepsilon^X_{o_X^*(h_{K^*},N)}\big) = \big(id_{h_{K^*}}, o_{X!}\big(\bar{\eta}_{o_X^*(N)}\big)\big)$. Since

$$\big(\bar{\eta}_{o_X^*(N)}\big)_{(R^*,\tau)} : (N\tilde{o}_X)(R^*,\tau)\longrightarrow \coprod_{\kappa\in X(R^*)} N(R^*,u_{R^*}) = o_X^*o_{X!}o_X^*(N)(R^*,\tau)$$

is the inclusion morphism to τ-component for $(R^*,\tau)\in\mathrm{Ob}\,\mathcal{C}_X$ and the following diagram commutes.

$$\begin{tikzcd}
(N\tilde{o}_X)(R^*,\tau) \arrow[r, "\left(\bar{\eta}_{o_X^*(N)}\right)_{(R^*,\tau)}"] \arrow[d, "i_X(N)_{(R^*,\tau)}"] & o_{X!}(N\tilde{o}_X)\tilde{o}_X(R^*,\tau) \arrow[d, "i_X(o_{X!}(N\tilde{o}_X))_{(R^*,\tau)}"] \\
o_{X!}(N\tilde{o}_X)(R^*,u_{R^*}) \arrow[r, "o_{X!}\left(\bar{\eta}_{o_X^*(N)}\right)_{(R^*,u_{R^*})}"] & o_{X!}(o_{X!}(N\tilde{o}_X)\tilde{o}_X)(R^*,u_{R^*})
\end{tikzcd}$$

Since $o_{X!}(o_{X!}(N\tilde{o}_X)\tilde{o}_X)(R^*,u_{R^*}) = \coprod_{\kappa\in X(R^*)}\coprod_{\sigma\in X(R^*)} N(R^*,u_{R^*})$ and $i_X(o_{X!}(N\tilde{o}_X))_{(R^*,\tau)}\big(\bar{\eta}_{o_X^*(N)}\big)_{(R^*,\tau)}$ is the inclusion morphism to "τ-τ-summand", $o_{X!}\big(\bar{\eta}_{o_X^*(N)}\big)_{(R^*,u_{R^*})}$ maps τ-summand of $o_{X!}(N\tilde{o}_X)(R^*,u_{R^*})$ to "τ-τ-summand" of $o_{X!}(o_{X!}(N\tilde{o}_X)\tilde{o}_X)(R^*,u_{R^*})$.

Proposition 10.3.3 *Suppose that $\mathcal{M}$ has coproducts. Then,*

$$\theta^{X,Y}(h_{K^*},N):((h_{K^*},N)^X)^Y = o_{Y!}o_Y^*o_{X!}o_X^*(h_{K^*},N)\to o_{X\times Y!}o_{X\times Y}^*(h_{K^*},N) = (h_{K^*},N)^{X\times Y}$$

is an isomorphism for any $(h_{K^*},N)\in\mathrm{Ob}\,\mathcal{MOD}_1$ and $X,Y\in\mathrm{Ob}\,\mathcal{T}$

Proof. We recall that $\theta^{X,Y}(h_{K^*},N)$ is defined to be the following composition.

$$((h_{K^*},N)^X)^Y \xrightarrow{((h_{K^*},N)^{\mathrm{pr}_X})^{\mathrm{pr}_Y}} ((h_{K^*},N)^{X\times Y})^{X\times Y} \xrightarrow{\epsilon^{X\times Y}_{(h_{K^*},N)}} (h_{K^*},N)^{X\times Y}$$

Note that we have the following equalities.

$$\begin{aligned}
((h_{K^*},N)^X)^Y &= o_{Y!}o_Y^*o_{X!}o_X^*(h_{K^*},N) = (h_{K^*},o_{Y!}(o_{X!}(N\tilde{o}_X)\tilde{o}_Y))\\
((h_{K^*},N)^{X\times Y})^{X\times Y} &= o_{X\times Y!}o_{X\times Y}^*o_{X\times Y!}o_{X\times Y}^*(h_{K^*},N) = (h_{K^*},o_{X\times Y!}(o_{X\times Y!}(N\tilde{o}_{X\times Y})\tilde{o}_{X\times Y}))\\
(h_{K^*},N)^{X\times Y} &= o_{X\times Y!}o_{X\times Y}^*(h_{K^*},N) = (h_{K^*},o_{X\times Y!}(N\tilde{o}_{X\times Y}))\\
\epsilon^{X\times Y}_{(h_{K^*},N)} &= o_{X\times Y!}\big(\varepsilon^{X\times Y}_{o_{X\times Y}^*(h_{K^*},N)}\big) = \big(id_{h_{K^*}},o_{X\times Y!}\big(\bar{\eta}_{o_{X\times Y}^*(N)}\big)\big)\\
((h_{K^*},N)^{\mathrm{pr}_X})^{\mathrm{pr}_Y} &= (id_{h_{K^*}},o_{Y!}o_Y^*(N^{\mathrm{pr}_X})o_{X\times Y!}(N\tilde{o}_{X\times Y})^{\mathrm{pr}_Y})
\end{aligned}$$

The following diagram is commutative for any $(\sigma,\tau)\in(X\times Y)(R^*)$.

$$\begin{tikzcd}
(N\tilde{o}_{X\times Y})(R^*,(\sigma,\tau)) \arrow[rr, "i_{X\times Y}(N)_{(R^*,(\sigma,\tau))}"] \arrow[d, "(\bar{\eta}_{o^*_{X\times Y}(N)})_{(R^*,(\sigma,\tau))}"] & & o_{X\times Y!}(N\tilde{o}_{X\times Y})(R^*,u_{R^*}) \arrow[d, "o_{X\times Y!}(\bar{\eta}_{o^*_{X\times Y}(N)})_{(R^*,u_{R^*})}"'] \\
o_{X\times Y!}(N\tilde{o}_{X\times Y})\tilde{o}_{X\times Y}(R^*,(\sigma,\tau)) \arrow[rr, "i_{X\times Y}(o_{X\times Y!}(N\tilde{o}_{X\times Y}))_{(R^*,(\sigma,\tau))}"] \arrow[d, "N^{\mathrm{pr}_X}_{\tilde{o}_{X\times Y}(R^*,(\sigma,\tau))}"] & & o_{X\times Y!}(o_{X\times Y!}(N\tilde{o}_{X\times Y})\tilde{o}_{X\times Y})(R^*,u_{R^*}) \arrow[d, "o_{Y!}o^*_Y(N^{\mathrm{pr}_X})o_{X\times Y!}(N\tilde{o}_{X\times Y})^{\mathrm{pr}_Y}_{(R^*,u_{R^*})}"'] \\
o_{Y!}(N\tilde{o}_Y)\tilde{o}_{X\times Y}(R^*,(\sigma,\tau)) \arrow[rr, "i_Y(o_{Y!}(N\tilde{o}_Y))_{(R^*,\tau)}"] & & o_{Y!}(o_{X!}(N\tilde{o}_X)\tilde{o}_Y)(R^*,u_{R^*})
\end{tikzcd}$$

Since $\left(\bar{\eta}_{o^*_{X\times Y}(N)}\right)_{(R^*,(\sigma,\tau))} : (N\tilde{o}_{X\times Y})(R^*,(\sigma,\tau)) \to o_{X\times Y!}(N\tilde{o}_{X\times Y})\tilde{o}_{X\times Y}(R^*,(\sigma,\tau))$ is the inclusion morphism to (σ,τ)-summand and $N^{\mathrm{pr}_X}_{(R^*,u_{R^*})} : o_{X\times Y!}(N\tilde{o}_{X\times Y})(R^*,u_{R^*}) \to o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*})$ maps (σ,τ)-summand to σ-summand which is mapped by $i_Y(o_{Y!}(N\tilde{o}_Y))_{(R^*,\tau)} : o_{Y!}(N\tilde{o}_Y)(R^*,u_{R^*}) \to o_{Y!}(o_{X!}(N\tilde{o}_X)\tilde{o}_Y)(R^*,u_{R^*})$ to "τ-σ-summand", the following composition is an isomorphism.

$$o_{X\times Y!}(N\tilde{o}_{X\times Y})(R^*,u_{R^*}) \xrightarrow{o_{X\times Y!}\left(\bar{\eta}_{o^*_{X\times Y}(N)}\right)_{(R^*,u_{R^*})}} o_{X\times Y!}(o_{X\times Y!}(N\tilde{o}_{X\times Y})\tilde{o}_{X\times Y})(R^*,u_{R^*})$$
$$\xrightarrow{o_{Y!}\left(N^{\mathrm{pr}_X}_{\tilde{o}_Y}\right)_{(R^*,u_{R^*})} o_{X\times Y!}(N\tilde{o}_{X\times Y})^{\mathrm{pr}_Y}_{(R^*,u_{R^*})}} o_{Y!}(o_{X!}(N\tilde{o}_X)\tilde{o}_Y)(R^*,u_{R^*})$$

In fact, the above composition is identified with the lower horizontal morphism of the following diagram which is an isomorphism

$$\begin{tikzcd}
(N\tilde{o}_{X\times Y})(R^*,(\sigma,\tau)) \arrow[r, "id_{N(R^*,u_{R^*})}"] \arrow[d, "i_{X\times Y}(N)_{(R^*,(\sigma,\tau))}"] & N(R^*,u_{R^*}) \arrow[d, "\text{inclusion to }\tau\text{-}\sigma\text{-summand}"] \\
\coprod_{(\sigma,\tau)\in(X\times Y)(R^*)} N(R^*,u_{R^*}) \arrow[r] & \coprod_{\tau\in Y(R^*)}\coprod_{\sigma\in X(R^*)} N(R^*,u_{R^*})
\end{tikzcd}$$

Therefore, $\theta^{X,Y}(h_{K^*},N)$ is an isomorphism. □

Proposition 10.3.4 *Suppose that $\mathcal{M}$ has coproducts and is complete. Then, $p_{\mathcal{T}} : \mathcal{MOD} \to \mathcal{T}$ is a cartesian closed fibered category.*

Proof. Clearly, $\mathcal{T}$ has finite products with terminal object $1 = h_{K^*}$. It follows from (10.2.8) and (6.4.1) that the presheaf F^X_N on $\mathcal{F}_1$ is representable for any $X \in \mathrm{Ob}\,\mathcal{T}$ and $N \in \mathrm{Ob}\,\mathcal{F}_1$. It follows from (10.2.11) and (6.3.1) that the presheaf $F_{X,M}$ on $\mathcal{F}_1^{op}$ is representable for any $X \in \mathrm{Ob}\,\mathcal{T}$ and $M \in \mathrm{Ob}\,\mathcal{F}_1$. Then, assertion follows from (6.5.6) and (10.3.3). □

10.4 Embedding of the fibered category of affine modules

Assume that $\mathcal{M}$ satisfies the condition (ii) of (10.1.1) below.

For an object $\boldsymbol{M}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$, we define an h_{R^*}-module $\widehat{\boldsymbol{M}} : \mathcal{C}_{h_{R^*}} \to \mathcal{M}od(\mathcal{C},\mathcal{M})$ as follows. Set $\widehat{\boldsymbol{M}}(T^*,\lambda) = \lambda^*(\boldsymbol{M})$ for $(T^*,\lambda) \in \mathrm{Ob}\,\mathcal{C}_{h_{R^*}}$. If $\xi \in \mathcal{C}_{h_{R^*}}((T^*,\lambda),(S^*,\nu))$, then we have $\nu = \xi\lambda$. We define $\widehat{\boldsymbol{M}}(\xi) : \lambda^*(\boldsymbol{M}) \to \nu^*(\boldsymbol{M})$ to be a composition $\lambda^*(\boldsymbol{M}) \xrightarrow{\alpha_\xi(\lambda^*(\boldsymbol{M}))} \xi^*(\lambda^*(\boldsymbol{M})) \xrightarrow{c_{\lambda,\xi}(\boldsymbol{M})^{-1}} (\xi\lambda)^*(\boldsymbol{M}) = \nu^*(\boldsymbol{M})$. In other words, $\widehat{\boldsymbol{M}}(\xi) = (\xi, id_{M^*}\,\widehat{\otimes}_{R^*}\,\xi)$ if $\boldsymbol{M} = (R^*,M^*,\alpha)$.

Proposition 10.4.1 *Let $\boldsymbol{M}$ be an object of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$. Then, $\widehat{\boldsymbol{M}} : \mathcal{C}_{h_{R^*}} \to \mathcal{M}od(\mathcal{C},\mathcal{M})$ is continuous.*

Proof. Since $p_{\mathcal{C}}\widehat{\boldsymbol{M}} = U_{h_{R^*}} : \mathcal{C}_{h_{R^*}} \to \mathcal{C}$, $p_{\mathcal{C}}\widehat{\boldsymbol{M}}$ is continuous. Since $p_{\mathcal{M}}\widehat{\boldsymbol{M}} = \widehat{F}_{M^*}U_{h_{R^*}} : \mathcal{C}_{h_{R^*}} \to \mathcal{M}$, $p_{\mathcal{M}}\widehat{\boldsymbol{M}} : \mathcal{C}_{h_{R^*}} \to \mathcal{M}$ is continuous by (8.2.3). □

Let $\boldsymbol{\varphi} : \boldsymbol{N} \to \boldsymbol{M}$ be a morphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})$ and put $\lambda = p_{\mathcal{C}}(\boldsymbol{\varphi})$. We denote by $\bar{\boldsymbol{\varphi}} : \lambda^*(\boldsymbol{N}) \to \boldsymbol{M}$ the unique morphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$ such that a composition $\boldsymbol{N} \xrightarrow{\alpha_\lambda(\boldsymbol{N})} \lambda^*(\boldsymbol{N}) \xrightarrow{\bar{\boldsymbol{\varphi}}} \boldsymbol{M}$ coincides with $\boldsymbol{\varphi}$. For $(T^*,\nu) \in \mathrm{Ob}\,\mathcal{C}_{h_{R^*}}$, we note that $h^*_\lambda(\widehat{\boldsymbol{N}})(T^*,\nu) = \widehat{\boldsymbol{N}}(T^*,\nu\lambda) = (\nu\lambda)^*(\boldsymbol{N})$. Define a morphism $\widehat{\boldsymbol{\varphi}}_\lambda : h^*_\lambda(\widehat{\boldsymbol{N}}) \to \widehat{\boldsymbol{M}}$ of h_{R^*}-modules by defining $\widehat{\boldsymbol{\varphi}}_{\lambda(T^*,\nu)} : h^*_\lambda(\widehat{\boldsymbol{N}})(T^*,\nu) \to \widehat{\boldsymbol{M}}(T^*,\nu)$ to be a composition

$$(\nu\lambda)^*(\boldsymbol{N}) \xrightarrow{c_{\lambda,\nu}(\boldsymbol{N})} \nu^*(\lambda^*(\boldsymbol{N})) \xrightarrow{\nu^*(\bar{\boldsymbol{\varphi}})} \nu^*(\boldsymbol{M}).$$

If $S^* = R^*$ and $\lambda = id_{R^*}$, we denote $\widehat{\boldsymbol{\varphi}}_\lambda$ by $\widehat{\boldsymbol{\varphi}}$, then, $\widehat{\boldsymbol{\varphi}}_{(T^*,\nu)} = \nu^*(\boldsymbol{\varphi}) : \nu^*(\boldsymbol{N}) \to \nu^*(\boldsymbol{M})$.

We define a functor $\hat{h} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \to \mathcal{MOD}$ as follows. For $\boldsymbol{M} \in \operatorname{Ob}\mathcal{M}od(\mathcal{C},\mathcal{M})$, put $\hat{h}(\boldsymbol{M}) = (h_{R^*}, \widehat{\boldsymbol{M}})$ if $p_{\mathcal{C}}(\boldsymbol{M}) = R^*$. For $\boldsymbol{\varphi} \in \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}(\boldsymbol{M},\boldsymbol{N})$, we set $\hat{h}(\boldsymbol{\varphi}) = (h_\lambda, \widehat{\boldsymbol{\varphi}}_\lambda) : (h_{R^*}, \widehat{\boldsymbol{M}}) \to (h_{S^*}, \widehat{\boldsymbol{N}})$ if $p_{\mathcal{C}}(\boldsymbol{\varphi}) = \lambda$. It is clear from the definition of $\hat{h}$ that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \arrow[r, "\hat{h}"] \arrow[d, "p_{\mathcal{C}}^{op}"] & \mathcal{MOD} \arrow[d, "p_{\mathcal{T}}"] \\
\mathcal{C}^{op} \arrow[r, "h"] & \mathcal{T}
\end{tikzcd}$$

Here, $h : \mathcal{C}^{op} \to \mathcal{T}$ is the Yoneda embedding given by $h(R^*) = h_{R^*}$.

Remark 10.4.2 *By restricting the domain of $\hat{h}$ to $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{R^*}$, we have a functor $\mathscr{S}_{R^*} : \mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*} \to \mathcal{M}od(h_{R^*})$. That is, $\mathscr{S}_{R^*}$ is defined by $\mathscr{S}_{R^*}(\boldsymbol{M}) = \widehat{\boldsymbol{M}}$ for $\boldsymbol{M} \in \operatorname{Ob}\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$ and $\mathscr{S}_{R^*}(\boldsymbol{\varphi}) = \widehat{\boldsymbol{\varphi}}$ for $\boldsymbol{\varphi} \in \operatorname{Mor}\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$.*

For a morphism $\lambda : S^* \to R^*$ of $\mathcal{C}$ and an object $\boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{S^*}$, there exists unique morphism $c_{\lambda,\hat{h}}(\boldsymbol{N}) : \hat{h}(\lambda^*(\boldsymbol{N})) \to h_\lambda^*(\hat{h}(\boldsymbol{N}))$ of $\mathcal{MOD}_{h_{R^*}}$ that makes the following diagram in $\mathcal{MOD}$ commute (See the paragraph above (6.1.15).).

$$\begin{tikzcd}
\hat{h}(\lambda^*(\boldsymbol{N})) \arrow[r, "\hat{h}(\alpha_\lambda(\boldsymbol{N}))"] \arrow[d, "c_{\lambda,\hat{h}}(\boldsymbol{N})"] & \hat{h}(\boldsymbol{N}) \\
h_\lambda^*(\hat{h}(\boldsymbol{N})) \arrow[ur, "\alpha_{h_\lambda}(\hat{h}(\boldsymbol{N}))"'] &
\end{tikzcd}$$

Proposition 10.4.3 *For any morphism $\lambda : S^* \to R^*$ of $\mathcal{C}$ and any object $\boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{S^*}$, $c_{\lambda,\hat{h}}(\boldsymbol{N})$ is an isomorphism of $\mathcal{MOD}_{h_{R^*}}$. Hence $\hat{h} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \to \mathcal{MOD}$ preserves cartesian morphisms. If $\hat{h}(\boldsymbol{\varphi})$ is a cartesian morphism of $\mathcal{MOD}$ for a morphism $\boldsymbol{\varphi}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}$, $\boldsymbol{\varphi}$ is a cartesian morphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}$.*

Proof. We first note that $\hat{h}(\boldsymbol{N}) = (h_{S^*}, \widehat{\boldsymbol{N}})$, $\hat{h}(\lambda^*(\boldsymbol{N})) = (h_{R^*}, \widehat{\lambda^*(\boldsymbol{N})})$, $h_\lambda^*(\hat{h}(\boldsymbol{N})) = (h_{R^*}, h_\lambda^*(\widehat{\boldsymbol{N}}))$ and that

$$\widehat{\lambda^*(\boldsymbol{N})}(T^*,\nu) = \nu^*(\lambda^*(\boldsymbol{N})), \qquad h_\lambda^*(\widehat{\boldsymbol{N}})(T^*,\nu) = \widehat{\boldsymbol{N}}\tilde{h}_\lambda(T^*,\nu) = \widehat{\boldsymbol{N}}(T^*, h_\lambda(\nu)) = (\nu\lambda)^*(\boldsymbol{N})$$

for $(T^*,\nu) \in \operatorname{Ob}\mathcal{C}_{h_{R^*}}$. If we put $\hat{h}(\alpha_\lambda(\boldsymbol{N})) = (h_\lambda, \psi)$ for a morphism $\psi : h_\lambda(\widehat{\boldsymbol{N}}) \to \widehat{\lambda^*(\boldsymbol{N})}$ of h_{R^*}-modules, ψ is given by $\psi_{(T^*,\nu)} = \boldsymbol{c}_{\lambda,\nu}(\boldsymbol{N}) : h_\lambda^*(\widehat{\boldsymbol{N}})(T^*,\nu) = (\nu\lambda)^*(\boldsymbol{N}) \to \nu^*(\lambda^*(\boldsymbol{N})) = \widehat{\lambda^*(\boldsymbol{N})}(T^*,\nu)$ for $(T^*,\nu) \in \operatorname{Ob}\mathcal{C}_{h_{R^*}}$. Since $\alpha_{h_\lambda}(\hat{h}(\boldsymbol{N})) = (h_\lambda, id_{h_\lambda^*(\widehat{\boldsymbol{N}})}) : h_\lambda^*(\hat{h}(\boldsymbol{N})) \to \hat{h}(\boldsymbol{N})$ by the proof of (10.2.2), $\hat{h}(\alpha_\lambda(\boldsymbol{N}))$ coincides with a composition $\hat{h}(\lambda^*(\boldsymbol{N})) \xrightarrow{(id_{h_{R^*}},\psi)} h_\lambda^*(\hat{h}(\boldsymbol{N})) \xrightarrow{\alpha_{h_\lambda}(\hat{h}(\boldsymbol{N}))} \hat{h}(\boldsymbol{N})$. Hence the unique morphism $c_{\lambda,\hat{h}}(\boldsymbol{N})$ of $\mathcal{MOD}_{h_{R^*}}$ that makes the diagram above commute coincides with $(id_{h_{R^*}}, \psi)$. Since ψ is an isomorphism of h_{R^*}-modules, $c_{\lambda,\hat{h}}(\boldsymbol{N})$ is an isomorphism. Suppose $\hat{h}(\boldsymbol{\varphi})$ is a cartesian morphism of $\mathcal{MOD}$ for a morphism $\boldsymbol{\varphi} : \boldsymbol{M} \to \boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}$. Put $\boldsymbol{\varphi} = (\lambda, \varphi) : \boldsymbol{M} \to \boldsymbol{N}$ and $\boldsymbol{M} = (R^*, M^*, \alpha)$, $\boldsymbol{N} = (S^*, N^*, \beta)$. Then, $\hat{h}(\boldsymbol{\varphi}) = (h_\lambda, \widehat{\boldsymbol{\varphi}}_\lambda)$ and

$$\hat{h}(\boldsymbol{\varphi})_* : \mathcal{MOD}_{h_{R^*}}((h_{R^*}, L), (h_{R^*}, \widehat{\boldsymbol{M}})) \to \mathcal{MOD}_{h_\lambda}((h_{R^*}, L), (h_{S^*}, \widehat{\boldsymbol{N}}))$$

is bijective for any h_{R^*}-module L. Consider the case $L = h_\lambda^*(\widehat{\boldsymbol{N}})$, then there exists a morphism $\boldsymbol{\psi} : \widehat{\boldsymbol{M}} \to h_\lambda^*(\widehat{\boldsymbol{N}})$ of h_{R^*}-modules satisfying $\boldsymbol{\psi}\widehat{\boldsymbol{\varphi}}_\lambda = id_{h_\lambda^*(\widehat{\boldsymbol{N}})}$. Next, consider the case $L = \widehat{\boldsymbol{M}}$. Since

$$\hat{h}(\boldsymbol{\varphi})_*(id_{h_{R^*}}, \widehat{\boldsymbol{\varphi}}_\lambda\boldsymbol{\psi}) = (h_\lambda, \widehat{\boldsymbol{\varphi}}_\lambda\boldsymbol{\psi}\widehat{\boldsymbol{\varphi}}_\lambda) = (h_\lambda, \widehat{\boldsymbol{\varphi}}_\lambda) = \hat{h}(\boldsymbol{\varphi})_*(id_{h_{R^*}}, id_{\widehat{\boldsymbol{M}}}),$$

we have $\widehat{\boldsymbol{\varphi}}_\lambda\boldsymbol{\psi} = id_{\widehat{\boldsymbol{M}}}$. Hence $\widehat{\boldsymbol{\varphi}}_\lambda : h_\lambda^*(\widehat{\boldsymbol{N}}) \to \widehat{\boldsymbol{M}}$ is an isomorphism of h_{R^*}-modules, in particular, $\widehat{\boldsymbol{\varphi}}_{\lambda(R^*,id_{R^*})}$ is an isomorphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$ from $h_\lambda^*(\widehat{\boldsymbol{N}})(R^*, id_{R^*}) = \lambda^*(\boldsymbol{N})$ to $\widehat{\boldsymbol{M}}(R^*, id_{R^*}) = \boldsymbol{M}$. On the other hand, if we denote by $\bar{\boldsymbol{\varphi}} : \lambda^*(\boldsymbol{N}) \to \boldsymbol{M}$ the unique morphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$ such that a composition $\boldsymbol{N} \xrightarrow{\alpha_\lambda(\boldsymbol{N})} \lambda^*(\boldsymbol{N}) \xrightarrow{\bar{\boldsymbol{\varphi}}} \boldsymbol{M}$ coincides with $\boldsymbol{\varphi}$, $\widehat{\boldsymbol{\varphi}}_{\lambda(R^*,id_{R^*})}$ is a composition

$$\lambda^*(\boldsymbol{N}) \xrightarrow{\boldsymbol{c}_{\lambda,id_{R^*}}(\boldsymbol{N})} id_{R^*}^*(\lambda^*(\boldsymbol{N})) \xrightarrow{id_{R^*}^*(\bar{\boldsymbol{\varphi}})} id_{R^*}^*(\boldsymbol{M}).$$

Hence $id_{R^*}^*(\bar{\boldsymbol{\varphi}})$ is an isomorphism and so is $\bar{\boldsymbol{\varphi}}$, which implies that $\boldsymbol{\varphi}$ is cartesian. □

Remark 10.4.4 *It follows from the definition of $\gamma_\lambda(\boldsymbol{N}) : h^*_\lambda(\widehat{\boldsymbol{N}}) \to \widehat{\lambda^*(\boldsymbol{N})}$ that the following diagram commutes for a morphism $\boldsymbol{\varphi} : \boldsymbol{M} \to \boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$.*

$$\begin{tikzcd}
h^*_\lambda \mathscr{S}_{S^*}(\boldsymbol{M}) \arrow[r, "\gamma_\lambda(\boldsymbol{M})"] \arrow[d, "h^*_\lambda \mathscr{S}_{S^*}(\boldsymbol{\varphi})"] & \mathscr{S}_{R^*}\lambda^*(\boldsymbol{M}) \arrow[d, "\mathscr{S}_{R^*}\lambda^*(\boldsymbol{\varphi})"] \\
h^*_\lambda \mathscr{S}_{S^*}(\boldsymbol{N}) \arrow[r, "\gamma_\lambda(\boldsymbol{N})"] & \mathscr{S}_{R^*}\lambda^*(\boldsymbol{N})
\end{tikzcd}$$

Hence we have a natural transformation $\gamma_\lambda : h^*_\lambda \mathscr{S}_{S^*} \to \mathscr{S}_{R^*}\lambda^*$.

Proposition 10.4.5 *$\hat{h} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \to \mathcal{MOD}$ is fully faithful.*

Proof. Suppose that $\hat{h}(\boldsymbol{\varphi}) = \hat{h}(\boldsymbol{\psi})$ for $\boldsymbol{\varphi},\boldsymbol{\psi} \in \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}(\boldsymbol{M},\boldsymbol{N})$. Put $p_{\mathcal{C}}(\boldsymbol{\varphi}) = \lambda$ and $p_{\mathcal{C}}(\boldsymbol{\psi}) = \nu$. Then, we have $h_\lambda = h_\nu : h_{R^*} \to h_{S^*}$ which implies $\lambda = h_\lambda(id_{R^*}) = h_\nu(id_{R^*}) = \nu : S^* \to R^*$. Thus we have $\widehat{\boldsymbol{\varphi}}_\lambda = \hat{\boldsymbol{\psi}}_\lambda : h^*_\lambda(\widehat{\boldsymbol{N}}) \to \widehat{\boldsymbol{M}}$. Let $\bar{\boldsymbol{\varphi}},\bar{\boldsymbol{\psi}} : \lambda^*(\boldsymbol{N}) \to \boldsymbol{M}$ be the morphisms such that $\boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}}\alpha_\lambda(\boldsymbol{N})$, $\boldsymbol{\psi} = \bar{\boldsymbol{\psi}}\alpha_\lambda(\boldsymbol{N})$. Since $\widehat{\boldsymbol{\varphi}}_{\lambda(R^*,id_{R^*})} = \bar{\boldsymbol{\varphi}}$ and $\hat{\boldsymbol{\psi}}_{\lambda(R^*,id_{R^*})} = \bar{\boldsymbol{\psi}}$, it follows that

$$\boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}}\alpha_\lambda(\boldsymbol{N}) = \widehat{\boldsymbol{\varphi}}_{\lambda(R^*,id_{R^*})}\alpha_\lambda(\boldsymbol{N}) = \hat{\boldsymbol{\psi}}_{\lambda(R^*,id_{R^*})}\alpha_\lambda(\boldsymbol{N}) = \bar{\boldsymbol{\psi}}\alpha_\lambda(\boldsymbol{N}) = \boldsymbol{\psi}.$$

Hence $\hat{h}$ is injective.

For objects $\boldsymbol{M}$ and $\boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})$ put $p_{\mathcal{C}}(\boldsymbol{M}) = R^*$ and $p_{\mathcal{C}}(\boldsymbol{N}) = S^*$. For $(f,\boldsymbol{\chi}) \in \mathcal{MOD}(\hat{h}(\boldsymbol{M}),\hat{h}(\boldsymbol{N}))$, put $\lambda = f_{R^*}(id_{R^*}) \in h_{S^*}(R^*) = \mathcal{C}(S^*,R^*)$. Then, we have $f = h_\lambda$ by Yoneda's lemma. We note that $\alpha_{id_{R^*}}(\boldsymbol{M}) : \boldsymbol{M} \to id^*_{R^*}(\boldsymbol{M})$ is an isomorphism. Define a morphism $\boldsymbol{\varphi} : \boldsymbol{N} \to \boldsymbol{M}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})$ to be the following composition.

$$\boldsymbol{N} \xrightarrow{\alpha_\lambda(\boldsymbol{N})} \lambda^*(\boldsymbol{N}) = h^*_\lambda(\widehat{\boldsymbol{N}})(R^*,id_{R^*}) \xrightarrow{\boldsymbol{\chi}_{(R^*,id_{R^*})}} \widehat{\boldsymbol{M}}(R^*,id_{R^*}) = id^*_{R^*}(\boldsymbol{M}) \xrightarrow{\alpha_{id_{R^*}}(\boldsymbol{M})^{-1}} \boldsymbol{M}$$

For $(T^*,\nu) \in \mathrm{Ob}\,\mathcal{C}_{h_{R^*}}$, consider a morphism $\nu : (R^*,id_{R^*}) \to (T^*,\nu)$ of $\mathcal{C}_{h_{R^*}}$. Then, the following diagram is commutative by the naturality of $\boldsymbol{\chi}$.

$$\begin{tikzcd}
 & & \nu^*(\lambda^*(\boldsymbol{N})) \arrow[r, "\boldsymbol{c}_{\lambda,\nu}(\boldsymbol{N})^{-1}"] & (\nu\lambda)^*(\boldsymbol{N}) \arrow[d, equal] \\
\boldsymbol{N} \arrow[r, "\alpha_\lambda(\boldsymbol{N})"] \arrow[d, "\boldsymbol{\varphi}"] \arrow[rrru, bend left, "\alpha_{\nu\lambda}(\boldsymbol{N})"] & \lambda^*(\boldsymbol{N}) \arrow[r, equal] \arrow[ru, "\alpha_\nu(\lambda^*(\boldsymbol{N}))"] & h^*_\lambda(\widehat{\boldsymbol{N}})(R^*,id_{R^*}) \arrow[r, "h^*_\lambda(\widehat{\boldsymbol{N}})(\nu)"] \arrow[d, "\boldsymbol{\chi}_{(R^*,id_{R^*})}"] & h^*_\lambda(\widehat{\boldsymbol{N}})(T^*,\nu) \arrow[d, "\boldsymbol{\chi}_{(T^*,\nu)}"] \\
\boldsymbol{M} \arrow[r, "\alpha_{id_{R^*}}(\boldsymbol{M})"] \arrow[rrrd, bend right, "\alpha_\nu(\boldsymbol{M})"'] & id^*_{R^*}(\boldsymbol{M}) \arrow[r, equal] \arrow[rd, "\alpha_\nu(id^*_{R^*}(\boldsymbol{M}))"'] & \widehat{\boldsymbol{M}}(R^*,id_{R^*}) \arrow[r, "\widehat{\boldsymbol{M}}(\nu)"] & \widehat{\boldsymbol{M}}(T^*,\nu) \arrow[d, equal] \\
 & & \nu^*(id^*_{R^*}(\boldsymbol{M})) \arrow[r, "\boldsymbol{c}_{id_{R^*},\nu}(\boldsymbol{N})^{-1}"] & \nu^*(\boldsymbol{M})
\end{tikzcd}$$

Put $\bar{\boldsymbol{\varphi}} = \alpha_{id_{R^*}}(\boldsymbol{M})^{-1}\boldsymbol{\chi}_{(R^*,id_{R^*})} : \lambda^*(\boldsymbol{N}) \to \boldsymbol{M}$. Then, it follows from the commutativity of the above diagram that we have

$$\begin{aligned}
\nu^*(\bar{\boldsymbol{\varphi}})\boldsymbol{c}_{\lambda,\nu}(\boldsymbol{N})\alpha_{\nu\lambda}(\boldsymbol{N}) &= \nu^*(\bar{\boldsymbol{\varphi}})\alpha_\nu(\lambda^*(\boldsymbol{N}))\alpha_\lambda(\boldsymbol{N}) = \alpha_\nu(\boldsymbol{M})\bar{\boldsymbol{\varphi}}\alpha_\lambda(\boldsymbol{N}) = \alpha_\nu(\boldsymbol{M})\alpha_{id_{R^*}}(\boldsymbol{M})^{-1}\boldsymbol{\chi}_{(R^*,id_{R^*})}\alpha_\lambda(\boldsymbol{N}) \\
&= \alpha_\nu(\boldsymbol{M})\boldsymbol{\varphi} = \boldsymbol{\chi}_{(T^*,\nu)}\alpha_{\nu\lambda}(\boldsymbol{N}).
\end{aligned}$$

Since $\alpha_{\nu\lambda}(\boldsymbol{N})$ is cartesian, the above equality implies $\boldsymbol{\chi}_{(T^*,\nu)} = \nu^*(\bar{\boldsymbol{\varphi}})\boldsymbol{c}_{\lambda,\nu}(\boldsymbol{N})$ which shows $\hat{h}(\boldsymbol{\varphi}) = (f,\boldsymbol{\chi})$. Therefore $\hat{h} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}(\boldsymbol{M},\boldsymbol{N}) \to \mathcal{MOD}(\hat{h}(\boldsymbol{M}),\hat{h}(\boldsymbol{N}))$ is surjective. □

For a fibered category $p : \mathcal{F} \to \mathcal{E}$, we denote by $\mathcal{F}_c$ a subcategory of $\mathcal{F}$ which has the same objects as $\mathcal{F}$ and whose morphisms are cartesian morphisms of $\mathcal{F}$. Then, (10.4.3) and (10.4.5) imply the following.

Proposition 10.4.6 *$\hat{h} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \to \mathcal{MOD}$ restricts to a fully faithful functor $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_c \to \mathcal{MOD}_c$.*

10.5 Quasi-coherent modules

Definition 10.5.1 *An F-module M is said to be quasi-coherent if the following condition (QC) is satisfied.*

(QC) If we put $M(S^*,\sigma)=(S^*,N^*,\beta)$, $M(R^*,\rho)=(R^*,M^*,\alpha)$ and $M(\lambda)=(\lambda,\tilde{\lambda}):M(S^*,\sigma)\to M(R^*,\rho)$ for a morphism $\lambda:(S^*,\sigma)\to(R^*,\rho)$ of F-models, then the following composition is an isomorphism.

$$N^*\,\widehat{\otimes}_{S^*}\,R^*\xrightarrow{\tilde{\lambda}\,\widehat{\otimes}_{S^*}\,id_{R^*}}M^*\,\widehat{\otimes}_{S^*}\,R^*\xrightarrow{\widehat{\otimes}_{\lambda}}M^*\,\widehat{\otimes}_{R^*}\,R^*$$

Here, $\widehat{\otimes}_{\lambda}:M^*\,\widehat{\otimes}_{S^*}\,R^*\to M^*\,\widehat{\otimes}_{R^*}\,R^*$ is the map induced by the map $\otimes_{\lambda}:M^*\otimes_{S^*}R^*\to M^*\otimes_{R^*}R^*$.

We denote by $\mathcal{QM}od(F)$ the full subcategory of $\mathcal{M}od(F)$ consisting of quasi-coherent F-modules.

Proposition 10.5.2 *Let $f:G\to F$ be a morphism of $\mathcal{T}$. If M is a quasi-coherent F-module, $f^*(M)$ is a quasi-coherent G-module.*

Proof. Let $\lambda:(S^*,\sigma)\to(R^*,\rho)$ be a morphism of G-models. By the naturality of f, λ is regarded as a morphism $\lambda:(S^*,f_{S^*}(\sigma))\to(R^*,f_{R^*}(\rho))$ of F-models. Hence if we put $f^*(M)(S^*,\sigma)=M(S^*,f_{S^*}(\sigma))=(S^*,N^*,\beta)$ and $f^*(M)(R^*,\rho)=M(R^*,f_{R^*}(\rho))=(R^*,M^*,\alpha)$, $f^*(M)(\lambda):f^*(M)(S^*,\sigma)\to f^*(M)(R^*,\rho)$ and $M(\lambda):M(S^*,f_{S^*}(\sigma))\to M(R^*,f_{R^*}(\rho))$ are the same maps from (S^*,N^*,β) to (R^*,M^*,α). Therefore (QC) of (10.5.1) are satisfied. □

Proposition 10.5.3 *For an object $\boldsymbol{M}=(R^*,M^*,\alpha)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})$, $\widehat{\boldsymbol{M}}$ is a quasi-coherent h_{R^*}-module.*

Proof. If $\xi\in\mathcal{C}_{h_{R^*}}((S^*,\lambda),(T^*,\nu))$, then $\nu=\xi\lambda$ and ξ is a morphism of R^*-algebras. Define maps $f:M^*\otimes_{R^*}S^*\otimes_{S^*}T^*\to M^*\otimes_{R^*}T^*$ and $g:M^*\otimes_{R^*}T^*\otimes_{T^*}T^*\to M^*\otimes_{R^*}T^*$ by $f(z\otimes x\otimes y)=z\otimes\xi(x)y$ and $g(z\otimes w\otimes y)=z\otimes wy$. Then, f and g are isomorphisms. Let $\hat{f}:M^*\,\widehat{\otimes}_{R^*}\,S^*\,\widehat{\otimes}_{S^*}\,T^*\to M^*\,\widehat{\otimes}_{R^*}\,T^*$ and $\hat{g}:M^*\,\widehat{\otimes}_{R^*}\,T^*\,\widehat{\otimes}_{T^*}\,T^*\to M^*\,\widehat{\otimes}_{R^*}\,T^*$ be isomorphisms induced by f and g, respectively. Since

$$\widehat{\boldsymbol{M}}(\xi)=(\xi,id_{M^*}\,\widehat{\otimes}_{R^*}\,\xi):\widehat{\boldsymbol{M}}(S^*,\lambda)=(S^*,M^*\,\widehat{\otimes}_{R^*}\,S^*,\alpha_{\lambda})\to(T^*,M^*\,\widehat{\otimes}_{R^*}\,T^*,\alpha_{\nu})=\widehat{\boldsymbol{M}}(T^*,\nu)$$

and the following diagram commutes,

$$\begin{tikzcd}
M^*\,\widehat{\otimes}_{R^*}\,S^*\,\widehat{\otimes}_{S^*}\,T^* \arrow[r, "id_{M^*}\,\widehat{\otimes}_{R^*}\,\xi\,\widehat{\otimes}_{S^*}\,id_{T^*}"] \arrow[d, "\hat{f}"] & M^*\,\widehat{\otimes}_{R^*}\,T^*\,\widehat{\otimes}_{S^*}\,T^* \arrow[d, "\widehat{\otimes}_{\xi}"] \\
M^*\,\widehat{\otimes}_{R^*}\,T^* & M^*\,\widehat{\otimes}_{R^*}\,T^*\,\widehat{\otimes}_{T^*}\,T^* \arrow[l, "\hat{g}"']
\end{tikzcd}$$

$M^*\,\widehat{\otimes}_{R^*}\,S^*\,\widehat{\otimes}_{S^*}\,T^*\xrightarrow{id_{M^*}\,\widehat{\otimes}_{R^*}\,\xi\,\widehat{\otimes}_{S^*}\,id_{T^*}}M^*\,\widehat{\otimes}_{R^*}\,T^*\,\widehat{\otimes}_{S^*}\,T^*\xrightarrow{\widehat{\otimes}_{\xi}}M^*\,\widehat{\otimes}_{R^*}\,T^*\,\widehat{\otimes}_{T^*}\,T^*$ is an isomorphism. □

We define a functor $\varGamma:\mathcal{M}od(h_{R^*})\to\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$ by $\varGamma(M)=M(R^*,id_{R^*})$ and $\varGamma(\varphi)=\varphi_{(R^*,id_{R^*})}$. For an h_{R^*}-module M, we define a morphism $\Phi_M:\mathscr{S}_{R^*}\varGamma(M)=\widehat{\varGamma(M)}\to M$ of h_{R^*}-modules as follows. We put $M(R^*,id_{R^*})=(R^*,M^*,\alpha)$ and $M(S^*,\sigma)=(S^*,M'^*,\alpha')$ for $(S^*,\sigma)\in\operatorname{Ob}\mathcal{C}_{h_{R^*}}$. Then, $\sigma\in h_{R^*}(S^*)$ defines a morphism $\sigma:(R^*,id_{R^*})\to(S^*,\sigma)$ of h_{R^*}-models and we put

$$M(\sigma)=(\sigma,\tilde{\sigma}):M(R^*,id_{R^*})=(R^*,M^*,\alpha)\to(S^*,M'^*,\alpha')=M(S^*,\sigma).$$

Since M'^* is complete Hausdorff, $\alpha':M'^*\otimes_{K^*}S^*\to M'^*$ induces an isomorphism $\hat{\alpha}':M'^*\,\widehat{\otimes}_{S^*}\,S^*\to M'^*$. We denote by $\hat{\sigma}:M^*\,\widehat{\otimes}_{R^*}\,S^*\to M'^*$ the following composition, which is an isomorphism if M is quasi-coherent.

$$M^*\,\widehat{\otimes}_{R^*}\,S^*\xrightarrow{\tilde{\sigma}\,\widehat{\otimes}_{R^*}\,id_{S^*}}M'^*\,\widehat{\otimes}_{R^*}\,S^*\xrightarrow{\widehat{\otimes}_{\sigma}}M'^*\,\widehat{\otimes}_{S^*}\,S^*\xrightarrow{\hat{\alpha}'}M'^*$$

Since $\tilde{\sigma}:M^*\to M'^*$ is a homomorphism of right R^*-modules if we regard M'^* as an right R^*-module by $\alpha'(id_{M'^*}\otimes_{K^*}\sigma):M'^*\otimes_{K^*}R^*\to M'^*$, the following diagram commutes. Here, α'_{σ} and α''_{σ} are maps induced by the multiplication of S^* and $\bar{\alpha}':M'^*\otimes_{S^*}S^*\to M'^*$ is the isomorphism induced by $\alpha':M'^*\otimes_{K^*}S^*\to M'^*$.

$$\begin{tikzcd}
(M^* \otimes_{R^*} S^*) \otimes_{K^*} S^* \arrow[r, "\eta_{M^* \otimes_{R^*} S^*} \otimes_{K^*} id_{S^*}"] \arrow[d, "(\tilde{\sigma} \otimes_{R^*} id_{S^*}) \otimes_{K^*} id_{S^*}"] & (M^* \widehat{\otimes}_{R^*} S^*) \otimes_{K^*} S^* \arrow[r, "\alpha_\sigma"] \arrow[d, "(\tilde{\sigma} \widehat{\otimes}_{R^*} id_{S^*}) \otimes_{K^*} id_{S^*}"] & M^* \widehat{\otimes}_{R^*} S^* \arrow[d, "\tilde{\sigma} \widehat{\otimes}_{R^*} id_{S^*}"] \\
(M'^* \otimes_{R^*} S^*) \otimes_{K^*} S^* \arrow[r, "\eta_{M^* \otimes_{R^*} S^*} \otimes_{K^*} id_{S^*}"] \arrow[d, "\otimes_\sigma \otimes_{K^*} id_{S^*}"] & (M'^* \widehat{\otimes}_{R^*} S^*) \otimes_{K^*} S^* \arrow[r, "\alpha'_\sigma"] \arrow[d, "\widehat{\otimes}_\sigma \otimes_{K^*} id_{S^*}"] & M'^* \widehat{\otimes}_{R^*} S^* \arrow[d, "\widehat{\otimes}_\sigma"] \\
(M'^* \otimes_{S^*} S^*) \otimes_{K^*} S^* \arrow[r, "\eta_{M^* \otimes_{S^*} S^*} \otimes_{K^*} id_{S^*}"] \arrow[dr, swap, "\tilde{\alpha}' \otimes_{K^*} id_{S^*}"] & (M'^* \widehat{\otimes}_{S^*} S^*) \otimes_{K^*} S^* \arrow[r, "\alpha''_\sigma"] \arrow[d, "\hat{\alpha}' \otimes_{K^*} id_{S^*}"] & M'^* \widehat{\otimes}_{S^*} S^* \arrow[d, "\hat{\alpha}'"] \\
& M'^* \otimes_{K^*} S^* \arrow[r, "\alpha'"] & M'^*
\end{tikzcd}$$

It follows that $(id_{S^*}, \hat{\sigma}) : \mathscr{S}_{R^*}\Gamma(M)(S^*, \sigma) = (S^*, M^* \widehat{\otimes}_{R^*} S^*, \alpha_\sigma) \to (S^*, M'^*, \alpha') = M(S^*, \sigma)$ is a homomorphism of $\mathcal{M}od(\mathcal{C}, \mathcal{M})$. We define $(\Phi_M)_{(S^*,\sigma)} : \mathscr{S}_{R^*}\Gamma(M)(S^*, \sigma) \to M(S^*, \sigma)$ by $(\Phi_M)_{(S^*,\sigma)} = (id_{S^*}, \hat{\sigma})$.

For a morphism $\xi : (S^*, \sigma) \to (T^*, \tau)$ of $\mathcal{C}_{h_{R^*}}$, we put $M(T^*, \tau) = (T^*, M''^*, \alpha'')$ and $M(\xi) = (\xi, \tilde{\xi}) : M(S^*, \sigma) \to M(T^*, \tau)$. Since $\tau = \xi\sigma$, we have $(\tau, \tilde{\tau}) = M(\tau) = M(\xi)M(\sigma) = (\xi\sigma, \tilde{\xi}\tilde{\sigma})$. Hence the following diagram commutes.

$$\begin{tikzcd}
M^* \widehat{\otimes}_{R^*} S^* \arrow[r, "\tilde{\sigma} \widehat{\otimes}_{R^*} id_{S^*}"] \arrow[d, "id_{M^*} \widehat{\otimes}_{R^*} \xi"] & M'^* \widehat{\otimes}_{R^*} S^* \arrow[r, "\widehat{\otimes}_\sigma"] \arrow[d, "\tilde{\xi} \widehat{\otimes}_{R^*} \xi"] & M'^* \widehat{\otimes}_{S^*} S^* \arrow[rr, "\hat{\alpha}'"] \arrow[d, "\tilde{\xi} \widehat{\otimes}_{S^*} \xi"] & & M'^* \arrow[d, "\tilde{\xi}"] \\
M^* \widehat{\otimes}_{R^*} T^* \arrow[r, "\tilde{\tau} \widehat{\otimes}_{R^*} id_{T^*}"] & M''^* \widehat{\otimes}_{R^*} T^* \arrow[r, "\widehat{\otimes}_\sigma"] & M''^* \widehat{\otimes}_{S^*} T^* \arrow[r, "\widehat{\otimes}_\xi"] & M''^* \widehat{\otimes}_{T^*} T^* \arrow[r, "\hat{\alpha}''"] & M''^*
\end{tikzcd}$$

Thus the following diagram commutes and this shows the naturality of Φ_M.

$$\begin{tikzcd}
\mathscr{S}_{R^*}\Gamma(M)(S^*, \sigma) \arrow[r, "(\Phi_M)_{(S^*,\sigma)}"] \arrow[d, "\mathscr{S}_{R^*}\Gamma(M)(\xi)"] & M(S^*, \sigma) \arrow[d, "M(\xi)"] \\
\mathscr{S}_{R^*}\Gamma(M)(T^*, \lambda) \arrow[r, "(\Phi_M)_{(T^*,\lambda)}"] & M(T^*, \lambda)
\end{tikzcd}$$

Let $\boldsymbol{\chi} : M \to N$ be a morphism of h_{R^*}-modules. Put $\Gamma(N) = (R^*, N^*, \beta)$, $\boldsymbol{\chi}_{(R^*, id_{R^*})} = (id_{R^*}, \chi)$ and $N(S^*, \sigma) = (S^*, N'^*, \beta')$, $\boldsymbol{\chi}_{(S^*,\sigma)} = (id_{S^*}, \chi')$ for $(S^*, \sigma) \in \mathcal{C}_{h_{R^*}}$. We also put

$$N(\sigma) = (\sigma, \tilde{\sigma}') : N(R^*, id_{R^*}) = (R^*, N^*, \beta) \to (S^*, N'^*, \beta') = N(S^*, \sigma).$$

Since the following left diagram commutes by the naturality of $\boldsymbol{\chi}$, so does the right one.

$$\begin{tikzcd}
M(R^*, id_{R^*}) \arrow[r, "M(\sigma)"] \arrow[d, "\boldsymbol{\chi}_{(R^*, id_{R^*})}"] & M(S^*, \sigma) \arrow[d, "\boldsymbol{\chi}_{(S^*,\sigma)}"] \\
N(R^*, id_{R^*}) \arrow[r, "N(\sigma)"] & N(S^*, \sigma)
\end{tikzcd}
\qquad
\begin{tikzcd}
M^* \arrow[r, "\tilde{\sigma}"] \arrow[d, "\chi"] & M'^* \arrow[d, "\chi'"] \\
N^* \arrow[r, "\tilde{\sigma}'"] & N'^*
\end{tikzcd}$$

Since χ' is a homomorphism of right S^*-modules, the following diagram commutes.

$$\begin{tikzcd}
M^* \widehat{\otimes}_{R^*} S^* \arrow[r, "\tilde{\sigma} \widehat{\otimes}_{R^*} id_{S^*}"] \arrow[d, "\chi \widehat{\otimes}_{R^*} id_{S^*}"] & M'^* \widehat{\otimes}_{R^*} S^* \arrow[r, "\widehat{\otimes}_\sigma"] \arrow[d, "\chi' \widehat{\otimes}_{R^*} id_{S^*}"] & M'^* \widehat{\otimes}_{S^*} S^* \arrow[r, "\hat{\alpha}'"] \arrow[d, "\chi' \widehat{\otimes}_{S^*} id_{S^*}"] & M'^* \arrow[d, "\chi'"] \\
N^* \widehat{\otimes}_{R^*} S^* \arrow[r, "\tilde{\sigma}' \widehat{\otimes}_{R^*} id_{S^*}"] & N'^* \widehat{\otimes}_{R^*} S^* \arrow[r, "\widehat{\otimes}_\sigma"] & N'^* \widehat{\otimes}_{S^*} S^* \arrow[r, "\hat{\beta}'"] & N'^*
\end{tikzcd}$$

It follows from the commutativity of the above diagram, we have

$$\begin{aligned}
(\boldsymbol{\chi}\Phi_M)_{(S^*,\sigma)} &= \boldsymbol{\chi}_{(S^*,\sigma)}(\Phi_M)_{(S^*,\sigma)} = (id_{S^*}, \chi')(id_{S^*}, \hat{\alpha}' \widehat{\otimes}_\sigma(\tilde{\sigma} \widehat{\otimes}_{R^*} id_{S^*})) = (id_{S^*}, \chi'\hat{\alpha}' \widehat{\otimes}_\sigma(\tilde{\sigma} \widehat{\otimes}_{R^*} id_{S^*})) \\
&= (id_{S^*}, \hat{\beta}' \widehat{\otimes}_\sigma(\tilde{\sigma}'\chi \widehat{\otimes}_{R^*} id_{S^*})) = (id_{S^*}, \hat{\beta}' \widehat{\otimes}_\sigma(\tilde{\sigma}' \widehat{\otimes}_{R^*} id_{S^*}))(id_{S^*}, \chi \widehat{\otimes}_{R^*} id_{S^*}) \\
&= (\Phi_N)_{(S^*,\sigma)}(\mathscr{S}_{R^*}\Gamma(\boldsymbol{\chi}))_{(S^*,\sigma)} = (\Phi_N \mathscr{S}_{R^*}\Gamma(\boldsymbol{\chi}))_{(S^*,\sigma)}
\end{aligned}$$

Thus the following diagram commutes and we have a natural transformation $\Phi : \mathscr{S}_{R^*}\Gamma \to id_{\mathcal{M}od(h_{R^*})}$.

$$
\begin{array}{ccc}
\mathscr{S}_{R^*}\mathit{\Gamma}(M) & \xrightarrow{\Phi_M} & M \\
\Big\downarrow{\scriptstyle \mathscr{S}_{R^*}\mathit{\Gamma}(\boldsymbol{\chi})} & & \Big\downarrow{\scriptstyle \boldsymbol{\chi}} \\
\mathscr{S}_{R^*}\mathit{\Gamma}(N) & \xrightarrow{\Phi_N} & N
\end{array}
$$

For an object $\boldsymbol{M} = (R^*, M^*, \alpha)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$, define a morphism $\Psi_{\boldsymbol{M}} : \boldsymbol{M} \to \mathit{\Gamma}\mathscr{S}_{R^*}(\boldsymbol{M})$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$ by $\Psi_{\boldsymbol{M}} = (id_{R^*}, \hat{\imath}_{M^*})$, where $\hat{\imath}_{M^*} : M^* \to M^* \mathbin{\widehat{\otimes}}_{R^*} R^*$ is the isomorphism induced by the map $M^* \to M^* \otimes_{R^*} R^*$ which maps x to $x \otimes 1$. Since $\Psi_{\boldsymbol{M}}$ is natural in $\boldsymbol{M}$, thus we have a natural equivalence $\Psi : id_{\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}} \to \mathit{\Gamma}\mathscr{S}_{R^*}$.

Proposition 10.5.4 *$\mathscr{S}_{R^*} : \mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*} \to \mathcal{M}od(h_{R^*})$ is a left adjoint of $\mathit{\Gamma} : \mathcal{M}od(h_{R^*}) \to \mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$ with unit $\Psi : id_{\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}} \to \mathit{\Gamma}\mathscr{S}_{R^*}$ and counit $\Phi : \mathscr{S}_{R^*}\mathit{\Gamma} \to id_{\mathcal{M}od(h_{R^*})}$. Moreover, $\Phi_M : \mathscr{S}_{R^*}\mathit{\Gamma}(M) \to M$ is an isomorphism if and only if M is a quasi-coherent h_{R^*}-module.*

Proof. Let M be a h_{R^*}-module and put $\mathit{\Gamma}(M) = (R^*, M^*, \alpha)$. Since $\hat{\imath}_{M^*} : M^* \to M^* \mathbin{\widehat{\otimes}}_{R^*} R^*$ is the inverse of the isomorphism $\hat{\alpha} : M^* \mathbin{\widehat{\otimes}}_{R^*} R^* \to M^*$ induced by $\alpha : M^* \otimes_{K^*} R^* \to M^*$, $\Psi_{\mathit{\Gamma}(M)} = (id_{R^*}, \hat{\imath}_{M^*})$ is the inverse of $\mathit{\Gamma}(\Phi_M) = (\Phi_M)_{(R^*, id_{R^*})} = (id_{R^*}, \hat{\alpha}) : \mathit{\Gamma}\mathscr{S}_{R^*}\mathit{\Gamma}(M) = (R^*, M^* \mathbin{\widehat{\otimes}}_{R^*} R^*, \alpha_{id_{R^*}}) \to (R^*, M^*, \alpha) = \Gamma(M)$. Hence we have $\mathit{\Gamma}(\Phi_M)\Psi_{\mathit{\Gamma}(M)} = id_{\mathit{\Gamma}(M)}$. Let $\boldsymbol{M} = (R^*, M^*, \alpha)$ be an object of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}$ and (S^*, σ) an h_{R^*}-model. Then, $\mathscr{S}_{R^*}(\boldsymbol{M})(S^*, \sigma) = (S^*, M^* \mathbin{\widehat{\otimes}}_{R^*} S^*, \alpha_\sigma)$, $\mathscr{S}_{R^*}\mathit{\Gamma}\mathscr{S}_{R^*}(\boldsymbol{M})(S^*, \sigma) = (S^*, (M^* \mathbin{\widehat{\otimes}}_{R^*} R^*) \mathbin{\widehat{\otimes}}_{R^*} S^*, (\alpha_{id_{R^*}})_\sigma)$ and $\mathscr{S}_{R^*}(\Psi_{\boldsymbol{M}})_{(S^*,\sigma)} = (id_{S^*}, \hat{\imath}_{M^*} \mathbin{\widehat{\otimes}}_{R^*} id_{S^*}) : \mathscr{S}_{R^*}(\boldsymbol{M})(S^*, \sigma) \to \mathscr{S}_{R^*}\mathit{\Gamma}\mathscr{S}_{R^*}(\boldsymbol{M})(S^*, \sigma)$. Since

$$\widehat{\boldsymbol{M}}(\sigma) = (\sigma, id_{M^*} \mathbin{\widehat{\otimes}}_{R^*} \sigma) : \widehat{\boldsymbol{M}}(R^*, id_{R^*}) = (R^*, M^* \mathbin{\widehat{\otimes}}_{R^*} R^*, \alpha_{id_{R^*}}) \to (S^*, M^* \mathbin{\widehat{\otimes}}_{R^*} S^*, \alpha_\sigma) = \widehat{\boldsymbol{M}}(S^*, \sigma),$$

it follows that $(\Phi_{\mathscr{S}_{R^*}(\boldsymbol{M})})_{(S^*,\sigma)} = (id_{S^*}, \check{\sigma})$, where $\check{\sigma}$ is the following composition.

$$(M^* \mathbin{\widehat{\otimes}}_{R^*} R^*) \mathbin{\widehat{\otimes}}_{R^*} S^* \xrightarrow{id_{M^*} \mathbin{\widehat{\otimes}}_{R^*} \sigma \mathbin{\widehat{\otimes}}_{R^*} id_{S^*}} (M^* \mathbin{\widehat{\otimes}}_{R^*} S^*) \mathbin{\widehat{\otimes}}_{R^*} S^* \xrightarrow{\widehat{\otimes}_\sigma} (M^* \mathbin{\widehat{\otimes}}_{R^*} S^*) \mathbin{\widehat{\otimes}}_{S^*} S^* \xrightarrow{\hat{\alpha}_\sigma} M^* \mathbin{\widehat{\otimes}}_{R^*} S^*$$

It is clear that $\check{\sigma}(\hat{\imath}_{M^*} \mathbin{\widehat{\otimes}}_{R^*} id_{S^*})$ is the identity map of $M^* \mathbin{\widehat{\otimes}}_{R^*} S^*$. Thus the following composition is the identity morphism of $\mathscr{S}_{R^*}(\boldsymbol{M})(S^*, \sigma)$ which implies $\Phi_{\mathscr{S}_{R^*}(\boldsymbol{M})}\mathscr{S}_{R^*}(\Psi_{\boldsymbol{M}}) = id_{\mathscr{S}_{R^*}(\boldsymbol{M})}$.

$$\mathscr{S}_{R^*}(\boldsymbol{M})(S^*, \sigma) \xrightarrow{\mathscr{S}_{R^*}(\Psi_{\boldsymbol{M}})_{(S^*,\sigma)}} \mathscr{S}_{R^*}\mathit{\Gamma}\mathscr{S}_{R^*}(\boldsymbol{M})(S^*, \sigma) \xrightarrow{(\Phi_{\mathscr{S}_{R^*}(\boldsymbol{M})})_{(S^*,\sigma)}} \mathscr{S}_{R^*}(\boldsymbol{M})(S^*, \sigma)$$

The last assertion is clear from the previous argument. □

Corollary 10.5.5 *Let M and N be h_{R^*}-modules. If M is quasi-coherent, then*

$$\mathit{\Gamma} : \mathcal{M}od(h_{R^*})(M, N) \to \mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}(\mathit{\Gamma}(M), \mathit{\Gamma}(N))$$

is bijective.

Proof. $\mathit{\Gamma} : \mathcal{M}od(h_{R^*})(M, N) \to \mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}(\mathit{\Gamma}(M), \mathit{\Gamma}(N))$ is a composition of $\Phi_M^* : \mathcal{M}od(h_{R^*})(M, N) \to \mathcal{M}od(h_{R^*})(\mathscr{S}_{R^*}\mathit{\Gamma}(M), N)$ and the adjunction $\mathcal{M}od(h_{R^*})(\mathscr{S}_{R^*}\mathit{\Gamma}(M), N) \to \mathcal{M}od(\mathcal{C},\mathcal{M})_{R^*}(\mathit{\Gamma}(M), \mathit{\Gamma}(N))$. Since M is quasi-coherent, Φ_M is an isomorphism, hence Φ_M^* is bijective. □

Proposition 10.5.6 *Let $D : \mathcal{D} \to \mathcal{T}$ be a functor and $(D(i) \xrightarrow{\iota_i} F)_{i \in \mathrm{Ob}\,\mathcal{D}}$ a colimiting cone of D. For an F-module M, we define a functor $D(M) : \mathcal{D} \to \mathcal{MOD}$ as follows. We put $D(M)(i) = (D(i), \iota_i^*(M))$. Let $\alpha : i \to j$ be a morphism of $\mathcal{D}$. Since $\iota_i = \iota_j D(\alpha)$, we have $D(\alpha)^*\iota_j^*(M) = \iota_i^*(M)$ and define a morphism $D(M)(\alpha) : (D(i), \iota_i^*(M)) \to (D(j), \iota_j^*(M))$ of $\mathcal{MOD}$ by $D(M)(\alpha) = (D(\alpha), id_{\iota_i^*(M)})$. Then,*

$$\left(D(M)(i) \xrightarrow{\left(\iota_i, id_{\iota_i^*(M)}\right)} (F, M) \right)_{i \in \mathrm{Ob}\,\mathcal{D}}$$

is a colimiting cone of $D(M)$.

Proof. Let $(D(M)(i) = (D(i), \iota_i^*(M)) \xrightarrow{(\eta_i, \zeta_i)} (G, N))_{i \in \mathrm{Ob}\,\mathcal{D}}$ be a cone of $D(M)$. By applying $p_{\mathcal{T}} : \mathcal{MOD} \to \mathcal{T}$ to $(D(M)(i) \xrightarrow{(\eta_i, \zeta_i)} (G, N))_{i \in \mathrm{Ob}\,\mathcal{D}}$, we have a cone $(D(i) \xrightarrow{\eta_i} G)_{i \in \mathrm{Ob}\,\mathcal{D}}$ in $\mathcal{T}$. Then, there exists unique morphism

$f : F \to G$ satisfying $f\iota_i = \eta_i$ for any $i \in \operatorname{Ob}\mathcal{D}$. For $(R^*, \rho) \in \operatorname{Ob}\mathcal{C}_F$, there exists $i \in \operatorname{Ob}\mathcal{D}$ and $\sigma \in D(i)(R^*)$ such that $(\iota_i)_{R^*}(\sigma) = \rho$. If $(\iota_j)_{R^*}(\tau) = \rho$ also holds for $j \in \operatorname{Ob}\mathcal{D}$ and $\tau \in D(j)(R^*)$, then we have

$$\begin{aligned}\eta_i^*(N)(R^*, \sigma) &= N(R^*, (\eta_i)_{R^*}(\sigma)) = N(R^*, f_{R^*}(\iota_i)_{R^*}(\sigma)) = N(R^*, f_{R^*}(\rho)) = N(R^*, f_{R^*}(\iota_j)_{R^*}(\tau)) \\ &= N(R^*, (\eta_j)_{R^*}(\tau)) = \eta_j^*(N)(R^*, \tau) \\ \iota_i^*(M)(R^*, \sigma) &= M(R^*, (\iota_i)_{R^*}(\sigma)) = M(R^*, \rho) = M(R^*, (\iota_j)_{R^*}(\tau)) = \iota_j^*(M)(R^*, \tau)\end{aligned}$$

and there exist morphisms $\alpha_{2k-1} : i_{2k-1} \to i_{2k}$, $\alpha_{2k} : i_{2k+1} \to i_{2k}$ of $\mathcal{D}$ for $1 \le k \le n$ and $\sigma_k \in D(i_k)$ for $1 \le k \le 2n+1$ such that $i_1 = 1$, $i_{2n+1} = j$, $\sigma_1 = \sigma$, $\sigma_{2k+1} = \tau$ and $D(\alpha_{2k-1})(\sigma_{2k-1}) = D(\alpha_{2k})(\sigma_{2k+1}) = \sigma_{2k}$ for $1 \le k \le n$. We note that, if $\alpha \in \mathcal{D}(i, j)$, we have $D(\alpha)^*(\zeta_j) = \zeta_i$, namely $(\zeta_i)_{(R^*,\rho)} = (\zeta_j)_{(R^*, D(\alpha)(\rho))}$ for $(R^*, \rho) \in \operatorname{Ob}\mathcal{C}_{D(i)}$. Therefore we have $(\zeta_{2k-1})_{(R^*,\sigma_{2k-1})} = (\zeta_{2k})_{(R^*,\sigma_{2k})} = (\zeta_{2k+1})_{(R^*,\sigma_{2k+1})}$ for $1 \le k \le n$, which implies $(\zeta_i)_{(R^*,\sigma)} = (\zeta_j)_{(R^*,\tau)}$ as morphisms from $\eta_i^*(N)(R^*, \sigma) = \eta_j^*(N)(R^*, \tau)$ to $\iota_i^*(M)(R^*, \sigma) = \iota_j^*(M)(R^*, \tau)$.

Define a morphism $\varphi : f^*(N) \to M$ of F-modules as follows. For an F-model (R^*, ρ), choose $i \in \operatorname{Ob}\mathcal{D}$ and $\sigma \in D(i)(R^*)$ such that $(\iota_i)_{R^*}(\sigma) = \rho$. Since $M(R^*, \rho) = M(R^*, (\iota_i)_{R^*}(\sigma)) = \iota_i^*(M)(R^*, \sigma)$ and

$$f^*(N)(R^*, \rho) = N(R^*, f_{R^*}(\rho)) = N(R^*, f_{R^*}((\iota_i)_{R^*}(\sigma))) = N(R^*, (\eta_i)_{R^*}(\sigma)) = \eta_i^*(N)(R^*, \sigma),$$

$\varphi_{(R^*,\rho)} : f^*(N)(R^*, \rho) \to M(R^*, \rho)$ is defined to be $(\zeta_i)_{(R^*,\sigma)} : \eta_i^*(N)(R^*, \sigma) \to \iota_i^*(M)(R^*, \sigma)$. This definition of $\varphi_{(R^*,\rho)}$ does not depend on the choice of either $i \in \operatorname{Ob}\mathcal{D}$ or $\sigma \in D(i)(R^*)$ by the above argument. Let $\xi : (R^*, \rho) \to (S^*, \lambda)$ be a morphism of $\mathcal{C}_F$ and choose $i \in \operatorname{Ob}\mathcal{D}$ and $\sigma \in D(i)(R^*)$ such that $(\iota_i)_{R^*}(\sigma) = \rho$. Then, $(\iota_i)_{S^*}(D(i)(\xi)(\sigma)) = F(\xi)((\iota_i)_{R^*}(\sigma)) = F(\xi)(\rho) = \lambda$ by the naturality of ι_i and $\varphi_{(S^*,\lambda)} : f^*(N)(S^*, \lambda) \to M(S^*, \lambda)$ is defined to be $(\zeta_i)_{(S^*, D(i)(\xi)(\sigma))} : \eta_i^*(N)(S^*, D(i)(\xi)(\sigma)) \to \iota_i^*(M)(S^*, D(i)(\xi)(\sigma))$. $\xi : R^* \to S^*$ is regarded as a morphism $(R^*, \sigma) \to (S^*, D(i)(\xi)(\sigma))$ of $\mathcal{C}_{D(i)}$. By the naturality of ζ_i, the following left diagram commutes and this implies the right diagram also commutes. Thus we see the naturality of φ.

$$\begin{tikzcd} \eta_i^*(N)(R^*, \sigma) \arrow[r, "(\zeta_i)_{(R^*,\sigma)}"] \arrow[d, "\eta_i^*(N)(\xi)"] & \iota_i^*(M)(R^*, \sigma) \arrow[d, "\iota_i^*(M)(\xi)"] \\ \eta_i^*(N)(S^*, D(i)(\xi)(\sigma)) \arrow[r, "(\zeta_i)_{(S^*,D(i)(\xi)(\sigma))}"] & \iota_i^*(M)(S^*, D(i)(\xi)(\sigma)) \end{tikzcd} \qquad \begin{tikzcd} f^*(N)(R^*, \rho) \arrow[r, "\varphi_{(R^*,\rho)}"] \arrow[d, "f^*(N)(\xi)"] & M(R^*, \rho) \arrow[d, "M(\xi)"] \\ f^*(N)(S^*, \lambda) \arrow[r, "\varphi_{(S^*,\lambda)}"] & M(S^*, \lambda) \end{tikzcd}$$

We verify that $(f, \varphi) : (F, M) \to (G, N)$ is the unique morphism of $\mathcal{MOD}$ satisfying $(f, \varphi)(\iota_i, id_{\iota_i^*(M)}) = (\eta_i, \zeta_i)$ for any $i \in \operatorname{Ob}\mathcal{D}$. For a $D(i)$-model (R^*, σ), since $\iota_i^*(\varphi)_{(R^*,\sigma)} = \varphi_{(R^*, D(i)(\sigma))} = (\zeta_i)_{(R^*,\sigma)}$ by the definition of φ, it follows that $(f, \varphi)(\iota_i, id_{\iota_i^*(M)}) = (\eta_i, \zeta_i)$. Suppose that $(g, \psi) : (F, M) \to (G, N)$ satisfy $(g, \psi)(\iota_i, id_{\iota_i^*(M)}) = (f, \varphi)(\iota_i, id_{\iota_i^*(M)})$ for any $i \in \operatorname{Ob}\mathcal{D}$. Then $g\iota_i = f\iota_i$ for any $i \in \operatorname{Ob}\mathcal{D}$ and it follows $g = f$ by the uniqueness of f. For any F-model (R^*, ρ), choose $i \in \operatorname{Ob}\mathcal{D}$ and $\sigma \in D(i)(R^*)$ satisfying $(\iota_i)_{R^*}(\sigma) = \rho$. Then, $\psi_{(R^*,\rho)} = \psi_{(R^*, (\iota_i)_{R^*}(\sigma))} = \iota_i^*(\psi)_{(R^*,\sigma)} = \iota_i^*(\varphi)_{(R^*,\sigma)} = \varphi_{(R^*, (\iota_i)_{R^*}(\sigma))} = \varphi_{(R^*,\rho)}$, that is $\psi = \varphi$. □

For $F \in \operatorname{Ob}\mathcal{T}$, since F is a colimit of representable functors, it follows from the proof of (7.2.11) that

$$\left(D(F)(R^*, \rho) \xrightarrow{\varphi(F)_{(R^*,\rho)}} F \right)_{(R^*,\rho) \in \operatorname{Ob}\mathcal{C}_F}$$

is a colimiting cone of the functor $D(F) : \mathcal{C}_F^{op} \to \operatorname{Funct}_c(\mathcal{C}, \mathcal{T}op)$ which is given by $D(F)(R^*, \rho) = h_{R^*}$ and $D(F)(f) = h_f$. For an F-module M, we define a functor $D(F; M) : \mathcal{C}_F^{op} \to \mathcal{MOD}$ as in (10.5.6). That is,

$$D(F; M)(R^*, \rho) = \left(h_{R^*}, \varphi(F)_{(R^*,\rho)}^*(M) \right)$$

for an F-model (R^*, ρ) and $D(F; M)(f) : D(F; M)(S^*, \sigma) \to D(F; M)(R^*, \rho)$ is defined to be

$$\left(h_f, id_{\varphi(F)_{(S^*,\sigma)}^*(M)} \right) : \left(h_{S^*}, \varphi(F)_{(S^*,\sigma)}^*(M) \right) \to \left(h_{R^*}, \varphi(F)_{(R^*,\rho)}^*(M) \right)$$

for a morphism of F-models $f : (R^*, \rho) \to (S^*, \sigma)$. We note that we have $\varphi(F)_{(R^*,\rho)}^*(M)(T^*, \tau) = M(T^*, F(\tau)(\rho))$ for an h_{R^*}-model (T^*, τ) by the definition of $\varphi(F)_{(R^*,\rho)}$. Therefore an h_{S^*}-module $h_f^*\varphi(F)_{(R^*,\rho)}^*(M)$ coincides with $\varphi(F)_{(S^*,\sigma)}^*(M)$. It follows from (10.5.6) that the following is a colimiting cone of $D(F; M)$.

$$\left(D(F; M)(R^*, \rho) \xrightarrow{\left(\varphi(F)_{(R^*,\rho)}, id_{\varphi(F)_{(R^*,\rho)}^*(M)} \right)} (F, M) \right)_{(R^*,\rho) \in \operatorname{Ob}\mathcal{C}_F}$$

We also define a functor $\hat{D}(F;M) : \mathcal{C}_F^{op} \to \mathcal{MOD}$ to be the composition

$$\mathcal{C}_F^{op} \xrightarrow{M^{op}} \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \xrightarrow{\hat{h}} \mathcal{MOD}.$$

That is, $\hat{D}(F;M)(R^*,\rho) = \left(h_{R^*}, \widehat{M(R^*,\rho)}\right)$ for $(R^*,\rho) \in \mathrm{Ob}\,\mathcal{C}_F$ and

$$\hat{D}(F;M)(\xi) : \left(h_{S^*}, \widehat{M(S^*,\sigma)}\right) \to \left(h_{R^*}, \widehat{M(R^*,\rho)}\right)$$

for a morphism of F-models $\xi : (R^*,\rho) \to (S^*,\sigma)$ is given as follows. Put $M(R^*,\rho) = (R^*, M^*, \alpha)$, $M(S^*,\sigma) = (S^*, N^*, \beta)$ and $M(\xi) = (\xi, \tilde{\xi}) : (R^*, M^*, \alpha) \to (S^*, N^*, \beta)$. For an h_{S^*}-model (T^*,τ), we note that

$$\begin{aligned}
h_\xi^*(\widehat{M(R^*,\rho)})(T^*,\tau) = \widehat{M(R^*,\rho)}(T^*,(h_\xi)_{T^*}(\tau)) = \widehat{M(R^*,\rho)}(T^*,\tau\xi) = (T^*, M^* \mathbin{\widehat{\otimes}}_{R^*} T^*, \alpha_{\tau\xi}) \\
\widehat{M(S^*,\sigma)}(T^*,\tau\xi) = (T^*, N^* \mathbin{\widehat{\otimes}}_{S^*} T^*, \beta_\xi),
\end{aligned}$$

$\boldsymbol{\xi}_{(T^*,\tau)} : h_\xi^*(\widehat{M(R^*,\rho)})(T^*,\tau) \to \widehat{M(S^*,\sigma)}(T^*,\tau)$ is given by $\boldsymbol{\xi}_{(T^*,\tau)} = \left(id_{T^*}, \widehat{\otimes}_\xi(\tilde{\xi} \mathbin{\widehat{\otimes}}_{R^*} id_{T^*})\right)$ and $\hat{D}(F;M)(\xi)$ is given by $\hat{D}(F;M)(\xi) = (h_\xi, \boldsymbol{\xi})$. Hence $\boldsymbol{\xi}$ is an isomorphism if M is quasi-coherent. It follows that $\hat{D}(F;M)(\xi)$ is cartesian for every morphism ξ of F-models if and only if M is quasi-coherent.

We define a natural transformation $\Phi_{(F,M)} : D(F;M) \to \hat{D}(F;M)$ as follows. For $(R^*,\rho) \in \mathrm{Ob}\,\mathcal{C}_F$, since $\Gamma\left(\varphi(F)^*_{(R^*,\rho)}(M)\right) = \varphi(F)^*_{(R^*,\rho)}(M)(R^*, id_{R^*}) = M(R^*,\rho)$, $\Phi_{\varphi(F)^*_{(R^*,\rho)}(M)}$ is a morphism of h_{R^*}-modules from $\widehat{M(R^*,\rho)}$ to $\varphi(F)^*_{(R^*,\rho)}(M)$. We set

$$\left(\Phi_{(F,M)}\right)_{(R^*,\rho)} = \left(id_{h_{R^*}}, \Phi_{\varphi(F)^*_{(R^*,\rho)}(M)}\right) : \left(h_{R^*}, \varphi(F)^*_{(R^*,\rho)}(M)\right) \to \left(h_{R^*}, \widehat{M(R^*,\rho)}\right).$$

It follows from (10.5.2) and (10.5.4) that $\Phi_{(F,M)}$ is a natural equivalence if M is quasi-coherent.

Let $E : \mathcal{D} \to \mathcal{MOD}$ be a functor and put $F = p_{\mathcal{C}}E$. Suppose that $\left(E(i) \xrightarrow{(f_i,\boldsymbol{\eta}_i)} (X,L)\right)$ is a colimiting cone of E. We put $E(i) = (F(i), M(i))$ for $i \in \mathrm{Ob}\,\mathcal{D}$ and $E(\tau) = (F(\tau), M(\tau))$ for $\tau \in \mathcal{D}(i,j)$. Then, $M(\tau) : F(\tau)^*(M(j)) \to M(i)$ and $\boldsymbol{\eta}_i : f_i^*(L) \to M(i)$ are morphisms of $F(i)$-modules. The following diagrams commute for morphisms $\tau : i \to j$, $\sigma : j \to k$ of $\mathcal{D}$.

$$\begin{tikzcd}
F(\tau)^*F(\sigma)^*(M(k)) \arrow[r, "F(\tau)^*(M(\sigma))"] \arrow[d, equal] & F(\tau)^*(M(j)) \arrow[d, "M(\tau)"] \\
F(\sigma\tau)^*(M(k)) \arrow[r, "M(\sigma\tau)"] & M(i)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(\tau)^*f_j^*(L) \arrow[r, "F(\tau)^*(\boldsymbol{\eta}_j)"] \arrow[d, equal] & F(\tau)^*(M(j)) \arrow[d, "M(\tau)"] \\
f_i^*(L) \arrow[r, "\boldsymbol{\eta}_i"] & M(i)
\end{tikzcd}$$

11 Representations of group objects

11.1 Representations of group objects

Assume that $\mathcal{T}$ is a category with finite products. Let $p : \mathcal{F} \to \mathcal{T}$ be a cloven fibered category and $f : Y \to X$ a morphism of $\mathcal{T}$. For objects M and N of $\mathcal{F}_1$ and a morphism $\xi : o_X^*(M) \to o_X^*(N)$ of $\mathcal{F}_X$, we denote $f^\sharp_{M,N}(\xi)$ by ξ_f for short. That is, ξ_f is the following composition.

$$o_Y^*(M) = (o_X f)^*(M) \xrightarrow{c_{o_X,f}(M)^{-1}} f^* o_X^*(M) \xrightarrow{f^*(\xi)} f^* o_X^*(N) \xrightarrow{c_{o_X,f}(N)} (o_X f)^*(N) = o_Y^*(N)$$

Definition 11.1.1 *Let $(G,\mu,\varepsilon,\iota)$ be a group object in $\mathcal{T}$. A pair (M,ξ) of an object M of $\mathcal{F}_1$ and a morphism $\xi : o_G^*(M) \to o_G^*(M)$ of $\mathcal{F}_G$ is called a left (resp. right) representation of G on M if the following conditions (i) (resp. (i′)) and (ii) are satisfied. If we say simply "a representation", this means a left representation. We denote by $\mathrm{pr}_1, \mathrm{pr}_2 : G \times G \to G$ the projections below.*

$$(i)\ \ \xi_\mu = \xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2} \qquad (i')\ \ \xi_\mu = \xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1} \qquad (ii)\ \ \xi_\varepsilon = id_M$$

Let $\xi : o_G^*(M) \to o_G^*(M)$ and $\zeta : o_G^*(N) \to o_G^*(N)$ be representations of G on M and N, respectively. A morphism $\varphi : M \to N$ of $\mathcal{F}_1$ is a called a morphism of representations of G from (M,ξ) to (N,ζ) if the following diagram commutes.

$$\begin{tikzcd} o_G^*(M) \arrow[r, "\xi"] \arrow[d, "o_G^*(\varphi)"] & o_G^*(M) \arrow[d, "o_G^*(\varphi)"] \\ o_G^*(N) \arrow[r, "\zeta"] & o_G^*(N) \end{tikzcd}$$

We denote by $\mathrm{Rep}(G\,;\mathcal{F})$ the category of representations of G and morphisms between them.

Example 11.1.2 *Let $(G,\mu,\varepsilon,\iota)$ be a group object in $\mathcal{T}$.*

(1) Let $p : \mathcal{T}^{(2)} \to \mathcal{T}$ be the fibered category given in (1) of (6.1.9). For $(1 \xrightarrow{\eta} X) \in \mathrm{Ob}\,\mathcal{T}_1^{(2)}$, a morphism $\xi = (id_G, \tilde{\xi}) : o_G^*(1 \xrightarrow{\eta} X) = (G \xrightarrow{\eta o_G} X) \to (G \xrightarrow{\eta o_G} X) = o_G^*(1 \xrightarrow{\eta} X)$ is a representation of G on $(1 \xrightarrow{\eta} X)$ if and only if $\tilde{\xi} : X \to X$ is the identity morphism of X by the condition (ii) of (11.1.1).

(2) Let $p : \mathcal{T}^{(2)} \to \mathcal{T}$ be the fibered category given in (2) of (6.1.9). For $(X \xrightarrow{o_X} 1) \in \mathrm{Ob}\,\mathcal{T}_1^{(2)}$, a morphism

$$\xi = ((\xi_1,\xi_2), id_G) : o_G^*(X \xrightarrow{o_X} 1) = (G \times X \xrightarrow{\mathrm{pr}_G} G) \to (G \times X \xrightarrow{\mathrm{pr}_G} G) = o_G^*(X \xrightarrow{o_X} 1)$$

is a representation of G on $(X \xrightarrow{o_X} 1)$ if and only if $\xi_1 = \mathrm{pr}_G : G \times X \to G$ and $\xi_2 : G \times X \to X$ is a left action of G. Let $\zeta = ((\mathrm{pr}_G, \zeta_2), id_G) : o_G^*(Y \xrightarrow{o_Y} 1) \to o_G^*(Y \xrightarrow{o_Y} 1)$ be a representation of G. For a morphism $f : X \to Y$ of $\mathcal{T}$, $(f, id_1) : (X \xrightarrow{o_X} 1) \to (Y \xrightarrow{o_Y} 1)$ is a morphism $\xi \to \zeta$ of representations if and only if $f : (X,\xi_2) \to (Y,\zeta_2)$ is a morphism of $\mathrm{Act}_l(G)$. Hence $\mathrm{Rep}(G\,;\mathcal{T}^{(2)})$ is identified with $\mathrm{Act}_l(G)$. We remark that the category of right representations of G is identified with $\mathrm{Act}_r(G)$.

Definition 11.1.3 *Let M be an object of $\mathcal{F}_1$.*

(1) The identity morphism of $o_G^*(M)$ is a representation of G on M and it is called the trivial representation of G on M.

(2) Let $\eta : N \to M$ be a subobject of M and (M,ξ), (N,ζ) representations of G on M, N, respectively. If η is a morphism of representations from (N,ζ) to (M,ξ), we call (N,ζ) a subrepresentation of (M,ξ).

Proposition 11.1.4 *Let $f : (H,\mu',\varepsilon',\iota') \to (G,\mu,\varepsilon,\iota)$ be a morphism of group objects in $\mathcal{T}$ and (M,ξ) a representation of G on M. Then, (M,ξ_f) is a representation of H on M.*

Proof. We denote by $\mathrm{pr}'_1, \mathrm{pr}'_2 : H \times H \to H$ the projections. Then, we have $f\mu' = \mu(f \times f)$, $f\mathrm{pr}'_2 = \mathrm{pr}_2(f \times f)$, $f\mathrm{pr}'_1 = \mathrm{pr}_1(f \times f)$ and $f\varepsilon' = \varepsilon$. Therefore (6.1.17) and (6.1.18) imply $(\xi_f)_{\varepsilon'} = \xi_{f\varepsilon'} = \xi_\varepsilon = id_M$ and

$$\begin{aligned}(\xi_f)_{\mathrm{pr}'_1}(\xi_f)_{\mathrm{pr}'_2} &= \xi_{f\mathrm{pr}'_1}\xi_{f\mathrm{pr}'_2} = \xi_{\mathrm{pr}_2(f\times f)}\xi_{\mathrm{pr}_1(f\times f)} = (\xi_{\mathrm{pr}_2})_{(f\times f)}(\xi_{\mathrm{pr}_1})_{(f\times f)} = (\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1})_{(f\times f)} \\ &= (\xi_\mu)_{f\times f} = \xi_{\mu(f\times f)} = \xi_{f\mu'} = (\xi_f)_{\mu'}.\end{aligned}$$

□

Let $\varphi : (M,\xi) \to (N,\zeta)$ be a morphism of $\mathrm{Rep}(G\,;\mathcal{F})$ and $f : H \to G$ a morphism of group objects in $\mathcal{T}$. It follows from (6.1.11) that the following diagram is commutative.

$$\begin{tikzcd}
o_H^*(M) \arrow[r, equal] \arrow[d, "o_H^*(\varphi)"] & (o_Gf)^*(M) \arrow[r, "c_{o_G,f}(M)"] \arrow[d, "(o_Gf)^*(\varphi)"] & f^*o_G^*(M) \arrow[r, "f^*(\xi)"] \arrow[d, "f^*o_G^*(\varphi)"] & f^*o_G^*(M) \arrow[r, "c_{o_G,f}(M)^{-1}"] \arrow[d, "f^*o_G^*(\varphi)"] & (o_Gf)^*(M) \arrow[r, equal] \arrow[d, "(o_Gf)^*(\varphi)"] & o_H^*(M) \arrow[d, "o_H^*(\varphi)"] \\
o_H^*(N) \arrow[r, equal] & (o_Gf)^*(N) \arrow[r, "c_{o_G,f}(N)"] & f^*o_G^*(N) \arrow[r, "f^*(\zeta)"] & f^*o_G^*(N) \arrow[r, "c_{o_G,f}(N)^{-1}"] & (o_Gf)^*(N) \arrow[r, equal] & o_H^*(N)
\end{tikzcd}$$

Thus we have a functor $f^\bullet : \mathrm{Rep}(G\,;\mathcal{F}) \to \mathrm{Rep}(H\,;\mathcal{F})$ given by $f^\bullet(M,\xi) = (M,\xi_f)$ and $f^\bullet(\varphi) = \varphi$.

Definition 11.1.5 *Let G be a group object of $\mathcal{T}$ and $i : H \to G$ a subgroup object of G. For a representation (M,ξ) of G, we call $i^\bullet(M,\xi)$ the restriction of (M,ξ) to H and denote this by $\mathrm{Res}_H^G(M,\xi)$.*

Lemma 11.1.6 *Let M, N be objects of $\mathcal{F}_1$ and $\xi : o_G^*(M) \to o_G^*(M)$, $\zeta : o_G^*(N) \to o_G^*(N)$ morphisms of $\mathcal{F}_G$. We assume that a morphism $\varphi : M \to N$ of $\mathcal{F}_1$ makes the following diagram commute.*

$$\begin{tikzcd}
o_G^*(M) \arrow[r, "\xi"] \arrow[d, "o_G^*(\varphi)"] & o_G^*(M) \arrow[d, "o_G^*(\varphi)"] \\
o_G^*(N) \arrow[r, "\zeta"] & o_G^*(N)
\end{tikzcd}$$

(1) Suppose that $\varphi : M \to N$ is an epimorphism of $\mathcal{F}_1$ and that

$$o_{G\times G}^*(\varphi)^* : \mathcal{F}_{G\times G}(o_{G\times G}^*(N), o_{G\times G}^*(N)) \to \mathcal{F}_{G\times G}(o_{G\times G}^*(M), o_{G\times G}^*(N))$$

is injective. If ξ is a representation of G on M, ζ is a representation of G on N.

(2) Suppose that $\varphi : M \to N$ is an monomorphism of $\mathcal{F}_1$ and that

$$o_{G\times G}^*(\varphi)_* : \mathcal{F}_{G\times G}(o_{G\times G}^*(M), o_{G\times G}^*(M)) \to \mathcal{F}_{G\times G}(o_{G\times G}^*(M), o_{G\times G}^*(N))$$

is injective. If ζ is a representation of G on N, ξ is a representation of G on M.

Proof. The following diagram commutes by the assumption and (6.1.17).

$$\begin{tikzcd}
o_{G\times G}^*(M) \arrow[d, "o_{G\times G}^*(\varphi)"] & o_{G\times G}^*(M) \arrow[l, "\xi_\mu"'] \arrow[r, "\xi_{\mathrm{pr}_2}"] \arrow[d, "o_{G\times G}^*(\varphi)"] & o_{G\times G}^*(M) \arrow[r, "\xi_{\mathrm{pr}_1}"] \arrow[d, "o_{G\times G}^*(\varphi)"] & o_{G\times G}^*(M) \arrow[d, "o_{G\times G}^*(\varphi)"] & o_1^*(M) \arrow[r, "\xi_\varepsilon"] \arrow[d, "o_1^*(\varphi)"] & o_1^*(M) \arrow[d, "o_1^*(\varphi)"] \\
o_{G\times G}^*(N) & o_{G\times G}^*(N) \arrow[l, "\zeta_\mu"'] \arrow[r, "\zeta_{\mathrm{pr}_2}"] & o_{G\times G}^*(N) \arrow[r, "\zeta_{\mathrm{pr}_1}"] & o_{G\times G}^*(N) & o_1^*(N) \arrow[r, "\zeta_\varepsilon"] & o_1^*(N)
\end{tikzcd}$$

(1) If ξ is a representation of G on M, it follows from the commutativity of the above diagrams that we have $\zeta_{\mathrm{pr}_1}\zeta_{\mathrm{pr}_2}o_{G\times G}^*(\varphi) = o_{G\times G}^*(\varphi)\xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2} = o_{G\times G}^*(\varphi)\xi_\mu = \zeta_\mu o_{G\times G}^*(\varphi)$ and $\zeta_\varepsilon o_1^*(\varphi) = o_1^*(\varphi)\xi_\varepsilon = o_1^*(\varphi)$. Hence $\zeta_{\mathrm{pr}_1}\zeta_{\mathrm{pr}_2} = \zeta_\mu$ and $\zeta_\varepsilon = id_N$ by the assumption.

(2) If ζ is a representation of G on N, it follows from the commutativity of the above diagrams that we have $o_{G\times G}^*(\varphi)\xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2} = \zeta_{\mathrm{pr}_1}\zeta_{\mathrm{pr}_2}o_{G\times G}^*(\varphi) = \zeta_\mu o_{G\times G}^*(\varphi) = o_{G\times G}^*(\varphi)\xi_\mu$ and $o_1^*(\varphi)\xi_\varepsilon = \zeta_\varepsilon o_1^*(\varphi) = o_1^*(\varphi)$. Hence $\xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2} = \xi_\mu$ and $\xi_\varepsilon = id_M$ by the assumption. □

Proposition 11.1.7 *Let $\varphi : M \to N$ be a morphism of $\mathcal{F}_1$.*

(1) If φ is an epimorphism and one of the following conditions is satisfied, the condition of (1) of (11.1.6) is satisfied.

(i) $o_{G\times G}^* : \mathcal{F}_1 \to \mathcal{F}_{G\times G}$ preserves epimorphisms.
(ii) The presheaf $F_N^{G\times G}$ on $\mathcal{F}_1$ is representable.
(iii) The presheaves $F_{G\times G,M}$, $F_{G\times G,N}$ on $\mathcal{F}_1^{op}$ are representable and $\varphi_{G\times G}^* : \mathcal{F}_1(N_{G\times G}, N) \to \mathcal{F}_1(M_{G\times G}, N)$ is injective.

(2) If φ is a monomorphism and one of the following conditions is satisfied, the condition of (2) of (11.1.6) is satisfied.

(i) $o_{G\times G}^* : \mathcal{F}_1 \to \mathcal{F}_{G\times G}$ preserves monomorphisms.
(ii) The presheaf $F_{G\times G,M}$ on $\mathcal{F}_1^{op}$ is representable.
(iii) The presheaves $F_M^{G\times G}$, $F_N^{G\times G}$ on $\mathcal{F}_1$ are representable and $\varphi_*^{G\times G} : \mathcal{F}_1(M, M^{G\times G}) \to \mathcal{F}_1(M, N^{G\times G})$ is injective.

Proof. (1) If (i) is satisfied, $o^*_{G\times G}(\varphi)$ is an epimorphism. If (ii) is satisfied, the assertion follows from (1) of (6.5.2). Assume that (iii) is satisfied. The following diagram is commutative by (6.3.3),

$$\begin{tikzcd}
\mathcal{F}_{G\times G}(o^*_{G\times G}(N), o^*_{G\times G}(N)) \arrow[r, "P_{G\times G}(N)_N"] \arrow[d, "o^*_{G\times G}(\varphi)^*"] & \mathcal{F}_1(N_{G\times G}, N) \arrow[d, "\varphi^*_{G\times G}"] \\
\mathcal{F}_{G\times G}(o^*_{G\times G}(M), o^*_{G\times G}(N)) \arrow[r, "P_{G\times G}(M)_N"] & \mathcal{F}_1(M_{G\times G}, N)
\end{tikzcd}$$

Since both $\varphi^*_{G\times G}$ and $P_{G\times G}(N)_N$ are injective, so is $o^*_{G\times G}(\varphi)^*$.

(2) If (i) is satisfied, $o^*_{G\times G}(\varphi)$ is a monomorphism. If (ii) is satisfied, the assertion follows from (2) of (6.5.2). Assume that (iii) is satisfied. The following diagram is commutative by (6.4.3),

$$\begin{tikzcd}
\mathcal{F}_{G\times G}(o^*_{G\times G}(M), o^*_{G\times G}(M)) \arrow[r, "E_{G\times G}(M)_M"] \arrow[d, "o^*_{G\times G}(\varphi)_*"] & \mathcal{F}_1(M, M^{G\times G}) \arrow[d, "\varphi^{G\times G}_*"] \\
\mathcal{F}_{G\times G}(o^*_{G\times G}(M), o^*_{G\times G}(N)) \arrow[r, "E_{G\times G}(N)_M"] & \mathcal{F}_1(M, N^{G\times G})
\end{tikzcd}$$

Since both $\varphi^{G\times G}_*$ and $E_{G\times G}(M)_M$ are injective, so is $o^*_{G\times G}(\varphi)^*$. □

Proposition 11.1.8 *Let (M,ξ) and (M,ζ) be representations of G on $M \in \mathrm{Ob}\,\mathcal{F}_1$. If $\xi_{\mathrm{pr}_2}\zeta_{\mathrm{pr}_1} = \zeta_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2}$, then $(M,\xi\zeta)$ is a representation of G on M.*

Proof. Since $\xi_\mu = \xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2}$ and $\zeta_\mu = \zeta_{\mathrm{pr}_1}\zeta_{\mathrm{pr}_2}$, it follows from (6.1.5) that we have

$$(\xi\zeta)_\mu = \xi_\mu\zeta_\mu = \xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2}\zeta_{\mathrm{pr}_1}\zeta_{\mathrm{pr}_2} = \xi_{\mathrm{pr}_1}\zeta_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2}\zeta_{\mathrm{pr}_2} = (\xi\zeta)_{\mathrm{pr}_1}(\xi\zeta)_{\mathrm{pr}_2}.$$

We also have $(\xi\zeta)_\varepsilon = \xi_\varepsilon\zeta_\varepsilon = id_M$. □

Corollary 11.1.9 *Let G and H be group objects of $\mathcal{T}$ and (M,ξ), (M,ζ) representations of G, H on $M \in \mathrm{Ob}\,\mathcal{F}_1$, respectively. We denote by $p_G : G\times H \to G$ and $p_H : G\times H \to H$ the projections. Suppose that representations (M,ξ_{p_G}) and (M,ζ_{p_H}) of $G\times H$ on M satisfy $\xi_{p_G}\zeta_{p_H} = \zeta_{p_H}\xi_{p_G}$. Then, $(M,\xi_{p_G}\zeta_{p_H})$ is a representation of $G\times H$ on M.*

Proof. Let us denote by $\mathrm{pr}_1, \mathrm{pr}_2 : G\times G \to G$, $\mathrm{pr}'_1, \mathrm{pr}'_2 : H\times H \to H$ and $\mathrm{pr}''_1, \mathrm{pr}''_2 : (G\times H)\times(G\times H) \to G\times H$ the projections. We put $\chi = (\mathrm{pr}_2 \times \mathrm{pr}'_1)(id_G \times T_{H,G} \times id_H)$. Then, we have $p_G\mathrm{pr}''_2 = p_G\chi$ and $p_H\mathrm{pr}''_1 = p_G\chi$. It follows from (6.1.17) and (6.1.18)

$$\xi_{p_G\mathrm{pr}''_2}\zeta_{p_H\mathrm{pr}''_1} = \xi_{p_G\chi}\zeta_{p_H\chi} = (\xi_{p_G})_\chi(\zeta_{p_H})_\chi = (\xi_{p_G}\zeta_{p_H})_\chi = (\zeta_{p_H}\xi_{p_G})_\chi = (\zeta_{p_H})_\chi(\xi_{p_G})_\chi = \zeta_{p_H\chi}\xi_{p_G\chi} = \zeta_{p_H\mathrm{pr}''_1}\xi_{p_G\mathrm{pr}''_2}.$$

Hence the assertion follows from (11.1.8). □

Let $p : \mathcal{F} \to \mathcal{T}$, $q : \mathcal{G} \to \mathcal{C}$ be normalized cloven fibered categories and $(G,\mu,\varepsilon,\iota)$ a group object in $\mathcal{T}$. Suppose that functors $F : \mathcal{T} \to \mathcal{C}$ and $\Phi : \mathcal{F} \to \mathcal{G}$ are given such that $q\Phi = Fp$ and Φ preserves cartesian morphisms. We assume that $F(G) \xleftarrow{F(\mathrm{pr}_1)} F(G\times G) \xrightarrow{F(\mathrm{pr}_2)} F(G)$ is a product of $F(G)$ and $F(G)$ and that $F(1)$ is a terminal object of $\mathcal{C}$. Then, $(F(G), F(\mu), F(\varepsilon), F(\iota))$ is a group object in $\mathcal{C}$.

Proposition 11.1.10 *Let M be an object of $\mathcal{F}_1$ and $\xi : o^*_G(M) \to o^*_G(M)$ a morphism of $\mathcal{F}_G$.*

(1) If (M,ξ) is a representation of G on M, $(\Phi(M), \Phi^G_{M,M}(\xi))$ is a representation of $F(G)$ on $\Phi(M)$.

(2) If Φ is faithful and $(\Phi(M), \Phi^G_{M,M}(\xi))$ is a representation of $F(G)$ on $\Phi(M)$, (M,ξ) is a representation of G on M.

Proof. (1) It follows from (6.1.20) and (6.1.19) that we have the following equality.

$$\Phi^G_{M,M}(\xi)_{F(\mathrm{pr}_1)}\Phi^G_{M,M}(\xi)_{F(\mathrm{pr}_2)} = \Phi^{G\times G}_{M,M}(\xi_{\mathrm{pr}_1})\Phi^{G\times G}_{M,M}(\xi_{\mathrm{pr}_2}) = \Phi^{G\times G}_{M,M}(\xi_{\mathrm{pr}_1})\Phi^{G\times G}_{M,M}(\xi_{\mathrm{pr}_2}) = \Phi^{G\times G}_{M,M}(\xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2})$$

Thus $\Phi^G_{M,M}(\xi)_{F(\mathrm{pr}_1)}\Phi^G_{M,M}(\xi)_{F(\mathrm{pr}_2)} = \Phi^{G\times G}_{M,M}(\xi_\mu) = \Phi^G_{M,M}(\xi)_{F(\mu)}$ by the assumption and (6.1.20). We also have $\Phi^G_{M,M}(\xi)_{F(\varepsilon)} = \Phi^1_{M,M}(\xi_\varepsilon) = \Phi^1_{M,M}(id_M) = id_{\Phi(M)}$ by (6.1.20) and the assumption. Hence $(\Phi(M), \Phi^G_{M,M}(\xi))$ is a representation of $F(\boldsymbol{C})$ on $\Phi(M)$.

(2) By (6.1.20), the assumption and the equality of (1) above, we have

$$\Phi^{G\times G}_{M,M}(\xi_\mu) = \Phi^G_{M,M}(\xi)_{F(\mu)} = \Phi^G_{M,M}(\xi)_{F(\mathrm{pr}_1)}\Phi^G_{M,M}(\xi)_{F(\mathrm{pr}_2)} = \Phi^{G\times G}_{M,M}(\xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2})$$
$$\Phi^1_{M,M}(\xi_\varepsilon) = \Phi^G_{M,M}(\xi)_{F(\varepsilon)} = id_{\Phi(M)} = \Phi^1_{M,M}(id_M)$$

Since Φ is faithful, $\Phi^{G\times G}_{M,M} : \mathcal{F}_{G\times G}(o^*_{G\times G}(M), o^*_{G\times G}(M)) \to \mathcal{G}_{F(G\times G)}(o^*_{F(G\times G)}(\Phi(M)), o^*_{F(G\times G)}(\Phi(M)))$ and $\Phi^1_{M,M} : \mathcal{F}_1(id^*_1(M), id^*_1(M)) \to \mathcal{G}_{F(1)}(id^*_{F(1)}(\Phi(M)), id^*_{F(1)}(\Phi(M)))$ are injective, which implies $\xi_\mu = \xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2}$ and $\xi_\varepsilon = id_M$. $\square$

Proposition 11.1.11 *Let $\varphi : M \to N$ be a morphism of $\mathcal{F}_1$ and (M,ξ), (N,ζ) representations of G.*
(1) If $\varphi : (M,\xi) \to (N,\zeta)$ is a morphism representations of G, $\Phi(\varphi) : (\Phi(M), \Phi^G_{M,M}(\xi)) \to (\Phi(N), \Phi^G_{N,N}(\zeta))$ is a morphism representations of $F(G)$.
(2) If Φ is faithful and $\Phi(\varphi) : (\Phi(M), \Phi^G_{M,M}(\xi)) \to (\Phi(N), \Phi^G_{N,N}(\zeta))$ is a morphism representations of $F(\boldsymbol{C})$, $\varphi : (M,\xi) \to (N,\zeta)$ is a morphism representations of $\boldsymbol{C}$.

Proof. By (6.1.15) that the left and the right rectangles of the following diagram $(*)$ are commutative.

$$\begin{tikzcd}
o^*_{F(G)}(\Phi(M)) \arrow[r, "c_{o_G,\Phi}(M)^{-1}"] \arrow[d, "o^*_{F(G)}(\Phi(\varphi))"] & \Phi(o^*_G(M)) \arrow[r, "\Phi(\xi)"] \arrow[d, "\Phi(o^*_G(\varphi))"] & \Phi(o^*_G(M)) \arrow[r, "c_{o_G,\Phi}(M)"] \arrow[d, "\Phi(o^*_G(\varphi))"] & o^*_{F(G)}(\Phi(M)) \arrow[d, "o^*_{F(G)}(\Phi(\varphi))"] & (*) \\
o^*_{F(G)}(\Phi(N)) \arrow[r, "c_{o_G,\Phi}(N)^{-1}"] & \Phi(o^*_G(N)) \arrow[r, "\Phi(\zeta)"] & \Phi(o^*_G(N)) \arrow[r, "c_{o_G,\Phi}(N)"] & o^*_{F(G)}(\Phi(N)) &
\end{tikzcd}$$

(1) Since $\Phi^G_{M,M}(\xi) = c_{o_G,\Phi}(M)\Phi(\xi)c_{o_G,\Phi}(M)^{-1}$, $\Phi^G_{N,N}(\zeta) = c_{o_G,\Phi}(N)\Phi(\zeta)c_{o_G,\Phi}(N)^{-1}$ and the middle rectangle of $(*)$ is commutative, the assertion follows.
(2) Since the outer rectangle of $(*)$ is commutative, we have

$$c_{o_G,\Phi}(N)\Phi(o^*_G(\varphi)\xi)c_{o_G,\Phi}(M)^{-1} = c_{o_G,\Phi}(N)\Phi(\zeta o^*_G(\varphi))c_{o_G,\Phi}(M)^{-1}.$$

Thus $\Phi(o^*_G(\varphi)\xi) = \Phi(\zeta o^*_G(\varphi))$ which implies $o^*_G(\varphi)\xi = \zeta o^*_G(\varphi)$ by the assumption. $\square$

We can define a functor $\Phi_G : \mathrm{Rep}(G;\mathcal{F}) \to \mathrm{Rep}(F(G);\mathcal{G})$ by $\Phi_G(M,\xi) = (\Phi(M), \Phi^G_{M,M}(\xi))$ and $\Phi_G(\varphi) = \Phi(\varphi)$ under the above situation. It follows from (11.1.11) that Φ_G is fully faithful if Φ is so.

Let $f : (H, \mu', \varepsilon', \iota') \to (G, \mu, \varepsilon, \iota)$ be a morphism of group objects in $\mathcal{T}$. We also assume that $F(H) \xleftarrow{F(\mathrm{pr}_1)} F(H\times H) \xrightarrow{F(\mathrm{pr}_2)} F(H)$ is a product of $F(H)$ and $F(H)$.

Proposition 11.1.12 *For a representation (M,ξ) of G, we have $\Phi^H_{M,M}(\xi_f) = \Phi^G_{M,M}(\xi)_{F(f)}$.*

Proof. The middle rectangle of the following diagram is commutative by (6.1.20).

$$\begin{tikzcd}
o^*_{F(H)}(\Phi(M)) \arrow[r, "c_{o_H,\Phi}(M)^{-1}"] \arrow[dr, "id_{o^*_{F(H)}(\Phi(M))}"'] & \Phi(o^*_H(M)) \arrow[r, "\Phi(f^\sharp(\xi))"] \arrow[d, "c_{o_H,\Phi}(M)"] & \Phi(o^*_H(M)) \arrow[r, "c_{o_H,\Phi}(M)"] \arrow[d, "c_{o_H,\Phi}(M)"] & o^*_{F(H)}(\Phi(M)) \\
& o^*_{F(H)}(\Phi(M)) \arrow[r, "F(f)^\sharp(\Phi^G_{M,M}(\xi))"] & o^*_{F(H)}(\Phi(M)) \arrow[ur, "id_{o^*_{F(H)}(\Phi(M))}"'] &
\end{tikzcd}$$

Since the upper horizontal composition of the above diagram is $\Phi^H_{M,M}(\xi_f)$, the assertion follows. $\square$

Let $\mathscr{F}_G : \mathrm{Rep}(G\,;\mathcal{F}) \to \mathcal{F}_1$ be a forgetful functor defined by $\mathscr{F}_G(M,\xi) = M$ and $\mathscr{F}_G(f) = f$.

Definition 11.1.13 *Let (M,ρ) be a representation of G on $M \in \mathrm{Ob}\,\mathcal{F}_1$.*
(1) (M,ρ) is called a left regular representation if there exist an object L of $\mathcal{F}_1$ and a bijection

$$\mathscr{A}^l_{(N,\xi)} : \mathrm{Rep}(G\,;\mathcal{F})((M,\rho),(N,\xi)) \to \mathcal{F}_1(L, \mathscr{F}_G(N,\xi))$$

for each $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(G\,;\mathcal{F})$ which is natural in (N,ξ).
(2) (M,ρ) is called a right regular representation if there exist an object R of $\mathcal{F}_1$ and a bijection

$$\mathscr{A}^r_{(N,\xi)} : \mathrm{Rep}(G\,;\mathcal{F})((N,\xi),(M,\rho)) \to \mathcal{F}_1(\mathscr{F}_G(N,\xi), R)$$

for each $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(G\,;\mathcal{F})$ which is natural in (N,ξ).

Proposition 11.1.14 *Let (M,ρ) be a representation of G on $M \in \mathrm{Ob}\,\mathcal{F}_1$.*

(1) (M,ρ) is a left regular representation if and only if there exists a morphism $\eta : L \to \mathscr{F}_G(M,\rho)$ of $\mathcal{F}_1$ such that, for any $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(G\,;\mathcal{F})$, the following composition is bijective.

$$\mathrm{Rep}(G\,;\mathcal{F})((M,\rho),(N,\xi)) \xrightarrow{\mathscr{F}_G} \mathcal{F}_1(\mathscr{F}_G(M,\rho),\mathscr{F}_G(N,\xi)) \xrightarrow{\eta^*} \mathcal{F}_1(L,\mathscr{F}_G(N,\xi))$$

(2) (M,ρ) is a right regular representation if and only if there exists a morphism $\varepsilon : \mathscr{F}_G(M,\rho) \to R$ of $\mathcal{F}_1$ such that, for any $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(G\,;\mathcal{F})$, the following composition is bijective.

$$\mathrm{Rep}(G\,;\mathcal{F})((N,\xi),(M,\rho)) \xrightarrow{\mathscr{F}_G} \mathcal{F}_1(\mathscr{F}_G(N,\xi),\mathscr{F}_G(M,\rho)) \xrightarrow{\varepsilon_*} \mathcal{F}_1(\mathscr{F}_G(N,\xi),R)$$

Proof. (1) Suppose that (M,ρ) is a left regular representation. We take $L \in \mathrm{Ob}\,\mathcal{F}_1$ and a natural bijection $\mathscr{A}^l_{(N,\xi)}$ as in (1) of (11.1.13). Put $\eta = \mathscr{A}^l_{(M,\rho)}(id_{(M,\rho)}) : L \to \mathscr{F}_G(M,\rho)$. For $f \in \mathrm{Rep}(G\,;\mathcal{F})((M,\rho),(N,\xi))$, the naturality of $\mathscr{A}^l$ implies $\mathscr{F}_G(f)\eta = \mathscr{F}_G(f)\mathscr{A}^l_{(M,\rho)}(id_{(M,\rho)}) = \mathscr{A}^l_{(N,\xi)}(f)$. Hence the composition $\eta^*\mathscr{F}_G : \mathrm{Rep}(G\,;\mathcal{F})((M,\rho),(N,\xi)) \to \mathcal{F}_1(L,\mathscr{F}_G(N,\xi))$ coincides with $\mathscr{A}^l_{(N,\xi)}$. The converse is obvious.

(2) Suppose that (M,ρ) is a right regular representation. We take $R \in \mathrm{Ob}\,\mathcal{F}_1$ and a natural bijection $\mathscr{A}^r_{(N,\xi)}$ as in (2) of (11.1.13). Put $\varepsilon = \mathscr{A}^r_{(M,\rho)}(id_{(M,\rho)}) : \mathscr{F}_G(M,\rho) \to R$. For $f \in \mathrm{Rep}(G\,;\mathcal{F})((N,\xi),(M,\rho))$, the naturality of $\mathscr{A}^r$ implies $\varepsilon\mathscr{F}_G(f) = \mathscr{A}^r_{(M,\rho)}(id_{(M,\rho)})\mathscr{F}_G(f) = \mathscr{A}^r_{(N,\xi)}(f)$. Hence the composition $\varepsilon_*\mathscr{F}_G : \mathrm{Rep}(G\,;\mathcal{F})((N,\xi),(M,\rho)) \to \mathcal{F}_1(\mathscr{F}_G(N,\xi),R)$ coincides with $\mathscr{A}^r_{(N,\xi)}$. The converse is obvious. □

Proposition 11.1.15 *The following assertions hold.*

(1) The forgetful functor $\mathscr{F}_G : \mathrm{Rep}(G\,;\mathcal{F}) \to \mathcal{F}_1$ has a left adjoint if and only if, for every $L \in \mathrm{Ob}\,\mathcal{F}_1$, there exist a representation (M_L,ρ_L) of G and a morphism $\eta_L : L \to \mathscr{F}_G(M_L,\rho_L)$ of $\mathcal{F}_1$ such that, for any $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(G\,;\mathcal{F})$, the following composition is bijective.

$$\mathrm{Rep}(G\,;\mathcal{F})((M_L,\rho_L),(N,\xi)) \xrightarrow{\mathscr{F}_G} \mathcal{F}_1(\mathscr{F}_G(M_L,\rho_L),\mathscr{F}_G(N,\xi)) \xrightarrow{\eta_L^*} \mathcal{F}_1(L,\mathscr{F}_G(N,\xi))$$

(2) The forgetful functor $\mathscr{F}_G : \mathrm{Rep}(G\,;\mathcal{F}) \to \mathcal{F}_1$ has a right adjoint if and only if, for every $R \in \mathrm{Ob}\,\mathcal{F}_1$, there exist a representation (M_R,ρ_R) of G and a morphism $\varepsilon_R : \mathscr{F}_G(M_R,\rho_R) \to R$ of $\mathcal{F}_1$ such that, for any $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(G\,;\mathcal{F})$, the following composition is bijective.

$$\mathrm{Rep}(G\,;\mathcal{F})((N,\xi),(M_R,\rho_R)) \xrightarrow{\mathscr{F}_G} \mathcal{F}_1(\mathscr{F}_G(N,\xi),\mathscr{F}_G(M_R,\rho_R)) \xrightarrow{\varepsilon_{R*}} \mathcal{F}_1(\mathscr{F}_G(N,\xi),R)$$

Proof. (1) Suppose that $\mathscr{F}_G$ has a left adjoint $\mathscr{L}_G : \mathcal{F}_1 \to \mathrm{Rep}(G\,;\mathcal{F})$. Let $\eta : id_{\mathcal{F}_1} \to \mathscr{F}_G\mathscr{L}_G$ be the unit of this adjunction. For $L \in \mathrm{Ob}\,\mathcal{F}_1$, a representation $\mathscr{L}_G(L)$ and a morphism $\eta_L : L \to \mathscr{F}_G\mathscr{L}_G(L)$ satisfies the condition. In fact, for $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(G\,;\mathcal{F})$, the composition

$$\mathrm{Rep}(G\,;\mathcal{F})(\mathscr{L}_G(L),(N,\xi)) \xrightarrow{\mathscr{F}_G} \mathcal{F}_1(\mathscr{F}_G\mathscr{L}_G(L),\mathscr{F}_G(N,\xi)) \xrightarrow{\eta_L^*} \mathcal{F}_1(L,\mathscr{F}_G(N,\xi))$$

is the adjoint bijection. We show the converse. Define a functor $\mathscr{L}_G : \mathcal{F}_1 \to \mathrm{Rep}(G\,;\mathcal{F})$ as follows. For an object L of $\mathcal{F}_1$, put $\mathscr{L}_G(L) = (M_L,\rho_L)$. For a morphism $\varphi : L \to K$ of $\mathcal{F}_1$, let $\mathscr{L}_G(\varphi) : (M_L,\rho_L) \to (M_K,\rho_K)$ be the morphism of $\mathrm{Rep}(G\,;\mathcal{F})$ which maps to $\eta_K\varphi$ by the composition

$$\mathrm{Rep}(G\,;\mathcal{F})((M_L,\rho_L),(M_K,\rho_K)) \xrightarrow{\mathscr{F}_G} \mathcal{F}_1(\mathscr{F}_G(M_L,\rho_L),\mathscr{F}_G(M_K,\rho_K)) \xrightarrow{\eta_L^*} \mathcal{F}_1(L,\mathscr{F}_G(M_K,\rho_K)).$$

It is easy to verify that $\mathscr{L}_G$ is a functor and that it is a left adjoint of $\mathscr{F}_G$.

(2) Suppose that $\mathscr{F}_G$ has right adjoint $\mathscr{R}_G : \mathcal{F}_1 \to \mathrm{Rep}(G\,;\mathcal{F})$. Let $\varepsilon : \mathscr{F}_G\mathscr{R}_G \to id_{\mathcal{F}_1}$ be the counit of this adjunction. For $R \in \mathrm{Ob}\,\mathcal{F}_1$, a representation $\mathscr{R}_G(R)$ and a morphism $\varepsilon_R : \mathscr{F}_G\mathscr{R}_G(R) \to R$ satisfies the condition. In fact, for $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(G\,;\mathcal{F})$, the composition

$$\mathrm{Rep}(G\,;\mathcal{F})((N,\xi),\mathscr{R}_G(R)) \xrightarrow{\mathscr{F}_G} \mathcal{F}_1(\mathscr{F}_G(N,\xi),\mathscr{F}_G\mathscr{R}_G(R)) \xrightarrow{\varepsilon_{R*}} \mathcal{F}_1(\mathscr{F}_G(N,\xi),R)$$

is the adjoint bijection. We show the converse. Define a functor $\mathscr{R}_G : \mathcal{F}_1 \to \mathrm{Rep}(G\,;\mathcal{F})$ as follows. For an object R of $\mathcal{F}_1$, put $\mathscr{R}_G(R) = (M_R,\rho_R)$. For a morphism $\varphi : Q \to R$ of $\mathcal{F}_1$, let $\mathscr{R}_G(\varphi) : (M_Q,\rho_Q) \to (M_R,\rho_R)$ be the morphism of $\mathrm{Rep}(G\,;\mathcal{F})$ which maps to $\varphi\varepsilon_Q$ by the composition

$$\mathrm{Rep}(G\,;\mathcal{F})((M_Q,\rho_Q),(M_R,\rho_R)) \xrightarrow{\mathscr{F}_G} \mathcal{F}_1(\mathscr{F}_G(M_Q,\rho_Q),\mathscr{F}_G(M_R,\rho_R)) \xrightarrow{\varepsilon_{R*}} \mathcal{F}_1(\mathscr{F}_G(M_Q,\rho_Q),R).$$

It is easy to verify that $\mathscr{R}_G$ is a functor and that it is a right adjoint of $\mathscr{F}_G$. □

11.2 Representations in fibered categories with products

Let $p : \mathcal{F} \to \mathcal{T}$ be a normalized cloven fibered category with products and $(G, \mu, \varepsilon, \iota)$ a group object in $\mathcal{T}$.

Proposition 11.2.1 *For $M \in \mathrm{Ob}\,\mathcal{F}_1$ and $\xi \in \mathcal{F}_G(o_G^*(M), o_G^*(M))$, we put $\hat{\xi} = P_G(M)_M(\xi) : G \times M \to M$. Then, (M, ξ) is a representation of G on M if and only if the following diagrams commute.*

$$\begin{tikzcd}
(G \times G) \times M \arrow[r, "\mu \times M"] \arrow[d, "\theta_{G,G}(M)"] & G \times M \arrow[r, "\hat{\xi}"] & M \\
G \times (G \times M) \arrow[r, "G \times \hat{\xi}"] & G \times M \arrow[ur, "\hat{\xi}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \times M \arrow[d, "\varepsilon \times M"'] \arrow[dr, bend left, "id_M"] & \\
G \times M \arrow[r, "\hat{\xi}"] & M
\end{tikzcd}$$

Proof. We have $P_{G \times G}(M)_M(\xi_\mu) = \hat{\xi}(\mu \times M)$ and $P_{G \times G}(M)_M(\xi_{\mathrm{pr}_i}) = \hat{\xi}(\mathrm{pr}_i \times M)$ for $i = 1, 2$ by (1) of (6.3.6). Hence (6.3.3), (6.3.6), (6.3.9), (6.3.18) imply

$$\begin{aligned}
P_{G \times G}(M)_M(\xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2}) &= \hat{\xi}(\mathrm{pr}_1 \times M)((G \times G) \times \hat{\xi}(\mathrm{pr}_2 \times M))\delta_{G \times G, M} = \hat{\xi}(\mathrm{pr}_1 \times \hat{\xi}(\mathrm{pr}_2 \times M))\delta_{G \times G, M} \\
&= \hat{\xi}(G \times \hat{\xi})(\mathrm{pr}_1 \times (\mathrm{pr}_2 \times M))\delta_{G \times G, M} = \hat{\xi}(G \times \hat{\xi})\theta_{G,G}(M)
\end{aligned}$$

and $P_1(M)_M(\xi_\varepsilon) = \hat{\xi}(\varepsilon \times M)$. Hence $\xi_\mu = \xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2}$ and $\xi_\varepsilon = id_M$ are equivalent to $\hat{\xi}(G \times \hat{\xi})\theta_{G,G}(M) = \hat{\xi}(\mu \times M)$ and $\hat{\xi}(\varepsilon \times M) = id_M$, respectively. □

Remark 11.2.2 *(1) Let $T_{G,G} : G \times G \to G \times G$ be the switching map. $\xi \in \mathcal{F}_G(o_G^*(M), o_G^*(M))$ is a right representation of G if and only if $\hat{\xi}(G \times \hat{\xi})\theta_{G,G}(M)(T_{G,G} \times M) = \hat{\xi}(\mu \times M)$ and $\hat{\xi}(\varepsilon \times M) = id_M$.*

(2) The image of the trivial representation of G on M by $P_G(M)_M$ is $o_G \times M : G \times M \to 1 \times M = M$ by (3) of (6.3.6).

(3) Let $f : (H, \mu', \varepsilon', \iota') \to (G, \mu, \varepsilon, \iota)$ be a morphism of group objects in $\mathcal{T}$ and (M, ξ) a representation of G. It follows from (1) of (6.3.6) that $P_G(M)_M(\xi_f) = \hat{\xi}(f \times M)$.

The following fact is a direct consequence of (6.3.5).

Proposition 11.2.3 *Let (M, ξ) and (N, ζ) be representations of G and $\varphi : M \to N$ a morphism of $\mathcal{F}_1$. We put $\hat{\xi} = P_G(M)_M(\xi)$ and $\hat{\zeta} = P_G(N)_N(\zeta)$. Then, φ is a morphism of representations if and only if the following diagram is commutative.*

$$\begin{tikzcd}
G \times M \arrow[r, "\hat{\xi}"] \arrow[d, "G \times \varphi"] & M \arrow[d, "\varphi"] \\
G \times N \arrow[r, "\hat{\zeta}"] & N
\end{tikzcd}$$

Let $\alpha : G \times X \to X$ be a left G-action on $X \in \mathrm{Ob}\,\mathcal{T}$. For an object M of $\mathcal{F}_1$, we assume that $\theta_{G,X}(M) : (G \times X) \times M \to G \times (X \times M)$ is an isomorphism and that $\theta_{G \times G, X}(M) : ((G \times G) \times X) \times M \to (G \times G) \times (X \times M)$ is an epimorphism. We define $\alpha_l(M) : G \times (X \times M) \to X \times M$ to be the following composition

$$G \times (X \times M) \xrightarrow{\theta_{G,X}(M)^{-1}} (G \times X) \times M \xrightarrow{\alpha \times M} X \times M$$

and put $\xi_l(\alpha, M) = P_G(X \times M)^{-1}_{X \times M}(\alpha_l(M)) = o_G^*(\alpha_l(M))\iota_G(X \times M) \in \mathcal{F}_G(o_G^*(X \times M), o_G^*(X \times M))$.

Proposition 11.2.4 *$(X \times M, \xi_l(\alpha, M))$ is a representation of G on $X \times M$.*

Proof. The following diagrams commute by (6.3.6), (6.3.20) and (6.3.21).

$$\begin{tikzcd}
G \times (G \times (X \times M)) \arrow[d, "G \times \theta_{G,X}(M)^{-1}"] & (G \times G) \times (X \times M) \arrow[l, "\theta_{G,X(X \times M)}"'] \arrow[r, "\mu \times (X \times M)"] & G \times (X \times M) \arrow[d, "\theta_{G,X}(M)^{-1}"] \\
G \times ((G \times X) \times M) \arrow[d, "G \times (\alpha \times M)"] & (G \times G \times X) \times M \arrow[l, "\theta_{G,G \times X}(M)"'] \arrow[u, "\theta_{G \times G,X}(M)"'] \arrow[r, "(\mu \times id_X) \times M"] \arrow[d, "(id_G \times \alpha) \times M"] & (G \times X) \times M \arrow[d, "\alpha \times M"] \\
G \times (X \times M) \arrow[r, "\theta_{G,X}(M)^{-1}"] & (G \times X) \times M \arrow[r, "\alpha \times M"] & X \times M
\end{tikzcd}$$

$$\begin{tikzcd}
1 \times (X \times M) \arrow[r, "\theta_{1,X}(M)^{-1}"] \arrow[d, "\varepsilon \times (X \times M)"] & (1 \times X) \times M \arrow[r, "\mathrm{pr}_X \times M"] \arrow[d, "(\varepsilon \times id_X) \times M"] & X \times M \\
G \times (X \times M) \arrow[r, "\theta_{G,X}(M)^{-1}"] & (G \times X) \times M \arrow[ur, "\alpha \times M"'] &
\end{tikzcd}$$

Hence we have $\alpha_l(M)(G\times\alpha_l(M))\theta_{G,X}(X\times M)=\alpha_l(M)(\mu\times(X\times M))$ and $\alpha_l(M)(\varepsilon\times(X\times M))=id_{X\times M}$ by (6.3.22). Then, the assertion follows from (11.2.1). □

Proposition 11.2.5 *Let $\alpha : G\times X\to X$ be a left G-action on $X\in\operatorname{Ob}\mathcal{T}$. We assume that $\theta_{G,X}(K)$ is an isomorphism for $K=M,N\in\operatorname{Ob}\mathcal{F}_1$ and that $\theta_{G\times G,X}(K)$ is an epimorphismfor $K=M,N\in\operatorname{Ob}\mathcal{F}_1$. For a morphism $\varphi : M\to N$ of $\mathcal{F}_1$, $X\times\varphi : X\times M\to X\times N$ is a morphism of representations from $(X\times M,\xi_l(\alpha,M))$ to $(X\times N,\xi_l(\alpha,N))$.*

Proof. The following diagram is commutative by (6.3.9) and (6.3.20).

$$\begin{tikzcd}
G\times(X\times M) \arrow[r, "\theta_{G,X}(M)^{-1}"] \arrow[d, "G\times(X\times\varphi)"] & (G\times X)\times M \arrow[r, "\alpha\times M"] \arrow[d, "(G\times X)\times\varphi"] & X\times M \arrow[d, "X\times\varphi"] \\
G\times(X\times N) \arrow[r, "\theta_{G,X}(N)^{-1}"] & (G\times X)\times N \arrow[r, "\alpha\times N"] & X\times N
\end{tikzcd}$$

Since $\alpha_l(M)=(\alpha\times M)\theta_{X,G}(M)^{-1}$ and $\alpha_l(N)=(\alpha\times N)\theta_{X,G}(N)^{-1}$, the result follows from (11.2.3). □

Proposition 11.2.6 *Let $\alpha : G\times X\to X$ and $\beta : G\times Y\to Y$ be left G-actions on $X,Y\in\operatorname{Ob}\mathcal{T}$. Assume that $\theta_{G,Z}(M)$ is an isomorphism for $Z=X,Y$ and that $\theta_{G\times G,Z}(M)$ is an epimorphism for $Z=X,Y$. If a morphism $f : X\to Y$ of $\mathcal{T}$ preserves G-actions, $f\times M : X\times M\to Y\times M$ is a morphism of representations from $(X\times M,\xi_l(\alpha,M))$ to $(Y\times M,\xi_l(\beta,M))$.*

Proof. The following diagram is commutative by (6.3.6) and (6.3.20).

$$\begin{tikzcd}
G\times(X\times M) \arrow[r, "\theta_{G,X}(M)^{-1}"] \arrow[d, "G\times(f\times M)"] & (G\times X)\times M \arrow[r, "\alpha\times M"] \arrow[d, "(id_G\times f)\times M"] & X\times M \arrow[d, "f\times M"] \\
G\times(Y\times M) \arrow[r, "\theta_{G,Y}(M)^{-1}"] & (G\times Y)\times M \arrow[r, "\beta\times M"] & Y\times M
\end{tikzcd}$$

Since $\alpha_l(M)=(\alpha\times M)\theta_{G,X}(M)^{-1}$, $\beta_l(M)=(\beta\times M)\theta_{G,X}(M)^{-1}$, the result follows from (11.2.3). □

We regard the multiplication $\mu : G\times G\to G$ as a left G-action of G on itself and, for $M\in\operatorname{Ob}\mathcal{F}_1$, assume that $\theta_{G,G}(M)$ is an isomorphism and that $\theta_{G\times G,G}(M)$ is an epimorphism.

Lemma 11.2.7 *For a representation (M,ζ) of G on $M\in\operatorname{Ob}\mathcal{F}_1$, put $\hat{\zeta}=P_G(M)_M(\zeta) : G\times M\to M$. Then, $\hat{\zeta} : (G\times M,\xi_l(\mu,M))\to(M,\zeta)$ is a morphism of representations.*

Proof. Since ζ is a representation of G on M, we have $\hat{\zeta}(G\times\hat{\zeta})\theta_{G,G}(M)=\hat{\zeta}(\mu\times M)$ by (11.2.1). Hence $\hat{\zeta}(G\times\hat{\zeta})=\hat{\zeta}\mu_l(M)$ by the definition of $\mu_l(M)$ and the result follows from (11.2.3). □

Theorem 11.2.8 *Let (N,ζ) be a representation of G on $N\in\operatorname{Ob}\mathcal{F}_1$. Assume that $\theta_{G,G}(K)$ is an isomorphism for $K=M,N\in\operatorname{Ob}\mathcal{F}_1$ that $\theta_{G\times G,G}(K)$ is an epimorphism for $K=M,N\in\operatorname{Ob}\mathcal{F}_1$. A map*

$$\Phi : \operatorname{Rep}(G\,;\mathcal{F})((G\times M,\xi_l(\mu,M)),(N,\zeta))\to\mathcal{F}_1(M,N)$$

defined by $\Phi(\varphi)=\varphi(\varepsilon\times M)$ is bijective. Hence, if $\theta_{G,G}(M)$ is an isomorphism and $\theta_{G\times G,G}(M)$ is an epimorphism for all $M\in\operatorname{Ob}\mathcal{F}_1$, a functor $\mathscr{L}_G : \mathcal{F}_1\to\operatorname{Rep}(G\,;\mathcal{F})$ defined by $\mathscr{L}_G(M)=(G\times M,\xi_l(\mu,M))$ for $M\in\operatorname{Ob}\mathcal{F}_1$ and $\mathscr{L}_G(\varphi)=G\times\varphi$ for $\varphi\in\operatorname{Mor}\mathcal{F}_1$ is a left adjoint of the forgetful functor $\mathscr{F}_G : \operatorname{Rep}(G\,;\mathcal{F})\to\mathcal{F}_1$.

Proof. We put $\hat{\zeta}=P_G(N)_N(\zeta) : G\times N\to N$. For $\psi\in\mathcal{F}_1(M,N)$, it follows from (11.2.5) that we have a morphism $G\times\psi : (G\times M,\xi_l(\mu,M))\to(G\times N,\xi_l(\mu,N))$ of representations. Since $\hat{\zeta} : (G\times N,\xi_l(\mu,N))\to(N,\zeta)$ is a morphism of representations by (11.2.7), $\hat{\zeta}(G\times\psi) : (G\times M,\xi_l(\mu,M))\to(N,\zeta)$ is a morphism of representations. It follows from (6.3.9) and (11.2.1) that $\Phi(\hat{\zeta}(G\times\psi))=\hat{\zeta}(G\times\psi)(\varepsilon\times M)=\hat{\zeta}(\varepsilon\times N)(1\times\psi)=\psi$. On the other hand, for $\varphi\in\operatorname{Rep}(G\,;\mathcal{F})((G\times M,\xi_l(\mu,M)),(N,\zeta))$, since $\hat{\zeta}(G\times\varphi)=\varphi(\mu\times M)\theta_{G,G}(M)^{-1}$ by (11.2.3) and the following diagram commutes by (6.3.6) and (6.3.20),

$$\begin{tikzcd}
G\times(1\times M) \arrow[d, "G\times(\varepsilon\times M)"] & (G\times 1)\times M \arrow[l, "\theta_{G,1}(M)"'] \arrow[r, "\mathrm{pr}_1\times M"] \arrow[d, "(id_G\times\varepsilon)\times M"] & G\times M \\
G\times(G\times M) & (G\times G)\times M \arrow[l, "\theta_{G,G}(M)"'] \arrow[ur, bend right, "\mu\times M"'] &
\end{tikzcd}$$

we have

$$\begin{aligned}\hat{\zeta}(G\times\varphi(\varepsilon\times M)) &= \hat{\zeta}(G\times\varphi)(G\times(\varepsilon\times M)) = \varphi(\mu\times M)\theta_{G,G}(M)^{-1}(G\times(\varepsilon\times M))\\ &= \varphi(\mathrm{pr}_1\times M)\theta_{G,1}(M)^{-1} = \varphi\end{aligned}$$

by (6.3.3) and (6.3.22). Therefore a correspondence $\psi\mapsto\hat{\zeta}(G\times\psi)$ gives the inverse map of Φ. □

For $X\in\operatorname{Ob}\mathcal{T}$, we denote by $\mathrm{prod}_X:\mathcal{F}_1\to\mathcal{F}_1$ the functor defined by $\mathrm{prod}_X(M)=X\times M$ for $M\in\operatorname{Ob}\mathcal{F}_1$ and $\mathrm{prod}_X(\varphi)=X\times\varphi$ for $\varphi\in\operatorname{Mor}\mathcal{F}_1$.

Proposition 11.2.9 *Let (M,ξ) and (M,ζ) be representations of G. Put $\hat{\xi}=P_G(M)_M(\xi)$ and $\hat{\zeta}=P_G(M)_M(\zeta)$. We assume that $\mathrm{prod}_G:\mathcal{F}_1\to\mathcal{F}_1$ preserves coequalizers (the presheaf F^G_K on $\mathcal{F}_1$ is representable for any $K\in\operatorname{Ob}\mathcal{F}_1$, for example. See (6.5.4).) and that $\theta_{G,G}(M)$ is an epimorphism. Let $\pi_{\xi,\zeta}:M\to M_{(\xi,\zeta)}$ be a coequalizer of $\hat{\xi},\hat{\zeta}:G\times M\to M$.*

(1) There exists unique morphism $\hat{\lambda}:G\times M_{(\xi,\zeta)}\to M_{(\xi,\zeta)}$ that makes the following diagram commute.

$$\begin{tikzcd}
G\times M \arrow[r, "G\times\pi_{\xi,\zeta}"] \arrow[d, "\hat{\xi}"] & G\times M_{(\xi,\zeta)} \arrow[d, "\hat{\lambda}"] & G\times M \arrow[l, "G\times\pi_{\xi,\zeta}"'] \arrow[d, "\hat{\zeta}"] \\
M \arrow[r, "\pi_{\xi,\zeta}"] & M_{(\xi,\zeta)} & M \arrow[l, "\pi_{\xi,\zeta}"']
\end{tikzcd}$$

(2) Moreover, we assume that $\mathrm{prod}_{G\times G}:\mathcal{F}_1\to\mathcal{F}_1$ maps coequalizers to epimorphisms (the presheaf $F^{G\times G}_K$ on $\mathcal{F}_1$ is representable for any $K\in\operatorname{Ob}\mathcal{F}_1$, for example. See (6.5.4).). If we put $\lambda=P_G(M_{(\xi,\zeta)})^{-1}_{M_{(\xi,\zeta)}}(\hat{\lambda})$, $(M_{(\xi,\zeta)},\lambda)$ is a representation of G on and $\pi_{\xi,\zeta}$ defines morphisms of representations $(M,\xi)\to(M_{(\xi,\zeta)},\lambda)$ and $(M,\zeta)\to(M_{(\xi,\zeta)},\lambda)$.

Proof. (1) Put $\chi=\pi_{\xi,\zeta}\hat{\xi}=\pi_{\xi,\zeta}\hat{\zeta}:G\times M\to M_{(\xi,\zeta)}$. Then, it follows from (11.2.1) that

$$\begin{aligned}\chi(G\times\hat{\xi})\theta_{G,G}(M) &= \pi_{\xi,\zeta}\hat{\xi}(G\times\hat{\xi})\theta_{G,G}(M) = \pi_{\xi,\zeta}\hat{\xi}(\mu\times M) = \pi_{\xi,\zeta}\hat{\zeta}(\mu\times M)\\ &= \pi_{\xi,\zeta}\hat{\zeta}(G\times\hat{\zeta})\theta_{G,G}(M) = \chi(G\times\hat{\zeta})\theta_{G,G}(M).\end{aligned}$$

Since $G\times\pi_{\xi,\zeta}:G\times M\to G\times M_{(\xi,\zeta)}$ is a coequalizer of $G\times\hat{\xi},G\times\hat{\zeta}:G\times(G\times M)\to G\times M$ by the assumption, there exists unique morphism $\hat{\lambda}:G\times M_{(\xi,\zeta)}\to M_{(\xi,\zeta)}$ that satisfies $\hat{\lambda}(G\times\pi_{\xi,\zeta})=\chi$.

(2) By (6.3.3), (6.3.8) and (6.3.20), the following diagrams are commutative.

$$\begin{tikzcd}
(G\times G)\times M \arrow[r, "\theta_{G,G}(M)"] \arrow[d, "(G\times G)\times\pi_{\xi,\zeta}"] & G\times(G\times M) \arrow[r, "G\times\hat{\xi}"] \arrow[d, "G\times(G\times\pi_{\xi,\zeta})"] & G\times M \arrow[r, "\hat{\xi}"] \arrow[d, "G\times\pi_{\xi,\zeta}"] & M \arrow[d, "\pi_{\xi,\zeta}"] \\
(G\times G)\times M_{(\xi,\zeta)} \arrow[r, "\theta_{G,G}(M_{(\xi,\zeta)})"] & G\times(G\times M_{(\xi,\zeta)}) \arrow[r, "G\times\hat{\lambda}"] & G\times M_{(\xi,\zeta)} \arrow[r, "\hat{\lambda}"] & M_{(\xi,\zeta)}
\end{tikzcd}$$

$$\begin{tikzcd}
(G\times G)\times M \arrow[r, "\mu\times M"] \arrow[d, "(G\times G)\times\pi_{\xi,\zeta}"] & G\times M \arrow[r, "\hat{\xi}"] \arrow[d, "G\times\pi_{\xi,\zeta}"] & M \arrow[d, "\pi_{\xi,\zeta}"] \\
(G\times G)\times M_{(\xi,\zeta)} \arrow[r, "\mu\times M_{(\xi,\zeta)}"] & G\times M_{(\xi,\zeta)} \arrow[r, "\hat{\lambda}"] & M_{(\xi,\zeta)}
\end{tikzcd}
\qquad
\begin{tikzcd}
1\times M \arrow[r, "\varepsilon\times M"] \arrow[d, "1\times\pi_{\xi,\zeta}"] & G\times M \arrow[r, "\hat{\xi}"] \arrow[d, "G\times\pi_{\xi,\zeta}"] & M \arrow[d, "\pi_{\xi,\zeta}"] \\
1\times M_{(\xi,\zeta)} \arrow[r, "\varepsilon\times M_{(\xi,\zeta)}"] & G\times M_{(\xi,\zeta)} \arrow[r, "\hat{\lambda}"] & M_{(\xi,\zeta)}
\end{tikzcd}$$

It follows from (11.2.1) that we have $\hat{\lambda}(\varepsilon\times M_{(\xi,\zeta)})\pi_{\xi,\zeta}=\pi_{\xi,\zeta}\hat{\xi}(\varepsilon\times M)=\pi_{\xi,\zeta}$ and

$$\hat{\lambda}(G\times\hat{\lambda})\theta_{G,G}(M_{(\xi,\zeta)})((G\times G)\times\pi_{\xi,\zeta})=pi_{\xi,\zeta}\hat{\xi}(G\times\hat{\xi})\theta_{G,G}(M)=\pi_{\xi,\zeta}\hat{\xi}(\mu\times M)=\hat{\lambda}(\mu\times M_{(\xi,\zeta)})((G\times G)\times\pi_{\xi,\zeta}).$$

Since $\pi_{\xi,\zeta}$ and $(G\times G)\times\pi_{\xi,\zeta}$ are epimorphisms, it follows that $\hat{\lambda}(G\times\hat{\lambda})\theta_{G,G}(M_{(\xi,\zeta)})=\hat{\lambda}(\mu\times M_{(\xi,\zeta)})$ and $\hat{\lambda}(\varepsilon\times M_{(\xi,\zeta)})=id_{M_{(\xi,\zeta)}}$. Therefore λ is a representation of G on $M_{(\xi,\zeta)}$ by (11.2.1). $\pi_{\xi,\zeta}:(M,\xi)\to(M_{(\xi,\zeta)},\lambda)$ and $\pi_{\xi,\zeta}:(M,\zeta)\to(M_{(\xi,\zeta)},\lambda)$ are morphisms of representations by the first assertion and (6.3.5). □

Remark 11.2.10 *For representations (M,ξ), (N,ζ) and (N,ζ') of G, suppose that there exists a morphism $\varphi:M\to N$ of $\mathcal{F}_1$ such that $\varphi:(M,\xi)\to(N,\zeta)$ and $\varphi:(M,\xi)\to(N,\zeta')$ are morphisms of $\operatorname{Rep}(\boldsymbol{C};\mathcal{F})$ and that $o^*_G(\varphi)^*:\mathcal{F}_G(o^*_G(N),o^*_G(N))\to\mathcal{F}_G(o^*_G(M),o^*_G(N))$ is injective (e.g. φ is an epimorphism and the presheaf F^G_N*

on $\mathcal{F}_1$ is representable. See (6.5.2)). Then, $\zeta o^*_G(\varphi) = o^*_G(\varphi)\xi = \zeta' o^*_G(\varphi)$ implies $\zeta = \zeta'$. In particular, since $\varphi : (M, id_{o^*_G(M)}) \to (N, id_{o^*_G(N)})$ is a morphism of representations for any morphism $\varphi : M \to N$ of $\mathcal{F}_1$, if there exists a morphism of representation $\varphi : (M, id_{o^*_G(M)}) \to (N, \zeta)$ such that φ is an epimorphism of $\mathcal{F}_1$, (N, ζ) is a trivial representation. Thus, if (M, ξ) or (M, ζ) is a trivial representation, so is $(M_{(\xi,\zeta)}, \lambda)$.

Proposition 11.2.11 *Let (M, ξ), (N, ξ'), (M, ζ) and (N, ζ') be objects of $\mathrm{Rep}(G\,; \mathcal{F})$. Put $\hat{\xi} = P_G(M)_M(\xi)$, $\hat{\xi}' = P_G(N)_N(\xi')$, $\hat{\zeta} = P_G(M)_M(\zeta)$ and $\hat{\zeta}' = P_G(N)_N(\zeta')$. Assume that $\mathrm{prod}_X : \mathcal{F}_1 \to \mathcal{F}_1$ maps coequalizers to epimorphisms for $X = G, G \times G$ (the presheaves F^G_K and $F^{G\times G}_K$ on $\mathcal{F}_1$ is representable for any $K \in \mathrm{Ob}\,\mathcal{F}_1$, for example). Suppose that $\pi_{\xi,\zeta} : M \to M_{(\xi,\zeta)}$ is an coequalizer of $\hat{\xi}, \hat{\zeta} : G \times M \to M$ and that $\pi_{\xi',\zeta'} : N \to N_{(\xi',\zeta')}$ is an coequalizer of $\hat{\xi}', \hat{\zeta}' : G \times N \to N$. We denote by $(M_{(\xi,\zeta)}, \lambda)$ and $(N_{(\xi',\zeta')}, \lambda')$ the representations of G given in (11.2.9). If a morphism $\varphi : M \to N$ defines morphisms of representations $(M, \xi) \to (N, \xi')$ and $(M, \zeta) \to (N, \zeta')$, then there exists unique morphism $\tilde{\varphi} : (M_{(\xi,\zeta)}, \lambda) \to (N_{(\xi',\zeta')}, \lambda')$ of representations that satisfies $\tilde{\varphi}\pi_{\xi,\zeta} = \pi_{\xi',\zeta'}\varphi$.*

Proof. Since $\pi_{\xi',\zeta'}\varphi\hat{\xi} = \pi_{\xi',\zeta'}\hat{\xi}'(G \times \varphi) = \pi_{\xi',\zeta'}\hat{\zeta}'(G \times \varphi) = \pi_{\xi',\zeta'}\varphi\hat{\zeta}$ by (11.2.3), there exists unique morphism $\tilde{\varphi} : M_{(\xi,\zeta)} \to N_{(\xi',\zeta')}$ that satisfies $\tilde{\varphi}\pi_{\xi,\zeta} = \pi_{\xi',\zeta'}\varphi$. Then, it follows from (11.2.9), (11.2.3) and (6.3.3) that

$$\tilde{\varphi}\hat{\lambda}(G \times \pi_{\xi,\zeta}) = \tilde{\varphi}\pi_{\xi,\zeta}\hat{\xi} = \pi_{\xi',\zeta'}\varphi\hat{\xi} = \pi_{\xi',\zeta'}\hat{\xi}'(G \times \varphi) = \hat{\lambda}'(G \times \pi_{\xi',\zeta'})(G \times \varphi) = \hat{\lambda}'(G \times \tilde{\varphi})(G \times \pi_{\xi,\zeta}).$$

Since $G \times \pi_{\xi,\zeta}$ is an epimorphism, it follows that $\tilde{\varphi}\hat{\lambda} = \hat{\lambda}'(G \times \tilde{\varphi})$, namely, $\tilde{\varphi} : (M_{(\xi,\zeta)}, \lambda) \to (N_{(\xi',\zeta')}, \lambda')$ is a morphism of representations by (11.2.3). □

We define a functor $\mathscr{T}_G : \mathcal{F}_1 \to \mathrm{Rep}(G\,; \mathcal{F})$ by $\mathscr{T}_G(M) = (M, id_{o^*_G(M)})$ and $\mathscr{T}_G(\varphi) = \varphi$. That is, $\mathscr{T}_G$ assigns each object M of $\mathcal{F}_1$ to the trivial representation of G on M.

Under assumptions that $\mathcal{F}_1$ has coequalizers and $\mathrm{prod}_G, \mathrm{prod}_{G\times G} : \mathcal{F}_1 \to \mathcal{F}_1$ map coequalizers to coequalizers and that $\theta_{G,G}(M)$ is an epimorphism, we define a functor $\mathscr{J}_G : \mathrm{Rep}(G\,; \mathcal{F}) \to \mathcal{F}_1$ as follows. We set $\mathscr{J}_G(M, \xi) = M_{(\xi, id_{o^*_G(M)})}$ for $(M, \xi) \in \mathrm{Ob}\,\mathrm{Rep}(G\,; \mathcal{F})$. For a morphism $\varphi : (M, \xi) \to (N, \zeta)$, it follows from (11.2.10) and (11.2.11) that there exists unique morphism $\tilde{\varphi} : \left(M_{(\xi, id_{o^*_G(M)})}, id_{o^*_G\left(M_{(\xi, id_{o^*_G(M)})}\right)}\right) \to \left(N_{(\zeta, id_{o^*_G(N)})}, id_{o^*_G\left(N_{(\zeta, id_{o^*_G(N)})}\right)}\right)$ of representations that satisfies $\tilde{\varphi}\,\pi_{\xi, id_{o^*_G(M)}} = \pi_{\zeta, id_{o^*_G(N)}}\varphi$. We set $\mathscr{J}_G(\varphi) = \tilde{\varphi}$.

Proposition 11.2.12 *$\mathscr{J}_G$ is a left adjoint of $\mathscr{T}_G$.*

Proof. We define a counit $\varepsilon : \mathscr{J}_G\mathscr{T}_G \to id_{\mathcal{F}_1}$ and a unit $\eta : id_{\mathrm{Rep}(G\,;\mathcal{F})} \to \mathscr{T}_G\mathscr{J}_G$ as follows. For $M \in \mathrm{Ob}\,\mathcal{F}_1$, since $\mathscr{J}_G(\mathscr{T}_G(M)) = M_{(id_{o^*_G(M)}, id_{o^*_G(M)})} = M$, let $\varepsilon_M : \mathscr{J}_G(\mathscr{T}_G(M)) \to M$ be the identity morphism of M. For $(M, \xi) \in \mathrm{Ob}\,\mathrm{Rep}(G\,; \mathcal{F})$, since $\mathscr{T}_G(\mathscr{J}_G(M, \xi)) = \left(M_{(\xi, id_{o^*_G(M)})}, id_{o^*_G\left(M_{(\xi, id_{o^*_G(M)})}\right)}\right)$ and

$$\pi_{\xi, id_{o^*_G(M)}} : (M, \xi) \to \left(M_{(\xi, id_{o^*_G(M)})}, id_{o^*_G\left(M_{(\xi, id_{o^*_G(M)})}\right)}\right)$$

is a morphism of representations by (11.2.9) and (11.2.10), $\eta_{(M,\xi)} : (M, \xi) \to \mathscr{T}_G(\mathscr{J}_G(M, \xi))$ is defined to be $\pi_{\xi, id_{o^*_G(M)}}$. Since $\mathscr{J}_G(M, \xi) = M$ if (M, ξ) is the trivial representation, the following morphism is the identity morphism of $M_{(\xi, id_{o^*_G(M)})}$.

$$\mathscr{J}_G(\eta_{(M,\xi)}) : \mathscr{J}_G(M, \xi) \to \mathscr{J}_G\left(M_{(\xi, id_{o^*_G(M)})}, id_{o^*_G\left(M_{(\xi, id_{o^*_G(M)})}\right)}\right)$$

Hence composition $\mathscr{T}_G(M) \xrightarrow{\eta_{\mathscr{T}_G(M)}} \mathscr{T}_G(\mathscr{J}_G(\mathscr{T}_G(M))) \xrightarrow{\mathscr{T}_G(\varepsilon_M)} \mathscr{T}_G(M)$ is the identity morphism of $(M, id_{o^*_G(M)})$ and composition $\mathscr{J}_G(M, \xi) \xrightarrow{\mathscr{J}_G(\eta_{(M,\xi)})} \mathscr{J}_G(\mathscr{T}_G(\mathscr{J}_G(M, \xi))) \xrightarrow{\varepsilon_{\mathscr{J}_G(M,\xi)}} \mathscr{J}_G(M, \xi)$ is the identity morphism of $M_{(\xi, id_{o^*_G(M)})}$. □

Remark 11.2.13 *We denote $\mathscr{J}_G(M, \xi) = M_{(\xi, id_{o^*_G(M)})}$ by M/ξ and call this the G-orbit of (M, ξ).*

Let $\alpha : X \times G \to X$ be a right G-action on $X \in \mathrm{Ob}\,\mathcal{T}$ and (M, ξ) a representation of G on $M \in \mathrm{Ob}\,\mathcal{F}_1$. We put $\hat{\xi} = P_G(M)_M(\xi) : G \times M \to M$ and denote by $P^{(X,\alpha)}_{(M,\xi)} : X \times M \to (X, \alpha) \times (M, \xi)$ a coequalizer of $\alpha \times M : (X \times G) \times M \to X \times M$ and a composition $(X \times G) \times M \xrightarrow{\theta_{X,G}(M)} X \times (G \times M) \xrightarrow{X \times \hat{\xi}} X \times M$.

Proposition 11.2.14 *Let $\alpha : X \times G \to X$, $\beta : Y \times G \to Y$ be right G-actions on $X, Y \in \mathrm{Ob}\,\mathcal{T}$ respectively and $f : X \to Y$ a morphism of $\mathcal{T}$ which preserves right G-actions. Let (M,ξ), (N,ζ) be representations of G on $M, N \in \mathrm{Ob}\,\mathcal{F}_1$ respectively and $\varphi : (M,\xi) \to (N,\zeta)$ a morphism of representations. There exist unique morphisms $f \times (M,\xi) : (X,\alpha) \times (M,\xi) \to (Y,\beta) \times (M,\xi)$ and $(X,\alpha) \times \varphi : (X,\alpha) \times (M,\xi) \to (X,\alpha) \times (N,\zeta)$ that make the following diagrams commute.*

$$\begin{tikzcd} X \times M \arrow[r, "P^{(X,\alpha)}_{(M,\xi)}"] \arrow[d, "f\times M"] & (X,\alpha) \times (M,\xi) \arrow[d, "f\times(M,\xi)"] \\ Y \times M \arrow[r, "P^{(Y,\beta)}_{(M,\xi)}"] & (Y,\beta) \times (M,\xi) \end{tikzcd} \qquad \begin{tikzcd} X \times M \arrow[r, "P^{(X,\alpha)}_{(M,\xi)}"] \arrow[d, "X\times\varphi"] & (X,\alpha) \times (M,\xi) \arrow[d, "(X,\alpha)\times\varphi"] \\ X \times N \arrow[r, "P^{(X,\alpha)}_{(N,\zeta)}"] & (X,\alpha) \times (N,\zeta) \end{tikzcd}$$

Proof. We put $\hat{\xi} = P_G(M)_M(\xi)$ and $\hat{\zeta} = P_G(N)_N(\zeta)$. The following diagram commute by (6.3.3), (6.3.8), (6.3.20), (6.3.20) and (6.3.5).

$$\begin{tikzcd} X \times N & (X\times G) \times N \arrow[l, "\alpha\times N"'] \arrow[r, "\theta_{X,G}(N)"] & X \times (G \times N) \arrow[r, "X\times\hat{\zeta}"] & X \times N \\ X \times M \arrow[u, "X\times\varphi"] \arrow[d, "f\times M"] & (X\times G) \times M \arrow[l, "\alpha\times M"'] \arrow[r, "\theta_{X,G}(M)"] \arrow[u, "(X\times G)\times\varphi"] \arrow[d, "(f\times id_G)\times M"] & X \times (G \times M) \arrow[r, "X\times\hat{\xi}"] \arrow[u, "X\times(G\times\varphi)"] \arrow[d, "f\times(G\times M)"] & X \times M \arrow[u, "X\times\varphi"] \arrow[d, "f\times M"] \\ Y \times M & (Y\times G) \times M \arrow[l, "\beta\times M"'] \arrow[r, "\theta_{Y,G}(M)"] & Y \times (G \times M) \arrow[r, "Y\times\hat{\xi}"] & Y \times M \end{tikzcd}$$

Hence we have the following equalities.

$$\begin{aligned} P^{(Y,\beta)}_{(M,\xi)}(f \times M)(\alpha \times M) &= P^{(Y,\beta)}_{(M,\xi)}(\beta \times M)((f \times id_G) \times M) \\ &= P^{(Y,\beta)}_{(M,\xi)}(Y \times \hat{\xi})\theta_{Y,G}(M)((f \times id_G) \times M) \\ &= P^{(Y,\beta)}_{(M,\xi)}(f \times M)(X \times \hat{\xi})\theta_{X,G}(M) \\ P^{(X,\alpha)}_{(N,\zeta)}(X \times \varphi)(\alpha \times M) &= P^{(X,\alpha)}_{(N,\zeta)}(\alpha \times N)((X \times G) \times \varphi) \\ &= P^{(X,\alpha)}_{(N,\zeta)}(X \times \hat{\zeta})\theta_{X,G}(N)((X \times G) \times \varphi) \\ &= P^{(X,\alpha)}_{(N,\zeta)}(X \times \varphi)(X \times \hat{\xi})\theta_{X,G}(M) \end{aligned}$$

Thus there exist unique morphisms

$$f \times (M,\xi) : (X,\alpha) \times (M,\xi) \to (Y,\beta) \times (M,\xi), \qquad (X,\alpha) \times \varphi : (X,\alpha) \times (M,\xi) \to (X,\alpha) \times (N,\zeta)$$

that satisfy $(f \times (M,\xi))P^{(X,\alpha)}_{(M,\xi)} = P^{(Y,\beta)}_{(M,\xi)}(f \times M)$ and $((X,\alpha) \times \varphi)P^{(X,\alpha)}_{(M,\xi)} = P^{(X,\alpha)}_{(N,\zeta)}(X \times \varphi)$. □

Lemma 11.2.15 *Let $\alpha : X \times G \to X$ be a right G-action on $X \in \mathrm{Ob}\,\mathcal{T}$ and M an object of $\mathcal{F}_1$. Then, $\alpha \times M : (X \times G) \times M \to X \times M$ is a coequalizer of $(\alpha \times id_G) \times M : (X \times G \times G) \times M \to (X \times G) \times M$ and $(id_X \times \mu) \times M : (X \times G \times G) \times M \to (X \times G) \times M$.*

Proof. Since $\alpha(id_X, \varepsilon o_X) = id_X$, we have $(\alpha \times M)((id_X, \varepsilon o_X) \times M) = id_{X\times M}$ which shows that $\alpha \times M$ is an epimorphism. Suppose that a morphism $\varphi : (X \times G) \times M \to N$ satisfies $\varphi((\alpha \times id_G) \times M) = \varphi((id_X \times \mu) \times M)$. We define $\psi : X \times M \to N$ by $\psi = \varphi((id_X, \varepsilon o_X) \times M)$. Since

$$\begin{aligned} &(\alpha \times id_G)(id_X \times (id_G, \varepsilon o_G)) = (\alpha \times id_G)(\mathrm{pr}_X, \mathrm{pr}_G, \varepsilon o_G \mathrm{pr}_G) = (\alpha, \varepsilon o_{X\times G}) = (id_X, \varepsilon o_X)\alpha, \\ &(id_X \times \mu)(id_X \times (id_G, \varepsilon o_G)) = (id_X \times \mu(id_G, \varepsilon o_G)) = id_{X\times G}, \end{aligned}$$

it follows from (6.3.3) that

$$\begin{aligned} \psi(\alpha \times M) &= \varphi((id_X, \varepsilon o_X) \times M)(\alpha \times M) = \varphi((id_X, \varepsilon o_X)\alpha \times M) = \varphi((\alpha \times id_G)(id_X \times (id_G, \varepsilon o_G)) \times M) \\ &= \varphi((\alpha \times id_G) \times M)((id_X \times (id_G, \varepsilon o_G)) \times M) = \varphi((id_X \times \mu) \times M)((id_X \times (id_G, \varepsilon o_G)) \times M) \\ &= \varphi((id_X \times \mu(id_G, \varepsilon o_G)) \times M) = \varphi. \end{aligned}$$

□

Proposition 11.2.16 *Let $\alpha : X \times G \to X$ be a right G-action on $X \in \operatorname{Ob}\mathcal{T}$ and M an object of $\mathcal{F}_1$. Suppose that $\theta_{Y,X}(M) : (Y \times G) \times M \to Y \times (G \times M)$ is an isomorphism for $Y = G, X, G \times G, X \times G$. Then, a composition $X \times (G \times M) \xrightarrow{\theta_{X,G}(M)^{-1}} (X \times G) \times M \xrightarrow{\alpha \times M} X \times M$ is a coequalizer of the following morphisms.*

$$(X \times G) \times (G \times M) \xrightarrow{\theta_{X,G}(G\times M)} X \times (G \times (G \times M)) \xrightarrow{X\times\theta_{G,G}(M)^{-1}} X \times ((G \times G) \times M) \xrightarrow{X\times(\mu\times M)} X \times (G \times M)$$

$$(X \times G) \times (G \times M) \xrightarrow{\alpha\times(G\times M)} X \times (G \times M)$$

Hence $(X, \alpha) \times \mathscr{L}_G(M)$ is isomorphic to $X \times M$.

Proof. Since $\mu_l(M)$ is a composition $G \times (G \times M) \xrightarrow{\theta_{G,G}(M)^{-1}} (G \times G) \times M \xrightarrow{\mu\times M} G \times M$,

$$P^{(X,\alpha)}_{\mathscr{L}_G(M)} : X \times (G \times M) \to (X, \alpha) \times \mathscr{L}_G(M) = (X, \alpha) \times (G \times M, \xi_l(\mu, M))$$

is a coequalizer of $\alpha \times (G \times M) : (X \times G) \times (G \times M) \to X \times (G \times M)$ and composition

$$(X \times G) \times (G \times M) \xrightarrow{\theta_{X,G}(G\times M)} X \times (G \times (G \times M)) \xrightarrow{X\times\theta_{G,G}(M)^{-1}} X \times ((G \times G) \times M) \xrightarrow{X\times(\mu\times M)} X \times (G \times M).$$

Since $\theta_{X\times G,G}(M) : (X \times G \times G) \times M \to (X \times G) \times (G \times M)$ is an isomorphism, $P^{(X,\alpha)}_{\mathscr{L}_G(M)}$ is a coequalizer of the following compositions by (6.3.21).

$$(X \times G \times G) \times M \xrightarrow{\theta_{X\times G,G}(M)} (X \times G) \times (G \times M) \xrightarrow{\alpha\times(G\times M)} X \times (G \times M)$$

$$(X \times G \times G) \times M \xrightarrow{\theta_{X,G\times G}(M)} X \times ((G \times G) \times M) \xrightarrow{X\times(\mu\times M)} X \times (G \times M)$$

Moreover, since $\theta_{X,G}(M) : (X \times G) \times M \to X \times (G \times M)$ is an isomorphism and compositions

$$(X \times G \times G) \times M \xrightarrow{\theta_{X\times G,G}(M)} (X \times G) \times (G \times M) \xrightarrow{\alpha\times(G\times M)} X \times (G \times M) \xrightarrow{\theta_{X,G}(M)^{-1}} (X \times G) \times M$$

$$(X \times G \times G) \times M \xrightarrow{\theta_{X,G\times G}(M)} X \times ((G \times G) \times M) \xrightarrow{X\times(\mu\times M)} X \times (G \times M) \xrightarrow{\theta_{X,G}(M)^{-1}} (X \times G) \times M$$

coincides with $(\alpha \times id_G) \times M : (X \times G \times G) \times M \to (X \times G) \times M$ and $(id_X \times \mu) \times M : (X \times G \times G) \to (X \times G) \times M$, respectively by (6.3.20), the assertion follows from (11.2.15). □

Proposition 11.2.17 *Let $i : N \to G$ a subgroup object of G such that a morphism $f : H \to G$ of group objects of $\mathcal{T}$ and normalizes N. For a representation (M, ξ) of G, consider the restriction $\operatorname{Res}^G_N(M, \xi) = (M, \xi_i)$ to N and put $\mathscr{J}_N(M, \xi_i) = M/\xi_i$. Under the following assumptions, there is a representation ζ of H on M/ξ_i such that the quotient morphism $\pi_{\xi_i, id_{o^*_N(M)}} : M \to M/\xi_i$ defines a morphism of representations from $f^\bullet(M, \xi)$ to $(M/\xi_i, \zeta)$.*

(i) The presheaf F^H_K on $\mathcal{F}_1$ is representable for any $K \in \operatorname{Ob}\mathcal{F}_1$.
(ii) The presheaf $F^{G\times G}_{M/\xi_i}$ on $\mathcal{F}_1$ is representable.
(iii) $\theta_{G,G}(M) : (G \times G) \times M \to G \times (G \times M)$ is an isomorphism.
(iv) $\theta_{H,N}(M) : (H \times N) \times M \to H \times (N \times M)$ is an epimorphism.

Proof. Put $\hat{\xi} = P_G(M)_M(\xi)$ and $\rho = \pi_{\xi_i, id_{o^*_N(M)}} : M \to M/\xi_i$. Then $P_N(M)_M(\xi_i) = \hat{\xi}(i \times M)$ by (6.3.6) and ρ is a coequalizer of $\hat{\xi}(i \times M) : N \times M \to M$ and $o_N \times M : N \times M \to M$ since $P_N(M)_M(id_{o^*_N(M)}) = o_N \times M$ by (2) of (11.2.2). It follows from (6.5.4) that $H \times \rho : H \times M \to H \times M/\xi_i$ is a coequalizer of $H \times \hat{\xi}(i \times M) : H \times (N \times M) \to H \times M$ and $H \times (o_N \times M) : H \times (N \times M) \to H \times M$. There exists a morphism $\alpha : H \times N \to N$ which satisfies $\mu(f \times i) = \mu(i \times id_G)(\alpha \times f)(id_H \times T_{H,N})(\Delta_H \times id_N)$ by (9.1.13). By (11.2.1) and (6.3.20), we have

$$
\begin{aligned}
\rho\hat{\xi}(f\times M)(H\times\hat{\xi})(H\times(i\times M))\theta_{H,N}(M) &= \rho\hat{\xi}(G\times\hat{\xi})(f\times(G\times M))(H\times(i\times M))\theta_{H,N}(M)\\
&= \rho\hat{\xi}(\mu\times M)\theta_{G,G}(M)^{-1}(f\times(i\times M))\theta_{H,N}(M) = \rho\hat{\xi}(\mu\times M)((f\times i)\times M)\\
&= \rho\hat{\xi}(\mu(f\times i)\times M) = \rho\hat{\xi}(\mu(i\times id_G)(\alpha\times f)(id_H\times T_{H,N})(\Delta_H\times id_N)\times M)\\
&= \rho\hat{\xi}(\mu\times M)((i\times id_G)(\alpha\times f)(id_H\times T_{H,N})(\Delta_H\times id_N)\times M)\\
&= \rho\hat{\xi}(G\times\hat{\xi})\theta_{G,G}(M)((i\times id_G)\times M)((\alpha\times f)(id_H\times T_{H,N})(\Delta_H\times id_N)\times M)\\
&= \rho\hat{\xi}(G\times\hat{\xi})(i\times(id_G\times M))\theta_{N,G}(M)((\alpha\times f)(id_H\times T_{H,N})(\Delta_H\times id_N)\times M)\\
&= \rho\hat{\xi}(i\times M)(N\times\hat{\xi})\theta_{N,G}(M)((\alpha\times f)(id_H\times T_{H,N})(\Delta_H\times id_N)\times M)\\
&= \rho(o_N\times M)(N\times\hat{\xi})\theta_{N,G}(M)((\alpha\times f)(id_H\times T_{H,N})(\Delta_H\times id_N)\times M)\\
&= \rho\hat{\xi}(o_N\times(G\times M))\theta_{N,G}(M)((\alpha\times f)(id_H\times T_{H,N})(\Delta_H\times id_N)\times M)\\
&= \rho\hat{\xi}\theta_{1,G}(M)((o_N\times G)\times M)((\alpha\times f)(id_H\times T_{H,N})(\Delta_H\times id_N)\times M)\\
&= \rho\hat{\xi}(f\times M)(\mathrm{pr}_H\times M) = \rho\hat{\xi}(\mathrm{pr}_G\times M)((f\times id_N)\times M)\\
&= \rho\hat{\xi}\theta_{G,1}(M)((id_G\times o_N)\times M)((f\times id_N)\times M)\\
&= \rho\hat{\xi}(id_G\times(o_N\times M))\theta_{G,N}(M)((f\times id_N)\times M)\\
&= \rho\hat{\xi}(id_G\times(o_N\times M))(f\times(id_N\times M))\theta_{H,N}(M)\\
&= \rho\hat{\xi}(f\times M)((H\times o_N)\times M))\theta_{H,N}(M)
\end{aligned}
$$

Since $\theta_{H,N}(M)$ is an epimorphism, the above implies

$$\rho\hat{\xi}(f\times M)(H\times\hat{\xi})(H\times(i\times M)) = \rho\hat{\xi}(f\times M)((H\times o_N)\times M)).$$

Thus there exists unique morphism $\hat{\zeta} : H\times M/\xi_i \to M/\xi_i$ that satisfies $\hat{\zeta}(H\times\rho) = \rho\hat{\xi}(f\times M)$. We put $\zeta = P_H(M/\xi_i)^{-1}_{M/\xi_i}(\hat{\zeta})$, then it follows from (6.3.5) and (1) of (11.1.6) that ζ is a representation of H on M/ξ_i and $\rho = \pi_{\xi_i, id_{o_N^*(M)}} : f^\bullet(M,\xi) \to (M/\xi_i,\zeta)$ is a morphism of representations. □

11.3 Representations in fibered categories with exponents

Let $p : \mathcal{F}\to\mathcal{T}$ be a normalized cloven fibered category with exponents and $(G,\mu,\varepsilon,\iota)$ a group object in $\mathcal{T}$.

Proposition 11.3.1 *For $M\in \mathrm{Ob}\,\mathcal{F}_1$ and $\xi\in\mathcal{F}_G(o_G^*(M), o_G^*(M))$, we put $\check{\xi} = E_G(M)_M(\xi) : M\to M^G$. Then, (M,ξ) is a representation of G on M if and only if the following diagrams commute.*

$$
\begin{tikzcd}
M \arrow[r, "\check{\xi}"] \arrow[dr, "\check{\xi}"'] & M^G \arrow[r, "(\check{\xi})^G"] & (M^G)^G \arrow[d, "\theta^{G,G}(M)"] \\
& M^G \arrow[r, "M^\mu"] & M^{G\times G}
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\check{\xi}"] \arrow[dr, "id_M"'] & M^G \arrow[d, "M^\varepsilon"] \\
& M^1
\end{tikzcd}
$$

Proof. We have $E_{G\times G}(M)_M(\xi_\mu) = M^\mu\check{\xi}$ and $E_{G\times G}(M)_M(\xi_{\mathrm{pr}_i}) = M^{\mathrm{pr}_i}\check{\xi}$ for $i=1,2$ by (6.4.6). Hence (6.4.3), (6.4.6), (6.4.9), (6.4.18) imply

$$
\begin{aligned}
E_{G\times G}(M)_M(\xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2}) &= \epsilon_M^{G\times G}(M^{\mathrm{pr}_1}\check{\xi})^{G\times G}M^{\mathrm{pr}_2}\check{\xi} = \epsilon_M^{G\times G}(M^{\mathrm{pr}_1})^{G\times G}\check{\xi}^{G\times G}M^{\mathrm{pr}_2}\check{\xi}\\
&= \epsilon_M^{G\times G}(M^{\mathrm{pr}_1})^{\mathrm{pr}_2}\check{\xi}^G\check{\xi} = \theta^{G,G}(M)\check{\xi}^G\check{\xi}\\
E_1(M)_M(\xi_\varepsilon) &= M^\varepsilon\check{\xi}.
\end{aligned}
$$

Hence $\xi_\mu = \xi_{\mathrm{pr}_1}\xi_{\mathrm{pr}_2}$ is equivalent to $\theta^{G,G}(M)\check{\xi}^G\check{\xi} = M^\mu\check{\xi}$ and $\xi_\varepsilon = id_M$ is equivalent to $M^\varepsilon\check{\xi} = id_M$. □

Remark 11.3.2 *(1) Let $T_{G,G} : G\times G\to G\times G$ be the switching map. $\xi\in\mathcal{F}_G(o_G^*(M), o_G^*(M))$ is a right representation of G if and only if $M^{T_{G,G}}\theta^{G,G}(M)\check{\xi}^G\check{\xi} = M^\mu\check{\xi}$ and $M^\varepsilon\check{\xi} = id_M$.*

(2) The image of the trivial representation of G on M by $E_G(M)_M$ is $M^{o_G} : M = M^1 \to M^G$ by (3) of (6.4.6).

(3) Let $f : (H,\mu',\varepsilon',\iota') \to (G,\mu,\varepsilon,\iota)$ be a morphism of group objects in $\mathcal{T}$ and (M,ξ) a representation of G. It follows from (1) of (6.4.6) that $E_G(M)_M(\xi_f) = M^f\check{\xi}$.

The following fact is a direct consequence of (6.4.5).

Proposition 11.3.3 *Let (M,ξ) and (N,ζ) be representations of G and $\varphi : M \to N$ a morphism of $\mathcal{F}_1$. We put $\check{\xi} = E_G(M)_M(\xi)$ and $\check{\zeta} = E_G(N)_N(\zeta)$. Then, φ is a morphism of representations if and only if the following diagram is commutative.*

$$\begin{tikzcd} M \arrow[r, "\check{\xi}"] \arrow[d, "\varphi"] & M^G \arrow[d, "\varphi^G"] \\ N \arrow[r, "\check{\zeta}"] & N^G \end{tikzcd}$$

Let $\alpha : X \times G \to X$ be a right G-action on $X \in \operatorname{Ob}\mathcal{T}$. For an object M of $\mathcal{F}_1$, we assume that $\theta^{X,G}(M) : (M^X)^G \to M^{X\times G}$ is an isomorphism that $\theta^{X,G\times G}(M) : (M^X)^{G\times G} \to M^{X\times G\times G}$ is a monomorphism. Define $\alpha_r(M) : M^X \to (M^X)^G$ to be a composition $M^X \xrightarrow{M^\alpha} M^{X\times G} \xrightarrow{\theta^{X,G}(M)^{-1}} (M^X)^G$ and put

$$\xi_r(\alpha, M) = E_G(M^X)^{-1}_{M^X}(\alpha_r(M)) = \pi_G(M^X)o^*_G(\alpha_r(M)) \in \mathcal{F}_G(o^*_G(M^X), o^*_G(M^X)).$$

Proposition 11.3.4 *$(M^X, \xi_r(\alpha, M))$ is a representation of G on M^X.*

Proof. The following diagrams commute by (6.4.6), (6.4.20), and (6.4.21).

$$\begin{tikzcd} M^X \arrow[r, "M^\alpha"] \arrow[d, "M^\alpha"] & M^{X\times G} \arrow[r, "\theta^{X,G}(M)^{-1}"] \arrow[d, "M^{id_X\times\mu}"] & (M^X)^G \arrow[d, "(M^X)^\mu"] \\ M^{X\times G} \arrow[r, "M^{\alpha\times id_G}"] \arrow[d, "\theta^{X,G}(M)^{-1}"] & M^{X\times G\times G} & (M^X)^{G\times G} \arrow[l, "\theta^{X,G\times G}(M)"'] \\ (M^X)^G \arrow[r, "(M^\alpha)^G"] & (M^{X\times G})^G \arrow[u, "\theta^{X\times G,G}(M)"] \arrow[r, "(\theta^{X,G}(M)^G)^{-1}"] & ((M^X)^G)^G \arrow[u, "\theta^{G,G}(M^X)"] \end{tikzcd}
\qquad
\begin{tikzcd} M^X \arrow[d, "M^\alpha"] \arrow[dr, "M^{\mathrm{pr}_X}"] & \\ M^{X\times G} \arrow[r, "M^{id_X\times\varepsilon}"] \arrow[d, "\theta^{X,G}(M)^{-1}"] & M^{X\times 1} \arrow[d, "\theta^{X,1}(M)^{-1}"] \\ (M^X)^G \arrow[r, "(M^X)^\varepsilon"] & (M^X)^1 \end{tikzcd}$$

Hence we have $\theta^{G,G}(M^X)\alpha_r(M)^G\alpha_r(M) = (M^X)^\mu\alpha_r(M)$ and $(M^X)^\varepsilon\alpha_r(M) = id_{M^X}$ by (6.4.22). Then, the assertion follows from (11.3.1). □

Proposition 11.3.5 *Let $\alpha : X \times G \to X$ be a right G-action on $X \in \operatorname{Ob}\mathcal{T}$. We assume that $\theta^{X,G}(K)$ is an isomorphism for $K = M, N \in \operatorname{Ob}\mathcal{F}_1$ and that $\theta^{X,G\times G}(K)$ is a monomorphism for $K = M, N \in \operatorname{Ob}\mathcal{F}_1$. For a morphism $\varphi : M \to N$ of $\mathcal{F}_1$, $\varphi^X : M^X \to N^X$ is a morphism of representations from $(M^X, \xi_r(\alpha, M))$ to $(N^X, \xi_r(\alpha, N))$.*

Proof. The following diagram is commutative by (6.4.9) and (6.4.20).

$$\begin{tikzcd} M^X \arrow[r, "M^\alpha"] \arrow[d, "\varphi^X"] & M^{X\times G} \arrow[r, "\theta^{X,G}(M)^{-1}"] \arrow[d, "\varphi^{X\times G}"] & (M^X)^G \arrow[d, "(\varphi^X)^G"] \\ N^X \arrow[r, "N^\alpha"] & N^{X\times G} \arrow[r, "\theta^{X,G}(N)^{-1}"] & (N^X)^G \end{tikzcd}$$

Since $\alpha_r(M) = \theta^{X,G}(M)^{-1}M^\alpha$ and $\alpha_r(N) = \theta^{X,G}(N)^{-1}N^\alpha$, the result follows from (11.3.3). □

Proposition 11.3.6 *Let $\alpha : X \times G \to X$ and $\beta : Y \times G \to Y$ be right G-actions on $X, Y \in \operatorname{Ob}\mathcal{T}$. Assume that $\theta^{Z,G}(M)$ is an isomorphism for $Z = X, Y$ and that $\theta^{Z,G\times G}(M)$ is a monomorphism for $Z = X, Y$. If a morphism $f : Y \to X$ of $\mathcal{T}$ preserves G-actions, $M^f : M^X \to M^Y$ is a morphism of representations from $(M^X, \xi_r(\alpha, M))$ to $(M^Y, \xi_r(\beta, M))$.*

Proof. The following diagram is commutative by (6.4.6) and (6.4.20).

$$\begin{tikzcd} M^X \arrow[r, "M^\alpha"] \arrow[d, "M^f"] & M^{X\times G} \arrow[r, "\theta^{X,G}(M)^{-1}"] \arrow[d, "M^{f\times id_G}"] & (M^X)^G \arrow[d, "(M^f)^G"] \\ M^Y \arrow[r, "M^\beta"] & M^{Y\times G} \arrow[r, "\theta^{Y,G}(M)^{-1}"] & (M^Y)^G \end{tikzcd}$$

Since $\alpha_r(M) = \theta^{X,G}(M)^{-1}M^{\alpha}$, $\beta_r(M) = \theta^{Y,G}(M)^{-1}M^{\beta}$, the result follows from (11.3.3). $\square$

We regard the multiplication $\mu : G \times G \to G$ as a right G-action of G on itself and, for $M \in \operatorname{Ob}\mathcal{F}_1$ assume that $\theta^{G,G}(M)$ is an isomorphism and that $\theta^{G,G\times G}(M)$ is a monomorphism.

Lemma 11.3.7 *For a representation (M,ζ) of G on $M \in \operatorname{Ob}\mathcal{F}_1$, put $\check{\zeta} = E_G(M)_M(\zeta) : M \to M^G$. Then, $\check{\zeta} : (M,\zeta) \to (M^G, \xi_r(\mu, M))$ is a morphism of representations.*

Proof. Since ζ is a representation of G on M, we have $\theta^{G,G}(M)\check{\zeta}^G\check{\zeta} = M^{\mu}\check{\zeta}$ by (11.3.1). Hence $\check{\zeta}^G\check{\zeta} = \mu_r(M)\check{\zeta}$ by the definition of $\mu_r(M)$ and the result follows from (11.3.3). $\square$

Theorem 11.3.8 *Let (M,ζ) be a representation of G on $M \in \operatorname{Ob}\mathcal{F}_1$. Assume that $\theta^{G,G}(K)$ is an isomorphism for $K = M, N \in \operatorname{Ob}\mathcal{F}_1$ and that $\theta^{G,G\times G}(K)$ is a monomorphism for $K = M, N \in \operatorname{Ob}\mathcal{F}_1$. A map*

$$\Phi : \operatorname{Rep}(G\,;\mathcal{F})((M,\zeta),(N^G, \xi_r(\mu, N))) \to \mathcal{F}_1(M,N)$$

defined by $\Phi(\varphi) = N^{\varepsilon}\varphi$ is bijective. Hence, if $\theta^{G,G}(N)$ is an isomorphism and $\theta^{G,G\times G}(N)$ is a monnomorphism for all $N \in \operatorname{Ob}\mathcal{F}_1$, a functor $\mathscr{R}_G : \mathcal{F}_1 \to \operatorname{Rep}(G\,;\mathcal{F})$ defined by $\mathscr{R}_G(N) = (N^G, \xi_r(\mu, N))$ for $N \in \operatorname{Ob}\mathcal{F}_1$ and $\mathscr{R}_G(\varphi) = \varphi^G$ for $\varphi \in \operatorname{Mor}\mathcal{F}_1$ is a right adjoint of the forgetful functor $\mathscr{F}_G : \operatorname{Rep}(G\,;\mathcal{F}) \to \mathcal{F}_1$.

Proof. We put $\check{\zeta} = E_G(M)_M(\zeta) : M \to M^G$. For $\psi \in \mathcal{F}_1(M,N)$, it follows from (11.3.5) that we have a morphism $\psi^G : (M^G, \xi_r(\mu, M)) \to (N^G, \xi_r(\mu, N))$ of representations. Since $\check{\zeta} : (M,\zeta) \to (M^G, \xi_r(\mu, M))$ is a morphism of representations by (11.3.7), $\psi^G\check{\zeta} : (M,\zeta) \to (N^G, \xi_r(\mu, N))$ is a morphism of representations. It follows from (6.4.9) and (11.3.1) that $\Phi(\psi^G\check{\zeta}) = N^{\varepsilon}\psi^G\check{\zeta} = \psi M^{\varepsilon}\check{\zeta} = \psi$. On the other hand, for $\varphi \in \operatorname{Rep}(G\,;\mathcal{F})((M,\zeta),(N^G, \xi_r(\mu, N)))$, since $\varphi^G\check{\zeta} = \theta^{G,G}(N)^{-1}N^{\mu}\varphi$ by (11.3.3) and the following diagram commutes by (6.4.6) and (6.4.20),

$$\begin{tikzcd}
(N^G)^G \arrow[r, "\theta^{G,G}(N)"] \arrow[d, "(N^{\varepsilon})^G"] & N^{G\times G} \arrow[d, "N^{\varepsilon\times id_G}"] & N^G \arrow[l, "N^{\mu}"'] \arrow[dl, bend left, "N^{\mathrm{pr}_2}"] \\
(N^1)^G \arrow[r, "\theta^{1,G}(N)"] & N^{1\times G} &
\end{tikzcd}$$

we have

$$(N^{\varepsilon}\varphi)^G\check{\zeta} = (N^{\varepsilon})^G\varphi^G\check{\zeta} = (N^{\varepsilon})^G\theta^{G,G}(N)^{-1}N^{\mu}\varphi = \theta^{1,G}(N)^{-1}N^{\varepsilon\times id_G}N^{\mu}\varphi = \theta^{1,G}(N)^{-1}N^{\mathrm{pr}_2}\varphi = \varphi$$

by (6.4.3) and (6.4.22). Therefore a correspondence $\varphi \mapsto \varphi^G\check{\zeta}$ gives the inverse map of Φ. $\square$

For $X \in \operatorname{Ob}\mathcal{T}$, we denote by $\exp_X : \mathcal{F}_1 \to \mathcal{F}_1$ the functor defined by $\exp_X(M) = M^X$ for $M \in \operatorname{Ob}\mathcal{F}_1$ and $\exp_X(\varphi) = \varphi^X$ for $\varphi \in \operatorname{Mor}\mathcal{F}_1$.

Proposition 11.3.9 *Let (M,ξ) and (M,ζ) be representations of G. Put $\check{\xi} = E_G(M)_M(\xi)$ and $\check{\zeta} = E_G(M)_M(\zeta)$. We assume that $\exp_G : \mathcal{F}_1 \to \mathcal{F}_1$ preserves equalizers (the presheaf $F_{G,K}$ on $\mathcal{F}_1^{op}$ is representable for any $K \in \operatorname{Ob}\mathcal{F}_1$, for example. See (6.5.4).) and that $\theta^{G,G}(M)$ is a monomorphism. Let $\iota_{\xi,\zeta} : M^{(\xi,\zeta)} \to M$ be an equalizer of $\check{\xi}, \check{\zeta} : M \to M^G$.*

(1) There exists unique morphism $\check{\lambda} : M^{(\xi,\zeta)} \to (M^{(\xi,\zeta)})^G$ that makes the following diagram commutes.

$$\begin{tikzcd}
M \arrow[d, "\check{\xi}"] & M^{(\xi,\zeta)} \arrow[l, "\iota_{\xi,\zeta}"'] \arrow[r, "\iota_{\xi,\zeta}"] \arrow[d, "\check{\lambda}"] & M \arrow[d, "\check{\zeta}"] \\
M^G & (M^{(\xi,\zeta)})^G \arrow[l, "\iota_{\xi,\zeta}^G"'] \arrow[r, "\iota_{\xi,\zeta}^G"] & M^G
\end{tikzcd}$$

(2) Moreover, assume that $\exp_{G\times G} : \mathcal{F}_1 \to \mathcal{F}_1$ maps equalizers to monomorphisms (the presheaf $F_{G\times G,K}$ on $\mathcal{F}_1$ is representable for any $K \in \operatorname{Ob}\mathcal{F}_1$, for example. See (6.5.4).). If we put $\lambda = E_G(M^{(\xi,\zeta)})^{-1}_{M^{(\xi,\zeta)}}(\check{\lambda})$, λ is a representation of G on $M^{(\xi,\zeta)}$ and $\iota_{\xi,\zeta}$ defines morphisms of representations $(M^{(\xi,\zeta)}, \lambda) \to (M,\xi)$ and $(M^{(\xi,\zeta)}, \lambda) \to (M,\zeta)$. Hence $(M^{(\xi,\zeta)}, \lambda)$ is a subrepresentation of both (M,ξ) and (M,ζ).

Proof. (1) Put $\chi = \check{\xi}\iota_{\xi,\zeta} = \check{\zeta}\iota_{\xi,\zeta} : M^{(\xi,\zeta)} \to M^G$. Then, it follows from (11.3.1) that

$$\theta^{G,G}(M)\check{\xi}^G\chi = \theta^{G,G}(M)\check{\xi}^G\check{\xi}\iota_{\xi,\zeta} = M^\mu\check{\xi}\iota_{\xi,\zeta} = M^\mu\check{\zeta}\iota_{\xi,\zeta} = \theta^{G,G}(M)\check{\zeta}^G\check{\zeta}\iota_{\xi,\zeta} = \theta^{G,G}(M)\check{\zeta}^G\chi.$$

Thus we have $\check{\xi}^G\chi = \check{\zeta}^G\chi$ by the assumption. Since $\iota^G_{\xi,\zeta} : (M^{(\xi,\zeta)})^G \to M^G$ is a equalizer of $\check{\xi}^G, \check{\zeta}^G : M^G \to (M^G)^G$ by the assumption, there exists unique morphism $\check{\lambda} : M^{(\xi,\zeta)} \to (M^{(\xi,\zeta)})^G$ that satisfies $\iota^G_{\xi,\zeta}\check{\lambda} = \chi$.

(2) By (6.4.3), (6.3.8), (6.3.20) and (11.3.1), the following diagrams are commutative.

$$\begin{tikzcd}
M^{(\xi,\zeta)} \arrow[r, "\check{\lambda}"] \arrow[d, "\iota_{\xi,\zeta}"] & (M^{(\xi,\zeta)})^G \arrow[r, "\check{\lambda}^G"] \arrow[d, "\iota^G_{\xi,\zeta}"] & ((M^{(\xi,\zeta)})^G)^G \arrow[r, "\theta^{G,G}(M^{(\xi,\zeta)})"] \arrow[d, "(\iota^G_{\xi,\zeta})^G"] & (M^{(\xi,\zeta)})^{G\times G} \arrow[d, "\iota^{G\times G}_{\xi,\zeta}"] \\
M \arrow[r, "\check{\xi}"] & M^G \arrow[r, "\check{\xi}^G"] & (M^G)^G \arrow[r, "\theta^{G,G}(M)"] & M^{G\times G}
\end{tikzcd}$$

$$\begin{tikzcd}
M^{(\xi,\zeta)} \arrow[r, "\check{\lambda}"] \arrow[d, "\iota_{\xi,\zeta}"] & (M^{(\xi,\zeta)})^G \arrow[r, "(M^{(\xi,\zeta)})^\mu"] \arrow[d, "\iota^G_{\xi,\zeta}"] & (M^{(\xi,\zeta)})^{G\times G} \arrow[d, "\iota^{G\times G}_{\xi,\zeta}"] \\
M \arrow[r, "\check{\xi}"] & M^G \arrow[r, "M^\mu"] & M^{G\times G}
\end{tikzcd}
\qquad
\begin{tikzcd}
M^{(\xi,\zeta)} \arrow[r, "\check{\lambda}"] \arrow[d, "\iota_{\xi,\zeta}"] & (M^{(\xi,\zeta)})^G \arrow[r, "(M^{(\xi,\zeta)})^\varepsilon"] \arrow[d, "\iota^G_{\xi,\zeta}"] & (M^{(\xi,\zeta)})^1 \arrow[d, "\iota^1_{\xi,\zeta}"] \\
M \arrow[r, "\check{\xi}"] & M^G \arrow[r, "M^\varepsilon"] & M^1
\end{tikzcd}$$

It follows from (11.3.1) that we have $\iota_{\xi,\zeta}(M^{(\xi,\zeta)})^\varepsilon\check{\lambda} = M^\varepsilon\check{\xi}\iota_{\xi,\zeta} = \iota_{\xi,\zeta}$ and

$$\iota^{G\times G}_{\xi,\zeta}\theta^{G,G}(M^{(\xi,\zeta)})\check{\lambda}^G\check{\lambda} = \theta^{G,G}(M)\check{\xi}^G\check{\xi}\iota_{\xi,\zeta} = M^\mu\check{\xi}\iota_{\xi,\zeta} = \iota^{G\times G}_{\xi,\zeta}(M^{(\xi,\zeta)})^\mu\check{\lambda}.$$

Since $\iota_{\xi,\zeta}$ and $\iota^G_{\xi,\zeta}$ are monomorphisms, it follows $\theta^{G,G}(M^{(\xi,\zeta)})\check{\lambda}^G\check{\lambda} = (M^{(\xi,\zeta)})^\mu\check{\xi}$ and $(M^{(\xi,\zeta)})^\varepsilon\check{\lambda} = id_{M^{(\xi,\zeta)}}$. Therefore λ is a representation of G on $M^{(\xi,\zeta)}$ by (11.3.1). $\iota_{\xi,\zeta} : (M^{(\xi,\zeta)},\lambda) \to (M,\xi)$ and $\iota_{\xi,\zeta} : (M^{(\xi,\zeta)},\lambda) \to (M,\zeta)$ are morphisms of representations by the first assertion and (6.4.5). □

Remark 11.3.10 *For representations (M,ξ), (N,ζ) and (N,ζ') of G, suppose that there exists a morphism $\varphi : N \to M$ of $\mathcal{F}_1$ such that $\varphi : (N,\zeta) \to (M,\xi)$ and $\varphi : (N,\zeta') \to (M,\xi)$ are morphisms of* $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$ *and that $o^*_G(\varphi)_* : \mathcal{F}_G(o^*_G(N),o^*_G(N)) \to \mathcal{F}_G(o^*_G(N),o^*_G(M))$ is injective (e.g. φ is a monomorphism and the presheaf $F_{G,N}$ on $\mathcal{F}_1^{op}$ is representable. See (6.5.2)). Then, $o^*_G(\varphi)\zeta = \xi o^*_G(\varphi) = o^*_G(\varphi)\zeta'$ implies $\zeta = \zeta'$. In particular, since $\varphi : (N, id_{o^*_G(N)}) \to (M, id_{o^*_G(M)})$ is a morphism of representations for any morphism $\varphi : N \to M$ of $\mathcal{F}_1$, if there exists a morphism of representation $\varphi : (N,\zeta) \to (M, id_{o^*_G(M)})$ such that φ is a monomorphism of $\mathcal{F}_1$, (N,ζ) is a trivial representation. Thus, if (M,ξ) or (M,ζ) is a trivial representation, so is $(M^{(\xi,\zeta)},\lambda)$.*

Proposition 11.3.11 *Let (M,ξ), (N,ξ'), (M,ζ) and (N,ζ') be objects of* $\mathrm{Rep}(G;\mathcal{F})$*. Put $\check{\xi} = E_G(M)_M(\xi)$, $\check{\xi}' = E_G(N)_N(\xi')$, $\check{\zeta} = E_G(M)_M(\zeta)$ and $\check{\zeta}' = E_G(N)_N(\zeta')$. Assume that $\exp_X : \mathcal{F}_1 \to \mathcal{F}_1$ maps equalizers to monomorphisms for $X = G, G\times G$ (the presheaves $F_{G,K}$ and $F_{G\times G,K}$ on $\mathcal{F}_1^{op}$ is representable for any $K \in \mathrm{Ob}\,\mathcal{F}_1$, for example). Suppose that $\iota_{\xi,\zeta} : M^{(\xi,\zeta)} \to M$ is an equalizer of $\check{\xi},\check{\zeta} : M \to M^G$ and that $\iota_{\xi',\zeta'} : N^{(\xi',\zeta')} \to N$ is an equalizer of $\check{\xi}',\check{\zeta}' : N \to N^G$. We denote by $(M^{(\xi,\zeta)},\lambda)$ and $(N^{(\xi',\zeta')},\lambda')$ the representations of G given in (11.3.9). If a morphism $\varphi : M \to N$ defines morphisms of representations $(M,\xi) \to (N,\xi')$ and $(M,\zeta) \to (N,\zeta')$, then there exists unique morphism $\tilde{\varphi} : (M^{(\xi,\zeta)},\lambda) \to (N^{(\xi',\zeta')},\lambda')$ of representations that satisfies $\varphi\iota_{\xi,\zeta} = \iota_{\xi',\zeta'}\tilde{\varphi}$.*

Proof. Since $\check{\xi}'\varphi\iota_{\xi,\zeta} = \varphi^G\check{\xi}\iota_{\xi,\zeta} = \varphi^G\check{\zeta}\iota_{\xi,\zeta} = \check{\zeta}'\varphi\iota_{\xi,\zeta}$ by (11.3.3), there exists unique morphism $\tilde{\varphi} : M^{(\xi,\zeta)} \to N^{(\xi',\zeta')}$ that satisfies $\varphi\iota_{\xi,\zeta} = \iota_{\xi',\zeta'}\tilde{\varphi}$. Then, it follows from (11.3.9), (11.3.3) and (6.4.3) that

$$\iota^G_{\xi',\zeta'}\check{\lambda}'\tilde{\varphi} = \check{\xi}'\iota_{\xi',\zeta'}\tilde{\varphi} = \check{\xi}'\varphi\iota_{\xi,\zeta} = \varphi^G\check{\xi}\iota_{\xi,\zeta} = \varphi^G\iota^G_{\xi,\zeta}\check{\lambda} = \iota^G_{\xi,\zeta}\tilde{\varphi}^G\check{\lambda}.$$

Since $\iota^G_{\xi',\zeta'}$ is a monomorphism, it follows $\check{\lambda}'\tilde{\varphi} = \tilde{\varphi}^G\check{\lambda}$, namely, $\tilde{\varphi} : (M^{(\xi,\zeta)},\lambda) \to (N^{(\xi',\zeta')},\lambda')$ is a morphism of representations by (11.3.3). □

Under an assumption that $\mathcal{F}_1$ has equalizers, we define a functor $\mathscr{I}_G : \mathrm{Rep}(G;\mathcal{F}) \to \mathcal{F}_1$ as follows. We set $\mathscr{I}_G(M,\xi) = M^{(\xi, id_{o^*_G(M)})}$ for $(M,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(G;\mathcal{F})$. For a morphism $\varphi : (M,\xi) \to (N,\zeta)$, it follows from (11.3.10) and (11.3.11) that there exists unique morphism

$$\tilde{\varphi} : \left(M^{(\xi, id_{o^*_G(M)})}, id_{o^*_G\left(M^{(\xi, id_{o^*_G(M)})}\right)}\right) \to \left(N^{(\zeta, id_{o^*_G(N)})}, id_{o^*_G\left(N^{(\zeta, id_{o^*_G(N)})}\right)}\right)$$

of representations that satisfies $\iota_{\zeta, id_{o^*_G(N)}}\tilde{\varphi} = \varphi\iota_{\xi, id_{o^*_G(M)}}$. We set $\mathscr{I}_G(\varphi) = \tilde{\varphi}$.

Proposition 11.3.12 *$\mathscr{I}_G$ is a right adjoint of $\mathscr{T}_G$.*

Proof. We define a unit $\eta : id_{\mathcal{F}_1} \to \mathscr{I}_G\mathscr{T}_G$ and a counit $\varepsilon : \mathscr{T}_G\mathscr{I}_G \to id_{\mathrm{Rep}(G\,;\mathcal{F})}$ as follows. For $M \in \mathrm{Ob}\,\mathcal{F}_1$, since $\mathscr{I}_G(\mathscr{T}_G(M)) = M^{(id_{o_G^*(M)}, id_{o_G^*(M)})} = M$, let $\eta_M : M \to \mathscr{I}_G(\mathscr{T}_G(M))$ be the identity morphism of M. For $(M,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(G\,;\mathcal{F})$, since $\mathscr{T}_G(\mathscr{I}_G(M,\xi)) = \left(M^{(\xi, id_{o_G^*(M)})}, id_{o_G^*\left(M^{(\xi,id_{o_G^*(M)})}\right)}\right)$ and

$$\iota_{\xi, id_{o_G^*(M)}} : \left(M^{(\xi, id_{o_G^*(M)})}, id_{o_G^*\left(M^{(\xi,id_{o_G^*(M)})}\right)}\right) \to (M,\xi)$$

is a morphism of representations by (11.3.9) and (11.3.10), $\varepsilon_{(M,\xi)} : \mathscr{T}_G(\mathscr{I}_G(M,\xi)) \to (M,\xi)$ is defined to be $\iota_{\xi, id_{o_G^*(M)}}$. Since $\mathscr{I}_G(M,\xi) = M$ if (M,ξ) is the trivial representation, the following morphism is the identity morphism of $M^{(\xi, id_{o_G^*(M)})}$.

$$\mathscr{I}_G(\varepsilon_{(M,\xi)}) : \mathscr{I}_G\left(M^{(\xi, id_{o_G^*(M)})}, id_{o_G^*\left(M^{(\xi,id_{o_G^*(M)})}\right)}\right) \to \mathscr{I}_G(M,\xi)$$

Hence composition $\mathscr{T}_G(M) \xrightarrow{\mathscr{T}_G(\eta_M)} \mathscr{T}_G(\mathscr{I}_G(\mathscr{T}_G(M))) \xrightarrow{\varepsilon_{\mathscr{T}_G(M)}} \mathscr{T}_G(M)$ is the identity morphism of $(M, id_{o_G^*(M)})$ and composition $\mathscr{I}_G(M,\xi) \xrightarrow{\eta_{\mathscr{I}_G(M,\xi)}} \mathscr{I}_G(\mathscr{T}_G(\mathscr{I}_G(M,\xi))) \xrightarrow{\mathscr{I}_G(\varepsilon_{(M,\xi)})} \mathscr{I}_G(M,\xi)$ is the identity morphism of $M^{(\xi, id_{o_G^*(M)})}$. □

Remark 11.3.13 *We denote $\mathscr{I}_G(M,\xi) = M^{(\xi, id_{o_G^*(M)})}$ by M^ξ and call this the G-fixed object of (M,ξ).*

Let $\alpha : G \times X \to X$ be a left G-action on $X \in \mathrm{Ob}\,\mathcal{T}$ and (M,ξ) a representation of G on $M \in \mathrm{Ob}\,\mathcal{F}_1$. We put $\check{\xi} = E_G(M)_M(\xi) : M \to M^G$ and denote by $E^{(X,\alpha)}_{(M,\xi)} : (M,\xi)^{(X,\alpha)} \to M^X$ an equalizer of $M^\alpha : M^X \to M^{G\times X}$ and composition $M^X \xrightarrow{\check{\xi}^X} (M^G)^X \xrightarrow{\theta^{G,X}(M)} M^{G\times X}$.

Proposition 11.3.14 *Let $\alpha : G \times X \to X$, $\beta : G \times Y \to Y$ be left G-actions on $X, Y \in \mathrm{Ob}\,\mathcal{T}$ respectively and $f : X \to Y$ a morphism of $\mathcal{T}$ which preserves left G-actions. Let (M,ξ), (N,ζ) be representations of G on $M, N \in \mathrm{Ob}\,\mathcal{F}_1$ respectively and $\varphi : (M,\xi) \to (N,\zeta)$ a morphism of representations. There exist unique morphisms $(M,\xi)^f : (M,\xi)^{(Y,\beta)} \to (M,\xi)^{(X,\alpha)}$ and $\varphi^{(X,\alpha)} : (M,\xi)^{(X,\alpha)} \to (N,\zeta)^{(X,\alpha)}$ that make the following diagrams commute.*

$$\begin{tikzcd}
(M,\xi)^{(Y,\beta)} \arrow[r, "E^{(Y,\beta)}_{(M,\xi)}"] \arrow[d, "(M,\xi)^f"] & M^Y \arrow[d, "M^f"] \\
(M,\xi)^{(X,\alpha)} \arrow[r, "E^{(X,\alpha)}_{(M,\xi)}"] & M^X
\end{tikzcd}
\qquad
\begin{tikzcd}
(M,\xi)^{(X,\alpha)} \arrow[r, "E^{(X,\alpha)}_{(M,\xi)}"] \arrow[d, "\varphi^{(X,\alpha)}"] & M^X \arrow[d, "\varphi^X"] \\
(N,\zeta)^{(X,\alpha)} \arrow[r, "E^{(X,\alpha)}_{(N,\zeta)}"] & N^X
\end{tikzcd}$$

Proof. We put $\check{\xi} = E_G(M)_M(\xi)$ and $\check{\zeta} = E_G(N)_N(\zeta)$. The following diagram commute by (6.4.3), (6.4.8), (6.4.20), (6.4.20) and (6.4.5).

$$\begin{tikzcd}
M^{G\times Y} \arrow[d, "M^{id_G\times f}"] & M^Y \arrow[l, "M^\beta"'] \arrow[r, "\check{\xi}^Y"] \arrow[d, "M^f"] & (M^G)^Y \arrow[r, "\theta^{G,Y}(M)"] \arrow[d, "(M^G)^f"] & M^{G\times Y} \arrow[d, "M^{id_G\times f}"] \\
M^{G\times X} \arrow[d, "\varphi^{G\times X}"] & M^X \arrow[l, "M^\alpha"'] \arrow[r, "\check{\xi}^X"] \arrow[d, "\varphi^X"] & (M^G)^X \arrow[r, "\theta^{G,X}(M)"] \arrow[d, "(\varphi^G)^X"] & M^{G\times X} \arrow[d, "\varphi^{G\times X}"] \\
N^{G\times X} & N^X \arrow[l, "N^\alpha"'] \arrow[r, "\check{\zeta}^X"] & (N^G)^X \arrow[r, "\theta^{G,X}(N)"] & N^{G\times X}
\end{tikzcd}$$

Hence we have the following equalities.

$$M^\alpha M^f E^{(Y,\beta)}_{(M,\xi)} = M^{id_G\times f} M^\beta E^{(Y,\beta)}_{(M,\xi)} = M^{id_G\times f}\theta^{G,Y}(M)\check{\zeta}^Y E^{(Y,\beta)}_{(M,\xi)} = \theta^{G,X}(M)\check{\xi}^X M^f E^{(Y,\beta)}_{(M,\xi)}$$
$$N^\alpha \varphi^X E^{(X,\alpha)}_{(M,\xi)} = \varphi^{G\times X} M^\alpha E^{(X,\alpha)}_{(M,\xi)} = \varphi^{G\times X}\theta^{G,X}(M)\check{\xi}^X E^{(X,\alpha)}_{(M,\xi)} = \theta^{G,X}(N)\check{\zeta}^X \varphi^G E^{(X,\alpha)}_{(M,\xi)}$$

Thus there exist unique morphisms $(M,\xi)^f : (M,\xi)^{(Y,\beta)} \to (M,\xi)^{(X,\alpha)}$ and $\varphi^{(X,\alpha)} : (M,\xi)^{(X,\alpha)} \to (N,\zeta)^{(X,\alpha)}$ that satisfy $E^{(X,\alpha)}_{(M,\xi)}(M,\xi)^f = M^f E^{(Y,\beta)}_{(M,\xi)}$ and $E^{(X,\alpha)}_{(N,\zeta)}\varphi^{(X,\alpha)} = \varphi^X E^{(X,\alpha)}_{(M,\xi)}$. □

Lemma 11.3.15 *Let $\alpha : G \times X \to X$ be a left G-action on $X \in \operatorname{Ob}\mathcal{T}$ and M an object of $\mathcal{F}_1$. Then, $M^{\alpha} : M^X \to M^{G\times X}$ is an equalizer of $M^{id_G\times\alpha} : M^{G\times X} \to M^{G\times G\times X}$ and $M^{\mu\times id_X} : M^{G\times X} \to M^{G\times G\times X}$.*

Proof. Since $\alpha(\varepsilon o_X, id_X) = id_X$, we have $M^{(\varepsilon o_X, id_X)}M^{\alpha} = id_{M^X}$ which shows that M^{α} is a monomorphism. Suppose that a morphism $\varphi : N \to M^{G\times X}$ satisfies $M^{id_G\times\alpha}\varphi = M^{\mu\times id_X}\varphi$. We define $\psi : N \to M^X$ by $\psi = M^{(\varepsilon o_X, id_X)}\varphi$. Since

$$(id_G \times \alpha)((\varepsilon o_G, id_G) \times id_X) = (id_G \times \alpha)(\varepsilon o_G \mathrm{pr}_G, \mathrm{pr}_G, \mathrm{pr}_X) = (\varepsilon o_{G\times X}, \alpha) = (\varepsilon o_X, id_X)\alpha,$$
$$(\mu \times id_X)((\varepsilon o_G, id_G) \times id_X) = (\mu(\varepsilon o_G, id_G) \times id_X) = id_{G\times X},$$

it follows from (6.3.3) that

$$\begin{aligned} M^{\alpha}\psi &= M^{\alpha}M^{(\varepsilon o_X, id_X)}\varphi = M^{(\varepsilon o_X, id_X)\alpha}\varphi = M^{(id_G\times\alpha)((\varepsilon o_G, id_G)\times id_X)}\varphi = M^{(\varepsilon o_G, id_G)\times id_X}M^{id_G\times\alpha}\varphi \\ &= M^{(\varepsilon o_G, id_G)\times id_X}M^{\mu\times id_X}\varphi = M^{(\mu\times id_X)((\varepsilon o_G, id_G)\times id_X)}\varphi = \varphi. \end{aligned}$$

□

Proposition 11.3.16 *Let $\alpha : G\times X \to X$ be a left G-action on $X \in \operatorname{Ob}\mathcal{T}$ and M an object of $\mathcal{F}_1$. Suppose that $\theta^{G,Y}(M) : (M^G)^Y \to M^{G\times Y}$ is an isomorphism for $Y = G, X, G\times G, G\times X$. Then, a composition $M^X \xrightarrow{M^{\alpha}} M^{G\times X} \xrightarrow{\theta^{G,X}(M)^{-1}} (M^G)^X$ is an equalizer of $(M^G)^{\alpha} : (M^G)^X \to (M^G)^{G\times X}$ and composition $(M^G)^X \xrightarrow{(M^{\mu})^X} (M^{G\times G})^X \xrightarrow{(\theta^{G,G}(M)^{-1})^X} ((M^G)^G)^X \xrightarrow{\theta^{G,X}(M^G)} (M^G)^{G\times X}$. Hence $\mathscr{R}_G(M)^{(X,\alpha)}$ is isomorphic to M^X.*

Proof. Since $\mu_r(M)$ is a composition $M^G \xrightarrow{M^{\mu}} M^{G\times G} \xrightarrow{\theta^{G,G}(M)^{-1}} (M^G)^G$, $E^{(X,\alpha)}_{\mathscr{R}_G(M)} : \mathscr{R}_G(M)^{(X,\alpha)} \to (M^G)^X$ is an equalizer of $(M^G)^{\alpha} : (M^G)^X \to (M^G)^{G\times X}$ and composition

$$(M^G)^X \xrightarrow{(M^{\mu})^X} (M^{G\times G})^X \xrightarrow{(\theta^{G,G}(M)^{-1})^X} ((M^G)^G)^X \xrightarrow{\theta^{G,X}(M^G)} (M^G)^{G\times X}.$$

Since $\theta^{G,G\times X}(M) : (M^G)^{G\times X} \to M^{G\times G\times X}$ is an isomorphism, $E^{(X,\alpha)}_{\mathscr{R}_G(M)} : \mathscr{R}_G(M)^{(X,\alpha)} \to (M^G)^X$ is an equalizer of the following compositions by (6.4.21).

$$(M^G)^X \xrightarrow{(M^G)^{\alpha}} (M^G)^{G\times X} \xrightarrow{\theta^{G,G\times X}(M)} M^{G\times G\times X} \qquad (M^G)^X \xrightarrow{(M^{\mu})^X} (M^{G\times G})^X \xrightarrow{\theta^{G\times G,X}(M)} M^{G\times G\times X}$$

Moreover, since $\theta^{G,X}(M)^{-1} : M^{G\times X} \to (M^G)^X$ is an isomorphism and compositions

$$M^{G\times X} \xrightarrow{\theta^{G,X}(M)^{-1}} (M^G)^X \xrightarrow{(M^G)^{\alpha}} (M^G)^{G\times X} \xrightarrow{\theta^{G,G\times X}(M)} M^{G\times G\times X}$$

$$M^{G\times X} \xrightarrow{\theta^{G,X}(M)^{-1}} (M^G)^X \xrightarrow{(M^{\mu})^X} (M^{G\times G})^X \xrightarrow{\theta^{G\times G,X}(M)} M^{G\times G\times X}$$

coincides with $M^{id_G\times\alpha} : M^{G\times X} \to M^{G\times G\times X}$ and $M^{\mu\times id_X} : M^{G\times X} \to M^{G\times G\times X}$, respectively by (6.3.20), the assertion follows from (11.3.15). □

Proposition 11.3.17 *Let $i : N \to G$ a subgroup object of G such that a morphism $f : H \to G$ of group objects of $\mathcal{T}$ and normalizes N. For a representation (M, ξ) of G, consider the restriction $\mathrm{Res}^G_N(M, \xi) = (M, \xi_i)$ to N and put $\mathscr{I}_N(M, \xi_i) = M^{\xi_i}$. Under the following assumptions, there is a representation ζ of H on M^{ξ_i} such that the monomorphism $\iota_{\xi_i, id_{o_N^*(M)}} : M^{\xi_i} \to M$ defines a morphism of representations from (M^{ξ_i}, ζ) to $f^{\bullet}(M, \xi)$.*

(i) The presheaf $F_{H,K}$ on $\mathcal{F}_1^{op}$ is representable for any $K \in \operatorname{Ob}\mathcal{F}_1$.
(ii) The presheaf $F_{G\times G, M^{\xi_i}}$ on $\mathcal{F}_1^{op}$ is representable.
(iii) $\theta^{G,G}(M) : (M^G)^G \to M^{G\times G}$ is an isomorphism.
(iv) $\theta^{N,H}(M) : (M^N)^H \to M^{N\times H}$ is a monomorphism.

Proof. Put $\check{\xi} = E_G(M)_M(\xi)$ and $\eta = \iota_{\xi_i, id_{o_N^*(M)}} : M^{\xi_i} \to M$. Then $E_N(M)_M(\xi_i) = M^i\check{\xi}$ by (6.4.6) and η is an equalizer of $M^i\check{\xi} : M \to M^N$ and $M^{o_N} : M \to M^N$ since $E_N(M)_M(id_{o_N^*(M)}) = M^{o_N}$ by (2) of (11.3.2). It follows from (6.5.4) that $\eta^H : (M^{\xi_i})^H \to M^H$ is an equalizer of $(M^i\check{\xi})^H : M^H \to (M^N)^H$ and $(M^{o_N})^H : M^H \to (M^N)^H$.

There exists a morphism $\beta : N \times H \to N$ which satisfies $\mu(i \times f) = \mu(id_G \times i)(f \times \beta)(T_{N,H} \times id_H)(id_N \times \Delta_H)$ by (9.1.13). By (11.3.1) and (6.4.20), we have

$$
\begin{aligned}
\theta^{N,H}(M)(M^i)^H\check{\xi}^H M^f\check{\xi}\eta &= \theta^{N,H}(M)(M^i)^H(M^G)^f\check{\xi}^G\check{\xi}\eta = \theta^{N,H}(M)(M^i)^H(M^G)^f\left(\theta^{G,H}(M)\right)^{-1}M^\mu\check{\xi}\eta \\
&= \theta^{N,H}(M)(M^i)^H\left(\theta^{G,H}(M)\right)^{-1}M^{id_G\times f}M^\mu\check{\xi}\eta = M^{i\times id_H}M^{id_G\times f}M^\mu\check{\xi}\eta = M^{\mu(i\times f)}\check{\xi}\eta \\
&= M^{\mu(id_G\times i)(f\times\beta)(T_{N,H}\times id_H)(id_N\times\Delta_H)}\check{\xi}\eta = M^{(id_G\times i)(f\times\beta)(T_{N,H}\times id_H)(id_N\times\Delta_H)}M^\mu\check{\xi}\eta \\
&= M^{(f\times\beta)(T_{N,H}\times id_H)(id_N\times\Delta_H)}M^{id_G\times i}\theta^{G,G}(M)\check{\xi}^G\check{\xi}\eta \\
&= M^{(f\times\beta)(T_{N,H}\times id_H)(id_N\times\Delta_H)}\theta^{G,N}(M)(M^G)^i\check{\xi}^G\check{\xi}\eta \\
&= M^{(f\times\beta)(T_{N,H}\times id_H)(id_N\times\Delta_H)}\theta^{G,N}(M)\check{\xi}^N M^i\check{\xi}\eta \\
&= M^{(f\times\beta)(T_{N,H}\times id_H)(id_N\times\Delta_H)}\theta^{G,N}(M)\check{\xi}^N M^{o_N}\eta \\
&= M^{(f\times\beta)(T_{N,H}\times id_H)(id_N\times\Delta_H)}\theta^{G,N}(M)(M^G)^{o_N}\check{\xi}\eta \\
&= M^{(f\times\beta)(T_{N,H}\times id_H)(id_N\times\Delta_H)}M^{id_G\times o_N}\theta^{G,1}(M)\check{\xi}\eta = M^{\mathrm{pr}_H}M^f\check{\xi}\eta \\
&= M^{id_N\times f}M^{\mathrm{pr}_G}\check{\xi}\eta = M^{id_N\times f}M^{o_N\times id_G}\theta^{1,G}(M)\check{\xi}\eta \\
&= M^{id_N\times f}\theta^{N,G}(M)(M^{o_N})^G\check{\xi}\eta = \theta^{N,H}(M)(M^{o_N})^H M^f\check{\xi}\eta.
\end{aligned}
$$

Since $\theta^{N,H}(M)$ is an isomorphism, the above implies $(M^i)^H\check{\xi}^H M^f\check{\xi}\eta = (M^{o_N})^H M^f\check{\xi}\eta$. Thus there exists unique morphism $\check{\zeta} : M^{\xi_i} \to (M^{\xi_i})^H$ that satisfies $\check{\zeta}\eta^H = M^f\check{\xi}\eta$. We put $\zeta = E_H(M^{\xi_i})^{-1}_{M^{\xi_i}}(\check{\zeta})$, then it follows from (6.4.5) and (2) of (11.1.6) that ζ is a representation of H on M^{ξ_i} and $\eta = \iota_{\xi_i, id_{o_N^*(M)}} : (M^{\xi_i}, \zeta) \to f^\bullet(M,\xi)$ is a morphism of representations. □

11.4 Left induced representations

Let $p : \mathcal{F} \to \mathcal{T}$ be a fibered category with products and $(G,\mu,\varepsilon,\iota)$, $(H,\mu',\varepsilon',\iota')$ group objects in $\mathcal{T}$. For a morphism $f : H \to G$ of group objects, define a right H-action $\mu_f^r : G \times H \to G$ on G by $\mu_f^r = \mu(id_G \times f)$.

Assumption 11.4.1 *For a representation (M,ξ) of H, we put $\hat{\xi} = P_H(M)_M(\xi) : H \times M \to M$. We assume the following.*

(i) A coequalizer $P^{(G,\mu_f^r)}_{(M,\xi)} : G \times M \to (G,\mu_f^r) \times (M,\xi)$ of $(G \times H) \times M \overset{\mu_f^r \times M}{\underset{(G\times\hat{\xi})\theta_{G,H}(M)}{\rightrightarrows}} G \times M$ *exists.*

(ii) $G \times P^{(G,\mu_f^r)}_{(M,\xi)} : G \times (G \times M) \to G \times ((G,\mu_f^r) \times (M,\xi))$ is a coequalizer of

$$G \times ((G \times H) \times M) \overset{G\times(\mu_f^r \times M)}{\underset{G\times((G\times\hat{\xi})\theta_{G,H}(M))}{\rightrightarrows}} G \times (G \times M).$$

(iii) The following map is injective.

$o^*_{G\times G}\left(P^{(G,\mu_f^r)}_{(M,\xi)}\right)^* : \mathcal{F}_{G\times G}(o^*_{G\times G}((G,\mu_f^r)\times(M,\xi)), o^*_{G\times G}((G,\mu_f^r)\times(M,\xi))) \to \mathcal{F}_{G\times G}(o^*_{G\times G}(G\times M), o^*_{G\times G}((G,\mu_f^r)\times(M,\xi)))$

(iv) $\theta_{G,G}(M) : G \times (G \times M) \to (G \times G) \times M$ is an isomorphism.

(v) The following morphisms are epimorphisms.

$$\theta_{G\times G,G}(M) : (G\times G\times G)\times M \to (G\times G)\times(G\times M), \quad \theta_{G,G\times H}(M) : (G\times G\times H)\times M \to G\times((G\times H)\times M)$$

Let (M,ξ) be a representation of H on $M \in \mathrm{Ob}\,\mathcal{F}_1$. We define a representation ξ_f^l of G on $(G,\mu_f^r)\times(M,\xi)$ as follows.

Put $\hat{\xi} = P_H(M)_M(\xi) : H \times M \to M$. The right rectangle of the following upper diagram commutes by the associativity of μ and (6.3.3). It follows from (6.3.20) that the left rectangle of the following upper diagram and the upper right and the lower left rectangles of the following lower diagram commute. The lower right rectangle of the following lower diagram commutes by (6.3.9) and the upper left rectangle of the following lower diagram commutes by (6.3.21).

$$\begin{tikzcd}
G\times((G\times H)\times M) \arrow[d, "G\times(\mu_f^r\times M)"] & (G\times G\times H)\times M \arrow[l, "\theta_{G,G\times H}(M)"'] \arrow[r, "(\mu\times id_H)\times M"] \arrow[d, "(id_G\times\mu_f^r)\times M"] & (G\times H)\times M \arrow[d, "\mu_f^r\times M"] \\
G\times(G\times M) \arrow[r, "\theta_{G,G}(M)^{-1}"] & (G\times G)\times M \arrow[r, "\mu\times M"] & G\times M
\end{tikzcd}$$

$$\begin{tikzcd}
G\times((G\times H)\times M) \arrow[d, "G\times\theta_{G,H}(M)"] & (G\times G\times H)\times M \arrow[l, "\theta_{G,G\times H}(M)"'] \arrow[r, "(\mu\times id_H)\times M"] \arrow[d, "\theta_{G\times G,H}(M)"] & (G\times H)\times M \arrow[d, "\theta_{G,H}(M)"] \\
G\times(G\times(H\times M)) \arrow[d, "G\times(G\times\hat{\xi})"] & (G\times G)\times(H\times M) \arrow[l, "\theta_{G,G}(H\times M)"'] \arrow[r, "\mu\times(H\times M)"] \arrow[d, "(G\times G)\times\hat{\xi}"] & G\times(H\times M) \arrow[d, "G\times\hat{\xi}"] \\
G\times(G\times M) \arrow[r, "\theta_{G,G}(M)^{-1}"] & (G\times G)\times M \arrow[r, "\mu\times M"] & G\times M
\end{tikzcd}$$

The commutativity of the above diagrams and the definition of $P_{(M,\xi)}^{(G,\mu_f^r)}$ imply that the following equalities.

$$\begin{aligned}
P_{(M,\xi)}^{(G,\mu_f^r)}\mu_l(M)(G\times(\mu_f^r\times M))\theta_{G,G\times H}(M) &= P_{(M,\xi)}^{(G,\mu_f^r)}(\mu\times M)\theta_{G,G}(M)^{-1}(G\times(\mu_f^r\times M))\theta_{G,G\times H}(M) \\
&= P_{(M,\xi)}^{(G,\mu_f^r)}(\mu_f^r\times M)((\mu\times id_H)\times M) \\
&= P_{(M,\xi)}^{(G,\mu_f^r)}(G\times\hat{\xi})\theta_{G,H}(M)((\mu\times id_H)\times M) \\
&= P_{(M,\xi)}^{(G,\mu_f^r)}(\mu\times M)\theta_{G,G}(M)^{-1}(G\times(G\times\hat{\xi}))(G\times\theta_{G,H}(M))\theta_{G,G\times H}(M) \\
&= P_{(M,\xi)}^{(G,\mu_f^r)}\mu_l(M)(G\times((G\times\hat{\xi})\theta_{G,H}(M)))\theta_{G,G\times H}(M)
\end{aligned}$$

Since $\theta_{G,G\times H}(M)$ is an epimorphism by the assumption (v) of (11.4.1), we have

$$P_{(M,\xi)}^{(G,\mu_f^r)}\mu_l(M)(G\times(\mu_f^r\times M)) = P_{(M,\xi)}^{(G,\mu_f^r)}\mu_l(M)(G\times((G\times\hat{\xi})\theta_{G,H}(M))).$$

It follows from (ii) of (11.4.1) that there exists unique morphism $\hat{\xi}_f : G\times((G,\mu_f^r)\times(M,\xi))\to(G,\mu_f^r)\times(M,\xi)$ that makes the following diagram commutes.

$$\begin{tikzcd}
G\times(G\times M) \arrow[r, "G\times P_{(M,\xi)}^{(G,\mu_f^r)}"] \arrow[d, "\mu_l(M)=(\mu\times M)\theta_{G,G}(M)^{-1}"] & G\times((G,\mu_f^r)\times(M,\xi)) \arrow[d, "\hat{\xi}_f"] \\
G\times M \arrow[r, "P_{(M,\xi)}^{(G,\mu_f^r)}"] & (G,\mu_f^r)\times(M,\xi)
\end{tikzcd}$$

We put $\xi_f^l = P_G((G,\mu_f^r)\times(M,\xi))^{-1}_{(G,\mu_f^r)\times(M,\xi)}(\hat{\xi}_f) : o_G^*((G,\mu_f^r)\times(M,\xi))\to o_G^*((G,\mu_f^r)\times(M,\xi))$.

Proposition 11.4.2 *$((G,\mu_f^r)\times(M,\xi),\xi_f^l)$ is a representation of G and*

$$P_{(M,\xi)}^{(G,\mu_f^r)} : (G\times M,\xi_l(\mu,M))\to((G,\mu_f^r)\times(M,\xi),\xi_f^l)$$

is a morphism of representations.

Proof. The following diagram commutes by (1) of (6.3.3) and the definition of $\hat{\xi}_f$.

$$\begin{tikzcd}
o_G^*(G\times M) \arrow[r, "\iota_G(G\times M)"] \arrow[d, "o_G^*\left(P_{(M,\xi)}^{(G,\mu_f^r)}\right)"] & o_G^*(G\times(G\times M)) \arrow[r, "o_G^*(\mu_l(M))"] \arrow[d, "o_G^*\left(G\times P_{(M,\xi)}^{(G,\mu_f^r)}\right)"] & o_G^*(G\times M) \arrow[d, "o_G^*\left(P_{(M,\xi)}^{(G,\mu_f^r)}\right)"] \\
o_G^*((G,\mu_f^r)\times(M,\xi)) \arrow[r, "\iota_G((G,\mu_f^r)\times(M,\xi))"] & o_G^*(G\times((G,\mu_f^r)\times(M,\xi))) \arrow[r, "o_G^*(\hat{\xi}_f)"] & o_G^*((G,\mu_f^r)\times(M,\xi))
\end{tikzcd}$$

The upper composition of the above diagram is $\xi_l(\mu,M)$ and the lower composition of the above diagram is ξ_f^l by (6.3.2). Thus the assertion follows from (iii) of (11.4.1) and (1) of (11.1.6). $\square$

Proposition 11.4.3 *Assume (11.4.1) for representations (M,ξ) and (N,ζ) of H. If $\varphi : (M,\xi) \to (N,\zeta)$ is a morphism of representations of H,*

$$(G,\mu_f^r)\times\varphi : (G,\mu_f^r)\times(M,\xi) \to (G,\mu_f^r)\times(N,\zeta)$$

defines a morphism $\big((G,\mu_f^r)\times(M,\xi),\xi_f^l\big) \to \big((G,\mu_f^r)\times(N,\zeta),\zeta_f^l\big)$ of representations.

Proof. The following diagram commutes by the definitions of $\hat{\xi}_f$, $\hat{\zeta}_f$, $(G,\mu_f^r)\times\varphi$ and (6.3.3).

$$\begin{tikzcd}
G\times M & G\times(G\times M) \arrow[l, "\mu_l(M)"'] \arrow[r, "G\times(G\times\varphi)"] \arrow[d, "G\times P^{(G,\mu_f^r)}_{(M,\xi)}"] & G\times(G\times N) \arrow[r, "\mu_l(N)"] \arrow[d, "G\times P^{(G,\mu_f^r)}_{(N,\zeta)}"] & G\times N \arrow[d, "P^{(G,\mu_f^r)}_{(N,\zeta)}"] \\
(G,\mu_f^r)\times(M,\xi) \arrow[from=u, "P^{(G,\mu_f^r)}_{(M,\xi)}"] & G\times\big((G,\mu_f^r)\times(M,\xi)\big) \arrow[l, "\hat{\xi}_f"'] \arrow[r, "G\times\left((G,\mu_f^r)\times\varphi\right)"] & G\times\big((G,\mu_f^r)\times(N,\zeta)\big) \arrow[r, "\hat{\zeta}_f"] & (G,\mu_f^r)\times(N,\zeta)
\end{tikzcd}$$

Hence it follows from (11.2.5) and (11.2.14) that

$$\begin{aligned}
\hat{\zeta}_f\big(G\times\big((G,\mu_f^r)\times\varphi\big)\big)\left(G\times P^{(G,\mu_f^r)}_{(M,\xi)}\right) &= P^{(G,\mu_f^r)}_{(N,\zeta)}\mu_l(N)(G\times(G\times\varphi)) = P^{(G,\mu_f^r)}_{(N,\zeta)}(G\times\varphi)\mu_l(M) \\
&= ((G,\mu_f^r)\times\varphi)P^{(G,\mu_f^r)}_{(M,\xi)}\mu_l(M) = ((G,\mu_f^r)\times\varphi)\hat{\xi}_f\left(G\times P^{(G,\mu_f^r)}_{(M,\xi)}\right).
\end{aligned}$$

Since $G\times P^{(G,\mu_f^r)}_{(M,\xi)}$ is an epimorphism by (ii) of (11.4.1), the above equality implies

$$\hat{\zeta}_f\big(G\times\big((G,\mu_f^r)\times\varphi\big)\big) = ((G,\mu_f^r)\times\varphi)\hat{\xi}_f.$$

Thus the result follows from (11.2.3). □

Assume that the assumptions of (11.4.1) is satisfied for any object (M,ξ) of $\mathrm{Rep}(H\,;\mathcal{F})$. By (11.4.2) and (11.4.3), we can define a functor $f_! : \mathrm{Rep}(H\,;\mathcal{F}) \to \mathrm{Rep}(G\,;\mathcal{F})$ by $f_!(M,\xi) = ((G,\mu_f^r)\times(M,\xi),\xi_f^l)$ and $f_!(\varphi) = (G,\mu_f^r)\times\varphi$.

Lemma 11.4.4 *Let (M,ξ) be a representation of H. The following diagram commutes.*

$$\begin{tikzcd}
& H\times M \arrow[dl, "\hat{\xi}"'] \arrow[r, "f\times M"] & G\times M \arrow[d, "P^{(G,\mu_f^r)}_{(M,\xi)}"] \\
M \arrow[r, "\varepsilon\times M"] & G\times M \arrow[r, "P^{(G,\mu_f^r)}_{(M,\xi)}"] & (G,\mu_f^r)\times(M,\xi)
\end{tikzcd}$$

Proof. Since the following diagram commutes,

$$\begin{tikzcd}
H\times M \arrow[r, "(o_H,id_H)\times M"] \arrow[d, "(o_H,id_H)\times M"] & (1\times H)\times M \arrow[r, "\theta_{1,H}(M)"] \arrow[d, "(\varepsilon\times id_G)\times M"] & 1\times(H\times M) \arrow[r, "1\times\hat{\xi}"] \arrow[d, "\varepsilon\times(H\times M)"] & 1\times M \arrow[d, "\varepsilon\times M"] \\
(1\times H)\times M \arrow[r, "(\varepsilon\times id_H)\times M"] \arrow[d, "(id_1\times f)\times M"] & (G\times H)\times M \arrow[r, "\theta_{G,H}(M)"] \arrow[d, "(id_G\times f)\times M"] & G\times(H\times M) \arrow[r, "G\times\hat{\xi}"] & G\times M \arrow[d, "P^{(G,\mu_f^r)}_{(M,\xi)}"] \\
(1\times G)\times M \arrow[r, "(\varepsilon\times id_G)\times M"] & (G\times G)\times M \arrow[r, "\mu\times M"] & G\times M \arrow[r, "P^{(G,\mu_f^r)}_{(M,\xi)}"] & (G,\mu_f^r)\times(M,\xi)
\end{tikzcd}$$

it follows from (6.3.22) that

$$\begin{aligned}
P^{(G,\mu_f^r)}_{(M,\xi)}(\varepsilon\times M)\hat{\xi} &= P^{(G,\mu_f^r)}_{(M,\xi)}(\varepsilon\times M)(1\times\hat{\xi})\theta_{1,H}(M)((o_H,id_H)\times M) \\
&= P^{(G,\mu_f^r)}_{(M,\xi)}(\mu\times M)((\varepsilon\times id_G)\times M)((id_1\times f)\times M)((o_H,id_H)\times M) \\
&= P^{(G,\mu_f^r)}_{(M,\xi)}(\mu(\varepsilon\times id_G)(id_1\times f)(o_H,id_H)\times M) \\
&= P^{(G,\mu_f^r)}_{(M,\xi)}(\mu(\varepsilon\times id_G)(o_G,id_G)f\times M) = P^{(G,\mu_f^r)}_{(M,\xi)}(f\times M).
\end{aligned}$$

□

Proposition 11.4.5 *Let (M,ξ) be a representation of H. A composition*

$$M = 1\times M \xrightarrow{\varepsilon\times M} G\times M \xrightarrow{P^{(G,\mu^r_f)}_{(M,\xi)}} (G,\mu^r_f)\times(M,\xi)$$

defines a morphism $(M,\xi)\to\big((G,\mu^r_f)\times(M,\xi),(\xi^l_f)_f\big) = f^\bullet f_!(M,\xi)$ of representations of H.

Proof. We put $\tilde{\xi}_f = P_H\big((G,\mu^r_f)\times(M,\xi)\big)_{(G,\mu^r_f)\times(M,\xi)}\big((\xi^l_f)_f\big)$ Then, it follows from (6.3.6) that

$$\tilde{\xi}_f = \hat{\xi}_f\big(f\times\big((G,\mu^r_f)\times(M,\xi)\big)\big).$$

The following diagram commutes by the definition of $\hat{\xi}_f$, (6.3.3), (6.3.9), (6.3.20).

$$\begin{tikzcd}
H\times(H\times M) \arrow[r, "H\times(f\times M)"] \arrow[d, "f\times(H\times M)"] & H\times(G\times M) \arrow[r, "H\times P^{(G,\mu^r_f)}_{(M,\xi)}"] \arrow[d, "f\times(G\times M)"] & H\times\big((G,\mu^r_f)\times(M,\xi)\big) \arrow[d, "f\times\left((G,\mu^r_f)\times(M,\xi)\right)"] \\
G\times(H\times M) \arrow[r, "G\times(f\times M)"] \arrow[d, "\theta_{G,H}(M)^{-1}"] & G\times(G\times M) \arrow[r, "G\times P^{(G,\mu^r_f)}_{(M,\xi)}"] \arrow[d, "\theta_{G,G}(M)^{-1}"] & G\times\big((G,\mu^r_f)\times(M,\xi)\big) \arrow[dd, "\hat{\xi}_f"] \\
(G\times H)\times M \arrow[r, "(id_G\times f)\times M"] & (G\times G)\times M \arrow[d, "\mu\times M"] & \\
 & G\times M \arrow[r, "P^{(G,\mu^r_f)}_{(M,\xi)}"] & (G,\mu^r_f)\times(M,\xi)
\end{tikzcd}$$

Recall that $P^{(G,\mu^r_f)}_{(M,\xi)} : G\times M\to(G,\mu^r_f)\times(M,\xi)$ is a coequalizer of $(\mu\times M)((id_G\times f)\times M) : (G\times H)\times M\to G\times M$ and $(G\times\hat{\xi})\theta_{G,H}(M) : (G\times H)\times M\to G\times M$. Hence by (11.4.4), we have

$$\begin{aligned}
\tilde{\xi}_f\Big(H\times P^{(G,\mu^r_f)}_{(M,\xi)}(\varepsilon\times M)\Big) &= \hat{\xi}_f\big(f\times\big((G,\mu^r_f)\times(M,\xi)\big)\big)\Big(H\times P^{(G,\mu^r_f)}_{(M,\xi)}\Big)(H\times(\varepsilon\times M))\\
&= \hat{\xi}_f\big(f\times\big((G,\mu^r_f)\times(M,\xi)\big)\big)\Big(H\times P^{(G,\mu^r_f)}_{(M,\xi)}\Big)(H\times(f\times M))(H\times(\varepsilon'\times M))\\
&= P^{(G,\mu^r_f)}_{(M,\xi)}(\mu\times M)((id_G\times f)\times M)\theta_{G,H}(M)^{-1}(f\times(H\times M))(H\times(\varepsilon'\times M))\\
&= P^{(G,\mu^r_f)}_{(M,\xi)}(G\times\hat{\xi})(G\times(\varepsilon'\times M))(f\times(1\times M))\\
&= P^{(G,\mu^r_f)}_{(M,\xi)}(G\times\hat{\xi}(\varepsilon'\times M))(f\times(1\times M)) = P^{(G,\mu^r_f)}_{(M,\xi)}(f\times M) = P^{(G,\mu^r_f)}_{(M,\xi)}(\varepsilon\times M)\hat{\xi}
\end{aligned}$$

Therefore $P^{(G,\mu^r_f)}_{(M,\xi)}(\varepsilon\times M) : (M,\xi)\to\big((G,\mu^r_f)\times(M,\xi),(\xi_f)_f\big) = f^\bullet f_!(M,\xi)$ is a morphism of representations of H. ∎

Theorem 11.4.6 *$f_! : \mathrm{Rep}(H\,;\mathcal{F})\to\mathrm{Rep}(G\,;\mathcal{F})$ is a left adjoint of $f^\bullet : \mathrm{Rep}(G\,;\mathcal{F})\to\mathrm{Rep}(H\,;\mathcal{F})$.*

Proof. Let (M,ξ) be an object of $\mathrm{Rep}(H\,;\mathcal{F})$ and (N,ζ) an object of $\mathrm{Rep}(G\,;\mathcal{F})$. For a morphism $\varphi : (M,\xi)\to(N,\zeta_f) = f^\bullet(N,\zeta)$ of $\mathrm{Rep}(H\,;\mathcal{F})$, we define a morphism ${}^t\varphi : (G,\mu^r_f)\times(M,\xi)\to(N,\zeta)$ as follows. Put $\hat{\xi} = P_H(M)_M(\xi)$ and $\hat{\zeta} = P_G(N)_N(\zeta)$. Then, $P_H(N)_N(\zeta_f) = \hat{\zeta}(f\times N)$ by (6.3.3) and it follows from (6.3.5) that $\varphi\hat{\xi} = \hat{\zeta}(f\times N)(H\times\varphi)$. By (11.2.1) and (6.3.20), we have

$$\hat{\zeta}(\mu\times N)((id_G\times f)\times N) = \hat{\zeta}(G\times\hat{\zeta})\theta_{G,G}(N)((id_G\times f)\times N) = \hat{\zeta}(G\times\hat{\zeta})(G\times(f\times N))\theta_{G,H}(N).$$

Hence

$$\begin{aligned}
\hat{\zeta}(G\times\varphi)(\mu^r_f\times M) &= \hat{\zeta}(\mu^r_f\times N)((G\times H)\times\varphi) = \hat{\zeta}(\mu\times N)((id_G\times f)\times N)((G\times H)\times\varphi)\\
&= \hat{\zeta}(G\times\hat{\zeta})(G\times(f\times N))\theta_{G,H}(N)((G\times H)\times\varphi)\\
&= \hat{\zeta}(G\times\hat{\zeta})(G\times(f\times N))(G\times(H\times\varphi))\theta_{G,H}(N) = \hat{\zeta}(G\times\hat{\zeta}(f\times N)(H\times\varphi))\theta_{G,H}(N)\\
&= \hat{\zeta}(G\times\varphi\hat{\xi})\theta_{G,H}(N) = \hat{\zeta}(G\times\varphi)(G\times\hat{\xi})\theta_{G,H}(N),
\end{aligned}$$

which implies that there exists a unique morphism ${}^t\varphi : (G,\mu_f^r)\times(M,\xi)\to N$ that satisfies ${}^t\varphi P_{(M,\xi)}^{(G,\mu_f^r)} = \hat{\zeta}(G\times\varphi)$. Then,

$$\begin{aligned}\hat{\zeta}(G\times {}^t\varphi)\Big(G\times P_{(M,\xi)}^{(G,\mu_f^r)}\Big) &= \hat{\zeta}(G\times\hat{\zeta}(G\times\varphi)) = \hat{\zeta}(G\times\hat{\zeta})(G\times(G\times\varphi)) = \hat{\zeta}(\mu\times N)\theta_{G,G}(N)^{-1}(G\times(G\times\varphi))\\ &= \hat{\zeta}(\mu\times N)((G\times G)\times\varphi)\theta_{G,G}(M)^{-1} = \hat{\zeta}(G\times\varphi)(\mu\times M)\theta_{G,G}(M)^{-1}\\ &= {}^t\varphi P_{(M,\xi)}^{(G,\mu_f^r)}(\mu\times M)\theta_{G,G}(M)^{-1} = {}^t\varphi\,\hat{\xi}_f\Big(G\times P_{(M,\xi)}^{(G,\mu_f^r)}\Big)\end{aligned}$$

by (11.2.1), (6.3.20) and the definition of $\hat{\xi}_f$. Since $G\times P_{(M,\xi)}^{(G,\mu_f^r)}$ is an epimorphism by (ii) of (11.4.1), the above implies $\hat{\zeta}(G\times {}^t\varphi) = {}^t\varphi\,\hat{\xi}_f$. Namely, ${}^t\varphi$ is a morphism $f_!(M,\xi) = \big((G,\mu_f^r)\times(M,\xi),\xi_f^l\big)\to(N,\zeta)$ of representations by (11.2.3). We define a map $\mathrm{ad}_{(N,\zeta)}^{(M,\xi)} : \mathrm{Rep}(H\,;\mathcal{F})((M,\xi),f^\bullet(N,\zeta))\to\mathrm{Rep}(G\,;\mathcal{F})(f_!(M,\xi),(N,\zeta))$ by $\mathrm{ad}_{(N,\zeta)}^{(M,\xi)}(\varphi) = {}^t\varphi$.

Let $\psi : (N,\zeta)\to(L,\lambda)$ be a morphism of $\mathrm{Rep}(G\,;\mathcal{F})$ and put $\hat{\lambda} = P_G(L)_L(\lambda)$. For a morphism $\varphi : (M,\xi)\to(N,\zeta_f) = f^\bullet(N,\zeta)$ of $\mathrm{Rep}(H\,;\mathcal{F})$, since $\psi{}^t\varphi P_{(M,\xi)}^{(G,\mu_f^r)} = \psi\hat{\zeta}(G\times\varphi) = \hat{\lambda}(G\times\psi)(G\times\varphi) = \hat{\lambda}(G\times\psi\varphi)$, we have $\mathrm{ad}_{(L,\lambda)}^{(M,\xi)}(\psi\varphi) = \psi{}^t\varphi$ by the definition of $\mathrm{ad}_{(L,\lambda)}^{(M,\xi)}$. This shows that the following diagram commutes.

$$\begin{tikzcd}\mathrm{Rep}(H\,;\mathcal{F})((M,\xi),f^\bullet(N,\zeta)) \arrow[r, "\mathrm{ad}_{(N,\zeta)}^{(M,\xi)}"] \arrow[d, "f^\bullet(\psi)_*"] & \mathrm{Rep}(G\,;\mathcal{F})(f_!(M,\xi),(N,\zeta)) \arrow[d, "\psi_*"] \\ \mathrm{Rep}(H\,;\mathcal{F})((M,\xi),f^\bullet(L,\lambda)) \arrow[r, "\mathrm{ad}_{(L,\lambda)}^{(M,\xi)}"] & \mathrm{Rep}(G\,;\mathcal{F})(f_!(M,\xi),(L,\lambda))\end{tikzcd}$$

Let $\gamma : (P,\chi)\to(M,\xi)$ be a morphism of $\mathrm{Rep}(H\,;\mathcal{F})$ and put $\hat{\chi} = P_H(P)_P(\chi)$. Since ${}^t\varphi P_{(M,\xi)}^{(G,\mu_f^r)} = \hat{\zeta}(G\times\varphi)$, we have ${}^t\varphi((G,\mu_f^r)\times\gamma)P_{(P,\chi)}^{(G,\mu_f^r)} = {}^t\varphi P_{(M,\xi)}^{(G,\mu_f^r)}(G\times\gamma) = \hat{\zeta}(G\times\varphi)(G\times\gamma) = \hat{\zeta}(G\times\varphi\gamma)$, which implies that $\mathrm{ad}_{(P,\chi)}^{(N,\zeta)}(\varphi\gamma) = {}^t\varphi((G,\mu_f^r)\times\gamma)$. Therefore the following diagram also commutes.

$$\begin{tikzcd}\mathrm{Rep}(H\,;\mathcal{F})((M,\xi),f^\bullet(N,\zeta)) \arrow[r, "\mathrm{ad}_{(N,\zeta)}^{(M,\xi)}"] \arrow[d, "\gamma^*"] & \mathrm{Rep}(G\,;\mathcal{F})(f_!(M,\xi),(N,\zeta)) \arrow[d, "f_!(\gamma)^*"] \\ \mathrm{Rep}(H\,;\mathcal{F})((P,\chi),f^\bullet(N,\zeta)) \arrow[r, "\mathrm{ad}_{(N,\zeta)}^{(P,\chi)}"] & \mathrm{Rep}(G\,;\mathcal{F})(f_!(P,\chi),(N,\zeta))\end{tikzcd}$$

If $\mathrm{ad}_{(N,\zeta)}^{(M,\xi)}(\varphi) = \mathrm{ad}_{(N,\zeta)}^{(M,\xi)}(\psi)$ for $\varphi,\psi\in\mathrm{Rep}(H\,;\mathcal{F})((M,\xi),f^\bullet(N,\zeta))$, then $\hat{\zeta}(G\times\varphi) = \hat{\zeta}(G\times\psi)$. It follows from (11.2.1) and (6.3.9) that $\varphi = \hat{\zeta}(\varepsilon\times N)\varphi = \hat{\zeta}(G\times\varphi)(\varepsilon\times M) = \hat{\zeta}(G\times\psi)(\varepsilon\times M) = \hat{\zeta}(\varepsilon\times N)\psi = \psi$. Hence $\mathrm{ad}_{(N,\zeta)}^{(M,\xi)}$ is injective.

For a morphism $\psi : f_!(M,\xi)\to(N,\zeta)$ of $\mathrm{Rep}(G\,;\mathcal{F})$, define $\varphi : M\to N$ to be a composition

$$M = 1\times M \xrightarrow{\varepsilon\times M} G\times M \xrightarrow{P_{(M,\xi)}^{(G,\mu_f^r)}} (G,\mu_f^r)\times(M,\xi) \xrightarrow{\psi} N$$

Since $\psi = f^\bullet(\psi) : \big((G,\mu_f^r)\times(M,\xi),(\xi_f^l)_f\big)\to(N,\zeta_f)$ is a morphism of representations of H, it follows from (11.4.5) that φ defines a morphism $(M,\xi)\to f^\bullet(N,\zeta) = (N,\zeta_f)$ of representations of H. We note that the following diagrams commute.

$$\begin{tikzcd}G\times(G\times M) \arrow[r, "(\mu\times M)\theta_{G,G}(M)^{-1}"] \arrow[d, "G\times P_{(M,\xi)}^{(G,\mu_f^r)}"] & G\times M \arrow[d, "P_{(M,\xi)}^{(G,\mu_f^r)}"] \\ G\times((G,\mu_f^r)\times(M,\xi)) \arrow[r, "\hat{\xi}_f"] \arrow[d, "G\times\psi"] & (G,\mu_f^r)\times(M,\xi) \arrow[d, "\psi"] \\ G\times N \arrow[r, "\hat{\zeta}"] & N\end{tikzcd} \qquad \begin{tikzcd}G\times(1\times M) \arrow[r, "G\times(\varepsilon\times M)"] \arrow[d, "\theta_{1,G}(M)^{-1}"] & G\times(G\times M) \arrow[d, "\theta_{G,G}(M)^{-1}"] \\ (G\times 1)\times M \arrow[r, "(id_G\times\varepsilon)\times M"] \arrow[dr, "\mathrm{pr}_2\times M"] & (G\times G)\times M \arrow[d, "\mu\times M"] \\ & G\times M\end{tikzcd}$$

It follows from (6.3.22) that

$$\begin{aligned}\hat{\zeta}(G\times\varphi) &= \hat{\zeta}(G\times\psi)\Big(G\times P^{(G,\mu^r_f)}_{(M,\xi)}\Big)(G\times(\varepsilon\times M)) = \psi P^{(G,\mu^r_f)}_{(M,\xi)}(\mu\times M)\theta_{G,G}(M)^{-1}(G\times(\varepsilon\times M))\\ &= \psi P^{(G,\mu^r_f)}_{(M,\xi)}(\mathrm{pr}_2\times M)\theta_{1,G}(M)^{-1} = \psi P^{(G,\mu^r_f)}_{(M,\xi)},\end{aligned}$$

which implies $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}(\varphi)=\psi$ and this shows that $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}$ is surjective. $\square$

Remark 11.4.7 *For $(N,\zeta)\in\mathrm{Rep}(G\,;\mathcal{F})$, the counit*

$$\varepsilon^l(f)_{(N,\zeta)} = \mathrm{ad}^{f^\bullet(N,\zeta)}_{(N,\zeta)}(id_{f^\bullet(N,\zeta)}) : f_!f^\bullet(N,\zeta) = \big((G,\mu^r_f)\times(N,\zeta_f),(\zeta_f)^l_f\big)\to(N,\zeta)$$

of the adjunction $f_!\dashv f^\bullet$ is the unique morphism that satisfies $\varepsilon^l(f)_{(N,\zeta)}P^{(G,\mu^r_f)}_{(N,\zeta_f)}=\hat{\zeta}$. For $(M,\xi)\in\mathrm{Rep}(H\,;\mathcal{F})$, the unit of the adjunction $f_!\dashv f^\bullet$

$$\eta^l(f)_{(M,\xi)} : (M,\xi)\to\big((G,\mu^r_f)\times(M,\xi),(\xi^l_f)_f\big) = f^\bullet f_!(M,\xi)$$

is given by the composition $M=1\times M\xrightarrow{\varepsilon\times M}G\times M\xrightarrow{P^{(G,\mu^r_f)}_{(M,\xi)}}(G,\mu^r_f)\times(M,\xi)$.

Definition 11.4.8 *Let G be a group object of $\mathcal{T}$ and $\iota:H\to G$ a subgroup object of G. For a representation (M,ξ) of H, we call $\iota_!(M,\xi)$ the left induced representation of (M,ξ) to G and denote this by $\mathrm{Lind}^G_H(M,\xi)$.*

11.5 Right induced representations

Let $p:\mathcal{F}\to\mathcal{T}$ be a fibered category with exponents and $(G,\mu,\varepsilon,\iota)$, $(H,\mu',\varepsilon',\iota')$ group objects in $\mathcal{T}$. For a morphism $f:H\to G$ of group objects, define a left H-action $\mu^l_f:H\times G\to G$ on G by $\mu^l_f=\mu(f\times id_G)$.

Assumption 11.5.1 *For a representation (M,ξ) of H, we put $\check{\xi}=E_H(M)_M(\xi):M\to M^H$. We assume the following.*

(i) An equalizer $E^{(G,\mu^l_f)}_{(M,\xi)}:(M,\xi)^{(G,\mu^l_f)}\to M^G$ of $M^G \underset{\theta^{H,G}(M)\check{\xi}^G}{\overset{M^{\mu^l_f}}{\rightrightarrows}} M^{H\times G}$ exists.

(ii) $\Big(E^{(G,\mu^l_f)}_{(M,\xi)}\Big)^G:\big((M,\xi)^{(G,\mu^l_f)}\big)^G\to(M^G)^G$ is an equalizer of $(M^G)^G \underset{\theta^{H,G}(M)^G(\check{\xi}^G)^G}{\overset{(M^{\mu^l_f})^G}{\rightrightarrows}} (M^{H\times G})^G$.

(iii) The following map is injective.

$$o^*_{G\times G}\big(E^{(G,\mu^l_f)}_{(M,\xi)}\big)_*:\mathcal{F}_{G\times G}\big(o^*_{G\times G}\big((M,\xi)^{(G,\mu^l_f)}\big),o^*_{G\times G}\big((M,\xi)^{(G,\mu^l_f)}\big)\big)\to\mathcal{F}_{G\times G}\big(o^*_{G\times G}\big((M,\xi)^{(G,\mu^l_f)}\big),o^*_{G\times G}(M^G)\big)$$

(iv) $\theta^{G,G}(M):(M^G)^G\to M^{G\times G}$ is an isomorphism.

(v) The following morphisms are monomorphisms.

$$\theta^{G,G\times G}(M):(M^G)^{G\times G}\to M^{G\times G\times G},\qquad \theta^{H\times G,G}(M):(M^{H\times G})^G\to M^{H\times G\times G}$$

Let (M,ξ) be a representation of H on $M\in\mathrm{Ob}\,\mathcal{F}_1$. We define a representation ξ^r_f of G on $(M,\xi)^{(G,\mu^l_f)}$ as follows.

Put $\check{\xi}=E_H(M)_M(\xi):M\to M^H$. The upper rectangle of the following left diagram commutes by the associativity of μ and (6.4.3). It follows from (6.4.20) that the lower rectangle of the following left diagram and the upper right and the lower left rectangles of the following right diagram commute. The upper left rectangle of the following right diagram commutes by (6.4.9) and the lower right rectangle of the following right diagram commutes by (6.4.21).

$$\begin{tikzcd}
M^G \arrow[r, "M^{\mu^l_f}"] \arrow[d, "M^\mu"] & M^{H\times G} \arrow[d, "M^{id_H\times\mu}"] \\
M^{G\times G} \arrow[r, "M^{\mu^l_f\times id_G}"] \arrow[d, "\theta^{G,G}(M)^{-1}"] & M^{H\times G\times G} \\
(M^G)^G \arrow[r, "(M^{\mu^l_f})^G"] & (M^{H\times G})^G \arrow[u, "\theta^{H\times G,G}(M)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M^G \arrow[r, "\check{\xi}^G"] \arrow[d, "M^\mu"] & (M^H)^G \arrow[r, "\theta^{H,G}(M)"] \arrow[d, "(M^H)^\mu"] & M^{H\times G} \arrow[d, "M^{id_H\times\mu}"] \\
M^{G\times G} \arrow[r, "\check{\xi}^{G\times G}"] \arrow[d, "\theta^{G,G}(M)^{-1}"] & (M^H)^{G\times G} \arrow[r, "\theta^{H,G\times G}(M)"] & M^{H\times G\times G} \\
(M^G)^G \arrow[r, "(\check{\xi}^G)^G"] & ((M^H)^G)^G \arrow[r, "\theta^{H,G}(M)^G"] \arrow[u, "\theta^{G,G}(M^H)"'] & (M^{H\times G})^G \arrow[u, "\theta^{H\times G,G}(M)"']
\end{tikzcd}$$

The commutativity of the above diagrams and the definition of $E^{(G,\mu^l_f)}_{(M,\xi)}$ imply that the following equalities.

$$\begin{aligned}
\theta^{H\times G,G}(M)(M^{\mu^l_f})^G\mu_r(M)E^{(G,\mu^l_f)}_{(M,\xi)} &= \theta^{H\times G,G}(M)(M^{\mu^l_f})^G\theta^{G,G}(M)^{-1}M^{\mu}E^{(G,\mu^l_f)}_{(M,\xi)}\\
&= M^{id_H\times\mu}M^{\mu^l_f}E^{(G,\mu^l_f)}_{(M,\xi)} = M^{id_H\times\mu}\theta^{H,G}(M)\check{\xi}^G E^{(G,\mu^l_f)}_{(M,\xi)}\\
&= \theta^{H\times G,G}(M)\theta^{H,G}(M)^G(\check{\xi}^G)^G\theta^{G,G}(M)^{-1}M^{\mu}E^{(G,\mu^l_f)}_{(M,\xi)}\\
&= \theta^{H\times G,G}(M)\theta^{H,G}(M)^G(\check{\xi}^G)^G\mu_r(M)E^{(G,\mu^l_f)}_{(M,\xi)}
\end{aligned}$$

Since $\theta^{H\times G,G}(M)$ is a monomorphism by the assumption (v) of (11.5.1), we have

$$(M^{\mu^l_f})^G\mu_r(M)E^{(G,\mu^l_f)}_{(M,\xi)} = \theta^{H,G}(M)^G(\check{\xi}^G)^G\mu_r(M)E^{(G,\mu^l_f)}_{(M,\xi)}$$

It follows from (ii) of (11.5.1) that there exists unique morphism $\check{\xi}_f : (M,\xi)^{(G,\mu^l_f)} \to \big((M,\xi)^{(G,\mu^l_f)}\big)^G$ that makes the following diagram commutes.

$$\begin{tikzcd}
(M,\xi)^{(G,\mu^l_f)} \arrow[r, "E^{(G,\mu^l_f)}_{(M,\xi)}"] \arrow[d, "\check{\xi}_f"] & M^G \arrow[d, "\mu_r(M)=\theta^{G,G}(M)^{-1}M^{\mu}"] \\
\big((M,\xi)^{(G,\mu^l_f)}\big)^G \arrow[r, "\big(E^{(G,\mu^l_f)}_{(M,\xi)}\big)^G"] & (M^G)^G
\end{tikzcd}$$

We put $\xi^r_f = E_G\big((M,\xi)^{(G,\mu^l_f)}\big)^{-1}_{(M,\xi)^{(G,\mu^l_f)}}(\check{\xi}_f) : o^*_G\big((M,\xi)^{(G,\mu^l_f)}\big) \to o^*_G\big((M,\xi)^{(G,\mu^l_f)}\big)$.

Proposition 11.5.2 *$\big((M,\xi)^{(G,\mu^l_f)},\xi^r_f\big)$ is a representation of G and*

$$E^{(G,\mu^l_f)}_{(M,\xi)} : \big((M,\xi)^{(G,\mu^l_f)},\xi^r_f\big) \to (M^G,\xi_r(\mu,M))$$

is a morphism of representations.

Proof. The following diagram commutes by (1) of (6.4.3) and the definition of $\check{\xi}_f$.

$$\begin{tikzcd}
o^*_G\big((M,\xi)^{(G,\mu^l_f)}\big) \arrow[r, "o^*_G(\check{\xi}_f)"] \arrow[d, "o^*_G\big(E^{(G,\mu^l_f)}_{(M,\xi)}\big)"] & o^*_G\Big(\big((M,\xi)^{(G,\mu^l_f)}\big)^G\Big) \arrow[r, "\pi_G\big((M,\xi)^{(G,\mu^l_f)}\big)"] \arrow[d, "o^*_G\Big(\big(E^{(G,\mu^l_f)}_{(M,\xi)}\big)^G\Big)"] & o^*_G\big((M,\xi)^{(G,\mu^l_f)}\big) \arrow[d, "o^*_G\big(E^{(G,\mu^l_f)}_{(M,\xi)}\big)"] \\
o^*_G(M^G) \arrow[r, "o^*_G(\mu_r(M))"] & o^*_G\big((M^G)^G\big) \arrow[r, "\pi_G(M^G)"] & o^*_G(M^G)
\end{tikzcd}$$

The upper composition of the above diagram is ξ^r_f and the lower composition of the above diagram is $\xi_r(\mu,M)$ by (6.4.2). Since $E^{(G,\mu^l_f)}_{(M,\xi)}$ is a monomorphism, the assertion follows from (2) of (11.1.6). □

Lemma 11.5.3 *Assume (11.5.1) for representations (M,ξ) and (N,ζ) of H. If $\varphi : (M,\xi) \to (N,\zeta)$ is a morphism of representations of H,*

$$\varphi^{(G,\mu^l_f)} : (M,\xi)^{(G,\mu^l_f)} \to (N,\zeta)^{(G,\mu^l_f)}$$

defines a morphism $\big((M,\xi)^{(G,\mu^l_f)},\xi^r_f\big) \to \big((N,\zeta)^{(G,\mu^l_f)},\zeta^r_f\big)$ of representations.

Proof. The following diagram commutes by the definitions of $\check{\xi}_f$, $\check{\zeta}_f$, $\varphi^{(G,\mu^l_f)}$ and (6.4.3).

$$\begin{tikzcd}
(M,\xi)^{(G,\mu^l_f)} \arrow[r, "\check{\xi}_f"] \arrow[d, "E^{(G,\mu^l_f)}_{(M,\xi)}"] & \big((M,\xi)^{(G,\mu^l_f)}\big)^G \arrow[r, "\big(\varphi^{(G,\mu^l_f)}\big)^G"] \arrow[d, "\big(E^{(G,\mu^l_f)}_{(M,\xi)}\big)^G"] & \big((N,\zeta)^{(G,\mu^l_f)}\big)^G \arrow[d, "\big(E^{(G,\mu^l_f)}_{(N,\zeta)}\big)^G"] & (N,\zeta)^{(G,\mu^l_f)} \arrow[l, "\check{\zeta}_f"'] \arrow[d, "E^{(G,\mu^l_f)}_{(N,\zeta)}"] \\
M^G \arrow[r, "\mu_r(M)"] & (M^G)^G \arrow[r, "(\varphi^G)^G"] & (N^G)^G & N^G \arrow[l, "\mu_r(N)"']
\end{tikzcd}$$

Hence it follows from (11.3.5) and (11.3.14) that

$$\begin{aligned}\left(E^{(G,\mu^l_f)}_{(N,\zeta)}\right)^G \left(\varphi^{(G,\mu^l_f)}\right)^G \check{\xi}_f &= (\varphi^G)^G \mu_r(M) E^{(G,\mu^l_f)}_{(M,\xi)} = \mu_r(N)\varphi^G E^{(G,\mu^l_f)}_{(M,\xi)} = \mu_r(N) E^{(G,\mu^l_f)}_{(N,\zeta)}\varphi^{(G,\mu^l_f)} \\ &= \left(E^{(G,\mu^l_f)}_{(N,\zeta)}\right)^G \check{\zeta}_f \varphi^{(G,\mu^l_f)}.\end{aligned}$$

Since $\left(E^{(G,\mu^l_f)}_{(N,\zeta)}\right)^G$ is a monomorphism by the assumption, the above equality implies $\left(\varphi^{(G,\mu^l_f)}\right)^G \check{\xi}_f = \check{\zeta}_f \varphi^{(G,\mu^l_f)}$. Thus the result follows from (11.3.3). $\square$

Assume that the assumption of (11.5.1) is satisfied for any object (M,ξ) of $\mathrm{Rep}(H\,;\mathcal{F})$. By (11.5.2) and (11.5.3), we can define a functor $f_\bullet : \mathrm{Rep}(H\,;\mathcal{F}) \to \mathrm{Rep}(G\,;\mathcal{F})$ by $f_\bullet(M,\xi) = \big((M,\xi)^{(G,\mu^l_f)}, \xi^r_f\big)$ and $f_\bullet(\varphi) = \varphi^{(G,\mu^l_f)}$.

Lemma 11.5.4 *Let (M,ξ) be a representation of H. The following diagram commutes.*

$$\begin{tikzcd}
(M,\xi)^{(G,\mu^l_f)} \arrow[r, "E^{(G,\mu^l_f)}_{(M,\xi)}"] \arrow[d, "E^{(G,\mu^l_f)}_{(M,\xi)}"] & M^G \arrow[r, "M^\varepsilon"] & M \arrow[dl, "\check{\xi}"] \\
M^G \arrow[r, "M^f"] & M^H &
\end{tikzcd}$$

Proof. Since the following diagram commutes,

$$\begin{tikzcd}
(M,\xi)^{(G,\mu^l_f)} \arrow[r, "E^{(G,\mu^l_f)}_{(M,\xi)}"] \arrow[d, "E^{(G,\mu^l_f)}_{(M,\xi)}"] & M^G \arrow[r, "M^\mu"] & M^{G\times G} \arrow[r, "M^{id_G\times\varepsilon}"] \arrow[d, "M^{f\times id_G}"] & M^{G\times 1} \arrow[d, "M^{f\times id_1}"] \\
M^G \arrow[r, "\check{\xi}^G"] \arrow[d, "M^\varepsilon"] & (M^H)^G \arrow[r, "\theta^{H,G}(M)"] \arrow[d, "(M^H)^\varepsilon"] & M^{H\times G} \arrow[r, "M^{id_H\times\varepsilon}"] \arrow[d, "M^{id_G\times\varepsilon}"] & M^{H\times 1} \arrow[d, "M^{(id_H,o_H)}"] \\
M^1 \arrow[r, "\check{\xi}^1"] & (M^H)^1 \arrow[r, "\theta^{H,1}(M)"] & M^{H\times 1} \arrow[r, "M^{(id_H,o_H)}"] & M^H
\end{tikzcd}$$

it follows from (6.4.22) that

$$\begin{aligned}\check{\xi}M^\varepsilon E^{(G,\mu^l_f)}_{(M,\xi)} &= M^{(id_H,o_H)}\theta^{H,1}(M)\check{\xi}^1 M^\varepsilon E^{(G,\mu^l_f)}_{(M,\xi)} = M^{(id_H,o_H)} M^{f\times id_1} M^{id_G\times\varepsilon} M^\mu E^{(G,\mu^l_f)}_{(M,\xi)} \\ &= M^{\mu(id_G\times\varepsilon)(f\times id_1)(id_H,o_H)} E^{(G,\mu^l_f)}_{(M,\xi)} = M^{\mu(id_G\times\varepsilon)(id_G,o_G)f} E^{(G,\mu^l_f)}_{(M,\xi)} = M^f E^{(G,\mu^l_f)}_{(M,\xi)}.\end{aligned}$$

$\square$

Proposition 11.5.5 *Let (M,ξ) be a representation of H. A composition*

$$(M,\xi)^{(G,\mu^l_f)} \xrightarrow{E^{(G,\mu^l_f)}_{(M,\xi)}} M^G \xrightarrow{M^\varepsilon} M^1 = M$$

defines a morphism $f^\bullet f_\bullet(M,\xi) = \big((M,\xi)^{(G,\mu^l_f)}, (\xi^r_f)_f\big) \to (M,\xi)$ of representations of H.

Proof. We put $\bar{\xi}_f = E_H\big((M,\xi)^{(G,\mu^l_f)}\big)_{(M,\xi)^{(G,\mu^l_f)}}\big((\xi^r_f)_f\big)$ Then, $\bar{\xi}_f = \big((M,\xi)^{(G,\mu^l_f)}\big)^f \check{\xi}_f$ by (6.4.3). The following diagram commutes by the definition of $\check{\xi}_f$, (6.4.3), (6.4.9), (6.4.20).

$$\begin{tikzcd}
(M,\xi)^{(G,\mu^l_f)} \arrow[rr, "\check{\xi}_f"] \arrow[d, "E^{(G,\mu^l_f)}_{(M,\xi)}"] & & \big((M,\xi)^{(G,\mu^l_f)}\big)^G \arrow[r, "\left((M,\xi)^{(G,\mu^l_f)}\right)^f"] \arrow[d, "\left(E^{(G,\mu^l_f)}_{(M,\xi)}\right)^G"] & \big((M,\xi)^{(G,\mu^l_f)}\big)^H \arrow[d, "\left(E^{(G,\mu^l_f)}_{(M,\xi)}\right)^H"] \\
M^G \arrow[r, "M^\mu"] & M^{G\times G} \arrow[r, "\theta^{G,G}(M)^{-1}"] \arrow[d, "M^{f\times id_G}"] & (M^G)^G \arrow[r, "(M^G)^f"] \arrow[d, "(M^f)^G"] & (M^G)^H \arrow[d, "(M^f)^H"] \\
& M^{H\times G} \arrow[r, "\theta^{H,G}(M)^{-1}"] & (M^H)^G \arrow[r, "(M^H)^f"] & (M^H)^H
\end{tikzcd}$$

Recall that $E^{(G,\mu^l_f)}_{(M,\xi)} : (M,\xi)^{(G,\mu^l_f)} \to M^G$ is an equalizer of $M^{f\times id_G}M^\mu : M^G \to M^{H\times G}$ and $\theta^{H,G}(M)\check{\xi}^G : M^G \to M^{H\times G}$. Hence by (11.5.4), we have

$$\begin{aligned}\Big(M^\varepsilon E^{(G,\mu^l_f)}_{(M,\xi)}\Big)^H\bar{\xi}_f &= (M^\varepsilon)^H\Big(E^{(G,\mu^l_f)}_{(M,\xi)}\Big)^H\big((M,\xi)^{(G,\mu^l_f)}\big)^f\check{\xi}_f = (M^{f\varepsilon'})^H(M^G)^f\theta^{G,G}(M)^{-1}M^\mu E^{(G,\mu^l_f)}_{(M,\xi)}\\ &= (M^{\varepsilon'})^H(M^H)^f\theta^{H,G}(M)^{-1}M^{f\times id_G}M^\mu E^{(G,\mu^l_f)}_{(M,\xi)} = M^f(M^{\varepsilon'})^G\check{\xi}^G E^{(G,\mu^l_f)}_{(M,\xi)}\\ &= M^f E^{(G,\mu^l_f)}_{(M,\xi)} = \check{\xi}M^\varepsilon E^{(G,\mu^l_f)}_{(M,\xi)}\end{aligned}$$

Therefore $M^\varepsilon E^{(G,\mu^l_f)}_{(M,\xi)} : f^\bullet f_\bullet(M,\xi) = \big((M,\xi)^{(G,\mu^l_f)}, (\xi^r_f)_f\big) \to (M,\xi)$ is a morphism of representations of H. □

Theorem 11.5.6 *$f_\bullet : \mathrm{Rep}(H\,;\mathcal{F}) \to \mathrm{Rep}(G\,;\mathcal{F})$ is a right adjoint of $f^\bullet : \mathrm{Rep}(G\,;\mathcal{F}) \to \mathrm{Rep}(H\,;\mathcal{F})$.*

Proof. Let (M,ξ) be an object of $\mathrm{Rep}(G\,;\mathcal{F})$ and (N,ζ) an object of $\mathrm{Rep}(H\,;\mathcal{F})$. For a morphism $\varphi : f^\bullet(M,\xi) = (M,\xi_f) \to (N,\zeta)$ of $\mathrm{Rep}(H\,;\mathcal{F})$, we define a morphism $\varphi^t : M \to (N,\zeta)^{(G,\mu^l_f)}$ as follows. Put $\check{\xi} = E_G(M)_M(\xi)$ and $\check{\zeta} = E_H(N)_N(\zeta)$. Then, $E_G(M)_M(\xi_f) = M^f\check{\xi}$ by (6.4.3) and it follows from (11.3.3) that $\check{\zeta}\varphi = \varphi^H M^f\check{\xi}$. By (11.3.1) and (6.4.20), we have $M^{f\times id_G}M^\mu\check{\xi} = M^{f\times id_G}\theta^{G,G}(M)\check{\xi}^G\check{\xi} = \theta^{H,G}(M)(M^f)^G\check{\xi}^G\check{\xi}$. Hence

$$\begin{aligned}N^{\mu^l_f}\varphi^G\check{\xi} &= \varphi^{H\times G}M^{\mu^l_f}\check{\xi} = \varphi^{H\times G}M^{f\times id_G}M^\mu\check{\xi} = \varphi^{H\times G}\theta^{H,G}(M)(M^f)^G\check{\xi}^G\check{\xi} = \theta^{H,G}(N)(\varphi^H)^G(M^f)^G\check{\xi}^G\check{\xi}\\ &= \theta^{H,G}(N)(\varphi^H M^f\check{\xi})^G\check{\xi} = \theta^{H,G}(N)(\check{\zeta}\varphi)^G\check{\xi} = \theta^{H,G}(N)\check{\zeta}^G\varphi^G\check{\xi}\end{aligned}$$

and this implies that there exists a unique morphism $\varphi^t : M \to (N,\zeta)^{(G,\mu^l_f)}$ that satisfies $E^{(G,\mu^l_f)}_{(N,\zeta)}\varphi^t = \varphi^G\check{\xi}$. Then,

$$\begin{aligned}\Big(E^{(G,\mu^l_f)}_{(N,\zeta)}\Big)^G(\varphi^t)^G\check{\xi} &= (\varphi^G)^G\check{\xi}^G\check{\xi} = (\varphi^G)^G\theta^{G,G}(M)^{-1}M^\mu\check{\xi} = \theta^{G,G}(N)^{-1}\varphi^{G\times G}M^\mu\check{\xi} = \theta^{G,G}(N)^{-1}N^\mu\varphi^G\check{\xi}\\ &= \theta^{G,G}(N)^{-1}N^\mu E^{(G,\mu^l_f)}_{(N,\zeta)}\varphi^t = \Big(E^{(G,\mu^l_f)}_{(N,\zeta)}\Big)^G\check{\zeta}_f\varphi^t\end{aligned}$$

by (11.3.1), (6.4.20) and the definition of $\check{\zeta}_f$. Since $\Big(E^{(G,\mu^l_f)}_{(N,\zeta)}\Big)^G$ is a monomorphism by (*ii*) of (11.5.1), the above implies $(\varphi^t)^G\check{\xi} = \check{\zeta}_f\varphi^t$. Namely, φ^t is a morphism $(M,\xi) \to \big((N,\zeta)^{(G,\mu^l_f)}, \zeta^r_f\big) = f_\bullet(N,\zeta)$ of representations by (11.3.3). We define a map $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)} : \mathrm{Rep}(H\,;\mathcal{F})(f^\bullet(M,\xi),(N,\zeta)) \to \mathrm{Rep}(G\,;\mathcal{F})((M,\xi), f_\bullet(N,\zeta))$ by $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}(\varphi) = \varphi^t$.

Let $\psi : (L,\lambda) \to (M,\xi)$ be a morphism of $\mathrm{Rep}(G\,;\mathcal{F})$ and $\varphi : f^\bullet(M,\xi) \to (N,\zeta)$ a morphism of $\mathrm{Rep}(H\,;\mathcal{F})$. We put $\check{\lambda} = E_G(L)_L(\lambda)$. Since $E^{(G,\mu^l_f)}_{(N,\zeta)}\varphi^t\psi = \varphi^G\check{\xi}\psi = \varphi^G\psi^G\check{\lambda} = (\varphi\psi)^G\check{\lambda}$, we have $\mathrm{ad}^{(L,\lambda)}_{(N,\zeta)}(\varphi\psi) = \varphi^t\psi$ by the definition of $\mathrm{ad}^{(L,\lambda)}_{(N,\zeta)}$. This shows that the following diagram commutes.

$$\begin{array}{ccc}\mathrm{Rep}(H\,;\mathcal{F})(f^\bullet(M,\xi),(N,\zeta)) & \xrightarrow{\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}} & \mathrm{Rep}(G\,;\mathcal{F})((M,\xi), f_\bullet(N,\zeta))\\ \big\downarrow{\scriptstyle f^\bullet(\psi)^*} & & \big\downarrow{\scriptstyle \psi^*}\\ \mathrm{Rep}(H\,;\mathcal{F})(f^\bullet(L,\lambda),(N,\zeta)) & \xrightarrow{\mathrm{ad}^{(L,\lambda)}_{(N,\zeta)}} & \mathrm{Rep}(G\,;\mathcal{F})((L,\lambda), f_\bullet(N,\zeta))\end{array}$$

Let $\gamma : (N,\zeta) \to (P,\chi)$ be a morphism of $\mathrm{Rep}(H\,;\mathcal{F})$ and put $\check{\chi} = E_H(P)_P(\chi)$. Since $E^{(G,\mu^l_f)}_{(N,\zeta)}\varphi^t = \varphi^G\check{\xi}$, we have $E^{(G,\mu^l_f)}_{(P,\chi)}\gamma^{(G,\mu^l_f)}\varphi^t = \gamma^G E^{(G,\mu^l_f)}_{(N,\zeta)}\varphi^t = \gamma^G\varphi^G\check{\xi} = (\gamma\varphi)^G\check{\xi}$, which implies $\mathrm{ad}^{(M,\xi)}_{(P,\chi)}(\gamma\varphi) = \gamma^{(G,\mu^l_f)}\varphi^t$. Therefore the following diagram also commutes.

$$\begin{array}{ccc}\mathrm{Rep}(H\,;\mathcal{F})(f^\bullet(M,\xi),(N,\zeta)) & \xrightarrow{\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}} & \mathrm{Rep}(G\,;\mathcal{F})((M,\xi), f_\bullet(N,\zeta))\\ \big\downarrow{\scriptstyle \gamma_*} & & \big\downarrow{\scriptstyle f_\bullet(\gamma)_*}\\ \mathrm{Rep}(H\,;\mathcal{F})(f^\bullet(M,\xi),(P,\chi)) & \xrightarrow{\mathrm{ad}^{(M,\xi)}_{(P,\chi)}} & \mathrm{Rep}(G\,;\mathcal{F})((M,\xi), f_\bullet(P,\chi))\end{array}$$

If $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}(\varphi) = \mathrm{ad}^{(M,\xi)}_{(N,\zeta)}(\psi)$ for $\varphi, \psi \in \mathrm{Rep}(H\,;\mathcal{F})(f^\bullet(M,\xi),(N,\zeta))$, then $\varphi^G\check{\xi} = \psi^G\check{\xi}$. It follows from (11.3.1) and (6.4.9) that $\varphi = \varphi M^\varepsilon\check{\xi} = N^\varepsilon\varphi^G\check{\xi} = N^\varepsilon\psi^G\check{\xi} = \psi M^\varepsilon\check{\xi} = \psi$. Hence $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}$ is injective.

For a morphism $\psi : (M,\xi) \to f_\bullet(N,\zeta)$ of $\mathrm{Rep}(G\,;\mathcal{F})$, define $\varphi : M \to N$ to be a composition

$$M \xrightarrow{\psi} (N,\zeta)^{(G,\mu^l_f)} \xrightarrow{E^{(G,\mu^l_f)}_{(N,\zeta)}} N^G \xrightarrow{N^\varepsilon} N^1 = N.$$

Since $\psi = f^\bullet(\psi) : (M,\xi_f) \to \Big((N,\zeta)^{(G,\mu^l_f)}, (\zeta^r_f)_f\Big)$ is a morphism of representations of H, it follows from (11.5.5) that φ defines a morphism $f^\bullet(M,\xi) = (M,\xi_f) \to (N,\zeta)$ of representations of H. We note that the following diagrams commutes.

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\check{\xi}"] & (N,\zeta)^{(G,\mu^l_f)} \arrow[r, "E^{(G,\mu^l_f)}_{(N,\zeta)}"] \arrow[d, "\check{\zeta}_f"] & N^G \arrow[d, "\theta^{G,G}(N)^{-1}N^\mu"] \\ M^G \arrow[r, "\psi^G"] & \big((N,\zeta)^{(G,\mu^l_f)}\big)^G \arrow[r, "\left(E^{(G,\mu^l_f)}_{(N,\zeta)}\right)^G"] & (N^G)^G \end{tikzcd}
\qquad
\begin{tikzcd} N^G \arrow[r, "N^\mu"] \arrow[dr, "N^{\mathrm{pr}_2}"'] & N^{G\times G} \arrow[r, "\theta^{G,G}(N)^{-1}"] \arrow[d, "N^{\varepsilon\times id_G}"] & (N^G)^G \arrow[d, "(N^\varepsilon)^G"] \\ & N^{1\times G} \arrow[r, "\theta^{1,G}(N)^{-1}"] & (N^1)^G \end{tikzcd}$$

It follows from (6.4.22) that

$$\varphi^G\check{\xi} = (N^\varepsilon)^G\Big(E^{(G,\mu^l_f)}_{(N,\zeta)}\Big)^G\psi^G\check{\xi} = (N^\varepsilon)^G\theta^{G,G}(N)^{-1}N^\mu E^{(G,\mu^l_f)}_{(N,\zeta)}\psi = \theta^{1,G}(N)^{-1}N^{\mathrm{pr}_2}E^{(G,\mu^l_f)}_{(N,\zeta)}\psi = E^{(G,\mu^l_f)}_{(N,\zeta)}\psi,$$

which implies $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}(\varphi) = \psi$ and this shows that $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}$ is surjective. □

Remark 11.5.7 *For $(M,\xi) \in \mathrm{Rep}(G\,;\mathcal{F})$, the unit*

$$\eta^r(f)_{(M,\xi)} = \mathrm{ad}^{(M,\xi)}_{f^\bullet(M,\xi)}\big(id_{f^\bullet(M,\xi)}\big) : (M,\xi) \to \Big((M,\xi_f)^{(G,\mu^l_f)}, (\xi_f)^f\Big) = f_\bullet f^\bullet(M,\xi)$$

of the adjunction $f^\bullet \dashv f_\bullet$ is the unique morphism that satisfies $E^{(G,\mu^l_f)}_{(M,\xi_f)}\eta^r(f)_{(M,\xi)} = \check{\xi}$. For $(N,\zeta) \in \mathrm{Rep}(H\,;\mathcal{F})$, the counit of the adjunction $f^\bullet \dashv f_\bullet$

$$\varepsilon^r(f)_{(N,\zeta)} : f^\bullet f_\bullet(N,\zeta) = \Big((N,\zeta)^{(G,\mu^l_f)}, (\zeta^r_f)_f\Big) \to (N,\zeta)$$

is given by the composition $(N,\zeta)^{(G,\mu^l_f)} \xrightarrow{E^{(G,\mu^l_f)}_{(N,\zeta)}} N^G \xrightarrow{N^\varepsilon} N^1 = N$.

Definition 11.5.8 *Let G be a group object of $\mathcal{T}$ and $\iota : H \to G$ a subgroup object of G. For a representation (M,ξ) of H, we call $\iota_\bullet(M,\xi)$ the right induced representation of (M,ξ) to G and denote this by $\mathrm{Rind}^G_H(M,\xi)$.*

12 Representations in fibered category of affine modules

12.1 Topological Hopf algebras and comodules

We call an internal group in $\mathcal{T}op\mathcal{A}lg_{cK^*}^{op}$ a topological Hopf algebra. Namely, a topological Hopf algebra consists of an object A^* of $\mathcal{T}op\mathcal{A}lg_{cK^*}$ with unit $u_{A^*} : K^* \to A^*$ and product $m_{A^*} : A^* \otimes_{K^*} A^* \to A^*$ and three morphisms $\varepsilon : A^* \to K^*$, $\mu : A^* \to A^* \widehat{\otimes}_{K^*} A^*$, $\iota : A^* \to A^*$ of $\mathcal{T}op\mathcal{A}lg_{K^*}$ which make the following diagrams commute.

$$\begin{tikzcd}
A^* \arrow[r, "\mu"] \arrow[d, "\mu"] & A^* \widehat{\otimes}_{K^*} A^* \arrow[d, "id_{A^*} \widehat{\otimes}_{K^*} \mu"] \\
A^* \widehat{\otimes}_{K^*} A^* \arrow[r, "\mu \widehat{\otimes}_{K^*} id_{A^*}"] & A^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} A^*
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A^* \arrow[dl, "j_1"'] \arrow[d, "\mu"] \arrow[dr, "j_2"] & \\
A^* \widehat{\otimes}_{K^*} K^* & A^* \widehat{\otimes}_{K^*} A^* \arrow[l, "id_{A^*} \widehat{\otimes}_{K^*} \varepsilon"'] \arrow[r, "\varepsilon \widehat{\otimes}_{K^*} id_{A^*}"] & K^* \widehat{\otimes}_{K^*} A^*
\end{tikzcd}$$

$$\begin{tikzcd}
A^* & K^* \arrow[l, "u_{A^*}"'] & A^* \arrow[l, "\varepsilon"'] \arrow[r, "\varepsilon"] \arrow[d, "\mu"] & K^* \arrow[r, "u_{A^*}"] & A^* \\
A^* \widehat{\otimes}_{K^*} A^* \arrow[u, "\hat{m}_{A^*}"] & & A^* \widehat{\otimes}_{K^*} A^* \arrow[ll, "id_{A^*} \widehat{\otimes}_{K^*} \iota"'] \arrow[rr, "\iota \widehat{\otimes}_{K^*} id_{A^*}"] & & A^* \widehat{\otimes}_{K^*} A^* \arrow[u, "\hat{m}_{A^*}"']
\end{tikzcd}$$

Here, $\hat{m}_{A^*} : A^* \widehat{\otimes}_{K^*} A^* \to A^*$ is the map induced by m_{A^*} and $j_1 : A^* \to K^* \widehat{\otimes}_{K^*} A^*$, $j_2 : A^* \to A^* \widehat{\otimes}_{K^*} K^*$ are maps defined by $j_1(a) = a \otimes 1$, $j_2(a) = 1 \otimes a$.

We assume that a subcategory $\mathcal{C}$ of $\mathcal{T}op\mathcal{A}lg_{cK^*}$ has finite coproducts. We also assume that a subcategory $\mathcal{M}$ of $\mathcal{T}op\mathcal{M}od_{cK^*}$ is additive, satisfies (10.1.1) and that every morphism of $\mathcal{M}$ has a kernel in $\mathcal{M}$ and consider the fibered category $p_{\mathcal{C}}^{op} : \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op} \to \mathcal{C}^{op}$ of affine modules (10.1.4).

We recall that, for morphisms $\lambda : R^* \to S^*$, $\nu : S^* \to T^*$ of $\mathcal{C}$ and an object $\boldsymbol{M} = (R^*, M^*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{R^*}$, we have $(\nu\lambda)^*(\boldsymbol{M}) = (T^*, M^* \widehat{\otimes}_{R^*} T^*, \hat{\alpha}_{\nu\lambda})$, $\nu^*(\lambda^*(\boldsymbol{M})) = (T^*, (M^* \widehat{\otimes}_{R^*} S^*) \widehat{\otimes}_{S^*} T^*, \widehat{(\hat{\alpha}_\lambda)}_\nu)$ and the canonical isomorphism $\boldsymbol{c}_{\lambda,\nu}(\boldsymbol{M}) : (\nu\lambda)^*(\boldsymbol{M}) \to \nu^*(\lambda^*(\boldsymbol{M}))$ is given by $\boldsymbol{c}_{\lambda,\nu}(\boldsymbol{M}) = (id_{T^*}, \hat{c}_{\lambda,\nu,M^*})$, where

$$\hat{c}_{\lambda,\nu,M^*} : M^* \widehat{\otimes}_{R^*} T^* \to (M^* \widehat{\otimes}_{R^*} S^*) \widehat{\otimes}_{S^*} T^*$$

is the map induced by a map $M^* \otimes_{R^*} T^* \to (M^* \otimes_{R^*} S^*) \otimes_{S^*} T^*$ which maps $x \otimes t$ to $(x \otimes 1) \otimes t$.

Notations 12.1.1 (1) *For a morphism $\lambda : R^* \to S^*$ of K^*-algebras, we define a left R^*-module structure $R^* \otimes_{K^*} S^* \to S^*$ on S^* by $r \otimes s \mapsto \lambda(r)s$ and denote by ${}_\lambda S^*$ the right R^*-module S^* with this structure map.*

(2) *For a K^*-module M^* and morphisms $\lambda, \nu : R^* \to S^*$ of K^*-algebras, we denote by*

$$\hat{\tau}_{\lambda,\nu} : (M^* \widehat{\otimes}_{K^*} R^*) \widehat{\otimes}_{R^*} {}_\lambda S^* \to (M^* \widehat{\otimes}_{K^*} R^*) \widehat{\otimes}_{R^*} {}_\nu S^*$$

a composition

$$(M^* \otimes_{K^*} R^*) \otimes_{R^*} {}_\lambda S^* \xrightarrow{\tilde{c}^{-1}_{u_{R^*},\lambda,M^*}} M^* \otimes_{K^*} {}_{\lambda u_{R^*}} S^* = M^* \otimes_{K^*} {}_{\nu u_{R^*}} S^* \xrightarrow{\tilde{c}_{u_{R^*},\nu,M^*}} (M^* \otimes_{K^*} R^*) \otimes_{R^*} {}_\nu S^*.$$

Let $(A^*, \mu, \varepsilon, \iota)$ be a topological Hopf algebra in $\mathcal{C}$ which is an group object in $\mathcal{C}^{op}$ which we denote by A^* for short We apply the definition (11.1.1) to G and a fibered category $p_{\mathcal{C}}^{op} : \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op} \to \mathcal{C}^{op}$, then we have the following.

Proposition 12.1.2 *Let $\boldsymbol{M} = (K^*, M^*, \alpha)$ be an object of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$ and $\boldsymbol{\xi} : u_{A^*}^*(\boldsymbol{M}) \to u_{A^*}^*(\boldsymbol{M})$ a morphism of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{A^*}$. We put $\boldsymbol{\xi} = (id_{A^*}, \xi)$ where $\xi : M^* \widehat{\otimes}_{K^*} A^* \to M^* \widehat{\otimes}_{K^*} A^*$ is a homomorphism of right A^*-modules. Then, $\boldsymbol{\xi}$ gives a representation of A^* on $\boldsymbol{M} = (K^*, M^*, \alpha)$ if and only if*

$$\xi \widehat{\otimes}_{A^*} id_{K^*} : (M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} {}_\varepsilon K^* \to (M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} {}_\varepsilon K^*$$

is the identity map of $(M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} {}_\varepsilon K^*$ and ξ makes the following diagram commute.

$$\begin{tikzcd}
(M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} {}_\mu (A^* \widehat{\otimes}_{K^*} A^*) \arrow[r, "\xi \widehat{\otimes}_{A^*} id_{A^* \widehat{\otimes}_{K^*} A^*}"] \arrow[d, "\hat{\tau}_{\mu,\hat{\imath}_1}"] & (M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} {}_\mu (A^* \widehat{\otimes}_{K^*} A^*) \arrow[d, "\hat{\tau}_{\mu,\hat{\imath}_2}"] \\
(M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} {}_{\hat{\imath}_1} (A^* \widehat{\otimes}_{K^*} A^*) \arrow[d, "\xi \widehat{\otimes}_{A^*} id_{A^* \widehat{\otimes}_{K^*} A^*}"] & (M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} {}_{\hat{\imath}_2} (A^* \widehat{\otimes}_{K^*} A^*) \\
(M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} {}_{\hat{\imath}_1} (A^* \widehat{\otimes}_{K^*} A^*) \arrow[r, "\hat{\tau}_{\hat{\imath}_1,\hat{\imath}_2}"] & (M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} {}_{\hat{\imath}_2} (A^* \widehat{\otimes}_{K^*} A^*) \arrow[u, "\xi \widehat{\otimes}_{A^*} id_{A^* \widehat{\otimes}_{K^*} A^*}"']
\end{tikzcd}$$

For a morphism $\boldsymbol{\xi} : u_{A^*}^*(\boldsymbol{M}) \to u_{A^*}^*(\boldsymbol{M})$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}^{op}$, we put

$$\hat{\boldsymbol{\xi}} = P_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) \in \mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}^{op}(A^* \times \boldsymbol{M}, \boldsymbol{M}).$$

If we put $\boldsymbol{\xi} = (id_{A^*}, \xi)$, ξ is a right A^*-module homomorphism $M^* \widehat{\otimes}_{K^*} A^* \to M^* \widehat{\otimes}_{K^*} A^*$. Let us denote by $i_{M^*} : M^* \to M^* \otimes_{K^*} A^*$ a map defined by $i_{M^*}(x) = x \otimes 1$ and by $\hat{\imath}_{M^*} : M^* \to M^* \widehat{\otimes}_{K^*} A^*$ a composition

$$M^* \xrightarrow{i_{M^*}} M^* \otimes_{K^*} A^* \xrightarrow{\text{completion}} M^* \widehat{\otimes}_{K^*} A^*.$$

Since $A^* \times \boldsymbol{M} = (K^*, M^* \widehat{\otimes}_{K^*} A^*, \alpha_{u_{A^*}}(id_{M^* \widehat{\otimes}_{K^*} A^*} \widehat{\otimes}_{K^*} u_{A^*}))$ and $\hat{\boldsymbol{\xi}} = (id_{K^*}, \hat{\xi})$ for a homomorphism $\hat{\xi} = \xi \hat{\imath}_{M^*} : M^* \to M^* \widehat{\otimes}_{K^*} A^*$ of right K^*-modules by (3) of (10.1.9), the following result follows from (11.2.1) and (10.1.9).

Proposition 12.1.3 *$\boldsymbol{\xi}$ defines a representation of A^* on $\boldsymbol{M}$ if and only if a composition*

$$M^* \xrightarrow{\hat{\xi}} M^* \widehat{\otimes}_{K^*} A^* \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} \varepsilon} M^* \widehat{\otimes}_{K^*} K^* = M^* \otimes_{K^*} K^* \xrightarrow{\alpha} M^*$$

is the identity morphism of M^* and the following diagram commute.

$$\begin{tikzcd}
M^* \arrow[r, "\hat{\xi}"] \arrow[dr, "\hat{\xi}"'] & M^* \widehat{\otimes}_{K^*} A^* \arrow[r, "\hat{\xi} \otimes_{K^*} id_{A^*}"] & (M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{K^*} A^* \arrow[d, "\tilde{\theta}_{A^*,A^*}(\boldsymbol{M})"] \\
& M^* \widehat{\otimes}_{K^*} A^* \arrow[r, "id_{M^*} \otimes_{K^*} \mu"] & M^* \widehat{\otimes}_{K^*} (A^* \widehat{\otimes}_{K^*} A^*)
\end{tikzcd}$$

Remark 12.1.4 *For an object $\boldsymbol{M} = (K^*, M^*, \alpha)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$, let $i_{M^*} : M^* \to M^* \otimes_{K^*} A^*$ be the map defined by $i_{M^*}(x) = x \otimes 1$ and $\hat{\imath}_{M^*} : M^* \to M^* \widehat{\otimes}_{K^*} A^*$ a composition $M^* \xrightarrow{i_{M^*}} M^* \otimes_{K^*} A^* \xrightarrow{\eta_{M^* \otimes_{K^*} A^*}} M^* \widehat{\otimes}_{K^*} A^*$. Then we have $P_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(id_{\boldsymbol{M}}) = (id_{K^*}, \hat{\imath}_{M^*})$ by (10.1.9). We call $(M^*, \hat{\imath}_{M^*})$ the trivial right A^*-comodule.*

The following result follows from (11.2.3) and (10.1.9).

Proposition 12.1.5 *Let $(\boldsymbol{M}, \boldsymbol{\xi})$ and $(\boldsymbol{N}, \boldsymbol{\zeta})$ be representations of A^* and we put $P_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{K^*}, \hat{\xi})$, $P_{A^*}(\boldsymbol{N})_{\boldsymbol{N}}(\boldsymbol{\zeta}) = (id_{K^*}, \hat{\zeta})$ Suppose $\boldsymbol{M} = (K^*, M^*, \alpha)$, $\boldsymbol{N} = (K^*, N^*, \beta)$. A morphism $\boldsymbol{\varphi} = (id_{K^*}, \varphi) : \boldsymbol{M} \to \boldsymbol{N}$, of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}^{op}$ gives a morphism $(\boldsymbol{M}, \boldsymbol{\xi}) \to (\boldsymbol{N}, \boldsymbol{\zeta})$ of representations of A^* if and only if the following diagram commutative.*

$$\begin{tikzcd}
N^* \arrow[r, "\hat{\zeta}"] \arrow[d, "\varphi"] & N^* \widehat{\otimes}_{K^*} A^* \arrow[d, "\varphi \widehat{\otimes}_{K^*} id_{A^*}"] \\
M^* \arrow[r, "\hat{\xi}"] & M^* \widehat{\otimes}_{K^*} A^*
\end{tikzcd}$$

If a morphism $\hat{\xi} : M^* \to M^* \otimes_{K^*} A^*$ of right K^*-modules satisfies the conditions of (12.1.3), a pair $(M^*, \hat{\xi})$ is usually called a right Γ-comodule. It follows from the above fact that, the category of representations of A^* is isomorphic to the opposite category of the category of right A^*-comodules.

Definition 12.1.6 *Assume that K^* is an object of $\mathcal{C}$ and that $\Sigma^n K^*$ is an object of $\mathcal{M}$ as a right K^*-module. We denote by $\Sigma^n \boldsymbol{K}$ an object $(K^*, \Sigma^n K^*, \Sigma^n m_{K^*})$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$ and consider the trivial representation $(\Sigma^n \boldsymbol{K}, id_{u_{A^*}^*(\Sigma^n \boldsymbol{K})})$ of A^* on $\Sigma^n \boldsymbol{K}$. For a representation $(\boldsymbol{M}, \boldsymbol{\xi})$ of A^*, we call a morphism $(\boldsymbol{M}, \boldsymbol{\xi}) \to (\Sigma^n \boldsymbol{K}, id_{u_{A^*}^*(\Sigma^n \boldsymbol{K})})$ of representations of A^* an n-dimensional primitive element of $(\boldsymbol{M}, \boldsymbol{\xi})$.*

Proposition 12.1.7 *For a representation $(\boldsymbol{M}, \boldsymbol{\xi})$ of A^*, put $\boldsymbol{M} = (K^*, M^*, \alpha)$ and $P_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{K^*}, \hat{\xi})$. For $x \in M^n$, we define a map $\varphi_x : \Sigma^n K^* \to M^*$ by $\varphi_x([n], r) = xr$. Then, a morphism $(id_{K^*}, \varphi_x) : \boldsymbol{M} \to \Sigma^n \boldsymbol{K}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}^{op}$ is an n-dimensional primitive element of $(\boldsymbol{M}, \boldsymbol{\xi})$ if and only if x satisfies $\hat{\xi}(x) = \hat{\imath}_{M^*}(x)$.*

Proof. We put $P_{A^*}(\Sigma^n \boldsymbol{K})_{\Sigma^n \boldsymbol{K}}(id_{u_{A^*}^*(\Sigma^n \boldsymbol{K})}) = (id_{K^*}, \hat{\zeta})$. Then, $\hat{\zeta} = \hat{\imath}_{\Sigma^n K^*} : \Sigma^n K^* \to \Sigma^n K^* \widehat{\otimes}_{K^*} A^*$ and the following diagram is commutative if and only if the following diagram is commutative.

$$\begin{tikzcd}
\Sigma^n K^* \arrow[r, "\hat{\varsigma}"] \arrow[d, "\varphi_x"] & \Sigma^n K^* \widehat{\otimes}_{K^*} A^* \arrow[d, "\varphi_x \widehat{\otimes}_{K^*} id_{A^*}"] \\
M^* \arrow[r, "\hat{\xi}"] & M^* \widehat{\otimes}_{K^*} A^*
\end{tikzcd}$$

Hence (id_{K^*}, φ_x) is a morphism of representations of A^* if and only if x satisfies $\hat{\xi}(x) = \hat{\imath}_{M^*}(x)$ by (12.1.5). $\square$

We define a subset $P_n(\boldsymbol{M}, \boldsymbol{\xi})$ of M^n by $P_n(\boldsymbol{M}, \boldsymbol{\xi}) = \{x \in M^n \,|\, \hat{\xi}(x) = \hat{\imath}_{M^*}(x)\}$. It follows from (12.1.7) that we have a bijection between $P_n(\boldsymbol{M}, \boldsymbol{\xi})$ and the set of all n-dimensional primitive elements of $(\boldsymbol{M}, \boldsymbol{\xi})$.

Proposition 12.1.8 *Let $f : A^* \to B^*$ be a morphism of topological Hopf algebras and a representation $(\boldsymbol{M}, \boldsymbol{\xi})$ of A^* on $\boldsymbol{M} = (K^*, M^*, \alpha)$.*

(1) We put $\boldsymbol{\xi} = (id_{A^*}, \xi)$ and define a map $\xi_f : M^* \widehat{\otimes}_{K^*} B^* \to M^* \widehat{\otimes}_{K^*} B^*$ to be the unique map that makes the following diagram commute.

$$\begin{tikzcd}
M^* \widehat{\otimes}_{K^*} B^* \arrow[r, "\xi_f"] \arrow[d, "\hat{c}_{u_{A^*}, f, M^*}"] & M^* \widehat{\otimes}_{K^*} B^* \arrow[d, "\hat{c}_{u_{A^*}, f, M^*}"] \\
(M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} B^* \arrow[r, "\xi \widehat{\otimes}_{K^*} id_{B^*}"] & (M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} B^*
\end{tikzcd}$$

Then, we have $\boldsymbol{\xi}_f = (id_{B^*}, \xi_f)$.

(2) We put $P_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{K^*}, \hat{\xi})$ and $P_{B^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}_f) = (id_{K^*}, \hat{\xi}_f)$. Then, $\hat{\xi}_f$ is the following composition.

$$M^* \xrightarrow{\hat{\xi}} M^* \widehat{\otimes}_{K^*} A^* \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} f} M^* \widehat{\otimes}_{K^*} B^*$$

Proof. (1) The assertion follows from (10.1.8) and (11.1.4).

(2) It follows from (11.2.2) and (5) of (10.1.9) that we have the following equalities in $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K_*}$.

$$P_{B^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}_f) = (f \times \boldsymbol{M})\hat{\boldsymbol{\xi}} = (id_{K^*}, id_{M^*} \widehat{\otimes}_{K^*} f)(id_{K^*}, \hat{\xi}) = (id_{K^*}, (id_{M^*} \widehat{\otimes}_{K^*} f)\hat{\xi})$$

Hence the assertion follows. $\square$

Proposition 12.1.9 *Let $f : A^* \to B^*$ be a morphism of Hopf algebras and $(\boldsymbol{M}, \boldsymbol{\xi})$, $(\boldsymbol{M}, \boldsymbol{\zeta})$ representations of A^*, B^*, respectively. Put $\boldsymbol{M} = (K^*, M^*, \alpha)$ and $\boldsymbol{\xi} = (id_{K^*}, \xi)$, $\boldsymbol{\zeta} = (id_{K^*}, \zeta)$. If the following diagram is commutative, $\boldsymbol{\zeta} = \boldsymbol{\xi}_f$ holds.*

$$\begin{tikzcd}
M^* \widehat{\otimes}_{K^*} A^* \arrow[r, "\xi"] \arrow[d, "id_{M^*} \widehat{\otimes}_{K^*} f"] & M^* \widehat{\otimes}_{K^*} A^* \arrow[d, "id_{M^*} \widehat{\otimes}_{K^*} f"] \\
M^* \widehat{\otimes}_{K^*} B^* \arrow[r, "\zeta"] & M^* \widehat{\otimes}_{K^*} B^*
\end{tikzcd}$$

Proof. The upper rectangle of the following diagram is commutative by the definition of ξ_f and the upper middle one is commutative by the assumption. Other rectangles and the semicircles on the both sides are commutative.

$$\begin{tikzcd}
M^* \widehat{\otimes}_{K^*} B^* \arrow[r, "\xi_f"] \arrow[d, "\hat{c}_{u_{A^*}, f, M^*}"] \arrow[ddd, bend right=60, "\hat{c}_{u_{B^*}, id_{B^*}, M^*}"'] & M^* \widehat{\otimes}_{K^*} B^* \arrow[d, "\hat{c}_{u_{A^*}, f, M^*}"'] \arrow[ddd, bend left=60, "\hat{c}_{u_{B^*}, id_{B^*}, M^*}"] \\
(M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} B^* \arrow[r, "\xi \widehat{\otimes}_{K^*} id_{B^*}"] \arrow[d, "(id_{M^*} \widehat{\otimes}_{K^*} f) \widehat{\otimes}_{A^*} id_{B^*}"] & (M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{A^*} B^* \arrow[d, "(id_{M^*} \widehat{\otimes}_{K^*} f) \widehat{\otimes}_{A^*} id_{B^*}"'] \\
(M^* \widehat{\otimes}_{K^*} B^*) \widehat{\otimes}_{A^*} B^* \arrow[r, "\zeta \widehat{\otimes}_{A^*} id_{B^*}"] \arrow[d, "\widehat{\otimes}_f"] & (M^* \widehat{\otimes}_{K^*} B^*) \widehat{\otimes}_{A^*} B^* \arrow[d, "\widehat{\otimes}_f"] \\
(M^* \widehat{\otimes}_{K^*} B^*) \widehat{\otimes}_{B^*} B^* \arrow[r, "\zeta \widehat{\otimes}_{B^*} id_{B^*}"] \arrow[d, "\hat{c}^{-1}_{u_{B^*}, id_{B^*}, M^*}"] & (M^* \widehat{\otimes}_{K^*} B^*) \widehat{\otimes}_{B^*} B^* \arrow[d, "\hat{c}^{-1}_{u_{B^*}, id_{B^*}, M^*}"] \\
M^* \widehat{\otimes}_{K^*} B^* \arrow[r, "\zeta"] & M^* \widehat{\otimes}_{K^*} B^*
\end{tikzcd}$$

Hence the composition of the vertical maps is the identity map of $M^* \widehat{\otimes}_{K^*} B^*$ and the assertion follows. $\square$

Definition 12.1.10 *For a topological Hopf algebra A^*, a left A^*-comodule algebra is a pair (R^*, γ) of object R^* of $\mathcal{C}$ and a morphism $\gamma : R^* \to A^* \widehat{\otimes}_{K^*} R^*$ of $\mathcal{C}$ which makes the following diagrams commute.*

$$\begin{tikzcd}
R^* \arrow[r, "\gamma"] \arrow[d, "\gamma"] & A^* \widehat{\otimes}_{K^*} R^* \arrow[d, "\mu \widehat{\otimes}_{K^*} id_{R^*}"] \\
A^* \widehat{\otimes}_{K^*} R^* \arrow[r, "id_{A^*} \widehat{\otimes}_{K^*} \gamma"] & A^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^*
\end{tikzcd}
\qquad
\begin{tikzcd}
R^* \arrow[r, "\gamma"] \arrow[dr, "j_2"] & A^* \widehat{\otimes}_{K^*} R^* \arrow[d, "\varepsilon \widehat{\otimes}_{K^*} id_{R^*}"] \\
& K^* \widehat{\otimes}_{K^*} R^*
\end{tikzcd}$$

Here, $j_2 : R^* \to K^* \otimes_{K^*} R^* = K^* \widehat{\otimes}_{K^*} R^*$ is a map defined by $j_2(r) = 1 \otimes r$. Similarly, a right A^*-comodule algebra is a pair (R^*, γ) of object R^* of $\mathcal{C}$ and a morphism $\gamma : R^* \to R^* \widehat{\otimes}_{K^*} A^*$ of $\mathcal{C}$ which makes the following diagrams commute.

$$\begin{tikzcd}
R^* \arrow[r, "\gamma"] \arrow[d, "\gamma"] & R^* \widehat{\otimes}_{K^*} A^* \arrow[d, "id_{R^*} \widehat{\otimes}_{K^*} \mu"] \\
R^* \widehat{\otimes}_{K^*} A^* \arrow[r, "\gamma \widehat{\otimes}_{K^*} id_{A^*}"] & R^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} A^*
\end{tikzcd}
\qquad
\begin{tikzcd}
R^* \arrow[r, "\gamma"] \arrow[dr, "j_1"] & R^* \widehat{\otimes}_{K^*} A^* \arrow[d, "id_{R^*} \widehat{\otimes}_{K^*} \varepsilon"] \\
& R^* \widehat{\otimes}_{K^*} K^*
\end{tikzcd}$$

Here, $j_1 : R^* \to R^* \otimes_{K^*} K^* = R^* \widehat{\otimes}_{K^*} K^*$ is a map defined by $j_1(r) = r \otimes 1$.

Proposition 12.1.11 *Let $(A^*, \mu, \varepsilon, \iota)$ be a topological Hopf algebra and (R^*, γ) a right A^*-comodule. Define a map $\tilde{\gamma} : R^* \to A^* \widehat{\otimes}_{K^*} R^*$ to be the following composition.*

$$R^* \xrightarrow{\gamma} R^* \widehat{\otimes}_{K^*} A^* \xrightarrow{\widehat{T}_{R^*,A^*}} A^* \widehat{\otimes}_{K^*} R^* \xrightarrow{\iota \widehat{\otimes}_{K^*} id_{R^*}} A^* \widehat{\otimes}_{K^*} R^*$$

Then, $(R^*, \tilde{\gamma})$ is a left A^*-comodule. If (R^*, γ) a right A^*-comodule algebra, $(R^*, \tilde{\gamma})$ is a left A^*-comodule algebra.

Proof. The following diagrams commute by the assumption.

$$\begin{tikzcd}
R^* \arrow[r, "\gamma"] \arrow[dr, "j_1"] & R^* \widehat{\otimes}_{K^*} A^* \arrow[r, "\widehat{T}_2"] \arrow[d, "id_{R^*} \widehat{\otimes}_{K^*} \varepsilon"] & A^* \widehat{\otimes}_{K^*} R^* \arrow[d, "\varepsilon \widehat{\otimes}_{K^*} id_{R^*}"] \\
& R^* \widehat{\otimes}_{K^*} K^* \arrow[r, "\widehat{T}_{R^*,K^*}"] & K^* \widehat{\otimes}_{K^*} R^*
\end{tikzcd}$$

$$\begin{tikzcd}
R^* \arrow[r, "\gamma"] \arrow[d, "\gamma"] & R^* \widehat{\otimes}_{K^*} A^* \arrow[r, "\widehat{T}_2"] \arrow[d, "id_{R^*} \widehat{\otimes}_{K^*} \mu"] & A^* \widehat{\otimes}_{K^*} R^* \arrow[r, "\iota \widehat{\otimes}_{K^*} id_{R^*}"] \arrow[d, "\mu \widehat{\otimes}_{K^*} id_{R^*}"] & A^* \widehat{\otimes}_{K^*} R^* \arrow[d, "\mu \widehat{\otimes}_{K^*} id_{R^*}"] \\
R^* \widehat{\otimes}_{K^*} A^* \arrow[r, "\gamma \widehat{\otimes}_{K^*} id_{A^*}"] \arrow[d, "\widehat{T}_2"] & R^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} A^* \arrow[r, "\widehat{T}_{R^*, A^* \widehat{\otimes}_{K^*} A^*}"] \arrow[d, "\widehat{T}_{R^* \widehat{\otimes}_{K^*} A^*, A^*}"] & A^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^* \arrow[r, "\widehat{T}_1(\iota \widehat{\otimes}_{K^*} \iota) \widehat{\otimes}_{K^*} id_{R^*}"] \arrow[d, "\widehat{T}_1 \widehat{\otimes}_{K^*} id_{R^*}"] & A^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^* \\
A^* \widehat{\otimes}_{K^*} R^* \arrow[r, "id_{A^*} \widehat{\otimes}_{K^*} \gamma"] \arrow[d, "\iota \widehat{\otimes}_{K^*} id_{R^*}"] & A^* \widehat{\otimes}_{K^*} R^* \widehat{\otimes}_{K^*} A^* \arrow[r, "id_{A^*} \widehat{\otimes}_{K^*} \widehat{T}_2"] \arrow[d, "\iota \widehat{\otimes}_{K^*} id_{R^*} \widehat{\otimes}_{K^*} id_{A^*}"] & A^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^* \arrow[d, "\iota \widehat{\otimes}_{K^*} id_{A^*} \widehat{\otimes}_{K^*} id_{R^*}"] & \\
A^* \widehat{\otimes}_{K^*} R^* \arrow[r, "id_{A^*} \widehat{\otimes}_{K^*} \gamma"] & A^* \widehat{\otimes}_{K^*} R^* \widehat{\otimes}_{K^*} A^* \arrow[r, "id_{A^*} \widehat{\otimes}_{K^*} \widehat{T}_2"] & A^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^* \arrow[uur, bend right, "id_{A^*} \widehat{\otimes}_{K^*} \iota \widehat{\otimes}_{K^*} id_{R^*}"'] &
\end{tikzcd}$$

Here we put $\widehat{T}_1 = \widehat{T}_{A^*,A^*}$ and $\widehat{T}_2 = \widehat{T}_{R^*,A^*}$. Since $j_2 = \widehat{T}_{R^*,K^*} j_1$, the assertion follows. $\square$

Definition 12.1.12 *We call $(R^*, \tilde{\gamma})$ the left A^*-comodule associated with (R^*, γ). If (R^*, γ) a right A^*-comodule algebra, we call $(R^*, \tilde{\gamma})$ a left A^*-comodule algebra associated with (R^*, γ).*

Let (R^*, γ) be a left A^*-comodule algebra. For an object $\boldsymbol{M} = (K^*, M^*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$, we define $\boldsymbol{\gamma}_l(\boldsymbol{M}) : R^* \times \boldsymbol{M} \to A^* \times (R^* \times \boldsymbol{M})$ to be the following composition.

$$R^* \times \boldsymbol{M} \xrightarrow{\gamma \times \boldsymbol{M}} (A^* \widehat{\otimes}_{K^*} R^*) \times \boldsymbol{M} \xrightarrow{\theta_{A^*,R^*}(\boldsymbol{M})^{-1}} A^* \times (R^* \times \boldsymbol{M})$$

Proposition 12.1.13 *If $\boldsymbol{M} = (K^*, M^*, \alpha)$, we define a map $\hat{\gamma}_{\boldsymbol{M}} : M^* \widehat{\otimes}_{K^*} R^* \to (M^* \widehat{\otimes}_{K^*} R^*) \widehat{\otimes}_{K^*} A^*$ to be the following composition.*

$$M^* \widehat{\otimes}_{K^*} R^* \xrightarrow{id_{M^*} \otimes_{K^*} \gamma} M^* \widehat{\otimes}_{K^*} (A^* \widehat{\otimes}_{K^*} R^*) \xrightarrow{\tilde{\theta}_{A^*,R^*}(\boldsymbol{M})^{-1}} (M^* \widehat{\otimes}_{K^*} R^*) \widehat{\otimes}_{K^*} A^*$$

Then, we have $\hat{\boldsymbol{\gamma}}_l(\boldsymbol{M}) = (id_{K^*}, \hat{\gamma}_{\boldsymbol{M}})$.

Proof. The assertion is a direct consequence of (10.1.9) and (10.1.11). $\square$

(10.1.12) and (11.2.9) implies the following result.

Proposition 12.1.14 *Assume that K^* is a field. Let $(\boldsymbol{M},\boldsymbol{\xi})$ and $(\boldsymbol{M},\boldsymbol{\zeta})$ be representations of A^* on $\boldsymbol{M} = (K^*, M^*, \alpha) \in \mathrm{Ob}\,\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$. We put $P_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{K^*}, \hat{\xi})$ and $P_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\zeta}) = (id_{K^*}, \hat{\zeta})$.*

(1) Let $\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}} : M^*_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} \to M^*$ be the kernel of $\hat{\xi} - \hat{\zeta} : M^* \to M^* \widehat{\otimes}_{K^*} A^*$. We denote by $\bar{\alpha}$ the right K^*-module structure of $M^*_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}$ as a submodule of M^*. There exists unique homomorphism $\hat{\lambda} : M^*_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} \to M^*_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} \widehat{\otimes}_{K^*} A^*$ of right A^*-modules that makes the following diagram commute.

$$\begin{CD} M^* @<{\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}}}<< M^*_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} @>{\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}}}>> M^* \\ @V{\hat{\xi}}VV @V{\hat{\lambda}}VV @V{\hat{\zeta}}VV \\ M^* \widehat{\otimes}_{K^*} A^* @<{\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}} \otimes_{K^*} id_{A^*}}<< M^*_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} \widehat{\otimes}_{K^*} A^* @>{\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}} \otimes_{K^*} id_{A^*}}>> M^* \widehat{\otimes}_{K^*} A^* \end{CD}$$

(2) We put $\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} = (K^*, M^*_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \bar{\alpha})$, $\hat{\boldsymbol{\lambda}} = (id_{K^*}, \hat{\lambda}) : \boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} \to A^* \times \boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}$ and $\boldsymbol{\lambda} = P_{A^*}(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})})^{-1}_{\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}}(\hat{\boldsymbol{\lambda}}) : u^*_{A^*}(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}) \to u^*_{A^*}(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})})$. Then, $(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \boldsymbol{\lambda})$ is a representation of A^* and a morphism $\boldsymbol{\kappa}_{\boldsymbol{\xi},\boldsymbol{\zeta}} = (id_{K^*}, \kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}}) : \boldsymbol{M} \to \boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{K^*}$ defines morphisms of representations $(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \boldsymbol{\lambda}) \to (\boldsymbol{M},\boldsymbol{\xi})$ and $(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \boldsymbol{\lambda}) \to (\boldsymbol{M},\boldsymbol{\zeta})$.

(3) Let $(\boldsymbol{N},\boldsymbol{\nu})$ be a representation of A^*. Suppose that a morphism $\boldsymbol{\varphi} : \boldsymbol{M} \to \boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{A^*}$ gives morphisms $(\boldsymbol{M},\boldsymbol{\xi}) \to (\boldsymbol{N},\boldsymbol{\nu})$ and $(\boldsymbol{M},\boldsymbol{\zeta}) \to (\boldsymbol{N},\boldsymbol{\nu})$ of representations of A^*. Then, there exists unique morphism $\tilde{\boldsymbol{\varphi}} : (\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \lambda) \to (\boldsymbol{N},\boldsymbol{\nu})$ of representations of A^* that satisfies $\tilde{\boldsymbol{\varphi}}\boldsymbol{\kappa}_{\boldsymbol{\xi},\boldsymbol{\zeta}} = \boldsymbol{\varphi}$.

Remark 12.1.15 *Recall that $\hat{\imath}_{M^*} : M^* \to M^* \widehat{\otimes}_{K^*} A^*$ is the composition of a map $M^* \to M^* \otimes_{K^*} A^*$ given by $x \mapsto x \otimes 1$ and $\eta_{M^* \otimes_{K^*} A^*} : M^* \otimes_{K^*} A^* \to M^* \widehat{\otimes}_{K^*} A^*$. If $\boldsymbol{\zeta} = id_{u^*_{A^*}(\boldsymbol{M})}$, namely $(\boldsymbol{M},\boldsymbol{\zeta})$ is the trivial representation of A^* on $\boldsymbol{M}$, $\hat{\zeta} : M^* \to M^* \widehat{\otimes}_{K^*} A^*$ coincides with $\hat{\imath}_{M^*}$. Hence if we put $P(\boldsymbol{M},\boldsymbol{\xi}) = \mathrm{Ker}(\hat{\xi} - \hat{\imath}_{M^*})$, then we have $P(\boldsymbol{M},\boldsymbol{\xi}) = \sum\limits_{n \in \boldsymbol{Z}} P_n(\boldsymbol{M},\boldsymbol{\xi})$ and the obit $\boldsymbol{M}/\boldsymbol{\xi}$ of $(\boldsymbol{M},\boldsymbol{\xi})$ is a K^*-module $P(\boldsymbol{M},\boldsymbol{\xi})$ of primitive elements of M^*.*

Let (R^*,γ) be a right A^*-comodule algebra and $(\boldsymbol{M},\boldsymbol{\xi})$ a representations of A^* on $\boldsymbol{M} = (K^*, M^*, \alpha)$. We put $P_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = \hat{\boldsymbol{\xi}}$ and denote by $P^{(R^*,\gamma)}_{(\boldsymbol{M},\boldsymbol{\xi})} : (R^*,\gamma) \times (\boldsymbol{M},\boldsymbol{\xi}) \to R^* \times \boldsymbol{M}$ an equalizer of $\gamma \times \boldsymbol{M} : R^* \times \boldsymbol{M} \to (R^* \widehat{\otimes}_{K^*} A^*) \times \boldsymbol{M}$ and a composition

$$R^* \times \boldsymbol{M} \xrightarrow{R^* \times \hat{\boldsymbol{\xi}}} R^* \times (A^* \times \boldsymbol{M}) \xrightarrow{\theta_{R^*,A^*}(\boldsymbol{M})} (R^* \widehat{\otimes}_{K^*} A^*) \times \boldsymbol{M}.$$

The following result is a direct consequence of (10.1.9).

Proposition 12.1.16 *We put $P_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{K^*}, \hat{\xi})$. Let $\widetilde{P}^{(R^*,\gamma)}_{(M^*,\hat{\xi})} : (M^*,\hat{\xi}) \,\square_{A^*} (R^*,\gamma) \to M^* \widehat{\otimes}_{K^*} R^*$ be an equalizer of $id_{M^*} \widehat{\otimes}_{K^*} \gamma : M^* \widehat{\otimes}_{K^*} R^* \to M^* \widehat{\otimes}_{K^*} (R^* \widehat{\otimes}_{K^*} A^*)$ and the following composition.*

$$M^* \widehat{\otimes}_{K^*} R^* \xrightarrow{\hat{\xi} \widehat{\otimes}_{K^*} id_{R^*}} (M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{K^*} R^* \xrightarrow{\tilde{\theta}_{R^*,A^*}(\boldsymbol{M})} M^* \widehat{\otimes}_{K^*} (R^* \widehat{\otimes}_{K^*} A^*)$$

Then, we have $(R^*,\gamma) \times (\boldsymbol{M},\boldsymbol{\xi}) = (K^*, (M^*,\hat{\xi}) \,\square_{A^*} (R^*,\gamma), \tilde{\alpha})$ and $P^{(R^*,\gamma)}_{(\boldsymbol{M},\boldsymbol{\xi})} = \left(id_{K^*}, \widetilde{P}^{(R^*,\gamma)}_{(M^*,\hat{\xi})}\right)$, where $\tilde{\alpha}$ is the K^*-module structure of $(M^*,\hat{\xi}) \,\square_{A^*} (R^*,\gamma)$ as a submodule of $M^* \widehat{\otimes}_{K^*} R^*$.

Remark 12.1.17 *Define a map $\gamma' : R^* \to A^* \widehat{\otimes}_{K^*} R^*$ to be the following composition.*

$$R^* \xrightarrow{\gamma} R^* \widehat{\otimes}_{K^*} A^* \xrightarrow{\widehat{T}_{R^*,A^*}} A^* \widehat{\otimes}_{K^*} R^*$$

We note that $\tilde{\theta}_{R^*,A^*}(\boldsymbol{M})$ is an isomorphism and the following diagram is commutative by the definition of $\tilde{\theta}_{R^*,A^*}(\boldsymbol{M})$.

$$\begin{CD} M^* \widehat{\otimes}_{K^*} R^* @>{id_{M^*} \widehat{\otimes}_{K^*} \gamma}>> M^* \widehat{\otimes}_{K^*} (R^* \widehat{\otimes}_{K^*} A^*) \\ @V{id_{M^*} \widehat{\otimes}_{K^*} \gamma'}VV @VV{\tilde{\theta}_{R^*,A^*}(\boldsymbol{M})^{-1}}V \\ M^* \widehat{\otimes}_{K^*} (A^* \widehat{\otimes}_{K^*} R^*) @>{\cong}>> (M^* \widehat{\otimes}_{K^*} A^*) \widehat{\otimes}_{K^*} R^* \end{CD}$$

It follows that $\widetilde{P}^{(R^*,\gamma)}_{(M^*,\hat{\xi})} : (M^*,\hat{\xi})\,\square_{A^*}(R^*,\gamma) \to M^*\,\widehat{\otimes}_{K^*}\,R^*$ *is an equalizer of*

$$M^*\,\widehat{\otimes}_{K^*}\,R^* \xrightarrow{id_{M^*}\,\widehat{\otimes}_{K^*}\,\gamma'} M^*\,\widehat{\otimes}_{K^*}(R^*\,\widehat{\otimes}_{K^*}\,A^*) \cong (M^*\,\widehat{\otimes}_{K^*}\,A^*)\,\widehat{\otimes}_{K^*}\,R^*$$

and $\hat{\xi}\,\widehat{\otimes}_{K^*}\,id_{R^*} : M^*\,\widehat{\otimes}_{K^*}\,R^* \to (M^*\,\widehat{\otimes}_{K^*}\,A^*)\,\widehat{\otimes}_{K^*}\,R^*$.

We also have the following result by (11.2.14).

Proposition 12.1.18 *Let* (R^*,γ), (S^*,δ) *be right* A^**-comodule algebras and* $f : S^* \to R^*$ *a morphism of right* A^**-comodule algebras. Let* $(M^*,\hat{\xi})$, $(N^*,\hat{\zeta})$ *be right* A^**-comodules and* $\varphi : M^* \to N^*$ *a morphism of right* A^**-comodules. We put* $\boldsymbol{M} = (K^*,M^*,\alpha)$, $\boldsymbol{\xi} = P_{A^*}(\boldsymbol{M})^{-1}_{\boldsymbol{M}}((id_{K^*},\hat{\xi}))$, $\boldsymbol{N} = (K^*,N^*,\beta)$, $\boldsymbol{\zeta} = P_{A^*}(\boldsymbol{N})^{-1}_{\boldsymbol{N}}((id_{K^*},\hat{\zeta}))$ *and* $\boldsymbol{\varphi} = (id_{K^*},\varphi)$*. There exist unique maps* $f_{(M^*,\hat{\xi})} : (M^*,\hat{\xi})\,\square_{A^*}(S^*,\delta) \to (M^*,\hat{\xi})\,\square_{A^*}(R^*,\gamma)$ *and* $\varphi_{(R^*,\gamma)} : (M^*,\hat{\xi})\,\square_{A^*}(R^*,\gamma) \to (N^*,\hat{\zeta})\,\square_{A^*}(S^*,\delta)$ *that make the following diagrams commute.*

$$\begin{tikzcd}
(M^*,\hat{\xi})\,\square_{A^*}(S^*,\delta) \arrow[r, "\widetilde{P}^{(S^*,\delta)}_{(M^*,\hat{\xi})}"] \arrow[d, "f_{(M^*,\hat{\xi})}"] & M^*\,\widehat{\otimes}_{K^*}\,S^* \arrow[d, "id_{M^*}\,\widehat{\otimes}_{K^*}\,f"] \\
(M^*,\hat{\xi})\,\square_{A^*}(R^*,\gamma) \arrow[r, "\widetilde{P}^{(R^*,\gamma)}_{(N^*,\hat{\zeta})}"] & M^*\,\widehat{\otimes}_{K^*}\,R^*
\end{tikzcd}
\qquad
\begin{tikzcd}
(M^*,\hat{\xi})\,\square_{A^*}(R^*,\gamma) \arrow[r, "\widetilde{P}^{(R^*,\gamma)}_{(M^*,\hat{\xi})}"] \arrow[d, "\varphi_{(R^*,\gamma)}"] & M^*\,\widehat{\otimes}_{K^*}\,R^* \arrow[d, "\varphi\,\widehat{\otimes}_{K^*}\,id_{R^*}"] \\
(N^*,\hat{\zeta})\,\square_{A^*}(S^*,\delta) \arrow[r, "\widetilde{P}^{(R^*,\gamma)}_{(N^*,\hat{\zeta})}"] & N^*\,\widehat{\otimes}_{K^*}\,R^*
\end{tikzcd}$$

12.2 Representations of topological Hopf algebras over a field

For the rest of this subsection, we assume that K^* is a field such that $K^i = \{0\}$ for $i \neq 0$ and that every object of $\mathcal{M}$ is profinite. When we consider $\boldsymbol{N}^{A^*}$ for an object A^* of $\mathcal{C}$ and an object $\boldsymbol{N} = (K^*,N^*,\beta)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$, we assume that A^* and N^* satisfy the conditions (i) and (ii) of (10.1.13), respectively.

Let $(A^*,\mu,\varepsilon,\iota)$ be a topological Hopf algebra in $\mathcal{C}$. For a morphism $\boldsymbol{\zeta} : u^*_{A^*}(\boldsymbol{N}) \to u^*_{A^*}(\boldsymbol{N})$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{K^*}$, we put $\check{\boldsymbol{\zeta}} = E_{A^*}(\boldsymbol{N})_{\boldsymbol{N}}(\boldsymbol{\zeta}) \in \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{K^*}(\boldsymbol{N},\boldsymbol{N}^{A^*})$. If we put $\boldsymbol{\zeta} = (id_{A^*},\zeta)$, ζ is a right A^*-module homomorphism $N^*\,\widehat{\otimes}_{K^*}\,A^* \to N^*\,\widehat{\otimes}_{K^*}\,A^*$. We note that $A^{**}\otimes_{K^*}N^*$ is complete by (2.3.2). It follows from (10.1.14) that we have $\boldsymbol{N}^{A^*} = (K^*,A^{**}\otimes_{K^*}N^*,\beta^{A^*})$ and that if we put $\check{\boldsymbol{\zeta}} = (id_{K^*},\check{\zeta})$, $\check{\zeta} : A^{**}\otimes_{K^*}N^* \to N^*$ is the following composition.

$$\begin{aligned}
A^{**}\otimes_{K^*}N^* &\xrightarrow{T_{A^{**},N^*}} N^*\otimes_{K^*}A^{**} \xrightarrow{\hat{\imath}_{N^*,u_{A^*}}\otimes_{K^*}id_{A^{**}}} (N^*\,\widehat{\otimes}_{K^*}\,A^*)\otimes_{K^*}A^{**} \xrightarrow{\zeta\otimes_{K^*}id_{A^{**}}} (N^*\,\widehat{\otimes}_{K^*}\,A^*)\otimes_{K^*}A^{**} \\
&\xrightarrow{\widehat{T}_{N^*,A^*}\otimes_{K^*}id_{A^{**}}} (A^*\,\widehat{\otimes}_{K^*}\,N^*)\otimes_{K^*}A^{**} \xrightarrow{(\chi_{A^*,K^*}\,\widehat{\otimes}_{K^*}\,id_{N^*})\otimes_{K^*}id_{A^{**}}} \mathcal{H}om^*(A^{**},K^*)\,\widehat{\otimes}_{K^*}\,N^*\otimes_{K^*}A^{**} \\
&\xrightarrow{\hat{\varphi}^{A^{**}}_{N^*}\otimes_{K^*}id_{A^{**}}} \mathcal{H}om^*(A^{**},N^*)\otimes_{K^*}A^{**} \xrightarrow{ev^{A^{**}}_{N^*}} N^*
\end{aligned}$$

We recall an isomorphism $\kappa_{N^*} : N^* \to \mathcal{H}om^*(K^*,N^*)$ given in (3.1.24). The following result follows from (11.3.1) and (10.1.15).

Proposition 12.2.1 $\boldsymbol{\zeta}$ *defines a representation of* A^* *on* $\boldsymbol{N}$ *if and only if a composition*

$$N^* \xrightarrow{i_{N^*}} K^*\otimes_{K^*}N^* \xrightarrow{\kappa_{K^*}\otimes_{K^*}id_{N^*}} K^{**}\otimes_{K^*}N^* \xrightarrow{\varepsilon^*\otimes_{K^*}id_{N^*}} A^{**}\otimes_{K^*}N^* \xrightarrow{\check{\zeta}} N^*$$

is the identity morphism of N^* *and the following diagram commute.*

$$\begin{tikzcd}
\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)\otimes_{K^*}N^* \arrow[r, "\mu^*\otimes_{K^*}id_{N^*}"] \arrow[d, "\tilde{\theta}^{A^*,A^*}(\boldsymbol{N})"] & A^{**}\otimes_{K^*}N^* \arrow[r, "\check{\zeta}"] & N^* \\
A^{**}\otimes_{K^*}(A^{**}\otimes_{K^*}N^*) \arrow[r, "id_{A^{**}}\otimes_{K^*}\check{\zeta}"] & A^{**}\otimes_{K^*}N^* \arrow[ur, "\check{\zeta}"'] &
\end{tikzcd}$$

Remark 12.2.2 *We denote by* $u_{A^{**}} : K^* \to A^{**}$ *a compsition* $K^* \xrightarrow{\kappa_{K^*}} K^{**} \xrightarrow{\varepsilon^*} A^{**}$ *and define a map* $\check{\mu} : A^{**}\otimes_{K^*}A^{**} \to A^{**}$ *to be the following composition.*

$$A^{**}\otimes_{K^*}A^{**} \xrightarrow{\phi} \mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*) \xrightarrow{\mu^*} A^{**}$$

Then, A^{**} *is a* K^**-algebra with product* $\check{\mu}$ *and unit* $u_{A^{**}}$*. (12.2.1) shows that* $\boldsymbol{\zeta}$ *defines a representation of* A^* *on* $\boldsymbol{N}$ *if and only if* $\check{\zeta}$ *is a left* A^{**}*-module structure of* N^**.*

Proposition 12.2.3 *Let $\boldsymbol{N} = (K^*, N^*, \beta)$ be an object of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$ and $\boldsymbol{\zeta} : u_{A^*}^*(\boldsymbol{N}) \to u_{A^*}^*(\boldsymbol{N})$ a morphism of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$. We put*

$$P_{A^*}(\boldsymbol{N})_{\boldsymbol{N}}(\boldsymbol{\zeta}) = (id_{K^*}, \hat{\zeta}) : \boldsymbol{N} \to A^* \times \boldsymbol{N} \quad \textit{and} \quad E_{A^*}(\boldsymbol{N})_{\boldsymbol{N}}(\boldsymbol{\zeta}) = (id_{K^*}, \check{\zeta}) : \boldsymbol{N}^{A^*} \to \boldsymbol{N}.$$

Then, $\check{\zeta} : A^{**} \otimes_{K^*} N^* \to N^*$ is mapped to $\hat{\zeta} : N^* \to N^* \mathbin{\widehat{\otimes}}_{K^*} A^*$ by the following composition.

$$\begin{aligned}\mathrm{Hom}^c_{K^*}(A^{**} \otimes_{K^*} N^*, N^*) &\xrightarrow{\Lambda_{A^{**},N^*,N^*}} \mathrm{Hom}^c_{K^*}(N^*, N^* \mathbin{\widehat{\otimes}}_{K^*} \mathcal{H}om^*(A^{**}, K^*)) \\ &\xrightarrow{(id_{N^*} \widehat{\otimes}_{K^*} \chi^{-1}_{A^*,K^*})_*} \mathrm{Hom}^c_{K^*}(N^*, N^* \mathbin{\widehat{\otimes}}_{K^*} A^*)\end{aligned}$$

Proof. Since $\hat{\zeta} = \zeta \hat{i}_{N^*,u_{A^*}} : N^* \to N^* \mathbin{\widehat{\otimes}}_{K^*} A^*$, $\check{\zeta}$ is the following composition.

$$\begin{aligned}A^{**} \otimes_{K^*} N^* &\xrightarrow{T_{A^{**},N^*}} N^* \otimes_{K^*} A^{**} \xrightarrow{\hat{\zeta} \otimes_{K^*} id_{A^{**}}} (N^* \mathbin{\widehat{\otimes}}_{K^*} A^*) \otimes_{K^*} A^{**} \xrightarrow{\widehat{T}_{N^*,A^*} \otimes_{K^*} id_{A^{**}}} (A^* \mathbin{\widehat{\otimes}}_{K^*} N^*) \otimes_{K^*} A^{**} \\ &\xrightarrow{(\chi_{A^*,K^*} \widehat{\otimes}_{K^*} id_{N^*}) \otimes_{K^*} id_{A^{**}}} (\mathcal{H}om^*(A^{**}, K^*) \mathbin{\widehat{\otimes}}_{K^*} N^*) \otimes_{K^*} A^{**} \\ &\xrightarrow{\hat{\varphi}^{A^{**}}_{N^*} \otimes_{K^*} id_{A^{**}}} \mathcal{H}om^*(A^{**}, N^*) \otimes_{K^*} A^{**} \xrightarrow{ev^{A^{**}}_{N^*}} N^*\end{aligned}$$

If we put $\psi = \hat{\varphi}^{A^{**}}_{N^*}(\chi_{A^*,K^*} \mathbin{\widehat{\otimes}}_{K^*} id_{N^*}) \widehat{T}_{N^*,A^*} \hat{\zeta} : N^* \to \mathcal{H}om^*(A^{**}, N^*)$,

$$\Phi_{N^*,A^{**},N^*} : \mathrm{Hom}^c_{K^*}\big(N^* \otimes_{K^*} A^{**}, N^*\big) \to \mathrm{Hom}^c_{K^*}\big(N^*, \mathcal{H}om^*(A^{**}, N^*)\big)$$

maps $ev^{A^{**}}_{N^*}(\psi \otimes_{K^*} id_{A^{**}}) : N^* \otimes_{K^*} A^{**} \to N^*$ to ψ. Hence Λ_{A^{**},N^*,N^*} maps $\check{\zeta}$ to

$$\widehat{T}_{\mathcal{H}om^*(A^{**},K^*),N^{**}}(\chi_{A^*,K^*} \mathbin{\widehat{\otimes}}_{K^*} id_{N^*}) \widehat{T}_{N^*,A^*} \hat{\zeta} = (id_{N^*} \mathbin{\widehat{\otimes}}_{K^*} \chi_{A^*,K^*}) \hat{\zeta}.$$

Thus the assertion follows. □

Remark 12.2.4 *The above result shows that $\hat{\zeta}$ is the Milnor coaction associated with $\check{\zeta}$.*

Proposition 12.2.5 *For an object $\boldsymbol{M} = (K^*, M^*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$, let $j_{M^*} : A^{**} \otimes_{K^*} M^* \to M^*$ be the following composition.*

$$A^{**} \otimes_{K^*} M^* \xrightarrow{u^*_{A^*} \otimes_{K^*} id_{M^*}} K^{**} \otimes_{K^*} M^* \xrightarrow{\kappa^{-1}_{K^*} \otimes_{K^*} id_{M^*}} K^* \otimes_{K^*} M^* \xrightarrow{\cong} M^*$$

Then we have $E_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(id_{\boldsymbol{M}}) = (id_{K^*}, j_{M^*})$.

Proof. Put $E_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(id_{\boldsymbol{M}}) = (id_{K^*}, \psi)$. It follows from (10.1.14) that $\psi : A^{**} \otimes_{K^*} M^* \to M^*$ is the following composition.

$$\begin{aligned}A^{**} \otimes_{K^*} M^* &\xrightarrow{T_{A^{**},M^*}} M^* \otimes_{K^*} A^{**} \xrightarrow{\hat{i}_{u_{A^*},M^*} \otimes_{K^*} id_{A^{**}}} (A^* \mathbin{\widehat{\otimes}}_{K^*} M^*) \otimes_{K^*} A^{**} \xrightarrow{(\chi_{A^*,K^*} \widehat{\otimes}_{K^*} id_{M^*}) \otimes_{K^*} id_{A^{**}}} \\ &(\mathcal{H}om^*(A^{**}, K^*) \mathbin{\widehat{\otimes}}_{K^*} M^*) \otimes_{K^*} A^{**} \xrightarrow{\hat{\varphi}^{A^{**}}_{M^*} \otimes_{K^*} id_{A^{**}}} \mathcal{H}om^*(A^{**}, M^*) \otimes_{K^*} A^{**} \xrightarrow{ev^{A^{**}}_{M^*}} M^*\end{aligned}$$

Since $\chi_{A^*,K^*}(1) : A^{**} = \mathcal{H}om^*(A^*, K^*) \to K^*$ maps f to $f([k], 1)$ for $f \in \mathcal{H}om^k(A^*, K^*)$,

$$\varphi^{A^{**}}_{M^*}(\chi_{A^*,K^*}(1) \otimes x) : \Sigma^m A^{**} = \mathcal{H}om^*(A^*, K^*) \to M^*$$

maps $([m], f)$ to $(-1)^{km}\chi_{A^*,K^*}(1)(f)x = (-1)^{km} f([k], 1)x$ for $x \in M^m$. By the commutativity of the following diagram, ψ maps $f \otimes x \in A^{**} \otimes_{K^*} M^*$ to $ev^{A^{**}}_{M^*}((-1)^{km}\varphi^{A^{**}}_{M^*}(\chi_{A^*,K^*}(1) \otimes x) \otimes f) = f([k], 1)x$.

$$\begin{tikzcd}
M^* \arrow[r, "i_{u_{A^*},M^*}"] \arrow[dr, "\hat{i}_{u_{A^*},M^*}"'] & A^* \otimes_{K^*} M^* \arrow[r, "\chi_{A^*,K^*} \otimes_{K^*} id_{M^*}"] \arrow[d, "\eta_{A^* \otimes_{K^*} M^*}"] & \mathcal{H}om^*(A^{**}, K^*) \otimes_{K^*} M^* \arrow[d, "\eta_{\mathcal{H}om^*(A^{**},K^*) \otimes_{K^*} M^*}"'] \arrow[dr, "\varphi^{A^{**}}_{M^*}"] & \\
& A^* \mathbin{\widehat{\otimes}}_{K^*} M^* \arrow[r, "\chi_{A^*,K^*} \widehat{\otimes}_{K^*} id_{M^*}"] & \mathcal{H}om^*(A^{**}, K^*) \mathbin{\widehat{\otimes}}_{K^*} M^* \arrow[r, "\hat{\varphi}^{A^{**}}_{M^*}"] & \mathcal{H}om^*(A^{**}, M^*)
\end{tikzcd}$$

Since $\kappa^{-1}_{K^*} : \mathcal{H}om^*(K^*, K^*) \to K^*$ maps $g \in \mathcal{H}om^k(K^*, K^*)$ to $g([k], 1)$, j_{A^*} maps $f \otimes x \in A^{**} \otimes_{K^*} M^*$ to $\kappa^{-1}_{K^*}(f \Sigma^k u_{A^*})x = (f \Sigma^k u_{A^*})([k], 1)x = f([k], 1)x$. Therefore we have $j_{A^*} = \psi$. □

Definition 12.2.6 *We call (M^*, j_{M^*}) the trivial left A^{**}-module.*

The following result follows from (11.3.3) and (10.1.15).

Proposition 12.2.7 *Let $(\boldsymbol{M}, \boldsymbol{\xi})$ and $(\boldsymbol{N}, \boldsymbol{\zeta})$ be representations of A^* and we put $E_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{K^*}, \check{\xi})$, $E_{A^*}(\boldsymbol{N})_{\boldsymbol{N}}(\boldsymbol{\zeta}) = (id_{K^*}, \check{\zeta})$. Suppose $\boldsymbol{M} = (K^*, M^*, \alpha)$, $\boldsymbol{N} = (K^*, N^*, \beta)$. A morphism $\boldsymbol{\varphi} = (id_{K^*}, \varphi) : \boldsymbol{M} \to \boldsymbol{N}$, of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{K^*}$ gives a morphism $(\boldsymbol{M}, \boldsymbol{\xi}) \to (\boldsymbol{N}, \boldsymbol{\zeta})$ of representations of A^* if and only if the following diagram commutative.*

$$\begin{array}{ccc}
A^{**} \otimes_{K^*} N^* & \xrightarrow{\check{\zeta}} & N^* \\
\Big\downarrow{\scriptstyle id_{A^{**}} \otimes_{K^*} \varphi} & & \Big\downarrow{\scriptstyle \varphi} \\
A^{**} \otimes_{K^*} M^* & \xrightarrow{\check{\xi}} & M^*
\end{array}$$

Let (R^*, γ) be a right A^*-comodule algebra. We note that if A^* and R^* satisfy the condition (i) of (10.1.13), then A^*, R^* and $R^* \otimes_{K^*} A^*$ are complete. For an object $\boldsymbol{N} = (K^*, N^*, \beta)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$, we define $\boldsymbol{\gamma}_r(\boldsymbol{N}) : \boldsymbol{N}^{R^*} \to (\boldsymbol{N}^{R^*})^{A^*}$ to be the following composition.

$$(\boldsymbol{N}^{R^*})^{A^*} \xrightarrow{\theta^{R^*, A^*}(\boldsymbol{N})^{-1}} \boldsymbol{N}^{R^* \otimes_{K^*} A^*} = \boldsymbol{N}^{R^* \widehat{\otimes}_{K^*} A^*} \xrightarrow{\boldsymbol{N}^{\gamma}} \boldsymbol{N}^{R^*}$$

The assertion is a direct consequence of (10.1.15) and (10.1.19).

Proposition 12.2.8 *If $\boldsymbol{N} = (K^*, M^*, \beta)$, we define a map $\check{\gamma}_{\boldsymbol{N}} : M^* \widehat{\otimes}_{K^*} R^* \to (M^* \widehat{\otimes}_{K^*} R^*) \widehat{\otimes}_{K^*} A^*$ to be the following composition.*

$$\begin{aligned}
A^{**} \otimes_{K^*} (R^{**} \otimes_{K^*} M^*) &\xrightarrow{\tilde{\theta}_{R^*, A^*}(\boldsymbol{N})^{-1}} \mathcal{H}om^*(R^* \otimes_{K^*} A^*, K^*) \otimes_{K^*} M^* = \mathcal{H}om^*(R^* \widehat{\otimes}_{K^*} A^*, K^*) \otimes_{K^*} M^* \\
&\xrightarrow{\gamma^* \otimes_{K^*} id_{M^*}} R^{**} \otimes_{K^*} M^*
\end{aligned}$$

Then, we have $\check{\gamma}_l(\boldsymbol{N}) = (id_{K^*}, \check{\gamma}_{\boldsymbol{N}})$.

Proposition 12.2.9 *Let $(\boldsymbol{M}, \boldsymbol{\xi})$ and $(\boldsymbol{M}, \boldsymbol{\zeta})$ be representations of A^* on $\boldsymbol{M} = (K^*, M^*, \alpha) \in \mathrm{Ob}\, \mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$. We put $E_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{K^*}, \check{\xi})$ and $E_{A^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\zeta}) = (id_{K^*}, \check{\zeta})$.*

(1) Let $\iota_{\boldsymbol{\xi}, \boldsymbol{\zeta}} : M^* \to M^{(\boldsymbol{\xi}:\boldsymbol{\zeta})*}$ be the cokernel of $\check{\xi} - \check{\zeta} : A^{**} \otimes_{K^*} M^* \to M^*$. We denote by $\bar{\alpha}$ the right K^*-module structure of $M^{(\boldsymbol{\xi}:\boldsymbol{\zeta})*}$ as a quotient module of M^*. There exists unique homomorphism $\check{\lambda} : A^{**} \otimes_{K^*} M^{(\boldsymbol{\xi}:\boldsymbol{\zeta})*} \to M^{(\boldsymbol{\xi}:\boldsymbol{\zeta})*}$ of right A^*-modules that makes the following diagram commute.

$$\begin{array}{ccccc}
A^{**} \otimes_{K^*} M^* & \xrightarrow{id_{A^{**}} \otimes_{K^*} \iota_{\boldsymbol{\xi}, \boldsymbol{\zeta}}} & A^{**} \otimes_{K^*} M^{(\boldsymbol{\xi}:\boldsymbol{\zeta})*} & \xleftarrow{id_{A^{**}} \otimes_{K^*} \iota_{\boldsymbol{\xi}, \boldsymbol{\zeta}}} & A^{**} \otimes_{K^*} M^* \\
\Big\downarrow{\scriptstyle \check{\xi}} & & \Big\downarrow{\scriptstyle \check{\lambda}} & & \Big\downarrow{\scriptstyle \check{\zeta}} \\
M^* & \xrightarrow{\iota_{\boldsymbol{\xi}, \boldsymbol{\zeta}}} & M^{(\boldsymbol{\xi}:\boldsymbol{\zeta})*} & \xleftarrow{\iota_{\boldsymbol{\xi}, \boldsymbol{\zeta}}} & M^*
\end{array}$$

(2) We put $\boldsymbol{M}^{(\boldsymbol{\xi}:\boldsymbol{\zeta})} = (K^*, M^{(\boldsymbol{\xi}:\boldsymbol{\zeta})*}, \bar{\alpha})$, $\check{\boldsymbol{\lambda}} = (id_{K^*}, \check{\lambda}) : (\boldsymbol{M}^{(\boldsymbol{\xi}:\boldsymbol{\zeta})})^{A^*} \to \boldsymbol{M}^{(\boldsymbol{\xi}:\boldsymbol{\zeta})}$ and $\boldsymbol{\lambda} = E_{A^*}(\boldsymbol{M}^{(\boldsymbol{\xi}:\boldsymbol{\zeta})})^{-1}_{\boldsymbol{M}^{(\boldsymbol{\xi}:\boldsymbol{\zeta})}}(\check{\boldsymbol{\lambda}}) : u^*_{A^*}(\boldsymbol{M}^{(\boldsymbol{\xi}:\boldsymbol{\zeta})}) \to u^*_{A^*}(\boldsymbol{M}^{(\boldsymbol{\xi}:\boldsymbol{\zeta})})$. Then, $(\boldsymbol{M}^{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \boldsymbol{\lambda})$ is a representation of A^* and a morphism $\boldsymbol{\iota}_{\boldsymbol{\xi}, \boldsymbol{\zeta}} = (id_{K^*}, \iota_{\boldsymbol{\xi}, \boldsymbol{\zeta}}) : \boldsymbol{M}^{(\boldsymbol{\xi}:\boldsymbol{\zeta})} \to \boldsymbol{M}$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{K^*}$ defines morphisms of representations $(\boldsymbol{M}^{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \boldsymbol{\lambda}) \to (\boldsymbol{M}, \boldsymbol{\xi})$ and $(\boldsymbol{M}^{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \boldsymbol{\lambda}) \to (\boldsymbol{M}, \boldsymbol{\zeta})$.

(3) Let $(\boldsymbol{N}, \boldsymbol{\nu})$ be a representation of A^*. Suppose that a morphism $\boldsymbol{\varphi} : \boldsymbol{M} \to \boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{A^*}$ gives morphisms $(\boldsymbol{M}, \boldsymbol{\xi}) \to (\boldsymbol{N}, \boldsymbol{\nu})$ and $(\boldsymbol{M}, \boldsymbol{\zeta}) \to (\boldsymbol{N}, \boldsymbol{\nu})$ of representations of A^*. Then, there exists unique morphism $\tilde{\boldsymbol{\varphi}} : (\boldsymbol{N}, \boldsymbol{\nu}) \to (\boldsymbol{M}^{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \lambda)$ of representations of A^* that satisfies $\boldsymbol{\iota}_{\boldsymbol{\xi}, \boldsymbol{\zeta}} \tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi}$.

Remark 12.2.10 *If $\boldsymbol{\zeta} = id_{u^*_{A^*}(\boldsymbol{M})}$, namely $(\boldsymbol{M}, \boldsymbol{\zeta})$ is the trivial representation of A^* on $\boldsymbol{M}$, we see in (12.2.5) that $\check{\zeta} : A^{**} \otimes_{K^*} M^* \to M^*$ coincides with j_{M^*}. Hence if we put $Q(\boldsymbol{M}, \boldsymbol{\xi}) = \mathrm{Coker}(\check{\xi} - j_{M^*})$ and denote by α_Q the right K^*-module structure of $Q(\boldsymbol{M}, \boldsymbol{\xi})$, then $\boldsymbol{M}^{\boldsymbol{\xi}} = (K^*, Q(\boldsymbol{M}, \boldsymbol{\xi}), \alpha_Q)$ is the A^*-fixed object of $(\boldsymbol{M}, \boldsymbol{\xi})$. We note that $Q(\boldsymbol{M}, \boldsymbol{\xi})$ is regarded as the module of "indecomposable elemems".*

Let (R^*, γ) be a left A^*-comodule algebra and $(\boldsymbol{N}, \boldsymbol{\zeta})$ a representations of A^* on $\boldsymbol{N} = (K^*, N^*, \beta)$. We assume that A^* and R^* satisfy the condition (i) of (10.1.13). Hence $A^* \widehat{\otimes}_{K^*} R^* = A^* \otimes_{K^*} R^*$ holds. We put $E_{A^*}(\boldsymbol{N})_{\boldsymbol{N}}(\zeta) = \check{\boldsymbol{\zeta}}$ and denote by $E^{(R^*,\gamma)}_{(\boldsymbol{N},\boldsymbol{\zeta})} : \boldsymbol{N}^{R^*} \to (\boldsymbol{N}, \boldsymbol{\zeta})^{(R^*,\gamma)}$ an coequalizer of $\boldsymbol{N}^\gamma : \boldsymbol{N}^{A^* \otimes_{K^*} R^*} \to \boldsymbol{N}^{R^*}$ and a composition

$$\boldsymbol{N}^{A^* \otimes_{K^*} R^*} \xrightarrow{\theta^{A^*, R^*}(\boldsymbol{N})} (\boldsymbol{N}^{A^*})^{R^*} \xrightarrow{\check{\boldsymbol{\zeta}}^{R^*}} \boldsymbol{N}^{R^*}.$$

The following result is a direct consequence of (10.1.15).

Proposition 12.2.11 *We put $E_{A^*}(\boldsymbol{N})_{\boldsymbol{N}}(\boldsymbol{\zeta}) = (id_{K^*}, \check{\zeta})$. Let $\widetilde{E}^{(R^*,\gamma)}_{(N^*,\check{\zeta})} : R^{**} \otimes_{K^*} N^* \to (R^*, \gamma) \otimes_{A^{**}} (N^*, \check{\zeta})$ be a coequalizer of $\gamma^* \otimes_{K^*} id_{N^*} : \mathcal{H}om^*(A^* \otimes_{K^*} R^*, K^*) \otimes_{K^*} N^* \to R^{**} \otimes_{K^*} N^*$ and the following composition.*

$$\mathcal{H}om^*(A^* \otimes_{K^*} R^*, K^*) \otimes_{K^*} N^* \xrightarrow{\tilde{\theta}^{A^*, R^*}(\boldsymbol{N})} R^{**} \otimes_{K^*} (A^{**} \otimes_{K^*} N^*) \xrightarrow{id_{R^{**}} \otimes_{K^*} \check{\zeta}} R^{**} \otimes_{K^*} N^*$$

Then, we have $(\boldsymbol{N}, \boldsymbol{\zeta})^{(R^*,\gamma)} = (K^*, (R^*, \gamma) \otimes_{A^{**}} (N^*, \check{\zeta}), \tilde{\beta})$ and $E^{(R^*,\gamma)}_{(\boldsymbol{N},\boldsymbol{\zeta})} = \left(id_{K^*}, \widetilde{E}^{(R^*,\gamma)}_{(N^*,\check{\zeta})}\right)$, where $\tilde{\beta}$ is the K^*-module structure of $(R^*, \gamma) \otimes_{A^{**}} (N^*, \check{\zeta})$ as a quotient module of $R^{**} \otimes_{K^*} N^*$.

Remark 12.2.12 *Define a map $\gamma'' : R^{**} \otimes_{K^*} A^{**} \to R^{**}$ to be the following composition.*

$$R^{**} \otimes_{K^*} A^{**} \xrightarrow{T_{R^{**}, A^{**}}} A^{**} \otimes_{K^*} R^{**} \xrightarrow{\phi} \mathcal{H}om^*(A^* \otimes_{K^*} R^*, K^*) \xrightarrow{\gamma^*} R^{**}$$

Then, γ'' is a right $(A^{**})^{op}$-module structure on R^{**}. We note that $\tilde{\theta}^{A^*, R^*}(\boldsymbol{N})$ is an isomorphism and the following diagram is commutative by the definition of $\tilde{\theta}^{A^*, R^*}(\boldsymbol{N})$.

$$\begin{tikzcd}
R^{**} \otimes_{K^*} (A^{**} \otimes_{K^*} N^*) \arrow[r, "\cong"] \arrow[d, "\tilde{\theta}^{A^*, R^*}(\boldsymbol{N})^{-1}"'] & (R^{**} \otimes_{K^*} A^{**}) \otimes_{K^*} N^* \arrow[d, "\gamma'' \otimes_{K^*} id_{N^*}"] \\
\mathcal{H}om^*(A^* \otimes_{K^*} R^*, K^*) \otimes_{K^*} N^* \arrow[r, "\gamma^* \otimes_{K^*} id_{N^*}"] & R^{**} \otimes_{K^*} N^*
\end{tikzcd}$$

It follows that $\widetilde{E}^{(R^*,\gamma)}_{(N^*,\check{\zeta})} : R^{**} \otimes_{K^*} N^* \to (R^*, \gamma) \otimes_{A^{**}} (N^*, \check{\zeta})$ is a coequalizer of

$$R^{**} \otimes_{K^*} (A^{**} \otimes_{K^*} N^*) \cong (R^{**} \otimes_{K^*} A^{**}) \otimes_{K^*} N^* \xrightarrow{\gamma'' \otimes_{K^*} id_{N^*}} R^{**} \otimes_{K^*} N^*$$

and $id_{R^{**}} \otimes_{K^*} \check{\zeta} : R^{**} \otimes_{K^*} (A^{**} \otimes_{K^*} N^*) \to R^{**} \otimes_{K^*} N^*$.

We also have the following result by (11.3.14).

Proposition 12.2.13 *Let (R^*, γ), (S^*, δ) be left A^*-comodule algebras and $f : S^* \to R^*$ a morphism of left A^*-comodule algebras. Let $(M^*, \check{\xi})$, $(N^*, \check{\zeta})$ be left A^{**}-modules and $\varphi : M^* \to N^*$ a morphism of left A^{**}-modules. We put $\boldsymbol{M} = (K^*, M^*, \alpha)$, $\boldsymbol{\xi} = E_{A^*}(\boldsymbol{M})^{-1}_{\boldsymbol{M}}((id_{K^*}, \check{\xi}))$, $\boldsymbol{N} = (K^*, N^*, \beta)$, $\boldsymbol{\zeta} = E_{A^*}(\boldsymbol{N})^{-1}_{\boldsymbol{N}}((id_{K^*}, \check{\zeta}))$ and $\boldsymbol{\varphi} = (id_{K^*}, \varphi)$. There exist unique maps $f^{(M^*,\check{\xi})} : (R^*, \gamma) \otimes_{A^{**}} (M^*, \check{\xi}) \to (S^*, \delta) \otimes_{A^{**}} (M^*, \check{\xi})$ and $\varphi^{(R^*,\gamma)} : (R^*, \gamma) \otimes_{A^{**}} (M^*, \check{\xi}) \to (R^*, \gamma) \otimes_{A^{**}} (N^*, \check{\zeta})$ that make the following diagrams commute.*

$$\begin{tikzcd}
R^{**} \otimes_{K^*} M^* \arrow[r, "\widetilde{E}^{(S^*,\delta)}_{(M^*,\check{\xi})}"] \arrow[d, "f^* \otimes_{K^*} id_{M^*}"'] & (R^*, \gamma) \otimes_{A^{**}} (M^*, \check{\xi}) \arrow[d, "f^{(M^*,\check{\xi})}"] \\
S^{**} \otimes_{K^*} M^* \arrow[r, "\widetilde{E}^{(R^*,\gamma)}_{(N^*,\check{\zeta})}"] & (S^*, \delta) \otimes_{A^{**}} (M^*, \check{\xi})
\end{tikzcd}
\qquad
\begin{tikzcd}
R^{**} \otimes_{K^*} M^* \arrow[r, "\widetilde{E}^{(R^*,\gamma)}_{(M^*,\check{\xi})}"] \arrow[d, "id_{R^{**}} \otimes_{K^*} \varphi"'] & (R^*, \gamma) \otimes_{A^{**}} (M^*, \check{\xi}) \arrow[d, "\varphi^{(R^*,\gamma)}"] \\
R^{**} \otimes_{K^*} N^* \arrow[r, "\widetilde{E}^{(R^*,\gamma)}_{(N^*,\check{\zeta})}"] & (R^*, \gamma) \otimes_{A^{**}} (N^*, \check{\zeta})
\end{tikzcd}$$

12.3 Left induced representations of topological Hopf algebras

Let $(A^*, \mu, \varepsilon, \iota)$, $(B^*, \mu', \varepsilon', \iota')$ be topological Hopf algebras over a field K^* and $f : A^* \to B^*$ a morphism of topological Hopf algebras. We denote by μ^r_f a composition $A^* \xrightarrow{\mu} A^* \widehat{\otimes}_{K^*} A^* \xrightarrow{id_{A^*} \widehat{\otimes}_{K^*} f} A^* \widehat{\otimes}_{K^*} B^*$ and consider a right B^*-comodule algebra (A^*, μ^r_f). We note that

$$\mathrm{prod}_{A^*} : \mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M}od_{cK^*})_{K^*} \to \mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M}od_{cK^*})_{K^*}$$

preserves equalizers by (10.1.12) and

$$\theta_{R^*,S^*}(\boldsymbol{M}) : R^* \times (S^* \times \boldsymbol{M}) \to (R^* \,\widehat{\otimes}_{K^*} S^*) \times \boldsymbol{M}$$

is an isomorphism for any $R^*, S^* \in \mathrm{Ob}\,\mathcal{TopAlg}_{cK^*}$ and $\boldsymbol{M} \in \mathrm{Ob}\mathcal{Mod}(\mathcal{TopAlg}_{cK^*}, \mathcal{Mod}_{cK^*})_{K^*}$ by (10.1.11). Since K^* is a field, $u^*_{R^*} : \mathcal{Mod}(\mathcal{TopAlg}_{cK^*}, \mathcal{Mod}_{cK^*})_{K^*} \to \mathcal{Mod}(\mathcal{TopAlg}_{cK^*}, \mathcal{Mod}_{cK^*})_{R^*}$ preserves monomorphisms for any $R^* \in \mathrm{Ob}\,\mathcal{TopAlg}_{cK^*}$ by (1.3.14). Hence the assumptions of (11.4.1) are all satisfied.

Let $(\boldsymbol{M}, \boldsymbol{\xi})$ be a representation of B^* on $\boldsymbol{M} = (K^*, M^*, \alpha)$ and put $P_{B^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = \hat{\boldsymbol{\xi}} = (id_{K^*}, \hat{\xi})$. We consider an equalizer $P^{(A^*,\mu^r_f)}_{(\boldsymbol{M},\boldsymbol{\xi})} : (A^*, \mu^r_f) \times (\boldsymbol{M}, \boldsymbol{\xi}) \to A^* \times \boldsymbol{M}$ of $\mu^r_f \times \boldsymbol{M} : A^* \times \boldsymbol{M} \to (A^* \,\widehat{\otimes}_{K^*} B^*) \times \boldsymbol{M}$ and the following composition.

$$A^* \times \boldsymbol{M} \xrightarrow{A^* \times \hat{\boldsymbol{\xi}}} A^* \times (B^* \times \boldsymbol{M}) \xrightarrow{\theta_{A^*,B^*}(\boldsymbol{M})} (A^* \,\widehat{\otimes}_{K^*} B^*) \times \boldsymbol{M}$$

Let $\widetilde{P}^{(A^*,\mu^r_f)}_{(M^*,\hat{\xi})} : (M^*, \hat{\xi}) \,\square_{B^*} (A^*, \mu^r_f) \to M^* \,\widehat{\otimes}_{K^*} A^*$ be an equalizer of the following compositions.

$$M^* \,\widehat{\otimes}_{K^*} A^* \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} \mu} M^* \,\widehat{\otimes}_{K^*} (A^* \,\widehat{\otimes}_{K^*} A^*) \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} (id_{A^*} \widehat{\otimes}_{K^*} f)} M^* \,\widehat{\otimes}_{K^*} (A^* \,\widehat{\otimes}_{K^*} B^*)$$

$$M^* \,\widehat{\otimes}_{K^*} A^* \xrightarrow{\hat{\xi} \widehat{\otimes}_{K^*} id_{A^*}} (M^* \,\widehat{\otimes}_{K^*} B^*) \,\widehat{\otimes}_{K^*} A^* \xrightarrow{\tilde{\theta}_{A^*,B^*}(\boldsymbol{M})} M^* \,\widehat{\otimes}_{K^*} (A^* \,\widehat{\otimes}_{K^*} B^*)$$

It follows from (12.1.16) that $(A^*, \mu^r_f) \times (\boldsymbol{M}, \boldsymbol{\xi}) = (K^*, (M^*, \hat{\xi}) \,\square_{B^*} (A^*, \mu^r_f), \tilde{\alpha})$ and $P^{(A^*,\mu^r_f)}_{(\boldsymbol{M},\boldsymbol{\xi})} = \left(id_{K^*}, \widetilde{P}^{(A^*,\mu^r_f)}_{(M^*,\hat{\xi})}\right)$, where $\tilde{\alpha}$ is the K^*-module structure of $(M^*, \hat{\xi}) \,\square_{B^*} (A^*, \mu^r_f)$ as a submodule of $M^* \,\widehat{\otimes}_{K^*} A^*$.

We regard (A^*, μ) as a left A^*-comodule algebra and consider a morphism $\boldsymbol{\mu}_l(\boldsymbol{M}) : A^* \times \boldsymbol{M} \to A^* \times (A^* \times \boldsymbol{M})$ of $\mathcal{Mod}(\mathcal{TopAlg}_{cK^*}, \mathcal{Mod}_{cK^*})_{K^*}$, that is, $\boldsymbol{\mu}_l(\boldsymbol{M})$ is the following composition.

$$A^* \times \boldsymbol{M} \xrightarrow{\mu \times \boldsymbol{M}} (A^* \,\widehat{\otimes}_{K^*} A^*) \times \boldsymbol{M} \xrightarrow{\theta_{A^*,A^*}(\boldsymbol{M})^{-1}} A^* \times (A^* \times \boldsymbol{M})$$

It follows from (12.1.13) that if we define a map $\mu_l(\boldsymbol{M}) : M^* \,\widehat{\otimes}_{K^*} A^* \to (M^* \,\widehat{\otimes}_{K^*} A^*) \,\widehat{\otimes}_{K^*} A^*$ to be the following composition, then we have $\boldsymbol{\mu}_l(\boldsymbol{M}) = (id_{K^*}, \mu_l(\boldsymbol{M}))$.

$$M^* \,\widehat{\otimes}_{K^*} A^* \xrightarrow{id_{M_*} \widehat{\otimes}_{K^*} \mu} M^* \,\widehat{\otimes}_{K^*} (A^* \,\widehat{\otimes}_{K^*} A^*) \xrightarrow{\tilde{\theta}_{A^*,A^*}(\boldsymbol{M})^{-1}} (M^* \,\widehat{\otimes}_{K^*} A^*) \,\widehat{\otimes}_{K^*} A^*$$

If we put $\boldsymbol{\xi}_l(\mu, \boldsymbol{M}) = P_{A^*}(A^* \times \boldsymbol{M})^{-1}_{A^* \times \boldsymbol{M}}(\boldsymbol{\mu}_l(\boldsymbol{M})) : u^*_{A^*}(A^* \times \boldsymbol{M}) \to u^*_{A^*}(A^* \times \boldsymbol{M})$, $(A^* \times \boldsymbol{M}, \boldsymbol{\xi}_l(\mu, \boldsymbol{M}))$ is a representation of A^* by (11.2.4).

There exists unique map $\hat{\xi}_f : (M^*, \hat{\xi}) \,\square_{B^*} (A^*, \mu^r_f) \to ((M^*, \hat{\xi}) \,\square_{B^*} (A^*, \mu^r_f)) \,\widehat{\otimes}_{K^*} A^*$ that makes the following diagram commute by the argument after (11.4.1).

$$\begin{tikzcd}
(M^*, \hat{\xi}) \,\square_{B^*} (A^*, \mu^r_f) \arrow[r, "\widetilde{P}^{(A^*,\mu^r_f)}_{(M^*,\hat{\xi})}"] \arrow[d, "\hat{\xi}_f"] & M^* \,\widehat{\otimes}_{K^*} A^* \arrow[d, "\mu_l(\boldsymbol{M})"] \\
((M^*, \hat{\xi}) \,\square_{B^*} (A^*, \mu^r_f)) \,\widehat{\otimes}_{K^*} A^* \arrow[r, "\widetilde{P}^{(A^*,\mu^r_f)}_{(M^*,\hat{\xi})} \widehat{\otimes}_{K^*} id_{A^*}"] & (M^* \,\widehat{\otimes}_{K^*} A^*) \,\widehat{\otimes}_{K^*} A^*
\end{tikzcd}$$

Then, we have the following result by (11.4.2)

Proposition 12.3.1 *$\hat{\xi}_f : (M^*, \hat{\xi}) \,\square_{B^*} (A^*, \mu^r_f) \to ((M^*, \hat{\xi}) \,\square_{B^*} (A^*, \mu^r_f)) \,\widehat{\otimes}_{K^*} A^*$ is a right A^*-comodule structure of $(M^*, \hat{\xi}) \,\square_{B^*} (A^*, \mu^r_f)$ and $\widetilde{P}^{(A^*,\mu^r_f)}_{(M^*,\hat{\xi})} : ((M^*, \hat{\xi}) \,\square_{B^*} (A^*, \mu^r_f), \hat{\xi}_f) \to (M^* \,\widehat{\otimes}_{K^*} A^*, \hat{\mu}_f(\boldsymbol{M}))$ is a morphism of right A^*-comodules.*

We put $\hat{\boldsymbol{\xi}}_f = (id_{K^*}, \hat{\xi}_f) : A^* \times ((A^*, \mu^r_f) \times (\boldsymbol{M}, \boldsymbol{\xi})) \to (A^*, \mu^r_f) \times (\boldsymbol{M}, \boldsymbol{\xi})$ and

$$\boldsymbol{\xi}^l_f = P_{A^*}((A^*, \mu^r_f) \times (\boldsymbol{M}, \boldsymbol{\xi}))^{-1}_{(A^*,\mu^r_f) \times (\boldsymbol{M},\boldsymbol{\xi})}(\hat{\boldsymbol{\xi}}_f) : u^*_{A^*}((A^*, \mu^r_f) \times (\boldsymbol{M}, \boldsymbol{\xi})) \to u^*_{A^*}((A^*, \mu^r_f) \times (\boldsymbol{M}, \boldsymbol{\xi})).$$

It follows from (12.3.1) that $((A^*, \mu^r_f) \times (\boldsymbol{M}, \boldsymbol{\xi}), \boldsymbol{\xi}^l_f)$ is a representation of A^* and that we have the following morphism of representations of A^*.

$$P^{(A^*,\mu^r_f)}_{(\boldsymbol{M},\boldsymbol{\xi})} : (A^* \times \boldsymbol{M}, \boldsymbol{\xi}_l(\mu, \boldsymbol{M})) \to ((A^*, \mu^r_f) \times (\boldsymbol{M}, \boldsymbol{\xi}), \boldsymbol{\xi}^l_f)$$

Let $\boldsymbol{\varphi} : (\boldsymbol{N}, \boldsymbol{\zeta}) \to (\boldsymbol{M}, \boldsymbol{\xi})$ be a morphism of representations of B^*. We put $P_{B^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{K^*}, \hat{\xi})$ and $P_{B^*}(\boldsymbol{N})_{\boldsymbol{N}}(\boldsymbol{\zeta}) = (id_{K^*}, \hat{\zeta})$. Then, if $\boldsymbol{M} = (K^*, M^*, \alpha)$, $\boldsymbol{N} = (K^*, N^*, \beta)$ and $\boldsymbol{\varphi} = (id_{K^*}, \varphi)$, $\varphi : (M^*, \hat{\xi}) \to (N^*, \hat{\zeta})$ is a morphism of right B^*-comodules. The following results is a special case of (11.4.3) and (12.1.18).

Proposition 12.3.2 *There exists unique morphism $\varphi_f : (M^*, \hat{\xi}) \Box_{B^*} (A^*, \mu^r_f) \to (N^*, \hat{\zeta}) \Box_{B^*} (A^*, \mu^r_f)$ that makes the following diagram commute.*

$$\begin{tikzcd}
(M^*, \hat{\xi}) \Box_{B^*} (A^*, \mu^r_f) \arrow[r, "\widetilde{P}^{(A^*, \mu^r_f)}_{(M^*, \hat{\xi})}"] \arrow[d, "\varphi_f"] & M^* \widehat{\otimes}_{K^*} A^* \arrow[d, "\varphi \widehat{\otimes}_{K^*} id_{A^*}"] \\
(N^*, \hat{\zeta}) \Box_{B^*} (A^*, \mu^r_f) \arrow[r, "\widetilde{P}^{(A^*, \mu^r_f)}_{(\boldsymbol{N}, \boldsymbol{\zeta})}"] & N^* \widehat{\otimes}_{K^*} A^*
\end{tikzcd}$$

Moreover φ_f is a morphism of right A^*-comodules.

The following result is a direct consequence of (11.4.5).

Proposition 12.3.3 *For a representation $(\boldsymbol{M}, \boldsymbol{\xi})$ of B^* on $\boldsymbol{M} = (K^*, M^*, \alpha)$ and a morphism $f : A^* \to B^*$ of Hopf algebras, we put $P_{B^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{K^*}, \hat{\xi})$ and regard $(M^*, \hat{\xi}) \Box_{B^*} (A^*, \mu^r_f)$ as a right B^*-comodule by*

$$(id_{(M^*, \hat{\xi}) \Box_{B^*} (A^*, \mu^r_f)} \widehat{\otimes}_{K^*} f) \hat{\xi}_f : (M^*, \hat{\xi}) \Box_{B^*} (A^*, \mu^r_f) \to (M^*, \hat{\xi}) \Box_{B^*} (A^*, \mu^r_f) \widehat{\otimes}_{K^*} B^*.$$

Then, the followin composition is a morphism of right B^*-comoddules.

$$(M^*, \hat{\xi}) \Box_{B^*} (A^*, \mu^r_f) \xrightarrow{\widetilde{P}^{(A^*, \mu^r_f)}_{(M^*, \hat{\xi})}} M^* \widehat{\otimes}_{K^*} A^* \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} \varepsilon} M^* \widehat{\otimes}_{K^*} K^* = M^* \otimes_{K^*} K^* \xrightarrow{\alpha} M^*$$

We denote by $(\eta_f)_{(M^*, \hat{\xi})} : (M^*, \hat{\xi}) \Box_{B^*} (A^*, \mu^r_f) \to M^*$ the morphism of right B^*-comodules given in (12.3.3). By (12.3.2), the following diagram is commutative for a morphism $\varphi : (M^*, \hat{\xi}) \to (N^*, \hat{\zeta})$ of right B^*-comodules.

$$\begin{tikzcd}
(M^*, \hat{\xi}) \Box_{B^*} (A^*, \mu^r_f) \arrow[r, "\widetilde{P}^{(A^*, \mu^r_f)}_{(M^*, \hat{\xi})}"] \arrow[d, "\varphi_f"] \arrow[rrr, bend left, "(\eta_f)_{(M^*, \hat{\xi})}"] & M^* \widehat{\otimes}_{K^*} A^* \arrow[r, "id_{M^*} \otimes_{K^*} \varepsilon"] \arrow[d, "\varphi \widehat{\otimes}_{K^*} id_{A^*}"] & M^* \widehat{\otimes}_{K^*} K^* \arrow[r, "\alpha"] \arrow[d, "\varphi \widehat{\otimes}_{K^*} id_{K^*}"] & M^* \arrow[d, "\varphi"] \\
(N^*, \hat{\zeta}) \Box_{B^*} (A^*, \mu^r_f) \arrow[r, "\widetilde{P}^{(A^*, \mu^r_f)}_{(\boldsymbol{N}, \boldsymbol{\zeta})}"] \arrow[rrr, bend right, "(\eta_f)_{(\boldsymbol{N}, \boldsymbol{\zeta})}"'] & N^* \widehat{\otimes}_{K^*} A^* \arrow[r, "id_{N^*} \otimes_{K^*} \varepsilon"] & N^* \widehat{\otimes}_{K^*} K^* \arrow[r, "\beta"] & N^*
\end{tikzcd}$$

We denote by $\mathrm{Comod}(A^*)$ the category of right A^*-comodules and recall that the opposite category of $\mathrm{Comod}(A^*)$ is isomorphic to the category of representations of A^*. We denote by $\mathrm{Rep}(A^*)$ the category of representations of A^* for short. For a representation $(\boldsymbol{M}, \boldsymbol{\xi})$ of B^* and a representation $(\boldsymbol{N}, \boldsymbol{\zeta})$ of A^*, we put $\boldsymbol{M} = (K^*, M^*, \alpha)$ and $\boldsymbol{N} = (K^*, N_*, \beta)$ and define a map

$$\mathrm{ad}^{(\boldsymbol{M}, \boldsymbol{\xi})}_{(\boldsymbol{N}, \boldsymbol{\zeta})} : \mathrm{Rep}(A^*)(((A^*, \mu^r_f) \times (\boldsymbol{M}, \boldsymbol{\xi}), \boldsymbol{\xi}^l_f), (\boldsymbol{N}, \boldsymbol{\zeta})) \to \mathrm{Rep}(B^*)((\boldsymbol{M}, \boldsymbol{\xi}), f^{\bullet}(\boldsymbol{N}, \boldsymbol{\zeta}))$$

by giving a map

$$\mathrm{Comod}(A^*)((N^*, \hat{\zeta}), ((M^*, \hat{\xi}) \Box_{B^*} (A^*, \mu^r_f), \hat{\xi}_f)) \to \mathrm{Comod}(B^*)((N^*, (id_{N^*} \widehat{\otimes}_{K^*} f)\hat{\zeta}), (M^*, \hat{\xi}))$$

which maps $\psi \in \mathrm{Comod}(A^*)((N^*, \hat{\zeta}), ((M^*, \hat{\xi}) \Box_{B^*} (A^*, \mu^r_f), \hat{\xi}_f))$ to the following composition.

$$N^* \xrightarrow{\psi} (M^*, \hat{\xi}) \Box_{B^*} (A^*, \mu^r_f) \xrightarrow{(\eta_f)_{(M^*, \hat{\xi})}} M^*$$

Finally, we have the following result by (11.4.6).

Theorem 12.3.4 $\mathrm{ad}^{(\boldsymbol{M}, \boldsymbol{\xi})}_{(\boldsymbol{N}, \boldsymbol{\zeta})} : \mathrm{Rep}(A^*)(((A^*, \mu^r_f) \times (\boldsymbol{M}, \boldsymbol{\xi}), \boldsymbol{\xi}^l_f), (\boldsymbol{N}, \boldsymbol{\zeta})) \to \mathrm{Rep}(B^*)((\boldsymbol{M}, \boldsymbol{\xi}), \boldsymbol{f}^{\bullet}(\boldsymbol{N}, \boldsymbol{\zeta}))$ *is a bijection. Hence a correspondence $(\boldsymbol{M}, \boldsymbol{\xi}) \mapsto ((A^*, \mu^r_f) \times (\boldsymbol{M}, \boldsymbol{\xi}), \boldsymbol{\xi}^l_f)$ gives a left adjoint of the restriction functor $\boldsymbol{f}^{\bullet} : \mathrm{Rep}(A^*) \to \mathrm{Rep}(B^*)$.*

12.4 Right induced representations of topological Hopf algebras

We assume that K^* is a field which satisfies $K^i = \{0\}$ for $i \neq 0$ and let $\mathcal{M}$ be a full subcategory of $\mathcal{M}od_{cK^*}$ consisting of objects which satisfy the condition (ii) of (10.1.13).

Let $(A^*, \mu, \varepsilon, \iota)$, $(B^*, \mu', \varepsilon', \iota')$ be topological Hopf algebras over K^* and $f : A^* \to B^*$ a morphism of topological Hopf algebras. We assume that A^* and B^* satisfy the condition (i) of (10.1.13). We denote by μ^l_f a composition $A^* \xrightarrow{\mu} A^* \otimes_{K^*} A^* \xrightarrow{f \otimes_{K^*} id_{A^*}} B^* \otimes_{K^*} A^*$ and consider a left B^*-comodule algebra (A^*, μ^l_f). We note that

$$\exp_{A^*} : \mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M})_{K^*} \to \mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M})_{K^*}$$

preserves equalizers by (10.1.20) and if $R^*, S^* \in \operatorname{Ob} \mathcal{T}op\mathcal{A}lg_{cK^*}$ satisfy the condition (i) of (10.1.13),

$$\theta^{R^*,S^*}(\boldsymbol{N}) : \boldsymbol{N}^{R^* \otimes_{K^*} S^*} \to (\boldsymbol{N}^{R^*})^{S^*}$$

is an isomorphism for any $\boldsymbol{N} \in \operatorname{Ob}\mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M})_{K^*}$ by (10.1.19). Since K^* is a field,

$$u^*_{R^*} : \mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M})_{K^*} \to \mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M})_{R^*}$$

preserves monomorphisms for any $R^* \in \operatorname{Ob} \mathcal{T}op\mathcal{A}lg_{cK^*}$ by (2.1.5) and (1.3.14). Hence the assumptions of (11.5.1) are all satisfied if A^* satisfies the condition (i) of (10.1.13) and $\boldsymbol{N}$ is an object of $\mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M})_{K^*}$.

Let $(\boldsymbol{N}, \boldsymbol{\zeta})$ be a representation of B^* on $\boldsymbol{N} = (K^*, N^*, \alpha)$ and put $E_{B^*}(\boldsymbol{N})_{\boldsymbol{N}}(\boldsymbol{\zeta}) = (id_{K^*}, \check{\zeta})$. We consider a coequalizer $E^{(A^*,\mu^l_f)}_{(\boldsymbol{N},\boldsymbol{\zeta})} : \boldsymbol{N}^{A^*} \to (\boldsymbol{N}, \boldsymbol{\zeta})^{(A^*,\mu^l_f)}$ of $\boldsymbol{N}^{\mu^l_f} : \boldsymbol{N}^{B^* \otimes_{K^*} A^*} \to \boldsymbol{N}^{A^*}$ and the following composition.

$$\boldsymbol{N}^{B^* \otimes_{K^*} A^*} \xrightarrow{\theta^{B^*,A^*}(\boldsymbol{N})} (\boldsymbol{N}^{B^*})^{A^*} \xrightarrow{\check{\zeta}^{A^*}} \boldsymbol{N}^{A^*}$$

Let $\widetilde{E}^{(A^*,\mu^l_f)}_{(N^*,\check{\zeta})} : A^{**} \otimes_{K^*} N^* \to (A^*, \mu^l_f) \otimes_{B^{**}} (N^*, \check{\zeta})$ be a coequalizer of

$$A^{**} \otimes_{K^*} (B^{**} \otimes_{K^*} N^*) \cong (A^{**} \otimes_{K^*} B^{**}) \otimes_{K^*} N^* \xrightarrow{\bar{\mu}^l_f \otimes_{K^*} id_{N^*}} A^{**} \otimes_{K^*} N^*$$

and $id_{A^{**}} \otimes_{K^*} \check{\zeta} : A^{**} \otimes_{K^*} (B^{**} \otimes_{K^*} N^*) \to A^{**} \otimes_{K^*} N^*$, where $\bar{\mu}^l_f : A^{**} \otimes_{K^*} B^{**} \to A^{**}$ is the following composition.

$$A^{**} \otimes_{K^*} B^{**} \xrightarrow{T_{A^{**},B^{**}}} B^{**} \otimes_{K^*} A^{**} \xrightarrow{\phi} \mathcal{H}om^*(B^* \otimes_{K^*} A^*, K^*) \xrightarrow{(\mu^l_f)^*} A^{**}$$

It follows from (12.2.12) that $(\boldsymbol{N}, \boldsymbol{\zeta})^{(A^*,\mu^l_f)} = (K^*, (A^*, \mu^l_f) \otimes_{B^{**}} (N^*, \check{\zeta}), \check{\alpha})$ and $E^{(A^*,\mu^l_f)}_{(\boldsymbol{N},\boldsymbol{\zeta})} = \left(id_{K^*}, \widetilde{E}^{(A^*,\mu^l_f)}_{(N^*,\check{\zeta})}\right)$, where $\check{\alpha}$ is the K^*-module structure of $(A^*, \mu^l_f) \otimes_{B^{**}} (N^*, \check{\zeta})$ as a quotient module of $A^{**} \otimes_{K^*} N^*$.

We regard (A^*, μ) as a right A^*-comodule algebra and consider a morphism $\boldsymbol{\mu}_r(\boldsymbol{N}) : (\boldsymbol{N}^{A^*})^{A^*} \to \boldsymbol{N}^{A^*}$ of $\mathcal{M}od(\mathcal{T}op\mathcal{A}lg_{cK^*}, \mathcal{M})_{K^*}$, that is, $\boldsymbol{\mu}_r(\boldsymbol{N})$ is the following composition.

$$(\boldsymbol{N}^{A^*})^{A^*} \xrightarrow{\theta^{A^*,A^*}(\boldsymbol{N})^{-1}} \boldsymbol{N}^{A^* \otimes_{K^*} A^*} \xrightarrow{\boldsymbol{N}^{\mu}} \boldsymbol{N}^{A^*}$$

It follows from (12.2.8) that if we define a map $\mu_r(\boldsymbol{N}) : A^{**} \otimes_{K^*} (A^{**} \otimes_{K^*} N^*) \to A^{**} \otimes_{K^*} N^*$ to be the following composition, then we have $\boldsymbol{\mu}_r(\boldsymbol{N}) = (id_{K^*}, \mu_r(\boldsymbol{N}))$.

$$A^{**} \otimes_{K^*} (A^{**} \otimes_{K^*} N^*) \xrightarrow{\tilde{\theta}^{A^*,A^*}(\boldsymbol{N})^{-1}} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^*) \otimes_{K^*} N^* \xrightarrow{\mu^* \otimes_{K^*} id_{M_*}} A^{**} \otimes_{K^*} N^*$$

If we put $\boldsymbol{\xi}_r(\mu, \boldsymbol{N}) = E_{A^*}(\boldsymbol{N}^{A^*})^{-1}_{\boldsymbol{N}^{A^*}}(\boldsymbol{\mu}_r(\boldsymbol{N})) : u^*_{A^*}(\boldsymbol{N}^{A^*}) \to u^*_{A^*}(\boldsymbol{N}^{A^*})$, $(\boldsymbol{N}^{A^*}, \boldsymbol{\xi}_r(\mu, \boldsymbol{N}))$ is a representation of A^* by (11.3.4).

There exists unique map $\check{\zeta}_f : A^{**} \otimes_{K^*} ((A^*, \mu^l_f) \otimes_{B^{**}} (N^*, \check{\zeta})) \to (A^*, \mu^l_f) \otimes_{B^{**}} (N^*, \check{\zeta})$ that makes the following diagram commute by the argument after (11.5.1).

$$\begin{tikzcd}
A^{**} \otimes_{K^*} (A^{**} \otimes_{K^*} N^*) \arrow[r, "id_{A^{**}} \otimes_{K^*} \widetilde{E}^{(A^*,\mu^l_f)}_{(N^*,\check{\zeta})}"] \arrow[d, "\mu_r(\boldsymbol{N})"] & A^{**} \otimes_{K^*} ((A^*, \mu^l_f) \otimes_{B^{**}} (N^*, \check{\zeta})) \arrow[d, "\check{\zeta}_f"] \\
A^{**} \otimes_{K^*} N^* \arrow[r, "\widetilde{E}^{(A^*,\mu^l_f)}_{(N^*,\check{\zeta})}"] & (A^*, \mu^l_f) \otimes_{B^{**}} (N^*, \check{\zeta})
\end{tikzcd}$$

Then, we have the following result by (11.5.2)

Proposition 12.4.1 *$\check{\zeta}_f : A^{**}\otimes_{K^*}((A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta})) \to (A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta})$ is a left A^{**}-module structure of $(A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta})$ and $\widetilde{E}^{(A^*,\mu_f^l)}_{(N^*,\check{\zeta})} : (A^{**}\otimes_{K^*} M^*, \check{\mu}_f(\boldsymbol{N})) \to ((A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta}), \check{\zeta}_f)$ is a morphism of right A^*-comodules.*

We put $\check{\boldsymbol{\zeta}}_f = (id_{K^*}, \check{\zeta}_f) : (\boldsymbol{N},\boldsymbol{\zeta})^{(A^*,\mu_f^l)} \to \left((\boldsymbol{N},\boldsymbol{\zeta})^{(A^*,\mu_f^l)}\right)^{A^*}$ and

$$\boldsymbol{\zeta}_f^l = E_{A^*}\left((\boldsymbol{N},\boldsymbol{\zeta})^{(A^*,\mu_f^l)}\right)^{-1}_{(\boldsymbol{N},\boldsymbol{\zeta})^{(A^*,\mu_f^l)}}(\check{\boldsymbol{\zeta}}_f) : u_{A^*}^*\left((\boldsymbol{N},\boldsymbol{\zeta})^{(A^*,\mu_f^l)}\right) \to u_{A^*}^*\left((\boldsymbol{N},\boldsymbol{\zeta})^{(A^*,\mu_f^l)}\right).$$

It follows from (12.4.1) that $((\boldsymbol{N},\boldsymbol{\zeta})^{(A^*,\mu_f^l)}, \boldsymbol{\zeta}_f^l)$ is a representation of A^* and that we have the following morphism of representations of A^*.

$$E^{(A^*,\mu_f^l)}_{(\boldsymbol{N},\boldsymbol{\zeta})} : ((\boldsymbol{N},\boldsymbol{\zeta})^{(A^*,\mu_f^l)}, \boldsymbol{\zeta}_f^l) \to (\boldsymbol{N}^{A^*}, \boldsymbol{\xi}_r(\mu,\boldsymbol{N}))$$

Let $\boldsymbol{\varphi} : (\boldsymbol{N},\boldsymbol{\zeta}) \to (\boldsymbol{M},\boldsymbol{\xi})$ be a morphism of representations of B^*. We put $E_{B^*}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{K^*}, \check{\xi})$ and $E_{B^*}(\boldsymbol{N})_{\boldsymbol{N}}(\boldsymbol{\zeta}) = (id_{K^*}, \check{\zeta})$. Then, if $\boldsymbol{M} = (K^*, M^*, \alpha)$, $\boldsymbol{N} = (K^*, N^*, \beta)$ and $\boldsymbol{\varphi} = (id_{K^*}, \varphi)$, $\varphi : (M^*, \check{\xi}) \to (N^*, \check{\zeta})$ is a morphism of left B^{**}-modules. The following results is a special case of (11.5.3) and (12.2.13).

Proposition 12.4.2 *There exists unique morphism $\varphi^f : (A^*,\mu_f^l)\otimes_{B^{**}}(M^*,\check{\xi}) \to (A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta})$ that makes the following diagram commute.*

$$\begin{tikzcd}
A^{**}\otimes_{K^*} M^* \arrow[r, "\widetilde{E}^{(A^*,\mu_f^l)}_{(M^*,\check{\xi})}"] \arrow[d, "id_{A^{**}}\otimes_{K^*}\varphi"] & (A^*,\mu_f^l)\otimes_{B^{**}}(M^*,\check{\xi}) \arrow[d, "\varphi^f"] \\
A^{**}\otimes_{K^*} N^* \arrow[r, "\widetilde{E}^{(A^*,\mu_f^l)}_{(N^*,\check{\zeta})}"] & (A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta})
\end{tikzcd}$$

Moreover φ^f is a morphism of left A^{**}-modules.

The following result is a direct consequence of (11.5.5).

Proposition 12.4.3 *For a representation $(\boldsymbol{N},\boldsymbol{\zeta})$ of B^* on $\boldsymbol{N} = (K^*, N^*, \beta)$ and a morphism $f : A^* \to B^*$ of Hopf algebras, we put $E_{B^*}(\boldsymbol{N})_{\boldsymbol{N}}(\boldsymbol{\zeta}) = (id_{K^*}, \check{\zeta})$ and regard $(A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta})$ as a left B^{**}-module by*

$$\check{\zeta}_f\,(f^*\otimes_{K^*} id_{(A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta})}) : B^{**}\otimes_{K^*}(A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta}) \to (A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta}).$$

Then, the following composition is a morphism of left B^{**}-modules.

$$N^* \xrightarrow{T_{N^*,K^*}\beta^{-1}} K^*\otimes_{K^*} N^* \xrightarrow{\kappa_{K^*}\otimes_{K^*} id_{N^*}} K^{**}\otimes_{K^*} N^* \xrightarrow{\varepsilon^*\otimes_{K^*} id_{N^*}} A^{**}\otimes_{K^*} N^* \xrightarrow{\widetilde{E}^{(A^*,\mu_f^l)}_{(N^*,\check{\zeta})}} (A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta})$$

We denote by $(\varepsilon_f)_{(N^*,\check{\zeta})} : N^* \to (A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta})$ the morphism of left B^{**}-modules given in (12.4.2). By (12.4.3), the following diagram is commutative for a morphism $\varphi : (M^*,\check{\xi}) \to (N^*,\check{\zeta})$ of left B^{**}-modules.

$$\begin{tikzcd}
M^* \arrow[r, "T_{M^*,K^*}\alpha^{-1}"'] \arrow[rrrr, bend left, "(\varepsilon_f)_{(M^*,\check{\xi})}"] \arrow[d, "\varphi"] & K^*\otimes_{K^*} M^* \arrow[r, "\kappa_{K^*}\otimes_{K^*} id_{M^*}"] \arrow[d, "id_{K^*}\otimes_{K^*}\varphi"] & K^{**}\otimes_{K^*} M^* \arrow[r, "\varepsilon^*\otimes_{K^*} id_{M^*}"] \arrow[d, "id_{K^{**}}\otimes_{K^*}\varphi"] & A^{**}\otimes_{K^*} M^* \arrow[r, "\widetilde{E}^{(A^*,\mu_f^l)}_{(\boldsymbol{M},\boldsymbol{\xi})}"'] \arrow[d, "id_{A^{**}}\otimes_{K^*}\varphi"] & (A^*,\mu_f^l)\otimes_{B^{**}}(M^*,\check{\xi}) \arrow[d, "\varphi^f"] \\
N^* \arrow[r, "T_{N^*,K^*}\beta^{-1}"] \arrow[rrrr, bend right, "(\varepsilon_f)_{(N^*,\check{\zeta})}"'] & K^*\otimes_{K^*} N^* \arrow[r, "\kappa_{K^*}\otimes_{K^*} id_{N^*}"] & K^{**}\otimes_{K^*} N^* \arrow[r, "\varepsilon^*\otimes_{K^*} id_{N^*}"] & A^{**}\otimes_{K^*} N^* \arrow[r, "\widetilde{E}^{(A^*,\mu_f^l)}_{(\boldsymbol{N},\boldsymbol{\zeta})}"] & (A^*,\mu_f^l)\otimes_{B^{**}}(N^*,\check{\zeta})
\end{tikzcd}$$

We denote by $\mathcal{M}od(A^{**})$ the category of left A^{**}-modules and recall that the opposite category of $\mathcal{M}od(A^{**})$ is isomorphic to the category of representations $\mathrm{Rep}(A^*)$ of A^*. For a representation $(\boldsymbol{N},\boldsymbol{\zeta})$ of B^* and a representation $(\boldsymbol{M},\boldsymbol{\xi})$ of A^*, we put $\boldsymbol{M}=(K^*,M^*,\alpha)$ and $\boldsymbol{N}=(K^*,N^*,\beta)$ and define a map

$$\mathrm{ad}^{(\boldsymbol{M},\boldsymbol{\xi})}_{(\boldsymbol{N},\boldsymbol{\zeta})} : \mathrm{Rep}(A^*)\big((\boldsymbol{M},\boldsymbol{\xi}),\big((\boldsymbol{N},\boldsymbol{\zeta})^{(A^*,\mu^l_f)},\boldsymbol{\zeta}^l_f\big)\big) \to \mathrm{Rep}(B^*)(f^{\bullet}(\boldsymbol{M},\boldsymbol{\xi}),(\boldsymbol{N},\boldsymbol{\zeta}))$$

by giving a map

$$\mathcal{M}od(A^{**})(((A^*,\mu^l_f)\otimes_{B^{**}}(N^*,\check{\zeta}),\check{\zeta}_f),(M^*,\check{\xi}))\to \mathcal{M}od(B^{**})((N^*,\check{\zeta}),(M^*,\check{\xi}(f^*\otimes_{K^*} id_{M^*})))$$

which maps $\psi\in\mathcal{M}od(A^{**})(((A^*,\mu^l_f)\otimes_{B^{**}}(N^*,\check{\zeta}),\check{\zeta}_f),(M^*,\check{\xi}))$ to the following composition.

$$N^*\xrightarrow{(\varepsilon_f)_{(N^*,\check{\zeta})}}(A^*,\mu^l_f)\otimes_{B^{**}}(N^*,\check{\zeta})\xrightarrow{\psi} M^*$$

Finally, we have the following result by (11.5.6).

Theorem 12.4.4 $\mathrm{ad}^{(\boldsymbol{M},\boldsymbol{\xi})}_{(\boldsymbol{N},\boldsymbol{\zeta})} : \mathrm{Rep}(A^*)\big((\boldsymbol{M},\boldsymbol{\xi}),\big((\boldsymbol{N},\boldsymbol{\zeta})^{(A^*,\mu^l_f)},\boldsymbol{\zeta}^l_f\big)\big) \to \mathrm{Rep}(B^*)(f^{\bullet}(\boldsymbol{M},\boldsymbol{\xi}),(\boldsymbol{N},\boldsymbol{\zeta}))$ *is a bijection. Hence a correspondence* $(\boldsymbol{N},\boldsymbol{\zeta})\mapsto\big((\boldsymbol{N},\boldsymbol{\zeta})^{(A^*,\mu^l_f)},\boldsymbol{\zeta}^l_f\big)$ *gives a right adjoint of the restriction functor* $\boldsymbol{f}^{\bullet}$: $\mathrm{Rep}(A^*)\to\mathrm{Rep}(B^*)$.

13 Representations in fibered category of functorial modules

13.1 Representation of topological group functors

We assume that a subcategory $\mathcal{C}$ of $\mathcal{T}op\mathcal{A}lg_{K^*}$ contains K^* as an object and fix a terminal object 1 of $\mathcal{T} = \mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$ which is the functor h_{K^*} represented by K^*. For an object F of $\mathcal{T}$, we denote by o_F the unique morphism from F to 1. We also assume that a subcategory $\mathcal{M}$ of $\mathcal{T}op\mathcal{M}od_{K^*}$ satisfies the condition (ii) of (10.1.1).

Consider the fibered category $p_{\mathcal{T}} : \mathcal{MOD} \to \mathcal{T}$ given in (10.2.2). As we see in the proof of (10.2.2), $f^*g^*(H, L) = (gf)^*(H, L)$ holds for morphisms $f : G \to F$, $g : F \to H$ of $\mathcal{T}$ and $(H, L) \in \mathrm{Ob}\,\mathcal{MOD}_H$. We also note that $o_G^*(1, M) = (G, M\tilde{o}_G)$ for $G \in \mathrm{Ob}\,\mathcal{T}$ and h_{K^*}-module $M : \mathcal{C}_{h_{K^*}} \to \mathcal{M}od(\mathcal{C}, \mathcal{M})$ and that $\mathcal{MOD}_F$ is isomorphic to the opposite category $\mathcal{M}od(F)^{op}$ of the category of F-modules.

We specialize the definition (11.1.1) of general group objects to topological group functor on $\mathcal{C}$ as follows.

Definition 13.1.1 *Let $(G, \mu, \varepsilon, \iota)$ be a group object in $\mathcal{T}$. A pair (M, ξ) of an h_{K^*}-module M and a morphism $\xi : M\tilde{o}_G \to M\tilde{o}_G$ of G-modules is called a (left) representation of G on M if the following conditions are satisfied.*

(i) $\xi_{(R^*, \mu_{R^*}(g,h))} = \xi_{(R^*, h)}\xi_{(R^*, g)} : M(R^*, u_{R^*}) \to M(R^*, u_{R^*})$ holds for any $(R^*, (g, h)) \in \mathrm{Ob}\,\mathcal{C}_{G\times G}$.

(ii) $\xi_{(R^*, e_{R^*})} = id_{M(R^*, u_{R^*})}$ holds for any $R^* \in \mathrm{Ob}\,\mathcal{C}$, where $e_{R^*} \in G(R^*)$ is the unit of $G(R^*)$.

Let (M, ξ) and (N, ζ) be representations of G on M and N, respectively. A morphism $\varphi : M \to N$ of h_{K^*}-modules is a called a morphism of representations of G from (N, ζ) to (M, ξ) if the following diagram commutes for any $(R^*, g) \in \mathrm{Ob}\,\mathcal{C}_G$.

$$\begin{array}{ccc} M(R^*, u_{R^*}) & \xrightarrow{\xi_{(R^*,g)}} & M(R^*, u_{R^*}) \\ \downarrow{\scriptstyle \varphi_{(R^*,g)}} & & \downarrow{\scriptstyle \varphi_{(R^*,g)}} \\ N(R^*, u_{R^*}) & \xrightarrow{\zeta_{(R^*,g)}} & N(R^*, u_{R^*}) \end{array}$$

We denote by $\mathrm{Rep}(G\,; \mathcal{M})$ the category of representations of G and morphisms between them.

Remark 13.1.2 *Let M, N be h_{K^*}-modules and $\xi : M\tilde{o}_G \to N\tilde{o}_G$ a morphism of G-modules. For an object (R^*, g) of $\mathcal{C}_G$, we put $\xi_{(R^*,g)} = (id_{R^*}, \xi(R^*, g)) : M(R^*, u_{R^*}) \to N(R^*, u_{R^*})$. If $M(R^*, u_{R^*}) = (R^*, M^*_{R^*}, \alpha_{R^*})$ and $N(R^*, u_{R^*}) = (R^*, N^*_{R^*}, \beta_{R^*})$ for an object R^* of $\mathcal{C}$, $\xi(R^*, g) : M^*_{R^*} \to N^*_{R^*}$ is a homomorphism of right R^*-modules. If $M = N$, the condition (i) of (13.1.1) is equivalent to $\xi(R^*, \mu_{R^*}(g, h)) = \xi(R^*, h)\xi(R^*, g)$ for any $(R^*, (g, h)) \in \mathrm{Ob}\,\mathcal{C}_{G\times G}$ and the condition (ii) of (13.1.1) is equivalent to $\xi(R^*, e_{R^*}) = id_{M^*_{R^*}}$ for any $R^* \in \mathrm{Ob}\,\mathcal{C}$. Thus a map $\alpha(\xi)_{R^*} : M^*_{R^*} \times G(R^*) \to M^*_{R^*}$ defined by $\alpha(\xi)_{R^*}(x, g) = \xi(R^*, g)(x)$ is a right action of $G(R^*)$ on $M^*_{R^*}$ for any $R^* \in \mathrm{Ob}\,\mathcal{C}$ if and only if ξ is a representation of G on M. We note that ξ is the trivial representation of G on M if and only if $\alpha(\xi)_{R^*}$ is the trivial $G(R^*)$ action on $M^*_{R^*}$ for any $R^* \in \mathrm{Ob}\,\mathcal{C}$.*

Remark 13.1.3 *A right representation of G on M is equivalent to $\xi(R^*, \mu_{R^*}(g, h)) = \xi(R^*, g)\xi(R^*, h)$ for any object $(R^*, (g, h))$ of $\mathcal{C}_{G\times G}$ and $\xi(R^*, e_{R^*}) = id_{M^*_{R^*}}$ for any object R^* of $\mathcal{C}$. Hence we have a left action $G(R^*) \times M^*_{R^*} \to M^*_{R^*}$ of $G(R^*)$ on $M^*_{R^*}$ which maps $(g, x) \in M^*_{R^*} \times G(R^*)$ to $\xi(R^*, g)(x)$.*

Since the Yoneda embedding $h : \mathcal{C}^{op} \to \mathcal{T}$ preserves finite limits and terminal objects, if $(A^*, \mu, \varepsilon, \iota)$ is a Hopf algebra in $\mathcal{C}$, $(h_{A^*}, h_\mu, h_\varepsilon, h_\iota)$ is a group object in $\mathcal{T}$. We denote h_{A^*} by G_{A^*}. Consider the functor $\hat{h} : \mathcal{M}od(\mathcal{C}, \mathcal{M})^{op} \to \mathcal{MOD}$ defined the paragraph above (10.4.3). We recall from (10.4.3) and (10.4.5) that $\hat{h}$ preserves and reflects cartesian morphisms and is fully faithful. By (10.4.3) and (10.4.3), the following results are special cases of (11.1.10) and (11.1.11).

Proposition 13.1.4 *Let $(A^*, \mu, \varepsilon, \iota)$ be a Hopf algebra in $\mathcal{C}$ and $\boldsymbol{M}$ an object of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K^*}$. We consider an affine group scheme G_{A^*} represented by A^*. For a morphism $\boldsymbol{\xi} : u^*_{A^*}(\boldsymbol{M}) \to u^*_{A^*}(\boldsymbol{M})$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{A^*}$, $\hat{h}^{A^*}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\xi}) : h^*_{u_{A^*}}(\hat{h}(\boldsymbol{M})) \to h^*_{u_{A^*}}(\hat{h}(\boldsymbol{M}))$ defines a representation of G_{A^*} on $\hat{h}(\boldsymbol{M})$ if and only if $(\boldsymbol{M}, \boldsymbol{\xi})$ is a representation of A^* on $\boldsymbol{M}$.*

Proposition 13.1.5 *Let $\boldsymbol{\varphi} : \boldsymbol{M} \to \boldsymbol{N}$ be a morphism of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{K^*}$ and $(\boldsymbol{M}, \boldsymbol{\xi})$, $(\boldsymbol{N}, \boldsymbol{\zeta})$ representations of A^*. Then, $\hat{h}(\boldsymbol{\varphi}) : \hat{h}(\boldsymbol{M}) \to \hat{h}(\boldsymbol{N})$ defines a morphism of representations of G_{A^*} from $(\hat{h}(\boldsymbol{M}), \hat{h}^{A^*}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\xi}))$ to $(\hat{h}(\boldsymbol{N}), \hat{h}^{A^*}_{\boldsymbol{N},\boldsymbol{N}}(\boldsymbol{\zeta}))$ if only if $\boldsymbol{\varphi}$ defines a morphism of representations of A^* from $(\boldsymbol{M}, \boldsymbol{\xi})$ to $(\boldsymbol{N}, \boldsymbol{\zeta})$.*

13.2 Representations as group actions

The following is a special case of (6.1.20).

Proposition 13.2.1 *Let $\boldsymbol{M}$, $\boldsymbol{N}$ be objects of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$. For a morphism $\gamma : A^* \to R^*$ of $\mathcal{C}$, the following diagram is commutative.*

$$\begin{CD}
\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{R^*}(u^*_{A^*}(\boldsymbol{N}), u^*_{A^*}(\boldsymbol{M})) @>{\gamma^\sharp_{\boldsymbol{N},\boldsymbol{M}}}>> \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{A^*}(u^*_{R^*}(\boldsymbol{N}), u^*_{R^*}(\boldsymbol{M}))\\
@V{\hat{h}^{A^*}_{\boldsymbol{N},\boldsymbol{M}}}VV @VV{\hat{h}^{R^*}_{\boldsymbol{N},\boldsymbol{M}}}V\\
\mathcal{MOD}_{h_{A^*}}(o^*_{h_{A^*}}(\hat{h}(\boldsymbol{N})), o^*_{h_{R^*}}(\hat{h}(\boldsymbol{M}))) @>{h^*_\gamma}>> \mathcal{MOD}_{h_{R^*}}(o^*_{h_{R^*}}(\hat{h}(\boldsymbol{N})), o^*_{h_{R^*}}(\hat{h}(\boldsymbol{M})))
\end{CD}$$

The following result is a direct consequence of the definitions of maps $\hat{h}^{A^*}_{\boldsymbol{N},\boldsymbol{M}}$, $\gamma^\sharp_{\boldsymbol{N},\boldsymbol{M}}$ and the proof of (10.4.3).

Proposition 13.2.2 *For a morphism $\boldsymbol{\varphi} : u^*_{A^*}(\boldsymbol{N}) \to u^*_{A^*}(\boldsymbol{M})$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{A^*}$, we put $\hat{h}^{A^*}_{\boldsymbol{N},\boldsymbol{M}}(\boldsymbol{\varphi}) = (id_{h_{A^*}}, \widetilde{\boldsymbol{\varphi}})$ for a morphism $\widetilde{\boldsymbol{\varphi}} : \widehat{\boldsymbol{M}}\tilde{o}_{h_{A^*}} \to \widehat{\boldsymbol{N}}\tilde{o}_{h_{A^*}}$ of h_{A^*}-modules. Then, $\widetilde{\boldsymbol{\varphi}}_{(R^*,\gamma)} = \gamma^\sharp_{\boldsymbol{N},\boldsymbol{M}}(\boldsymbol{\varphi}) : u^*_{R^*}(\boldsymbol{M}) \to u^*_{R^*}(\boldsymbol{N})$ for any object (R^*,γ) of $\mathcal{C}_{h_{A^*}}$.*

Remark 13.2.3 *Suppose that $\boldsymbol{M} = (K^*, M^*, \alpha)$ and $\boldsymbol{N} = (K^*, N^*, \beta)$. If we use the notation of (13.1.2), we have $\widetilde{\boldsymbol{\varphi}}_{(R^*,\gamma)} = (id_{R^*}, \widetilde{\boldsymbol{\varphi}}(R^*,\gamma))$ for a morphism $\widetilde{\boldsymbol{\varphi}}(R^*,\gamma) : M^* \widehat{\otimes}_{K^*} R^* \to N^* \widehat{\otimes}_{K^*} R^*$ of right R^*-modules. We denote $\widetilde{\boldsymbol{\varphi}}(R^*,\gamma)$ by $\gamma_{\boldsymbol{\varphi}}$ for short below. It follows from (13.2.2) that $\widetilde{\boldsymbol{\varphi}}_{(A^*,id_{A^*})} = \boldsymbol{\varphi}$.*

Proposition 13.2.4 *For a morphism $\boldsymbol{\varphi} = (id_{A^*}, \varphi) : u^*_{A^*}(\boldsymbol{M}) \to u^*_{A^*}(\boldsymbol{N})$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{A^*}$ and an object (R^*,γ) of $\mathcal{C}_{h_{A^*}}$, the following diagram is commutative.*

$$\begin{CD}
M^*\widehat{\otimes}_{K^*}A^*\widehat{\otimes}_{A^*}R^* @>{\varphi\,\widehat{\otimes}_{A^*}\,id_{R^*}}>> N^*\widehat{\otimes}_{K^*}A^*\widehat{\otimes}_{A^*}R^*\\
@V{id_{M^*}\widehat{\otimes}_{K^*}\hat{\chi}_\gamma}VV @VV{id_{M^*}\widehat{\otimes}_{K^*}\hat{\chi}_\gamma}V\\
M^*\widehat{\otimes}_{K^*}R^* @>{\gamma_{\boldsymbol{\varphi}}}>> N^*\widehat{\otimes}_{K^*}R^*\\
@V{\hat{\imath}_{M^*}\widehat{\otimes}_{K^*}id_{R^*}}VV @AA{id_{N^*}\widehat{\otimes}_{K^*}\hat{m}_{R^*}}A\\
M^*\widehat{\otimes}_{K^*}A^*\widehat{\otimes}_{K^*}R^* @>{\varphi\,\widehat{\otimes}_{K^*}\,id_{R^*}}>> N^*\widehat{\otimes}_{K^*}A^*\widehat{\otimes}_{K^*}R^* @>{id_{N^*}\widehat{\otimes}_{K^*}\gamma\widehat{\otimes}_{K^*}id_{R^*}}>> N^*\widehat{\otimes}_{K^*}R^*\widehat{\otimes}_{K^*}R^*
\end{CD}$$

Here, $\hat{m}_{R^*} : R^*\widehat{\otimes}_{K^*}R^* \to R^*$ is the map induced by the multiplication of R^* and $\hat{\chi}_\gamma : A^*\widehat{\otimes}_{A^*}R^* \to R*$ is the isomorphism induced by an isomorphism $A^*\otimes_{A^*}R^* \to R*$ given by $a\otimes r \to \gamma(a)r$.

Proof. Since $\widetilde{\boldsymbol{\varphi}}_{(R^*,\gamma)} = \gamma^\sharp_{\boldsymbol{N},\boldsymbol{M}}(\boldsymbol{\varphi})$ by (13.2.2), the upper rectangle is commutative. Let $q : A^*\otimes_{K^*}S^* \to A^*\otimes_{A^*}S^*$ be the quotient map and $\hat{q} : A^*\widehat{\otimes}_{K^*}S^* \to A^*\widehat{\otimes}_{A^*}S^*$ the map induced by q. Then, the following diagram is commutative.

$$\begin{CD}
M^*\widehat{\otimes}_{K^*}A^*\widehat{\otimes}_{A^*}R^* @>{\varphi\,\widehat{\otimes}_{A^*}\,id_{R^*}}>> N^*\widehat{\otimes}_{K^*}A^*\widehat{\otimes}_{A^*}R^*\\
@V{id_{M^*}\widehat{\otimes}_{K^*}\hat{\chi}_\gamma}VV @VV{id_{M^*}\widehat{\otimes}_{K^*}\hat{\chi}_\gamma}V\\
M^*\widehat{\otimes}_{K^*}R^* @. N^*\widehat{\otimes}_{K^*}R^*\\
@V{\hat{\imath}_{M^*}\widehat{\otimes}_{K^*}id_{R^*}}VV @AA{id_{N^*}\widehat{\otimes}_{K^*}\hat{m}_{R^*}}A\\
M^*\widehat{\otimes}_{K^*}A^*\widehat{\otimes}_{K^*}R^* @>{\varphi\,\widehat{\otimes}_{K^*}\,id_{R^*}}>> N^*\widehat{\otimes}_{K^*}A^*\widehat{\otimes}_{K^*}R^* @>{id_{N^*}\widehat{\otimes}_{K^*}\gamma\widehat{\otimes}_{K^*}id_{R^*}}>> N^*\widehat{\otimes}_{K^*}R^*\widehat{\otimes}_{K^*}R^*
\end{CD}$$

$$N^*\widehat{\otimes}_{K^*}A^*\widehat{\otimes}_{K^*}R^* \xrightarrow{id_{N^*}\widehat{\otimes}_{K^*}\hat{q}} N^*\widehat{\otimes}_{K^*}A^*\widehat{\otimes}_{A^*}R^*$$

Since χ_γ is an isomorphism, the lower rectangle is also commutative. $\square$

Proposition 13.2.5 *For a morphism a morphism $\lambda : (R^*,\gamma) \to (S^*,\lambda\gamma)$ of $\mathcal{C}_{h_{A^*}}$, the following diagram is commutative.*

$$\begin{CD}
M^*\widehat{\otimes}_{K^*}R^* @>{id_{M^*}\widehat{\otimes}_{K^*}\lambda}>> M^*\widehat{\otimes}_{K^*}S^*\\
@V{\gamma_{\boldsymbol{\varphi}}}VV @VV{(\lambda\gamma)_{\boldsymbol{\varphi}}}V\\
N^*\widehat{\otimes}_{K^*}R^* @>{id_{N^*}\widehat{\otimes}_{K^*}\lambda}>> N^*\widehat{\otimes}_{K^*}S^*
\end{CD}$$

Proof. Since $\widetilde{\varphi} : \widehat{\boldsymbol{M}}\tilde{o}_{h_{A^*}} \to \widehat{\boldsymbol{N}}\tilde{o}_{h_{A^*}}$ is a morphism of h_{A^*}-modules, the following diagram is commutative.

$$\begin{tikzcd}
u^*_{R^*}(\boldsymbol{M}) = \widehat{\boldsymbol{M}}\tilde{o}_{h_{A^*}}(R^*,\gamma) \arrow[r, "\widehat{\boldsymbol{M}}\tilde{o}_{h_{A^*}}(\lambda)"] \arrow[d, "\widetilde{\varphi}_{(R^*,\gamma)} = (id_{R^*}, \gamma_{\varphi})"] & \widehat{\boldsymbol{M}}\tilde{o}_{h_{A^*}}(S^*,\lambda\gamma) = u^*_{S^*}(\boldsymbol{M}) \arrow[d, "\widetilde{\varphi}_{(S^*,\lambda\gamma)} = (id_{S^*}, (\lambda\gamma)_{\varphi})"] \\
u^*_{R^*}(\boldsymbol{N}) = \widehat{\boldsymbol{N}}\tilde{o}_{h_{A^*}}(R^*,\gamma) \arrow[r, "\widehat{\boldsymbol{N}}\tilde{o}_{h_{A^*}}(\lambda)"] & \widehat{\boldsymbol{N}}\tilde{o}_{h_{A^*}}(S^*,\lambda\gamma) = u^*_{S^*}(\boldsymbol{N})
\end{tikzcd}$$

The assertion follows from the commutativity of the above diagram. □

Let $(A^*, \mu, \varepsilon, \iota)$ be a topological Hopf algebra. We denote by G_{A^*} the presheaf on $\mathcal{C}^{op}$ represented by A^* instead of h_{A^*}. Then, G_{A^*} is a group object in $\mathcal{T} = \mathrm{Funct}_r(\mathcal{C}, \mathcal{T}op)$. The following assertion is a direct consequence of (13.2.5).

Proposition 13.2.6 *Let $\boldsymbol{M}$ be an object of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$ and $\boldsymbol{\xi} : u_{A^*}(\boldsymbol{M}) \to u_{A^*}(\boldsymbol{M})$ a morphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{A^*}$. For an object R^* of $\mathcal{C}$, define a map $\alpha(\boldsymbol{\xi})_{R^*} : (M^* \widehat{\otimes}_{K^*} R^*) \times G_{A^*}(R^*) \to M^* \widehat{\otimes}_{K^*} R^*$ by $\alpha(\boldsymbol{\xi})_{R^*}(x,g) = g_{\boldsymbol{\xi}}(x)$. For a morphism $\lambda : R^* \to S^*$ of $\mathcal{C}$, the following diagram is commutative.*

$$\begin{tikzcd}
(M^* \widehat{\otimes}_{K^*} R^*) \times G_{A^*}(R^*) \arrow[r, "\alpha(\boldsymbol{\xi})_{R^*}"] \arrow[d, "(id_{M^*} \widehat{\otimes}_{K^*} \lambda) \times G_{A^*}(\lambda)"] & M^* \widehat{\otimes}_{K^*} R^* \arrow[d, "id_{M^*} \widehat{\otimes}_{K^*} \lambda"] \\
(M^* \widehat{\otimes}_{K^*} S^*) \times G_{A^*}(S^*) \arrow[r, "\alpha(\boldsymbol{\xi})_{S^*}"] & M^* \widehat{\otimes}_{K^*} S^*
\end{tikzcd}$$

Proposition 13.2.7 *The map $\alpha(\boldsymbol{\xi})_{R^*} : (M^* \widehat{\otimes}_{K^*} R^*) \times G_{A^*}(R^*) \to M^* \widehat{\otimes}_{K^*} R^*$ defined in (13.2.6) is a right action of $G_{A^*}(R^*)$ on $M^* \widehat{\otimes}_{K^*} R^*$ for any $R^* \in \mathrm{Ob}\,\mathcal{C}$ if and only if $(\boldsymbol{M}, \boldsymbol{\xi})$ is a representation of A^* on $\boldsymbol{M}$.*

Proof. Suppose that $\alpha(\boldsymbol{\xi})_{R^*}$ is a right action of $G_{A^*}(R^*)$ on $M^* \widehat{\otimes}_{K^*} R^*$ for any object R^* of $\mathcal{C}$. If we put $\hat{h}^{A^*}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{G_{A^*}}, \widetilde{\boldsymbol{\xi}})$ for a morphism $\widetilde{\boldsymbol{\xi}} : \widehat{\boldsymbol{M}}\tilde{o}_{G_{A^*}} \to \widehat{\boldsymbol{M}}\tilde{o}_{G_{A^*}}$ of G_{A^*}-modules, we have $\widetilde{\boldsymbol{\xi}}_{(R^*,g)} = g^{\sharp}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{R^*}, g_{\boldsymbol{\xi}})$ for $(R^*, g) \in \mathrm{Ob}\,\mathcal{C}_{h_{A^*}}$ by (13.2.2). It follows from (13.1.2) and (13.1.4) that $(\boldsymbol{M}, \boldsymbol{\xi})$ is a representation of A^*.

Suppose that $(\boldsymbol{M}, \boldsymbol{\xi})$ is a representation of A^* on $\boldsymbol{M}$. It follows from (13.1.4) that $(\hat{h}(\boldsymbol{M}), \hat{h}^{A^*}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\xi}))$ is a representation of G_{A^*} on $\hat{h}(\boldsymbol{M}) = (h_{K^*}, \widehat{\boldsymbol{M}})$. Hence, for each object R^* of $\mathcal{C}$, $\alpha(\boldsymbol{\xi})_{R^*}$ is a right of $G_{A^*}(R^*)$ on $M^* \widehat{\otimes}_{K^*} R^*$ by (13.1.2). □

Remark 13.2.8 *Put $\boldsymbol{\xi} = (id_{A^*}, \xi)$. $\alpha(\boldsymbol{\xi})_{A^*} : (M^* \widehat{\otimes}_{K^*} A^*) \times G_{A^*}(A^*) \to M^* \widehat{\otimes}_{K^*} A^*$ is regarded as a "generic action" in the sence that the map $M^* \widehat{\otimes}_{K^*} A^* \to M^* \widehat{\otimes}_{K^*} A^*$ defined from the right action of $id_{A^*} \in G_{A^*}(A^*)$ coincides with ξ by (13.2.3).*

Proposition 13.2.9 *Let $\boldsymbol{M} = (K^*, M^*, \kappa)$ be an object of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K^*}$ and*

$$\beta_{R^*} : (M^* \widehat{\otimes}_{K^*} R^*) \times G_{A^*}(R^*) \to M^* \widehat{\otimes}_{K^*} R^*$$

a right R^*-linear $G_{A^*}(R^*)$-action on $M^* \widehat{\otimes}_{K^*} R^*$ which is natural in $R^* \in \mathrm{Ob}\,\mathcal{C}$. Suppose that $M^* \otimes_{K^*} R^*$ is complete (M^* is finite dimensional for example) or both β_{A^*} and $\alpha_{R^*}(\boldsymbol{\xi})$ are continuous. If we define a map $\xi : M^* \widehat{\otimes}_{K^*} A^* \to M^* \widehat{\otimes}_{K^*} A^*$ by $\xi(x) = \beta_{A^*}(x, id_{A^*})$ and put $\boldsymbol{\xi} = (id_{A^*}, \xi) : u^*_{A^*}(\boldsymbol{M}) \to u^*_{A^*}(\boldsymbol{M})$, then the map $\alpha_{R^*}(\boldsymbol{\xi}) : (M^* \widehat{\otimes}_{K^*} R^*) \times G_{A^*}(R^*) \to M^* \widehat{\otimes}_{K^*} R^*$ defined in (13.2.6) coincides with β_{R^*}.

Proof. It follow from (13.2.8) that $\beta_{A^*}(x, id_{A^*}) = \xi(x) = \alpha_{A^*}(\boldsymbol{\xi})(x, id_{A^*})$ for $x \in M^* \widehat{\otimes}_{K^*} A^*$. For $R^* \in \mathrm{Ob}\,\mathcal{C}$ and $g \in G_{A^*}(R^*)$, the following diagram is commutative by the naturality of β_{R^*} and α_{R^*}.

$$\begin{tikzcd}
M^* \widehat{\otimes}_{K^*} A^* \arrow[d, "id_{M^*} \widehat{\otimes}_{K^*} g"] & (M^* \widehat{\otimes}_{K^*} A^*) \times G_{A^*}(A^*) \arrow[l, "\beta_{A^*}"'] \arrow[r, "\alpha(\boldsymbol{\xi})_{A^*}"] \arrow[d, "(id_{M^*} \widehat{\otimes}_{K^*} g) \times G_{A^*}(g)"] & M^* \widehat{\otimes}_{K^*} A^* \arrow[d, "id_{M^*} \widehat{\otimes}_{K^*} g"] \\
M^* \widehat{\otimes}_{K^*} R^* & (M^* \widehat{\otimes}_{K^*} R^*) \times G_{A^*}(R^*) \arrow[l, "\beta_{R^*}"'] \arrow[r, "\alpha(\boldsymbol{\xi})_{R^*}"] & M^* \widehat{\otimes}_{K^*} R^*
\end{tikzcd}$$

For $a \in M^*$ and $r \in R*$, we have the following equality. Here $\eta_{M^*\otimes_{K^*}R^*} : M^* \otimes_{K^*} R^* \to M^* \widehat{\otimes}_{K^*} R^*$ is the canonical map.

$$\begin{aligned}
\beta_{R^*}(\eta_{M^*\otimes_{K^*}R^*}(a \otimes r), g) &= \beta_{R^*}(\eta_{M^*\otimes_{K^*}R^*}(a \otimes 1)r, g) = \beta_{R^*}(\eta_{M^*\otimes_{K^*}R^*}(a \otimes g(1)), G_{A^*}(g)(id_{A^*}))r \\
&= \beta_{R^*}((id_{M^*} \widehat{\otimes}_{K^*} g) \times G_{A^*}(g))(\eta_{M^*\otimes_{K^*}R^*}(a \otimes 1), id_{A^*})r \\
&= (id_{M^*} \widehat{\otimes}_{K^*} g)\beta_{A^*}(\eta_{M^*\otimes_{K^*}R^*}(a \otimes 1), id_{A^*})r \\
&= (id_{M^*} \widehat{\otimes}_{K^*} g)\alpha(\boldsymbol{\xi})_{A^*}(\eta_{M^*\otimes_{K^*}R^*}(a \otimes 1), id_{A^*})r \\
&= \alpha_{R^*}(\boldsymbol{\xi})((id_{M^*} \widehat{\otimes}_{K^*} g) \times G_{A^*}(g))(\eta_{M^*\otimes_{K^*}R^*}(a \otimes 1), id_{A^*})r \\
&= \alpha_{R^*}(\boldsymbol{\xi})(\eta_{M^*\otimes_{K^*}R^*}(a \otimes g(1)), G_{A^*}(g)(id_{A^*}))r \\
&= \alpha_{R^*}(\boldsymbol{\xi})(\eta_{M^*\otimes_{K^*}R^*}(a \otimes 1)r, g) = \alpha_{R^*}(\boldsymbol{\xi})(\eta_{M^*\otimes_{K^*}R^*}(a \otimes r), g)
\end{aligned}$$

Hence the restriction of β_{R^*} to $\operatorname{Im}\eta_{M^*\otimes_{K^*}R^*} \times G_{A^*}(R^*)$ which is a dense subspace of $M^* \widehat{\otimes}_{K^*} R^*$ coincides with that of $\alpha_{R^*}(\boldsymbol{\xi})$. Hence the assertion follows if $M^* \otimes_{K^*} R^*$ is complete or both β_{R^*} and $\alpha_{R^*}(\boldsymbol{\xi})$ are continuous.□

Proposition 13.2.10 *Let $(\boldsymbol{M}, \boldsymbol{\xi})$ and $(\boldsymbol{N}, \boldsymbol{\zeta})$ be representations of A^* on $\boldsymbol{M} = (K^*, M^*, \alpha)$ and $\boldsymbol{N} = (K^*, N^*, \beta)$, respectively. For a morphism $\boldsymbol{\varphi} = (id_{K^*}, \varphi) : (\boldsymbol{N}, \boldsymbol{\zeta}) \to (\boldsymbol{M}, \boldsymbol{\xi})$ of representations of A^* and an object R^* of $\mathcal{C}$, the following diagram is commutative.*

$$\begin{CD}
(M^* \widehat{\otimes}_{K^*} R^*) \times G_{A^*}(R^*) @>{\alpha(\boldsymbol{\xi})_{R^*}}>> M^* \widehat{\otimes}_{K^*} R^* \\
@V{(\varphi \widehat{\otimes}_{K^*} id_{R^*}) \times id_{G_{A^*}(R^*)}}VV @VV{\varphi \widehat{\otimes}_{K^*} id_{R^*}}V \\
(N^* \widehat{\otimes}_{K^*} R^*) \times G_{A^*}(R^*) @>{\alpha(\boldsymbol{\zeta})_{R^*}}>> N^* \widehat{\otimes}_{K^*} R^*
\end{CD}$$

Proof. It follows from (13.1.5) that $\hat{h}(\boldsymbol{\varphi}) : (\hat{h}(\boldsymbol{M}), \hat{h}^{A^*}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\xi})) \to (\hat{h}(\boldsymbol{N}), \hat{h}^{A^*}_{\boldsymbol{N},\boldsymbol{N}}(\boldsymbol{\zeta}))$ is a morphism of representations of G_{A^*}. Recall that we put $\hat{h}^{A^*}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{G_{A^*}}, \widetilde{\boldsymbol{\xi}})$, $\hat{h}^{A^*}_{\boldsymbol{N},\boldsymbol{N}}(\boldsymbol{\zeta})) = (id_{G_{A^*}}, \widetilde{\boldsymbol{\zeta}})$ and $\hat{h}(\boldsymbol{\varphi}) = (id_{G_{A^*}}, \widehat{\boldsymbol{\varphi}})$. Then, the following diagram is commutative for any object (R^*, g) of $\mathcal{C}_{G_{A^*}}$.

$$\begin{CD}
\widehat{\boldsymbol{M}}\tilde{o}_{G_{A^*}}(R^*, g) @>{\widetilde{\boldsymbol{\xi}}_{(R^*,g)}}>> \widehat{\boldsymbol{N}}\tilde{o}_{G_{A^*}}(R^*, g) \\
@V{\widehat{\boldsymbol{\varphi}}_{(R^*,g)}}VV @VV{\widehat{\boldsymbol{\varphi}}_{(R^*,g)}}V \\
\widehat{\boldsymbol{M}}\tilde{o}_{G_{A^*}}(R^*, g) @>{\widetilde{\boldsymbol{\zeta}}_{(R^*,g)}}>> \widehat{\boldsymbol{N}}\tilde{o}_{G_{A^*}}(R^*, g)
\end{CD}$$

Since $\widetilde{\boldsymbol{\xi}}_{(R^*,g)} = (id_{R^*}, g_{\boldsymbol{\xi}})$, $\widetilde{\boldsymbol{\zeta}}_{(R^*,g)} = (id_{R^*}, g_{\boldsymbol{\zeta}})$ and $\widehat{\boldsymbol{\varphi}}_{(R^*,g)} = (id_{R^*}, \varphi \widehat{\otimes}_{K^*} id_{R^*})$, we have $(\varphi \widehat{\otimes}_{K^*} id_{R^*})g_{\boldsymbol{\xi}} = g_{\boldsymbol{\zeta}}(\varphi \widehat{\otimes}_{K^*} id_{R^*})$ by the commutativity of the above diagram. This implies the result. □

Proposition 13.2.11 *Let $f : A^* \to B^*$ be a morphism of Hopf algebras and $(\boldsymbol{M}, \boldsymbol{\xi})$ a representation of A^* on $\boldsymbol{M} = (K^*, M^*, \alpha)$. We denote by $G_f : G_{B^*} \to G_{A^*}$ the morphism h_f of presheaves induced by f. Then, for each object R^* of $\mathcal{C}$, the following diagram is commutative.*

$$\begin{array}{ccc}
(M^* \widehat{\otimes}_{K^*} R^*) \times G_{B^*}(R^*) & \xrightarrow{\alpha(\boldsymbol{\xi}_f)_{R^*}} & M^* \widehat{\otimes}_{K^*} R^* \\
\Big\downarrow{\scriptstyle id_{M^* \widehat{\otimes}_{K^*} R^*} \times G_{fR^*}} & \nearrow{\scriptstyle \alpha(\boldsymbol{\xi})_{R^*}} & \\
(M^* \widehat{\otimes}_{K^*} R^*) \times G_{A^*}(R^*) & &
\end{array}$$

Proof. We have $\hat{h}^{B^*}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\xi}_f) = \hat{h}^{A^*}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\xi})_{G_f} : h^*_{u_{B^*}}(\hat{h}(\boldsymbol{M})) \to h^*_{u_{B^*}}(\hat{h}(\boldsymbol{M}))$ by (11.1.12). On the other hand, since $\hat{h}^{B^*}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\xi}_f) = (id_{G_{B^*}}, \widetilde{\boldsymbol{\xi}_f})$ and $\hat{h}^{A^*}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\xi})_{G_f} = (id_{G_{B^*}}, G^*_f(\widetilde{\boldsymbol{\xi}}))$, it follows $\widetilde{\boldsymbol{\xi}_f}_{(R^*,g)} = G^*_f(\widetilde{\boldsymbol{\xi}})_{(R^*,g)} = \widetilde{\boldsymbol{\xi}}_{(R^*, G_{fR^*}(g))}$ for any object (R^*, g) of $\mathcal{C}_{G_{B^*}}$. Hence we have $g_{\boldsymbol{\xi}_f} = G_{fR^*}(g)_{\boldsymbol{\xi}}$ which implies the assertion. □

Theorem 13.2.12 *Suppose that K^* is a field. Let A^* be a complete Hopf algebra over K^* such that the coproduct $\mu : A^* \to A^* \widehat{\otimes}_{K^*} A^*$ lifts to $\tilde{\mu} : A^* \to A^* \otimes_{K^*} A^*$ and $(\boldsymbol{M}, \boldsymbol{\xi})$ a representation of A^* on $\boldsymbol{M} = (K^*, M^*, \alpha)$ such that M^* is profinite. Suppose that there exists a family $\{(\boldsymbol{M}_i, \boldsymbol{\xi}_i)\}_{i\in I}$ of subrepresentations of $(\boldsymbol{M}, \boldsymbol{\xi})$ such that if we put $\boldsymbol{M}_i = (K^*, M^*_i, \alpha_i)$, $\{M^*_i \,|\, i \in I\}$ is a cofinal subset of $\mathcal{N}_{M^*}$. Then, the right action $\alpha(\boldsymbol{\xi})_{R^*} : (M^* \widehat{\otimes}_{K^*} R^*) \times G_{A^*}(R^*) \to M^* \widehat{\otimes}_{K^*} R^*$ is continuous for any profinite K^*-algebra R^*.*

Proof. If M^* and R^* are both finite and A^* is finitely generated, then both $G_{A^*}(R^*)$ and $M^* \mathbin{\widehat{\otimes}}_{K^*} R^*$ are discrete, hence $\alpha(\boldsymbol{\xi})_{R^*}$ is continuous. Let $\bar{\xi}_i : M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} A^* \to M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} A^*$ be the unique map that makes the following diagram commute, where $p_i : M^* \to M^*/M_i^*$ is the quotient map.

$$\begin{tikzcd}
M^* \mathbin{\widehat{\otimes}}_{K^*} A^* \arrow[r, "\xi"] \arrow[d, "p_i \mathbin{\widehat{\otimes}}_{K^*} id_{A^*}"] & M^* \mathbin{\widehat{\otimes}}_{K^*} A^* \arrow[d, "p_i \mathbin{\widehat{\otimes}}_{K^*} id_{A^*}"] \\
M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} A^* \arrow[r, "\bar{\xi}_i"] & M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} A^*
\end{tikzcd}$$

Let us denote by $\bar{\alpha}_i$ the K^*-module structure of M^*/M_i^*. We have a representation $(\bar{\boldsymbol{M}}_i, \bar{\boldsymbol{\xi}}_i)$ of A^* on $\bar{\boldsymbol{M}}_i = (K^*, M^*/M_i^*, \bar{\alpha}_i)$ such that $(id_{K^*}, p_{N^*}) : (\boldsymbol{M}, \boldsymbol{\xi}) \to (\bar{\boldsymbol{M}}_i, \bar{\boldsymbol{\xi}}_i)$ is a morphism of representations of A^*. Hence the following diagram is commutative by (13.2.10).

$$\begin{tikzcd}
(M^* \mathbin{\widehat{\otimes}}_{K^*} R^*) \times G_{A^*}(R^*) \arrow[r, "\alpha(\boldsymbol{\xi})_{R^*}"] \arrow[d, "(p_i \mathbin{\widehat{\otimes}}_{K^*} id_{R^*}) \times id_{G_{A^*}(R^*)}"] & M^* \mathbin{\widehat{\otimes}}_{K^*} R^* \arrow[d, "p_i \mathbin{\widehat{\otimes}}_{K^*} id_{R^*}"] \\
(M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} R^*) \times G_{A^*}(R^*) \arrow[r, "\alpha(\bar{\boldsymbol{\xi}}_i)_{R^*}"] & M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} R^*
\end{tikzcd} \quad \cdots (i)$$

By (5.2.11), there exists a finitely generated Hopf subalgebra A_i^* of A^* such that if we denote by $\iota_i : A_i^* \to A^*$ the inclusion map, there exists a map $\tilde{\xi}_i : M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} A_i^* \to M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} A_i^*$ which makes the following diagram commute and $\tilde{\boldsymbol{\xi}}_i = (id_{A_i}, \tilde{\xi}_i)$ defines a representation of A_i^* on $\bar{\boldsymbol{M}}_i$.

$$\begin{tikzcd}
M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} A_i^* \arrow[r, "\tilde{\xi}_i"] \arrow[d, "id_{M^*/M_i^*} \mathbin{\widehat{\otimes}}_{K^*} \iota_i"] & M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} A_i^* \arrow[d, "id_{M^*/M_i^*} \mathbin{\widehat{\otimes}}_{K^*} \iota_i"] \\
M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} A^* \arrow[r, "\bar{\xi}_i"] & M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} A^*
\end{tikzcd}$$

It follows from (12.1.9) that $(id_{A^*}, \bar{\xi}_i)$ is the restriction of $(id_{A^*}, \tilde{\xi}_i)$ by ι_i. Hence the following diagram is commutative by (13.2.11).

$$\begin{tikzcd}
(M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} R^*) \times G_{A^*}(R^*) \arrow[r, "\alpha(\bar{\boldsymbol{\xi}}_i)_{R^*}"] \arrow[d, "id_{M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} R^*} \times G_{\iota_i R^*}"] & M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} R^* \\
(M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} R^*) \times G_{A_i^*}(R^*) \arrow[ur, "\alpha(\tilde{\boldsymbol{\xi}}_i)_{R^*}"'] &
\end{tikzcd} \quad \cdots (ii)$$

Then, the following diagram commutes by (i) and (ii).

$$\begin{tikzcd}
(M^* \mathbin{\widehat{\otimes}}_{K^*} R^*) \times G_{A^*}(R^*) \arrow[r, "\alpha(\boldsymbol{\xi})_{R^*}"] \arrow[d, "p_i \mathbin{\widehat{\otimes}}_{K^*} id_{R^*} \times G_{\iota_i R^*}"] & M^* \mathbin{\widehat{\otimes}}_{K^*} R^* \arrow[d, "p_i \mathbin{\widehat{\otimes}}_{K^*} id_{R^*}"] \\
(M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} R^*) \times G_{A_i^*}(R^*) \arrow[r, "\alpha(\tilde{\boldsymbol{\xi}}_i)_{R^*}"] & M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} R^*
\end{tikzcd} \quad \cdots (iii)$$

As we noted first, $\alpha(\tilde{\boldsymbol{\xi}}_i)_{R^*}$ is continuous if R^* is finite. By (13.2.6) and (2.3.10), $\alpha(\tilde{\boldsymbol{\xi}}_i)_{R^*}$ is continuous if R^* is profinite. We give a partial order $\leqq$ to I by "$i \leqq j$ if and only if $M_i^* \subset M_j^*$". Since

$$\left(M^* \mathbin{\widehat{\otimes}}_{K^*} R^* \xrightarrow{p_i \mathbin{\widehat{\otimes}}_{K^*} id_{R^*}} M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} R^*\right)_{i \in I}$$

is a limiting cone of a functor $D : I \to \mathcal{T}op^*$ which maps $i \in I$ to $M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} R^*$ by (2.3.11) and

$$\left(M^* \mathbin{\widehat{\otimes}}_{K^*} R^* \times G_{A^*}(R^*) \xrightarrow{\alpha(\tilde{\boldsymbol{\xi}}_i)_{R^*}(p_i \mathbin{\widehat{\otimes}}_{K^*} id_{R^*} \times G_{\iota_i R^*})} M^*/M_i^* \mathbin{\widehat{\otimes}}_{K^*} R^*\right)_{i \in I}$$

is a cone of D by the commutativity of (iii), the unique map $\alpha(\boldsymbol{\xi})_{R^*} : M^* \mathbin{\widehat{\otimes}}_{K^*} R^* \times G_{A^*}(R^*) \to M^* \mathbin{\widehat{\otimes}}_{K^*} R^*$ that makes diagram (iii) above commute for any $i \in I$ is continuous. □

Remark 13.2.13 *Let A^* be a 0-connected Hopf algebra with the skeletal topology and M^* a coconnective right A^*-comodule of finite type with the skeletal topology. Since $A^* \otimes_{K^*} A^*$ has the skeletal topology by (2.3.3), A^* satisfies the condition of (13.2.12). Suppose $M^n = \{0\}$ for $n \geq N$. Since $M^{[n]*}$ is an open subcomodule if $n > |N|$ and $\{M^{[n]*} \,|\, n > |N|\}$ is a cofinal subset of $\mathcal{N}_{M^*}$, M^* satisfies the condition of (13.2.12).*

The following is another case that a group action is continuous.

Proposition 13.2.14 *Assume that K^* is a field. Let G be a topological K^*-group functor and M^* a K^*-module. Suppose that $G(R^*)$ and $M^* \otimes_{K^*} R^*$ are discrete if R^* is finite. Let $\alpha_{R^*} : (M^* \widehat{\otimes}_{K^*} R^*) \times G(R^*) \to M^* \widehat{\otimes}_{K^*} R^*$ be a right $G(R^*)$-action which is natural in R^*. If R^* is profinite, α_{R^*} is continuous.*

Proof. If R^* is finite, then both $M^* \otimes_{K^*} R^*$ and $G(R^*)$ are discrete, hence α_{R^*} is continuous. For an open ideal $\mathfrak{a}$ of R^*, let $\pi_{\mathfrak{a}} : R^* \to R^*/\mathfrak{a}$ be the quotient map. The following diagram is commutative by the naturality of α_{R^*}.

$$\begin{array}{ccc}
(M^* \widehat{\otimes}_{K^*} R^*) \times G(R^*) & \xrightarrow{\alpha_{R^*}} & M^* \widehat{\otimes}_{K^*} R^* \\
\Big\downarrow{\scriptstyle (id_{M^*} \widehat{\otimes}_{K^*} \pi_{\mathfrak{a}}) \times G(\pi_{\mathfrak{a}})} & & \Big\downarrow{\scriptstyle id_{M^*} \widehat{\otimes}_{K^*} \pi_{\mathfrak{a}}} \\
(M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a}) \times G(R^*/\mathfrak{a}) & \xrightarrow{\alpha_{R^*/\mathfrak{a}}} & M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a}
\end{array}$$

Hence $(id_{M^*} \widehat{\otimes}_{K^*} \pi_{\mathfrak{a}})\alpha_{R^*}$ is continuous for any open ideal $\mathfrak{a}$ of R^*. If R^* is profinite, it follows from (2.3.9) that

$$\big(M^* \widehat{\otimes}_{K^*} R^* \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} \pi_{R^*}} M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a}\big)_{\mathfrak{a} \in \mathcal{I}_{R^*}}$$

is a limiting cone, which implies the continuity of α_{R^*}. □

Remark 13.2.15 *Assume that G is represented by an topological Hopf algebra and that M^* is finite dimensional. Then, $G(R^*)$ and $M^* \otimes_{K^*} R^*$ are discrete R^* is finite.*

13.3 Fixed points

Let G be a group object in $\mathcal{T}$ and ξ, ζ representations of G on an $h_{\boldsymbol{K}^*}$-module M. We put $M(R^*, u_{R^*}) = (R^*, M^*_{R^*}, \alpha_{R^*})$ for $R^* \in \operatorname{Ob}\mathcal{C}$. For a morphism $\lambda : R^* \to S^*$ of $\mathcal{C}$, we put $M_\lambda = p_{\mathcal{M}}(M(\lambda)) : M^*_{R^*} \to M^*_{S^*}$. Define a K^*-submodule $(M^*_{R^*})^{\xi,\zeta}$ of $M^*_{R^*}$ by

$$(M^*_{R^*})^{\xi,\zeta} = \{x \in M^*_{R^*} \,|\, \xi(S^*, g)(M_\lambda(x)) = \zeta(S^*, g)(M_\lambda(x)) \text{ for any } (S^*, g) \in \operatorname{Ob}\mathcal{C}_G \text{ and } \lambda \in \mathcal{C}(R^*, S^*)\}.$$

Lemma 13.3.1 *Let G be a group object in $\mathcal{T}$ represented by a Hopf algebra $(A^*, \mu, \varepsilon, \iota)$ and ξ, ζ representations of G on an $h_{\boldsymbol{K}^*}$-module M. For $R^* \in \operatorname{Ob}\mathcal{C}$, let us denote by $\hat{i}_{A^*} : A^* \to A^* \widehat{\otimes}_{K^*} R^*$ and $\hat{i}_{R^*} : R^* \to A^* \widehat{\otimes}_{K^*} R^*$ the maps given by $\hat{i}_{A^*}(z) = \eta_{A^* \otimes_{K^*} R^*}(z \otimes 1)$ and $\hat{i}_{R^*}(r) = \eta_{A^* \otimes_{K^*} R^*}(1 \otimes r)$, respectively. Then, we have*

$$(M^*_{R^*})^{\xi,\zeta} = \{x \in M^*_{R^*} \,|\, \xi(A^* \widehat{\otimes}_{K^*} R^*, \hat{i}_{A^*})(M_{\hat{i}_{R^*}}(x)) = \zeta(A^* \widehat{\otimes}_{K^*} R^*, \hat{i}_{A^*})(M_{\hat{i}_{R^*}}(x))\}.$$

Proof. It is clear that the right hand side contains the left hand side. Suppose that $x \in M^*_{R^*}$ satisfies $\xi(A^* \widehat{\otimes}_{K^*} R^*, \hat{i}_{A^*})(M_{\hat{i}_{R^*}}(x)) = \zeta(A^* \widehat{\otimes}_{K^*} R^*, \hat{i}_{A^*})(M_{\hat{i}_{R^*}}(x))$. For $(S^*, g) \in \operatorname{Ob}\mathcal{C}_G$ and $\lambda \in \mathcal{C}(R^*, S^*)$, there exists unique K^*-algebra homomorphism $\sigma : A^* \widehat{\otimes}_{K^*} R^* \to S^*$ that satisfies $\sigma\hat{i}_{A^*} = g$ and $\sigma\hat{i}_{R^*} = \lambda$. Then, σ is regarded as a morphism $\sigma : (A^* \widehat{\otimes}_{K^*} R^*, \hat{i}_{A^*}) \to (S^*, g)$ and the following diagrams commute.

$$\begin{array}{ccc}
M^*_{A^* \widehat{\otimes}_{K^*} R^*} & \xrightarrow{\xi(A^* \widehat{\otimes}_{K^*} R^*, \hat{i}_{A^*})} & M^*_{A^* \widehat{\otimes}_{K^*} R^*} \\
\Big\downarrow{\scriptstyle M_\sigma} & & \Big\downarrow{\scriptstyle M_\sigma} \\
M^*_{S^*} & \xrightarrow{\xi(S^*, g)} & M^*_{S^*}
\end{array}
\qquad
\begin{array}{ccc}
M^*_{A^* \widehat{\otimes}_{K^*} R^*} & \xrightarrow{\zeta(A^* \widehat{\otimes}_{K^*} R^*, \hat{i}_{A^*})} & M^*_{A^* \widehat{\otimes}_{K^*} R^*} \\
\Big\downarrow{\scriptstyle M_\sigma} & & \Big\downarrow{\scriptstyle M_\sigma} \\
M^*_{S^*} & \xrightarrow{\zeta(S^*, g)} & M^*_{S^*}
\end{array}$$

Hence we have

$$\begin{aligned}
\xi(S^*, g)(M_\lambda(x)) &= \xi(S^*, g)(M_\sigma(M_{\hat{i}_{R^*}}(x))) = M_\sigma\xi(A^* \widehat{\otimes}_{K^*} R^*, \hat{i}_{A^*})(M_{\hat{i}_{R^*}}(x)) \\
&= M_\sigma\zeta(A^* \widehat{\otimes}_{K^*} R^*, \hat{i}_{A^*})(M_{\hat{i}_{R^*}}(x)) = \zeta(S^*, g)(M_\sigma(M_{\hat{i}_{R^*}}(x))) = \zeta(S^*, g)(M_\lambda(x)),
\end{aligned}$$

which implies $x \in (M^*_{R^*})^{\xi,\zeta}$. □

Proposition 13.3.2 *(1) $(M^*_{R^*})^{\xi,\zeta}$ is an R^*-submodule of $M^*_{R^*}$.*

(2) For a morphism $\lambda : R^* \to S^*$ of $\mathcal{C}$, $M_\lambda : M^*_{R^*} \to M^*_{S^*}$ maps $(M^*_{R^*})^{\xi,\zeta}$ into $(M^*_{S^*})^{\xi,\zeta}$.

(3) For $(R^*, g) \in \mathrm{Ob}\,\mathcal{C}_G$, $\xi(R^*, g) : M^*_{R^*} \to M^*_{R^*}$ maps $(M^*_{R^*})^{\xi,\zeta}$ into $(M^*_{R^*})^{\xi,\zeta}$.

Proof. (1) If $x \in (M^*_{R^*})^{\xi,\zeta}$ and $r \in R^*$, since a diagram

$$\begin{tikzcd}
M^*_{R^*} \otimes_{K^*} R^* \arrow[r, "M_\lambda \otimes_{K^*} \lambda"] \arrow[d, "\alpha_{R^*}"] & M^*_{S^*} \otimes_{K^*} S^* \arrow[r, "\chi(S^*, g) \otimes_{K^*} id_{S^*}"] \arrow[d, "\alpha_{S^*}"] & M^*_{S^*} \otimes_{K^*} S^* \arrow[d, "\alpha_{S^*}"] \\
M^*_{R^*} \arrow[r, "M_\lambda"] & M^*_{S^*} \arrow[r, "\chi(S^*, g)"] & M^*_{S^*}
\end{tikzcd}$$

commutes for $\chi = \xi, \zeta$, $(S^*, g) \in \mathrm{Ob}\,\mathcal{C}_G$ and $\lambda \in \mathcal{C}(R^*, S^*)$, we have

$$\begin{aligned}
\xi(S^*, g)(M_\lambda(\alpha_{R^*}(x \otimes r))) &= \xi(S^*, g)(\alpha_{S^*}(M_\lambda(x) \otimes \lambda(r))) = \alpha_{S^*}(\xi(S^*, g)(M_\lambda(x)) \otimes \lambda(r)) \\
&= \alpha_{S^*}(\zeta(S^*, g)(M_\lambda(x)) \otimes \lambda(r)) = \zeta(S^*, g)(\alpha_{S^*}(M_\lambda(x) \otimes \lambda(r))) \\
&= \zeta(S^*, g)(M_\lambda(\alpha_{R^*}(x \otimes r))).
\end{aligned}$$

Hence $(M^*_{R^*})^{\xi,\zeta}$ is a R^*-submodule of $M^*_{R^*}$.

(2) It is clear from the definition of $(M^*_{R^*})^{\xi,\zeta}$ that $M_\lambda(x) \in (M^*_{S^*})^{\xi,\zeta}$ if $x \in (M^*_{R^*})^{\xi,\zeta}$ and $\lambda \in \mathcal{C}(R^*, S^*)$.

(3) Suppose $x \in (M^*_{R^*})^{\xi,\zeta}$. For a morphism $\lambda : R^* \to S^*$ of $\mathcal{C}$ and $(R^*, g), (S^*, h) \in \mathcal{C}_G$, it follows from

$$\begin{aligned}
\xi(S^*, h)(M_\lambda(\xi(R^*, g)(x))) &= \xi(S^*, h)(\xi(S^*, G(\lambda)(g))(M_\lambda(x)) = \xi(S^*, \mu_{S^*}(h, G(\lambda)(g)))(M_\lambda(x)) \\
&= \zeta(S^*, \mu_{S^*}(h, G(\lambda)(g)))(M_\lambda(x)) = \zeta(S^*, h)(\zeta(S^*, G(\lambda)(g))(M_\lambda(x))) \\
&= \zeta(S^*, h)(M_\lambda(\zeta(R^*, g)(x))) = \zeta(S^*, h)(M_\lambda(\xi(R^*, g)(x)))
\end{aligned}$$

that $M_\lambda(\xi(R^*, g)(x)) \in (M^*_{R^*})^{\xi,\zeta}$. $\square$

Proposition 13.3.3 *Let ξ, ζ representations of G on M and ξ', ζ' representations of G on N. If a morphism $\boldsymbol{\varphi} : M \to N$ of h_{K^*}-modules gives morphisms $\xi \to \xi'$ and $\zeta \to \zeta'$, then $\varphi_{R^*} = p_{\mathcal{M}}(\boldsymbol{\varphi}_{R^*})$ maps $(M^*_{R^*})^{\xi,\zeta}$ into $(N^*_{R^*})^{\xi',\zeta'}$ for any $R^* \in \mathrm{Ob}\,\mathcal{C}$.*

Proof. If $x \in (M^*_{R^*})^{\xi,\zeta}$, since diagrams

$$\begin{tikzcd}
M^*_{R^*} \arrow[r, "M_\lambda"] \arrow[d, "\varphi_{R^*}"] & M^*_{S^*} \arrow[r, "\xi(S^*, g)"] \arrow[d, "\varphi_{S^*}"] & M^*_{S^*} \arrow[d, "\varphi_{S^*}"] \\
N^*_{R^*} \arrow[r, "N_\lambda"] & N^*_{S^*} \arrow[r, "\xi'(S^*, g)"] & N^*_{S^*}
\end{tikzcd}
\qquad
\begin{tikzcd}
M^*_{R^*} \arrow[r, "M_\lambda"] \arrow[d, "\varphi_{R^*}"] & M^*_{S^*} \arrow[r, "\zeta(S^*, g)"] \arrow[d, "\varphi_{S^*}"] & M^*_{S^*} \arrow[d, "\varphi_{S^*}"] \\
N^*_{R^*} \arrow[r, "N_\lambda"] & N^*_{S^*} \arrow[r, "\zeta'(S^*, g)"] & N^*_{S^*}
\end{tikzcd}$$

commute for $(S^*, g) \in \mathrm{Ob}\,\mathcal{C}_G$ and $\lambda \in \mathcal{C}(R^*, S^*)$, we have

$$\begin{aligned}
\xi'(S^*, g)(N_\lambda(\varphi_{R^*}(x))) &= \xi'(S^*, g)(\varphi_{S^*}(M_\lambda(x))) = \varphi_{S^*}(\xi(S^*, g)(M_\lambda(x))) = \varphi_{S^*}(\zeta(S^*, g)(M_\lambda(x))) \\
&= \zeta'(S^*, g)(\varphi_{S^*}(M_\lambda(x))) = \zeta'(S^*, g)(M_\lambda(\varphi_{R^*}(x))) = \zeta'(S^*, g)(\varphi_{S^*}(N_\lambda(x))).
\end{aligned}$$

Hence φ_{R^*} maps $(M^*_{R^*})^{\xi,\zeta}$ into $(N^*_{R^*})^{\xi',\zeta'}$. $\square$

Let us denote by $\alpha^{\xi,\zeta}_{R^*} : (M^*_{R^*})^{\xi,\zeta} \otimes_{K^*} R^* \to (M^*_{R^*})^{\xi,\zeta}$ the right R^*-module structure of $(M^*_{R^*})^{\xi,\zeta}$ and by $M^{\xi,\zeta}_\lambda : (M^*_{R^*})^{\xi,\zeta} \to (M^*_{S^*})^{\xi,\zeta}$ the map induced by $M_\lambda : M^*_{R^*} \to M^*_{S^*}$. Thus we have an h_{K^*}-module $M^{\xi,\zeta}$ defined by $M^{\xi,\zeta}(R^*, u_{R^*}) = (R^*, (M^*_{R^*})^{\xi,\zeta}, \alpha^{\xi,\zeta}_{R^*})$ and $M^{\xi,\zeta}(\lambda) = (\lambda, M^{\xi,\zeta}_\lambda)$. The inclusion maps $\iota_{R^*} : (M^*_{R^*})^{\xi,\zeta} \to M^*_{R^*}$ for $R^* \in \mathrm{Ob}\,\mathcal{C}$ define a morphism $\iota : M^{\xi,\zeta} \to M$ of h_{K^*}-modules.

For $(R^*, g) \in \mathrm{Ob}\,\mathcal{C}_G$, define a map $\mu_{\xi,\zeta}(R^*, g) : (M^*_{R^*})^{\xi,\zeta} \to (M^*_{R^*})^{\xi,\zeta}$ by $\mu_{\xi,\zeta}(R^*, g)(x) = \xi(R^*, g)(x)$ and define a morphism $\mu_{\xi,\zeta} : M^{\xi,\zeta}\tilde{o}_G \to M^{\xi,\zeta}\tilde{o}_G$ of G-modules by $(\mu_{\xi,\zeta})_{(R^*,g)} = (id_{R^*}, \mu_{\xi,\zeta}(R^*, g))$. Then, $\mu_{\xi,\zeta}$ is a representation of G on $M^{\xi,\zeta}$ and $\boldsymbol{\iota} : M^{\xi,\zeta} \to M$ defines morphisms of representations $\boldsymbol{\iota}_\xi : \mu_{\xi,\zeta} \to \xi$ and $\boldsymbol{\iota}_\zeta : \mu_{\xi,\zeta} \to \zeta$. $\mu_{\xi,\zeta}$ is a subrepresentation of both ξ and ζ.

Proposition 13.3.4 *Let ξ and ζ be representations of G on M. If θ is a subrepresentation of both ξ and ζ, θ is a subrepresentation of $\mu_{\xi,\zeta}$.*

Proof. Suppose that θ is a representation on a submodule N of M and let $\boldsymbol{\eta} : N \to M$ be the inclusion morphism. Then, the following diagrams commute by the assumption.

$$\begin{tikzcd} M\tilde{o}_G \arrow[d, "\xi"] & N\tilde{o}_G \arrow[l, "\boldsymbol{\eta}_{\tilde{o}_G}"'] \arrow[r, "\boldsymbol{\eta}_{\tilde{o}_G}"] \arrow[d, "\theta"] & M\tilde{o}_G \arrow[d, "\zeta"] \\ M\tilde{o}_G & N\tilde{o}_G \arrow[l, "\boldsymbol{\eta}_{\tilde{o}_G}"'] \arrow[r, "\boldsymbol{\eta}_{\tilde{o}_G}"] & M\tilde{o}_G \end{tikzcd}$$

We set $p_{\mathcal{M}}(N(R^*, u_{R^*})) = N^*_{R^*}$ and $p_{\mathcal{M}}(\boldsymbol{\eta}_{R^*}) = \eta_{R^*}$ for $R^* \in \operatorname{Ob}\mathcal{C}$. If $x \in N^*_{R^*}$, $(S^*, g) \in \mathcal{C}_G$ and $\lambda \in \mathcal{C}(R^*, S^*)$, it follows from the commutativity of the above diagram that

$$\begin{aligned}\xi(S^*, g)(M_\lambda(\eta_{R^*}(x))) &= \xi(S^*, g)(\eta_{S^*}(M_\lambda(x))) = \eta_{S^*}(\theta(S^*, g)(M_\lambda(x))) = \zeta(S^*, g)(\eta_{S^*}(M_\lambda(x))) \\ &= \zeta(S^*, g)(M_\lambda(\eta_{R^*}(x))).\end{aligned}$$

Hence we have $\eta_{R^*}(x) \in (M^*_{R^*})^{\xi,\zeta}$. □

For a group object G in $\mathcal{T}$, we define a functor $T_G : \mathcal{M}od(h_{K^*}) \to \operatorname{Rep}(G\,;\mathcal{M})$ as follows. For an object M of $\mathcal{M}od(h_{K^*})$, $T_G(M) : M\tilde{o}_G \to M\tilde{o}_G$ is the trivial representation. For a morphism $\boldsymbol{\varphi} : M \to N$ of $\mathcal{M}od(h_{K^*})$, it is clear that $\boldsymbol{\varphi}$ gives a morphism from $T_G(M)$ to $T_G(N)$ and set $T_G(\boldsymbol{\varphi}) = \boldsymbol{\varphi}$. We denote by $\pi : \operatorname{Rep}(G\,;\mathcal{M}) \to \mathcal{M}od(h_{K^*})$ the forgetful functor given by $\pi(\xi) = M$ for $(\xi : M\tilde{o}_G \to M\tilde{o}_G) \in \operatorname{Ob}\operatorname{Rep}(G\,;\mathcal{M})$ and $\pi(\boldsymbol{\varphi}) = \boldsymbol{\varphi}$ for $\boldsymbol{\varphi} \in \operatorname{Mor}\operatorname{Rep}(G\,;\mathcal{M})$. Note that πT_G is the identity functor of $\mathcal{M}od(h_{K^*})$.

If τ is the trivial representation of G on M, we denote $M^{\xi,\tau}$ by M^ξ or M^G.

Let ξ and ζ be representations of G on h_{K^*}-modules M and N, respectively. For a morphism $\boldsymbol{\varphi} : \xi \to \zeta$ of $\operatorname{Rep}(G\,;\mathcal{M})$, it follows from (13.3.3) that $\boldsymbol{\varphi}$ induces a homomorphism $\varphi^G_{R^*} : (M^*_{R^*})^{\xi,\tau} \to (N^*_{R^*})^{\zeta,\tau}$ of R^*-modules for each $R^* \in \operatorname{Ob}\mathcal{C}$ and this defines a morphism $\boldsymbol{\varphi}^G : M^\xi \to N^\zeta$ of h_{K^*}-modules. We define a functor $I_G : \operatorname{Rep}(G\,;\mathcal{M}) \to \mathcal{M}od(h_{K^*})$ as follows. Set $I_G(\xi) = \pi(\mu_{\xi,\tau}) = M^\xi$ for $(\xi : M\tilde{o}_G \to M\tilde{o}_G) \in \operatorname{Ob}\operatorname{Rep}(G\,;\mathcal{M})$ and $I_G(\boldsymbol{\varphi}) = \boldsymbol{\varphi}^G$ for $\boldsymbol{\varphi} \in \operatorname{Mor}\operatorname{Rep}(G\,;\mathcal{M})$.

Proposition 13.3.5 *I_G is a right adjoint of T_G.*

Proof. Let M be an h_{K^*}-module and put $M(R^*, u_{R^*}) = (R^*, M^*_{R^*}, \alpha_{R^*})$ for $R^* \in \operatorname{Ob}\mathcal{C}$. Consider the trivial representation of G on M. Then, $(M^*_{R^*})^\xi = M^*_{R^*}$ for any $R^* \in \operatorname{Ob}\mathcal{C}$ thus we have $I_G(T_G(M)) = M$. Let $\xi : M\tilde{o}_G \to M\tilde{o}_G$ be a representation of G on $M \in \operatorname{Ob}\mathcal{M}od(h_{K^*})$. For $R^* \in \operatorname{Ob}\mathcal{C}$, we denote by $\iota_{R^*} : (M^*_{R^*})^\xi \to M^*_{R^*}$ the inclusion map. If $x \in (M^*_{R^*})^\xi$ and $(R^*, g) \in \mathcal{C}_G$, then

$$\xi(R^*, g)(\iota_{R^*}(x)) = \iota_{R^*}(x) = \iota_{R^*}(T_G(M^\xi)(R^*, g)(x)).$$

Therefore a morphism $M^\xi \to M$ of h_{K^*}-modules induced by ι_{R^*} gives a morphism $\varepsilon_\xi : T_G(I_G(\xi)) \to \xi$ of $\operatorname{Rep}(G\,;\mathcal{M})$. It is clear that ε_ξ is natural in ξ and that $I_G(\varepsilon_\xi)$ is the identity morphism of $I_G(\xi)$. Moreover, $\varepsilon_{T_G(M)}$ is the identity morphism of $T_G(M)$ for $M \in \operatorname{Ob}\mathcal{M}od(h_{K^*})$. Hence I_G is a right adjoint of T_G. □

Let G_{A^*} be a topological affine group scheme represented by a Hopf algebra $(A^*, \mu, \varepsilon, \iota)$ and $(\boldsymbol{M}, \boldsymbol{\xi})$, $(\boldsymbol{M}, \boldsymbol{\zeta})$ representations of A^* on $\boldsymbol{M} = (K^*, M^*, \alpha)$. We put $\boldsymbol{\xi} = (id_{A^*}, \xi)$, $\boldsymbol{\zeta} = (id_{A^*}, \zeta)$ and denote by $\hat{\xi}, \hat{\zeta} : M^* \to M^* \mathbin{\widehat{\otimes}}_{K^*} A^*$ the right A^*-comodule structure associated with $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, respectively. We put $\hat{h}^{A^*}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{h_{A^*}}, \widetilde{\boldsymbol{\xi}})$ and $\hat{h}^{A^*}_{\boldsymbol{M},\boldsymbol{M}}(\boldsymbol{\zeta}) = (id_{h_{A^*}}, \widetilde{\boldsymbol{\zeta}})$. Then, for $(R^*, g) \in \operatorname{Ob}\mathcal{C}_{h_{A^*}}$, we have $\widetilde{\boldsymbol{\xi}}_{(R^*,g)} = (id_{R^*}, \tilde{\xi}(R^*, g))$, $\widetilde{\zeta}_{(R^*,g)} = (id_{R^*}, \tilde{\zeta}(R^*, g))$ where $\tilde{\xi}(R^*, g), \tilde{\zeta}(R^*, g) : M^* \mathbin{\widehat{\otimes}}_{K^*} R^* \to M^* \mathbin{\widehat{\otimes}}_{K^*} R^*$ are homomorphisms right R^*-modules which are the following compositions, respectively.

$$M^* \mathbin{\widehat{\otimes}}_{K^*} R^* \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} \hat{\chi}_g^{-1}} M^* \mathbin{\widehat{\otimes}}_{K^*} A^* \mathbin{\widehat{\otimes}}_{A^*} R^* \xrightarrow{\xi \widehat{\otimes}_{A^*} id_{R^*}} M^* \mathbin{\widehat{\otimes}}_{K^*} A^* \mathbin{\widehat{\otimes}}_{A^*} R^* \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} \hat{\chi}_g} M^* \mathbin{\widehat{\otimes}}_{K^*} R^*$$

$$M^* \mathbin{\widehat{\otimes}}_{K^*} R^* \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} \hat{\chi}_g^{-1}} M^* \mathbin{\widehat{\otimes}}_{K^*} A^* \mathbin{\widehat{\otimes}}_{A^*} R^* \xrightarrow{\zeta \widehat{\otimes}_{A^*} id_{R^*}} M^* \mathbin{\widehat{\otimes}}_{K^*} A^* \mathbin{\widehat{\otimes}}_{A^*} R^* \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} \hat{\chi}_g} M^* \mathbin{\widehat{\otimes}}_{K^*} R^*$$

Here, $\hat{\chi}_g : A^* \mathbin{\widehat{\otimes}}_{A^*} R^* \to R^*$ is the isomorphism induced by the isomorphism $A^* \otimes_{A^*} R^* \to R^*$ which maps $a \otimes r$ to $g(a)r$.

Proposition 13.3.6 *For $R^* \in \operatorname{Ob}\mathcal{C}$, $(M^* \mathbin{\widehat{\otimes}}_{K^*} R^*)^{\boldsymbol{\xi},\boldsymbol{\zeta}}$ is the kernel of*

$$(\hat{\xi} - \hat{\zeta}) \mathbin{\widehat{\otimes}}_{K^*} id_{R^*} : M^* \mathbin{\widehat{\otimes}}_{K^*} R^* \to M^* \mathbin{\widehat{\otimes}}_{K^*} A^* \mathbin{\widehat{\otimes}}_{K^*} R^*.$$

Proof. Recall that $x \in (M^* \widehat{\otimes}_{K^*} R^*)^{\xi,\zeta}$ if and only if $\tilde{\xi}(S^*,g)((id_{M^*} \widehat{\otimes}_{K^*} \lambda)(x)) = \tilde{\zeta}(S^*,g)((id_{M^*} \widehat{\otimes}_{K^*} \lambda)(x))$ for any $(S^*,g) \in \mathrm{Ob}\,\mathcal{C}_{h_{A^*}}$ and $\lambda \in \mathcal{C}(R^*,S^*)$.

For $(S^*,g) \in \mathrm{Ob}\,\mathcal{C}_{h_{A^*}}$, let $\hat{q} : A^* \widehat{\otimes}_{K^*} S^* \to A^* \widehat{\otimes}_{A^*} S^*$ the map given in the proof of (13.2.4). Then the following diagrams are commutative for $\lambda \in \mathcal{C}(R^*,S^*)$.

$$\begin{tikzcd}
M^* \widehat{\otimes}_{K^*} R^* \arrow[r, "\hat{\xi}\,\widehat{\otimes}_{K^*} id_{R^*}"] \arrow[d, "id_{M^*} \widehat{\otimes}_{K^*} \lambda"] & M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^* \arrow[r, "id_{M^*} \widehat{\otimes}_{K^*} id_{A^*} \widehat{\otimes}_{K^*} \lambda"] & M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} S^* \arrow[d, "id_{M^*} \widehat{\otimes}_{K^*} \hat{q}"] \\
M^* \widehat{\otimes}_{K^*} S^* \arrow[r, "id_{M^*} \widehat{\otimes}_{K^*} \hat{\chi}_g^{-1}"] & M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{A^*} S^* \arrow[r, "\xi\,\widehat{\otimes}_{A^*} id_{S^*}"] & M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{A^*} S^*
\end{tikzcd}$$

$$\begin{tikzcd}
M^* \widehat{\otimes}_{K^*} R^* \arrow[r, "\hat{\zeta}\,\widehat{\otimes}_{K^*} id_{R^*}"] \arrow[d, "id_{M^*} \widehat{\otimes}_{K^*} \lambda"] & M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^* \arrow[r, "id_{M^*} \widehat{\otimes}_{K^*} id_{A^*} \widehat{\otimes}_{K^*} \lambda"] & M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} S^* \arrow[d, "id_{M^*} \widehat{\otimes}_{K^*} \hat{q}"] \\
M^* \widehat{\otimes}_{K^*} S^* \arrow[r, "id_{M^*} \widehat{\otimes}_{K^*} \hat{\chi}_g^{-1}"] & M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{A^*} S^* \arrow[r, "\zeta\,\widehat{\otimes}_{A^*} id_{S^*}"] & M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{A^*} S^*
\end{tikzcd}$$

It follows from the commutativity of the above diagram and the fact that $\hat{\chi}_g$ is an isomorphism that $x \in M^* \widehat{\otimes}_{K^*} R^*$ satisfies $\tilde{\xi}(S^*,g)((id_{M^*} \widehat{\otimes}_{K^*} \lambda)(x)) = \tilde{\zeta}(S^*,g)((id_{M^*} \widehat{\otimes}_{K^*} \lambda)(x))$ if and only if

$$(id_{M^*} \widehat{\otimes}_{K^*} \hat{q}(id_{A^*} \widehat{\otimes}_{K^*} \lambda))(\hat{\xi}\,\widehat{\otimes}_{K^*} id_{R^*})(x) = (id_{M^*} \widehat{\otimes}_{K^*} \hat{q}(id_{A^*} \widehat{\otimes}_{K^*} \lambda))(\hat{\zeta}\,\widehat{\otimes}_{K^*} id_{R^*})(x) \quad \cdots (*)$$

Therefore $x \in (M^* \widehat{\otimes}_{K^*} R^*)^{\xi,\zeta}$ if $x \in M^* \widehat{\otimes}_{K^*} R^*$ satisfies $(\hat{\xi}\,\widehat{\otimes}_{K^*} id_{R^*})(x) = (\hat{\zeta}\,\widehat{\otimes}_{K^*} id_{R^*})(x)$. Conversely, suppose that $x \in (M^* \widehat{\otimes}_{K^*} R^*)^{\xi,\zeta}$. Let $\lambda : R^* \to A^* \widehat{\otimes}_{K^*} R^*$ be the map induced by a map $R^* \to A^* \otimes_{K^*} R^*$ given by $r \mapsto 1 \otimes r$. Then, the following composition is the identity map of $A^* \widehat{\otimes}_{K^*} R^*$.

$$A^* \widehat{\otimes}_{K^*} R^* \xrightarrow{id_{A^*} \widehat{\otimes}_{K^*} \lambda} A^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^* \xrightarrow{\hat{q}} A^* \widehat{\otimes}_{A^*} A^* \widehat{\otimes}_{K^*} R^* \xrightarrow{\hat{\chi}_{id_{A^*}} \widehat{\otimes}_{K^*} id_{R^*}} A^* \widehat{\otimes}_{K^*} R^*$$

Hence $(*)$ implies $(\hat{\xi}\,\widehat{\otimes}_{K^*} id_{R^*})(x) = (\hat{\zeta}\,\widehat{\otimes}_{K^*} id_{R^*})(x)$. □

In particular, $x \in M^*$ belongs to $(M^*)^{\xi,\zeta}$ if and only if $\hat{\xi}(x) = \hat{\zeta}(x)$. Hence the map $\hat{\imath}_{M^*} : M^* \to M^* \widehat{\otimes}_{K^*} R^*$ given by $\hat{\imath}_{M^*}(x) = \eta_{M^* \widehat{\otimes}_{K^*} R^*}(x \otimes 1)$ maps $(M^*)^{\xi,\zeta}$ into $(M^* \widehat{\otimes}_{K^*} R^*)^{\xi,\zeta}$. Since $(M^* \widehat{\otimes}_{K^*} R^*)^{\xi,\zeta}$ is a closed subspace of $M^* \widehat{\otimes}_{K^*} R^*$ by the continuity of $\hat{\xi}$ and $\hat{\zeta}$, $(M^* \widehat{\otimes}_{K^*} R^*)^{\xi,\zeta}$ is complete Hausdorff. Thus we have a map $\hat{\imath}^{\xi,\zeta}_{R^*} : (M^*)^{\xi,\zeta} \widehat{\otimes}_{K^*} R^* \to (M^* \widehat{\otimes}_{K^*} R^*)^{\xi,\zeta}$.

If R^* is flat over K^*, since $0 \to (M^*)^{\xi,\zeta} \xrightarrow{\hat{\imath}_{K^*}} M^* \xrightarrow{\hat{\xi}-\hat{\zeta}} M^* \widehat{\otimes}_{K^*} A^*$ is exact, we have an exact sequence

$$0 \longrightarrow (M^*)^{\xi,\zeta} \otimes_{K^*} R^* \xrightarrow{\hat{\imath}_{K^*} \otimes_{K^*} id_{R^*}} M^* \otimes_{K^*} R^* \xrightarrow{(\hat{\xi}-\hat{\zeta}) \otimes_{K^*} id_{R^*}} (M^* \widehat{\otimes}_{K^*} A^*) \otimes_{K^*} R^*.$$

Proposition 13.3.7 *Let R^* be an object of $\mathcal{C}$. Suppose that there exists a cofinal subset J of $\mathcal{I}_{R^*}$ such that $R^*/\mathfrak{a}$ is a finitely generated and free K^*-module for any $\mathfrak{a} \in J$. Then $\hat{\imath}^{\xi,\zeta}_{R^*} : (M^*)^{\xi,\zeta} \widehat{\otimes}_{K^*} R^* \to (M^* \widehat{\otimes}_{K^*} R^*)^{\xi,\zeta}$ is an isomorphism.*

Proof. If R^* is discrete, $\hat{\imath}^{\xi,\zeta}_{R^*}$ is regarded as a map from $(M^*)^{\xi,\zeta} \otimes_{K^*} R^*$ to $(M^* \otimes_{K^*} R^*)^{\xi,\zeta}$. Hence it follows from the above exact sequence that $\hat{\imath}^{\xi,\zeta}_{R^*/\mathfrak{a}} : (M^*)^{\xi,\zeta} \widehat{\otimes}_{K^*} R^*/\mathfrak{a} \to (M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a})^{\xi,\zeta}$ is an isomorphism for any $\mathfrak{a} \in J$. Let $q_{\mathfrak{a}} : R^* \to R^*/\mathfrak{a}$ be the quotient map. Since the right rectangle of the following diagram commutes and the upper and lower raws are exact, there exists a morphism $\hat{q}_{\mathfrak{a}} : (M^* \widehat{\otimes}_{K^*} R^*)^{\xi,\zeta} \to (M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a})^{\xi,\zeta}$ which makes the left rectangle of the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & (M^* \widehat{\otimes}_{K^*} R^*)^{\xi,\zeta} \arrow[r, "j_{R^*}"] \arrow[d, "\hat{q}_{\mathfrak{a}}"] & M^* \widehat{\otimes}_{K^*} R^* \arrow[r, "(\hat{\xi}-\hat{\zeta})\,\widehat{\otimes}_{K^*} id_{R^*}"] \arrow[d, "id_{M^*} \widehat{\otimes}_{K^*} q_{\mathfrak{a}}"] & M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^* \arrow[d, "id_{M^*} \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} q_{\mathfrak{a}}"] \\
0 \arrow[r] & (M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a})^{\xi,\zeta} \arrow[r, "j_{R^*/\mathfrak{a}}"] & M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a} \arrow[r, "(\hat{\xi}-\hat{\zeta})\,\widehat{\otimes}_{K^*} id_{R^*/\mathfrak{a}}"] & M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a}
\end{tikzcd}$$

Consider functors $D_1, D_2, D_3 : J \to \mathcal{T}op\mathcal{M}od_{K^*}$ defined by $D_1(\mathfrak{a}) = M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a}$, $D_2(\mathfrak{a}) = M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a}$ and $D_3(\mathfrak{a}) = (M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a})^{\xi,\zeta}$. Then, $\left((M^* \widehat{\otimes}_{K^*} R^*)^{\xi,\zeta} \xrightarrow{\hat{q}_{\mathfrak{a}}} (M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a})^{\xi,\zeta}\right)_{\mathfrak{a} \in J}$ is a cone of D_3. By (2.3.11),

$$\left(M^* \widehat{\otimes}_{K^*} R^* \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} q_{\mathfrak{a}}} M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a}\right)_{\mathfrak{a} \in J}$$

and

$$\left(M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^* \xrightarrow{id_{M^* \widehat{\otimes}_{K^*} A^*} \widehat{\otimes}_{K^*} q_{\mathfrak{a}}} M^* \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a}\right)_{\mathfrak{a}\in J}$$

are limiting cones of D_1 and D_2, respectively. Let $\left(N^* \xrightarrow{\rho_{\mathfrak{a}}} (M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a})^{\boldsymbol{\xi},\boldsymbol{\zeta}}\right)_{\mathfrak{a}\in J}$ be a cone of D_3. Then, $\left(N^* \xrightarrow{j_{R^*/\mathfrak{a}}\rho_{\mathfrak{a}}} M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a}\right)_{\mathfrak{a}\in J}$ is a cone of D_1 and there exists unique morphism $\varphi : N^* \to M^* \widehat{\otimes}_{K^*} R^*$ satisfying $(id_{M^*} \widehat{\otimes}_{K^*} q_{\mathfrak{a}})\varphi = j_{R^*/\mathfrak{a}}\rho_{\mathfrak{a}}$ for any $\mathfrak{a} \in J$. Since

$$\begin{aligned}(id_{M^*} \widehat{\otimes}_{K^*} A^* \widehat{\otimes}_{K^*} q_{\mathfrak{a}})((\hat{\xi} - \hat{\zeta}) \widehat{\otimes}_{K^*} id_{R^*})\varphi &= ((\hat{\xi} - \hat{\zeta}) \widehat{\otimes}_{K^*} id_{R^*/\mathfrak{a}})(id_{M^*} \widehat{\otimes}_{K^*} q_{\mathfrak{a}})\varphi \\ &= ((\hat{\xi} - \hat{\zeta}) \widehat{\otimes}_{K^*} id_{R^*/\mathfrak{a}})j_{R^*/\mathfrak{a}}\rho_{\mathfrak{a}} = 0\end{aligned}$$

for any $\mathfrak{a} \in J$, we have $((\hat{\xi} - \hat{\zeta}) \widehat{\otimes}_{K^*} id_{R^*})\varphi = 0$ and this implies that there exists unique morphism $\tilde{\varphi} : N^* \to (M^* \widehat{\otimes}_{K^*} R^*)^{\boldsymbol{\xi},\boldsymbol{\zeta}}$ that satisfies $j_{R^*}\tilde{\varphi} = \varphi$. Then, $j_{R^*/\mathfrak{a}}\hat{q}_{\mathfrak{a}}\tilde{\varphi} = (id_{M^*} \widehat{\otimes}_{K^*} q_{\mathfrak{a}})j_{R^*}\tilde{\varphi} = (id_{M^*} \widehat{\otimes}_{K^*} q_{\mathfrak{a}})\varphi = j_{R^*/\mathfrak{a}}\rho_{\mathfrak{a}}$, hence $\hat{q}_{\mathfrak{a}}\tilde{\varphi} = \rho_{\mathfrak{a}}$ for any $\mathfrak{a} \in J$. It is clear that $\tilde{\varphi}$ is unique morphism that satisfies $\hat{q}_{\mathfrak{a}}\tilde{\varphi} = \rho_{\mathfrak{a}}$ for any $\mathfrak{a} \in J$. Therefore $\left((M^* \widehat{\otimes}_{K^*} R^*)^{\boldsymbol{\xi},\boldsymbol{\zeta}} \xrightarrow{\hat{q}_{\mathfrak{a}}} (M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a})^{\boldsymbol{\xi},\boldsymbol{\zeta}}\right)_{\mathfrak{a}\in J}$ is a limiting cone of D_3. It follows from (2.3.11) that $\left((M^*)^{\boldsymbol{\xi},\boldsymbol{\zeta}} \widehat{\otimes}_{K^*} R^* \xrightarrow{id_{(M^*)^{\boldsymbol{\xi},\boldsymbol{\zeta}}} \widehat{\otimes}_{K^*} q_{\mathfrak{a}}} (M^*)^{\boldsymbol{\xi},\boldsymbol{\zeta}} \widehat{\otimes}_{K^*} R^*/\mathfrak{a}\right)_{\mathfrak{a}\in J}$ is a limiting cone of a functor $D_4 : J \to \mathcal{TopMod}_{K^*}$ defined by $D_4(\mathfrak{a}) = (M^*)^{\boldsymbol{\xi},\boldsymbol{\zeta}} \widehat{\otimes}_{K^*} R^*/\mathfrak{a}$. Since $\hat{\imath}_{R^*/\mathfrak{a}} : (M^*)^{\boldsymbol{\xi},\boldsymbol{\zeta}} \widehat{\otimes}_{K^*} R^*/\mathfrak{a} \to (M^* \widehat{\otimes}_{K^*} R^*/\mathfrak{a})^{\boldsymbol{\xi},\boldsymbol{\zeta}}$ defines a natutal equivalence $D_4 \to D_3$, $\hat{\imath}^{\boldsymbol{\xi},\boldsymbol{\zeta}}_{R^*} : (M^*)^{\boldsymbol{\xi},\boldsymbol{\zeta}} \widehat{\otimes}_{K^*} R^* \to (M^* \widehat{\otimes}_{K^*} R^*)^{\boldsymbol{\xi},\boldsymbol{\zeta}}$ is an isomorphism. $\square$

14 Examples of representations of topological group schemes

14.1 Representations of general linear groups

We consider the case $\mathcal{C} = \mathcal{TopAlg}_{pfK^*}$ and $\mathcal{M} = \mathcal{TopMod}_{pfK^*}$ below and assume that K^* is a field such that $K^i = \{0\}$ if $i \neq 0$. For a non-increasing sequence $\boldsymbol{v} = (s_1, s_2, \dots, s_n)$ of integers, let us denote by $\beta_{\boldsymbol{v}}$ the K^*-module structure of $V_{\boldsymbol{v}}^*$ (8.6.2) and consider an object $\boldsymbol{V_v} = (K^*, V_{\boldsymbol{v}}^*, \beta_{\boldsymbol{v}})$ of $\mathcal{Mod}(\mathcal{C}, \mathcal{M})_{K^*}$. We note that $V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^* = V_{\boldsymbol{v}}^* \otimes_{K^*} R^*$ holds for an object R^* of $\mathcal{C}$ since $V_{\boldsymbol{v}}^*$ is a finite dimensional vector space over K^*. Recall that $\mathcal{GL}_{\boldsymbol{v}}$ is a group object in $\mathcal{T} = \mathrm{Funct}_r(\mathcal{C}, \mathcal{Top})$ represented by the topological Hopf algebra $\hat{A}_{\boldsymbol{v}}^*$ which is given in the proof of (8.6.15). Define a map

$$\beta_{R^*} : (V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*) \times \mathcal{GL}_{\boldsymbol{v}}(R^*) = (V_{\boldsymbol{v}}^* \otimes_{K^*} R^*) \times \mathcal{GL}_{R^*}(V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*) \to V_{\boldsymbol{v}}^* \otimes_{K^*} R^* = V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*$$

by $\beta_{R^*}(x, g) = g(x)$. It is clear that β_{R^*} is a right $\mathcal{GL}_{\boldsymbol{v}}(R^*)$-action on $V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*$.

Proposition 14.1.1 *For a morphism $\varphi : R^* \to S^*$, the following diagram is commutative.*

$$\begin{tikzcd}
(V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*) \times \mathcal{GL}_{\boldsymbol{v}}(R^*) \arrow[r, "\beta_{R^*}"] \arrow[d, "(id_{V_{\boldsymbol{v}}^*} \widehat{\otimes}_{K^*} \varphi) \times \mathcal{GL}_{\boldsymbol{v}}(\varphi)"] & V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^* \arrow[d, "id_{V_{\boldsymbol{v}}^*} \widehat{\otimes}_{K^*} \varphi"] \\
(V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} S^*) \times \mathcal{GL}_{\boldsymbol{v}}(S^*) \arrow[r, "\beta_{S^*}"] & V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*
\end{tikzcd}$$

Proof. Define a map $i_{R^*} : R^* \to R^* \otimes_{R^*} S^*$ by $i_{R^*}(r) = r \otimes 1$. We denote by $\chi_\varphi : R^* \otimes_{R^*} S^* \to S^*$ the isomorphism defined by $\chi_\varphi(r \otimes s) = \varphi(r)s$. Then, we have $\chi_\varphi i_{R^*} = \varphi$. Hence the following diagram is commutative for $g \in \mathcal{GL}_{\boldsymbol{v}}(R^*) = \mathcal{GL}_{R^*}(V_{\boldsymbol{v}}^* \otimes_{K^*} R^*)$.

$$\begin{tikzcd}
 & V_{\boldsymbol{v}}^* \otimes_{K^*} R^* \arrow[r, "g"] \arrow[dl, "id_{V_{\boldsymbol{v}}^*} \otimes_{K^*} \varphi"'] \arrow[d, "id_{V_{\boldsymbol{v}}^*} \otimes_{K^*} i_{R^*}"] & V_{\boldsymbol{v}}^* \otimes_{K^*} R^* \arrow[d, "id_{V_{\boldsymbol{v}}^*} \otimes_{K^*} i_{R^*}"] \arrow[dr, "id_{V_{\boldsymbol{v}}^*} \otimes_{K^*} \varphi"] & \\
V_{\boldsymbol{v}}^* \otimes_{K^*} S^* \arrow[r, "id_{V_{\boldsymbol{v}}^*} \otimes_{K^*} \chi_\varphi^{-1}"] & V_{\boldsymbol{v}}^* \otimes_{K^*} R^* \otimes_{R^*} S^* \arrow[r, "g \otimes_{K^*} id_{S^*}"] & V_{\boldsymbol{v}}^* \otimes_{K^*} R^* \otimes_{R^*} S^* \arrow[r, "id_{V_{\boldsymbol{v}}^*} \otimes_{K^*} \chi_\varphi"] & V_{\boldsymbol{v}}^* \otimes_{K^*} S^*
\end{tikzcd}$$

Since the composition of lower horizontal maps of the above diagram is $\mathcal{GL}_{\boldsymbol{v}}(\varphi)(g) = T_\varphi(g)$, the assertion follows from the commutativity of the above diagram. □

We see the following fact from (14.1.1) and (13.2.14).

Proposition 14.1.2 *$\beta_{R^*} : (V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*) \times \mathcal{GL}_{\boldsymbol{v}}(R^*) \to V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*$ is continuous.*

Let us denote by $\xi_{\boldsymbol{v}} \in \mathcal{GL}_{\hat{A}_{\boldsymbol{v}}}(V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} \hat{A}_{\boldsymbol{v}})$ the image of the identity map of $\hat{A}_{\boldsymbol{v}}^*$ by the natural isomorphism $(\varphi_{\boldsymbol{v}} h_{\eta_{A_{\boldsymbol{v}}^*}})_{\hat{A}_{\boldsymbol{v}}^*} : h_{\hat{A}_{\boldsymbol{v}}^*}(\hat{A}_{\boldsymbol{v}}^*) \to \mathcal{GL}_{\boldsymbol{v}}(\hat{A}_{\boldsymbol{v}}^*)$ (*c.f.* the proof of (8.6.15)). We put $\boldsymbol{\xi_v} = (id_{\hat{A}_{\boldsymbol{v}}^*}, \xi_{\boldsymbol{v}}) : u_{\hat{A}_{\boldsymbol{v}}^*}^*(\boldsymbol{V_v}) \to u_{\hat{A}_{\boldsymbol{v}}^*}^*(\boldsymbol{V_v})$. Then, we have a map $\alpha(\boldsymbol{\xi_v})_{R^*} : (V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*) \times \mathcal{GL}_{\boldsymbol{v}}(R^*) \to V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*$ defined in (13.2.6).

Proposition 14.1.3 *$\alpha(\boldsymbol{\xi_v})_{R^*}$ coincides with a right $\mathcal{GL}_{\boldsymbol{v}}(R^*)$-action β_{R^*} above for each $R^* \in \mathrm{Ob}\,\mathcal{C}$ and $\boldsymbol{\xi_v}$ is a representation of $\hat{A}_{\boldsymbol{v}}^*$ on $\boldsymbol{V_v}$.*

Proof. The first assertion is a direct consequence of (13.2.9). Hence $\alpha(\boldsymbol{\xi_v})_{R^*}$ is a right $\mathcal{GL}_{\boldsymbol{v}}(R^*)$-action and it follows from (13.2.7) that $\boldsymbol{\xi_v}$ is a representation of $\hat{A}_{\boldsymbol{v}}^*$. □

Let us denote by $\hat{x}_{ij}$ the image of x_{ij} by $\eta_{A_{\boldsymbol{v}}^*} : A_{\boldsymbol{v}}^* \to \hat{A}_{\boldsymbol{v}}^*$. The next result follows from the definition of the natural equivalence $h_{\hat{A}_{\boldsymbol{v}}^*} \to \mathcal{GL}_{\boldsymbol{v}}$ (8.6.15)

Proposition 14.1.4 *$\xi_{\boldsymbol{v}} : V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} \hat{A}_{\boldsymbol{v}}^* \to V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} \hat{A}_{\boldsymbol{v}}^*$ maps $\boldsymbol{v}_j \otimes 1$ to $\sum_{i=1}^{n} \boldsymbol{v}_i \otimes \hat{x}_{ij}$. Hence the right $\hat{A}_{\boldsymbol{v}}^*$-comodule structure $\hat{\xi}_{\boldsymbol{v}} : V_{\boldsymbol{v}}^* \to V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} \hat{A}_{\boldsymbol{v}}^*$ is given by $\hat{\xi}_{\boldsymbol{v}}(\boldsymbol{v}_j) = \sum_{i=1}^{n} \boldsymbol{v}_i \otimes \hat{x}_{ij}$.*

Let G be a topological K^*-group functor and $\alpha_{R^*} : (V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*) \times G(R^*) \to V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*$ a right $G(R^*)$-action of on $V_{\boldsymbol{v}}^* \widehat{\otimes}_{K^*} R^*$ which is continuous and natural in $R^* \in \mathrm{Ob}\,\mathcal{C}$. For $g \in G(R^*)$, let $\tilde{\alpha}_{R^*}(g) \in \mathcal{GL}_{\boldsymbol{v}}(R^*) = \mathcal{GL}_{R^*}(V_{\boldsymbol{v}}^* \otimes_{K^*} R^*)$ be the linear map given by $x \mapsto \alpha_{R^*}(x, g)$.

Proposition 14.1.5 *$\tilde\alpha_{R^*} : G(R^*) \to \mathcal{GL}_{\boldsymbol v}(R^*)$ is natural in R^* and continuous homomorphism of groups.*

Proof. For a morphism $\varphi : R^* \to S^*$ of $\mathcal C$ and $g \in G(R^*)$, put $f = \tilde\alpha_{R^*}(g)$. Then, we have $\mathcal{GL}_{\boldsymbol v}(\varphi)\tilde\alpha_{R^*}(g) = \mathcal{GL}_{\boldsymbol v}(\varphi)(f) = f_\varphi$. On the other hand, for $x \in V^*_{\boldsymbol v}\widehat\otimes_{K^*}R^* = V^*_{\boldsymbol v}\otimes_{K^*}R^*$, we have the following equalities by the naturality of α_{R^*} and the definition of $f_\varphi = \mathcal{GL}_{\boldsymbol v}(\varphi)(f)$.

$$\begin{aligned}\tilde\alpha_{S^*}(G(\varphi)(g))((id_{V^*_{\boldsymbol v}}\widehat\otimes_{K^*}\varphi)(x)) &= \alpha_{S^*}(G(\varphi)(g),(id_{V^*_{\boldsymbol v}}\widehat\otimes_{K^*}\varphi)(x)) = (id_{V^*_{\boldsymbol v}}\widehat\otimes_{K^*}\varphi)\alpha_{R^*}(g,x) = (id_{V^*_{\boldsymbol v}}\widehat\otimes_{K^*}\varphi)f(x)\\ &= f_\varphi(id_{V^*_{\boldsymbol v}}\widehat\otimes_{K^*}\varphi)(x) = \mathcal{GL}_{\boldsymbol v}(\varphi)(\tilde\alpha_{R^*}(g))((id_{V^*_{\boldsymbol v}}\widehat\otimes_{K^*}\varphi)(x))\end{aligned}$$

Thus we have $\tilde\alpha_{S^*}(G(\varphi)(g))(v\otimes 1) = \mathcal{GL}_{\boldsymbol v}(\varphi)(\tilde\alpha_{R^*}(g))(v\otimes 1)$ for any $v \in V^*_{\boldsymbol v}$. Since both $\tilde\alpha_{S^*}(G(\varphi)(g))$ and $\mathcal{GL}_{\boldsymbol v}(\varphi)(\tilde\alpha_{R^*}(g))$ are isomorphisms of right S^*-modules, $\tilde\alpha_{S^*}(G(\varphi)(g)) = \mathcal{GL}_{\boldsymbol v}(\varphi)(\tilde\alpha_{R^*}(g))$, namely $\tilde\alpha_{R^*}$ is natural. For $g, h \in G(R^*)$ and $x \in V^*_{\boldsymbol v}\widehat\otimes_{K^*}R^*$, since

$$\tilde\alpha_{R^*}(gh)(x) = \alpha_{R^*}(x, gh) = \alpha_{R^*}(\alpha_{R^*}(x,g),h) = \tilde\alpha_{R^*}(h)(\tilde\alpha_{R^*}(g)(x)) = \mu_{\boldsymbol v}(\tilde\alpha_{R^*}(g),\tilde\alpha_{R^*}(h))(x),$$

$\tilde\alpha_{R^*}$ is a homomorphism of groups.

For $v \in V^k_{\boldsymbol v}$ and $\mathfrak a \in \mathcal I_{R^*}$, by the continuity of α_{R^*}, there exists an open neighborhood U of the unit of $G(R^*)$ satisfying $\alpha_{R^*}(\{v\otimes 1\}\times U) \subset v\otimes_{K^*}1 + V^*_{\boldsymbol v}\otimes_{K^*}\mathfrak a$. Thus $\tilde\alpha_{R^*}(U) \subset id_{V^*_{\boldsymbol v}\otimes_{K^*}R^*} + O(K^*v\otimes_{K^*}1, V^*_{\boldsymbol v}\otimes_{K^*}\mathfrak a)^0$. It follows from (3) of (3.1.4) that $\tilde\alpha_{R^*} : G(R^*) \to \mathcal{GL}_{\boldsymbol v}(R^*)$ is continuous. □

Let A^* be a topological Hopf algebra in $\mathcal C$ and $\boldsymbol\xi = (id_{A^*}, \xi) : u^*_{A^*}(\boldsymbol V_{\boldsymbol v}) \to u^*_{A^*}(\boldsymbol V_{\boldsymbol v})$ a representation of A^* on $\boldsymbol V_{\boldsymbol v} = (K^*, V^*_{\boldsymbol v}, \beta_{\boldsymbol v})$. We denote by G_{A^*} the topological affine group scheme represented by A^*. There is a right $G_{A^*}(R^*)$-action $\alpha(\boldsymbol\xi)_{R^*} : (V^*_{\boldsymbol v}\widehat\otimes_{K^*}R^*)\times G_{A^*}(R^*) \to V^*_{\boldsymbol v}\widehat\otimes_{K^*}R^*$ on $V^*_{\boldsymbol v}\widehat\otimes_{K^*}R^*$ given in (13.2.6). Hence we have a morphism $\tilde\alpha(\boldsymbol\xi) : G_{A^*} \to \mathcal{GL}_{\boldsymbol v}$ of topological affine K^*-group schemes. Put $\bar{\boldsymbol\xi} = \tilde\alpha(\boldsymbol\xi)_{A^*}(id_{A^*}) : A^*_{\boldsymbol v} \to A^*$, then $\bar{\boldsymbol\xi}$ induces $\tilde\alpha(\boldsymbol\xi)$. Therefore $\bar{\boldsymbol\xi}$ is a morphism of Hopf algebras.

Let $\hat\xi : V^*_{\boldsymbol v} \to V^*_{\boldsymbol v}\widehat\otimes_{K^*}A^*$ the right A^*-comodule structure associated with $\boldsymbol\xi$, namely $\hat\xi$ is a composition $V^*_{\boldsymbol v} \xrightarrow{i_{V^*_{\boldsymbol v}}} V^*_{\boldsymbol v}\widehat\otimes_{K^*}A^* \xrightarrow{\xi} V^*_{\boldsymbol v}\widehat\otimes_{K^*}A^*$.

Proposition 14.1.6 *If $\hat\xi(\boldsymbol v_j) = \sum\limits_{i=1}^n \boldsymbol v_i\otimes a_{ij}$ for $j = 1,2,\dots,n$, then $\bar\xi : A^*_{\boldsymbol v} \to A^*$ is given by $\bar\xi(x_{ij}) = a_{ij}$ (hence $\bar\xi(y_{ij}) = \iota_{A^*}(a_{ij})$, where $\iota_{A^*} : A^* \to A^*$ denotes the conjugation of A^*.*

Proof. By the assumption, we have

$$\bar{\boldsymbol\xi}(\boldsymbol v_j\otimes 1) = (\tilde\alpha(\boldsymbol\xi)_{A^*}(id_{A^*}))(\boldsymbol v_j\otimes 1) = \alpha_{A^*}(\boldsymbol\xi)(\boldsymbol v_j\otimes 1, id_{A^*}) = \xi(\boldsymbol v_j\otimes 1) = \hat\xi(\boldsymbol v_j) = \sum_{i=1}^n \boldsymbol v_j\otimes a_{ij}$$

for $j = 1,2,\dots,n$. Hence the map $A^*_{\boldsymbol v} \to A^*$ given by $x_{ij} \mapsto a_{ij}$ corresponds to the linear map $\bar{\boldsymbol\xi}$ by the isomorphism $h_{A^*_{\boldsymbol v}}(A^*) \to \mathcal{GL}_{\boldsymbol v}(A^*) = \mathcal{GL}_{A^*}(V^*_{\boldsymbol v}\otimes_{K^*}A^*)$. □

14.2 Embedding of the affine group represented by the dual Steenrod algebra

For a non-increasing sequence $\boldsymbol v = (s_1, s_2, \dots, s_n)$ of integers, let $\mathfrak u_{\boldsymbol v}$ be the ideal of $A^*_{\boldsymbol v}$ generated by a set

$$\{x_{jj}-1 \,|\, j = 1,2,\dots,n\}\cup\{y_{jj}-1 \,|\, j = 1,2,\dots,n\}\cup\{x_{ij} \,|\, s_i \geq s_j,\ i\neq j\}\cup\{y_{ij} \,|\, s_i \geq s_j,\ i\neq j\}.$$

Then, $\mu_{\boldsymbol v}(\mathfrak u_{\boldsymbol v}) \subset \mathfrak u_{\boldsymbol v}\otimes A^*_{\boldsymbol v} + A^*_{\boldsymbol v}\otimes\mathfrak u_{\boldsymbol v}$. Hence $A^*_{\boldsymbol v}/\mathfrak u_{\boldsymbol v}$ has a structure of a Hopf algebra. Put $\tilde A^*_{\boldsymbol v} = A^*_{\boldsymbol v}/\mathfrak u_{\boldsymbol v}$. Then, it is easy to verify that $\tilde A^*_{\boldsymbol v}$ is a polynomial algebra generated by $\{x_{ij} \,|\, s_i < s_j\}$. Hence $\tilde A^k_{\boldsymbol v} = \{0\}$ if $k < 0$, $\dim \tilde A^0_{\boldsymbol v} = 1$. It follows that $\tilde A^*_{\boldsymbol v}$ is finite type and has the cofinite topology by (1.4.11). It follows from (1.4.3) that $\tilde A^*_{\boldsymbol v}$ has the skeletal topology. Moreover, if we put $\mathfrak a_{\boldsymbol v} = \operatorname{Ker}\varepsilon_{\tilde A^*_{\boldsymbol v}} = (x_{ij} \,|\, s_i < s_j)$, the topology on $\tilde A^*_{\boldsymbol v}$ coincides with the $\mathfrak a_{\boldsymbol v}$-adic topology. Let us denote by $\mathcal{UL}_{\boldsymbol v}$ the topological group scheme represented by $\tilde A^*_{\boldsymbol v}$. Then, $\mathcal{UL}_{\boldsymbol v}$ is regarded as a "closed subgroup scheme" of $\mathcal{GL}_{\boldsymbol v}$ by the morphism induced by the quotient map $A^*_{\boldsymbol v} \to A^*_{\boldsymbol v}/\mathfrak u_{\boldsymbol v} = \tilde A^*_{\boldsymbol v}$.

Suppose that a topological Hopf algebra A^* in $\mathcal C$ satisfies $A^k = \{0\}$ for $k < 0$ and $\dim A^0 = 1$. Then, each algebra homomorphism $A^*_{\boldsymbol v} \to A^*$ factors though the quotient map $A^*_{\boldsymbol v} \to \tilde A^*_{\boldsymbol v}$. Hence a homomorphism from the topological group scheme represented by A^* to $\mathcal{GL}_{\boldsymbol v}$ lifts to $\mathcal{UL}_{\boldsymbol v}$.

Example 14.2.1 *Let p be an odd prime and $\mathcal{A}_{p*} = E(\tau_0, \tau_1, \dots) \otimes \boldsymbol{F}_p[\xi_1, \xi_2, \dots]$ the dual of the mod p Steenrod algebra. Put $\boldsymbol{v}_p(n) = (-1, -2, \dots, -2n-1)$ and regard $V^*_{\boldsymbol{v}_p(n)}$ as the reduced mod p cohomology group $\widetilde{H}^*(L_{2n+1}(p))$ of the $(2n+1)$-dimensional mod p lens space. Recall that $H^*(L_{2n+1}(p)) = E(t) \otimes \boldsymbol{F}_p[s]/(s^{n+1})$ for $t \in H^{-1}(L_{2n+1}(p))$, $s \in H^{-2}(L_{2n+1}(p))$ and that the Milnor coaction $\varphi : H^*(L_{2n+1}(p)) \to H^*(L_{2n+1}(p)) \otimes \mathcal{A}_{p*}$ is given by $\varphi(t) = t \otimes 1 - \sum\limits_{k \geq 0} s^{p^k} \otimes \tau_k$ and $\varphi(s) = \sum\limits_{k \geq 0} s^{p^k} \otimes \xi_k$. Thus we have a representation of the topological affine group scheme represented by $\mathcal{A}_{p*}$ on $V^*_{\boldsymbol{v}_p(n)} = \widetilde{H}^*(L_{2n+1}(p))$. We put $\boldsymbol{v}_{2j-1} = ts^{j-1}$, $\boldsymbol{v}_{2j} = s^j$, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{2n+1}\}$ is a basis of $V^*_{\boldsymbol{v}_p(n)}$ and*

$$\varphi(\boldsymbol{v}_{2j}) = \sum_{i \geq j} \boldsymbol{v}_{2i} \otimes \left(\sum_{\substack{\sum_{l \geq 1} k_l \leq j \\ \sum_{l \geq 1}(p^l - 1)k_l = i - j}} \frac{j!}{\left(j - \sum_{l \geq 1} k_l\right)! k_1! k_2! \cdots} \xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3} \cdots \right),$$

$$\varphi(\boldsymbol{v}_{2j+1}) = \sum_{i \geq j} \boldsymbol{v}_{2i+1} \otimes \left(\sum_{\substack{\sum_{l \geq 1} k_l \leq j \\ \sum_{l \geq 1}(p^l - 1)k_l = i - j}} \frac{j!}{\left(j - \sum_{l \geq 1} k_l\right)! k_1! k_2! \cdots} \xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3} \cdots \right)$$
$$- \sum_{i > j} \boldsymbol{v}_{2i} \otimes \left(\sum_{\substack{\sum_{l \geq 1} k_l \leq j \\ p^k + \sum_{l \geq 1}(p^l - 1)k_l = i - j}} \frac{j!}{\left(j - \sum_{l \geq 1} k_l\right)! k_1! k_2! \cdots} \tau_k \xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3} \cdots \right)$$

It follows from (14.1.6) that the map $\rho_{p,n} : \tilde{A}^*_{\boldsymbol{v}_p(n)} \to \mathcal{A}_{p*}$ is given by

$$\rho_{p,n}(x_{2i\,2j}) = \sum_{\substack{\sum_{l \geq 1} k_l \leq j \\ \sum_{l \geq 1}(p^l - 1)k_l = i - j}} \frac{j!}{\left(j - \sum_{l \geq 1} k_l\right)! k_1! k_2! \cdots} \xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3} \cdots$$
$$\rho_{p,n}(x_{2i+1\,2j+1}) = \sum_{\substack{\sum_{l \geq 1} k_l \leq j \\ \sum_{l \geq 1}(p^l - 1)k_l = i - j}} \frac{j!}{\left(j - \sum_{l \geq 1} k_l\right)! k_1! k_2! \cdots} \xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3} \cdots$$
$$\rho_{p,n}(x_{2i\,2j+1}) = \sum_{\substack{\sum_{l \geq 1} k_l \leq j \\ p^k + \sum_{l \geq 1}(p^l - 1)k_l = i - j}} \frac{j!}{\left(j - \sum_{l \geq 1} k_l\right)! k_1! k_2! \cdots} \tau_k \xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3} \cdots$$
$$\rho_{p,n}(x_{2i+1\,2j}) = 0$$

Example 14.2.2 *Let $\mathcal{A}_{2*} = \boldsymbol{F}_2[\zeta_1, \zeta_2, \dots]$ be the dual of the mod 2 Steenrod algebra. Put $\boldsymbol{v}_2(n) = (-1, -2, \dots, -n)$ and regard $V^*_{\boldsymbol{v}_2(n)}$ as the reduced mod 2 cohomology group $\widetilde{H}^*(\boldsymbol{R}P^n)$ of the n-dimensional real projective space. Recall that $H^*(\boldsymbol{R}P^n) = \boldsymbol{F}_2[t]/(t^{n+1})$ for $t \in H^{-1}(\boldsymbol{R}P^n)$ and that the Milnor coaction $\varphi : H^*(\boldsymbol{R}P^n) \to H^*(\boldsymbol{R}P^n) \otimes \mathcal{A}_{2*}$ is given by $\varphi(t) = \sum\limits_{k \geq 0} t^{2^k} \otimes \zeta_k$. Thus we have a representation of the topological affine group scheme represented by $\mathcal{A}_{2*}$ on $V^*_{\boldsymbol{v}_2(n)} = \widetilde{H}^*(\boldsymbol{R}P^n)$. We put $\boldsymbol{v}_j = t^j$, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ is a basis of $V^*_{\boldsymbol{v}_2(n)}$ and*

$$\varphi(\boldsymbol{v}_j) = \sum_{i \geq j} \boldsymbol{v}_i \otimes \left(\sum_{\substack{\sum_{s \geq 1} k_s \leq j \\ \sum_{s \geq 1}(2^s - 1)k_s = i - j}} \frac{j!}{\left(j - \sum_{s \geq 1} k_s\right)! k_1! k_2! \cdots} \zeta_1^{k_1} \zeta_2^{k_2} \zeta_3^{k_3} \cdots \right).$$

It follows from (14.1.6) that the map $\rho_{2,n} : \tilde{A}^*_{\boldsymbol{v}_2(n)} \to \mathcal{A}_{2*}$ is given by

$$\rho_{2,n}(x_{ij}) = \sum_{\substack{\sum_{s\geq 1} k_s \leq j \\ \sum_{s\geq 1}(2^s-1)k_s = i-j}} \frac{j!}{\left(j - \sum_{s\geq 1} k_s\right)! k_1! k_2! \cdots} \zeta_1^{k_1}\zeta_2^{k_2}\zeta_3^{k_3}\cdots.$$

Example 14.2.3 *Put $\tilde{\boldsymbol{v}}_p(n) = (-1, -2, \ldots, -2p^{j-2}, \ldots, -2p^n)$ and regard $V^*_{\tilde{\boldsymbol{v}}_p(n)}$ as a subspace of $V^*_{\boldsymbol{v}_p(p^n)}$. Put $\tilde{\boldsymbol{v}}_1 = t$ and $\tilde{\boldsymbol{v}}_j = s^{p^{j-2}}$ $(2 \leq j \leq n+2)$. $V^*_{\tilde{\boldsymbol{v}}_p(n)}$ is a subcomodule of $V^*_{\boldsymbol{v}_p(p^n)}$ and*

$$\varphi(\tilde{\boldsymbol{v}}_1) = \tilde{\boldsymbol{v}}_1 \otimes 1 - \sum_{i=2}^{n+2} \tilde{\boldsymbol{v}}_i \otimes \tau_{i-2}, \qquad \varphi(\tilde{\boldsymbol{v}}_j) = \sum_{i=j}^{n+2} \tilde{\boldsymbol{v}}_i \otimes \xi_{i-j}^{p^{j-2}}.$$

On the other hand, $\tilde{A}^*_{\tilde{\boldsymbol{v}}_p(n)}$ is a polynomial algebra generated by $\{x_{ij} \,|\, 1 \leq j < i \leq n+2\}$ with $x_{i1} \in \tilde{A}^{2p^{i-2}-1}_{\tilde{\boldsymbol{v}}_p(n)}$ and $x_{ij} \in \tilde{A}^{2p^{j-2}(p^{i-j}-1)}_{\tilde{\boldsymbol{v}}_p(n)}$ $(2 \leq j < i)$. It follows from (14.1.6) that the map $\tilde{\rho}_{p,n} : \tilde{A}^*_{\tilde{\boldsymbol{v}}_p(n)} \to \mathcal{A}_{p*}$ is given by

$$\tilde{\rho}_{p,n}(x_{i1}) = -\tau_{i-2}, \qquad \tilde{\rho}_{p,n}(x_{ij}) = \xi_{i-j}^{p^{j-2}} \ (2 \leq j < i).$$

Example 14.2.4 *Put $\tilde{\boldsymbol{v}}_2(n) = (-1, -2, \ldots, -2^{j-1}, \ldots, -2^n)$ and regard $V^*_{\tilde{\boldsymbol{v}}_2(n)}$ as a subspace of $V^*_{\boldsymbol{v}_2(2^n)}$. Then, $\tilde{\boldsymbol{v}}_j = t^{2^{j-1}}$ $(1 \leq j \leq n+1)$. $V^*_{\tilde{\boldsymbol{v}}_2(n)}$ is a subcomodule of $V^*_{\boldsymbol{v}_2(2^n)}$ and*

$$\varphi(\tilde{\boldsymbol{v}}_j) = \sum_{i=j}^{n+1} \tilde{\boldsymbol{v}}_i \otimes \zeta_{i-j}^{2^{j-1}}.$$

On the other hand, $\tilde{A}^*_{\tilde{\boldsymbol{v}}_2(n)}$ is a polynomial algebra generated by $\{x_{ij} \,|\, 1 \leq j < i \leq n+1\}$ with $x_{ij} \in \tilde{A}^{2^{j-1}(2^{i-j}-1)}_{\tilde{\boldsymbol{v}}_2(n)}$ $(1 \leq j < i)$. It follows from (14.1.6) that the map $\tilde{\rho}_{2,n} : \tilde{A}^*_{\boldsymbol{v}_2(n)} \to \mathcal{A}_{2*}$ is given by

$$\tilde{\rho}_{2,n}(x_{ij}) = \zeta_{i-j}^{2^{j-1}} \ (1 \leq j < i).$$

Example 14.2.5 *Let $\tilde{A}^*_{p\infty}$ be the colimit of the direct system of the inclusion maps $\tilde{A}^*_{\boldsymbol{v}_p(n)} \hookrightarrow \tilde{A}^*_{\boldsymbol{v}_p(n+1)}$. Then, $\tilde{A}^*_{p\infty} = E(x_{ij} \,|\, i > j \geq 0,\ i+j \text{ is odd}) \otimes \boldsymbol{F}_p[x_{ij} \,|\, i > j > 0, \text{ is odd}]$ with $\deg x_{ij} = i - j$. Put $V^*_p = \widetilde{H}^*(B\boldsymbol{Z}/p\boldsymbol{Z})$ and give V^*_p the skeletal topology. We also put $\boldsymbol{v}_{2j-1} = ts^{j-1}$ and $\boldsymbol{v}_{2j} = s^j$. Define $\varphi_\infty : V^*_p \to V^*_p \mathbin{\widehat{\otimes}} \tilde{A}^*_{p\infty}$ by $\varphi_\infty(\boldsymbol{v}_j) = \sum_{i\geq j} \boldsymbol{v}_i \otimes x_{ij}$. Then, $\tilde{A}^*_{p\infty}$ and V^*_p satisfy the conditions of (13.2.12). Hence we have a representation of the toplogical affine group scheme represented by $\tilde{A}^*_{p\infty}$. We note that the maps $\rho_{p,n} : \tilde{A}^*_{\boldsymbol{v}_p(n)} \to \mathcal{A}_{p*}$ in (14.2.1) extend to the map $\rho_p : \tilde{A}^*_{p\infty} \to \mathcal{A}_{p*}$ given by*

$$\begin{aligned}
\rho_p(x_{2i\,2j}) &= \sum_{\substack{\sum_{l\geq 1} k_l \leq j \\ \sum_{l\geq 1}(p^l-1)k_l = i-j}} \frac{j!}{\left(j - \sum_{l\geq 1} k_l\right)! k_1! k_2! \cdots} \xi_1^{k_1}\xi_2^{k_2}\xi_3^{k_3}\cdots \\
\rho_p(x_{2i+1\,2j+1}) &= \sum_{\substack{\sum_{l\geq 1} k_l \leq j \\ \sum_{l\geq 1}(p^l-1)k_l = i-j}} \frac{j!}{\left(j - \sum_{l\geq 1} k_l\right)! k_1! k_2! \cdots} \xi_1^{k_1}\xi_2^{k_2}\xi_3^{k_3}\cdots \\
\rho_p(x_{2i\,2j+1}) &= \sum_{\substack{\sum_{l\geq 1} k_l \leq j \\ p^k + \sum_{l\geq 1}(p^l-1)k_l = i-j}} \frac{j!}{\left(j - \sum_{l\geq 1} k_l\right)! k_1! k_2! \cdots} \tau_k \xi_1^{k_1}\xi_2^{k_2}\xi_3^{k_3}\cdots \\
\rho_p(x_{2i+1\,2j}) &= 0.
\end{aligned}$$

Since ρ_p is an epimorphism, we can regard the group scheme represented by $\mathcal{A}_{p*}$ as a closed subgroup scheme of the group scheme represented by $\tilde{A}^*_{p\infty}$.

Example 14.2.6 *Let $\tilde{A}^*_{2\infty}$ be the colimit of the direct system of the inclusion maps $\tilde{A}^*_{\boldsymbol{v}_2(n)} \hookrightarrow \tilde{A}^*_{\boldsymbol{v}_2(n+1)}$. Then, $\tilde{A}^*_{2\infty}$ is a graded polynomial algebra over $\boldsymbol{F}_2$ generated by $\{x_{ij}|, i > j \geq 1\}$ with $\deg x_{ij} = i - j$. Put $V^*_2 = \widetilde{H}^*(\boldsymbol{R}P^\infty)$ and give V^*_2 the skeletal topology. We also put $\boldsymbol{v}_j = t^j$. Define $\varphi_\infty : V^*_2 \to V^*_2 \widehat{\otimes} \tilde{A}^*_{2\infty}$ by $\varphi_\infty(\boldsymbol{v}_j) = \sum_{i\geq 1} \boldsymbol{v}_i \otimes x_{ij}$. Then, $\tilde{A}^*_{2\infty}$ and V^*_2 satisfy the conditions of (13.2.12). Hence we have a representation of the toplogical affine group scheme represented by $\tilde{A}^*_{2\infty}$. We note that the maps $\rho_{2,n} : \tilde{A}^*_{\boldsymbol{v}_2(n)} \to \mathcal{A}_{2*}$ in (14.2.2) extend to the map $\rho_2 : \tilde{A}^*_{2\infty} \to \mathcal{A}_{2*}$ given by*

$$\rho_2(x_{ij}) = \sum_{\substack{\sum_{s\geq 1} k_s \leq j \\ \sum_{s\geq 1}(2^s-1)k_s = i-j}} \frac{j!}{\left(j - \sum_{s\geq 1} k_s\right)! k_1! k_2! \cdots} \zeta_1^{k_1}\zeta_2^{k_2}\zeta_3^{k_3}\cdots .$$

Since ρ_2 is an epimorphism, we can regard the group scheme represented by $\mathcal{A}_{2*}$ as a closed subgroup scheme of the group scheme represented by $\tilde{A}^*_{2\infty}$.

Example 14.2.7 *Let $A^*_{p\infty}$ be the colimit of the direct system of the inclusion maps $\tilde{A}^*_{\tilde{\boldsymbol{v}}_p(n)} \hookrightarrow \tilde{A}^*_{\tilde{\boldsymbol{v}}_p(n+1)}$. Then, $A^*_{p\infty} = E(x_{i1}|\, i > 1) \otimes \boldsymbol{F}_p[x_{ij}|\, i > j > 1]$ with $\deg x_{i1} = 2p^{i-2} - 1$, $\deg x_{ij} = 2p^{j-2}(p^{i-j} - 1)$. Let W^*_p be the colimit of the direct system of the inclusion maps $V^*_{\tilde{\boldsymbol{v}}_p(n)} \hookrightarrow V^*_{\tilde{\boldsymbol{v}}_p(n+1)}$ and give W^*_p the skeletal topology. Define $\psi_\infty : W^*_p \to W^*_p \widehat{\otimes} \tilde{A}^*_{p\infty}$ by $\psi_\infty(\tilde{\boldsymbol{v}}_j) = \sum_{i\geq j} \tilde{\boldsymbol{v}}_i \otimes x_{ij}$. Then, $A^*_{p\infty}$ and W^*_p satisfy the conditions of (13.2.12). Hence we have a representation of the toplogical affine group scheme represented by $A^*_{p\infty}$. We note that the maps $\tilde{\rho}_{p,n} : \tilde{A}^*_{\tilde{\boldsymbol{v}}_p(n)} \to \mathcal{A}_{p*}$ in (14.2.1) extend to the map $\tilde{\rho}_p : A^*_{p\infty} \to \mathcal{A}_{p*}$ given by*

$$\tilde{\rho}_p(x_{i1}) = -\tau_{i-2}, \qquad \tilde{\rho}_p(x_{ij}) = \xi_{i-j}^{p^{j-2}} \ (2 \leq j < i).$$

Since $\tilde{\rho}_p$ is an epimorphism, we can regard the group scheme represented by $\mathcal{A}_{p*}$ as a closed subgroup scheme of the group scheme represented by $A^*_{p\infty}$.

Example 14.2.8 *Let $A^*_{2\infty}$ be the colimit of the direct system of the inclusion maps $\tilde{A}^*_{\tilde{\boldsymbol{v}}_2(n)} \hookrightarrow \tilde{A}^*_{\tilde{\boldsymbol{v}}_2(n+1)}$. Then, $A^*_{2\infty}$ is a graded polynomial algebra over $\boldsymbol{F}_2$ generated by $\{x_{ij}|, i > j \geq 1\}$ with $\deg x_{ij} = 2^{i-1} - 2^{j-1}$. Let W^*_2 be the colimit of the direct system of the inclusion maps $V^*_{\tilde{\boldsymbol{v}}_2(n)} \hookrightarrow V^*_{\tilde{\boldsymbol{v}}_2(n+1)}$ and give W^*_2 the skeletal topology. Define $\psi_\infty : W^*_2 \to W^*_2 \widehat{\otimes} \tilde{A}^*_{2\infty}$ by $\varphi_\infty(\tilde{\boldsymbol{v}}_j) = \sum_{i\geq 1} \tilde{\boldsymbol{v}}_i \otimes x_{ij}$. Then, $\tilde{A}^*_{2\infty}$ and W^*_2 satisfy the conditions of (13.2.12). Hence we have a representation of the toplogical affine group scheme represented by $A^*_{2\infty}$. We note that the maps $\tilde{\rho}_{p,n} : \tilde{A}^*_{\boldsymbol{v}_2(n)} \to \mathcal{A}_{2*}$ in (14.2.2) extend to the map $\tilde{\rho}_2 : A^*_{2\infty} \to \mathcal{A}_{2*}$ given by*

$$\tilde{\rho}_2(x_{ij}) = \zeta_{i-j}^{2^{j-1}}.$$

Since $\tilde{\rho}_2$ is an epimorphism, we can regard the group scheme represented by $\mathcal{A}_{2*}$ as a closed subgroup scheme of the group scheme represented by $A^*_{2\infty}$.

14.3 Representations on cyclic modules

Recall the definitions of Seq and Seq^b in (8.7.27) and the Milnor basis (8.7.28).

Theorem 14.3.1 *([16]) For $E = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, \dots) \in \mathrm{Seq}^b$ and $R = (r_1, r_2, \dots, r_n, \dots) \in \mathrm{Seq}$, let k be the number of non-negative integer n which satisfies $\varepsilon_n = 1$. Then, we have*

$$Q(E)\wp(R) = (-1)^{\frac{k(k-1)}{2}} \rho(E, R).$$

Hence $\{Q(E)\wp(R)\,|\, E \in \mathrm{Seq}^b,\, R \in \mathrm{Seq}\}$ is a basis of $\mathcal{A}^*_p$. Clearly $\{Sq(R)\,|\, R \in \mathrm{Seq}\}$ is a basis of $\mathcal{A}^*_2$.

There are the following relations.

$$Q_jQ_k + Q_kQ_j = 0, \quad \wp(R)Q_k - Q_k\wp(R) = \sum_{i\geq 1} Q_{k+i}\wp(R - (\overbrace{0, \dots, 0}^{i-1}, p^k, 0, \dots))$$

Here, we set $\wp(R)=0$ if $R\notin \mathrm{Seq}$. We also have $Q_0=\beta$, $Q_{k+1}=\wp^{p^k}Q_k-Q_k\wp^{p^k}$.

Let X range over all infinite matrices

$$\left\|\begin{array}{cccccc} * & x_{01} & x_{02} & \cdot & \cdot & \cdot \\ x_{10} & x_{11} & \cdot & \cdot & \cdot & \cdot \\ x_{20} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right\|$$

of non-negative integers, almost all zero, with leading entry omitted. For each such matrix X, let us define $R(X)=(r_1,r_2,\dots,r_n,\dots)$, $S(X)=(s_1,s_2,\dots,s_n,\dots)$, $T(X)=(t_1,t_2,\dots,t_n,\dots)$ and $b(X)$ as follows.

$r_i=\sum_{j\geqq 0}p^jx_{ij}$ *(weighted row sum)*, $s_j=\sum_{i\geqq 0}x_{ij}$ *(column sum)*, $t_n=\sum_{i+j=n}x_{ij}$ *(diagonal sum)*, $b(X)=\dfrac{\prod_{n\geqq 1}t_n!}{\prod_{i,j\geqq 0}x_{ij}!}$

For $R,S\in \mathrm{Seq}$, $Sq(R)Sq(S)=\sum_{R(X)=R,S(X)=S}b(X)Sq(T(X))$ and $\wp(R)\wp(S)=\sum_{R(X)=R,S(X)=S}b(X)\wp(T(X))$ hold.

The coproduct $\varphi:\mathcal{A}_p^*\to\mathcal{A}_p^*\otimes\mathcal{A}_p^*$ is given as follows.

$$\varphi(Sq(R))=\sum_{S+T=R}Sq(S)\otimes Sq(T),\quad \varphi(\wp(R))=\sum_{S+T=R}\wp(S)\otimes\wp(T),\quad \varphi(Q_k)=1\otimes Q_k+Q_k\otimes 1$$

For $R=(r_1,r_2,\dots,r_k)\in\mathrm{Seq}$, consider the following linear equations $(*)$ on the unknowns y_α which are non-negative integers for $\alpha\in\bigcup_{n\geqq 1}\mathrm{Part}(n)$. Here, we denote by $\delta_{i,j}$ the Kronecker delta.

$$\sum_{n\geqq 1}\sum_{\alpha\in\mathrm{Part}(n)}\sum_{j=1}^{l(\alpha)}\delta_{i,\alpha(j)}p^{\alpha[j]}y_\alpha=r_i\quad (i=1,2,\dots,k)\ \cdots(*)$$

For each solution $Y=(y_\alpha)$ of $(*)$, we put $u_n=\sum_{\alpha\in\mathrm{Part}(n)}y_\alpha$, $U(Y)=(u_1,u_2,\dots)$ and $c(Y)=\dfrac{\prod_{n\geqq 1}u_n!}{\prod_{n\geqq 1}\prod_{\alpha\in\mathrm{Part}(n)}y_\alpha!}$.
Then, the conjugation $\chi:\mathcal{A}_p^*\to\mathcal{A}_p^*$ is given by $\chi(Q_n)=-Q_n$ and

$$\chi(Sq(R))=\sum c(Y)Sq(U(Y)),\qquad \chi(\wp(R))=(-1)^{r_1+r_2+\cdots+r_k}\sum c(Y)\wp(U(Y))$$

where the summations extends over all solutions Y to the equations $(*)$.

Remark 14.3.2 *Let us denote by E_i the sequence $(\overbrace{0,\dots,0}^{i-1},1,0,\dots)$. If p is odd, the subspace $P(\mathcal{A}_p^*)$ of $\mathcal{A}_2^*$ consisting of primitive elements has a basis $\{Q_n\,|\,n\geqq 0\}\cup\{\wp(E_i)\,|\,i\geqq 1\}$. If we define $Q_n\in\mathcal{A}_2^*$ for $n\geqq 0$ inductively by $Q_0=Sq^1$ and $Q_{n+1}=Sq^{2^n}Q_n+Q_n\,Sq^{2^n}$, then we have $Q_n^2=0$ and $Q_n=Sq(E_{n+1})$. Moreover, $\{Q_0,Q_1,\dots,Q_n,\dots\}$ is a basis of the subspace $P(\mathcal{A}_2^*)$ of primitive elements of $\mathcal{A}_2^*$.*

Let $\mathcal{A}_2(n)^*$ be a subalgebra of $\mathcal{A}_2^*$ generated by $\{Sq^{2^s}\,|\,s=0,1,2,\dots,n-1\}$ and $\mathcal{A}_p(n)^*$ a subalgebra of $\mathcal{A}_p^*$ generated by $\{\beta\}\cup\{\wp^{p^s}\,|\,s=0,1,2,\dots,n-1\}$. Define a subset $\mathrm{Seq}(p,n)$ of Seq for a prime p and a non-negative integer n by $\mathrm{Seq}(p,n)=\{(r_1,r_2,\dots,r_n)\in\mathrm{Seq}\,|\,r_i<2^{n+1-i}\ (i=1,2,\dots,n)\}$.

For $R=(i_1,i_2,\dots,i_n,\dots)\in\mathrm{Seq}$, put $\ell(R)=\max\{k\,|\,i_k\neq 0\}$ if $R\neq\mathbf{0}$ and $\ell(\mathbf{0})=0$. We call $\ell(R)$ the length of R. The following fact is also shown in [16].

Proposition 14.3.3 *$\mathcal{A}_2(n)^*$ has a basis $\{Sq(R)\,|\,R\in\mathrm{Seq}(2,n)\}$. If p is an odd prime, $\mathcal{A}_p(n)^*$ has a basis $\{Q(E)\wp(R)\,|\,\ell(E)\leqq n,\ R\in\mathrm{Seq}(p,n)\}$. $\mathcal{A}_p(n)^*$ is a Hopf subalgebra of $\mathcal{A}_p^*$ whose dual Hopf algebra $\mathcal{A}_p(n)_*$ is given as follows.*

$$\mathcal{A}_2(n)_*=\boldsymbol{F}_2[\zeta_1,\zeta_2,\dots,\zeta_n]/(\zeta_1^{2^n},\zeta_2^{2^{n-1}},\dots,\zeta_n^2)$$
$$\mathcal{A}_p(n)_*=E(\tau_0,\tau_1,\dots,\tau_n)\otimes\boldsymbol{F}_p[\xi_1,\xi_2,\dots,\xi_n]/(\xi_1^{p^n},\xi_2^{p^{n-1}},\dots,\xi_n^p)$$

Example 14.3.4 *(1) $\mathcal{A}_2(1)^*$ has a basis $\mathcal{A}_2(1)^0 = \langle 1\rangle$, $\mathcal{A}_2(1)^{-1} = \langle Sq^1\rangle$ with relations $Sq^1Sq^1 = 0$. Hence $\mathcal{A}_2(1)^*$ is a 2-dimensional vector space.*

(2) $\mathcal{A}_p(0)^*$ has a basis $\mathcal{A}_p(0)^0 = \langle 1\rangle$, $\mathcal{A}_p(0)^{-1} = \langle Q_0\rangle$ with relations $Q_0^2 = 0$. Hence $\mathcal{A}_p(0)^*$ is a 2-dimensional vector space.

Example 14.3.5 *(1) $\mathcal{A}_2(2)^*$ has the following basis*

$$\mathcal{A}_2(2)^0 = \langle 1\rangle,\quad \mathcal{A}_2(2)^{-1} = \langle Sq^1\rangle,\quad \mathcal{A}_2(2)^{-2} = \langle Sq^2\rangle,\quad \mathcal{A}_2(2)^{-3} = \langle Sq^2Sq^1, Sq^1Sq^2\rangle,$$
$$\mathcal{A}_2(2)^{-4} = \langle Sq^1Sq^2Sq^1\rangle,\quad \mathcal{A}_2(2)^{-5} = \langle Sq^2Sq^1Sq^2\rangle,\quad \mathcal{A}_2(2)^{-6} = \langle Sq^1Sq^2Sq^1Sq^2\rangle$$

with relations $Sq^1Sq^1 = 0$, $Sq^2Sq^2 = Sq^1Sq^2Sq^1$. Hence $\mathcal{A}_2(2)^*$ is an 8-dimensional vector space.

(2) $\mathcal{A}_p(1)^*$ has the following basis for $i = 1, 2, \dots, p-1$

$$\mathcal{A}_p(1)^0 = \langle 1\rangle,\ \mathcal{A}_p(1)^{-1} = \langle Q_0\rangle,\ \mathcal{A}_p(1)^{-2i(p-1)} = \langle \wp^i\rangle,\ \mathcal{A}_p(1)^{-2i(p-1)-1} = \langle Q_0\wp^i, Q_1\wp^{i-1}\rangle,$$
$$\mathcal{A}_p(1)^{-2i(p-1)-2} = \langle Q_0Q_1\wp^{i-1}\rangle,\ \mathcal{A}_p(1)^{-2p(p-1)-1} = \langle Q_1\wp^{p-1}\rangle,\ \mathcal{A}_p(1)^{-2p(p-1)-2} = \langle Q_0Q_1\wp^{p-1}\rangle$$

with relations $Q_0^2 = Q_1^2 = 0$, $Q_1Q_0 = -Q_0Q_1$, $(\wp^1)^p = 0$, $(\wp^1)^i = i!\wp^i$, $\wp^iQ_0 = Q_0\wp^i + Q_1\wp^{i-1}$, $\wp^iQ_1 = Q_1\wp^i$ for $1 \leqq i \leqq p-1$. Hence $\mathcal{A}_p(1)^*$ is a $4p$-dimensional vector space.

The following facts are direct applications of (4.2.9).

Proposition 14.3.6 *Let $\hat{\gamma} : M^* \to M^* \widehat{\otimes}_{\boldsymbol{F}_p} \mathcal{A}_{p*}$ be a right $\mathcal{A}_{p*}$-comodule structure on M^*. Suppose that $\hat{\gamma}$ is the Milnor coaction associated with a left $\mathcal{A}_p^*$-module structure $\check{\gamma} : \mathcal{A}_p^* \otimes_{\boldsymbol{F}_p} M^* \to M^*$ on M^*, that is, $\Lambda(\check{\gamma}) = \hat{\gamma}$. Then, we have the following equality for $x \in M^*$.*

$$\hat{\gamma}(x) = \begin{cases} \displaystyle\sum_{E\in \mathrm{Seq}^b,\, R\in \mathrm{Seq}} (-1)^{\deg x \deg \tau(E)\xi(R)} \rho(E,R)(x) \otimes \tau(E)\xi(R) & p \text{ is an odd prime} \\ \displaystyle\sum_{R\in \mathrm{Seq}} \rho(R)(x) \otimes \zeta(R) & p = 2 \end{cases}$$

Proposition 14.3.7 *Let $\hat{\gamma} : M^* \to M^* \otimes_{\boldsymbol{F}_p} \mathcal{A}_p(n)_*$ be a right $\mathcal{A}_p(n)_*$-comodule structure on M^*. Suppose that $\hat{\gamma}$ is the Milnor coaction associated with a left $\mathcal{A}_p(n)^*$-module structure $\check{\gamma} : \mathcal{A}_p(n)^* \otimes_{\boldsymbol{F}_p} M^* \to M^*$ on M^*, that is, $\Lambda(\check{\gamma}) = \hat{\gamma}$. Then, we have the following equality for $x \in M^*$.*

$$\hat{\gamma}(x) = \begin{cases} \displaystyle\sum_{E\in \mathrm{Seq}^b,\, \ell(E)\leqq n,\, R\in \mathrm{Seq}(p,n)} (-1)^{\deg x \deg \tau(E)\xi(R)} \rho(E,R)(x) \otimes \tau(E)\xi(R) & p \text{ is an odd prime} \\ \displaystyle\sum_{R\in \mathrm{Seq}(2,n)} \rho(R)(x) \otimes \zeta(R) & p = 2 \end{cases}$$

For an $\mathcal{A}_p^*$-module M^* and a submodule N^* of M^*, let us denote by $[\theta]$ the class of $\theta \in M^*$ in M^*/N^*.

Example 14.3.8 *(1) $\mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^3$ has the following basis*

$$(\mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^3)^0 = \langle [1]\rangle,\quad (\mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^3)^{-1} = \langle [Sq^1]\rangle,\quad (\mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^3)^{-2} = \langle [Sq^2]\rangle,$$
$$(\mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^3)^{-3} = \langle [Sq^2Sq^1]\rangle,\quad (\mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^3)^{-4} = \langle [Sq^1Sq^2Sq^1]\rangle$$

with relations $Sq^1[Sq^1] = Sq^1[Sq^2] = Sq^2Sq^1[Sq^2Sq^1] = Sq^2[Sq^1Sq^2Sq^1] = 0$, $Sq^2[Sq^2] = Sq^1[Sq^2Sq^1] = [Sq^1Sq^2Sq^1]$. Hence $\mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^3$ is a 5-dimensional vector space.

(2) Since $\wp^i\beta\wp^1 = (-1)^i\binom{p-1}{i}\big((i+1)Q_0\wp^{i+1} + iQ_1\wp^i\big)$ for $1 \leqq i \leqq p-2$ and $\wp^{p-1}\beta\wp^1 = -Q_1\wp^{p-1}$, $\mathcal{A}_p(1)^*/\mathcal{A}_p(1)^*\beta\wp^1$ has the following basis for $i = 1, 2, \dots, p-1$.

$$(\mathcal{A}_p(1)^*/\mathcal{A}_p(1)^*\beta\wp^1)^0 = \langle [1]\rangle,\ (\mathcal{A}_p(1)^*/\mathcal{A}_p(1)^*\beta\wp^1)^{-1} = \langle [Q_0]\rangle,\ (\mathcal{A}_p(1)^*/\mathcal{A}_p(1)^*\beta\wp^1)^{-2i(p-1)} = \langle [\wp^i]\rangle,$$
$$(\mathcal{A}_p(1)^*/\mathcal{A}_p(1)^*\beta\wp^1)^{-2i(p-1)-1} = \langle [Q_1\wp^{i-1}]\rangle,\ (\mathcal{A}_p(1)^*/\mathcal{A}_p(1)^*\beta\wp^1)^{-2p} = \langle [Q_0Q_1]\rangle$$

Hence $\mathcal{A}_p(1)^*/\mathcal{A}_p(1)^*\beta\wp^1$ is a $(2p+1)$-dimensional vector space.

Proposition 14.3.9 *The $\mathcal{A}_2(2)_*$-comodule structure $\hat{\xi} : \mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^3 \to \mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^3 \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(2)_*$ is given as follows.*

$$\begin{aligned}
\hat{\xi}([1]) &= [1]\otimes 1 + [Sq^1]\otimes\zeta_1 + [Sq^2]\otimes\zeta_1^2 + [Sq^2Sq^1]\otimes\zeta_2 + [Sq^1Sq^2Sq^1]\otimes\zeta_1\zeta_2 \\
\hat{\xi}([Sq^1]) &= [Sq^1]\otimes 1 + [Sq^2Sq^1]\otimes\zeta_1^2 + [Sq^1Sq^2Sq^1]\otimes\zeta_1^3 + [Sq^1Sq^2Sq^1]\otimes\zeta_2 \\
\hat{\xi}([Sq^2]) &= [Sq^2]\otimes 1 + [Sq^1Sq^2Sq^1]\otimes\zeta_1^2 \\
\hat{\xi}([Sq^2Sq^1]) &= [Sq^2Sq^1]\otimes 1 + [Sq^1Sq^2Sq^1]\otimes\zeta_1 \\
\hat{\xi}([Sq^1Sq^2Sq^1]) &= [Sq^1Sq^2Sq^1]\otimes 1
\end{aligned}$$

Proof. We consider the Milnor basis $\{Sq(R)\,|\,R\in\mathrm{Seq}(2,2)\}$ of $\mathcal{A}_2(2)^*$ which is the dual basis of $\{\zeta(R)\,|\,,R\in\mathrm{Seq}(2,2)\}$ of $\mathcal{A}_2(2)_*$. Since $Sq(i,0)=Sq^i$, $Sq(0,1)=Sq^2Sq^1+Sq^3$, $Sq(1,1)=Sq^1Sq^2Sq^1$, $Sq(2,1)=Sq^2Sq^1Sq^2$, $Sq(3,1)=Sq^1Sq^2Sq^1Sq^2$, we have the following table of actions of the Milnor basis of $\mathcal{A}_2(2)^*$ on the basis of $\mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^3$.

	$[1]$	$[Sq^1]$	$[Sq^2]$	$[Sq^2Sq^1]$	$[Sq^1Sq^2Sq^1]$
1	$[1]$	$[Sq^1]$	$[Sq^2]$	$[Sq^2Sq^1]$	$[Sq^1Sq^2Sq^1]$
$Sq(1,0)$	$[Sq^1]$	0	0	$[Sq^1Sq^2Sq^1]$	0
$Sq(2,0)$	$[Sq^2]$	$[Sq^2Sq^1]$	$[Sq^1Sq^2Sq^1]$	0	0
$Sq(3,0)$	0	$[Sq^1Sq^2Sq^1]$	0	0	0
$Sq(0,1)$	$[Sq^2Sq^1]$	$[Sq^1Sq^2Sq^1]$	0	0	0
$Sq(1,1)$	$[Sq^1Sq^2Sq^1]$	0	0	0	0
$Sq(2,1)$	0	0	0	0	0
$Sq(3,1)$	0	0	0	0	0

Hence the assertion follows from (14.3.7). $\square$

Let $E(\tau)$ be an exterior algebra generated by a single element τ of degree d over a field K^* such that $K^i=\{0\}$ for $i\neq 0$. Define a coproduct $\mu : E(\tau)\to E(\tau)\otimes_{K^*}E(\tau)$ and counit $\varepsilon : E(\tau)\to K^*$ by $\mu(\tau)=1\otimes\tau+\tau\otimes 1$ and $\varepsilon(\tau)=0$, respectively. Thus we have a Hopf algebra $E(\tau)$.

Proposition 14.3.10 *For a graded K^*-module M^*, we put $D_d(M^*)=\{\delta\in\mathrm{Hom}_{K^*}(M^*,\Sigma^dM^*)\,|\,(\Sigma^d\delta)\delta=0\}$. For $\delta\in D_d(M^*)$, define $\bar{\delta} : M^*\to M^*\otimes_{K^*}E(\tau)$ by $\bar{\delta}(x)=x\otimes 1+\delta(x)\otimes\tau$. Then, $\bar{\delta}$ is a right $E(\tau)$-comodule structure map of M^*. Conversely, if $\varphi : M^*\to M^*\otimes_{K^*}E(\tau)$ is a right $E(\tau)$-comodule structure map of M^*, there exists unique map $\tilde{\varphi} : M^*\to\Sigma^dM^*$ that satisfies $\varphi(x)=x\otimes 1+\tilde{\varphi}(x)\otimes\tau$ for any $x\in M^*$ and φ belongs to $D_d(M^*)$.*

Proof. Suppose that a map $\varphi : M^*\to M^*\otimes_{K^*}E(\tau)$ satisfies $(id_{M^*}\otimes_{K^*}\varepsilon)\varphi(x)=x\otimes 1$ for $x\in M^*$. Then, $\varphi(x)-x\otimes 1\in\mathrm{Ker}\,(id_{M^*}\otimes_{K^*}\varepsilon)=M^*\otimes_{K^*}\mathrm{Ker}\,\varepsilon=M^*\otimes_{K^*}K^*\tau$. Hence there exists unique $y\in M^*$ such that $\varphi(x)-x\otimes 1=y\otimes\tau$ for each $x\in M^*$ which implies that there exists unique map $\tilde{\varphi} : M^*\to\Sigma^dM^*$ that satisfies $\varphi(x)=x\otimes 1+\tilde{\varphi}(x)\otimes\tau$ for any $x\in M^*$. We have the following equalities.

$$\begin{aligned}
(id_{M^*}\otimes_{K^*}\mu)\varphi(x) &= (id_{M^*}\otimes_{K^*}\mu)(x\otimes 1+\tilde{\varphi}(x)\otimes\tau) = x\otimes 1\otimes 1+\tilde{\varphi}(x)\otimes(1\otimes\tau+\tau\otimes 1) \\
&= x\otimes 1\otimes 1+\tilde{\varphi}(x)\otimes 1\otimes\tau+\tilde{\varphi}(x)\otimes\tau\otimes 1 \\
(\varphi\otimes_{K^*}id_{E(\tau)})\varphi(x) &= (\varphi\otimes_{K^*}id_{E(\tau)})(x\otimes 1+\tilde{\varphi}(x)\otimes\tau) = \varphi(x)\otimes 1+\varphi(\tilde{\varphi}(x))\otimes\tau \\
&= x\otimes 1\otimes 1+\tilde{\varphi}(x)\otimes\tau\otimes 1+\tilde{\varphi}(x)\otimes 1\otimes\tau+\tilde{\varphi}(\tilde{\varphi}(x))\otimes\tau\otimes\tau
\end{aligned}$$

It follows that $\varphi : M^*\to M^*\otimes_{K^*}E(\tau)$ is a right $E(\tau)$-comodule structure map of M^* if and only if $\tilde{\varphi}\in D_d(M^*)$. $\square$

We define $E(\tau)$-comodules $K^*\langle v\rangle$ and $K^*\langle v,w\rangle$ as follows. $K^*\langle v\rangle$ is generated by a single element v whose comodule structure is given by $v\mapsto v\otimes 1$. $K^*\langle v,w\rangle$ is generated by v and w whose comodule structure is given by $v\mapsto v\otimes 1+w\otimes\tau$ and $w\mapsto w\otimes 1$.

Proposition 14.3.11 *Every right $E(\tau)$-comodule is a direct sum of comodules of the form $K^*\langle v\rangle$ and $K^*\langle v,w\rangle$.*

Proof. Let $M*$ be a right $E(\tau)$-comodule with structure map $\varphi : M^* \to M^* \otimes_{K^*} E(\tau)$. There exists $\tilde{\varphi} \in D_d(M^*)$ satisfying $\varphi(x) = x \otimes 1 + \tilde{\varphi}(x) \otimes \tau$ for any $x \in M^*$ by (14.3.10). We choose a set $\{v_i\}_{i\in I}$ of elements of M^* so that $\{\tilde{\varphi}(v_i)\}_{i\in I}$ is a basis of the image of $\tilde{\varphi}$. We also choose a set $\{w_j\}_{j\in J}$ of elements of M^* so that $\{\tilde{\varphi}(v_i)\}_{i\in I} \cup \{w_j\}_{j\in J}$ is a basis of the kernel of $\tilde{\varphi}$. Then, $\{v_i\}_{i\in I} \cup \{\tilde{\varphi}(v_i)\}_{i\in I} \cup \{w_j\}_{j\in J}$ is a basis of M^*. The subspace spanned by w_j is a subcomodule of M^* which is isomorphic to $K^*\langle w_j\rangle$ and the subspace spanned by v_i and $\tilde{\varphi}(v_i)$ is also a subcomodule of M^* which is isomorphic to $K^*\langle v_i, \tilde{\varphi}(v_i)\rangle$. Hence M^* is isomorphic to $\bigoplus_{j\in J} K^*\langle w_j\rangle \oplus \bigoplus_{j\in J} K^*\langle v_i, \tilde{\varphi}(v_i)\rangle$. □

Consider the case $K^* = \boldsymbol{F}_2$ and $\tau = \zeta_1$, then $E(\tau) = \mathcal{A}_2(1)_*$. We denote by $i_v : \boldsymbol{F}_2\langle v\rangle \to \boldsymbol{F}_2\langle v\rangle \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(1)_*$ and $j_{v,w} : \boldsymbol{F}_2\langle v, w\rangle \to \boldsymbol{F}_2\langle v, w\rangle \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(1)_*$ the structure maps of right comodules. We also denote by $\rho_2 : \mathcal{A}_2(2)_* \to \mathcal{A}_2(1)_*$ the quotient map and put $\mu^r_{\rho_2} = (id_{\mathcal{A}_2(2)_*} \otimes_{\boldsymbol{F}_2} \rho_2)\mu : \mathcal{A}_2(2)_* \to \mathcal{A}_2(2)_* \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(1)_*$. Then, we have the following table.

$$
\begin{array}{lll}
\mu^r_{\rho_2}(1) = 1 \otimes 1 & \mu^r_{\rho_2}(\zeta_1) = 1 \otimes \zeta_1 + \zeta_1 \otimes 1 & \mu^r_{\rho_2}(\zeta_1^2) = \zeta_1^2 \otimes 1 \\
\mu^r_{\rho_2}(\zeta_1^3) = \zeta_1^2 \otimes \zeta_1 + \zeta_1^3 \otimes 1 & \mu^r_{\rho_2}(\zeta_2) = \zeta_1^2 \otimes \zeta_1 + \zeta_2 \otimes 1 & \mu^r_{\rho_2}(\zeta_1\zeta_2) = (\zeta_1^3 + \zeta_2) \otimes \zeta_1 + \zeta_1\zeta_2 \otimes 1 \\
\mu^r_{\rho_2}(\zeta_1^2\zeta_2) = \zeta_1^2\zeta_2 \otimes 1 & \mu^r_{\rho_2}(\zeta_1^3\zeta_2) = \zeta_1^2\zeta_2 \otimes \zeta_1 + \zeta_1^3\zeta_2 \otimes 1 &
\end{array}
$$

Let M^* be a right $\mathcal{A}_2(1)_*$-comodule with structure map $\varphi : M^* \to M^* \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(1)_*$ and put $\boldsymbol{M} = (\boldsymbol{F}_2, M^*, \alpha)$ where α is a $\boldsymbol{F}_2$-structure map of M^*. Recall that we denote the kernel of

$$id_{M^*} \otimes_{\boldsymbol{F}_2} \mu^r_{\rho_2} - \tilde{\theta}_{\mathcal{A}_2(2)_*,\mathcal{A}_2(1)_*}(\boldsymbol{M})(\varphi \otimes_{\boldsymbol{F}_2} id_{\mathcal{A}_2(2)_*}) : M^* \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(2)_* \to M^* \otimes_{\boldsymbol{F}_2} (\mathcal{A}_2(2)_* \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(1)_*)$$

by $\widetilde{P}^{(\mathcal{A}_2(2)_*,\mu^r_{\rho_2})}_{(M^*,\varphi)} : (M^*, \varphi)\,\square_{\mathcal{A}_2(1)_*}(\mathcal{A}_2(2)_*, \mu^r_{\rho_2}) \to M^* \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(2)_*$ and that we denote by φ_{ρ_2} the right $\mathcal{A}_2(2)_*$-comodule structure map of $(M^*, \varphi)\,\square_{\mathcal{A}_2(1)_*}(\mathcal{A}_2(2)_*, \mu^r_{\rho_2})$.

Proposition 14.3.12 *If we put $v_0 = v \otimes 1$, $v_2 = v \otimes \zeta_1^2$, $v_3 = v \otimes (\zeta_1^3 + \zeta_2)$, $v_5 = v \otimes \zeta_1^2\zeta_2$, then $\{v_0, v_2, v_3, v_5\}$ is a basis of $(\boldsymbol{F}_2\langle v\rangle, i_v)\,\square_{\mathcal{A}_2(1)_*}(\mathcal{A}_2(2)_*, \mu^r_{\rho_2})$ and the following equalities hold.*

$$
\begin{array}{ll}
i_{v\rho_2}(v_0) = v_0 \otimes 1 & i_{v\rho_2}(v_2) = v_2 \otimes 1 + v_0 \otimes \zeta_1^2 \\
i_{v\rho_2}(v_3) = v_3 \otimes 1 + v_2 \otimes \zeta_1 + v_0 \otimes (\zeta_1^3 + \zeta_2) & i_{v\rho_2}(v_5) = v_5 \otimes 1 + v_3 \otimes \zeta_1^2 + v_2 \otimes \zeta_2 + v_0 \otimes \zeta_1^2\zeta_2
\end{array}
$$

Let us denote by $\tilde{i}_{v\rho_2} : \mathcal{A}_2(2)^* \otimes_{\boldsymbol{F}_2} : (\boldsymbol{F}_2\langle v\rangle, i_v)\,\square_{\mathcal{A}_2(1)_*}(\mathcal{A}_2(2)_*, \mu^r_{\rho_2}) \to (\boldsymbol{F}_2\langle v\rangle, i_v)\,\square_{\mathcal{A}_2(1)_*}(\mathcal{A}_2(2)_*, \mu^r_{\rho_2})$ the left $\mathcal{A}_2(2)^*$-module structure map such that the Milnor coaction associated with $\tilde{i}_{v\rho_2}$ is $i_{v\rho_2}$. Then the following equalities hold.

$$
\begin{array}{lll}
\tilde{i}_{v\rho_2}(Sq(2,0) \otimes v_2) = v_0 & \tilde{i}_{v\rho_2}(Sq(1,0) \otimes v_3) = v_2 & \tilde{i}_{v\rho_2}(Sq(3,0) \otimes v_3) = \tilde{i}_{v\rho_2}(Sq(0,1) \otimes v_3) = v_0 \\
\tilde{i}_{v\rho_2}(Sq(2,0) \otimes v_5) = v_3 & \tilde{i}_{v\rho_2}(Sq(0,1) \otimes v_5) = v_2 & \tilde{i}_{v\rho_2}(Sq(2,1) \otimes v_5) = v_0
\end{array}
$$

For $R \in \mathrm{Seq}(2,2) - \{\boldsymbol{0}\}$, we have $\tilde{i}_{v\rho_2}(Sq(R) \otimes v_j) = 0$ if the pair (j, R) is not in the above table. Moreover, $(\boldsymbol{F}_2\langle v\rangle, i_v)\,\square_{\mathcal{A}_2(1)_*}(\mathcal{A}_2(2)_*, \mu^r_{\rho_2})$ is isomophic to $\Sigma^{\deg v+5}\mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^1$.

Lemma 14.3.13 *If we put $w_0 = w\otimes 1$, $w_1 = v\otimes 1 + w\otimes\zeta_1$, $w_2 = w\otimes\zeta_1^2$, $w_3 = v\otimes\zeta_1^2 + w\otimes\zeta_1^3$, $\tilde{w}_3 = v\otimes\zeta_1^2 + w\otimes\zeta_2$, $w_4 = v \otimes (\zeta_1^3 + \zeta_2) + w \otimes \zeta_1\zeta_2$, $w_5 = w \otimes \zeta_1^2\zeta_2$, $w_6 = v \otimes \zeta_1^2\zeta_2 + w \otimes \zeta_1^3\zeta_2$, then $\{w_0, w_1, w_2, w_3, \tilde{w}_3, w_4, w_5, w_6\}$ is a basis of $(\boldsymbol{F}_2\langle v, w\rangle, j_{v,w})\,\square_{\mathcal{A}_2(1)_*}(\mathcal{A}_2(2)_*, \mu^r_{\rho_2})$ and the following equalities holds.*

$$
\begin{aligned}
j_{v,w\rho_2}(w_0) &= w_0 \otimes 1 \\
j_{v,w\rho_2}(w_1) &= w_1 \otimes 1 + w_0 \otimes \zeta_1 \\
j_{v,w\rho_2}(w_2) &= w_2 \otimes 1 + w_0 \otimes \zeta_1^2 \\
j_{v,w\rho_2}(w_3) &= w_3 \otimes 1 + w_2 \otimes \zeta_1 + w_1 \otimes \zeta_1^2 + w_0 \otimes \zeta_1^3 \\
j_{v,w\rho_2}(\tilde{w}_3) &= \tilde{w}_3 \otimes 1 + w_1 \otimes \zeta_1^2 + w_0 \otimes \zeta_2 \\
j_{v,w\rho_2}(w_4) &= w_4 \otimes 1 + \bar{w}_3 \otimes \zeta_1 + w_2 \otimes \zeta_1^2 + w_1 \otimes (\zeta_1^3 + \zeta_2) + w_0 \otimes \zeta_1\zeta_2 \\
j_{v,w\rho_2}(w_5) &= w_5 \otimes 1 + (w_3 + \tilde{w}_3) \otimes \zeta_1^2 + w_2 \otimes \zeta_2 + w_0 \otimes \zeta_1^2\zeta_2 \\
j_{v,w\rho_2}(w_6) &= w_6 \otimes 1 + w_5 \otimes \zeta_1 + w_4 \otimes \zeta_1^2 + w_3 \otimes \zeta_2 + (w_3 + \tilde{w}_3) \otimes \zeta_1^3 + w_2 \otimes \zeta_1\zeta_2 + w_1 \otimes \zeta_1^2\zeta_2 + w_0 \otimes \zeta_1^3\zeta_2
\end{aligned}
$$

Proposition 14.3.14 *For a graded $\boldsymbol{F}_2$-module M^*, we put*

$$\mathcal{D}(M^*) = \{(\varphi,\psi) \in \mathrm{Hom}_{\boldsymbol{F}_2}(M^*, \Sigma M^*) \times \mathrm{Hom}_{\boldsymbol{F}_2}(M^*, \Sigma^2 M^*) \,|\, (\Sigma\varphi)\varphi = 0,\ (\Sigma^3\varphi)(\Sigma\psi)\varphi = (\Sigma^2\psi)\psi\}.$$

For $(\varphi,\psi) \in \mathcal{D}(M^*)$, define $\delta_{(\varphi,\psi)} : M^* \to M^* \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(2)_*$ by

$$\begin{aligned}\delta_{(\varphi,\psi)}(x) = {} & x\otimes 1 + \varphi(x)\otimes\zeta_1 + \psi(x)\otimes\zeta_1^2 + \varphi(\psi(x))\otimes\zeta_1^3 + (\varphi(\psi(x)) + \psi(\varphi(x)))\otimes\zeta_2 \\ & + \varphi(\psi(\varphi(x)))\otimes\zeta_1\zeta_2 + \psi(\varphi(\psi(x)))\otimes\zeta_1^2\zeta_2 + \varphi(\psi(\varphi(\psi(x))))\otimes\zeta_1^3\zeta_2\end{aligned}$$

Then, $\delta_{(\varphi,\psi)}$ is a right $\mathcal{A}_2(2)_*$-comodule structure map of M^*. Conversely, if $\gamma : M^* \to M^* \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(2)_*$ is a right $\mathcal{A}_2(2)_*$-comodule structure map of M^*, there exists unique $(\varphi,\psi) \in \mathcal{D}(M^*)$ that satisfies $\gamma = \delta_{(\varphi,\psi)}$.

Proof. Suppose that a map $\gamma : M^* \to M^* \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(2)_*$ satisfies $(id_{M^*} \otimes_{\boldsymbol{F}_2} \varepsilon)\gamma(x) = x \otimes 1$ for $x \in M^*$. Then, $\gamma(x) - x\otimes 1 \in \mathrm{Ker}\,(id_{M^*} \otimes_{\boldsymbol{F}_2} \varepsilon) = M^* \otimes_{\boldsymbol{F}_2} \mathrm{Ker}\,\varepsilon$. Since $\mathrm{Ker}\,\varepsilon$ has a basis $\{\zeta_1, \zeta_1^2, \zeta_1^3, \zeta_2, \zeta_1\zeta_2, \zeta_1^2\zeta_2, \zeta_1^3\zeta_2\}$, there exist unique $v_i \in M^{\deg x - i}$ $(i = 1,2,3)$ and $w_j \in M^{\deg x - j}$ $(j = 3,4,5,6)$ such that

$$\gamma(x) - x\otimes 1 = v_1\otimes\zeta_1 + v_2\otimes\zeta_1^2 + v_3\otimes\zeta_1^3 + w_3\otimes\zeta_2 + w_4\otimes\zeta_1\zeta_2 + w_5\otimes\zeta_1^2\zeta_2 + w_6\otimes\zeta_1^3\zeta_2$$

for each $x \in M^*$ which implies that there exists unique maps $\varphi_i : M^* \to \Sigma^i M^*$ $(i = 1,2,3)$ and $\psi_j : M^* \to \Sigma^j M^*$ $(i = 3,4,5,6)$ that satisfy

$$\gamma(x) = x\otimes 1 + \varphi_1(x)\otimes\zeta_1 + \varphi_2(x)\otimes\zeta_1^2 + \varphi_3(x)\otimes\zeta_1^3 + \psi_3(x)\otimes\zeta_2 + \psi_4(x)\otimes\zeta_1\zeta_2 + \psi_5(x)\otimes\zeta_1^2\zeta_2 + \psi_6(x)\otimes\zeta_1^3\zeta_2$$

for any $x \in M^*$. We have the following equality.

$$\begin{aligned}&(\gamma\otimes_{\boldsymbol{F}_2} id_{\mathcal{A}_2(2)_*})\gamma(x) - (id_{M^*}\otimes_{\boldsymbol{F}_2}\mu)\gamma(x) \\
&\quad = \varphi_1(\varphi_1(x))\otimes\zeta_1\otimes\zeta_1 + (\varphi_3(x)+\varphi_1(\varphi_2(x)))\otimes\zeta_1\otimes\zeta_1^2 + (\varphi_3(x)+\psi_3(x)+\varphi_2(\varphi_1(x)))\otimes\zeta_1^2\otimes\zeta_1 \\
&\qquad + (\psi_4(x)+\varphi_3(\varphi_1(x)))\otimes\zeta_1^3\otimes\zeta_1 + (\psi_4(x)+\varphi_2(\varphi_2(x)))\otimes\zeta_1^2\otimes\zeta_1^2 + \varphi_1(\varphi_3(x))\otimes\zeta_1\otimes\zeta_1^3 \\
&\qquad + (\psi_4(x)+\varphi_1(\psi_3(x)))\otimes\zeta_1\otimes\zeta_2 + (\psi_4(x)+\psi_3(\varphi_1(x)))\otimes\zeta_2\otimes\zeta_1 + (\psi_5(x)+\varphi_2(\varphi_3(x)))\otimes\zeta_1^2\otimes\zeta_1^3 \\
&\qquad + (\psi_5(x)+\varphi_2(\psi_3(x)))\otimes\zeta_1^2\otimes\zeta_2 + (\psi_5(x)+\psi_3(\varphi_2(x)))\otimes\zeta_2\otimes\zeta_1^2 + \varphi_3(\varphi_2(x))\otimes\zeta_1^3\otimes\zeta_1^2 \\
&\qquad + \psi_4(\varphi_1(x))\otimes\zeta_1\zeta_2\otimes\zeta_1 + \varphi_1(\psi_4(x))\otimes\zeta_1\otimes\zeta_1\zeta_2 + (\psi_6(x)+\varphi_3(\varphi_3(x)))\otimes\zeta_1^3\otimes\zeta_1^3 \\
&\qquad + (\psi_6(x)+\varphi_3(\psi_3(x)))\otimes\zeta_1^3\otimes\zeta_2 + (\psi_6(x)+\psi_3(\varphi_3(x)))\otimes\zeta_2\otimes\zeta_1^3 + (\psi_6(x)+\psi_4(\varphi_2(x)))\otimes\zeta_1\zeta_2\otimes\zeta_1^2 \\
&\qquad + (\psi_6(x)+\varphi_1(\psi_5(x)))\otimes\zeta_1\otimes\zeta_1^2\zeta_2 + (\psi_6(x)+\varphi_2(\psi_4(x)))\otimes\zeta_1^2\otimes\zeta_1\zeta_2 + (\psi_6(x)+\psi_5(\varphi_1(x)))\otimes\zeta_1^2\zeta_2\otimes\zeta_1 \\
&\qquad + \psi_3(\psi_3(x))\otimes\zeta_2\otimes\zeta_2 + \varphi_1(\psi_6(x))\otimes\zeta_1\otimes\zeta_1^3\zeta_2 + \psi_6(\varphi_1(x))\otimes\zeta_1^3\zeta_2\otimes\zeta_1 + \varphi_2(\psi_5(x))\otimes\zeta_1^2\otimes\zeta_1^2\zeta_2 \\
&\qquad + \psi_5(\varphi_2(x))\otimes\zeta_1^2\zeta_2\otimes\zeta_1^2 + \varphi_3(\psi_4(x))\otimes\zeta_1^3\otimes\zeta_1\zeta_2 + \psi_4(\varphi_3(x))\otimes\zeta_1\zeta_2\otimes\zeta_1^3 + \psi_3(\psi_4(x))\otimes\zeta_2\otimes\zeta_1\zeta_2 \\
&\qquad + \psi_4(\psi_3(x))\otimes\zeta_1\zeta_2\otimes\zeta_2 + \varphi_2(\psi_6(x))\otimes\zeta_1^2\otimes\zeta_1^3\zeta_2 + \psi_6(\varphi_2(x))\otimes\zeta_1^3\zeta_2\otimes\zeta_1^2 + \varphi_3(\psi_5(x))\otimes\zeta_1^3\otimes\zeta_1^2\zeta_2 \\
&\qquad + \psi_5(\varphi_3(x))\otimes\zeta_1^2\zeta_2\otimes\zeta_1^3 + \psi_3(\psi_5(x))\otimes\zeta_2\otimes\zeta_1^2\zeta_2 + \psi_5(\psi_3(x))\otimes\zeta_1^2\zeta_2\otimes\zeta_2 + \psi_4(\psi_4(x))\otimes\zeta_1\zeta_2\otimes\zeta_1\zeta_2 \\
&\qquad + \varphi_3(\psi_6(x))\otimes\zeta_1^3\otimes\zeta_1^3\zeta_2 + \psi_6(\varphi_3(x))\otimes\zeta_1^3\zeta_2\otimes\zeta_1^3 + \psi_3(\psi_6(x))\otimes\zeta_2\otimes\zeta_1^3\zeta_2 + \psi_6(\psi_3(x))\otimes\zeta_1^3\zeta_2\otimes\zeta_2 \\
&\qquad + \psi_4(\psi_5(x))\otimes\zeta_1\zeta_2\otimes\zeta_1^2\zeta_2 + \psi_5(\psi_4(x))\otimes\zeta_1^2\zeta_2\otimes\zeta_1\zeta_2 + \psi_4(\psi_6(x))\otimes\zeta_1\zeta_2\otimes\zeta_1^3\zeta_2 + \psi_6(\psi_4(x))\otimes\zeta_1^3\zeta_2\otimes\zeta_1\zeta_2 \\
&\qquad + \psi_5(\psi_5(x))\otimes\zeta_1^2\zeta_2\otimes\zeta_1^2\zeta_2 + \psi_5(\psi_6(x))\otimes\zeta_1^2\zeta_2\otimes\zeta_1^3\zeta_2 + \psi_6(\psi_5(x))\otimes\zeta_1^3\zeta_2\otimes\zeta_1^2\zeta_2 + \psi_6(\psi_6(x))\otimes\zeta_1^3\zeta_2\otimes\zeta_1^3\zeta_2\end{aligned}$$

It can be verified from the above equality that $\gamma : M^* \to M^* \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(2)_*$ is a right $\mathcal{A}_2(2)_*$-comodule structure map of M^* if and only if the following equalities hold.

$$\varphi_1\varphi_1 = 0,\ \varphi_3 = \varphi_1\varphi_2,\ \psi_3 = \varphi_1\varphi_2 + \varphi_2\varphi_1,\ \psi_4 = \varphi_1\varphi_2\varphi_1 = \varphi_2\varphi_2,\ \psi_5 = \varphi_2\varphi_1\varphi_2,\ \psi_6 = \varphi_1\varphi_2\varphi_1\varphi_2$$

Put $\varphi = \varphi_1$ and $\psi = \varphi_2$. Then, $\gamma : M^* \to M^* \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(2)_*$ is a right $\mathcal{A}_2(2)_*$-comodule structure map of M^* if and only if $(\varphi,\psi) \in \mathcal{D}(M^*)$. Moreover, if $\gamma : M^* \to M^* \otimes_{\boldsymbol{F}_2} \mathcal{A}_2(2)_*$ is a right $\mathcal{A}_2(2)_*$-comodule structure map of M^*, $\gamma = \delta_{(\varphi,\psi)}$ holds and the uniqueness of (φ,ψ) is clear. □

Put $\boldsymbol{J} = (\boldsymbol{F}_2, \mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^3, \alpha_2)$ and let $\boldsymbol{\xi} = (id_{\mathcal{A}_2(2)^*}, \xi) : u^*_{\mathcal{A}_2(2)_*}(\boldsymbol{J}) \to u^*_{\mathcal{A}_2(2)_*}(\boldsymbol{J})$ be the representation of $\mathcal{A}_2(2)_*$ on $\mathcal{A}_2(2)^*/\mathcal{A}_2(2)^*Sq^3$ defined from $\hat{\xi}$.

15 Unstable representations

15.1 Filtered modules

We assume that K^* is a field such that $K^i = \{0\}$ if $i \neq 0$ in this section.

For an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ with an increasing filtration $(F_iM^*)_{i\in\boldsymbol{Z}}$, we put $E_i^*M^* = F_iM^*/F_{i-1}M^*$. Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ with filtrations $(F_iM^*)_{i\in\boldsymbol{Z}}$ and $(F_iN^*)_{i\in\boldsymbol{Z}}$, respectively. We define a filtration $(F_i(M^*\otimes_{K^*}N^*))_{i\in\boldsymbol{Z}}$ of $M^*\otimes_{K^*}N^*$ by

$$F_i(M^*\otimes_{K^*}N^*) = \sum_{j+k=i} F_jM^*\otimes_{K^*}F_kN^*.$$

We denote by $\rho_{M^*,i} : F_iM^* \to E_i^*M^*$ the quotient map and by $\eta_{j,k} : F_jM^*\otimes_{K^*}F_kN^* \to F_{j+k}(M^*\otimes_{K^*}N^*)$ the inclusion map for $j,k\in\boldsymbol{Z}$. Then a composition

$$F_jM^*\otimes_{K^*}F_kN^* \xrightarrow{\eta_{j,k}} F_{j+k}(M^*\otimes_{K^*}N^*) \xrightarrow{\rho_{M^*\otimes_{K^*}N^*,j+k}} E_{j+k}^*(M^*\otimes_{K^*}N^*)$$

induces a map $\varphi_{j,k} : E_j^*M^*\otimes_{K^*}E_k^*N^* \to E_{j+k}^*(M^*\otimes_{K^*}N^*)$.

Proposition 15.1.1 *For $u\in\boldsymbol{Z}$, we define a map $\Phi_u = \Phi_{M^*,N^*,u} : \bigoplus\limits_{j+k=u}(E_j^*M^*\otimes_{K^*}E_k^*N^*) \to E_u^*(M^*\otimes_{K^*}N^*)$ by $\Phi_{M^*,N^*,u}((x_{jk})) = \sum\limits_{j+k=u}\varphi_{j,k}(x_{jk})$. Then, $\Phi_{M^*,N^*,u}$ is an isomorphism.*

Proof. Let $(v_s)_{s\in S}$ be a basis of M^* indexed by a set S with filtration $(S_j)_{j\in\boldsymbol{Z}}$ such that $v_s\in F_jM^* - F_{j-1}M^*$ for $s\in S_j - S_{j-1}$. Similarly, let $(w_t)_{t\in T}$ be a basis of N^* indexed by a set T with filtration $(T_k)_{k\in\boldsymbol{Z}}$ such that $w_t\in F_kN^* - F_{k-1}N^*$ for $t\in T_k - T_{k-1}$. Suppose $(x_{jk})\in\bigoplus\limits_{j+k=u}(E_jM^*\otimes_{K^*}E_kN^*)$, where $x_{jk}\in E_jM^*\otimes_{K^*}E_kN^*$. Take $\bar{x}_{jk}\in F_jM^*\otimes_{K^*}F_kN^*$ which is mapped to x_{jk} by $\rho_{M^*,j}\otimes_{K^*}\rho_{N^*,k} : F_jM^*\otimes_{K^*}F_kN^* \to E_jM^*\otimes_{K^*}E_kN^*$. We may assume that $\bar{x}_{jk} = \sum\limits_{s\in S_j-S_{j-1},t\in T_k-T_{k-1}} a_{st}v_s\otimes w_t$ for $a_{st}\in K^*$. Since

$$\begin{aligned}\Phi_{M^*,N^*,u}((x_{jk})) &= \sum_{j+k=u}\varphi_{j,k}(x_{jk}) = \sum_{j+k=u}\rho_{M^*\otimes_{K^*}N^*,u}\eta_{j,k}(\bar{x}_{jk}) \\ &= \rho_{M^*\otimes_{K^*}N^*,u}\Biggl(\sum_{j+k=u}\ \sum_{s\in S_j-S_{j-1},t\in T_k-T_{k-1}} a_{st}v_s\otimes w_t\Biggr),\end{aligned}$$

$(x_{jk})\in\operatorname{Ker}\Phi_{M^*,N^*,u}$ if and only if $\sum\limits_{j+k=u}\ \sum\limits_{s\in S_j-S_{j-1},t\in T_k-T_{k-1}} a_{st}v_s\otimes w_t\in F_{u-1}(M^*\otimes_{K^*}N^*)$. On the other hand, since $(v_s\otimes w_t)_{(s,t)\in\bigcup\limits_{j+k=u}(S_j\times T_k)}$ is a basis of $F_u(M^*\otimes_{K^*}N^*)$ and $v_s\otimes w_t\notin F_{u-1}(M^*\otimes_{K^*}N^*)$ if $s\in S_j - S_{j-1}$, $t\in T_k - T_{k-1}$ and $j+k=u$, we see that $\sum\limits_{j+k=u}\ \sum\limits_{s\in S_j-S_{j-1},t\in T_k-T_{k-1}} a_{st}v_s\otimes w_t\in F_{u-1}(M^*\otimes_{K^*}N^*)$ implies $a_{st}=0$ for all $(s,t)\in\bigcup\limits_{j+k=u}(S_j-S_{j-1})\times(T_k-T_{k-1})$. Hence $\Phi_{M^*,N^*,u}$ is injective. It is clear that $\Phi_{M^*,N^*,u}$ is surjective. $\square$

Definition 15.1.2 *For an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ with an increasing filtration $(F_iM^*)_{i\in\boldsymbol{Z}}$, we denote by $\kappa_{M^*,i} : F_iM^* \to M^*$ the inclusion map. For an object P^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, define a filtration $(F_s\mathcal{H}om^*(M^*,P^*))_{s\in\boldsymbol{Z}}$ of $\mathcal{H}om^*(M^*,P^*)$ by*

$$F_s\mathcal{H}om^*(M^*,P^*) = \operatorname{Ker}(\kappa_{M^*,-s-1}^* : \mathcal{H}om^*(M^*,P^*) \to \mathcal{H}om^*(F_{-s-1}M^*,P^*))$$

Note that $(F_s\mathcal{H}om^*(M^*,P^*))_{s\in\boldsymbol{Z}}$ is an increasing filtration. If $P^* = K^*$, we denote $(F_s\mathcal{H}om^*(M^*,K^*))_{s\in\boldsymbol{Z}}$ by $(F_sM^{**})_{s\in\boldsymbol{Z}}$ and call this the dual filtration of $(F_iM^*)_{i\in\boldsymbol{Z}}$.

We denote by $\iota_{M^*,i} : F_{i-1}M^* \to F_iM^*$, $\tilde{\kappa}_{M^*,i} : E_i^*M^* \to M^*/F_{i-1}M^*$ the inclusion maps and also denote by $\pi_{M^*,i} : M^* \to M^*/F_{i-1}M^*$, $\tilde{\iota}_{M^*,i} : M^*/F_{i-1}M^* \to M^*/F_iM^*$ the quotient maps. We also denote by

$$\bar{\kappa}_{M^*,P^*,i} : \mathcal{H}om^*(M^*,P^*)/F_{-i-1}\mathcal{H}om^*(M^*,P^*) \to \mathcal{H}om^*(F_iM^*,P^*)$$

be the map induced by $\kappa_{M^*,i}^* : \mathcal{H}om^*(M^*,P^*) \to \mathcal{H}om^*(F_iM^*,P^*)$, that is, $\bar{\kappa}_{M^*,P^*,i}$ is unique map that satisfies $\bar{\kappa}_{M^*,P^*,i}\pi_{\mathcal{H}om^*(M^*,P^*),-i} = \kappa_{M^*,i}^*$.

Remark 15.1.3 *(1) If $\kappa^*_{M^*,i}$ is a quotient map, $\bar\kappa_{M^*,i}$ is an isomorphism. If $\kappa_{M^*,i}$ has a continuous left inverse, $\kappa^*_{M^*,i}$ is a quotient map by (3.1.7). For example, if M^* has skeletal topology, $\kappa_{M^*,i}$ has a continuous left inverse.*

(2) Since $0 \to \mathcal{H}om^*(M^*/F_{-i-1}M^*, P^*) \xrightarrow{\pi^*_{M^*,-i}} \mathcal{H}om^*(M^*, P^*) \xrightarrow{\kappa^*_{M^*,-i-1}} \mathcal{H}om^*(F_{-i-1}M^*, P^*)$ is exact, $\pi^*_{M^*,-i} : \mathcal{H}om^*(M^*/F_{-i-1}M^*, P^*) \to \mathcal{H}om^*(M^*, P^*)$ is an isomorphism onto $F_i\mathcal{H}om^*(M^*, P^*)$. We denote by $\hat\pi_{M^*,P^*,i} : \mathcal{H}om^*(M^*/F_{-i-1}M^*, P^*) \to F_i\mathcal{H}om^*(M^*, P^*)$ the isomorphism obtained from $\pi^*_{M^*,-i}$.

Since the vertical columns of the following diagram is exact and the lower rectangle is commutative, there exist unique map $\tilde\rho_{M^*,P^*,i} : E^*_i\mathcal{H}om^*(M^*, P^*) \to \mathcal{H}om^*(E^*_{-i}M^*, P^*)$ that makes the upper rectangle commute.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
E^*_i\mathcal{H}om^*(M^*, P^*) \arrow[r, dashed, "\tilde\rho_{M^*,P^*,i}"] \arrow[d, "\tilde\kappa_{\mathcal{H}om^*(M^*,P^*),i}"] & \mathcal{H}om^*(E^*_{-i}M^*, P^*) \arrow[d, "\rho^*_{M^*,-i}"] \\
\mathcal{H}om^*(M^*, P^*)/F_{i-1}\mathcal{H}om^*(M^*, P^*) \arrow[r, "\bar\kappa_{M^*,P^*,-i}"] \arrow[d, "\iota_{\mathcal{H}om^*(M^*,P^*),i}"] & \mathcal{H}om^*(F_{-i}M^*, P^*) \arrow[d, "\iota^*_{M^*,-i}"] \\
\mathcal{H}om^*(M^*, P^*)/F_i\mathcal{H}om^*(M^*, P^*) \arrow[r, "\bar\kappa_{M^*,P^*,-i-1}"] & \mathcal{H}om^*(F_{-i-1}M^*, P^*)
\end{tikzcd}$$

Thus we have the following result by (15.1.3).

Proposition 15.1.4 *If $\bar\kappa_{M^*,P^*,i} : \mathcal{H}om^*(M^*, P^*)/F_{-i-1}\mathcal{H}om^*(M^*, P^*) \to \mathcal{H}om^*(F_iM^*, P^*)$ is a quotient map for any $i \in \boldsymbol{Z}$, $\tilde\rho_{M^*,P^*,i} : E^*_i\mathcal{H}om^*(M^*, P^*) \to \mathcal{H}om^*(E^*_{-i}M^*, P^*)$ is an isomorphism for any $i \in \boldsymbol{Z}$. In particular, if M^* has skeletal topology, $\tilde\rho_{M^*,P^*,i}$ is an isomorphism.*

Remark 15.1.5 *By the definitions of $\bar\kappa_{M^*,P^*,-i} : \mathcal{H}om^*(M^*, P^*)/F_{i-1}\mathcal{H}om^*(M^*, P^*) \to \mathcal{H}om^*(F_{-i}M^*, P^*)$ and $\hat\pi_{M^*,P^*,i} : \mathcal{H}om^*(M^*/F_{-i-1}M^*, P^*) \to F_i\mathcal{H}om^*(M^*, P^*)$, we have $\bar\kappa_{M^*,P^*,-i}\pi_{\mathcal{H}om^*(M^*,P^*),i} = \kappa^*_{M^*,-i}$ and $\kappa_{\mathcal{H}om^*(M^*,P^*),i}\hat\pi_{M^*,P^*,i} = \pi^*_{M^*,-i}$. Hence it follows from the definition of $\tilde\rho_{M^*,P^*,i}$, the following equalities hold.*

$$\begin{aligned}
\rho^*_{M^*,-i}\tilde\rho_{M^*,P^*,i}\rho_{\mathcal{H}om^*(M^*,P^*),i}\hat\pi_{M^*,P^*,i} &= \bar\kappa_{M^*,P^*,-i}\tilde\kappa_{\mathcal{H}om^*(M^*,P^*),i}\rho_{\mathcal{H}om^*(M^*,P^*),i}\hat\pi_{M^*,P^*,i} \\
&= \bar\kappa_{M^*,P^*,-i}\pi_{\mathcal{H}om^*(M^*,P^*),i}\kappa_{\mathcal{H}om^*(M^*,P^*),i}\hat\pi_{M^*,P^*,i} \\
&= \kappa^*_{M^*,-i}\pi^*_{M^*,-i} = (\pi_{M^*,-i}\kappa_{M^*,-i})^* \\
&= (\tilde\kappa_{M^*,-i}\rho_{M^*,-i})^* = \rho^*_{M^*,-i}\tilde\kappa^*_{M^*,-i}
\end{aligned}$$

Since $\rho^*_{M^*,-i} : \mathcal{H}om^*(E^*_{-i}M^*, P^*) \to \mathcal{H}om^*(F_{-i}M^*, P^*)$ is injective, we see that $\tilde\rho_{M^*,P^*,i}$makes the following diagram commute.

$$\begin{tikzcd}
F_i\mathcal{H}om^*(M^*, P^*) \arrow[r, "\rho_{\mathcal{H}om^*(M^*,P^*),i}"] & E^*_i\mathcal{H}om^*(M^*, P^*) \arrow[d, "\tilde\rho_{M^*,P^*,i}"] \\
\mathcal{H}om^*(M^*/F_{-i-1}M^*, P^*) \arrow[u, "\hat\pi_{M^*,P^*,i}"] \arrow[r, "\tilde\kappa^*_{M^*,-i}"] & \mathcal{H}om^*(E^*_{-i}M^*, P^*)
\end{tikzcd}$$

Proposition 15.1.6 *Let M^*, N^* and P^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$. Suppose that filtrations $(F_iM^*)_{i\in\boldsymbol{Z}}$ of M^* and $(F_iN^*)_{i\in\boldsymbol{Z}}$ of N^* are given such that a morphism $f : M^* \to N^*$ of $\mathcal{T}op\mathcal{M}od_{K^*}$ satisfies $f(F_iM^*) \subset F_iN^*$ for any $i \in \boldsymbol{Z}$. Then $f^* : \mathcal{H}om^*(N^*.P^*) \to \mathcal{H}om^*(M^*, P^*)$ satisfies $f^*(F_s\mathcal{H}om^*(N^*, P^*)) \subset F_s\mathcal{H}om^*(M^*, P^*)$ for any $s \in \boldsymbol{Z}$.*

Proof. For $\varphi \in (F_s\mathcal{H}om^*(N^*, P^*))^n$, since $\varphi(\Sigma^nF_{-s-1}N^*) = \{0\}$, we have

$$(f^*(\varphi))(\Sigma^nF_{-s-1}M^*) = \varphi\Sigma^nf(\Sigma^nF_{-s-1}M^*) = \varphi\Sigma^n(f(F_{-s-1}M^*)) \subset \varphi(\Sigma^nF_{-s-1}N^*) = \{0\}$$

which implies $f^*(\varphi) \in F_s\mathcal{H}om^*(M^*, P^*)$. □

Proposition 15.1.7 *Let $g : P^* \to Q^*$ be a morphism of $\mathcal{T}op\mathcal{M}od_{K^*}$ and M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ with a filtration $(F_iM^*)_{i\in\boldsymbol{Z}}$. Then, $g_* : \mathcal{H}om^*(M^*, P^*) \to \mathcal{H}om^*(M^*, Q^*)$ maps $F_i\mathcal{H}om^*(M^*, P^*)$ into $F_i\mathcal{H}om^*(M^*, Q^*)$.*

Proof. In fact, for $\varphi \in (F_i\mathcal{H}om^*(M^*,P^*))^m$, we have $(g_*(\varphi))(\Sigma^m F_{-i-1}M^*) = g(\varphi(\Sigma^m F_{-i-1}M^*)) = \{0\}$ since $\varphi(\Sigma^m F_{-i-1}M^*) = \{0\}$, which shows $g_*(\varphi) \in (F_i\mathcal{H}om^*(M^*,Q^*))^m$. □

Proposition 15.1.8 *Suppose that an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ has skeletal topology. Let $\mathfrak{F} = (F_iM^*)_{i\in\boldsymbol{Z}}$ be a filtration of M^* and $\mathfrak{F}^* = (F_i\mathcal{H}om^*(M^*,K^*))_{i\in\boldsymbol{Z}}$ the dual filtration of $\mathfrak{F}$. We denote by*

$$\mathfrak{F}^{**} = (F_i\mathcal{H}om^*(\mathcal{H}om^*(M^*,K^*),K^*))_{i\in\boldsymbol{Z}}$$

the dual filtration of $\mathfrak{F}^*$. Then the map $\chi_{M^*,K^*} : M^* \to \mathcal{H}om^*(\mathcal{H}om^*(M^*,K^*),K^*)$ defined in (3.3.4) maps F_iM^* bijectively onto $F_i\mathcal{H}om^*(\mathcal{H}om^*(M^*,K^*),K^*)$.

Proof. We recall that, for $x \in M^n$, $\chi_{M^*,K^*}(x) \in \mathcal{H}om^n(\mathcal{H}om^*(M^*,K^*),K^*)$ maps $([n],f) \in \Sigma^n\mathcal{H}om^*(M^*,K^*)$ to $(-1)^{n(k-n)}f([k-n],x)$ for $f \in \mathcal{H}om^{k-n}(M^*,K^*) = \mathrm{Hom}^c_{K^*}(\Sigma^{k-n}M^*,K^*)$.

For $x \in M^n$, since $F_i\mathcal{H}om^*(\mathcal{H}om^*(M^*,K^*),K^*)$ is the kernel of

$$\kappa^*_{\mathcal{H}om^*(M^*,K^*),-i-1} : \mathcal{H}om^*(\mathcal{H}om^*(M^*,K^*),K^*) \to \mathcal{H}om^*(F_{-i-1}\mathcal{H}om^*(M^*,K^*),K^*),$$

$\chi_{M^*,K^*}(x) \in F_i\mathcal{H}om^*(\mathcal{H}om^*(M^*,K^*),K^*)$ if and only if the following composition is trivial.

$$\Sigma^nF_{-i-1}\mathcal{H}om^*(M^*,K^*) \xrightarrow{\Sigma^n\kappa_{\mathcal{H}om^*(M^*,K^*),-i-1}} \Sigma^n\mathcal{H}om^*(M^*,K^*) \xrightarrow{\chi_{M^*,K^*}(x)} K^*$$

If $([n],f) \in \Sigma^nF_{-i-1}\mathcal{H}om^*(M^*,K^*)$ for $f \in \mathcal{H}om^{k-n}(M^*,K^*)$, $f\Sigma^{k-n}\kappa_{M^*,i} : \Sigma^{k-n}F_iM^* \to K^*$ is trivial. Hence, if $x \in (F_iM^*)^n$, we have the following equality which shows that $\chi_{M^*,K^*}(x)\Sigma^n\kappa_{\mathcal{H}om^*(M^*,K^*),-i-1}$ is trivial.

$$\chi_{M^*,K^*}(x)\Sigma^n\kappa_{\mathcal{H}om^*(M^*,K^*),-i-1}([n],f) = \chi_{M^*,K^*}(x)([n],f) = (-1)^{n(k-n)}f\Sigma^{k-n}\kappa_{M^*,i}([k-n],x)$$

It follows that χ_{M^*,K^*} maps F_iM^* into $F_i\mathcal{H}om^*(\mathcal{H}om^*(M^*,K^*),K^*)$. Since M^* has skeletal topology, so does $\mathcal{H}om^*(M^*,K^*)$ by (3.1.36). Hence $\kappa^*_{M^*,i} : \mathcal{H}om^*(M^*,K^*) \to \mathcal{H}om^*(F_iM^*,K^*)$ and $\kappa^*_{\mathcal{H}om^*(M^*,K^*),-i-1} : \mathcal{H}om^*(\mathcal{H}om^*(M^*,K^*),K^*) \to \mathcal{H}om^*(F_{-i-1}\mathcal{H}om^*(M^*,K^*),K^*)$ are surjective by (15.1.3). Since M^* is finite type, we have the following equalities.

$$\begin{aligned}\dim(F_i\mathcal{H}om^*(\mathcal{H}om^*(M^*,K^*),K^*))^n &= \dim\mathcal{H}om^n(\mathcal{H}om^*(M^*,K^*),K^*) - \dim\mathcal{H}om^n(F_{-i-1}\mathcal{H}om^*(M^*,K^*),K^*)\\ &= \dim M^n - (\dim\mathcal{H}om^{-n}(M^*,K^*) - \dim\mathcal{H}om^{-n}(F_iM^*,K^*))\\ &= \dim M^n - (\dim M^n - \dim(F_iM^*)^n) = \dim(F_iM^*)^n\end{aligned}$$

Since χ_{M^*,K^*} is injective by (3.3.6), the above equalities show that χ_{M^*,K^*} is an isomorphism. □

For objects P^* and Q^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, consider filtrations $(F_u(\mathcal{H}om^*(M^*,P^*)\otimes_{K^*}\mathcal{H}om^*(N^*,Q^*))_{u\in\boldsymbol{Z}}$ and $(F_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,P^*\otimes_{K^*}Q^*))_{u\in\boldsymbol{Z}}$ of $\mathcal{H}om^*(M^*,P^*)\otimes_{K^*}\mathcal{H}om^*(N^*,Q^*)$ and $\mathcal{H}om^*(M^*\otimes_{K^*}N^*,P^*\otimes_{K^*}Q^*)$, respectively below.

Lemma 15.1.9 $\phi = \phi(M^*,N^*;P^*,Q^*) : \mathcal{H}om^*(M^*,P^*)\otimes_{K^*}\mathcal{H}om^*(N^*,Q^*) \to \mathcal{H}om^*(M^*\otimes_{K^*}N^*,P^*\otimes_{K^*}Q^*)$ *preserves filtrations.*

Proof. For $s,t \in \boldsymbol{Z}$, $f \in (F_s\mathcal{H}om^*(M^*,P^*))^m$ and $g \in (F_t\mathcal{H}om^*(N^*,Q^*))^n$, since f maps $\Sigma^mF_{-s-1}M^*$ to zero and g maps $\Sigma^nF_{-t-1}N^*$ to zero, $\phi(f\otimes g) = (f\otimes_{K^*}g)\big(\tau^{m,n}_{M^*,N^*}\big)^{-1} : \Sigma^{m+n}(M^*\otimes_{K^*}N^*) \to P^*\otimes_{K^*}Q^*$ maps $\Sigma^{m+n}(F_jM^* * \otimes_{K^*}F_kN^*)$ to zero if $j+k < -s-t$. Hence ϕ maps $F_u(\mathcal{H}om^*(M^*,P^*)\otimes_{K^*}\mathcal{H}om^*(N^*,Q^*))$ to $F_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,P^*\otimes_{K^*}Q^*)$. □

Let $\phi_u = \phi_u(M^*,N^*;P^*,Q^*) : F_u(\mathcal{H}om^*(M^*,P^*)\otimes_{K^*}\mathcal{H}om^*(N^*,Q^*)) \to F_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,P^*\otimes_{K^*}Q^*)$ be the map obtained from ϕ by restricting the source and the target of ϕ. There exists unique map

$$\bar\phi_u = \bar\phi_u(M^*,N^*;P^*,Q^*) : E^*_u(\mathcal{H}om^*(M^*,P^*)\otimes_{K^*}\mathcal{H}om^*(N^*,Q^*)) \to E^*_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,P^*\otimes_{K^*}Q^*)$$

that makes the following diagram commute.

$$\begin{CD} F_u(\mathcal{H}om^*(M^*,P^*)\otimes_{K^*}\mathcal{H}om^*(N^*,Q^*)) @>{\rho_{\mathcal{H}om^*(M^*,P^*)\otimes_{K^*}\mathcal{H}om^*(N^*,Q^*),u}}>> E^*_u(\mathcal{H}om^*(M^*,P^*)\otimes_{K^*}\mathcal{H}om^*(N^*,Q^*))\\ @V{\phi_u(M^*,N^*;P^*,Q^*)}VV @VV{\bar\phi_u(M^*,N^*;P^*,Q^*)}V\\ F_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,P^*\otimes_{K^*}Q^*) @>{\rho_{\mathcal{H}om^*(M^*\otimes_{K^*}N^*,P^*\otimes_{K^*}Q^*),u}}>> E^*_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,P^*\otimes_{K^*}Q^*) \end{CD}$$

For $j, k, u \in \boldsymbol{Z}$ such that $j + k = -u$, define a map $p_{j,k} : \bigoplus_{s+t=-u} (E_s^* M^* \otimes_{K^*} E_t^* N^*) \to E_j^* M^* \otimes_{K^*} E_k^* N^*$ by $p_{j,k}((x_{st})) = x_{jk}$ where $x_{st} \in E_s^* M^* \otimes_{K^*} E_t^* N^*$. We define a map

$$\Psi_u : \bigoplus_{j+k=u} \mathcal{H}om^*(E_{-j}^* M^* \otimes_{K^*} E_{-k}^* N^*, P^* \otimes_{K^*} Q^*) \to \mathcal{H}om^*\Bigl(\bigoplus_{j+k=-u} (E_j^* M^* \otimes_{K^*} E_k^* N^*), P^* \otimes_{K^*} Q^*\Bigr)$$

by $\Psi_u((\varphi_j)_{j\in\boldsymbol{Z}}) = \sum_{j\in\boldsymbol{Z}} \varphi_j \Sigma^n p_{-j,j-u} = \sum_{j\in\boldsymbol{Z}} p^*_{-j,j-u}(\varphi_j)$ where $\varphi_j \in \mathcal{H}om^n(E_{-j}^* M^* \otimes_{K^*} E_{j-u}^* N^*, P^* \otimes_{K^*} Q^*)$ and $\varphi_j = 0$ except for finite number of j's.

Suppose that M^* and N^* have skeletal topology. Recall from (2.1.20) that, if M^* and N^* are both connective or both coconnective, $M^* \otimes_{K^*} N^*$ has skeletal topology. Hence it follows from (15.1.4) that there is an isomorphism $\tilde{\rho}_{M^* \otimes_{K^*} N^*, P^* \otimes_{K^*} Q^*, u} : E_u^* \mathcal{H}om^*(M^* \otimes_{K^*} N^*, P^* \otimes_{K^*} Q^*) \to \mathcal{H}om^*(E_{-u}^*(M^* \otimes_{K^*} N^*), P^* \otimes_{K^*} Q^*)$.

Lemma 15.1.10 *If M^* and N^* have skeletal topology and M^* and N^* are both connective or both coconnective, then the following diagram is commutative.*

$$\begin{tikzcd}
\bigoplus_{j+k=u} (E_j^* \mathcal{H}om^*(M^*, P^*) \otimes_{K^*} E_k^* \mathcal{H}om^*(N^*, Q^*)) \arrow[d, "\bigoplus_{j+k=u} (\tilde{\rho}_{M^*,P^*,j} \otimes_{K^*} \tilde{\rho}_{N^*,Q^*,k})"] \arrow[dr, "\Phi_{\mathcal{H}om^*(M^*,P^*), \mathcal{H}om^*(N^*,Q^*), u}"] & \\
\bigoplus_{j+k=u} (\mathcal{H}om^*(E_{-j}^* M^*, P^*) \otimes_{K^*} \mathcal{H}om^*(E_{-k}^* N^*, Q^*)) \arrow[d, "\bigoplus_{j+k=u} \phi(E_{-j}^* M^*, E_{-k}^* N^*; P^*, Q^*)"] & E_u^*(\mathcal{H}om^*(M^*, P^*) \otimes_{K^*} \mathcal{H}om^*(N^*, Q^*)) \arrow[d, "\bar{\phi}_u(M^*, N^*; P^*, Q^*)"] \\
\bigoplus_{j+k=u} \mathcal{H}om^*(E_{-j}^* M^* \otimes_{K^*} E_{-k}^* N^*, P^* \otimes_{K^*} Q^*) \arrow[d, "\Psi_u"] & E_u^* \mathcal{H}om^*(M^* \otimes_{K^*} N^*, P^* \otimes_{K^*} Q^*) \arrow[d, "\tilde{\rho}_{M^* \otimes_{K^*} N^*, P^* \otimes_{K^*} Q^*, u}"] \\
\mathcal{H}om^*\Bigl(\bigoplus_{j+k=-u} (E_j^* M^* \otimes_{K^*} E_k^* N^*), P^* \otimes_{K^*} Q^*\Bigr) & \mathcal{H}om^*(E_{-u}^*(M^* \otimes_{K^*} N^*), P^* \otimes_{K^*} Q^*) \arrow[l, "\Phi^*_{M^*,N^*,-u}"']
\end{tikzcd}$$

Proof. For each $\alpha \in E_j^* \mathcal{H}om^*(M^*, P^*)$ and $\beta \in E_k^* \mathcal{H}om^*(N^*, Q^*)$ ($j+k = u$), we choose $f \in (F_j \mathcal{H}om^*(M^*, P^*))^m$ and $g \in (F_k \mathcal{H}om^*(N^*, Q^*))^n$ which satisfy $\rho_{\mathcal{H}om^*(M^*,P^*),j}(f) = \alpha$ and $\rho_{\mathcal{H}om^*(N^*,Q^*),k}(g) = \beta$, respectively. Let $\bar{f} : \Sigma^m M^*/F_{-j-1} M^* \to P^*$ and $\bar{g} : \Sigma^n N * /F_{-k-1} N^* \to Q^*$ be the maps which satisfy $\bar{f} \Sigma^m \pi_{M^*,-j} = f$ and $\bar{g} \Sigma^n \pi_{N^*,-k} = g$, respectively. Then, we have $\tilde{\rho}_{M^*,P^*,j}(\alpha) = \bar{f} \Sigma^m \tilde{\kappa}_{M^*,-j}$ and $\tilde{\rho}_{N^*,Q^*,k}(\beta) = \bar{g} \Sigma^n \tilde{\kappa}_{N^*,-k}$. Hence the image of $\alpha \otimes \beta \in E_j^* \mathcal{H}om^*(M^*, P^*) \otimes_{K^*} E_k^* \mathcal{H}om^*(N^*, Q^*)$ by the composition of the vertical maps is given as follows.

$$\begin{aligned}
\Psi_u(\phi(\tilde{\rho}_{M^*,P^*,j}(\alpha) \otimes \tilde{\rho}_{N^*,Q^*,k}(\beta))) &= \Psi_u((\bar{f} \Sigma^m \tilde{\kappa}_{M^*,-j} \otimes_{K^*} \bar{g} \Sigma^n \tilde{\kappa}_{N^*,-k})(\tau^{m,n}_{E_{-j}^* M^*, E_{-k}^* N^*})^{-1}) \\
&= (\bar{f} \otimes_{K^*} \bar{g})(\Sigma^m \tilde{\kappa}_{M^*,-j} \otimes_{K^*} \Sigma^n \tilde{\kappa}_{N^*,-k})(\tau^{m,n}_{E_{-j}^* M^*, E_{-k}^* N^*})^{-1} \Sigma^{m+n} p_{-j,-k} \\
&= (\bar{f} \otimes_{K^*} \bar{g})(\tau^{m,n}_{M^*/F_{-j-1} M^*, N^*/F^*_{-k-1} N^*})^{-1} \Sigma^{m+n} (\tilde{\kappa}_{M^*,-j} \otimes_{K^*} \tilde{\kappa}_{N^*,-k}) \Sigma^{m+n} p_{-j,-k}
\end{aligned}$$

Let $\overline{f \otimes g} : \Sigma^{m+n}(M^* \otimes_{K^*} N^*)/F_{-u-1}(M^* \otimes_{K^*} N^*) \to P^* \otimes_{K^*} Q^*$ be uniqe map that satisfes

$$\overline{f \otimes g}\, \Sigma^{m+n} \pi_{M^* \otimes_{K^*} N^*, -u} = \phi_u(f \otimes g).$$

Then, we have the following.

$$\begin{aligned}
\tilde{\rho}_{M^* \otimes_{K^*} N^*, P^* \otimes_{K^*} Q^*, u}(\bar{\phi}_u(\Phi_u(\alpha \otimes \beta))) &= \tilde{\rho}_{M^* \otimes_{K^*} N^*, P^* \otimes_{K^*} Q^*, u}(\bar{\phi}_u(\rho_{\mathcal{H}om^*(M^*,P^*) \otimes_{K^*} \mathcal{H}om^*(N^*,Q^*), u}(f \otimes g))) \\
&= \tilde{\rho}_{M^* \otimes_{K^*} N^*, P^* \otimes_{K^*} Q^*, u}(\rho_{\mathcal{H}om^*(M^* \otimes_{K^*} N^*, P^* \otimes_{K^*} Q^*), u}(\phi_u(f \otimes g))) \\
&= \tilde{\kappa}^*_{M^* \otimes_{K^*} N^*, -u}(\hat{\pi}^{-1}_{M^* \otimes_{K^*} N^*, P^* \otimes_{K^*} Q^*, u})(\phi_u(f \otimes g))) \\
&= \overline{f \otimes g}\, \Sigma^{m+n} \tilde{\kappa}_{M^* \otimes_{K^*} N^*, -u}
\end{aligned}$$

Thus $\Phi^*_{M^*,N^*,-u}(\tilde{\rho}_{M^* \otimes_{K^*} N^*, P^* \otimes_{K^*} Q^*, u}(\bar{\phi}_u(\Phi_u(\alpha \otimes \beta)))) = \overline{f \otimes g}(\Sigma^{m+n} \tilde{\kappa}_{M^* \otimes_{K^*} N^*, -u}) \Phi_{M^*,N^*,-u}$ holds. Take

$x \in F_jM^*$ and $y \in F_kN^*$. Then, we have

$$\begin{aligned}
&(\bar{f} \otimes_{K^*} \bar{g})(\tau^{m,n}_{M^*/F_{-j-1}M^*,N^*/F^*_{-k-1}N^*})^{-1}\Sigma^{m+n}(\tilde{\kappa}_{M^*,-j} \otimes_{K^*} \tilde{\kappa}_{N^*,-k})\Sigma^{m+n}p_{-j,-k}([m+n], \rho_{M^*,j}(x) \otimes \rho_{N^*,k}(y)) \\
&\qquad = (\bar{f} \otimes_{K^*} \bar{g})(\tau^{m,n}_{M^*/F_{-j-1}M^*,N^*/F^*_{-k-1}N^*})^{-1}([m+n], \tilde{\kappa}_{M^*,-j}\rho_{M^*,j}(x) \otimes \tilde{\kappa}_{N^*,-k}\rho_{N^*,k}(y)) \\
&\qquad = (\bar{f} \otimes_{K^*} \bar{g})(\tau^{m,n}_{M^*/F_{-j-1}M^*,N^*/F^*_{-k-1}N^*})^{-1}([m+n], \pi_{M^*,j}\kappa_{M^*,-j}(x) \otimes \pi_{N^*,k}\kappa_{N^*,-k}(y)) \\
&\qquad = (\bar{f} \otimes_{K^*} \bar{g})(\Sigma^m\pi_{M^*,j} \otimes_{K^*} \Sigma^n\pi_{N^*,k})(\tau^{m,n}_{M^*,N^*})^{-1}([m+n], x \otimes y) \\
&\qquad = (\bar{f}\Sigma^m\pi_{M^*,j} \otimes_{K^*} \bar{g}\Sigma^n\pi_{N^*,k})(\tau^{m,n}_{M^*,N^*})^{-1}([m+n], x \otimes y) \\
&\qquad = (f \otimes_{K^*} g)(\tau^{m,n}_{M^*,N^*})^{-1}([m+n], x \otimes y) = \phi_u(f \otimes g)([m+n], x \otimes y) \\
&\quad \overline{f \otimes g}(\Sigma^{m+n}\tilde{\kappa}_{M^*\otimes_{K^*}N^*,-u})\Phi_{M^*,N^*,-u}([m+n], \rho_{M^*,j}(x) \otimes \rho_{N^*,k}(y)) \\
&\qquad = \overline{f \otimes g}(\Sigma^{m+n}\tilde{\kappa}_{M^*\otimes_{K^*}N^*,-u})([m+n], \rho_{M^*\otimes_{K^*}N^*,-u}(x \otimes y)) \\
&\qquad = \overline{f \otimes g}(\Sigma^{m+n}\pi_{M^*\otimes_{K^*}N^*,-u})([m+n], \kappa_{M^*\otimes_{K^*}N^*,-u}(x \otimes y)) = \phi_u(f \otimes g)([m+n], x \otimes y).
\end{aligned}$$

The above equalities show the assertion. □

Lemma 15.1.11 *If $(F_iM^*)_{i\in\boldsymbol{Z}}$ and $(F_iN^*)_{i\in\boldsymbol{Z}}$ satisfy "$F_rM^* = M^*$ and $F_rN^* = N^*$ for some $r \in \boldsymbol{Z}$" or "$F_rM^* = \{0\}$ and $F_rN^* = \{0\}$ for some $r \in \boldsymbol{Z}$", then Ψ_u is an isomorphism.*

Proof. If $F_rM^* = M^*$ and $F_rN^* = N^*$ for some $r \in \boldsymbol{Z}$, then $E^*_iM^* = \{0\}$ and $E^*_iN^* = \{0\}$ holds for $i > r$. Hence $\bigoplus\limits_{j+k=-u}(E^*_jM^*\otimes_{K^*}E^*_kN^*) = \bigoplus\limits_{j=-u-r}^{r}(E^*_jM^*\otimes_{K^*}E^*_{-u-j}N^*)$ is a finite sum of $E^*_jM^*\otimes_{K^*}E^*_kN^*$'s. Similarly, $\bigoplus\limits_{j+k=u}\mathcal{H}om^*(E^*_{-j}M^*\otimes_{K^*}E^*_{-k}N^*, P^*\otimes_{K^*}Q^*) = \bigoplus\limits_{j=-u-r}^{r}\mathcal{H}om^*(E^*_jM^*\otimes_{K^*}E^*_{-u-j}N^*, P^*\otimes_{K^*}Q^*)$ is also finite sum of $\mathcal{H}om^*(E^*_jM^*\otimes_{K^*}E^*_{-u-j}N^*, P^*\otimes_{K^*}Q^*)$'s. It follows from (3.1.12) that Ψ_u is an isomorphism.

If $F_rM^* = \{0\}$ and $F_rN^* = \{0\}$ for some $r \in \boldsymbol{Z}$, then $E^*_iM^* = \{0\}$ and $E^*_iN^* = \{0\}$ holds for $i \leqq r$. Hence $\bigoplus\limits_{j+k=-u}(E^*_jM^*\otimes_{K^*}E^*_kN^*) = \bigoplus\limits_{j=r+1}^{-u-r-1}(E^*_jM^*\otimes_{K^*}E^*_{-u-j}N^*)$ is a finite sum of $E^*_jM^*\otimes_{K^*}E^*_kN^*$'s. Similarly, $\bigoplus\limits_{j+k=u}\mathcal{H}om^*(E^*_{-j}M^*\otimes_{K^*}E^*_{-k}N^*, P^*\otimes_{K^*}Q^*) = \bigoplus\limits_{j=r+1}^{-u-r-1}\mathcal{H}om^*(E^*_jM^*\otimes_{K^*}E^*_{-u-j}N^*, P^*\otimes_{K^*}Q^*)$ is also finite sum of $\mathcal{H}om^*(E^*_jM^*\otimes_{K^*}E^*_{-u-j}N^*, P^*\otimes_{K^*}Q^*)$'s. It follows from (3.1.12) that Ψ_u is also an isomorphism this case. □

Proposition 15.1.12 *Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ with filtrations $(F_iM^*)_{i\in\boldsymbol{Z}}$ and $(F_iN^*)_{i\in\boldsymbol{Z}}$. Assume that M^* and N^* are finite type, both connective or both coconnective and have skeletal topology. We also assume that "$F_rM^* = M^*$ and $F_rN^* = N^*$ for some $r \in \boldsymbol{Z}$" or "$F_rM^* = \{0\}$ and $F_rN^* = \{0\}$ for some $r \in \boldsymbol{Z}$" For objects P^* and Q^* of $\mathcal{T}op\mathcal{M}od_{K^*}$, the following map is an isomorphism.*

$$\hat{\bar{\phi}}_u = \hat{\bar{\phi}}_u(M^*, N^*; P^*, Q^*) : E_u(\mathcal{H}om^*(M^*, P^*)\otimes_{K^*}\mathcal{H}om^*(N^*, Q^*))^\frown \to E_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*, P^*\otimes_{K^*}Q^*)^\frown$$

Proof. It follows from (15.1.10) and (1.3.11) that the following diagram is commutative.

$$\begin{tikzcd}
\bigoplus\limits_{j+k=u}(E^*_j\mathcal{H}om^*(M^*, P^*)\hat{\otimes}_{K^*}E^*_k\mathcal{H}om^*(N^*, Q^*)) \arrow[d, "\bigoplus\limits_{j+k=u}(\tilde{\rho}_{M^*,P^*,j}\hat{\otimes}_{K^*}\tilde{\rho}_{N^*,Q^*,k})"] \arrow[dr, "\hat{\Phi}_{\mathcal{H}om^*(M^*,P^*),\mathcal{H}om^*(N^*,Q^*),u}"] & \\
\bigoplus\limits_{j+k=u}(\mathcal{H}om^*(E^*_{-j}M^*, P^*)\hat{\otimes}_{K^*}\mathcal{H}om^*(E^*_{-k}N^*, Q^*)) \arrow[d, "\bigoplus\limits_{j+k=u}\hat{\phi}(E^*_{-j}M^*, E^*_{-k}N^*; P^*, Q^*)"] & E^*_u(\mathcal{H}om^*(M^*, P^*)\otimes_{K^*}\mathcal{H}om^*(N^*, Q^*))^\frown \arrow[d, "\hat{\bar{\phi}}_u(M^*, N^*; P^*, Q^*)"] \\
\bigoplus\limits_{j+k=u}\mathcal{H}om^*(E^*_{-j}M^*\otimes_{K^*}E^*_{-k}N^*, P^*\otimes_{K^*}Q^*)^\frown \arrow[d, "\hat{\Psi}_u"] & E^*_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*, P^*\otimes_{K^*}Q^*)^\frown \arrow[d, "\hat{\bar{\rho}}_{M^*\otimes_{K^*}N^*,P^*\otimes_{K^*}Q^*,u}"] \\
\mathcal{H}om^*\left(\bigoplus\limits_{j+k=-u}(E^*_jM^*\otimes_{K^*}E^*_kN^*), P^*\otimes_{K^*}Q^*\right)^\frown & \mathcal{H}om^*(E^*_{-u}(M^*\otimes_{K^*}N^*), P^*\otimes_{K^*}Q^*)^\frown \arrow[l, "\hat{\Phi}^*_{M^*,N^*,-u}"']
\end{tikzcd}$$

$\hat{\phi} : \mathcal{H}om^*(E^*_{-j}M^*, P^*)\widehat{\otimes}_{K^*}\mathcal{H}om^*(E^*_{-k}N^*, Q^*) \to \mathcal{H}om^*(E^*_{-j}M^*\otimes_{K^*}E^*_{-k}N^*, P^*\otimes_{K^*}Q^*)\widehat{\ }$ is an isomorphism by (4.1.7). Hence the assertion follows from (15.1.1), (15.1.4), (15.1.11) and the commutativity of the above diagram. □

Remark 15.1.13 *Under the conditions of (15.1.12), assume moreover that P^* and Q^* are bounded and discrete. Then, $P^*\otimes_{K^*}Q^*$ is bounded and discrete. It follows from (3.1.36) that $\mathcal{H}om^*(M^*, P^*)$, $\mathcal{H}om^*(N^*, Q^*)$ and $\mathcal{H}om^*(M^*\otimes_{K^*}N^*, P^*\otimes_{K^*}Q^*)$ have skeletal topology. Since M^* and N^* are both connective or both coconnective, $\mathcal{H}om^*(M^*, P^*)$ and $\mathcal{H}om^*(N^*, Q^*)$ are both coconnective or both connective. Hence it follows from (2) of (2.1.20) that $\mathcal{H}om^*(M^*, P^*)\otimes_{K^*}\mathcal{H}om^*(N^*, Q^*)$ has skeletal topology. Therefore $E^*_u(\mathcal{H}om^*(M^*, P^*)\otimes_{K^*}\mathcal{H}om^*(N^*, Q^*))$ and $E^*_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*, P^*\otimes_{K^*}Q^*)$ also have skeletal topology, hence these are complete. Thus we see that*

$$\bar{\phi}_u = \bar{\phi}_u(M^*, N^*; P^*, Q^*) : E^*_u(\mathcal{H}om^*(M^*, P^*)\otimes_{K^*}\mathcal{H}om^*(N^*, Q^*)) \to E^*_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*, P^*\otimes_{K^*}Q^*)$$

is an isomorphism.

Condition 15.1.14 *For an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ with an increasing filtration $(F_iM^*)_{i\in\boldsymbol{Z}}$ of subspaces of M^*, consider the following conditions on $(F_iM^*)_{i\in\boldsymbol{Z}}$.*

$$(f1)\ \bigcap_{i\in\boldsymbol{Z}} F_iM^* = \{0\}. \qquad\qquad (f1^*)\ \bigcup_{i\in\boldsymbol{Z}} F_iM^* = M^*.$$

Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ with filtration $(F_iM^*)_{i\in\boldsymbol{Z}}$ and N^* a submodule of M^*. We denote by $p_{N^*} : M^* \to M^*/N^*$ the quotient map. We define filtrations $(F_iN^*)_{i\in\boldsymbol{Z}}$ of N^* and $(F_i(M^*/N^*))_{i\in\boldsymbol{Z}}$ of M^*/N^* by $F_iN^* = N^*\cap F_iM^*$ and $F_i(M^*/N^*) = p_{N^*}(F_iM^*)$, respectively.

Proposition 15.1.15 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ with filtration $(F_iM^*)_{i\in\boldsymbol{Z}}$ and N^* a submodule of M^*.*

(1) Suppose that $(F_iM^*)_{i\in\boldsymbol{Z}}$ satisfies $(f1)$. Then, $(F_iN^*)_{i\in\boldsymbol{Z}}$ satisfies $(f1)$. Moreover if M^* is finite type, $(F_i(M^*/N^*))_{i\in\boldsymbol{Z}}$ satisfies $(f1)$.

(2) If $(F_iM^*)_{i\in\boldsymbol{Z}}$ satisfies $(f1^*)$, $(F_iN^*)_{i\in\boldsymbol{Z}}$ and $(F_i(M^*/N^*))_{i\in\boldsymbol{Z}}$ satisfy $(f1^*)$.

Proof. (1) The first assertion is obvious. If M^* is finite type, for $n\in\boldsymbol{Z}$, there exists $i_n\in\boldsymbol{Z}$ such that $(F_{i_n}M^*)^n = \{0\}$. Hence we have $(F_{i_n}(M^*/N^*))^n = \{0\}$ which shows that $(F_i(M^*/N^*))_{i\in\boldsymbol{Z}}$ satisfies $(f1)$.

(2) The assertion is straightforward. □

Proposition 15.1.16 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ with filtration $(F_iM^*)_{i\in\boldsymbol{Z}}$. Assume that M^* is finite type.*

(1) If $(F_iM^*)_{i\in\boldsymbol{Z}}$ satisfies $(f1)$, $(F_i\mathcal{H}om^*(M^*, K^*))_{i\in\boldsymbol{Z}}$ satisfies $(f1^*)$.

(2) If $(F_iM^*)_{i\in\boldsymbol{Z}}$ satisfies $(f1^*)$, $(F_i\mathcal{H}om^*(M^*, K^*))_{i\in\boldsymbol{Z}}$ satisfies $(f1)$.

Proof. (1) For each $n\in\boldsymbol{Z}$, since M^* is finite type, there exists $a_n\in\boldsymbol{Z}$ such that $(F_{a_n}M^*)^n = \{0\}$ by the assumption. Since $\mathcal{H}om^n(F_{-i-1}M^*, K^*) = \mathrm{Hom}^c_{K^*}(\Sigma^nF_{-i-1}M^*, K^*)$ is isomorphic to $\mathrm{Hom}^c_{K^0}((F_{-i-1}M^*)^{-n}, K^0)$, $\mathcal{H}om^n(F_{-i-1}M^*, K^*) = \{0\}$ if $i \geqq -a_{-n}-1$. Hence if $i\geqq -a_{-n}-1$, we have

$$(F_i\mathcal{H}om^*(M^*, K^*))^n = \mathrm{Ker}\,(\kappa^*_{M^*,-i-1} : \mathcal{H}om^n(M^*, K^*) \to \mathcal{H}om^n(F_{-i-1}M^*, K^*)) = \mathcal{H}om^n(M^*, K^*)$$

which implies the assertion.

(2) For each $n\in\boldsymbol{Z}$, since M^* is finite type, there exists $b_n\in\boldsymbol{Z}$ such that $(F_{b_n}M^*)^n = M^n$ by the assumption. Since $\mathcal{H}om^n(F_{-i-1}M^*, K^*) = \mathrm{Hom}^c_{K^*}(\Sigma^nF_{-i-1}M^*, K^*)$ is isomorphic to $\mathrm{Hom}^c_{K^0}((F_{-i-1}M^*)^{-n}, K^0)$, $\mathcal{H}om^n(F_{-i-1}M^*, K^*) = \mathcal{H}om^n(M^*, K^*)$ if $i\leqq -b_{-n}-1$. Hence if $i\leqq -b_{-n}-1$, we have

$$(F_i\mathcal{H}om^*(M^*, K^*))^n = \mathrm{Ker}\,(\kappa^*_{M^*,-i-1} : \mathcal{H}om^n(M^*, K^*) \to \mathcal{H}om^n(F_{-i-1}M^*, K^*)) = \{0\}$$

which implies the assertion. □

Lemma 15.1.17 *Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ with increasing filtrations $(F_iM^*)_{i\in\boldsymbol{Z}}$ and $(F_iN^*)_{i\in\boldsymbol{Z}}$, respectively. Assume that M^* and N^* are finite type and both connective or coconnective. Then, if both $(F_iM^*)_{i\in\boldsymbol{Z}}$ and $(F_iN^*)_{i\in\boldsymbol{Z}}$ satisfy $(f1)$, $(F_i(M^*\otimes_{K^*}N^*))_{i\in\boldsymbol{Z}}$ also satisfies $(f1)$.*

Proof. Since M^* and N^* are finite type, it follows from $(f1)$ that, for each $n \in \boldsymbol{Z}$, there exists $a_n \in \boldsymbol{Z}$ which satisfies $(F_{a_n}M^*)^n = \{0\}$ and $(F_{a_n}N^*)^n = \{0\}$. We have the following equality for $i, n \in \boldsymbol{Z}$.

$$(F_i(M^* \otimes_{K^*} N^*))^n = \sum_{j\in\boldsymbol{Z}} (F_jM^* \otimes_{K^*} F_{i-j}N^*)^n = \sum_{j,m\in\boldsymbol{Z}} (F_jM^*)^m \otimes_{K^*} (F_{i-j}N^*)^{n-m} \cdots (*)$$

If both M^* and N^* are connective, there exists $c \in \boldsymbol{Z}$ such that $M^m = \{0\}$ and $N^m = \{0\}$ for $m < c$. We may assume that $c \leqq \frac{n}{2}$. Then, we have $(*) = \sum_{j\in\boldsymbol{Z}} \sum_{c\leqq m\leqq n-c} (F_jM^*)^m \otimes_{K^*} (F_{i-j}N^*)^{n-m}$. Suppose that $i \leqq \min\{a_m + a_{n-m} \,|\, m = c, c+1, \dots, n-c\}$. If there exist integers j' and m' $(c \leqq m' \leqq n-c)$ which satisfy $j' > a_{m'}$ and $i-j' > a_{n-m'}$, then $i > a_{m'} + a_{n-m'}$ which contradicts the assumption. Thus $j \leqq a_m$ or $i-j \leqq a_{n-m}$ holds for each $c \leqq m \leqq n-c$. Hence $(F_i(M^* \otimes_{K^*} N^*))^n = \{0\}$ if $i \leqq \min\{a_m + a_{n-m} \,|\, m = c, c+1, \dots, n-c\}$. If both M^* and N^* are coconnective, there exists $c \in \boldsymbol{Z}$ such that $M^m = \{0\}$ and $N^m = \{0\}$ for $m > c$. We may assume that $c \geqq \frac{n}{2}$. Then, we have $(*) = \sum_{j\in\boldsymbol{Z}} \sum_{n-c\leqq m\leqq c} (F_jM^*)^m \otimes_{K^*} (F_{i-j}N^*)^{n-m}$. Suppose that $i \leqq \min\{a_m + a_{n-m} \,|\, m = n-c, n-c+1, \dots, c\}$. If there exist integers j' and m' $(n-c \leqq m' \leqq c)$ which satisfy $j' > a_{m'}$ and $i-j' > a_{n-m'}$, then $i > a_{m'} + a_{n-m'}$ which contradicts the assumption. Thus $j \leqq a_m$ or $i-j \leqq a_{n-m}$ holds for $c \leqq m \leqq n-c$. Hence $(F_i(M^* \otimes_{K^*} N^*))^n = \{0\}$ if $i \leqq \min\{a_m + a_{n-m} \,|\, m = n-c, n-c+1, \dots, c\}$. □

Remark 15.1.18 *If both $(F_iM^*)_{i\in\boldsymbol{Z}}$ and $(F_iN^*)_{i\in\boldsymbol{Z}}$ satisfy $(f1^*)$, $(F_i(M^* \otimes_{K^*} N^*))_{i\in\boldsymbol{Z}}$ also satisfies $(f1^*)$. In fact, for $x \in M^* \otimes_{K^*} N^*$, there exist $y_l \in M^*$ and $z_l \in N^*$ $(l = 1, 2, \dots, n)$ such that $x = \sum_{l=1}^{n} y_l \otimes z_l$. There exist $j_l, k_l \in \boldsymbol{Z}$ such that $y_l \in F_{j_l}M^*$ and $z_l \in F_{k_l}N^*$ for $l = 1, 2, \dots, n$ by $(f1^*)$. Put $u = \max\{j_l + k_l \,|\, l = 1, 2, \dots, n\}$, then $x \in F_u(M^* \otimes_{K^*} N^*)$.*

$\tilde{\mu}_{K^*} : K^* \otimes_{K^*} K^* \to K^*$ denotes the isomorphism defined from the multiplication of K^*.

Proposition 15.1.19 *Suppose that M^* and N^* are finite type, both connective or both coconnective and have skeletal topology. If the following conditions are satisfied,*

$$\tilde{\mu}_{K^* *}\phi : \mathcal{H}om^*(M^*, K^*) \otimes_{K^*} \mathcal{H}om^*(N^*, K^*) \to \mathcal{H}om^*(M^* \otimes_{K^*} N^*, K^*)$$

maps $F_u(\mathcal{H}om^*(M^*, K^*) \otimes_{K^*} \mathcal{H}om^*(N^*, K^*))$ onto $F_u\mathcal{H}om^*(M^* \otimes_{K^*} N^*, K^*)$.

(i) *"$F_rM^* = M^*$ and $F_rN^* = N^*$ for some $r \in \boldsymbol{Z}$" or "$F_rM^* = \{0\}$ and $F_rN^* = \{0\}$ for some $r \in \boldsymbol{Z}$".*
(ii) *"Both $(F_iM^*)_{i\in\boldsymbol{Z}}$ and $(F_iN^*)_{i\in\boldsymbol{Z}}$ satisfy $(f1)$." or "both $(F_iM^*)_{i\in\boldsymbol{Z}}$ and $(F_iN^*)_{i\in\boldsymbol{Z}}$ satisfy $(f1^*)$."*

Proof. We first note that the following maps are isomorphisms by (4.1.7) and (15.1.13), respectively.

$$\phi = \phi(M^*, N^*; K^*, K^*) : \mathcal{H}om^*(M^*, K^*) \otimes_{K^*} \mathcal{H}om^*(N^*, K^*) \to \mathcal{H}om^*(M^* \otimes_{K^*} N^*, K^* \otimes_{K^*} K^*)$$
$$\bar{\phi}_u = \bar{\phi}_u(M^*, N^*; K^*, K^*) : E_u^*(\mathcal{H}om^*(M^*, K^*) \otimes_{K^*} \mathcal{H}om^*(N^*, K^*)) \to E_u^*\mathcal{H}om^*(M^* \otimes_{K^*} N^*, K^* \otimes_{K^*} K^*)$$

We also have the following commutative diagram whose vertical columns are exact.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
(F_{u-1}(\mathcal{H}om^*(M^*, K^*) \otimes_{K^*} \mathcal{H}om^*(N^*, K^*)))^n \arrow[r, "\phi_{u-1}(M^*, N^*; K^*, K^*)"] \arrow[d, "\iota_{\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*),u}"] & (F_{u-1}\mathcal{H}om^*(M^* \otimes_{K^*} N^*, K^* \otimes_{K^*} K^*))^n \arrow[d, "\iota_{\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*\otimes_{K^*}K^*),u}"'] \\
(F_u(\mathcal{H}om^*(M^*, K^*) \otimes_{K^*} \mathcal{H}om^*(N^*, K^*)))^n \arrow[r, "\phi_u(M^*, N^*; K^*, K^*)"] \arrow[d, "\rho_{\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*),u}"] & (F_u\mathcal{H}om^*(M^* \otimes_{K^*} N^*, K^* \otimes_{K^*} K^*))^n \arrow[d, "\rho_{\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*\otimes_{K^*}K^*),u}"'] \\
E_u^n(\mathcal{H}om^*(M^*, K^*) \otimes_{K^*} \mathcal{H}om^*(N^*, K^*)) \arrow[r, "\bar{\phi}_u(M^*, N^*; K^*, K^*)"] \arrow[d] & E_u^n\mathcal{H}om^*(M^* \otimes_{K^*} N^*, K^* \otimes_{K^*} K^*) \arrow[d] \\
0 & 0
\end{tikzcd}$$

diagram $(*)$

Suppose that $(F_iM^*)_{i\in\boldsymbol{Z}}$ and $(F_iN^*)_{i\in\boldsymbol{Z}}$ satisfy $(f1)$. It follows from (15.1.16) that $(F_i\mathcal{H}om^*(M^*,K^*))_{i\in\boldsymbol{Z}}$ and $(F_i\mathcal{H}om^*(N^*,K^*))_{i\in\boldsymbol{Z}}$ satisfies $(f1^*)$. Hence $(F_u(\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)))_{u\in\boldsymbol{Z}}$ also satisfies $(f1^*)$ by (15.1.18). Thus, for $n\in\boldsymbol{Z}$, there exists $a_n\in\boldsymbol{Z}$ such that

$$(F_{a_n}(\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)))^n=(\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*))^n.$$

On the other hand, since $(F_i(M^*\otimes_{K^*}N^*))_{i\in\boldsymbol{Z}}$ satisfies $(f1)$ by (15.1.17), $(F_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*))_{u\in\boldsymbol{Z}}$ satisfies $(f1^*)$ by (15.1.16). Hence, for $n\in\boldsymbol{Z}$, there exists $b_n\in\boldsymbol{Z}$ such that

$$(F_{b_n}\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*))^n=\mathcal{H}om^n(M^*\otimes_{K^*}N^*,K^*).$$

Therefore if $u\geqq\max\{a_n,b_n\}$, $\phi:\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)\to\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*\otimes_{K^*}K^*)$ maps $(F_u(\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)))^n$ onto $(F_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*\otimes_{K^*}K^*))^n$. Assume that ϕ_u maps $(F_u(\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)))^n$ onto $(F_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*\otimes_{K^*}K^*))^n$. Since the middle and the lower horizontal maps of diagram $(*)$ are isomorphism, so is the top horizontal map. Therefore $\phi_u(M^*,N^*;K^*,K^*):(F_u(\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)))^n\to(F_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*\otimes_{K^*}K^*))^n$ is an isomorphism for any $u\in\boldsymbol{Z}$.

Suppose that $(F_iM^*)_{i\in\boldsymbol{Z}}$ and $(F_iN^*)_{i\in\boldsymbol{Z}}$ satisfy $(f1^*)$. It follows from (15.1.16) that $(F_i\mathcal{H}om^*(M^*,K^*))_{i\in\boldsymbol{Z}}$ and $(F_i\mathcal{H}om^*(N^*,K^*))_{i\in\boldsymbol{Z}}$ satisfies $(f1)$. Hence $(F_u(\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)))_{u\in\boldsymbol{Z}}$ also satisfies $(f1)$ by (15.1.17). Thus, for $n\in\boldsymbol{Z}$, there exists $c_n\in\boldsymbol{Z}$ such that

$$(F_{c_n}(\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)))^n=\{0\}.$$

On the other hand, since $(F_i(M^*\otimes_{K^*}N^*))_{i\in\boldsymbol{Z}}$ satisfies $(f1^*)$ by (15.1.18), $(F_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*))_{u\in\boldsymbol{Z}}$ satisfies $(f1)$ by (15.1.16). Hence, for $n\in\boldsymbol{Z}$, there exists $d_n\in\boldsymbol{Z}$ such that

$$(F_{d_n}\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*))^n=\{0\}.$$

Therefore if $u\leqq\min\{c_n,d_n\}$, $\phi:\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)\to\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*\otimes_{K^*}K^*)$ maps $(F_u(\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)))^n$ onto $(F_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*\otimes_{K^*}K^*))^n$. Assume that ϕ_{u-1} maps $(F_{u-1}(\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)))^n$ onto $(F_{u-1}\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*\otimes_{K^*}K^*))^n$. Since the top and the lower horizontal maps of diagram $(*)$ are isomorphism, so is the middle horizontal map. Therefore $\phi_u(M^*,N^*;K^*,K^*):(F_u(\mathcal{H}om^*(M^*,K^*)\otimes_{K^*}\mathcal{H}om^*(N^*,K^*)))^n\to(F_u\mathcal{H}om^*(M^*\otimes_{K^*}N^*,K^*\otimes_{K^*}K^*))^n$ is an isomorphism for any $u\in\boldsymbol{Z}$. □

Condition 15.1.20 *For an object M^* of $\mathcal{T}op\mathcal{M}od_{K^*}$ with an increasing filtration $(F_iM^*)_{i\in\boldsymbol{Z}}$ of subspaces of M^*, consider the following conditions on $(F_iM^*)_{i\in\boldsymbol{Z}}$.*

$(f2)$ $F_iM^*=M^*$ *if* $i\geqq 0$. $\qquad$ $(f2^*)$ $F_iM^*=\{0\}$ *if* $i<0$.

The following assertion is clear.

Proposition 15.1.21 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ with filtration $(F_iM^*)_{i\in\boldsymbol{Z}}$ and N^* a submodule of M^*.*
(1) If $(F_iM^*)_{i\in\boldsymbol{Z}}$ satisfies $(f2)$, $(F_iN^*)_{i\in\boldsymbol{Z}}$ and $(F_i(M^*/N^*))_{i\in\boldsymbol{Z}}$ satisfy $(f2)$.
(2) If $(F_iM^*)_{i\in\boldsymbol{Z}}$ satisfies $(f2^*)$, $(F_iN^*)_{i\in\boldsymbol{Z}}$ and $(F_i(M^*/N^*))_{i\in\boldsymbol{Z}}$ satisfy $(f2^*)$.

Proposition 15.1.22 *If a filtration $(F_iM^*)_{i\in\boldsymbol{Z}}$ of M^* satisfies $(f2)$, $(F_sM^{**})_{s\in\boldsymbol{Z}}$ satisfies $(f2^*)$. If M^* is a T_1-space and $(F_sM^{**})_{s\in\boldsymbol{Z}}$ satisfies $(f2^*)$, then $(F_iM^*)_{i\in\boldsymbol{Z}}$ satisfies $(f2)$.*

Proof. Let $\pi_{M^*,0}:M^*\to M^*/F_0M^*$ be the quotient map. Then we have an exact sequence

$$0\to\mathcal{H}om^*(M^*/F_0M^*,K^*)\xrightarrow{\pi^*_{M^*,0}}\mathcal{H}om^*(M^*,K^*)\xrightarrow{\kappa^*_{M^*,0}}\mathcal{H}om^*(F_0M^*,K^*).$$

If $F_0M^*=M^*$, then $\kappa_{M^*,0}$ is the identity map which implies that $F_{-1}M^{**}=\operatorname{Ker}\kappa^*_{M^*,0}=\{0\}$. Assume that $F_{-1}M^{**}=\operatorname{Ker}\kappa_{M^*,0^*}=\{0\}$ and that M^* is a T_1-space. Then, we have $\mathcal{H}om^*(M^*/F_0M^*,K^*)=\{0\}$ by the above exact sequence. Since $\chi_{M^*/F_0M^*,K^*}:M^*/F_0M^*\to\mathcal{H}om^*(\mathcal{H}om^*(M^*/F_0M^*,K^*),K^*)$ is injective by (3.3.5), we have $M^*/F_0M^*=\{0\}$, namely $F_0M^*=M^*$. □

Proposition 15.1.23 *If a filtration $(F_iM^*)_{i\in\boldsymbol{Z}}$ of M^* satisfies $(f2^*)$, $(F_iM^{**})_{i\in\boldsymbol{Z}}$ satisfies $(f2)$. If M^* is a T_1-space and $(F_iM^{**})_{i\in\boldsymbol{Z}}$ satisfies $(f2)$, then $(F_iM^*)_{i\in\boldsymbol{Z}}$ satisfies $(f2^*)$.*

Proof. We have $F_0M^{**} = \mathrm{Ker}(\kappa^*_{M^*,-1} : M^{**} = \mathcal{H}om^*(M^*,K^*) \to \mathcal{H}om^*(F_{-1}M^*,K^*)) = M^{**}$ if $F_{-1}M^* = \{0\}$. Assume that $F_{-1}M^* \neq \{0\}$ and that M^* is a T_1-space. There exists an open subspace U^* of M^* such that $F_{-1}M^* \not\subset U^*$. Let $p : M^* \to M^*/U^*$ be the quotient map. Since M^*/U^* is discrete, there exist $n \in \boldsymbol{Z}$ and a continuous linear map $\varphi : \Sigma^n(M^*/U^*) \to K^*$ whose restriction to $\Sigma^n((F_{-1}M^* + U^*)/U^*)$ is not trivial. Then, $\kappa^*_{M^*,-1} : \mathcal{H}om^*(M^*,K^*) \to \mathcal{H}om^*(F_{-1}M^*,K^*))$ maps $\varphi(\Sigma^n p) \in \mathcal{H}om^n(M^*,K^*)$ to a non-trivial element of $\mathcal{H}om^*(F_{-1}M^*,K^*)$. This implies that $F_0M^{**} \neq M^{**}$. Hence $F_{-1}M^* = \{0\}$ if $F_0M^{**} = M^{**}$. □

The following assertion is clear.

Proposition 15.1.24 *Let M^* and N^* be objects of $\mathcal{T}op\mathcal{M}od_{K^*}$ with filtrations $(F_iM^*)_{i\in\boldsymbol{Z}}$ and $(F_iN^*)_{i\in\boldsymbol{Z}}$, respectively. If $(F_iM^*)_{i\in\boldsymbol{Z}}$ and $(F_iN^*)_{i\in\boldsymbol{Z}}$ satisfy (f2) (resp. (f2*)), so does $(F_i(M^* \otimes_{K^*} N^*))_{i\in\boldsymbol{Z}}$.*

Definition 15.1.25 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ with an increasing filtration $\mathfrak{F} = (F_iM^*)_{i\in\boldsymbol{Z}}$ which is not trivial, that is, $E^*_iM^* \neq \{0\}$ for some $i \in \boldsymbol{Z}$. We put $E^j_iM^* = (F_iM^*/F_{i-1}M^*)^j$ and define a subset $S(\mathfrak{F})$ of $\boldsymbol{Z}$ by $S(\mathfrak{F}) = \{i \in \boldsymbol{Z} \mid E^*_iM^* \neq \{0\}\}$.*

(1) Put $c_{\mathfrak{F}}(i) = \max\{j \in \boldsymbol{Z} \mid E^j_iM^* \neq \{0\}\}$ for $i \in S(\mathfrak{F})$ and $I(\mathfrak{F}) = \{n \in \boldsymbol{Z} \mid n = i + c_{\mathfrak{F}}(i)$ for some $i \in S(\mathfrak{F})\}$ if M^* is coconnective.

(2) Put $c^*_{\mathfrak{F}}(i) = \min\{j \in \boldsymbol{Z} \mid E^j_iM^* \neq \{0\}\}$ for $i \in S(\mathfrak{F})$ and $I^*(\mathfrak{F}) = \{n \in \boldsymbol{Z} \mid n = i + c^*_{\mathfrak{F}}(i)$ for some $i \in S(\mathfrak{F})\}$ if M^* is connective.

Remark 15.1.26 *(1) If $\mathfrak{F}$ satisfies (f1) and (f2), then $c_{\mathfrak{F}}(i) \leqq 0$ for each i. If $\mathfrak{F}$ satisfies (f1*) and (f2*), then $c^*_{\mathfrak{F}}(i) \geqq 0$ for each i.*

(2) Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ with filtration $\mathfrak{F}_{M^*} = (F_iM^*)_{i\in\boldsymbol{Z}}$ and N^* a submodule of M^*. Put $\mathfrak{F}_{N^*} = (F_iN^*)_{i\in\boldsymbol{Z}}$ and $\mathfrak{F}_{M^*/N^*} = (F_i(M^*/N^*))_{i\in\boldsymbol{Z}}$. Then, the inclusion map $N^* \to M^*$ induces an injection $E^*_iN^* \to E^*_iM^*$ and the quotient map $M^* \to M^*/N^*$ induces a surjection $E^*_iM^* \to E^*_i(M^*/N^*)$ for each $i \in \boldsymbol{Z}$. Hence $S(\mathfrak{F}_{N^*})$ and $S(\mathfrak{F}_{M^*/N^*})$ are subsets of $S(\mathfrak{F}_{M^*})$. If M^* is coconnective, then we have $c_{\mathfrak{F}_{N^*}}(i) \leqq c_{\mathfrak{F}_{M^*}}(i)$ for any $i \in S(\mathfrak{F}_{N^*})$ and $c_{\mathfrak{F}_{M^*/N^*}}(i) \leqq c_{\mathfrak{F}_{M^*}}(i)$ for any $i \in S(\mathfrak{F}_{M^*/N^*})$. If M^* is connective, then $c^*_{\mathfrak{F}_{N^*}}(i) \geqq c^*_{\mathfrak{F}_{M^*}}(i)$ for any $i \in S(\mathfrak{F}_{N^*})$ and $c_{\mathfrak{F}_{M^*/N^*}}(i) \geqq c_{\mathfrak{F}_{M^*}}(i)$ for any $i \in S(\mathfrak{F}_{M^*/N^*})$.

Proposition 15.1.27 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ with an increasing filtration $\mathfrak{F} = (F_iM^*)_{i\in\boldsymbol{Z}}$. Assume that M^* is finite type and has skeletal topology. We consider the dual filtration $\mathfrak{F}^* = (F_iM^{**})_{i\in\boldsymbol{Z}}$ of $\mathfrak{F}$.*

(1) $E^j_iM^{**} \neq \{0\}$ if and only if $E^{-j}_{-i}M^* \neq \{0\}$. Hence we have $S(\mathfrak{F}^*) = \{j \in \boldsymbol{Z} \mid -j \in S(\mathfrak{F})\}$.

(2) $c^*_{\mathfrak{F}^*}(i) = -c_{\mathfrak{F}}(-i)$ holds if M^* is coconnective and $c_{\mathfrak{F}^*}(i) = -c^*_{\mathfrak{F}}(-i)$ holds if M^* is connective.

Proof. (1) Note that since M^* is finite type and has skeletal topology, $\kappa^*_{M^*,i} : M^{**} \to \mathcal{H}om^*(F_iM^*,K^*)$ is surjective for each $i \in \boldsymbol{Z}$. Then the following diagram is commutative and both horizontal rows are exact.

$$\begin{array}{ccccccc}
0 \to (F_{i-1}M^{**})^j & \xrightarrow{\kappa_{M^{**},i-1}} & (M^{**})^j & \xrightarrow{\kappa^*_{M^*,-i}} & \mathcal{H}om^j(F_{-i}M^*,K^*) \cong \mathrm{Hom}^c_{K^*}((F_{-i}M^*)^{-j},K^0) \to 0 \\
\big\downarrow{\scriptstyle \iota_{M^{**},i}} & & \Big\| & & \big\downarrow{\scriptstyle \iota^*_{M^*,-i-1}} \\
0 \to (F_iM^{**})^j & \xrightarrow{\kappa_{M^{**},i}} & (M^{**})^j & \xrightarrow{\kappa^*_{M^*,-i-1}} & \mathcal{H}om^j(F_{-i-1}M^*,K^*) \cong \mathrm{Hom}^c_{K^*}((F_{-i-1}M^*)^{-j},K^0) \to 0
\end{array}$$

If $E^{-j}_{-i}M^* = \{0\}$, then $\iota^*_{M^*,-i-1}$ of the above diagram is the identity map. Hence so is $\iota_{M^{**},i}$ which means $E^j_iM^{**} = \{0\}$.

Conversely, assume that $E^j_iM^{**} = \{0\}$. Then, $\iota^*_{M^*,-i-1}$ of the above diagram is bijective. Hence if we denote by $\rho_{M^*,-i} : F_{-i}M^* \to E^*_{-i}M^*$ the quotient map, the exactness of the following diagram implies that $\mathrm{Hom}^c_{K^0}(E^{-j}_{-i}M^*,K^0) = \{0\}$.

$$0 \to \mathrm{Hom}^c_{K^0}(E^{-j}_{-i}M^*,K^0) \xrightarrow{\rho^*_{M^*,-i}} \mathrm{Hom}^c_{K^0}((F_{-i}M^*)^{-j},K^0) \xrightarrow{\iota^*_{-i-1}} \mathrm{Hom}^c_{K^0}((F_{-i-1}M^*)^{-j},K^0)$$

Thus we have $E^{-j}_{-i}M^* = \{0\}$. Therefore $E^j_iM^{**} \neq \{0\}$ if and only if $E^{-j}_{-i}M^* \neq \{0\}$, the assertion follows.

(2) Since a correspondence $j \mapsto -j$ gives a bijection between $S(\mathfrak{F})$ and $S(\mathfrak{F}^*)$ by (1), the assertion follows. □

We consider the following conditions for later sections.

Condition 15.1.28 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ with an increasing filtration : $\mathfrak{F} = (F_iM^*)_{i\in\boldsymbol{Z}}$ which is not trivial. Assume that M^* is coconnective.*

($f3$) $E_i^j M^* = \{0\}$ *if* $i+j \notin I(\mathfrak{F})$.
($f4$) *A map* $S(\mathfrak{F}) \to \boldsymbol{Z}$ *which assigns* $i \in S(\mathfrak{F})$ *to* $i + c_{\mathfrak{F}}(i)$ *is injective.*

Condition 15.1.29 *Let* M^* *be an object of* $\mathcal{T}op\mathcal{M}od_{K^*}$ *with an increasing filtration* $\mathfrak{F} = (F_i M^*)_{i\in \boldsymbol{Z}}$ *which is not trivial. Assume that* M^* *is connective.* $\mathfrak{F} = (F_i M^*)_{i\in \boldsymbol{Z}}$ *which is not trivial, we consider the following conditions.*

($f3^*$) $E_i^j M^* = \{0\}$ *if* $i+j \notin I^*(\mathfrak{F})$.
($f4^*$) *A map* $S(\mathfrak{F}) \to \boldsymbol{Z}$ *which assigns* $i \in S(\mathfrak{F})$ *to* $i + c^*_{\mathfrak{F}}(i)$ *is injective.*

Proposition 15.1.30 *Let* M^* *be an object of* $\mathcal{T}op\mathcal{M}od_{K^*}$ *with an increasing filtration* $\mathfrak{F} = (F_i M^*)_{i\in \boldsymbol{Z}}$ *which is not trivial. Let* $\mathfrak{F}^* = (F_i M^{**})_{i\in \boldsymbol{Z}}$ *be the dual filtration of* $\mathfrak{F}$.
(1) Assume that M^* *is coconnective.* $\mathfrak{F}$ *satisfies* ($f3$) *if and only if* $\mathfrak{F}^*$ *satisfies* ($f3^*$).
(2) Assume that M^* *is connective.* $\mathfrak{F}$ *satisfies* ($f3^*$) *if and only if* $\mathfrak{F}^*$ *satisfies* ($f3$).

Proof. (1) Suppose that $\mathfrak{F}$ satisfies ($f3$) and that $i+j \neq k + c^*_{\mathfrak{F}^*}(k)$ for any $k \in S(\mathfrak{F}^*)$. If $k \in S(\mathfrak{F})$, then $-k \in S(\mathfrak{F}^*)$ and $c^*_{\mathfrak{F}^*}(-k) = -c_{\mathfrak{F}}(k)$ by (15.1.27). Hence we have $(-i)+(-j) \neq k + c_{\mathfrak{F}}(k)$ for any $k \in S(\mathfrak{F})$, which implies $E_{-i}^{-j} M^* = \{0\}$. It follows from (15.1.27) that $E_i^j M^{**} = \{0\}$.

Suppose that $\mathfrak{F}^*$ satisfies ($f3^*$) and that $i+j \neq k + c_{\mathfrak{F}}(k)$ for any $k \in S(\mathfrak{F})$. If $k \in S(\mathfrak{F}^*)$, then $-k \in S(\mathfrak{F})$ and $c_{\mathfrak{F}}(-k) = -c^*_{\mathfrak{F}^*}(k)$ by (15.1.27). Hence we have $(-i)+(-j) \neq k + c^*_{\mathfrak{F}^*}(k)$ for any $k \in S(\mathfrak{F}^*)$, which implies $E_{-i}^{-j} M^{**} = \{0\}$. It follows from (15.1.27) that $E_i^j M^* = \{0\}$.

(2) Suppose that $\mathfrak{F}$ satisfies ($f3^*$) and that $i+j \neq k + c_{\mathfrak{F}^*}(k)$ for any $k \in S(\mathfrak{F}^*)$. If $k \in S(\mathfrak{F})$, then $-k \in S(\mathfrak{F}^*)$ and $-c_{\mathfrak{F}^*}(k) = c^*_{\mathfrak{F}}(-k)$ by (15.1.27). Hence we have $(-i)+(-j) \neq k + c^*_{\mathfrak{F}}(k)$ for any $k \in S(\mathfrak{F})$, which implies $E_{-i}^{-j} M^* = \{0\}$. It follows from (15.1.27) that $E_i^j M^{**} = \{0\}$.

Suppose that $\mathfrak{F}^*$ satisfies ($f3$) and that $i+j \neq k + c^*_{\mathfrak{F}}(k)$ for any $k \in S(\mathfrak{F})$. If $k \in S(\mathfrak{F}^*)$, then $-k \in S(\mathfrak{F})$ and $-c^*_{\mathfrak{F}}(k) = c_{\mathfrak{F}^*}(-k)$ by (15.1.27). Hence we have $(-i)+(-j) \neq k + c_{\mathfrak{F}^*}(k)$ for any $k \in S(\mathfrak{F}^*)$, which implies $E_{-i}^{-j} M^{**} = \{0\}$. It follows from (15.1.27) that $E_i^j M^* = \{0\}$. □

Proposition 15.1.31 *Let* M^* *be an object of* $\mathcal{T}op\mathcal{M}od_{K^*}$ *with an increasing filtration* $\mathfrak{F} = (F_i M^*)_{i\in \boldsymbol{Z}}$ *which is not trivial. Let* $\mathfrak{F}^* = (F_i M^{**})_{i\in \boldsymbol{Z}}$ *be the dual filtration of* $\mathfrak{F}$.
(1) Assume that M^* *is coconnective.* $\mathfrak{F}$ *satisfies* ($f4$) *if and only if* $\mathfrak{F}^*$ *satisfies* ($f4^*$).
(2) Assume that M^* *is connective.* $\mathfrak{F}$ *satisfies* ($f4^*$) *if and only if* $\mathfrak{F}^*$ *satisfies* ($f4$).

Proof. We define a map $d_{\mathfrak{F}} : S(\mathfrak{F}) \to \boldsymbol{Z}$ by $d_{\mathfrak{F}}(i) = i + c_{\mathfrak{F}}(i)$ if M^* is coconnective and a map $d^*_{\mathfrak{F}} : S(\mathfrak{F}) \to \boldsymbol{Z}$ by $d^*_{\mathfrak{F}}(i) = i + c^*_{\mathfrak{F}}(i)$ if M^* is connective.

(1) Since $d^*_{\mathfrak{F}^*}(i) = i + c^*_{\mathfrak{F}^*}(i) = -((-i) + c_{\mathfrak{F}}(-i)) = -d_{\mathfrak{F}}(-i)$ if $i \in S(\mathfrak{F}^*)$, $d^*_{\mathfrak{F}^*}$ is injective if and only if $d_{\mathfrak{F}}$ is injective.

(2) Since $d_{\mathfrak{F}^*}(i) = i + c_{\mathfrak{F}^*}(i) = -((-i) + c^*_{\mathfrak{F}}(-i)) = -d^*_{\mathfrak{F}}(-i)$ if $i \in S(\mathfrak{F}^*)$, $d_{\mathfrak{F}^*}$ is injective if and only if $d^*_{\mathfrak{F}}$ is injective. □

Remark 15.1.32 *Let* $M_1^*, M_2^*, \dots, M_n^*$ *be objects of* $\mathcal{T}op\mathcal{M}od_{K^*}$ *and* $\mathfrak{F}_k = (F_i M_k^*)_{i\in \boldsymbol{Z}}$ *a filtration of* M_k^*. *We define a filtration* $\mathfrak{F}_\otimes = (F_i(M_1^* \otimes_{K^*} M_2^* \otimes_{K^*} \cdots \otimes_{K^*} M_n^*))_{i\in \boldsymbol{Z}}$ *of* $M_1^* \otimes_{K^*} M_2^* \otimes_{K^*} \cdots \otimes_{K^*} M_n^*$ *by*

$$F_i(M_1^* \otimes_{K^*} M_2^* \otimes_{K^*} \cdots \otimes_{K^*} M_n^*) = \sum_{j_1+j_2+\cdots+j_n = i} F_{j_1} M_1^* \otimes_{K^*} F_{j_2} M_2^* \otimes_{K^*} \cdots \otimes_{K^*} F_{j_n} M_n^*.$$

It follows from (15.1.1) that there is an isomorphism

$$\bigoplus_{j_1+j_2+\cdots+j_n = i} (E^*_{j_1} M_1^* \otimes_{K^*} E^*_{j_2} M_2^* \otimes_{K^*} \cdots \otimes_{K^*} E^*_{j_n} M_n^*) \longrightarrow E_i^*(M_1^* \otimes_{K^*} M_2^* \otimes_{K^*} \cdots \otimes_{K^*} M_n^*).$$

(1) $S(\mathfrak{F}_\otimes) = \left\{ i \in \boldsymbol{Z} \,\middle|\, i = \sum_{k=1}^{n} j_k \text{ for } j_k \in S(\mathfrak{F}_k)\ (k = 1, 2, \dots, n) \right\}$ *holds.*
(2) If $M_1^*, M_2^*, \dots, M_n^*$ *are coconnective, then* $c_{\mathfrak{F}_\otimes} : S(\mathfrak{F}_\otimes) \to \boldsymbol{Z}$ *is given by*

$$c_{\mathfrak{F}_\otimes}(i) = \max\left\{ m \in \boldsymbol{Z} \,\middle|\, m = \sum_{k=1}^{n} c_{\mathfrak{F}_k}(j_k) \text{ for } j_k \in S(\mathfrak{F}_k)\ (k = 1, 2, \dots, n) \text{ satisfying } \sum_{k=1}^{n} j_k = i \right\}.$$

If $M_1^*, M_2^*, \dots, M_n^*$ *are connective, then* $c^*_{\mathfrak{F}_\otimes} : S(\mathfrak{F}_\otimes) \to \boldsymbol{Z}$ *is given by*

$$c^*_{\mathfrak{F}_\otimes}(i) = \min\left\{ m \in \boldsymbol{Z} \,\middle|\, m = \sum_{k=1}^{n} c_{\mathfrak{F}_k}(j_k) \text{ for } j_k \in S(\mathfrak{F}_k)\ (k = 1, 2, \dots, n) \text{ satisfying } \sum_{k=1}^{n} j_k = i \right\}.$$

We define a filtration $(F_iE(\tau))_{i\in\boldsymbol{Z}}$ of an exterior algebra $E(\tau)$ generated by τ by $F_iE(\tau)=\{0\}$ for $i<0$, $F_0E(\tau)=K^*$ and $F_iE(\tau)=E(\tau)$ for $i\geqq 1$. Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ with filtration $\mathfrak{F}=(F_iM^*)_{i\in\boldsymbol{Z}}$. Then, we have a filtration $\mathfrak{F}_\tau=(F_i(E(\tau)\otimes_{K^*}M^*))_{i\in\boldsymbol{Z}}$ of $E(\tau)\otimes_{K^*}M^*$ which is given by

$$F_i(E(\tau)\otimes_{K^*}M^*)=F_0E(\tau)\otimes_{K^*}F_iM^*+F_1E(\tau)\otimes_{K^*}F_{i-1}M^*.$$

Proposition 15.1.33 *(1) $S(\mathfrak{F}_\tau)=S(\mathfrak{F})\cup\{i\in\boldsymbol{Z}\,|\,i-1\in S(\mathfrak{F})\}$.*
(2) If M^* is connective, $c^*_{\mathfrak{F}_\tau}:S(\mathfrak{F}_\tau)\to\boldsymbol{Z}$ is given as follows.

$$c^*_{\mathfrak{F}_\tau}(i)=\begin{cases} c^*_{\mathfrak{F}}(i) & i\in S(\mathfrak{F})\text{ and } i-1\notin S(\mathfrak{F})\\ \min\{c^*_{\mathfrak{F}}(i),c^*_{\mathfrak{F}}(i-1)+\deg\tau\} & i\in S(\mathfrak{F})\text{ and } i-1\in S(\mathfrak{F})\\ c^*_{\mathfrak{F}}(i-1)+\deg\tau & i\notin S(\mathfrak{F})\text{ and } i-1\in S(\mathfrak{F})\end{cases}$$

Proof. (1) Since $E^*_i(E(\tau)\otimes_{K^*}M^*)$ is isomorphic to $(E^*_0E(\tau)\otimes_{K^*}E^*_iM^*)\oplus(E^*_1E(\tau)\otimes_{K^*}E^*_{i-1}M^*)$ by (15.1.1), $E^*_i(E(\tau)\otimes_{K^*}M^*)\neq\{0\}$ if and only if $E^*_iM^*\neq\{0\}$ or $E^*_{i-1}M^*\neq\{0\}$.

(2) The fact that $E^*_i(E(\tau)\otimes_{K^*}M^*)$ is isomorphic to $(E^*_0E(\tau)\otimes_{K^*}E^*_iM^*)\oplus(E^*_1E(\tau)\otimes_{K^*}E^*_{i-1}M^*)$ also implies the equality. □

Proposition 15.1.34 *Assume that $S(\mathfrak{F})\cap\{i\in\boldsymbol{Z}\,|\,i-1\in S(\mathfrak{F})\}=\emptyset$.*
(1) If $\mathfrak{F}$ satisfies $(f3^*)$, $\mathfrak{F}_\tau$ satisfies $(f3^*)$.
(2) If $\mathfrak{F}$ satisfies $(f4^*)$ and $a-b\neq\deg\tau+1$ for any $a,b\in I^*(\mathfrak{F})$, $\mathfrak{F}_\tau$ satisfies $(f4^*)$.

Proof. (1) Since $k+c^*_{\mathfrak{F}_\tau}(k)=k+c^*_{\mathfrak{F}}(k)$ if $k\in S(\mathfrak{F})$ and $k+c^*_{\mathfrak{F}_\tau}(k)=k-1+c^*_{\mathfrak{F}}(k-1)+\deg\tau+1$ if $k-1\in S(\mathfrak{F})$, we have $I^*(\mathfrak{F}_\tau)=I^*(\mathfrak{F})\cup\{n\in\boldsymbol{Z}\,|\,n-\deg\tau-1\in I^*(\mathfrak{F})\}$. Hence if $i+j\notin I^*(\mathfrak{F}_\tau)$, then we have $i+j\notin I^*(\mathfrak{F})$ and $i+j-\deg\tau-1\notin I^*(\mathfrak{F})$. This implies that $E^j_iM^*=E^{j-\deg\tau}_{i-1}M^*=\{0\}$ by the assumption. Since there is the following isomorphism by (15.1.1), $E^j_i(E(\tau)\otimes_{K^*}M^*)=\{0\}$ holds if $i+j\notin I^*(\mathfrak{F}_\tau)$.

$$\Phi^j_{E(\tau),M^*,i}:(E^0_0E(\tau)\otimes_{K^*}E^j_iM^*)\oplus(E^{\deg\tau}_1E(\tau)\otimes_{K^*}E^{j-\deg\tau}_{i-1}M^*)\to E^j_i(E(\tau)\otimes_{K^*}M^*)$$

(2) Define a map $f:S(\mathfrak{F}_\tau)\to\boldsymbol{Z}$ by $f(k)=k+c^*_{\mathfrak{F}_\tau}(k)$. Since $\mathfrak{F}$ satisfies $(f4^*)$ and $f(k)=k+c^*_{\mathfrak{F}}(k)$ if $k\in S(\mathfrak{F})$, the restriction of f to $S(\mathfrak{F})$ is injective. Similarly, since $f(k)=k-1+c^*_{\mathfrak{F}}(k-1)+\deg\tau+1$ if $k\in\{i\in\boldsymbol{Z}\,|\,i-1\in S(\mathfrak{F})\}$, the restriction of f to $\{i\in\boldsymbol{Z}\,|\,i-1\in S(\mathfrak{F})\}$ is also injective. Suppose that $f(k)=f(l)$ for $k\in S(\mathfrak{F})$ and $l\in\{i\in\boldsymbol{Z}\,|\,i-1\in S(\mathfrak{F})\}$. Then, we have $(k+c^*_{\mathfrak{F}}(k))-(l-1+c^*_{\mathfrak{F}}(l-1))=\deg\tau+1$ which contradicts the assumption. Hence f is injective. □

Let p be a prime number and K^* be a field of characteristic p such that $K^i=\{0\}$ if $i\neq 0$. For an object A^* of $\mathcal{T}op\mathcal{A}lg_{K^*}$, we denote by $A(k)^*$ the subalgebra of A^* generated by $\{x^{p^k}\,|\,x\in A^*\}$. Let $\mathfrak{F}=(F_iA^*)_{i\in\boldsymbol{Z}}$ be a filtration of A^*. Put $F_iA(k)^*=A(k)^*\cap F_iA^*$ and consider a filtration $\mathfrak{F}(k)=(F_iA(k)^*)_{i\in\boldsymbol{Z}}$ of $A(k)^*$.

Proposition 15.1.35 *Suppose that $\mathfrak{F}$ satisfies a condition "If $x\in F_iA^*-F_{i-1}A^*$, then $x^p\in F_{ip}A^*-F_{ip-1}A^*$.".*
(1) If $x\in F_iA(k)^*-F_{i-1}A(k)^*$ and l is a non-negative integer, then $x^{p^l}\in F_{ip^l}A(k)^*-F_{ip^l-1}A(k)^*$.
(2) $S(\mathfrak{F}(k))=\{ip^k\,|\,i\in S(\mathfrak{F})\}$ holds. $c^*_{\mathfrak{F}(k)}:S(\mathfrak{F}(k))\to\boldsymbol{Z}$ is given by $c^*_{\mathfrak{F}(k)}(ip^k)=p^kc^*_{\mathfrak{F}}(i)$ if A^* is connective.

Proof. (1) For $x\in F_iA(k)^*-F_{i-1}A(k)^*$, we assume inductively that $x^{p^l}\in F_{ip^l}A(k)^*-F_{ip^l-1}A(k)^*$. Then, we have $x^{p^{l+1}}\in F_{ip^{l+1}}A^*-F_{ip^{l+1}-1}A^*$ by the assumpsion. Since $x^{p^{l+1}}\in A(k)^*$, $x^{p^{l+1}}\in F_{ip^{l+1}}A(k)^*-F_{ip^{l+1}-1}A(k)^*$ holds.

(2) If $i\in S(\mathfrak{F})$, there exists $x\in F_iA^*-F_{i-1}A^*$. Hence $x^{p^k}\in F_{ip^k}A^*-F_{ip^k-1}A^*$ represents a non-zero element of $E^*_{ip^k}A(k)^*$ which shows $ip^k\in S(\mathfrak{F}(k))$. Since $A(k)^*$ is spanned by elements of A^* of the form x^{p^k}, it follows from the assumption that $E^*_iA(k)^*=\{0\}$ if i is not a multiple of p^k. If $ip^k\in S(\mathfrak{F}(k))$, there exists $y\in F_{ip^k}A(k)^*-F_{ip^k-1}A(k)^*$ such that $y=x^{p^k}$ for some $x\in F_iA^*-F_{i-1}A^*$. Thus $E^*_iA^*\neq\{0\}$, namely, $i\in S(\mathfrak{F})$. Assume that A^* is connective. The p^k-th power map $x\mapsto x^{p^k}$ from $(F_iA^*)^j$ to $(F_{ip^k}A(k)^*)^{jp^k}$ induces an injective additive map $E^j_iA^*\to E^{jp^k}_{ip^k}A(k)^*$ by the assumption. This implies that $c^*_{\mathfrak{F}(k)}(ip^k)\leqq p^kc^*_{\mathfrak{F}}(i)$. Since $A(k)^j=\{0\}$ if j is not a multiple of p^k, $c^*_{\mathfrak{F}(k)}(ip^k)$ is a multiple of p^k. There exists $y\in(F_{ip^k}A^*)^{c^*_{\mathfrak{F}(k)}(ip^k)}-(F_{ip^k-1}A^*)^{c^*_{\mathfrak{F}(k)}(ip^k)}$ such that $y=x^{p^k}$ for some $x\in(F_iA^*)^j-(F_{i-1}A^*)^j$, where $c^*_{\mathfrak{F}(k)}(ip^k)=jp^k$. Then, we have $E^j_iA^*\neq\{0\}$ which means $j\geqq c^*_{\mathfrak{F}}(i)$. Hence $c^*_{\mathfrak{F}(k)}(ip^k)=p^kc^*_{\mathfrak{F}}(i)$ holds. □

Proposition 15.1.36 *Assume that $\mathfrak{F}$ satisfies the condition of (15.1.35). If $\mathfrak{F}$ satisfies $(f3^*)$, so does $\mathfrak{F}(k)$ and if $\mathfrak{F}$ satisfies $(f4^*)$, so does $\mathfrak{F}(k)$.*

Proof. Assume that $\mathfrak{F}$ satisfies $(f3^*)$ It is clear that $E_i^j A(k)^* = \{0\}$ if i or j is not a multiple of p^k. Suppose that $E_{ip^k}^{jp^k} A(k)^* \neq \{0\}$, then we have $E_i^j A^* \neq \{0\}$. Hence $i + j = s + c^*_{\mathfrak{F}}(s)$ for some $s \in S(\mathfrak{F})$. Thus $ip^k + jp^k = sp^k + p^k c^*_{\mathfrak{F}}(k) = sp^k + c^*_{\mathfrak{F}(k)}(sp^k)$ by (15.1.35) and $\mathfrak{F}(k)$ satisfies $(f3^*)$.

If $ip^k + c^*_{\mathfrak{F}(k)}(ip^k) = jp^k + c^*_{\mathfrak{F}(k)}(jp^k)$, we have $i + c^*_{\mathfrak{F}}(i) = j + c^*_{\mathfrak{F}}(j)$ by (15.1.35). Hence we have $i = j$ if $\mathfrak{F}$ satisfies $(f4^*)$. □

Lemma 15.1.37 *Suppose that $x^p - a = 0$ has a root in K^* for any $a \in K^*$ and that a filtration $\mathfrak{F} = (F_i A^*)_{i \in \boldsymbol{Z}}$ of A^* satisfies $(f1^*)$ and $(f2^*)$. Let k be a positive integer.*

(1) If the p-th power map $x \mapsto x^p$ of A^* is injective and $x_1, x_2, \dots, x_n$ are linearly independent, then $x_1^{p^k}, x_2^{p^k}, \dots, x_n^{p^k}$ are also linearly independent.

(2) Let $\{x_\alpha \,|\, \alpha \in J_i\}$ be a basis of $F_i A^*$ for $i \in \boldsymbol{Z}$ such that $J_{i-1} \subset J_i$. If $\mathfrak{F}$ satisfies the condition of (15.1.35), $\{x_\alpha^{p^k} \,|\, \alpha \in J_i\}$ is a basis of $F_{ip^k} A(k)^*$ and $\{x_\alpha^{p^k} \,|\, \alpha \in J_{i-1}\}$ is a basis of $F_{ip^k - 1} A(k)^* = F_{(i-1)p^k} A(k)^*$.

(3) Let $f : A^* \to A(k)^*$ be a map defined by $f(x) = x^{p^k}$. f induces an additive bijection $E_i^j A^* \to E_{ip^k}^{jp^k} A(k)^*$.

Proof. (1) Suppose that $\sum_{i=1}^n c_i x_i^p = 0$ for $c_i \in K^*$. Let b_i be a p-th root of c_i. Then we have $\left(\sum_{i=1}^n b_i x_i\right)^p = 0$ which implies $\sum_{i=1}^n b_i x_i = 0$ by the assumption. Hence $b_i = 0$ for $i = 1, 2, \dots, n$ and $x_1^p, x_2^p, \dots, x_n^p$ are linearly independent. Thus the assertion follows from the induction on k.

(2) $\{x_\alpha^{p^k} \,|\, \alpha \in J_i\}$ is linearly independent by (1). For $x \in F_{ip^k} A(k)^* - F_{ip^k - 1} A(k)^*$, there exists $y \in A^*$ such that $x = y^{p^k}$ by the definition of $A(k)^*$. Then, $y \in F_i A^* - F_{i-1} A^*$ by the assumption, hence $y = \sum_{\alpha \in I} c_\alpha x_\alpha$ for a finite subset I of J_i and $c_\alpha \in K^*$. Thus we have $x = y^{p^k} = \sum_{\alpha \in I} c_\alpha^{p^k} x_\alpha^{p^k}$. In particular, $\{x_\alpha^{p^k} \,|\, \alpha \in J_0\}$ is a basis of $F_0 A(k)^*$. We note that $F_{ip^k - 1} A(k)^* = F_{(i-1)p^k} A(k)^*$ holds for $i = 1, 2, \dots$ by the assumption. Assume inductively that $\{x_\alpha^{p^k} \,|\, \alpha \in J_{i-1}\}$ is a basis of $F_{ip^k - 1} A(k)^* = F_{(i-1)p} A(k)^*$. Since each element of $F_{ip^k} A(k)^* - F_{ip^k - 1} A(k)^*$ is a linear combination of elements of $\{x_\alpha^{p^k} \,|\, \alpha \in J_i\}$ and $J_{i-1} \subset J_i$, $\{x_\alpha^{p^k} \,|\, \alpha \in J_i\}$ spans $F_{ip^k} A(k)^*$.

(3) We note that $\{\rho_{A^*, i}(x_\alpha) \,|\, \alpha \in J_i - J_{i-1}, \deg x_\alpha = j\}$ is a basis of $E_i^j A^*$. It follows from (2) that $\{\rho_{A(k)^*, i}(x_\alpha^{p^k}) \,|\, \alpha \in J_i - J_{i-1}, \deg x_\alpha = j\}$ is a basis of $E_{ip^k}^{jp^k} A^*$. Hence f induces a bijective correspondence between these basis. Since the p^k-th power map $a \mapsto a^{p^k}$ gives an automorphism of K^*, the map $E_i^j A^* \to E_{ip^k}^{jp^k} A(k)^*$ induced by f is an additive bijection. □

15.2 Filtered algebras and unstable modules

Let A^* be an algebra in $\mathcal{TopMod}_{K^*}$ filtered by $\mathfrak{F} = (F_i A^*)_{i \in \boldsymbol{Z}}$.

Definition 15.2.1 *Let M^* be a left A^*-module with a multiplication $\alpha : A^* \otimes_{K^*} M^* \to M^*$. M^* is called an unstable A^*-module with respect to $\mathfrak{F}$ or unstable A^*-module for short if $\alpha(F_{n-1} A^* \otimes_{K^*} M^n) = \{0\}$ for $n \in \boldsymbol{Z}$. We denote by $\mathcal{UMod}(A^*)$ the full subcategory of $\mathcal{Mod}(A^*)$ consisting of unstable A^*-modules.*

Remark 15.2.2 *An unstable A^*-module M^* is coconnective if $\mathfrak{F}$ satisfies $(f2)$ of (15.1.20). In fact, since $F_{n-1} A^* = A^*$ if $n \geqq 1$, we have $M^n \subset \alpha(A^* \otimes_{K^*} M^n) = \alpha(F_{n-1} A^* \otimes_{K^*} M^n) = \{0\}$.*

It is clear that submodules and quotient modules of an unstable module are also unstable and that the sum and the product of unstable modules are unstable. Hence $\mathcal{UMod}(A^*)$ is complete and cocomplete and the inclusion functor $I_{A^*} : \mathcal{UMod}(A^*) \to \mathcal{Mod}(A^*)$ preserves limits and colimits.

Proposition 15.2.3 *The inclusion functor $I_{A^*} : \mathcal{UMod}(A^*) \to \mathcal{Mod}(A^*)$ has a right adjoint.*

Proof. Let M^* be an object of $\mathcal{Mod}(A^*)$ and let us denote by $U_{A^*}(M^*)$ the set of all unstable submodules of M^*. Since $\{0\} \in U_{A^*}(M^*)$, $U_{A^*}(M^*)$ is not empty. If $(M_i^*)_{i \in I}$ is a family of elements of $U_{A^*}(M^*)$, the sum $\sum_{i\in I} M_i^*$ is contained in $U_{A^*}(M^*)$. Hence there exists the largest unstable submodule M_u^* of M^*. For a homomorphism $f : M^* \to N^*$ of A^*-modules, since the image of an unstable submodule of M^* is also unstable, f induces a homomorphism $f_u : M_u^* \to N_u^*$. Thus we have a functor $R_{A^*} : \mathcal{M}od(A^*) \to \mathcal{UM}od(A^*)$ defined by $R_{A^*}(M^*) = M_u^*$ and $R_{A^*}(f) = f_u$. It is clear that $R_{A^*}I_{A^*} = id_{\mathcal{UM}od(A^*)}$. Let $\eta : id_{\mathcal{UM}od(A^*)} \to R_{A^*}I_{A^*}$ be the identity natural transformation. We denote by $\varepsilon : I_{A^*}R_{A^*} \to id_{\mathcal{M}od(A^*)}$ the natural transformation defined from the inclusion maps $M_u^* \to M^*$. R_{A^*} is a right adjoint of I_{A^*} whose unit and counit are η and ε, respectively. □

Condition 15.2.4 *For an algebra A^* with a multiplication $\mu : A^* \otimes_{K^*} A^* \to A^*$ and an increasing filtration $\mathfrak{F} = (F_iA^*)_{i\in \boldsymbol{Z}}$ of subspaces of A^*, we consider the following conditions.*

($f5$) F_iA^*'s are left ideals of A^* for $i \in \boldsymbol{Z}$.
($f6$) $\mu(F_iA^* \otimes_{K^*} A^j) \subset F_{i-j}A^*$ for $i, j \in \boldsymbol{Z}$.

Remark 15.2.5 *Suppose that $\mathfrak{F}$ satisfies ($f5$) above. Then $\mathfrak{F}$ satisfies ($f6$) if and only if $\Sigma^n(A^*/F_{n-1}A^*)$ is an unstable A^*-module for any $n \in \boldsymbol{Z}$.*

The following assertion is obvious.

Proposition 15.2.6 *Let A^* be an algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with a multiplication $\mu : A^* \otimes_{K^*} A^* \to A^*$ and an increasing filtration $(F_iA^*)_{i\in\boldsymbol{Z}}$ of subspaces of A^*. Let B^* be a subalgebra of A^* and $\mathfrak{a}$ a two-sided ideal of A^*. We consider a filtration $(F_iB^*)_{i\in\boldsymbol{Z}}$ of B^* given by $F_iB^* = B^* \cap F_iA^*$ and a filtration $(F_i(A^*/\mathfrak{a}))_{i\in\boldsymbol{Z}}$ of $A^*/\mathfrak{a}$ given by $F_i(A^*/\mathfrak{a}) = \pi_{\mathfrak{a}}(F_iA^*)$, where $\pi_{\mathfrak{a}} : A^* \to A^*/\mathfrak{a}$ the quotient map.*

(1) If $(F_iA^*)_{i\in\boldsymbol{Z}}$ satisfies ($f5$), $(F_iB^*)_{i\in\boldsymbol{Z}}$ and $(F_i(A^*/\mathfrak{a}))_{i\in\boldsymbol{Z}}$ satisfy ($f5$).
(2) If $(F_iA^*)_{i\in\boldsymbol{Z}}$ satisfies ($f6$), $(F_iB^*)_{i\in\boldsymbol{Z}}$ and $(F_i(A^*/\mathfrak{a}))_{i\in\boldsymbol{Z}}$ satisfy ($f6$).

Proposition 15.2.7 *Let A^* and B^* be algebras in $\mathcal{T}op\mathcal{M}od_{K^*}$ with products $\mu_{A^*} : A^* \otimes_{K^*} A^* \to A^*$ and $\mu_{B^*} : B^* \otimes_{K^*} B^* \to B^*$, respectively. Define $\mu_{A^*\otimes_{K^*}B^*} : A^* \otimes_{K^*} B^* \otimes_{K^*} A^* \otimes_{K^*} B^* \to A^* \otimes_{K^*} B^*$ to be the following composition.*

$$A^* \otimes_{K^*} B^* \otimes_{K^*} A^* \otimes_{K^*} B^* \xrightarrow{id_{A^*}\otimes_{K^*}T_{A^*,B^*}\otimes_{K^*}id_{B^*}} A^* \otimes_{K^*} A^* \otimes_{K^*} B^* \otimes_{K^*} B^* \xrightarrow{\mu_{A^*}\otimes_{K^*}\mu_{B^*}} A^* \otimes_{K^*} B^*$$

Let $(F_iA^*)_{i\in\boldsymbol{Z}}$ and $(F_iB^*)_{i\in\boldsymbol{Z}}$ be filtrations of A^* and B^*, respectively.

(1) If $(F_iA^*)_{i\in\boldsymbol{Z}}$ and $(F_iB^*)_{i\in\boldsymbol{Z}}$ satisfies ($f5$), so does $(F_i(A^* \otimes_{K^*} B^*))_{i\in\boldsymbol{Z}}$.
(2) If $(F_iA^*)_{i\in\boldsymbol{Z}}$ and $(F_iB^*)_{i\in\boldsymbol{Z}}$ satisfies ($f6$), so does $(F_i(A^* \otimes_{K^*} B^*))_{i\in\boldsymbol{Z}}$.

Proof. (1) The assertion follows from the following relation.

$$\mu_{A^*\otimes_{K^*}B^*}(A^* \otimes_{K^*} B^* \otimes_{K^*} F_jA^* \otimes_{K^*} F_kB^*) = \mu_{A^*}(A^* \otimes_{K^*} F_jA^*) \otimes_{K^*} \mu_{B^*}(B^* \otimes_{K^*} F_kB^*) \subset F_jA^* \otimes_{K^*} F_kB^*$$

(2) For $i, j, k, l, m, n \in \boldsymbol{Z}$ which satisfy $j + k = i$ and $l + m = n$, we have the following by the assumption.

$$\begin{aligned}\mu_{A^*\otimes_{K^*}B^*}(F_jA^* \otimes_{K^*} F_kB^* \otimes_{K^*} A^l \otimes_{K^*} B^m) &= \mu_{A^*}(F_jA^* \otimes_{K^*} A^l) \otimes_{K^*} \mu_{B^*}(F_kB^* \otimes_{K^*} B^m)\\ &\subset F_{j-l}A^* \otimes_{K^*} F_{k-m}B^* \subset F_{i-n}(A^* \otimes_{K^*} B^*)\end{aligned}$$

Hence $\mu_{A^*\otimes_{K^*}B^*}(F_i(A^* \otimes_{K^*} B^*) \otimes_{K^*} (A^* \otimes_{K^*} B^*)^n) \subset F_{i-n}(A^* \otimes_{K^*} B^*)$ holds. □

Proposition 15.2.8 *For a left A^*-module M^* with structure map $\alpha : A^* \otimes_{K^*} M^* \to M^*$, define a subspace $\bar{M}^n$ of M^n by $\bar{M}^n = \{x \in M^n \,|\, \alpha(a \otimes x) = 0$ for any $a \in F_{n-1}M^*\}$ and put $\bar{M}^* = \sum_{n\in\boldsymbol{Z}} \bar{M}^n$. If $(F_iA^*)_{i\in\boldsymbol{Z}}$ satisfies ($f6$), $\bar{M}^*$ is the largest unstable submodule of M^*. Hence we have $R_{A^*}(M^*) = \bar{M}^*$.*

Proof. For $x \in \bar{M}^n$, $b \in A^m$ and $a \in F_{m+n-1}A^*$, since $\mu(a \otimes b) \in F_{n-1}A^*$ holds by ($f6$), we have an equality $\alpha(a \otimes \alpha(b \otimes x)) = \alpha(\mu(a \otimes b) \otimes x) = 0$ which shows $\alpha(b \otimes x) \in \bar{M}^{m+n}$. Hence $\bar{M}^*$ is an unstable submodule of M^*. It is clear that $\bar{M}^*$ is the largest submodule among unstable submodules of M^*. □

Let M^* be a left A^*-module with structure map $\alpha : A^* \otimes_{K^*} M^* \to M^*$ and suppose that $\mathfrak{F}$ satisfies ($f5$) of (15.2.4). We put $\mathcal{N}(M^*) = \sum_{n\in\boldsymbol{Z}} \alpha(F_{n-1}A^* \otimes_{K^*} M^n)$. In other words, $\mathcal{N}(M^*)$ is a submodule of M^* generated by $\{ax \in M^* \,|\, a \in F_{n-1}A^*,\ x \in M^n$ for some $n \in \boldsymbol{Z}\}$. Then, M^* is an unstable if and only if $\mathcal{N}(M^*) = \{0\}$. Put $L_{A^*}(M^*) = M^*/\mathcal{N}(M^*)$, then $L_{A^*}(M^*)$ is an unstable A^*-module. If $f : M^* \to N^*$ is a homomorphism of left A^*-modules, then f maps $\mathcal{N}(M^*)$ into $\mathcal{N}(N^*)$. Hence f induces a homomorphism of left A^*-modules $L_{A^*}(M^*) \to L_{A^*}(N^*)$ which we denote by $L_{A^*}(f)$. Thus we have a functor $L_{A^*} : \mathcal{M}od(A^*) \to \mathcal{UM}od(A^*)$.

Proposition 15.2.9 *If $\mathfrak{F}$ satisfies $(f5)$ of (15.2.4), the inclusion functor $I_{A^*} : \mathcal{UMod}(A^*) \to \mathcal{Mod}(A^*)$ has a left adjoint L_{A^*}.*

Proof. Since $L_{A^*}I_{A^*}(M^*) = M^*$ if M^* is an object of $\mathcal{UMod}(A^*)$, $L_{A^*}I_{A^*}$ is the identity functor and we define $\varepsilon : L_{A^*}I_{A^*} \to id_{\mathcal{UMod}(A^*)}$ to be the identity natural transformation. For an object M^* of $\mathcal{Mod}(A^*)$, let $\eta_{M^*} : M^* \to I_{A^*}L_{A^*}(M^*)$ be the quotient map $M^* \to M^*/\mathcal{N}(M^*)$. It is clear that η_{M^*} is natural in M^* and that compositions

$$L_{A^*}(M^*) \xrightarrow{L_{A^*}(\eta_{M^*})} L_{A^*}I_{A^*}L_{A^*}(M^*) \xrightarrow{\varepsilon_{L_{A^*}(M^*)}} L_{A^*}(M^*),$$
$$I_{A^*}(M^*) \xrightarrow{\eta_{I_{A^*}(M^*)}} I_{A^*}L_{A^*}I_{A^*}(M^*) \xrightarrow{I_{A^*}(\varepsilon_{M^*})} I_{A^*}(M^*)$$

are identity morphisms of $L_{A^*}(M^*)$ and $I_{A^*}(M^*)$, respectively. □

We note that the forgetful functor $O : \mathcal{Mod}(A^*) \to \mathcal{TopMod}_{K^*}$ has a left adjoint $F : \mathcal{TopMod}_{K^*} \to \mathcal{Mod}(A^*)$ given by $F(M^*) = A^* \otimes_{K^*} M^*$ and $F(f) = id_{A^*} \otimes_{K^*} f$. Let us denote by $\mathcal{F} : \mathcal{TopMod}_{K^*} \to \mathcal{UMod}(A^*)$ the composition of F and L_{A^*}, by $\mathcal{O} : \mathcal{UMod}(A^*) \to \mathcal{TopMod}_{K^*}$ the composition of I_{A^*} and O. By (15.2.9), we have the following result.

Proposition 15.2.10 *If $\mathfrak{F}$ satisfies $(f5)$ of (15.2.4), $\mathcal{F}$ is a left adjoint of $\mathcal{O}$. In particular, $\mathcal{F}(\Sigma^n K^*)$ represents a functor $\epsilon_n\mathcal{O} : \mathcal{UMod}(A^*) \to \mathcal{TopMod}_{K^*}$.*

Remark 15.2.11 (1) *Suppose that $\mathfrak{F}$ satisfies $(f5)$ and $(f6)$ of (15.2.4). Then, for an object M^* of $\mathcal{TopMod}_{K^*}$, we have $\mathcal{N}(F(M^*)) = \sum_{n\in\boldsymbol{Z}} F_{n-1}A^* \otimes_{K^*} M^n$. Hence $\mathcal{F}(M^*)$ is isomorphic to $\sum_{n\in\boldsymbol{Z}} A^*/F_{n-1}A^* \otimes_{K^*} M^n$ as a left A^*-module.*

(2) *For an object M^* of $\mathcal{UMod}(A^*)$, the left A^*-module structure map $\mu : A^* \otimes_{K^*} M^* \to M^*$ of M^* factors through the quotient map $A^* \otimes_{K^*} \mathcal{O}(M^*) \to \mathcal{FO}(M^*)$ and μ induces a map $\varepsilon_{M^*} : \mathcal{FO}(M^*) \to M^*$. For an object M^* of $\mathcal{TopMod}_{K^*}$, let $\eta_{M^*} : M^* \to \mathcal{OF}(M^*)$ be the composition of a map $M^* \to A^* \otimes_{K^*} M^*$ given by $x \mapsto 1 \otimes x$ and the the quotient map $A^* \otimes_{K^*} M^* \to \mathcal{F}(M^*)$. It is easy to verify that $\eta : id_{\mathcal{TopMod}_{K^*}} \to \mathcal{OF}$ and $\varepsilon : \mathcal{FO} \to id_{\mathcal{TopMod}_{K^*}}$ are the unit and the counit of the adjunction $\mathcal{F} \vdash \mathcal{O}$, respectively.*

Let $f : A^* \to B^*$ be a homomorphism of algebras in $\mathcal{TopMod}_{K^*}$. For a left B^*-module N^* with structure map $\beta : B^* \otimes_{K^*} N^* \to N^*$, we denote by $f_*(N^*)$ a left A^*-module N^* with a structure map $\beta(f \otimes_{K^*} id_{N^*}) : A^* \otimes_{K^*} N^* \to N^*$. Define a functor $f_* : \mathcal{Mod}(B^*) \to \mathcal{Mod}(A^*)$ by $N^* \mapsto f_*(N^*)$ for $N^* \in \operatorname{Ob}\mathcal{Mod}(B^*)$ and $f_*(\varphi) = \varphi$ for a morphism φ of $\mathcal{Mod}(B^*)$. We note that $f_* : \mathcal{Mod}(B^*) \to \mathcal{Mod}(A^*)$ has a left adjoint $f^* : \mathcal{Mod}(A^*) \to \mathcal{Mod}(B^*)$ defined as follows. Put $f^*(M^*) = B^* \otimes_{A^*} M^*$ for $M^* \in \operatorname{Ob}\mathcal{Mod}(A^*)$ and the left B^*-module structure of $f^*(M^*)$ is defined from the product of B^*. For a homomorphism $\varphi : M^* \to L^*$, put $f^*(\varphi) = id_{B^*} \otimes_{A^*} \varphi$. Then, f^* is a left adjoint of f_*.

Proposition 15.2.12 *Suppose that increasing filtrations $\mathfrak{F}_{A^*} = (F_iA^*)_{i\in\boldsymbol{Z}}$ of A^* and $\mathfrak{F}_{B^*} = (F_iB^*)_{i\in\boldsymbol{Z}}$ of B^* are given. If f preserves filtration, that is, $f(F_iA^*) \subset F_iB^*$ holds for $i \in \boldsymbol{Z}$, $f_* : \mathcal{Mod}(B^*) \to \mathcal{Mod}(A^*)$ maps each object of $\mathcal{UMod}(B^*)$ to that of $\mathcal{UMod}(A^*)$.*

Proof. For $M^* \in \operatorname{Ob}\mathcal{UMod}(B^*)$, let $\beta : B^* \otimes_{K^*} M^* \to M^*$ the structure map of M^*. Then, we have $\beta(f(F_{n-1}A^*) \otimes_{K^*} M^n) \subset \beta(F_{n-1}B^* \otimes_{K^*} M^n) = \{0\}$. □

Thus $f_* : \mathcal{Mod}(B^*) \to \mathcal{Mod}(A^*)$ restricts to a functor $f_{u*} : \mathcal{UMod}(B^*) \to \mathcal{UMod}(A^*)$.

Proposition 15.2.13 *If $\mathfrak{F}_{B^*}$ satisfies $(f5)$ of (15.2.4), $f_{u*} : \mathcal{UMod}(B^*) \to \mathcal{UMod}(A^*)$ has a left adjoint.*

Proof. Define $f_u^* : \mathcal{UMod}(A^*) \to \mathcal{UMod}(B^*)$ to be the following composition.

$$\mathcal{UMod}(A^*) \xrightarrow{I_{A^*}} \mathcal{Mod}(A^*) \xrightarrow{f^*} \mathcal{Mod}(B^*) \xrightarrow{L_{B^*}} \mathcal{UMod}(B^*)$$

Let M^* be an object of $\mathcal{UMod}(A^*)$ and N^* an object of $\mathcal{UMod}(B^*)$. Since $I_{B^*} : \mathcal{UMod}(B^*) \to \mathcal{Mod}(B^*)$ has a left adjoint $L_{B^*} : \mathcal{Mod}(B^*) \to \mathcal{UMod}(B^*)$ by (15.2.9) and $f^* : \mathcal{Mod}(A^*) \to \mathcal{Mod}(B^*)$ has a right adjoint $f_* : \mathcal{Mod}(B^*) \to \mathcal{Mod}(A^*)$, we have the following chain of natural bijections.

$$\begin{aligned}\mathcal{UMod}(B^*)(f_u^*(M^*), N^*) &= \mathcal{UMod}(B^*)(L_{B^*}f^*I_{A^*}(M^*), N^*) \cong \mathcal{Mod}(B^*)(f^*I_{A^*}(M^*), I_{B^*}(N^*)) \\ &\cong \mathcal{Mod}(A^*)(I_{A^*}(M^*), f_*I_{B^*}(N^*)) = \mathcal{Mod}(A^*)(I_{A^*}(M^*), I_{A^*}f_{u*}(N^*)) \\ &\cong \mathcal{UMod}(A^*)(M^*, f_{u*}(N^*))\end{aligned}$$

Hence $f_u^* : \mathcal{UMod}(A^*) \to \mathcal{UMod}(B^*)$ is a left adjoint of $f_{u*} : \mathcal{UMod}(B^*) \to \mathcal{UMod}(A^*)$. □

Condition 15.2.14 *For a coconnective algebra A^* with a multiplication $\mu : A^* \otimes_{K^*} A^* \to A^*$ and an increasing filtration $\mathfrak{F} = (F_iA^*)_{i \in \boldsymbol{Z}}$ of subspaces of A^*, we consider the following condition.*

($f7$) *$\mu((F_iA^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} A^j) + (F_{i-j-1}A^*)^{j+c_{\mathfrak{F}}(i)} = (F_{i-j}A^*)^{j+c_{\mathfrak{F}}(i)}$ holds for $i \in S(\mathfrak{F})$ and $j \in \boldsymbol{Z}$.*

Proposition 15.2.15 *Assume that $\mathfrak{F} = (F_iA^*)_{i \in \boldsymbol{Z}}$ satisfies ($f1$), ($f3$) and ($f7$). A left A^*-module M^* with structure map $\alpha : A^* \otimes_{K^*} M^* \to M^*$ is unstable if and only if $\alpha((F_iA^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} M^k) = \{0\}$ for any $i \in S(\mathfrak{F})$ and $k > i$.*

Proof. Assume that $\alpha((F_iA^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} M^k) = \{0\}$ for any $i \in S(\mathfrak{F})$ and $k > i$. Then, by the associativity of α, we have $\alpha(\mu((F_iA^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} A^j) \otimes_{K^*} M^{k-j}) = \alpha((F_iA^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} \alpha(A^j \otimes_{K^*} M^{k-j})) \subset \alpha((F_iA^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} M^k) = \{0\}$. Hence $\alpha(\mu((F_iA^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} A^j) \otimes_{K^*} M^{k-j}) = \{0\}$ holds. It follows from ($f7$) that we have

$$\begin{aligned}\alpha((F_{i-j}A^*)^{j+c_{\mathfrak{F}}(i)} \otimes_{K^*} M^{k-j}) &= \alpha((\mu((F_iA^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} A^j) + (F_{i-j-1}A^*)^{j+c_{\mathfrak{F}}(i)}) \otimes_{K^*} M^{k-j}) \\ &= \alpha(\mu((F_iA^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} A^j) \otimes_{K^*} M^{k-j}) + \alpha((F_{i-j-1}A^*)^{j+c_{\mathfrak{F}}(i)} \otimes_{K^*} M^{k-j}) \\ &= \alpha((F_{i-j-1}A^*)^{j+c_{\mathfrak{F}}(i)} \otimes_{K^*} M^{k-j})\end{aligned}$$

for $i \in S(\mathfrak{F})$ and $k > i$. Put $n = k - j$ and $s = i - j$, we see that the following equality holds for any $n \in \boldsymbol{Z}$, $i \in S(\mathfrak{F})$ and $s < n$.

$$\alpha((F_{s-1}A^*)^{i-s+c_{\mathfrak{F}}(i)} \otimes_{K^*} M^n) = \alpha((F_sA^*)^{i-s+c_{\mathfrak{F}}(i)} \otimes_{K^*} M^n)$$

Since $(F_{s-1}A^*)^t = (F_sA^*)^t$ if $s + t \neq i + c_{\mathfrak{F}}(i)$ for any $i \in S(\mathfrak{F})$ by ($f3$), it follows from the above equality that $\alpha((F_{s-1}A^*)^t \otimes_{K^*} M^n) = \alpha((F_sA^*)^t \otimes_{K^*} M^n)$ holds for any $t \in \boldsymbol{Z}$ if $s < n$. Hence we have

$$\alpha((F_mA^*)^t \otimes_{K^*} M^n) = \alpha((F_{n-1}A^*)^t \otimes_{K^*} M^n)$$

for any $t \in \boldsymbol{Z}$ if $m < n$. Since A^* is finite type, $(F_mA^*)^t = \{0\}$ for sufficently large m. Hence we have $\alpha((F_{n-1}A^*)^t \otimes_{K^*} M^n) = \{0\}$ for $n, t \in \boldsymbol{Z}$.

The converse follows from $\alpha((F_iA^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} M^k) \subset \alpha(F_{k-1}A^* \otimes_{K^*} M^k) = \{0\}$ for $i \in S(\mathfrak{F})$ and $k > i$. □

Condition 15.2.16 *For an algebra A^* with multiplication $\mu : A^* \otimes_{K^*} A^* \to A^*$, we assume that an increasing filtration $\mathfrak{F} = (F_iA^*)_{i \in \boldsymbol{Z}}$ of subspaces of A^* satisfies ($f6$). Then, μ defines $\mu_i^{k,j} : (F_iA^*)^k \otimes_{K^*} A^j \to (F_{i-j}A^*)^{j+k}$.*

($f8$) *$(\mu_i^{c_{\mathfrak{F}}(i),j})^{-1}((F_{i-j-1}A^*)^{j+c_{\mathfrak{F}}(i)}) = (F_{i-1}A^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} A^j + (F_iA^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} (F_{i-j-1}A^*)^j$ holds for $i \in S(\mathfrak{F})$, $j \in \boldsymbol{Z}$.*

Assume that $\mathfrak{F}$ satisfies ($f6$). Since the horizontal rows of the following diagram are exact, there exists unique map $\bar{\mu}_i^{k,j} : E_i^kA^* \otimes_{K^*} A^j \to E_{i-j}^kA^*$ that make the following diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & (F_{i-1}A^*)^k \otimes_{K^*} A^j \arrow[r, "\iota_{A^*,i} \otimes_{K^*} id_{A^j}"] \arrow[d, "\mu_{i-1}^{k,j}"] & (F_iA^*)^k \otimes_{K^*} A^j \arrow[r, "\rho_{A^*,i} \otimes_{K^*} id_{A^j}"] \arrow[d, "\mu_i^{k,j}"] & E_i^kA^* \otimes_{K^*} A^j \arrow[r] \arrow[d, dashed, "\bar{\mu}_i^{k,j}"] & 0 \\ 0 \arrow[r] & (F_{i-j-1}A^*)^{j+k} \arrow[r, "\iota_{A^*,i-j}"] & (F_{i-j}A^*)^{j+k} \arrow[r, "\rho_{A^*,i-j}"] & E_{i-j}^{j+k}A^* \arrow[r] & 0 \end{tikzcd}$$

If moreover $\mathfrak{F}$ satisfies ($f5$), since μ maps $A^* \otimes_{K*} F_{i-j-1}A^*$ into $F_{i-j-1}A^*$, $\bar{\mu}_i^{k,j}$ maps $E_i^kA^* \otimes_{K^*} (F_{i-j-1}A^*)^j$ to zero. Hence the exists unique map $\tilde{\mu}_i^{k,j} : E_i^kA^* \otimes_{K^*} (A^*/F_{i-j-1}A^*)^j \to E_{i-j}^{j+k}A^*$ that make the following diagram commute.

$$\begin{tikzcd} E_i^kA^* \otimes_{K^*} A^j \arrow[r, "id_{E_i^kA^*} \otimes_{K^*} \pi_{A^*,i-j}"] \arrow[dr, "\bar{\mu}_i^{k,j}"'] & E_i^kA^* \otimes_{K^*} (A^*/F_{i-j-1}A^*)^j \arrow[d, dashed, "\tilde{\mu}_i^{k,j}"] \\ & E_{i-j}^{j+k}A^* \end{tikzcd}$$

Remark 15.2.17 *$\mathfrak{F}$ satisfies ($f7$) if and only if $\tilde{\mu}_i^{c_{\mathfrak{F}}(i),j} : E_i^{c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} (A^*/F_{i-j-1}A^*)^j \to E_{i-j}^{j+c_{\mathfrak{F}}(i)}A^*$ is surjective for $i \in S(\mathfrak{F})$, $j \in \boldsymbol{Z}$. $\mathfrak{F}$ satisfies ($f8$) if and only if $\tilde{\mu}_i^{c_{\mathfrak{F}}(i),j} : E_i^{c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} (A^*/F_{i-j-1}A^*)^j \to E_{i-j}^{j+c_{\mathfrak{F}}(i)}A^*$ is injective for $i \in S(\mathfrak{F})$, $j \in \boldsymbol{Z}$. Thus $\mathfrak{F}$ satisfies ($f7$) and ($f8$) if and only if $\tilde{\mu}_i^{c_{\mathfrak{F}}(i),j} : E_i^{c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} (A^*/F_{i-j-1}A^*)^j \to E_{i-j}^{j+c_{\mathfrak{F}}(i)}A^*$ is an isomorphism for $i \in S(\mathfrak{F})$, $j \in \boldsymbol{Z}$.*

Let A^* be an algebra over a field K^* with an increasing filtration $\mathfrak{F} = (F_iA^*)_{i\in \boldsymbol{Z}}$. Suppose that $\mathfrak{F}$ satisfies $(f3)$, $(f4)$, $(f5)$, $(f6)$, $(f7)$ and $(f8)$. For an unstable A^*-module M^*, define an A^*-module ΦM^* as follows. Put

$$\Phi M^* = \sum_{i\in S(\mathfrak{F})} E_i^{c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} M^i.$$

In other words, $(\Phi M^*)^k = \{0\}$ if $k \neq i+c_{\mathfrak{F}}(i)$ for any $i \in S(\mathfrak{F})$ and $(\Phi M^*)^k = E_i^{c_{\mathfrak{F}}(i)}A^*\otimes_{K^*}M^i$ if $k = i+c_{\mathfrak{F}}(i)$ for $i \in S(\mathfrak{F})$ which is uniquely determined by $(f4)$. The topology on ΦM^* is the one as a subspace of $E_*^*A^*\otimes_{K^*}M^*$. The product $\mu : A^* \otimes_{K^*} A^* \to A^*$ of A^* defines maps $\bar{\mu}_i : A^* \otimes_{K^*} F_iA^* \to F_iA^*$ for $i \in \boldsymbol{Z}$ by $(f5)$. Since the horizontal rows of the following diagram are exact, there exists unique map $\mu_i : A^* \otimes_{K^*} E_i^*A^* \to E_i^*A^*$ that make the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & A^* \otimes_{K^*} F_{i-1}A^* \arrow[r, "id_{A^*}\otimes_{K^*}\iota_{A^*,\,i}"] \arrow[d, "\bar{\mu}_{i-1}"] & A^* \otimes_{K^*} F_iA^* \arrow[r, "id_{A^*}\otimes_{K^*}\rho_{A^*,\,i}"] \arrow[d, "\bar{\mu}_i"] & A^* \otimes_{K^*} E_iA^* \arrow[r] \arrow[d, dashed, "\mu_i"] & 0 \\
0 \arrow[r] & F_{i-1}A^* \arrow[r, "\iota_{A^*,\,i}"] & F_iA^* \arrow[r, "\rho_{A^*,\,i}"] & E_i^*A^* \arrow[r] & 0
\end{tikzcd}$$

It is easy to verify that μ_i gives $E_i^*A^*$ a structure of a left A^*-module.

Let $\alpha : A^* \otimes_{K^*} M^* \to M^*$ be the A^*-module structure map of M^*. Since M^* is unstable, α induces $\alpha_i : A^*/F_{i-1}A^* \otimes_{K^*} M^i \to M^*$. We define maps $\alpha_{j,k} : A^j \otimes_{K^*} (\Phi M^*)^k \to (\Phi M^*)^{j+k}$ for $j, k \in \boldsymbol{Z}$ as follows. If there exist $i, s \in S(\mathfrak{F})$ which satisfy $k = i + c_{\mathfrak{F}}(i)$ and $j + k = s + c_{\mathfrak{F}}(s)$, then such i and s are unique by $(f4)$. In this case, define $\alpha_{j,k}$ to be the following composition.

$$A^j\otimes_{K^*}(\Phi M^*)^{i+c_{\mathfrak{F}}(i)}=A^j\otimes_{K^*}E_i^{c_{\mathfrak{F}}(i)}A^*\otimes_{K^*}M^i\xrightarrow{\mu_i\otimes_{K^*}id_{M^i}}E_i^{j+c_{\mathfrak{F}}(i)}A^*\otimes_{K^*}M^i=E_i^{s-i+c_{\mathfrak{F}}(s)}A^*\otimes_{K^*}M^i$$

$$\xrightarrow{(\tilde{\mu}_s^{c_{\mathfrak{F}}(s),s-i})^{-1}\otimes_{K^*}id_{M^i}}E_s^{c_{\mathfrak{F}}(s)}A^*\otimes_{K^*}(A^*/F_{i-1}A^*)^{s-i}\otimes_{K^*}M^i\xrightarrow{id_{E_s^{c_{\mathfrak{F}}(s)}A^*}\otimes_{K^*}\alpha_i}E_s^{c_{\mathfrak{F}}(s)}A^*\otimes_{K^*}M^s=(\Phi M^*)^{s+c_{\mathfrak{F}}(s)}$$

Since $(\Phi M^*)^k = \{0\}$ if $k \neq i+c_{\mathfrak{F}}(i)$ for any $i \in S(\mathfrak{F})$, $\alpha_{j,k}$ is zero map other than the above case. Let us denote by $\alpha_\Phi : A^* \otimes_{K^*} \Phi M^* \to \Phi M^*$ the map induced by $\alpha_{j,k}$'s which gives a left A^*-module strucutre of M^*. Since μ_i maps $F_{i+c_{\mathfrak{F}}(i)-1}A^* \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^*$ into $\{0\}$ by $(f6)$, ΦM^* is an unstable A^*-module.

For a homomorphism $\varphi : M^* \to N^*$ between unstable A^*-modules, let $\Phi\varphi : \Phi M^* \to \Phi N^*$ be the map induced by $id_{E_i^{c_{\mathfrak{F}}(i)}A^*} \otimes_{K^*} \varphi$. Then, $\Phi\varphi$ is a homomorphism of left A^*-modules and Φ is an endofunctor of $\mathcal{UM}od(A^*)$.

For an unstable A^*-module $M^*$with structure map $\alpha : A^* \otimes_{K^*} M^* \to M^*$, let $\bar{\alpha}_i^j : E_i^jA^* \otimes_{K^*} M^i \to M^{i+j}$ be a restriction of $\alpha_i : A^*/F_{i-1}A^* \otimes_{K^*} M^i \to M^*$. We define a map $\lambda_{M^*} : \Phi M^* \to M^*$ as follows. If $k = i + c_{\mathfrak{F}}(i)$ for $i \in S(\mathfrak{F})$, we put $\lambda_{M^*}^k = \bar{\alpha}_i^{c_{\mathfrak{F}}(i)} : (\Phi M^*)^k = E_i^{c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} M^i \to M^{i+c_{\mathfrak{F}}(i)} = M^k$. If $k \neq i + c_{\mathfrak{F}}(i)$ for any $i \in S(\mathfrak{F})$, $\lambda_{M^*}^k : (\Phi M^*)^k \to M^k$ is the trivial map. Let λ_{M^*} be the map induced by $\lambda_{M^*}^k$'s.

Proposition 15.2.18 *λ_{M^*} is a homomorphism of left A^*-modules and natural in M^*.*

Proof. Suppose that $i, s \in S(\mathfrak{F})$ satisfy $i+j+c_{\mathfrak{F}}(i) = s+c_{\mathfrak{F}}(s)$. Then, the following diagram $(*)$ is commutative by the associativity of α.

$$\begin{tikzcd}
A^j \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} M^i \arrow[r, "id_{A^j}\otimes_{K^*}\bar{\alpha}_i^{c_{\mathfrak{F}}(i)}"] \arrow[d, "\mu_i\otimes_{K^*}id_{M^i}"] & A^j \otimes_{K^*} M^{i+c_{\mathfrak{F}}(i)} \arrow[d, "\alpha"] \\
E_i^{j+c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} M^i \arrow[r, "\bar{\alpha}_i^{j+c_{\mathfrak{F}}(i)}"] \arrow[d, equal] & M^{i+j+c_{\mathfrak{F}}(i)} \arrow[d, equal] \\
E_i^{s-i+c_{\mathfrak{F}}(s)}A^* \otimes_{K^*} M^i \arrow[r, "\bar{\alpha}_i^{s-i+c_{\mathfrak{F}}(s)}"] \arrow[d, "(\tilde{\mu}_s^{c_{\mathfrak{F}}(s),s-i})^{-1}\otimes_{K^*}id_{M^i}"] & M^{s+c_{\mathfrak{F}}(s)} \\
E_s^{c_{\mathfrak{F}}(s)}A^* \otimes_{K^*} (A^*/F_{i-1}A^*)^{s-i} \otimes_{K^*} M^i \arrow[r, "id_{E_s^{c_{\mathfrak{F}}(s)}A^*}\otimes_{K^*}\alpha_i"] & E_s^{c_{\mathfrak{F}}(s)} \otimes_{K^*} M^s \arrow[u, "\bar{\alpha}_s^{c_{\mathfrak{F}}(s)}"]
\end{tikzcd} \qquad (*)$$

Hence the following diagram is commutative.

$$\begin{array}{ccc} A^j \otimes_{K^*} (\Phi M^*)^{i+c_{\mathfrak{F}}(i)} & \xrightarrow{id_{A^j} \otimes_{K^*} \lambda_{M^*}^{i+c_{\mathfrak{F}}(i)}} & A^j \otimes_{K^*} M^{i+c_{\mathfrak{F}}(i)} \\ \downarrow{\scriptstyle \alpha_\Phi} & & \downarrow{\scriptstyle \alpha} \\ (\Phi M^*)^{s+c_{\mathfrak{F}}(s)} & \xrightarrow{\lambda_{M^*}^{s+c_{\mathfrak{F}}(s)}} & M^{s+c_{\mathfrak{F}}(s)} \end{array}$$

If $i+j+c_{\mathfrak{F}}(i) \neq s+c_{\mathfrak{F}}(s)$ for any $s \in S(\mathfrak{F})$, then $E_i^{j+c_{\mathfrak{F}}(i)}A^* = \{0\}$ by $(f3)$. Hence the commutativity of the top rectangle of diagram $(*)$ implies that the following composition is a trivial map.

$$A^j \otimes_{K^*} (\Phi M^*)^{i+c_{\mathfrak{F}}(i)} \xrightarrow{id_{A^j} \otimes_{K^*} \lambda_{M^*}^{i+c_{\mathfrak{F}}(i)}} A^j \otimes_{K^*} M^{i+c_{\mathfrak{F}}(i)} \xrightarrow{\alpha} M^{i+j+c_{\mathfrak{F}}(i)}$$

Thus we see that λ_{M^*} is a homomorphism of left A^*-modules. The naturality of λ_{M^*} follows from the definitions of λ_{M^*} and Φf for a homomorphism f of left A^*-modules. □

Recall that $\tilde{\iota}_{A^*,n} : A^*/F_{n-1}A^* \to A^*/F_nA^*$ denotes the quotient map. For a K^*-module M^*, we define a map $\sigma_{M^*} : \mathcal{F}(M^*) \to \Sigma^{-1}\mathcal{F}(\Sigma M^*)$ by $\sigma_{M^*}(x \otimes y) = ([-1], \tilde{\iota}_{A^*,n}(x) \otimes ([1], y))$ for $x \in A^*/F_{n-1}A^*$ and $y \in M^*$.

Proposition 15.2.19 *If $\mathfrak{F}$ satisfies $(f3)$, $(f5)$, $(f6)$, $(f7)$ and $(f8)$, the following is a short exact sequence.*

$$0 \to \Phi\mathcal{F}(M^*) \xrightarrow{\lambda_{\mathcal{F}(M^*)}} \mathcal{F}(M^*) \xrightarrow{\sigma_{M^*}} \Sigma^{-1}\mathcal{F}(\Sigma M^*) \to 0$$

Proof. Recall that $\mathcal{F}(M^*) = \sum\limits_{n \in \boldsymbol{Z}} A^*/F_{n-1}A^* \otimes_{K^*} M^n$. Hence we have

$$\Phi\mathcal{F}(M^*) = \sum_{i \in S(\mathfrak{F})} E_i^{c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} \mathcal{F}(M^*)^i = \sum_{i \in S(\mathfrak{F})} \sum_{n \in \boldsymbol{Z}} E_i^{c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} (A^*/F_{n-1}A^*)^{i-n} \otimes_{K^*} M^n.$$

By $(f3)$, $(f7)$ and $(f8)$, $\lambda_{\mathcal{F}(M^*)}$ is an injection onto $\sum\limits_{n \in \boldsymbol{Z}} E_n^*A^* \otimes_{K^*} M^n$, which is the kernel of σ_{M^*}. □

Proposition 15.2.20 *Let M^* be an unstable A^*-module. If $\mathfrak{F}$ satisfies $(f1) \sim (f8)$, then $\Sigma\mathrm{Coker}\,\lambda_{M^*}$ is an unstable A^*-module.*

Proof. Let $\alpha : A^* \otimes_{K^*} M^* \to M^*$ the structure map of M^*. Since $\mathrm{Im}\,\lambda_{M^*}^{i+c_{\mathfrak{F}}(i)} = \alpha((F_iA^*)^{c_{\mathfrak{F}}(i)} \otimes_{K^*} M^i)$, we have $(F_iA^*)^{c_{\mathfrak{F}}(i)}(\mathrm{Coker}\,\lambda_{M^*})^i = \{0\}$ for $i \in S(\mathfrak{F})$. If $i \in S(\mathfrak{F})$ and $k > i$, the instability of M^* and (15.2.15) imply $(F_iA^*)^{c_{\mathfrak{F}}(i)}(\mathrm{Coker}\,\lambda_{M^*})^k = \{0\}$. Thus the assertion follows from (15.2.15). □

Suppose that $\mathfrak{F}$ satisfies $(f1) \sim (f8)$. Define a functor $\Omega : \mathcal{UMod}(A^*) \to \mathcal{UMod}(A^*)$ as follows. For an objevt M^* of $\mathcal{UMod}(A^*)$, we put $\Omega M^* = \Sigma\mathrm{Coker}\lambda_{M^*}$ and denote by $\tilde{\eta}_{M^*} : M^* \to \mathrm{Coker}\,\lambda_{M^*} = \Sigma^{-1}\Omega M^*$ the quotient map. It follows from (15.2.20) that ΩM^* is an object of $\mathcal{UMod}(A^*)$. For a morphism $\varphi : M^* \to N^*$ of $\mathcal{UMod}(A^*)$, there exists unique map $\bar{\varphi} : \mathrm{Coker}\,\lambda_{M^*} \to \mathrm{Coker}\,\lambda_{N^*}$ that makes the following diagram commute. We put $\Omega\varphi = \Sigma\bar{\varphi} : \Omega M^* \to \Omega N^*$.

$$\begin{array}{ccccccc} \Phi M^* & \xrightarrow{\lambda_{M^*}} & M^* & \xrightarrow{\tilde{\eta}_{M^*}} & \mathrm{Coker}\,\lambda_{M^*} = \Sigma^{-1}\Omega M^* & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Phi\varphi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}} & & \\ \Phi N^* & \xrightarrow{\lambda_{N^*}} & N^* & \xrightarrow{\tilde{\eta}_{N^*}} & \mathrm{Coker}\,\lambda_{N^*} = \Sigma^{-1}\Omega N^* & \longrightarrow & 0 \end{array}$$

Proposition 15.2.21 *Suppose that $\mathfrak{F}$ satisfies $(f1) \sim (f8)$. Ω is a left adjoint of the desuspension functor Σ^{-1}.*

Proof. We first note that $\lambda_{\Sigma^{-1}M^*} : \Phi\Sigma^{-1}M^* \to \Sigma^{-1}M^*$ is trivial by the instability of M^*. It follows that $\tilde{\eta}_{\Sigma^{-1}M^*} : \Sigma^{-1}M^* \to \Sigma^{-1}\Omega\Sigma^{-1}M^*$ is an isomorphism. Define $\tilde{\varepsilon}_{M^*} : \Omega\Sigma^{-1}M^* \to M^*$ by $\tilde{\varepsilon}_{M^*} = \Sigma\tilde{\eta}_{\Sigma^{-1}M^*}^{-1}$. It is clear that $\Sigma^{-1}\tilde{\varepsilon}_{M^*}\tilde{\eta}_{\Sigma^{-1}M^*} = id_{\Sigma^{-1}M^*}$ holds. The following diagram commutes by the definition of $\Omega\tilde{\eta}_{M^*}$.

$$\begin{array}{ccccccc} \Phi M^* & \xrightarrow{\lambda_{M^*}} & M^* & \xrightarrow{\tilde{\eta}_{M^*}} & \Sigma^{-1}\Omega M^* & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Phi\tilde{\eta}_{M^*}} & & \downarrow{\scriptstyle \tilde{\eta}_{M^*}} & & \downarrow{\scriptstyle \Sigma^{-1}\Omega\tilde{\eta}_{M^*}} & & \\ \Phi\Sigma^{-1}\Omega M^* & \xrightarrow{\lambda_{\Sigma^{-1}\Omega M^*}} & \Sigma^{-1}\Omega M^* & \xrightarrow{\tilde{\eta}_{\Sigma^{-1}\Omega M^*}} & \Sigma^{-1}\Omega\Sigma^{-1}\Omega M^* & \longrightarrow & 0 \end{array}$$

Hence we have $\Sigma^{-1}(\tilde{\varepsilon}_{\Omega M^*}\Omega\tilde{\eta}_{M^*})\tilde{\eta}_{M^*} = \Sigma^{-1}\tilde{\varepsilon}_{\Omega M^*}(\Sigma^{-1}\Omega\tilde{\eta}_{M^*})\tilde{\eta}_{M^*} = \tilde{\eta}^{-1}_{\Sigma^{-1}\Omega M^*}\tilde{\eta}_{\Sigma^{-1}\Omega M^*}\tilde{\eta}_{M^*} = \tilde{\eta}_{M^*}$ by the definition of $\tilde{\varepsilon}$. Since $\tilde{\eta}_{M^*}$ is surjective, it follows $\tilde{\varepsilon}_{\Omega M^*}\Omega\tilde{\eta}_{M^*} = id_{\Omega M^*}$ and Ω is a left adjoint of Σ^{-1}. $\square$

Lemma 15.2.22 *Assume that a filtration $\mathfrak{F} = (F_iA^*)_{i\in\boldsymbol{Z}}$ satisfies $(f5)$, $(f6)$, $(f7)$ and $(f8)$. If $i, i+c_{\mathfrak{F}}(i) \in S(\mathfrak{F})$, the following composition maps $(F_{i+c_{\mathfrak{F}}(i)}A^*)^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))} \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^*$ onto $E_{i+c_{\mathfrak{F}}(i)}^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))}A^* \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^*$.*

$$A^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))} \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^* \xrightarrow{\mu_i} E_i^{c_{\mathfrak{F}}(i)+c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))}A^* \xrightarrow{\left(\tilde{\mu}_{i+c_{\mathfrak{F}}(i)}^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i)),c_{\mathfrak{F}}(i)}\right)^{-1}} E_{i+c_{\mathfrak{F}}(i)}^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))}A^* \otimes_{K^*} (A^*/F_{i-1}A^*)^{c_{\mathfrak{F}}(i)}$$

Proof. The image of $(F_{i+c_{\mathfrak{F}}(i)}A^*)^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))} \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^*$ by μ_i is a subspace

$$(\mu((F_{i+c_{\mathfrak{F}}(i)}A^*)^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))} \otimes_{K^*} (F_iA^*)^{c_{\mathfrak{F}}(i)}) + (F_{i-1}A^*)^{c_{\mathfrak{F}}(i)+c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))})/(F_{i-1}A^*)^{c_{\mathfrak{F}}(i)+c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))}$$

of $E_i^{c_{\mathfrak{F}}(i)+c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))}A^*$ which coincides with the image of $E_{i+c_{\mathfrak{F}}(i)}^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i))}A^* \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^*$ by an isomorphism $\tilde{\mu}_{i+c_{\mathfrak{F}}(i)}^{c_{\mathfrak{F}}(i+c_{\mathfrak{F}}(i)),c_{\mathfrak{F}}(i)}$. Thus the assertion follows. $\square$

For $i, s \in S(\mathfrak{F})$, let $\gamma_{s,i} : A^{s+c_{\mathfrak{F}}(s)-i-c_{\mathfrak{F}}(i)} \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^* \to E_s^{c_{\mathfrak{F}}(s)}A^* \otimes_{K^*} (A^*/F_{i-1}A^*)^{s-i}$ be a composition

$$A^{s+c_{\mathfrak{F}}(s)-i-c_{\mathfrak{F}}(i)} \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^* \xrightarrow{\mu_i} E_i^{j+c_{\mathfrak{F}}(i)}A^* = E_i^{s-i+c_{\mathfrak{F}}(s)}A^* \xrightarrow{(\tilde{\mu}_s^{c_{\mathfrak{F}}(s),s-i})^{-1}} E_s^{c_{\mathfrak{F}}(s)}A^* \otimes_{K^*} (A^*/F_{i-1}A^*)^{s-i}.$$

Proposition 15.2.23 *Let M^* be an unstable A^*-module. If $\mathfrak{F}$ satisfies $(f1) \sim (f8)$, $\Sigma\mathrm{Ker}\,\lambda_{M^*}$ is an unstable A^*-module.*

Proof. Put $N^i = \{x \in M^i |\, (F_iA^*)^{c_{\mathfrak{F}}(i)}x = \{0\}\}$. Then we have $(\mathrm{Ker}\,\lambda_{M^*})^{i+c_{\mathfrak{F}}(i)} = E_i^{c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} N^i$. By (15.2.15), it suffices to show the following equality for $i, j \in S(\mathfrak{F})$ satisfying $j \leqq i + c_{\mathfrak{F}}(i)$.

$$\alpha_{\Phi}((F_jA^*)^{c_{\mathfrak{F}}(j)} \otimes_{K^*} (E_i^{c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} N^i)) = \{0\} \;\cdots (*)$$

We may assume $i + c_{\mathfrak{F}}(i) + c_{\mathfrak{F}}(j) = s + c_{\mathfrak{F}}(s)$ for some $s \in S(\mathfrak{F})$ by the definition of α_{Φ} which is given by the following composition.

$$A^{c_{\mathfrak{F}}(j)} \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} N^i \xrightarrow{\gamma_{s,i}\otimes_{K^*} id_{N^i}} E_s^{c_{\mathfrak{F}}(s)}A^* \otimes_{K^*} (A^*/F_{i-1}A^*)^{s-i} \otimes_{K^*} N^i \xrightarrow{id_{E_s^{c_{\mathfrak{F}}(s)}A^*}\otimes_{K^*}\alpha_i} E_s^{c_{\mathfrak{F}}(s)} \otimes_{K^*} N^s$$

Since $\mu_i : A^* \otimes_{K^*} E_i^*A^* \to E_i^*A^*$ maps $(F_jA^*)^{c_{\mathfrak{F}}(j)} \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^*$ into

$$(F_{\max\{i-1,\min\{i,j-c_{\mathfrak{F}}(i)\}\}}A^*/F_{i-1}A^*)^{c_{\mathfrak{F}}(i)+c_{\mathfrak{F}}(j)}$$

by $(f5)$ and $(f6)$, μ_i maps $(F_jA^*)^{c_{\mathfrak{F}}(j)} \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^*$ to $\{0\}$ if $j < i + c_{\mathfrak{F}}(i)$. Hence $(*)$ holds if $j < i + c_{\mathfrak{F}}(i)$.

We consider the case $j = i + c_{\mathfrak{F}}(i)$. Then, we have $j + c_{\mathfrak{F}}(j) = s + c_{\mathfrak{F}}(s)$ which implies $s = j$ by $(f4)$. Since $\gamma_{j,i} : A^{c_{\mathfrak{F}}(j)} \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^* \to E_j^{c_{\mathfrak{F}}(j)}A^* \otimes_{K^*} (A^*/F_{i-1}A^*)^{j-i}$ maps $(F_jA^*)^{c_{\mathfrak{F}}(j)} \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^*$ onto $E_j^{c_{\mathfrak{F}}(j)}A^* \otimes_{K^*} E_i^{c_{\mathfrak{F}}(i)}A^*$ by (15.2.22) and $\alpha_i(E_i^{c_{\mathfrak{F}}(i)}A^* \otimes_{K^*} N^i) = \{0\}$ by the definition of N^i, $(*)$ holds. $\square$

Suppose that $\mathfrak{F}$ satisfies $(f1) \sim (f8)$. Define a functor $\Omega^1 : \mathcal{UMod}(A^*) \to \mathcal{UMod}(A^*)$ as follows. For an object M^* of $\mathcal{UMod}(A^*)$, we put $\Omega^1(M^*) = \Sigma\mathrm{Ker}\,\lambda_{M^*}$ and denote by $\iota_{M^*} : \mathrm{Ker}\,\lambda_{M^*} \to \Phi M^*$ the inclusion map. It follows from (15.2.23) that Ω^1M^* is an object of $\mathcal{UMod}(A^*)$. For a morphism $\varphi : M^* \to N^*$ of $\mathcal{UMod}(A^*)$, there exists unique map $\hat{\varphi} : \mathrm{Ker}\,\lambda_{M^*} \to \mathrm{Ker}\,\lambda_{N^*}$ that makes the following diagram commute. We put $\Omega^1\varphi = \Sigma\hat{\varphi} : \Omega^1M^* \to \Omega^1N^*$.

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^{-1}\Omega^1M^* = \mathrm{Ker}\,\lambda_{M^*} \arrow[r, "\iota_{M^*}"] \arrow[d, "\hat{\varphi}"] & \Phi M^* \arrow[r, "\lambda_{M^*}"] \arrow[d, "\Phi\varphi"] & M^* \arrow[d, "\varphi"] \\
0 \arrow[r] & \Sigma^{-1}\Omega^1N^* = \mathrm{Ker}\,\lambda_{N^*} \arrow[r, "\iota_{N^*}"] & \Phi N^* \arrow[r, "\lambda_{N^*}"] & N^*
\end{tikzcd}$$

Proposition 15.2.24 *Suppose that $\mathfrak{F}$ satisfies $(f1) \sim (f8)$. Ω^1 is the first left derived functor of Ω and all the higher derived functors of Ω are trivial.*

Proof. Let $M^* \xleftarrow{\varepsilon_{M^*}} B_0^* \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_{n-1}} B_{n-1}^* \xleftarrow{\partial_n} B_n^* \xleftarrow{\partial_{n+1}} \cdots$ be the bar resolution of M^*. Consider chain complexes $B. = (B_n^*, \partial_n)_{n\in\boldsymbol{Z}}$, $\Phi B. = (\Phi B_n^*, \Phi(\partial_n))_{n\in\boldsymbol{Z}}$ and $\Sigma^{-1}\Omega B. = (\Sigma^{-1}\Omega B_n^*, \Sigma^{-1}\Omega(\partial_n))_{n\in\boldsymbol{Z}}$. We denote by $\lambda. : \Phi B. \to B.$ and $\eta. : B. \to \Sigma^{-1}\Omega B.$ the chain maps given by $\lambda_{B_n^*}$'s and $\eta_{B_n^*}$'s, respectively. Since $0 \to \Phi B_n^* \xrightarrow{\lambda_{B_n^*}} B_n^* \xrightarrow{\eta_{B_n^*}} \Sigma^{-1}\Omega B_n^* \to 0$ is exact by (15.2.19). we have a short exact sequence of complexes $0 \to \Phi B. \xrightarrow{\lambda.} B. \xrightarrow{\eta.} \Sigma^{-1}\Omega B. \to 0$. Consider the long exact sequence associated with this short exact sequence. Clearly, Φ is an exact functor. We deduce that $\Sigma^{-1}H^n(\Omega B.) = H^n(\Sigma^{-1}\Omega B.)$ is trivial and that there is an exact sequence

$$0 \to \Sigma^{-1}H^1(\Omega B.) = H^1(\Sigma^{-1}\Omega B.) \to \Phi M_* \xrightarrow{\lambda_{M^*}} M^* \xrightarrow{\eta_{M^*}} \Sigma^{-1}\Omega M^* \to 0.$$

Thus $\Omega^n M^* = H^n(\Omega B.)$ is trivial if $n > 1$ and Ω^1 defined above is the first left derived functor of Ω. □

Suppose that A^* is a coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with coproduct $\delta : A^* \to A^* \widehat{\otimes}_{K^*} A^*$ and that A^* has skeletal topology. We consider the following condition under this assumption.

Condition 15.2.25 *Let $\mathfrak{F} = (F_iA^*)_{i\in\boldsymbol{Z}}$ be a filtration on A^*.*
(f9) $\delta(F_iA^*) \subset \sum\limits_{j+k=i} F_jA^* \widehat{\otimes}_{K^*} F_kA^*$ for $i \in \boldsymbol{Z}$.

Proposition 15.2.26 *Let A^* be a coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with coproduct $\delta : A^* \to A^* \widehat{\otimes}_{K^*} A^*$ and an increasing filtration $(F_iA^*)_{i\in\boldsymbol{Z}}$ of subspaces of A^*. Assume that A^* has skeletal topology. Let B^* be a subcoalgebra of A^* and I a two-sided coideal of A^*. If $(F_iA^*)_{i\in\boldsymbol{Z}}$ satisfies $(f9)$, $(F_iB^*)_{i\in\boldsymbol{Z}}$ and $(F_i(A^*/I))_{i\in\boldsymbol{Z}}$ satisfy $(f9)$.*

Proof. Since $\delta((F_iA^*)^n) \subset \sum\limits_{j+k=i} \prod\limits_{s+t=n} (F_jA^*)^s \otimes_{K^*} (F_kA^*)^t$ and $\delta(B^n) \subset \prod\limits_{s+t=n} B^s \otimes_{K^*} B^t$, we have

$$\delta((F_iB^*)^n) = \delta(B^n \cap (F_iA^*)^n) \subset \sum_{j+k=i} \prod_{s+t=n} B^s \cap (F_jA^*)^s \otimes_{K^*} B^t \cap (F_kA^*)^t = \sum_{j+k=i} \prod_{s+t=n} (F_jB^*)^s \otimes_{K^*} (F_kB^*)^t.$$

Hence $(F_iB^*)_{i\in\boldsymbol{Z}}$ satisfies $(f9)$. Since the quotient map $\pi_I : A^* \to A^*/I$ is a morphism of coagebras, $(F_i(A^*/I))_{i\in\boldsymbol{Z}}$ satisfies $(f9)$ by the definition of $F_i(A^*/I)$. □

Let A^* be a coconnective Hopf algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with coproduct δ and that A^* has skeletal topology. For left A^*-modules M^* and N^* with structure maps $\alpha : A^* \otimes_{K^*} M^* \to M^*$ $\beta : A^* \otimes_{K^*} N^* \to N^*$, define a map $\gamma : A^* \otimes_{K^*} M^* \otimes_{K^*} N^* \to M^* \otimes_{K^*} N^*$ to be the following composition.

$$\begin{aligned} A^* \otimes_{K^*} M^* \otimes_{K^*} N^* &\xrightarrow{\delta \otimes_{K^*} id_{M^*} \otimes_{K^*} id_{N^*}} A^* \otimes_{K^*} A^* \otimes_{K^*} M^* \otimes_{K^*} N^* \xrightarrow{id_{A^*} \otimes_{K^*} T_{A^*,\, M^*} \otimes_{K^*} id_{N^*}} \\ &A^* \otimes_{K^*} M^* \otimes_{K^*} A^* \otimes_{K^*} N^* \xrightarrow{\alpha \otimes_{K^*} \beta} M^* \otimes_{K^*} N^* \end{aligned}$$

Proposition 15.2.27 *Let M^* and N^* be unstable A^*-modules. If $\mathfrak{F}$ satisfies $(f9)$, $M^* \otimes_{K^*} N^*$ is an unstable A^*-module.*

Proof. For integers n and j, if integers s, t satisfy $s+t = n-1$, then $s \leqq j-1$ or $t \leqq n-j-1$ holds. Otherwise, "$s \geqq j$ and $t \geqq n-j$" implies $s+t \geqq n$ which contradicts to $s+t = n-1$. Since $\alpha(F_sA^* \otimes_{K^*} M^j) = \{0\}$ if $s \leqq j-1$ and $\beta(F_tA^* \otimes_{K^*} N^{n-j}) = \{0\}$ if $t \leqq n-j-1$, we have the following.

$$\begin{aligned} \gamma(F_{n-1}A^* \otimes_{K^*} M^j \otimes_{K^*} N^{n-j}) &= (\alpha \otimes_{K^*} \beta)(id_{A^*} \otimes_{K^*} T_{A^*,\, M^*} \otimes_{K^*} id_{N^*})(\delta(F_{n-1}A^*) \otimes_{K^*} M^j \otimes_{K^*} N^{n-j}) \\ &\subset \sum_{s+t=n-1} (\alpha \otimes_{K^*} \beta)(id_{A^*} \otimes_{K^*} T_{A^*,\, M^*} \otimes_{K^*} id_{N^*})(F_sA^* \otimes_{K^*} F_tA^* \otimes_{K^*} M^j \otimes_{K^*} N^{n-j}) \\ &= \sum_{s+t=n-1} (\alpha \otimes_{K^*} \beta)(F_sA^* \otimes_{K^*} M^j \otimes_{K^*} F_tA^* \otimes_{K^*} N^{n-j}) \\ &= \sum_{s+t=n-1} \alpha(F_sA^* \otimes_{K^*} M^j) \otimes_{K^*} \beta(F_tA^* \otimes_{K^*} N^{n-j}) = \{0\} \end{aligned}$$

Hence $M^* \otimes_{K^*} N^*$ is an unstable A^*-module. □

15.3 Filtered coalgebras and unstable comodules

Condition 15.3.1 *Let M^* be an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ with an increasing filtration $(F_iM^*)_{i\in\boldsymbol{Z}}$ of subspaces of M^*. We consider the following condition on $(F_iM^*)_{i\in\boldsymbol{Z}}$.*

(f0) $\hat{\phi}:\mathcal{H}om^*(M^*,K^*)\,\widehat{\otimes}_{K^*}\mathcal{H}om^*(F_iM^*,K^*)\to\mathcal{H}om^*(M^*\otimes_{K^*}F_iM^*,K^*)$ is an isomorphism for each $i\in\boldsymbol{Z}$.

Remark 15.3.2 *If $M^*\otimes_{K^*}F_iM^*$ is supercofinite, $(F_iM^*)_{i\in\boldsymbol{Z}}$ satisfies $(f0)$. For example, if M^* is finite type and has the skeletal topology, then $(F_iM^*)_{i\in\boldsymbol{Z}}$ satisfies $(f0)$.*

Condition 15.3.3 *Let C^* be a coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with a comultiplication $\delta:C^*\to C^*\,\widehat{\otimes}_{K^*}C^*$. Suppose that an increasing filtration $(F_iC^*)_{i\in\boldsymbol{Z}}$ of subspaces of C^* is given. Recall that $\pi_{C^*,\,j+1}:C^*\to C^*/F_jC^*$ denotes the quotient map and that $u_j:id_{\mathcal{T}op\mathcal{M}od_{K^*}}\iota_j\epsilon_j$ denotes the unit of the adjunction $\iota_j\vdash\epsilon_j$. (See (1.2.5).) Consider the following conditions.*

$(f5^*)$ $\delta(F_iC^*)\subset C^*\,\widehat{\otimes}_{K^*}F_iC^*$ for $i\in\boldsymbol{Z}$, that is, F_iC^*'s are left coideals of C^*.
$(f6^*)$ $\delta(F_iC^*)\subset\bigcap_{j\in\boldsymbol{Z}}\operatorname{Ker}(\pi_{C^*,\,i+j+1}\,\widehat{\otimes}_{K^*}u_j:C^*\,\widehat{\otimes}_{K^*}C^*\to C^*/F_{i+j}C^*\,\widehat{\otimes}_{K^*}\iota_j\epsilon_j(C^*))$ for $i\in\boldsymbol{Z}$.

Proposition 15.3.4 *Let C^* be an algebra with a multiplication $\delta:C^*\to C^*\,\widehat{\otimes}_{K^*}C^*$ and an increasing filtration $(F_iC^*)_{i\in\boldsymbol{Z}}$ of subspaces of C^*. Let D^* be a subcoalgebra of C^* and I a two-sided coideal of C^*. We consider a filtration $(F_iD^*)_{i\in\boldsymbol{Z}}$ of B^* given by $F_iD^*=D^*\cap F_iC^*$ and a filtration $(F_i(C^*/I))_{i\in\boldsymbol{Z}}$ of C^*/I given by $F_i(C^*/I)=\pi_I(F_iC^*)$, where $\pi_I:C^*\to C^*/I$ the quotient map.*

(1) Assume that $(F_iC^*)_{i\in\boldsymbol{Z}}$ satisfies $(f5^*)$. Then $(F_i(C^*/I))_{i\in\boldsymbol{Z}}$ satisfies $(f5^*)$. Moreover if C^* has skeletal topology, $(F_iD^*)_{i\in\boldsymbol{Z}}$ satisfies $(f5^*)$.

(2) Assume that $(F_iC^*)_{i\in\boldsymbol{Z}}$ satisfies $(f6^*)$. Then, $(F_iD^*)_{i\in\boldsymbol{Z}}$ satisfies $(f6^*)$. Moreover if C^* has skeletal topology, $(F_i(C^*/I))_{i\in\boldsymbol{Z}}$ satisfy $(f6^*)$.

Proof. (1) Let $\bar{\delta}:C^*/I\to C^*/I\,\widehat{\otimes}_{K^*}C^*/I$ be the coproduct of C^*/I induced by δ. It follows from the commutativity of the following diagram that $\pi_I\,\widehat{\otimes}_{K^*}\pi_I$ maps $C^*\,\widehat{\otimes}_{K^*}F_iC^*$ to $C^*/I\,\widehat{\otimes}_{K^*}F_i(C^*/I)$.

$$\begin{array}{ccc}
C^* & \xrightarrow{\ \delta\ } & C^*\,\widehat{\otimes}_{K^*}C^* \\
\downarrow{\scriptstyle \pi_I} & & \downarrow{\scriptstyle \pi_I\,\widehat{\otimes}_{K^*}\pi_I} \\
C^*/I & \xrightarrow{\ \bar{\delta}\ } & C^*/I\,\widehat{\otimes}_{K^*}C^*/I
\end{array}$$

For $j,k\in\boldsymbol{Z}$, we choose a basis $\{v_\alpha\}_{\alpha\in I}$ of C^j such that $\{v_\alpha\}_{\alpha\in I'}$ is a basis of D^j. We also choose a basis $\{w_\beta\}_{\beta\in J}$ of $(F_iC^*)^k$ and a basis $\{w_\beta\}_{\beta\in J'}$ of D^k such that $\{w_\beta\}_{\beta\in J\cap J'}$ is a basis of $(F_iD^*)^k=D^k\cap(F_iC^*)^k$. Then, $\{v_\alpha\otimes w_\beta\}_{\alpha\in I,\beta\in J}$ is a basis of $C^j\otimes_{K^*}(F_iC^*)^k$ and $\{v_\alpha\otimes w_\beta\}_{\alpha\in I',\beta\in J'}$ is a basis of $D^j\otimes_{K^*}D^k$. Hence $\{v_\alpha\otimes w_\beta\}_{\alpha\in I',\beta\in J\cap J'}$ is a basis of $(C^j\otimes_{K^*}(F_iC^*)^k)\cap(D^j\otimes_{K^*}D^k)$. Since $\{v_\alpha\otimes w_\beta\}_{\alpha\in I',\beta\in J\cap J'}$ is also a basis of $D^j\otimes_{K^*}(F_iD^*)^k$, we have $(C^j\otimes_{K^*}(F_iC^*)^k)\cap(D^j\otimes_{K^*}D^k)=D^j\otimes_{K^*}(F_iD^*)^k$. Since $\delta(D^n)\subset\prod_{j+k=n}D^j\otimes_{K^*}D^k$ and $\delta((F_iC^*)^n)\subset\prod_{j+k=n}C^j\otimes_{K^*}(F_iC^*)^k$ for $i\in\boldsymbol{Z}$ hold by the assumption and (2.3.2), we have $\delta((F_iD^*)^n)\subset\prod_{j+k=n}(C^j\otimes_{K^*}(F_iC^*)^k)\cap(D^j\otimes_{K^*}D^k)=\prod_{j+k=n}D^j\otimes_{K^*}(F_iD^*)^k=(D^*\,\widehat{\otimes}_{K^*}F_iD^*)^n$.

(2) We denote by $s:D^*\to C^*$ the inclusion map. We also denote by $\bar{s}_i:D^*/F_iD^*\to C^*/F_iC^*$ and $\bar{\pi}_{I,i}:C^*/F_iC^*\to(C^*/I)/F_i(C^*/I)$ maps induced by s and π_I, respectively. For $i,j\in\boldsymbol{Z}$, the following diagram is commutative.

$$\begin{array}{ccccc}
D^* & \xrightarrow{\ \delta\ } & D^*\,\widehat{\otimes}_{K^*}D^* & \xrightarrow{\ \pi_{D^*,i+j+1}\,\widehat{\otimes}_{K^*}u_j\ } & D^*/F_{i+j}D^*\,\widehat{\otimes}_{K^*}\iota_j\epsilon_j(D^*) \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s\,\widehat{\otimes}_{K^*}s} & & \downarrow{\scriptstyle \bar{s}_{i+j}\,\widehat{\otimes}_{K^*}\iota_j\epsilon_j(s)} \\
C^* & \xrightarrow{\ \delta\ } & C^*\,\widehat{\otimes}_{K^*}C^* & \xrightarrow{\ \pi_{C^*,i+j+1}\,\widehat{\otimes}_{K^*}u_j\ } & C^*/F_{i+j}C^*\,\widehat{\otimes}_{K^*}\iota_j\epsilon_j(C^*) \\
\downarrow{\scriptstyle \pi_I} & & \downarrow{\scriptstyle \pi_I\,\widehat{\otimes}_{K^*}\pi_I} & & \downarrow{\scriptstyle \bar{\pi}_{I,i+j}\,\widehat{\otimes}_{K^*}\iota_j\epsilon_j(\bar{\pi}_I)} \\
C^*/I & \xrightarrow{\ \bar{\delta}\ } & C^*/I\,\widehat{\otimes}_{K^*}C^*/I & \xrightarrow{\ \pi_{C^*/I,i+j+1}\,\widehat{\otimes}_{K^*}u_j\ } & (C^*/I)/F_{i+j}(C^*/I)\,\widehat{\otimes}_{K^*}\iota_j\epsilon_j(C^*/I)
\end{array}$$

Since $\bar{s}_{i+j}\otimes_{K^*}\iota_j\epsilon_j(s):D^*/F_{i+j}D^*\otimes_{K^*}\iota_j\epsilon_j(D^*))\to C^*/F_{i+j}C^*\otimes_{K^*}\iota_j\epsilon_j(C^*))$ is an isomorphism onto its image, $\bar{s}_{i+j}\,\widehat{\otimes}_{K^*}\iota_j\epsilon_j(s):D^*/F_{i+j}D^*\,\widehat{\otimes}_{K^*}\iota_j\epsilon_j(D^*))\to C^*/F_{i+j}C^*\,\widehat{\otimes}_{K^*}\iota_j\epsilon_j(C^*))$ is injective by (1.3.14). Hence

$(F_iD^*)_{i\in\boldsymbol{Z}}$ satisfies $(f6^*)$ by the assumption and the commutativity of the above diagram. If C^* has skeletal topology, it follows from (2.3.2) that the degree n component of $\bar{\pi}_{I,i+j}\,\widehat{\otimes}_{K^*}\,\iota_j\epsilon_j(\bar{\pi}_I)$ is

$$\bar{\pi}_{I,i+j}\otimes_{K^*}\bar{\pi}_I:(C^*/F_{i+j}C^*)^{n-j}\otimes_{K^*}C^j\to((C^*/I)/F_{i+j}(C^*/I))^{n-j}\otimes_{K^*}(C^*/I)^j.$$

Hence $\bar{\pi}_{I,i+j}\,\widehat{\otimes}_{K^*}\,\iota_j\epsilon_j(\bar{\pi}_I)$ is surjective which implies that $(F_i(C^*/I))_{i\in\boldsymbol{Z}}$ satisfies $(f6^*)$. □

Proposition 15.3.5 *Let A^* be an algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with multiplication $\mu:A^*\otimes_{K^*}A^*\to A^*$. If A^* is proper, we can consider the dual coalgebra A^{**} of A^* with a comultiplication $\hat{\mu}$ which which is the following composition.*

$$A^{**}\xrightarrow{\mu^*}\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)\xrightarrow{\tilde{\mu}_{K^*{}_*}^{-1}}\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*\otimes_{K^*}K^*)\xrightarrow{\hat{\phi}^{-1}}A^{**}\,\widehat{\otimes}_{K^*}\,A^{**}$$

Let $\mathfrak{F}=(F_iA^*)_{i\in\boldsymbol{Z}}$ be a filtration of A^* and $\mathfrak{F}^*=(F_iA^{**})_{i\in\boldsymbol{Z}}$ the dual filtration of $\mathfrak{F}$. If $\mathfrak{F}$ satisfies $(f0)$ of (15.3.1) and $(f5)$ of (15.2.4), then $\mathfrak{F}^*$ satisfies $(f5^*)$ of (15.3.3).

Proof. Since F_iA^* is a left ideal of A^*, μ induces a map $\mu':A^*\otimes_{K^*}F_iA^*\to F_iA^*$. The upper and the middle horizontal rows of the following diagram are exact and the bottom row is also exact by (1.3.12). Hence there are unique maps $\bar{\mu}:F_iA^{**}\to\operatorname{Ker}(id_{A^*}\otimes_{K^*}\kappa_{A^*,\,-i-1})^*$ and $\hat{\phi}':A^{**}\,\widehat{\otimes}_{K^*}\,F_iA^{**}\to\operatorname{Ker}(id_{A^*}\otimes_{K^*}\kappa_{A^*,\,-i-1})^*$ that make the following diagram commute. Here $\nu:\operatorname{Ker}(id_{A^*}\otimes_{K^*}\kappa_{A^*,\,-i-1})^*\to\mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*)$ denotes the inclusion map.

$$\begin{tikzcd}
0\to F_iA^{**} \arrow[r, "\kappa_{A^{**},i}"] \arrow[d, "\bar{\mu}"] & A^{**} \arrow[r, "\kappa^*_{A^*,\,-i-1}"] \arrow[d, "\mu^*"] & \mathcal{H}om^*(F_{-i-1}A^*,K^*) \arrow[d, "(\mu')^*"] \\
0\to\operatorname{Ker}(id_{A^*}\otimes_{K^*}\kappa_{A^*,\,-i-1})^* \arrow[r, "\nu"] & \mathcal{H}om^*(A^*\otimes_{K^*}A^*,K^*) \arrow[r, "(id_{A^*}\otimes_{K^*}\kappa_{A^*,\,-i-1})^*"] & \mathcal{H}om^*(A^*\otimes_{K^*}F_{-i-1}A^*,K^*) \\
0\to A^{**}\,\widehat{\otimes}_{K^*}\,F_iA^{**} \arrow[r, "id_{A^*}\widehat{\otimes}_{K^*}\kappa_{A^{**},i}"] \arrow[u, "\hat{\phi}'"] & A^{**}\,\widehat{\otimes}_{K^*}\,A^{**} \arrow[r, "id_{A^*}\otimes_{K^*}\kappa^*_{A^*,\,-i-1}"] \arrow[u, "\tilde{\mu}_{K^*{}_*}\hat{\phi}"] & A^{**}\,\widehat{\otimes}_{K^*}(F_{-i-1}A^*)^* \arrow[u, "\tilde{\mu}_{K^*{}_*}\hat{\phi}"]
\end{tikzcd}$$

Since $\tilde{\mu}_{K^*{}_*}\hat{\phi}$'s in the diagram are isomorphism by the assumption, $\hat{\phi}'$ is also an isomorphism. Thus the assertion follows from the commutativity of the above diagram. □

Proposition 15.3.6 *Let C^* be an coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with comultiplication $\delta:C^*\to C^*\,\widehat{\otimes}_{K^*}\,C^*$. Consider the dual algebra C^{**} of C^* with a multiplication μ which is the following composition.*

$$C^{**}\otimes_{K^*}C^{**}\xrightarrow{\phi}\mathcal{H}om^*(C^*\otimes_{K^*}C^*,K^*)\xrightarrow{(\eta^*_{C^*\otimes_{K^*}C^*})^{-1}}\mathcal{H}om^*(C^*\,\widehat{\otimes}_{K^*}\,C^*,K^*)\xrightarrow{\delta^*}C^{**}$$

Let $\mathfrak{F}=(F_iC^*)_{i\in\boldsymbol{Z}}$ be a filtration of C^* and $\mathfrak{F}^*=(F_iC^{**})_{i\in\boldsymbol{Z}}$ the dual filtration of $\mathfrak{F}$. If $\mathfrak{F}$ satisfies $(f5^*)$ of (15.3.3), then $\mathfrak{F}^*$ satisfies $(f5)$ of (15.2.4).

Proof. By $(f5^*)$, the coproduct δ induces $\bar{\delta}:F_{-i-1}C^*\to C^*\,\widehat{\otimes}_{K^*}\,F_{-i-1}C^*$. For $f\in(C^{**})^n$ and $g\in(F_iC^{**})^m$, we put $h=(\eta^*_{C^*\otimes_{K^*}C^*})^{-1}\phi(f\otimes g)$. Since $g\Sigma^m\kappa_{C^*,\,-i-1}=0$, the following diagram is commutative.

$$\begin{tikzcd}
\Sigma^{m+n}(C^*\otimes_{K^*}F_{-i-1}C^*) \arrow[r, "(\tau^{n,m}_{C^*,\,F_{-i-1}C^*})^{-1}"] \arrow[d, "\Sigma^{m+n}(id_{C^*}\otimes_{K^*}\kappa_{C^*,\,-i-1})"'] & \Sigma^nC^*\otimes_{K^*}\Sigma^mF_{-i-1}C^* \arrow[d, "\Sigma^nid_{C^*}\otimes_{K^*}\Sigma^m\kappa_{C^*,\,-i-1}"'] \arrow[dr, "\text{trivial map}"] & & \\
\Sigma^{m+n}(C^*\otimes_{K^*}C^*) \arrow[r, "(\tau^{n,m}_{C^*,\,C^*})^{-1}"] \arrow[dr, "\Sigma^{m+n}\eta_{C^*\otimes_{K^*}C^*}"'] & \Sigma^nC^*\otimes_{K^*}\Sigma^mC^* \arrow[r, "f\otimes_{K^*}g"] & K^*\otimes_{K^*}K^* \arrow[r, "\tilde{\mu}_{K^*}"] & K^* \\
& \Sigma^{m+n}(C^*\,\widehat{\otimes}_{K^*}\,C^*) \arrow[urr, "h"'] & &
\end{tikzcd}$$

Hence the following composition is trivial.

$$\Sigma^{m+n}(C^*\otimes_{K^*}F_{-i-1}C^*)\xrightarrow{\Sigma^{m+n}(id_{C^*}\otimes_{K^*}\kappa_{C^*,\,-i-1})}\Sigma^{m+n}(C^*\otimes_{K^*}C^*)\xrightarrow{\Sigma^{m+n}\eta_{C^*\otimes_{K^*}C^*}}\Sigma^{m+n}(C^*\,\widehat{\otimes}_{K^*}\,C^*)\xrightarrow{h}K^*$$

Since the above composition coincides with a composition

$$\Sigma^{m+n}(C^*\otimes_{K^*}F_{-i-1}C^*)\xrightarrow{\Sigma^{m+n}\eta_{C^*\otimes_{K^*}F_{-i-1}C^*}}\Sigma^{m+n}(C^*\,\widehat{\otimes}_{K^*}\,F_{-i-1}C^*)$$
$$\xrightarrow{\Sigma^{m+n}(id_{C^*}\,\widehat{\otimes}_{K^*}\,\kappa_{C^*,\,-i-1})}\Sigma^{m+n}(C^*\,\widehat{\otimes}_{K^*}\,C^*)\xrightarrow{h}K^*$$

and the image of $\Sigma^{m+n}\eta_{C^*\otimes_{K^*}F_{-i-1}C^*}$ is dense, a composition

$$\Sigma^{m+n}(C^*\mathbin{\widehat{\otimes}}_{K^*}F_{-i-1}C^*)\xrightarrow{\Sigma^{m+n}(id_{C^*}\mathbin{\widehat{\otimes}}_{K^*}\kappa_{C^*,\,-i-1})}\Sigma^{m+n}(C^*\mathbin{\widehat{\otimes}}_{K^*}C^*)\xrightarrow{h}K^*$$

is trivial. We also have the following commutative diagram.

$$\begin{tikzcd}
\Sigma^{m+n}F_{-i-1}C^* \arrow[rr, "\Sigma^{m+n}\kappa_{C^*,\,-i-1}"] \arrow[d, "\Sigma^{m+n}\bar{\delta}"] & & \Sigma^{m+n}C^* \arrow[d, "\Sigma^{m+n}\delta"] & \\
\Sigma^{m+n}(C^*\mathbin{\widehat{\otimes}}_{K^*}F_{-i-1}C^*) \arrow[rr, "\Sigma^{m+n}(id_{C^*}\mathbin{\widehat{\otimes}}_{K^*}\kappa_{C^*,\,-i-1})"] & & \Sigma^{m+n}(C^*\mathbin{\widehat{\otimes}}_{K^*}C^*) \arrow[r, "h"] & K^*
\end{tikzcd}$$

Since $\mu(f\otimes g)=\Sigma^{m+n}\delta h\in(C^{**})^{m+n}$, it follows that $\mu(f\otimes g)\in F_iC^{**}=\operatorname{Ker}\kappa^*_{C^*,\,-i-1}$. □

Proposition 15.3.7 *Let A^* be an algebra with a filtration $\mathfrak{F}=(F_iA^*)_{i\in\boldsymbol{Z}}$. Suppose that A^* is finite type, supercofinite and proper. If $\mathfrak{F}$ satisfies $(f6)$, the dual filtration of $\mathfrak{F}$ satisfies $(f6^*)$.*

Proof. First note that A^j has discrete topology for every $j\in\boldsymbol{Z}$ by the assumption and (1) of (1.4.3). There is an isomorphism $\theta_{A^*,\,j-i}:\iota_{j-i}\epsilon_{j-i}(A^{**})\to\mathcal{H}om^*(\iota_{i-j}\epsilon_{i-j}(A^*),K^*)$ by (3.1.14). $\kappa^*_{A^*,\,-j-1}:A^{**}\to\mathcal{H}om^*(F_{-j-1}A^*,K^*)$ induces an monomorphism $\bar{\kappa}_{A^*,\,K^*,-j-1}:A^{**}/F_jA^{**}\to\mathcal{H}om^*(F_{-j-1}A^*,K^*)$. Hence

$$\bar{\kappa}_{A^*,\,K^*,-j-1}\mathbin{\widehat{\otimes}}_{K^*}\theta_{A^*,\,j-i}:A^{**}/F_jA^{**}\mathbin{\widehat{\otimes}}_{K^*}\iota_{j-i}\epsilon_{j-i}(A^{**})\to\mathcal{H}om^*(F_{-j-1}A^*,K^*)\mathbin{\widehat{\otimes}}_{K^*}\mathcal{H}om^*(\iota_{i-j}\epsilon_{i-j}(A^*),K^*)$$

is a monomorphism. Since $F_{-j-1}A^*\otimes_{K^*}\iota_{i-j}\epsilon_{i-j}(A^*)$ is supercofinite, it follows from (4.1.7) that

$$\hat{\phi}:\mathcal{H}om^*(F_{-j-1}A^*,K^*)\mathbin{\widehat{\otimes}}_{K^*}\mathcal{H}om^*(\iota_{i-j}\epsilon_{i-j}(A^*),K^*)\to\mathcal{H}om^*(F_{-j-1}A^*\otimes_{K^*}\iota_{i-j}\epsilon_{i-j}(A^*),K^*\otimes_{K^*}K^*)$$

is an isomorphism. For $\alpha\in(F_iA^{**})^k$, it follows from (3.1.15) and the naturality of $\hat{\phi}$ that we have

$$\begin{aligned}
\tilde{\mu}_{K^**}\hat{\phi}(\bar{\kappa}_{A^*,\,K^*,-j-1}\mathbin{\widehat{\otimes}}_{K^*}\theta_{A^*,\,j-i})(\pi_{A^{**},j+1}\mathbin{\widehat{\otimes}}_{K^*}u_{j-i})\hat{\mu}(\alpha)&=\tilde{\mu}_{K^**}\hat{\phi}(\kappa^*_{A^*,\,-j-1}\mathbin{\widehat{\otimes}}_{K^*}c^*_{i-j})\hat{\phi}^{-1}\tilde{\mu}^{-1}_{K^**}\mu^*(\alpha)\\
&=(\kappa_{A^*,\,-j-1}\mathbin{\widehat{\otimes}}_{K^*}c_{i-j})^*\mu^*(\alpha)\\
&=\alpha\Sigma^k\mu(\kappa_{A^*,\,-j-1}\mathbin{\widehat{\otimes}}_{K^*}c_{i-j}).
\end{aligned}$$

Since α maps $\Sigma^kF_{-i-1}A^*$ to zero and the image of $\mu(\kappa_{A^*,\,-j-1}\mathbin{\widehat{\otimes}}_{K^*}c_{i-j})$ is contained in $F_{-i-1}A^*$ by $(f6)$, $\alpha\Sigma^k\mu(\kappa_{A^*,\,-j-1}\mathbin{\widehat{\otimes}}_{K^*}c_{i-j})=0$. Thus we have $(\pi_{A^{**},j+1}\mathbin{\widehat{\otimes}}_{K^*}u_{j-i})\hat{\mu}(\alpha)=0$, in other words, $\hat{\mu}(\alpha)$ is contained in $\operatorname{Ker}(\pi_{A^{**},j+1}\mathbin{\widehat{\otimes}}_{K^*}c_{j-i})$. □

For a coalgebra C^* in $\mathcal{T}op\mathcal{M}od_{K^*}$ with coproduct $\delta:C^*\to C^*\mathbin{\widehat{\otimes}}_{K^*}C^*$, recall from (5.1.1) that the product $\tilde{\delta}:C^{**}\otimes_{K^*}C^{**}\to C^{**}$ of the dual algebra C^{**} is defined to be the following composition.

$$\begin{aligned}
C^{**}\otimes_{K^*}C^{**}=\mathcal{H}om^*(C^*,K^*)\otimes_{K^*}\mathcal{H}om^*(C^*,K^*)&\xrightarrow{\phi}\mathcal{H}om^*(C^*\otimes_{K^*}C^*,K^*\otimes_{K^*}K^*)\\
\xrightarrow{\tilde{\mu}_{K^**}}\mathcal{H}om^*(C^*\otimes_{K^*}C^*,K^*)&\xrightarrow{(\eta^*_{C^*\otimes_{K^*}C^*})^{-1}}\mathcal{H}om^*(C^*\mathbin{\widehat{\otimes}}_{K^*}C^*,K^*)\xrightarrow{\delta^*}\mathcal{H}om^*(C^*,K^*)=C^{**}
\end{aligned}$$

Proposition 15.3.8 *Let C^* be a coalgebra with a filtration $\mathfrak{F}=(F_iC^*)_{i\in\boldsymbol{Z}}$. If $\mathfrak{F}$ satisfies $(f6^*)$, the dual filtration of $\mathfrak{F}$ satisfies $(f6)$.*

Proof. Take $f\in(F_iC^{**})^k$ and $g\in(C^{**})^j$. Since f maps $\Sigma^kF_{-i-1}C^*$ to 0, f induces $\bar{f}:\Sigma^k(C^*/F_{-i-1}C^*)\to K^*$ which satisfies $\bar{f}\Sigma^k\pi_{C^*,\,-i}=f$. Similarly, g induces $\bar{g}:\Sigma^j\iota_{-j}\epsilon_{-j}(C^*)\to K^*$ which satisfies $\bar{g}\Sigma^ju_{-j}=g$. Put $h=(\eta^*_{C^*\otimes_{K^*}C^*})^{-1}(\tilde{\mu}_{K^**}(\phi(f\otimes g)))$. Then, the following diagram is commutative.

$$\begin{tikzcd}
 & \Sigma^{j+k}(C^*\otimes_{K^*}C^*) \arrow[r, "\Sigma^{j+k}(\pi_{C^*,\,-i}\otimes_{K^*}u_{-j})"] \arrow[dl, "\Sigma^{j+k}\eta_{C^*\otimes_{K^*}C^*}"'] \arrow[d, "(\tau^{k,j})^{-1}_{C^*,\,C^*}"] & \Sigma^{j+k}((C^*/F_{-i-1}C^*)\otimes_{K^*}\iota_{-j}\epsilon_{-j}(C^*)) \arrow[d, "(\tau^{k,j}_{C^*/F_{-i-1}C^*,\iota_{-j}\epsilon_{-j}(C^*)})^{-1}"'] \\
\Sigma^{j+k}(C^*\mathbin{\widehat{\otimes}}_{K^*}C^*) \arrow[d, "h"] & \Sigma^jC^*\otimes_{K^*}\Sigma^kC^* \arrow[r, "\Sigma^k\pi_{C^*,\,-i}\otimes_{K^*}\Sigma^ju_{-j}"] \arrow[d, "f\otimes_{K^*}g"] & \Sigma^j(C^*/F_{-i-1}C^*)\otimes_{K^*}\Sigma^k\iota_{-j}\epsilon_{-j}(C^*) \arrow[dl, "\bar{f}\otimes_{K^*}\bar{g}"] \\
K^* & K^*\otimes_{K^*}K^* \arrow[l, "\tilde{\mu}_{K^*}"'] &
\end{tikzcd}$$

Hence we have the following commutative diagram by completing the above diagram.

$$
\begin{tikzcd}
\Sigma^{j+k}C^* \arrow[r, "\delta"] \arrow[dd, "\tilde{\delta}(f\otimes g)"'] & \Sigma^{j+k}(C^* \widehat{\otimes}_{K^*} C^*) \arrow[r, "\Sigma^{j+k}(\pi_{C^*,\,-i} \widehat{\otimes}_{K^*} u_{-j})"] \arrow[d, "(\hat{\tau}^{k,j})^{-1}_{C^*,\,C^*}"] \arrow[ddl, "h"'] & \Sigma^{j+k}((C^*/F_{-i-1}C^*) \widehat{\otimes}_{K^*} \iota_{-j}\epsilon_{-j}(C^*)) \arrow[d, "(\hat{\tau}^{k,j}_{C^*/F_{-i-1}C^*,\iota_{-j}\epsilon_{-j}(C^*)})^{-1}"'] \\
 & \Sigma^j C^* \widehat{\otimes}_{K^*} \Sigma^k C^* \arrow[r, "\Sigma^k \pi_{C^*,\,-i} \widehat{\otimes}_{K^*} \Sigma^j u_{-j}"] \arrow[d, "f \widehat{\otimes}_{K^*} g"] & \Sigma^j (C^*/F_{-i-1}C^*) \widehat{\otimes}_{K^*} \Sigma^k \iota_{-j}\epsilon_{-j}(C^*) \arrow[dl, "\bar{f} \widehat{\otimes}_{K^*} \bar{g}"] \\
K^* & K^* \widehat{\otimes}_{K^*} K^* \arrow[l, "\hat{\tilde{\mu}}_{K^*}"'] &
\end{tikzcd}
$$

Thus we have $(\tilde{\delta}(f \otimes g))(F_{j-i-1}C^*) = \hat{\tilde{\mu}}_{K^*}(\bar{f} \widehat{\otimes}_{K^*} \bar{g})(\hat{\tau}^{k,j})^{-1}\Sigma^{k+j}((\pi_{C^*,\,-i} \widehat{\otimes}_{K^*} u_{-j})\delta)(F_{j-i-1}C^*) = \{0\}$ by $(f6^*)$. This implies $\tilde{\delta}(f \otimes_{K^*} g) \in F_{i-j}C^{**}$. ∎

Proposition 15.3.9 *Let C^* and D^* be coalgebras in $\mathcal{T}op\mathcal{M}od_{K^*}$ with coproducts $\delta_{C^*} : C^* \to C^* \widehat{\otimes}_{K^*} C^*$ and $\delta_{D^*} : D^* \to D^* \widehat{\otimes}_{K^*} D^*$, respectively. Define $\delta_{C^* \otimes_{K^*} D^*} : C^* \otimes_{K^*} D^* \to (C^* \otimes_{K^*} D^*) \widehat{\otimes}_{K^*} (C^* \otimes_{K^*} D^*)$ to be the following composition. (See (2.3.7) for the definition of sh_{C^*,C^*,D^*,D^*}.)*

$$
C^* \otimes_{K^*} D^* \xrightarrow{\delta_{C^*} \otimes_{K^*} \delta_{D^*}} (C^* \widehat{\otimes}_{K^*} C^*) \otimes_{K^*} (D^* \widehat{\otimes}_{K^*} D^*) \xrightarrow{sh_{C^*,C^*,D^*,D^*}} (C^* \otimes_{K^*} D^*) \widehat{\otimes}_{K^*} (C^* \otimes_{K^*} D^*)
$$

Let $(F_iC^*)_{i\in \boldsymbol{Z}}$ and $(F_iD^*)_{i\in \boldsymbol{Z}}$ be filtrations of C^* and D^*, respectively.

(1) If $(F_iC^*)_{i\in \boldsymbol{Z}}$ and $(F_iD^*)_{i\in \boldsymbol{Z}}$ satisfies $(f5^*)$, so does $(F_i(C^* \otimes_{K^*} D^*))_{i\in \boldsymbol{Z}}$.

(2) Suppose that C^* and D^* satisfy the assumptions of (15.1.19). If $(F_iC^*)_{i\in \boldsymbol{Z}}$ and $(F_iD^*)_{i\in \boldsymbol{Z}}$ satisfies $(f6^*)$, so does $(F_i(C^* \otimes_{K^*} D^*))_{i\in \boldsymbol{Z}}$.

Proof. (1) Since $\delta_{C^*}(F_jC^*) \subset C^* \widehat{\otimes}_{K^*} F_jC^*$ and $\delta_{D^*}(F_kD^*) \subset D^* \widehat{\otimes}_{K^*} F_kD^*$, $\delta_{C^*} \otimes_{K^*} \delta_{D^*}$ maps $F_i(C^* \otimes_{K^*} D^*)$ into $\sum\limits_{j+k=i} (C^* \widehat{\otimes}_{K^*} F_jC^*) \otimes_{K^*} (D^* \widehat{\otimes}_{K^*} F_kD^*)$ which is mapped into $\sum\limits_{j+k=i} (C^* \otimes_{K^*} D^*) \widehat{\otimes}_{K^*} (F_jC^* \otimes_{K^*} F_kD^*)$ by sh_{C^*,C^*,D^*,D^*}.

(2) It follows from (15.3.8) that the dual filtrations $(F_iC^{**})_{i\in \boldsymbol{Z}}$ and $(F_iD^{**})_{i\in \boldsymbol{Z}}$ satisfy $(f6)$. Therefore $(F_i(C^{**} \otimes_{K^*} D^{**}))_{i\in \boldsymbol{Z}}$ also satisfies $(f6)$ by (15.2.7). We note that $C^* \otimes_{K^*} C^*$ and $D^* \otimes_{K^*} D^*$ are complete by the assumption and that the following diagram is commutative.

$$
\begin{tikzcd}
C^{**} \otimes_{K^*} D^{**} \otimes_{K^*} C^{**} \otimes_{K^*} D^{**} \arrow[r, "\tilde{\mu}_{K^**}\phi \otimes_{K^*} \tilde{\mu}_{K^**}\phi"] \arrow[d, "id_{C^{**}} \otimes_{K^*} T_{C^{**},D^{**}} \otimes_{K^*} id_{D^{**}}"] & \mathcal{H}om^*(C^* \otimes_{K^*} D^*, K^*) \otimes_{K^*} \mathcal{H}om^*(C^* \otimes_{K^*} D^*, K^*) \arrow[d, "\tilde{\mu}_{K^**}\phi"] \\
C^{**} \otimes_{K^*} C^{**} \otimes_{K^*} D^{**} \otimes_{K^*} D^{**} \arrow[d, "\tilde{\mu}_{K^**}\phi \otimes_{K^*} \tilde{\mu}_{K^**}\phi"] & \mathcal{H}om^*(C^* \otimes_{K^*} D^* \otimes_{K^*} C^* \otimes_{K^*} D^*, K^*) \arrow[d, "(id_{C^*} \otimes_{K^*} T_{C^*,D^*} \otimes_{K^*} id_{D^*})^*"] \\
\mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^*) \otimes_{K^*} \mathcal{H}om^*(D^* \otimes_{K^*} D^*, K^*) \arrow[d, "\delta^*_{C^*} \otimes_{K^*} \delta^*_{D^*}"] & \mathcal{H}om^*(C^* \otimes_{K^*} C^* \otimes_{K^*} D^* \otimes_{K^*} D^*, K^*) \arrow[d, "(\delta_{C^*} \otimes_{K^*} \delta_{D^*})^*"] \\
\mathcal{H}om^*(C^*, K^*) \otimes_{K^*} \mathcal{H}om^*(D^*, K^*) \arrow[r, "\tilde{\mu}_{K^**}\phi"] & \mathcal{H}om^*(C^* \otimes_{K^*} D^*, K^*)
\end{tikzcd}
$$

Hence the dual filtration $(F_i\mathcal{H}om^*(C^* \otimes_{K^*} D^*, K^*))_{i\in \boldsymbol{Z}}$ of $(F_i(C^* \otimes_{K^*} D^*))_{i\in \boldsymbol{Z}}$ satisfies $(f6)$ by (15.1.19). Thus the dual filtration $(F_i\mathcal{H}om^*(\mathcal{H}om^*(C^* \otimes_{K^*} D^*, K^*), K^*))_{i\in \boldsymbol{Z}}$ of $(F_i\mathcal{H}om^*(C^* \otimes_{K^*} D^*, K^*))_{i\in \boldsymbol{Z}}$ satisfies $(f6^*)$ by (15.3.7). It follow from (15.1.8) that $(F_i(C^* \otimes_{K^*} D^*))_{i\in \boldsymbol{Z}}$ satisfies $(f6^*)$. ∎

Let C^* be a K^*-coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with a filtration $\mathfrak{F} = (F_iC^*)_{i\in \boldsymbol{Z}}$. We denote by $\delta : C^* \to C^* \widehat{\otimes}_{K^*} C^*$ the coproduct of C^*. If $\mathfrak{F}$ satisfies $(f5^*)$ of (15.3.3), δ gives a map $\bar{\delta}_i : F_iC^* \to C^* \widehat{\otimes}_{K^*} F_iC^*$ for each $i \in \boldsymbol{Z}$. Since the upper horizontal row of the following diagram is exact, there exists unique map $\delta_i : E^*_iC^* \to C^* \widehat{\otimes}_{K^*} E^*_iC^*$ that makes the diagram commute.

$$
\begin{tikzcd}
0 \arrow[r] & F_{i-1}C^* \arrow[r, "\iota_{C^*,\,i}"] \arrow[d, "\bar{\delta}_{i-1}"] & F_iC^* \arrow[r, "\rho_{C^*,\,i}"] \arrow[d, "\bar{\delta}_i"] & E^*_iC^* \arrow[r] \arrow[d, dashed, "\delta_i"] & 0 \\
 & C^* \widehat{\otimes}_{K^*} F_{i-1}C^* \arrow[r, "id_{C^*} \widehat{\otimes}_{K^*} \iota_{C^*,\,i}"] & C^* \widehat{\otimes}_{K^*} F_iC^* \arrow[r, "id_{C^*} \widehat{\otimes}_{K^*} \rho_{C^*,\,i}"] & C^* \widehat{\otimes}_{K^*} E^*_iC^* &
\end{tikzcd}
$$

It is easy to verify that δ_i gives a left C^*-comodule structure of $E_i^*C^*$.

If $\mathfrak{F}$ satisfies $(f6^*)$ of (15.3.3), a composition $C^* \xrightarrow{\delta} C^* \widehat{\otimes}_{K^*} C^* \xrightarrow{\pi_{C^*,\,i+1} \widehat{\otimes}_{K^*} u_j} C^*/F_iC^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(C^*)$ maps $F_{i-j}C^*$ to zero. Thus we have unique map $\delta_{i,j} : C^*/F_{i-j}C^* \to C^*/F_iC^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(C^*)$ that makes the following diagram commute.

$$\begin{tikzcd}
C^* \arrow[r, "\delta"] \arrow[d, "\pi_{C^*,i-j+1}"] & C^* \widehat{\otimes}_{K^*} C^* \arrow[d, "\pi_{C^*,\,i+1} \widehat{\otimes}_{K^*} u_j"] \\
C^*/F_{i-j}C^* \arrow[r, dashed, "\delta_{i,j}"] & C^*/F_iC^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(C^*)
\end{tikzcd}$$

By the exactness of the vertical columns and the commutativity of the lower rectangle of the following diagram, there exists unique map $\bar{\delta}_{i,j} : E_{i-j}^*C^* \to E_i^*C^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(C^*)$ that makes the upper rectangle of the following diagram commute.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
E_{i-j}^*C^* \arrow[r, dashed, "\bar{\delta}_{i,j}"] \arrow[d, "\tilde{\kappa}_{C^*,\,i-j}"] & E_i^*C^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(C^*) \arrow[d, "\tilde{\kappa}_{C^*,\,i} \widehat{\otimes}_{K^*} id_{\iota_j\epsilon_j(C^*)}"] \\
C^*/F_{i-j-1}C^* \arrow[r, "\delta_{i-1,j}"] \arrow[d, "\tilde{\iota}_{C^*,\,i-j}"] & C^*/F_{i-1}C^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(C^*) \arrow[d, "\tilde{\iota}_{C^*,\,i} \widehat{\otimes}_{K^*} id_{\iota_j\epsilon_j(C^*)}"] \\
C^*/F_{i-j}C^* \arrow[r, "\delta_{i,j}"] & C^*/F_iC^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(C^*)
\end{tikzcd}$$

Suppose that $\mathfrak{F}$ satisfies both $(f5^*)$ and $(f6^*)$, then the image of $\bar{\delta}_{i,j}$ is contained in $E_i^*C^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(F_{i-j}C^*)$. Thus we have a map $\tilde{\delta}_{i,j} : E_{i-j}^*C^* \to E_i^*C^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(F_{i-j}C^*)$ which makes the following diagram commute.

$$\begin{tikzcd}
 & E_i^*C^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(F_{i-j}C^*) \arrow[d, "id_{E_i^*C^*} \widehat{\otimes}_{K^*} \iota_j\epsilon_j(\kappa_{C^*,i-j})"] \\
E_{i-j}^*C^* \arrow[ur, dashed, "\tilde{\delta}_{i,j}"] \arrow[r, "\bar{\delta}_{i,j}"] & E_i^*C^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(C^*)
\end{tikzcd}$$

If C^* has skeletal topology, the k-dimensional component of $\tilde{\delta}_{i,j}$ is expressed as follows by (2.3.2).

$$\tilde{\delta}_{i,j}^k : E_{i-j}^kC^* \to E_i^{k-j}C^* \otimes_{K^*} \iota_j\epsilon_j(F_{i-j}C^*)^j$$

Proposition 15.3.10 *Let A^* be an algebra in $\mathcal{TopMod}_{K^*}$ with an increasing filtration $(F_iA^*)_{i\in \boldsymbol{Z}}$ which satisfies $(f5)$ and $(f6)$. Suppose that A^* is finite type and has skeletal topology. Let $\delta : A^{**} \to A^{**} \widehat{\otimes}_{K^*} A^{**}$ be the coproduct of A^{**} defined from the product μ of A^*. We give A^{**} the dual filtration $(F_iA^{**})_{i\in \boldsymbol{Z}}$ and consider maps $\tilde{\mu}_i^{k,j} : E_i^kA^* \otimes_{K^*} (A^*/F_{i-j-1}A^*)^j \to E_{i-j}^{j+k}A^*$ and $\tilde{\delta}_{-i,-j}^{-j-k} : E_{j-i}^{-j-k}A^{**} \to E_{-i}^{-k}A^{**} \otimes_{K^*} (F_{j-i}A^{**})^{-j}$ for $i, j, k \in \boldsymbol{Z}$.*

(1) $\tilde{\mu}_i^{k,j}$ is surjective if and only if $\tilde{\delta}_{-i,-j}^{-j-k}$ is injective. *(2) $\tilde{\mu}_i^{k,j}$ is injective if and only if $\tilde{\delta}_{-i,-j}^{-j-k}$ is surjective.*

Proof. It follows from (3.1.14), (15.1.3) and (15.1.4) that we have the following isomorphisms.

$$\begin{aligned}
\theta_{A^*/F_{i-j-1}A^*,-j} &: \iota_{-j}\epsilon_{-j}(\mathcal{H}om^*(A^*/F_{i-j-1}A^*, K^*)) \to \mathcal{H}om^*(\iota_j\epsilon_j(A^*/F_{i-j-1}A^*), K^*) \\
\hat{\pi}_{A^*,\,K^*,j-i} &: \mathcal{H}om^*(A^*/F_{i-j-1}A^*, K^*) \to F_{j-i}\mathcal{H}om^*(A^*, K^*) = F_{j-i}A^{**} \\
\tilde{\rho}_{A^*,\,K^*,l} &: E_l^*A^{**} \to \mathcal{H}om^*(E_{-l}^*A^*, K^*) \;\; (l = -i, j-i)
\end{aligned}$$

We also have the following isomorphism by the assumption and (4.1.7).

$$\begin{aligned}
&\mathcal{H}om^*(E_i^*A^*, K^*) \widehat{\otimes}_{K^*} \mathcal{H}om^*(\iota_{-j}\epsilon_{-j}(A^*/F_{i-j-1}A^*), K^*) \xrightarrow{\hat{\phi}} \\
&\qquad \mathcal{H}om^*(E_i^*A^* \otimes_{K^*} \iota_{-j}\epsilon_{-j}(A^*/F_{i-j-1}A^*), K^* \otimes_{K^*} K^*) \xrightarrow{\tilde{\mu}_{K^{**}}} \mathcal{H}om^*(E_i^*A^* \otimes_{K^*} \iota_{-j}\epsilon_{-j}(A^*/F_{i-j-1}A^*), K^*)
\end{aligned}$$

Since both $\mathcal{H}om^*(E_i^* A^*, K^*)$ and $\mathcal{H}om^*(\iota_{-j}\epsilon_{-j}(A^*/F_{i-j-1}A^*), K^*)$ have skeletal topology by (3.1.36) and

$$\mathcal{H}om^n(\iota_{-j}\epsilon_{-j}(A^*/F_{i-j-1}A^*), K^*) = \{0\}$$

if $n \neq -j$, the domain of the above isomorphism $\tilde{\mu}_{K^* *}\hat{\phi}$ in degree $-j-k$ is

$$\mathcal{H}om^{-k}(E_i^* A^*, K^*) \otimes_{K^*} \mathcal{H}om^{-j}(\iota_{-j}\epsilon_{-j}(A^*/F_{i-j-1}A^*), K^*).$$

$\tilde{\mu}_i^{k,j} : E_i^k A^* \otimes_{K^*} (A^*/F_{i-j-1}A^*)^j \to E_{i-j}^{j+k} A^*$ defines a morphism $\tilde{\mu}_i^{*,j} : E_i^* A^* \otimes_{K^*} \iota_j \epsilon_j (A^*/F_{i-j-1}A^*) \to E_{i-j}^* A^*$ in $\mathcal{T}op\mathcal{M}od_{K^*}$. We note that $\mathcal{H}om^{-j-k}(E_{i-j}^* A^*, K^*)$ is naturally isomorphic to $\mathrm{Hom}_{K^0}(E_{i-j}^{j+k} A^*, K^0)$ and $\mathcal{H}om^{-j-k}(E_i^* A^* \otimes_{K^*} \iota_j \epsilon_j (A^*/F_{i-j-1}A^*), K^*)$ is naturally isomorphic to $\mathrm{Hom}_{K^0}(E_i^k A^* \otimes_{K^*} (A^*/F_{i-j-1}A^*)^j, K^0)$ as vector spaces over K^0. Hence we have the following commutative diagram.

$$\begin{CD}
\mathcal{H}om^{-j-k}(E_{i-j}^* A^*, K^*) @>{(\tilde{\mu}_i^{*,j})^*}>> \mathcal{H}om^{-j-k}(E_i^* A^* \otimes_{K^*} \iota_j \epsilon_j (A^*/F_{i-j-1}A^*), K^*) \\
@V{\cong}VV @VV{\cong}V \\
\mathrm{Hom}_{K^0}(E_{i-j}^{j+k} A^*, K^0) @>{(\tilde{\mu}_i^{k,j})^*}>> \mathrm{Hom}_{K^0}(E_i^k A^* \otimes_{K^*} (A^*/F_{i-j-1}A^*)^j, K^0)
\end{CD}$$

It follows that $\tilde{\mu}_i^{k,j}$ is surjective (resp. injective) if and only if $(\tilde{\mu}_i^{*,j})^*$ is injective (resp. surjective).

Since $(F_i A^{**})_{i \in \boldsymbol{Z}}$ satisfies $(f5^*)$ and $(f6^*)$ by (15.3.5) and (15.3.7),

$$(\pi_{A^{**},-i} \widehat{\otimes}_{K^*} u_{-j})\delta : A^{**} \to A^{**}/F_{-i-1}A^{**} \widehat{\otimes}_{K^*} \iota_{-j}\epsilon_{-j}(A^{**})$$

induces the map $\tilde{\delta}_{-i,-j}^{-j-k} : E_{j-i}^{-j-k} A^{**} \to E_{-i}^{-k} A^{**} \otimes_{K^*} \iota_{-j}\epsilon_{-j}(F_{j-i}A^{**})^{-j}$. We claim that the following diagram $(*)$ is commutative.

$$\begin{CD}
E_{-i}^{-k} A^{**} \otimes_{K^*} \iota_{-j}\epsilon_{-j}(F_{j-i}A^{**})^{-j} @<{\tilde{\delta}_{-i,-j}^{-j-k}}<< E_{j-i}^{-j-k} A^{**} \\
@V{id_{E_{-i}^{-k} A^{**}} \otimes_{K^*} \iota_{-j}\epsilon_{-j}(\hat{\pi}_{A^*,\,K^*,i-j})^{-1}}VV @VV{\tilde{\rho}_{A^*,\,K^*,j-i}}V \\
E_{-i}^{-k} A^{**} \otimes_{K^*} \iota_{-j}\epsilon_{-j}(\mathcal{H}om^*(A^*/F_{i-j-1}A^*, K^*))^{-j} @. \mathcal{H}om^{-j-k}(E_{i-j}^* A^*, K^*) \\
@V{\tilde{\rho}_{A^*,\,K^*,-i} \otimes_{K^*} \theta_{A^*/F_{i-j-1}A^*,-j}}VV @VV{(\tilde{\mu}_i^{*,j})^*}V \\
\mathcal{H}om^{-k}(E_i^* A^*, K^*) \otimes_{K^*} \mathcal{H}om^{-j}(\iota_j\epsilon_j(A^*/F_{i-j-1}A^*), K^*) @>{\tilde{\mu}_{K^* *}\hat{\phi}}>> \mathcal{H}om^{-j-k}(E_i^* A^* \otimes_{K^*} \iota_j \epsilon_j (A^*/F_{i-j-1}A^*), K^*)
\end{CD}$$

diagram $(*)$

For $\alpha \in E_{j-i}^{-j-k} A^{**}$, take $f \in (F_{j-i}A^{**})^{-j-k}$ such that $\alpha = \rho_{A^*,\,j-i}(f)$. We put

$$\delta(f) = \sum_{s \in S} g_s \otimes h_s \in \prod_{n \in \boldsymbol{Z}} \mathcal{H}om^n(A^*, K^*) \otimes_{K^*} \mathcal{H}om^{-j-k-n}(A^*, K^*)$$

for $g_s \in F_{-i}\mathcal{H}om^{n_s}(A^*, K^*)$ and $h_s \in \mathcal{H}om^{-j-k-n_s}(A^*, K^*)$. Also put $\{s_1, s_2, \dots, s_n\} = \{s \in S \,|\, n_s = -k\}$, then $\tilde{\delta}_{-i,-j}^{-j-k}(\alpha) = \sum_{\nu=1}^{n} \rho_{A^{**},-i}(g_{s_\nu}) \otimes h_{s_\nu}$ and $h_{s_\nu} \in (F_{j-i}A^{**})^{-j}$. Let $\bar{g}_\nu : \Sigma^{-k} E_i^* A^* \to K^*$ and $\bar{h}_\nu : \Sigma^{-j} A^*/F_{i-j}A^* \to K^*$ be the unique maps that satisfies $\bar{g}_\nu \Sigma^{-k}\rho_{A^*,\,i} = g_{s_\nu}\Sigma^{-k}\kappa_{A^*,\,i}$ and $\bar{h}_\nu \Sigma^{-j}\pi_{A^*,\,i-j} = h_{s_\nu}$ for $\nu = 1, 2, \dots, n$, respectively. Then, we have $\tilde{\rho}_{A^*,\,K^*,-i}(\rho_{A^{**},-i}(g_{s_\nu})) = \bar{g}_\nu$ and $\iota_{-j}\epsilon_{-j}(\hat{\pi}_{A^*,\,K^*,i-j})^{-1}(h_{s_\nu}) = \bar{h}_\nu$. Since a composition

$$\Sigma^{-j}\iota_j\epsilon_j(A^*/F_{j-i}A^*) = \iota_0\epsilon_0(\Sigma^{-j}A^*/F_{j-i}A^*) \xrightarrow{\iota_0\epsilon_0(\bar{h}_\nu)} \iota_0\epsilon_0(K^*) \xrightarrow{c_{0\,K^*}} K^*$$

is the image of $\bar{h}_\nu$ by $\theta_{A^*/F_{i-j-1}A^*,-j}$, we have

$$\begin{aligned}
&\hat{\phi}(\tilde{\rho}_{A^*,\,K^*,-i} \otimes_{K^*} \theta_{A^*/F_{i-j-1}A^*,-j})(id_{E_{-i}^* A^{**}} \otimes_{K^*} \iota_{-j}\epsilon_{-j}(\hat{\pi}_{A^*,\,K^*,i-j})^{-1})\tilde{\delta}_{-i,-j}^{-j-k}(\alpha) \\
&\qquad = \sum_{\nu=1}^{n} \hat{\phi}(\tilde{\rho}_{A^*,\,K^*,-i} \otimes_{K^*} \theta_{A^*/F_{i-j-1}A^*,-j})(\rho_{A^{**},-i}(g_{s_\nu}) \otimes \bar{h}_\nu) = \sum_{\nu=1}^{n} \hat{\phi}(\bar{g}_\nu \otimes c_{0\,K^*}\iota_0\epsilon_0(\bar{h}_\nu)).
\end{aligned}$$

On the other hand, let $\bar{f} : \Sigma^{-j-k}E^*_{i-j}A^* \to K^*$ be the map that satisfies $\bar{f}\Sigma^{-j-k}\rho_{A^*,\,i-j} = f\Sigma^{-j-k}\kappa_{A^*,\,i-j}$, then we have $(\tilde{\mu}^{*,j}_i)^*(\tilde{\rho}_{A^*,\,K^*,j-i}(\alpha)) = \bar{f}\Sigma^{-j-k}\tilde{\mu}^{*,j}_i$. Therefore it remains to verify

$$\sum_{\nu=1}^{n} \tilde{\mu}_{K^*}\hat{\phi}(\bar{g}_\nu \otimes c_{0\,K^*}\iota_0\epsilon_0(\bar{h}_\nu)) = \bar{f}\Sigma^{-j-k}\tilde{\mu}^{*,j}_i \quad \cdots (**)$$

to show the commutativity of $(*)$.

Let us denote by $\mu^{*,j}_i : F_iA^* \otimes_{K^*} \iota_j\epsilon_j(A^*) \to F_{i-j}A^*$ the map defined from μ by restricting its source and target. The following diagram is commutative by the definitions of $\tilde{\mu}^{*,j}_i$ and $\bar{f}$.

$$\begin{tikzcd}
\Sigma^{-j-k}F_iA^* \otimes_{K^*} \iota_j\epsilon_j(A^*) \arrow[r, "\Sigma^{-j-k}\mu^{*,j}_i"] \arrow[d, "\Sigma^{-j-k}(\rho_{A^*,\,i}\otimes_{K^*}\iota_j\epsilon_j(\pi_{A^*,\,i-j}))"] & \Sigma^{-j-k}F_{i-j}A^* \arrow[r, "\Sigma^{-j-k}\kappa_{A^*,\,i-j}"] \arrow[d, "\Sigma^{-j-k}\rho_{A^*,\,i-j}"] & \Sigma^{-j-k}A^* \arrow[d, "f"] \\
\Sigma^{-j-k}E^*_iA^* \otimes_{K^*} \iota_j\epsilon_j(A^*/F_{i-j-1}A^*) \arrow[r, "\Sigma^{-j-k}\tilde{\mu}^{*,j}_i"] & \Sigma^{-j-k}E_{i-j}A^* \arrow[r, "\bar{f}"] & K^*
\end{tikzcd}$$

By (1.2.6) and the definition of $\bar{h}_\nu$ we have the following equality.

$$\iota_0\epsilon_0(\bar{h}_\nu)\Sigma^{-j}\iota_j\epsilon_j(\pi_{A^*,\,i-j}) = \iota_0\epsilon_0(\bar{h}_\nu)\iota_0\epsilon_0(\Sigma^{-j}\pi_{A^*,\,i-j}) = \iota_0\epsilon_0(\bar{h}_\nu\Sigma^{-j}\pi_{A^*,\,i-j}) = \iota_0\epsilon_0(h_{s_\nu})$$

Thus we have the following.

$$\begin{aligned}
&\sum_{\nu=1}^{n} \hat{\phi}(\bar{g}_\nu \otimes c_{0\,K^*}\iota_0\epsilon_0(\bar{h}_\nu))(\Sigma^{-j-k}(\rho_{A^*,\,i} \otimes_{K^*} \iota_j\epsilon_j(\pi_{A^*,\,i-j}))\\
&\qquad = \sum_{\nu=1}^{n} (\bar{g}_\nu \otimes_{K^*} c_{0\,K^*}\iota_0\epsilon_0(\bar{h}_\nu))(\tau^{-k,-j}_{E^*_iA^*,\iota_j\epsilon_j(A^*/F_{i-j-1}A^*)})^{-1}(\Sigma^{-j-k}(\rho_{A^*,\,i} \otimes_{K^*} \iota_j\epsilon_j(\pi_{A^*,\,i-j})))\\
&\qquad = \sum_{\nu=1}^{n} (\bar{g}_\nu \otimes_{K^*} c_{0\,K^*}\iota_0\epsilon_0(\bar{h}_\nu))((\Sigma^{-k}\rho_{A^*,\,i}) \otimes_{K^*} (\Sigma^{-j}\iota_j\epsilon_j(\pi_{A^*,\,i-j})))(\tau^{-k,-j}_{F^*_iA^*,\iota_j\epsilon_j(A^*)})^{-1}\\
&\qquad = \sum_{\nu=1}^{n} ((\bar{g}_\nu\Sigma^{-k}\rho_{A^*,\,i}) \otimes_{K^*} (c_{0\,K^*}\iota_0\epsilon_0(\bar{h}_\nu)\Sigma^{-j}\iota_j\epsilon_j(\pi_{A^*,\,i-j})))(\tau^{-k,-j}_{F^*_iA^*,\iota_j\epsilon_j(A^*)})^{-1}\\
&\qquad = \sum_{\nu=1}^{n} (g_{s_\nu}\Sigma^{-k}\kappa_{A^*,\,i} \otimes_{K^*} c_{0\,K^*}\iota_0\epsilon_0(h_{s_\nu}))(\tau^{-k,-j}_{F^*_iA^*,\iota_j\epsilon_j(A^*)})^{-1}
\end{aligned}$$

Since $\Sigma^{-j-k}(\rho_{A^*,\,i} \otimes_{K^*} \iota_j\epsilon_j(\pi_{A^*,\,i-j}) : \Sigma^{-j-k}F_iA^* \otimes_{K^*} \iota_j\epsilon_j(A^*) \to \Sigma^{-j-k}E^*_iA^* \otimes_{K^*} \iota_j\epsilon_j(A^*/F_{i-j-1}A^*)$ is surjective, it suffices to show

$$\sum_{\nu=1}^{n} \tilde{\mu}_{K^*}(g_{s_\nu}\Sigma^{-k}\kappa_{A^*,\,i} \otimes_{K^*} c_{0\,K^*}\iota_0\epsilon_0(h_{s_\nu}))(\tau^{-k,-j}_{F^*_iA^*,\iota_j\epsilon_j(A^*)})^{-1} = f\Sigma^{-j-k}(\kappa_{A^*,\,i-j}\mu^{*,j}_i)$$

to verify $(**)$. The following diagram is commutative by the definition of $\mu^{*,j}_i$.

$$\begin{tikzcd}
\Sigma^{-j-k}F_iA^* \otimes_{K^*} \iota_j\epsilon_j(A^*) \arrow[r, "\Sigma^{-j-k}\mu^{*,j}_i"] \arrow[d, "\Sigma^{-j-k}(\kappa_{A^*,\,i}\otimes_{K^*}c_j)"] & \Sigma^{-j-k}F_{i-j}A^* \arrow[d, "\Sigma^{-j-k}\kappa_{A^*,\,i-j}"] \\
\Sigma^{-j-k}A^* \otimes_{K^*} A^* \arrow[r, "\Sigma^{-j-k}\mu"] & \Sigma^{-j-k}A^*
\end{tikzcd}$$

Since $(\Sigma^{-j-k-n_s}\iota_j\epsilon_j(A^*))^0 = \{0\}$ unless $n_s = -k$, $h_s\Sigma^{-j-k-n_s}c_j$ is trivial map. We remark that we have $h_s\Sigma^{-j}c_j = c_{0\,K^*}\iota_0\epsilon_0(h_{s_\nu})$ if $n_s = -k$, that is, $s = s_\nu$ for some ν. Since $f\Sigma^{-j-k}\mu = \sum_{s\in S} \tilde{\mu}_{K^*}\hat{\phi}(g_s \otimes h_s)$ by the definition of δ, we have the following equality which verifies $(**)$ by the commutativity of the above diagram.

$$\begin{aligned}
f\Sigma^{-j-k}(\kappa_{A^*,\,i-j}\mu_i^{*,j}) &= f\Sigma^{-j-k}\mu\Sigma^{-j-k}(\kappa_{A^*,\,i}\otimes_{K^*} c_j) = \sum_{s\in S}\tilde{\mu}_{K^*}\hat{\phi}(g_s\otimes h_s)\Sigma^{-j-k}(\kappa_{A^*,\,i}\otimes_{K^*} c_j)\\
&= \sum_{s\in S}\tilde{\mu}_{K^*}(g_s\otimes_{K^*} h_s)\big(\tau_{A^*,\,A^*}^{n_s,-j-k-n_s}\big)^{-1}\Sigma^{-j-k}(\kappa_{A^*,\,i}\otimes_{K^*} c_j)\\
&= \sum_{s\in S}\tilde{\mu}_{K^*}(g_s\otimes_{K^*} h_s)(\Sigma^{n_s}\kappa_{A^*,\,i}\otimes_{K^*}\Sigma^{-j-k-n_s}c_j)\big(\tau_{F_iA^*,\iota_j\epsilon_j(A^*)}^{n_s,-j-k-n_s}\big)^{-1}\\
&= \sum_{s\in S}\tilde{\mu}_{K^*}((g_s\Sigma^{n_s}\kappa_{A^*,\,i})\otimes_{K^*}(h_s\Sigma^{-j-k-n_s}c_j))\big(\tau_{F_iA^*,\iota_j\epsilon_j(A^*)}^{n_s,-j-k-n_s}\big)^{-1}\\
&= \sum_{\nu=1}^{n}\tilde{\mu}_{K^*}(g_{s_\nu}\Sigma^{-k}\kappa_{A^*,\,i}\otimes_{K^*} c_{0\,K^*}\iota_0\epsilon_0(h_{s_\nu}))(\tau_{F_i^*A^*,\iota_j\epsilon_j(A^*)}^{-k,-j})^{-1}
\end{aligned}$$

Since $\theta_{A^*/F_{i-j-1}A^*,-j}$, $\hat{\pi}_{A^*,\,K^*,j-i}$, $\tilde{\rho}_{A^*,\,K^*,-i}$ and $\tilde{\rho}_{A^*,\,K^*,j-i}$ are isomorphisms, the commutativity of diagram $(*)$ implies that $(\tilde{\mu}_i^{*,j})^*$ is surjective (resp. injective) if and only if $\tilde{\mu}_i^{k,j}$ is surjective (resp. injective). □

Proposition 15.3.11 *Let C^* be a coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with an increasing filtration $(F_iC^*)_{i\in\boldsymbol{Z}}$ which satisfies $(f5^*)$ and $(f6^*)$. Suppose that C^* is finite type and has skeletal topology. Let $\mu : C^{**}\otimes_{K^*} C^{**}\to C^{**}$ be the product of C^{**} defined from the coproduct δ of C^*. We give C^{**} the dual filtration $(F_iC^{**})_{i\in\boldsymbol{Z}}$ and consider maps $\tilde{\delta}_{i,j}^{j+k} : E_{i-j}^{j+k}C^*\to E_i^kC^*\otimes_{K^*}(F_{i-j}C^*)^j$ and $\tilde{\mu}_{-i}^{-k,-j} : E_{-i}^{-k}C^{**}\otimes_{K^*}(C^{**}/F_{j-i-1}C^{**})^{-j}\to E_{j-i}^{-j-k}C^{**}$ for $i,j,k\in\boldsymbol{Z}$.*

(1) $\tilde{\delta}_{i,j}^{j+k}$ is injective if and only if $\tilde{\mu}_{-i}^{-k,-j}$ is surjective .

(2) $\tilde{\delta}_{i,j}^{j+k}$ is surjective if and only if $\tilde{\mu}_{-i}^{-k,-j}$ is injective.

Proof. It follows from (3.1.14), (15.1.3) and (15.1.4) that we have the following isomorphisms.

$$\begin{aligned}
\theta_{F_{i-j}C^*,-j} &: \iota_{-j}\epsilon_{-j}(\mathcal{H}om^*(F_{i-j}C^*,K^*))\to\mathcal{H}om^*(\iota_j\epsilon_j(F_{i-j}C^*),K^*)\\
\bar{\kappa}_{C^*,\,K^*,i-j} &: C^{**}/F_{j-i-1}C^{**}\to\mathcal{H}om^*(F_{i-j}C^*,K^*)\\
\tilde{\rho}_{C^*,\,K^*,l} &: E_l^*C^{**}\to\mathcal{H}om^*(E_{-l}^*C^*,K^*)\ \ (l=-i,j-i)
\end{aligned}$$

We also have the following isomorphism by the assumption and (4.1.7).

$$\hat{\phi} : \mathcal{H}om^*(E_i^*C^*,K^*)\,\widehat{\otimes}_{K^*}\,\mathcal{H}om^*(\iota_j\epsilon_j(F_{i-j}C^*),K^*)\to\mathcal{H}om^*(E_i^*C^*\otimes_{K^*}\iota_j\epsilon_j(F_{i-j}C^*),K^*\otimes_{K^*}K^*)$$

Since both $\mathcal{H}om^*(E_i^*C^*,K^*)$ and $\mathcal{H}om^*(\iota_{-j}\epsilon_{-j}(F_{i-j}C^*),K^*)$ have skeletal topology by (3.1.36) and

$$\mathcal{H}om^n(\iota_{-j}\epsilon_{-j}(F_{i-j}C^*),K^*)=\{0\}$$

if $n\neq -j$, the domain of the above isomorphism $\hat{\phi}$ in degree $-j-k$ is

$$\mathcal{H}om^{-k}(E_i^*C^*,K^*)\otimes_{K^*}\mathcal{H}om^{-j}(\iota_{-j}\epsilon_{-j}(F_{i-j}C^*),K^*).$$

Note that $\mathcal{H}om^{-j-k}(E_i^*C^*\otimes_{K^*}\iota_j\epsilon_j(F_{i-j}C^*),K^*)$ is naturally isomorphic to $\mathrm{Hom}_{K^0}(E_i^kC^*\otimes_{K^*}(F_{i-j}C^*)^j,K^0)$ and $\mathcal{H}om^{-j-k}(E_{i-j}^*C^*,K^*)$ is naturally isomorphic to $\mathrm{Hom}_{K^0}(E_{i-j}^{j+k}C^*,K^0)$ as vector spaces over K^0. Hence we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{H}om^{-j-k}(E_i^*C^*\otimes_{K^*}\iota_j\epsilon_j(F_{i-j}C^*),K^*) \arrow[r, "\tilde{\delta}_{i,j}^*"] \arrow[d, "\cong"] & \mathcal{H}om^{-j-k}(E_{i-j}^*C^*,K^*) \arrow[d, "\cong"]\\
\mathrm{Hom}_{K^0}(E_i^kC^*\otimes_{K^*}(F_{i-j}C^*)^j,K^0) \arrow[r, "(\tilde{\delta}_{i,j}^{j+k})^*"] & \mathrm{Hom}_{K^0}(E_{i-j}^{j+k}C^*,K^0)
\end{tikzcd}$$

It follows that $\tilde{\delta}_{i,j}^{j+k}$ is injective (resp. surjective) if and only if $\tilde{\delta}_{i,j}^*$ is surjective (resp. injective).

Since $(F_iC^{**})_{i\in\boldsymbol{Z}}$ satisfies $(f5)$ and $(f6)$ by (15.3.6) and (15.3.8), $\mu : C^{**}\otimes_{K^*} C^{**}\to C^{**}$ induces the map $\tilde{\mu}_{-i}^{-k,-j} : E_{-i}^{-k}C^{**}\otimes_{K^*}\iota_{-j}\epsilon_{-j}(C^{**}/F_{j-i-1}C^{**})^{-j}\to E_{j-i}^{-j-k}C^{**}$. We claim that the following diagram (i) is commutative.

$$
\begin{tikzcd}
E^{-k}_{-i}C^{**}\otimes_{K^*}\iota_{-j}\epsilon_{-j}(C^{**}/F_{j-i-1}C^{**})^{-j} \arrow[r, "\tilde{\mu}^{-k,-j}_{-i}"] \arrow[d, "id_{E^*_{-i}C^{**}}\otimes_{K^*}\iota_{-j}\epsilon_{-j}(\bar{\kappa}_{C^*,\,K^*,i-j})"] & E^{-j-k}_{j-i}C^{**} \arrow[d, "\tilde{\rho}_{C^*,\,K^*,j-i}"] \\
E^{-k}_{-i}C^{**}\otimes_{K^*}\iota_{-j}\epsilon_{-j}(\mathcal{H}om^*(F_{i-j}C^*,K^*))^{-j} \arrow[d, "\tilde{\rho}_{C^*,\,K^*,-i}\otimes_{K^*}\theta_{F_{i-j}C^*,-j}"] & \mathcal{H}om^{-j-k}(E^*_{i-j}C^*,K^*) \\
\mathcal{H}om^{-k}(E^*_iC^*,K^*)\otimes_{K^*}\mathcal{H}om^{-j}(\iota_j\epsilon_j(F_{i-j}C^*),K^*) \arrow[r, "\tilde{\mu}_{K^*}\hat{\phi}"] & \mathcal{H}om^{-j-k}(E^*_iC^*\otimes_{K^*}\iota_j\epsilon_j(F_{i-j}C^*),K^*) \arrow[u, "\tilde{\delta}^*_{i,j}"']
\end{tikzcd}
$$

diagram (i)

For $\alpha\in E^{-k}_{-i}C^{**}$ and $\beta\in\iota_{-j}\epsilon_{-j}(C^{**}/F_{j-i-1}C^{**})^{-j}$, we take $f\in(F_{-i}C^{**})^{-k}$ and $g\in(C^{**})^{-j}$ such that $\alpha=\rho_{C^{**},-i}(f)$ and $\beta=\pi_{C^{**},j-i}(g)$. Since f maps $\Sigma^{-k}F_{i-1}C^*$ to zero, there exists unique map $\bar{f}:\Sigma^{-k}C^*/F_{i-1}C^*\to K^*$ that satisfies $\bar{f}\Sigma^{-k}\pi_{C^*,\,i}=f$. Then, we have $\tilde{\rho}_{C^*,\,K^*,-i}(\alpha)=\bar{f}\Sigma^{-k}\tilde{\kappa}_{C^*,\,-i}$ and

$$\theta_{F_{i-j}C^*,-j}(\iota_{-j}\epsilon_{-j}(\bar{\kappa}_{C^*,\,K^*,i-j})(\beta))=\theta_{F_{i-j}C^*,-j}(g\Sigma^{-j}\kappa_{C^*,\,i-j})=c_{0\,K^*}\iota_0\epsilon_0(g\Sigma^{-j}\kappa_{C^*,\,i-j})$$

which imply the following equality.

$$
\begin{aligned}
&\tilde{\delta}^*_{i,j}\tilde{\mu}_{K^*}\hat{\phi}(\tilde{\rho}_{C^*,\,K^*,-i}\otimes_{K^*}\theta_{F_{i-j}C^*,-j})(id_{E^*_{-i}C^{**}}\otimes_{K^*}\iota_{-j}\epsilon_{-j}(\bar{\kappa}_{C^*,\,K^*,i-j}))(\alpha\otimes\beta)\\
&\quad=\tilde{\delta}^*_{i,j}\tilde{\mu}_{K^*}\hat{\phi}(\Sigma^{-k}\tilde{\kappa}_{C^*,\,-i}\bar{f}\otimes c_{0\,K^*}\iota_0\epsilon_0(g\Sigma^{-j}\kappa_{C^*,\,i-j}))\\
&\quad=\tilde{\mu}_{K^*}(\bar{f}\Sigma^{-k}\tilde{\kappa}_{C^*,\,-i}\otimes_{K^*}c_{0\,K^*}\iota_0\epsilon_0(g\Sigma^{-j}\kappa_{C^*,\,i-j}))\big(\tau^{-k,-j}_{E^*_iC^*,\iota_j\epsilon_j(F_{i-j}C^*)}\big)^{-1}\Sigma^{-j-k}\tilde{\delta}_{i,j}\\
&\quad=\tilde{\mu}_{K^*}(\bar{f}\otimes_{K^*}c_{0\,K^*}\iota_0\epsilon_0(g))(\Sigma^{-k}\tilde{\kappa}_{C^*,\,-i}\otimes_{K^*}c_{0\,K^*}\iota_0\epsilon_0(\Sigma^{-j}\kappa_{C^*,\,i-j}))\big(\tau^{-k,-j}_{E^*_iC^*,\iota_j\epsilon_j(F_{i-j}C^*)}\big)^{-1}\Sigma^{-j-k}\tilde{\delta}_{i,j}\\
&\quad=\tilde{\mu}_{K^*}(\bar{f}\otimes_{K^*}c_{0\,K^*}\iota_0\epsilon_0(g))\big(\tau^{-k,-j}_{C^*/F^*_{i-1}C^*,\iota_j\epsilon_j(C^*)}\big)^{-1}\Sigma^{-j-k}(\tilde{\kappa}_{C^*,\,-i}\otimes_{K^*}\iota_j\epsilon_j(\kappa_{C^*,\,i-j}))\Sigma^{-j-k}\tilde{\delta}_{i,j}\quad\cdots(ii)
\end{aligned}
$$

It follows from the commutativity of the following diagram and (3) of (1.2.6) that

$$\tilde{\mu}_{K^*}(\bar{f}\otimes_{K^*}c_{0\,K^*}\iota_0\epsilon_0(g))\big(\tau^{-k,-j}_{C^*/F^*_{i-1}C^*,\iota_j\epsilon_j(C^*)}\big)^{-1}\Sigma^{-j-k}(\tilde{\kappa}_{C^*,\,-i}\otimes_{K^*}\iota_j\epsilon_j(\kappa_{C^*,\,i-j}))\Sigma^{-j-k}\tilde{\delta}_{i,j}\Sigma^{-j-k}\rho_{C^*,\,i-j}$$

coincides with

$$
\begin{aligned}
&\tilde{\mu}_{K^*}(\bar{f}\otimes_{K^*}c_{0\,K^*}\iota_0\epsilon_0(g))(\tau^{-k,-j}_{C^*/F_{i-1}C^*,\iota_j\epsilon_j(C^*)})^{-1}\Sigma^{-j-k}(\pi_{C^*,\,i}\mathbin{\widehat{\otimes}}_{K^*}u_j)\Sigma^{-j-k}\delta\Sigma^{-j-k}\kappa_{C^*,\,i-j}\\
&\quad=\tilde{\mu}_{K^*}(\bar{f}\otimes_{K^*}c_{0\,K^*}\iota_0\epsilon_0(g))(\Sigma^{-k}\pi_{C^*,\,i}\otimes_{K^*}\Sigma^{-j}u_j)(\tau^{-k,-j}_{C^*,\,C^*})^{-1}\Sigma^{-j-k}\delta\Sigma^{-j-k}\kappa_{C^*,\,i-j}\\
&\quad=\tilde{\mu}_{K^*}(\bar{f}\Sigma^{-k}\pi_{C^*,\,i}\otimes_{K^*}c_{0\,K^*}\iota_0\epsilon_0(g)\Sigma^{-j}u_j)(\tau^{-k,-j}_{C^*,\,C^*})^{-1}\Sigma^{-j-k}\delta\Sigma^{-j-k}\kappa_{C^*,\,i-j}\\
&\quad=\tilde{\mu}_{K^*}(f\otimes_{K^*}c_{0\,K^*}u_{0\,K^*}g)(\tau^{-k,-j}_{C^*,\,C^*})^{-1}\Sigma^{-j-k}\delta\Sigma^{-j-k}\kappa_{C^*,\,i-j}\\
&\quad=\tilde{\mu}_{K^*}(f\mathbin{\widehat{\otimes}}_{K^*}g)(\tau^{-k,-j}_{C^*,\,C^*})^{-1}\Sigma^{-j-k}\delta\Sigma^{-j-k}\kappa_{C^*,\,i-j}.
\end{aligned}
$$

$$
\begin{tikzcd}
\Sigma^{-j-k}F_{i-j}C^* \arrow[r, "\Sigma^{-j-k}\kappa_{C^*,\,i-j}"] \arrow[d, "\Sigma^{-j-k}\rho_{C^*,\,i-j}"] & \Sigma^{-j-k}C^* \arrow[r, "\Sigma^{-j-k}\delta"] \arrow[d, "\Sigma^{-j-k}\pi_{C^*,\,i-j}"] & \Sigma^{-j-k}C^*\mathbin{\widehat{\otimes}}_{K^*}C^* \arrow[dd, "\Sigma^{-j-k}(\pi_{C^*,\,i}\mathbin{\widehat{\otimes}}_{K^*}u_j)"] \\
\Sigma^{-j-k}E_{i-j}C^* \arrow[r, "\Sigma^{-j-k}\tilde{\kappa}_{C^*,\,i-j}"] \arrow[d, "\Sigma^{-j-k}\tilde{\delta}_{i,j}"] & \Sigma^{-j-k}C^*/F_{i-j}C^* \arrow[dr, "\Sigma^{-j-k}\delta_{i-1,j}"] & \\
\Sigma^{-j-k}E^*_iC^*\otimes_{K^*}\iota_j\epsilon_j(F_{i-j}C^*) \arrow[rr, "\Sigma^{-j-k}\tilde{\kappa}_{C^*,\,i}\otimes_{K^*}\iota_j\epsilon_j(\kappa_{C^*,\,i-j})"] & & \Sigma^{-j-k}C^*/F_{i-1}C^*\otimes_{K^*}\iota_j\epsilon_j(C^*)
\end{tikzcd}
$$

On the other hand, the following diagram is commutative by the definition of $\tilde{\mu}^{-k,-j}_{-i}$ and (15.1.5).

$$
\begin{tikzcd}
(F_{-i}C^{**})^{-k}\otimes_{K^*}\iota_{-j}\epsilon_{-j}(C^{**})^{-j} \arrow[r, "\mu^{-k,-j}_{-i}"] \arrow[d, "\rho_{C^{**},-i}\otimes_{K^*}\iota_{-j}\epsilon_{-j}(\pi_{C^{**},j-i})"] & (F_{j-i}C^{**})^{-j-k} \arrow[r, "\hat{\pi}^{-1}_{A^*,\,K^*,j-i}"] \arrow[d, "\rho_{C^{**},j-i}"] & \mathcal{H}om^{-j-k}(C^*/F_{i-j-1}C^*,K^*) \arrow[d, "\tilde{\kappa}^*_{C^*,\,i-j}"] \\
E^{-k}_{-i}C^{**}\otimes_{K^*}\iota_{-j}\epsilon_{-j}(C^{**}/F_{j-i-1}C^{**})^{-j} \arrow[r, "\tilde{\mu}^{-k,-j}_{-i}"] & E^{-j-k}_{j-i}C^{**} \arrow[r, "\tilde{\rho}_{C^*,\,K^*,j-i}"] & \mathcal{H}om^{-j-k}(E^*_{i-j}C^*,K^*)
\end{tikzcd}
$$

Since $\mu_{-i}^{-k,-j}(f\otimes g)\in(F_{j-i}C^{**})^{-j-k}$, there exists a map $h:\Sigma^{-j-k}C^*/F_{i-j-1}C^*\to K^*$ which satisfies $\mu_{-i}^{-k,-j}(f\otimes g)=h\Sigma^{-j-k}\pi_{C^*,i-j}$. Hence we have the following equality

$$\begin{aligned}\tilde\rho_{C^*,K^*,j-i}\tilde\mu_{-i}^{-k,-j}(\alpha\otimes\beta)&=\tilde\rho_{C^*,K^*,j-i}\,\tilde\mu_{-i}^{-k,-j}(\rho_{C^{**},-i}(f)\otimes\iota_{-j}\epsilon_{-j}(\pi_{C^{**},j-i})(g))\\&=\tilde\rho_{C^*,K^*,j-i}\,\rho_{C^{**},j-i}\,\mu_{-i}^{-k,-j}(f\otimes g)\\&=\tilde\kappa^*_{C^*,i-j}\hat\pi^{-1}_{C^*,K^*,j-i}(h\Sigma^{-j-k}\pi_{C^*,i-j})=h\Sigma^{-j-k}\tilde\kappa_{C^*,i-j}\quad\cdots(iii)\end{aligned}$$

Recall from (5.1.3) that the product μ of C^{**} is a composition

$$\begin{aligned}C^{**}\otimes_{K^*}C^{**}\xrightarrow{\phi}\mathcal{H}om^*(C^*\otimes_{K^*}C^*,K^*\otimes_{K^*}K^*)&\xrightarrow{c_{C^*\otimes_{K^*}C^*,K^*\otimes_{K^*}K^*}}\mathcal{H}om^*(C^*\mathbin{\widehat\otimes}_{K^*}C^*,K^*\mathbin{\widehat\otimes}_{K^*}K^*)\\&\xrightarrow{\tilde\mu_{K^*_*}}\mathcal{H}om^*(C^*\mathbin{\widehat\otimes}_{K^*}C^*,K^*)\xrightarrow{\delta^*}\mathcal{H}om^*(C^*,K^*).\end{aligned}$$

Here we identify $K^*\mathbin{\widehat\otimes}_{K^*}K^*$ with $K^*\otimes_{K^*}K^*$. Hence $\mu_{-i}^{-k,-j}(f\otimes g)\in F_{j-i}C^{**}$ is the following composition.

$$\Sigma^{-j-k}C^*\xrightarrow{\Sigma^{-j-k}\delta}\Sigma^{-j-k}C^*\mathbin{\widehat\otimes}_{K^*}C^*\xrightarrow{(\hat\tau^{-k,-j}_{C^*,C^*})^{-1}}\Sigma^{-k}C^*\mathbin{\widehat\otimes}_{K^*}\Sigma^{-j}C^*\xrightarrow{f\mathbin{\widehat\otimes}_{K^*}g}K^*\mathbin{\widehat\otimes}_{K^*}K^*\xrightarrow{\tilde\mu_{K^*}}K^*$$

Therefore we have

$$\begin{aligned}h\Sigma^{-j-k}\tilde\kappa_{C^*,i-j}\Sigma^{-j-k}\rho_{C^*,i-j}&=h\Sigma^{-j-k}\pi_{C^*,i-j}\Sigma^{-j-k}\kappa_{C^*,i-j}=\mu_{-i}^{-k,-j}(f\otimes g)\Sigma^{-j-k}\kappa_{C^*,i-j}\\&=\tilde\mu_{K^*}(f\mathbin{\widehat\otimes}_{K^*}g)(\hat\tau^{-k,-j}_{C^*,C^*})^{-1}\Sigma^{-j-k}\delta\Sigma^{-j-k}\kappa_{C^*,i-j},\end{aligned}$$

which shows that $h\Sigma^{-j-k}\tilde\kappa_{C^*,i-j}\Sigma^{-j-k}\rho_{C^*,i-j}$ is equal to

$$\tilde\mu_{K^*}(\bar f\otimes_{K^*}c_{0\,K^*}\iota_0\epsilon_0(g))\big(\tau^{-k,-j}_{C^*/F^*_{i-1}C^*,\iota_j\epsilon_j(C^*)}\big)^{-1}\Sigma^{-j-k}(\tilde\kappa_{C^*,-i}\otimes_{K^*}\iota_j\epsilon_j(\kappa_{C^*,i-j}))\Sigma^{-j-k}\tilde\delta_{i,j}\Sigma^{-j-k}\rho_{C^*,i-j}.$$

Since $\Sigma^{-j-k}\rho_{C^*,i-j}$ is surjective, we have

$$\tilde\mu_{K^*}(\bar f\otimes_{K^*}c_{0\,K^*}\iota_0\epsilon_0(g))\big(\tau^{-k,-j}_{C^*/F^*_{i-1}C^*,\iota_j\epsilon_j(C^*)}\big)^{-1}\Sigma^{-j-k}(\tilde\kappa_{C^*,-i}\otimes_{K^*}\iota_j\epsilon_j(\kappa_{C^*,i-j}))\Sigma^{-j-k}\tilde\delta_{i,j}=h\Sigma^{-j-k}\tilde\kappa_{C^*,i-j}.$$

Hence the commutativity of diagram (i) follows from (ii) and (iii).

Since $\bar\kappa_{C^*,K^*,i-j}$, $\theta_{F_{i-j-1}C^*,-j}$, $\tilde\rho_{C^*,K^*,-i}$ and $\tilde\rho_{C^*,K^*,j-i}$ are isomorphisms, the commutativity of diagram (i) implies that $\tilde\delta^*_{i,j}$ is injective (resp. surjective) if and only if $\tilde\mu_{-i}^{-k,-j}$ is injective (resp. surjective). □

Condition 15.3.12 *For a connective coalgebra C^* over K^* with a coproduct $\delta:C^*\to C^*\otimes_{K^*}C^*=C^*\mathbin{\widehat\otimes}_{K^*}C^*$ and an increasing filtration $\mathfrak{F}=(F_iC^*)_{i\in\boldsymbol{Z}}$, we consider the following conditions.*

- $(f7^*)$ $\tilde\delta_{i,j}^{c^*_{\mathfrak{F}}(i)+j}:E_{i-j}^{c^*_{\mathfrak{F}}(i)+j}C^*\to(E_i^*C^*\mathbin{\widehat\otimes}_{K^*}\iota_j\epsilon_j(F_{i-j}C^*))^{c^*_{\mathfrak{F}}(i)+j}$ *is injective for $i\in S(\mathfrak{F})$, $j\in\boldsymbol{Z}$.*
- $(f8^*)$ $\tilde\delta_{i,j}^{c^*_{\mathfrak{F}}(i)+j}:E_{i-j}^{c^*_{\mathfrak{F}}(i)+j}C^*\to(E_i^*C^*\mathbin{\widehat\otimes}_{K^*}\iota_j\epsilon_j(F_{i-j}C^*))^{c^*_{\mathfrak{F}}(i)+j}$ *is surjective for $i\in S(\mathfrak{F})$, $j\in\boldsymbol{Z}$.*

Proposition 15.3.13 *Let A^* be an algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with an increasing filtration $\mathfrak{F}=(F_iA^*)_{i\in\boldsymbol{Z}}$ which satisfies $(f5)$ and $(f6)$. Suppose that A^* is coconnective and finite type and has skeletal topology. Let δ be the coproduct of A^{**} defined from the product μ of A^*. We give A^{**} the dual filtration $\mathfrak{F}^*=(F_iA^{**})_{i\in\boldsymbol{Z}}$. $\mathfrak{F}$ satisfies $(f7)$ (resp. $(f8)$) if and only if $\mathfrak{F}^*$ satisfies $(f7^*)$ (resp. $(f8^*)$).*

Proof. It follows from (15.3.10) and (15.1.27) that $\tilde\mu_i^{c_{\mathfrak{F}}(i),j}:E_i^{c_{\mathfrak{F}}(i)}A^*\otimes_{K^*}(A^*/F_{i-j-1}A^*)^j\to E_{i-j}^{j+c_{\mathfrak{F}}(i)}A^*$ is surjective (resp. injective) if and only if $\tilde\delta_{-i,-j}^{-j+c^*_{\mathfrak{F}^*}(-i)}:E_{j-i}^{-j+c^*_{\mathfrak{F}^*}(-i)}A^{**}\to E_{-i}^{c^*_{\mathfrak{F}^*}(-i)}A^{**}\otimes_{K^*}(F_{j-i}A^{**})^{-j}$ is injective (resp. surjective). Thus $\mathfrak{F}$ satisfies $(f7)$ (resp. $(f8)$) if and only if $\mathfrak{F}^*$ satisfies $(f7^*)$ (resp. $(f8^*)$). □

Proposition 15.3.14 *Let C^* be an coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with an increasing filtration $\mathfrak{F}=(F_iC^*)_{i\in\boldsymbol{Z}}$ which satisfies $(f5^*)$ and $(f6^*)$. Suppose that C^* is connective and finite type and has skeletal topology. Let μ be the product of C^{**} defined from the coproduct δ of C^*. We give C^{**} the dual filtration $\mathfrak{F}^*=(F_iC^{**})_{i\in\boldsymbol{Z}}$. $\mathfrak{F}$ satisfies $(f7^*)$ (resp. $(f8^*)$) if and only if $\mathfrak{F}^*$ satisfies $(f7)$ (resp. $(f8)$).*

Proof. It follows from (15.3.11) and (15.1.27) that $\tilde{\delta}_{i,j}^{j+c^*_{\mathfrak{F}}(i)} : E_{i-j}^{j+c^*_{\mathfrak{F}}(i)}C^* \to E_i^{c^*_{\mathfrak{F}}(i)}C^* \otimes_{K^*} (F_{i-j}C^*)^j$ is injective (resp. surjective) if and only if $\tilde{\mu}_{-i}^{c_{\mathfrak{F}^*}(-i),-j} : E_{-i}^{c_{\mathfrak{F}^*}(-i)}C^{**} \otimes_{K^*} (C^{**}/F_{j-i-1}C^{**})^{-j} \to E_{j-i}^{-j+c_{\mathfrak{F}^*}(-i)}C^{**}$ is surjective (resp. injective). Thus $\mathfrak{F}$ satisfies $(f7^*)$ (resp. $(f8^*)$) if and only if $\mathfrak{F}^*$ satisfies $(f7)$ (resp. $(f8)$). □

We give an exterior algebra $E(\tau)$ a structure of Hopf algebra by declaring that τ is primitive. Let C^* be a filtered coalgebra with coproduct $\delta : C^* \to C^* \widehat{\otimes}_{K^*} C$ and filtration $\mathfrak{F} = (F_iC^*)_{i\in \boldsymbol{Z}}$. We assume that C^* is connective and has skeletal topology and that $E(\tau)$ has skeletal topology. Then, $E(\tau) \otimes_{K^*} C^*$ is connective and has skeletal topology. Define a coproduct $\delta_\tau : E(\tau) \otimes_{K^*} C^* \to (E(\tau) \otimes_{K^*} C^*) \otimes_{K^*} (E(\tau) \otimes_{K^*} C^*)$ by $\delta_\tau(1\otimes x) = \sum_k (1\otimes x_k) \otimes (1\otimes x'_k)$ and $\delta_\tau(\tau\otimes x) = \sum_k (\tau\otimes x_k)\otimes(1\otimes x'_k) + \sum_k (-1)^{\deg\tau\deg x_k}(1\otimes x_k)\otimes(\tau\otimes x'_k)$ if $\delta(x) = \sum_k x_k \otimes x'_k$.

Proposition 15.3.15 *Assume that $\mathfrak{F}$ satisfies $(f3^*)$, $(f4^*)$, $(f5^*)$ and $(f6^*)$. We also assume that $a - b \neq \deg\tau + 1$ for any $a, b \in I^*(\mathfrak{F})$ and that $S(\mathfrak{F}) \cap \{i \in \boldsymbol{Z}\,|\, i-1 \in S(\mathfrak{F})\} = \emptyset$. If $\mathfrak{F}$ satisfies $(f7^*)$ (resp. $(f8^*)$), so does the filtration $\mathfrak{F}_\tau$ of $E(\tau)\otimes_{K^*} C^*$ which is considered in (15.1.33).*

Proof. Put $d = \deg\tau$. Since $S(\mathfrak{F}_\tau)$ is the disjoint union of $S(\mathfrak{F})$ and $\{i \in \boldsymbol{Z}\,|\, i-1 \in S(\mathfrak{F})\}$, it follows from (15.1.33), $(f3^*)$ and the assumption that

$$E_{i-j-1}^{c^*_{\mathfrak{F}_\tau}(i)+j-d}C^* = E_{i-j-1}^{c^*_{\mathfrak{F}}(i)+j-d}C^* = \{0\} \text{ if } i \in S(\mathfrak{F}) \text{ and } E_{i-j}^{c^*_{\mathfrak{F}_\tau}(i)+j}C^* = E_{i-j}^{c^*_{\mathfrak{F}}(i-1)+j+d}C^* = \{0\} \text{ if } i-1 \in S(\mathfrak{F}).$$

Hence (15.1.1) implies that $\varphi_{0,i-j} : E_0^0E(\tau)\otimes_{K^*} E_{i-j}^{c^*_{\mathfrak{F}}(i)+j}C^* \to E_{i-j}^{c^*_{\mathfrak{F}_\tau}(i)+j}(E(\tau)\otimes_{K^*} C^*)$ is an isomorphism if $i\in S(\mathfrak{F})$ and $\varphi_{1,i-j-1} : E_1^dE(\tau)\otimes_{K^*} E_{i-j-1}^{c^*_{\mathfrak{F}}(i-1)+j}C^* \to E_{i-j}^{c^*_{\mathfrak{F}_\tau}(i)+j}(E(\tau)\otimes_{K^*} C^*)$ is an isomorphism if $i-1\in S(\mathfrak{F})$. □

Proposition 15.3.16 *Suppose that K^* a field of non-zero characteristic p such that $x^p - a = 0$ has a root in K^* for any $a \in K^*$. Let A^* be a connective Hopf algebra in $\mathcal{T}op\mathcal{A}lg_{K^*}$ with skeletal topology. We denote by $\delta : A^* \to A^* \otimes_{K^*} A^*$ the coproduct of A^*. Let $\mathfrak{F} = (F_iA^*)_{i\in\boldsymbol{Z}}$ be a filtration of A^* which satisfies $(f1^*)$ and $(f2^*)$ and consider the filtration $\mathfrak{F}(k) = (F_iA(k)^*)_{i\in\boldsymbol{Z}}$ of $A(k)^*$ obtained from $\mathfrak{F}$. We assume that $\mathfrak{F}$ satisfies the condition of (15.1.35). If $\mathfrak{F}$ satisfies $(f7^*)$ (resp. $(f8^*)$), so does $\mathfrak{F}(k)$.*

Proof. We can take a basis $\{x_\alpha\,|\,\alpha \in J_i\}$ of F_iA^* for $i \in \boldsymbol{Z}$ such that $J_{i-1} \subset J_i$, then $\{x_\alpha^{p^k}\,|\,\alpha\in J_i\}$ is a basis of $F_{ip^k}A(k)^*$. Let us denote by $\bar{f}_i^j : E_i^jA^* \to E_{ip^k}^{jp^k}A(k)^*$ the map induced by the p^k-th power map $f : A^* \to A(k)^*$ $x \mapsto x^{p^k}$. We denote by $f_s : (F_sA^*)^j \to (F_{sp^k}A(k)^*)^{jp^k}$ the map obtained from f by restricting the source and the target of f. Since $\bar{f}_i^j(\lambda u) = \lambda^{p^k}\bar{f}_i^j(u)$ and $f_s(\lambda v) = \lambda^{p^k}f_i(v)$ for $\lambda\in K^*$, $u \in E_i^jA^*$ and $v\in F_sA^*$, $\bar{f}_i^j \times f_s : E_i^jA^*\times(F_sA^*)^t \to E_{ip^k}^{jp^k}A^*\otimes_{K^*}(F_sA^*)^t$ induces a map $\bar{f}_i^j\otimes_{K^*} f_s : E_i^jA^*\otimes_{K^*}(F_sA^*)^t \to E_{ip^k}^{jp^k}A^*\otimes_{K^*}(F_sA^*)^t$. Then, the following diagram is commutative.

$$\begin{CD}
E_{i-j}^{c^*_{\mathfrak{F}}(i)+j}A^* @>{\tilde{\delta}_{i,j}^{c^*_{\mathfrak{F}}(i)+j}}>> E_i^{c^*_{\mathfrak{F}}(i)}A^*\otimes_{K^*}(F_{i-j}A^*)^j \\
@V{\bar{f}_{i-j}^{c^*_{\mathfrak{F}}(i)+j}}VV @VV{\bar{f}_i^{c^*_{\mathfrak{F}}(i)}\otimes_{K^*} f_{i-j}}V \\
E_{ip^k-jp^k}^{c^*_{\mathfrak{F}(k)}(ip^k)+jp^k}A(k)^* @>{\tilde{\delta}_{ip^k,jp^k}^{c^*_{\mathfrak{F}(k)}(ip^k)+jp^k}}>> E_{ip^k}^{c^*_{\mathfrak{F}(k)}(ip^k)}A(k)^*\otimes_{K^*}(F_{ip^k-jp^k}A(k)^*)^{jp^k}
\end{CD}$$

Since $\bar{f}_{i-j}^{c^*_{\mathfrak{F}}(i)+j}$, $\bar{f}_i^{c^*_{\mathfrak{F}}(i)}$ and f_{i-j} are bijections by (15.1.37), $\tilde{\delta}_{ip^k,jp^k}^{c^*_{\mathfrak{F}(k)}(ip^k)+jp^k}$ is injective (resp.surjective) if $\tilde{\delta}_{i,j}^{c^*_{\mathfrak{F}}(i)+j}$ is injective (resp. surjective). □

Suppose that C^* is an algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with product μ and that C^* has the skeletal topology. We consider the following condition under this assumption.

Condition 15.3.17 *Let $\mathfrak{F} = (F_iC^*)_{i\in\boldsymbol{Z}}$ be a filtration on C^*.*

$(f9^*)$ $\mu(F_iC^*\otimes_{K^*} F_jC^*) \subset F_{i+j}C^*$ for $i, j \in \boldsymbol{Z}$.

The following assertion is clear.

Proposition 15.3.18 *Let C^* be an algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with product $\mu : C^* \otimes_{K^*} C^* \to C^*$ and an increasing filtration $(F_iC^*)_{i\in \boldsymbol{Z}}$ of subspaces of C^*. Let D^* be a subalgebra of C^* and $\mathfrak{a}$ a two-sided ideal of C^*. If $(F_iC^*)_{i\in \boldsymbol{Z}}$ satisfies $(f9)$, $(F_iD^*)_{i\in \boldsymbol{Z}}$ and $(F_i(C^*/\mathfrak{a}))_{i\in \boldsymbol{Z}}$ satisfy $(f9^*)$.*

Proposition 15.3.19 *Let A^* be a coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with coproduct δ. Assume that A^* is connective or coconnective and that A^* has skeletal topology. If a filtration $(F_iA^*)_{i\in \boldsymbol{Z}}$ on A^* satisfies $(f9)$ of (15.2.25), then the dual filtration $(F_iA^{**})_{i\in \boldsymbol{Z}}$ of the dual algebra A^{**} satisfies $(f9^*)$ of (15.3.17).*

Proof. For $f \in (F_iA^{**})^m$ and $g \in (F_jA^{**})^n$, $\phi(f \otimes g) \in F_{i+j}\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^* \otimes_{K^*} K^*)$ holds by (15.1.9). We also have $\delta^*(F_{i+j}\mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^* \otimes_{K^*} K^*)) \subset F_{i+j}\mathcal{H}om^*(A^*, K^* \otimes_{K^*} K^*)$ by (15.1.6). Since the product $\tilde{\delta} : A^{**} \otimes_{K^*} A^{**} \to A^{**}$ is a composition

$$A^{**} \otimes_{K^*} A^{**} \xrightarrow{\phi} \mathcal{H}om^*(A^* \otimes_{K^*} A^*, K^* \otimes_{K^*} K^*) \xrightarrow{\delta^*} \mathcal{H}om^*(A^*, K^* \otimes_{K^*} K^*) \xrightarrow{\tilde{\mu}_{K^*}} \mathcal{H}om^*(A^*, K^*) = A^{**},$$

we deduce that $\tilde{\delta}(f \otimes g) \in F_{i+j}A^{**}$. □

Proposition 15.3.20 *Let C^* be an algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with product μ. Assume that C^* is finite type, connective or coconecctive and has skeletal topology. Suppose that a filtration $(F_iC^*)_{i\in \boldsymbol{Z}}$ of C^* satisfies $(f1)$ or $(f1^*)$ and that "$F_rC^* = C^*$ for some $r \in \boldsymbol{Z}$" or "$F_rC^* = \{0\}$ for some $r \in \boldsymbol{Z}$" holds. If a filtration $\mathfrak{F} = (F_iC^*)_{i\in \boldsymbol{Z}}$ on C^* satisfies $(f9^*)$ of (15.3.17), then the dual filtration $(F_iC^{**})_{i\in \boldsymbol{Z}}$ of $\mathfrak{F}$ satisfies $(f9)$ of (15.2.25).*

Proof. We first note that $\phi : C^{**} \otimes_{K^*} C^{**} \to \mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^* \otimes_{K^*} K^*)$ is an isomorphism and that the coproduct $\tilde{\mu} : C^{**} \to C^{**} \otimes_{K^*} C^{**}$ is defined to be the following composition.

$$C^{**} \xrightarrow{\mu^*} \mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^*) \xrightarrow{(\tilde{\mu}_{K^*}^{-1})_*} \mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^* \otimes_{K^*} K^*) \xrightarrow{\phi^{-1}} C^{**} \otimes_{K^*} C^{**}$$

It follows from (15.1.6), (15.1.7) and (15.1.19) that the following relations hold.

$$\begin{aligned}
\tilde{\mu}(F_iC^{**}) &\subset F_i\mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^*)\\
(\tilde{\mu}_{K^*}^{-1})_*(F_i\mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^*)) &\subset F_i\mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^* \otimes_{K^*} K^*)\\
\phi^{-1}(F_i\mathcal{H}om^*(C^* \otimes_{K^*} C^*, K^* \otimes_{K^*} K^*)) &= F_i(C^{**} \otimes_{K^*} C^{**})
\end{aligned}$$

Hence we have $\tilde{\mu}(F_iC^{**}) \subset F_i(C^{**} \otimes_{K^*} C^{**}) = \sum_{j+k=i} F_jC^{**} \otimes_{K^*} F_kC^{**}$. □

Proposition 15.3.21 *Let A^* be a filtered algebra with filtration $\mathfrak{F} = (F_iA^*)_{i\in \boldsymbol{Z}}$ and M^* a left A^*-module with structure map $\alpha : A^* \otimes_{K^*} M^* \to M^*$. Suppose that A^* and M^* are finite type and have skeletal topology so that $\hat{\varphi}^{A^*}_{M^*} : A^{**} \widehat{\otimes}_{K^*} M^* \to \mathcal{H}om^*(A^*, M^*)$ is an isomorphism. We give A^{**} the dual filtration $(F_iA^{**})_{i\in \boldsymbol{Z}}$ of $\mathfrak{F}$. M^* is an unstable A^*-module if and only if $\Lambda(\alpha) : M^* \to M^* \widehat{\otimes}_{K^*} A^{**}$ satisfies $\Lambda(\alpha)(M^j) \subset M^* \widehat{\otimes}_{K^*} F_{-j}A^{**}$ for any $j \in \boldsymbol{Z}$.*

Proof. $\alpha(F_{j-1}A^* \otimes_{K^*} M^j) = \{0\}$ if and only if $\Phi(\alpha T_{M^*,A^{**}}(id_{M^*} \otimes_{K^*} \kappa_{A^*,j-1}))(M^j) = \{0\}$. Hence the assertion follows from the exactness of the bottom row and the commutativity of the following diagram.

$$\begin{tikzcd}
 & M^j \arrow[r] & M^* \arrow[d, "\Phi(\alpha T_{M^*,A^*})"'] \arrow[dr, "\Phi(\alpha T_{M^*,A^*}(id_{M^*} \otimes_{K^*} \kappa_{A^*,j-1}))"] & \\
 & & \mathcal{H}om^*(A^*, M^*) \arrow[r, "\kappa^*_{A^*,j-1}"] \arrow[d, "(\hat{\varphi}^{A^*}_{M^*})^{-1}"', "\cong"] & \mathcal{H}om^*(F_{j-1}A^*, M^*) \arrow[d, "(\hat{\varphi}^{F_{j-1}A^*}_{M^*})^{-1}"', "\cong"] \\
0 \arrow[r] & F_{-j}A^{**} \widehat{\otimes}_{K^*} M^* \arrow[r, "\kappa_{A^{**},-j} \widehat{\otimes}_{K^*} id_{M^*}"] & A^{**} \widehat{\otimes}_{K^*} M^* \arrow[r, "\kappa^*_{A^*,j-1} \widehat{\otimes}_{K^*} id_{M^*}"] & \mathcal{H}om^*(F_{j-1}A^*, K^*) \widehat{\otimes}_{K^*} M^*
\end{tikzcd}$$

□

Now, we give the definition of unstable comodules.

Definition 15.3.22 *Let C^* be a coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with an increasing filtration $\mathfrak{F} = (F_iC^*)_{i\in \boldsymbol{Z}}$ of subspaces of C^*. A right C^*-comodule $\lambda : M^* \to M^* \widehat{\otimes}_{K^*} C^*$ is called an unstable C^*-comodule with respect to $\mathfrak{F}$ if $\lambda(M^{-j}) \subset M^* \widehat{\otimes}_{K^*} F_jC^*$ for any $j \in \boldsymbol{Z}$. We denote by $\mathcal{UC}omod(C^*)$ the full subcategory of the category right C^*-comodules consisting of unstable C^*-comodules.*

Remark 15.3.23 *An unstable C^*-comodule M^* is coconnective if $\mathfrak{F} = (F_iC^*)_{i\in \boldsymbol{Z}}$ satisfies $(f2^*)$ of (15.1.20) and M^* is complete. In fact, since $F_{-n}C^* = \{0\}$ if $n \geqq 1$, we have $\lambda(M^n) \subset M^* \widehat{\otimes}_{K^*} F_{-n}C^* = \{0\}$. Since $(id_{M^*} \widehat{\otimes}_{K^*} \varepsilon)\lambda : M^* \to M^* \widehat{\otimes}_{K^*} K^*$ coincides with the map given by $x \mapsto x \widehat{\otimes} 1$ which is an isomorphism, $\lambda(M^n) = \{0\}$ implies $M^n = \{0\}$.*

Let C^* be a coalgebra with a filtration $(F_iC^*)_{i\in \boldsymbol{Z}}$. It is clear that subcomodules and quotient comodules of an unstable comodule are also unstable and that the sum and the product of unstable comodules are unstable. Hence $\mathcal{U}\mathcal{C}omod(C^*)$ is complete and cocomplete and the inclusion functor $J_{C^*} : \mathcal{U}\mathcal{C}omod(C^*) \to \mathcal{C}omod(C^*)$ preserves limits and colimits.

Proposition 15.3.24 *The inclusion functor $J_{C^*} : \mathcal{U}\mathcal{C}omod(C^*) \to \mathcal{C}omod(C^*)$ has a right adjoint.*

Proof. Let M^* be an object of $\mathcal{C}omod(C^*)$ and let us denote by $U_{C^*}(M^*)$ the set of all unstable subcomodules of M^*. Since $\{0\} \in U_{C^*}(M^*)$, $U_{C^*}(M^*)$ is not empty. If $(M_i^*)_{i\in I}$ is a family of elements of $U_{C^*}(M^*)$, the sum $\sum\limits_{i\in I} M_i^*$ is contained in $U_{C^*}(M^*)$. Hence there exists the largest unstable subcomodule M_u^* of M^*. For a homomorphism $f : M^* \to N^*$ of C^*-comodules, since the image of an unstable subcomodule of M^* is also unstable, f induces a homomorphism $f_u : M_u^* \to W_u^*$. Thus we have a functor $R_{C^*} : \mathcal{C}omod(C^*) \to \mathcal{U}\mathcal{C}omod(C^*)$ defined by $R_{C^*}(M^*) = M_u^*$ and $R_{C^*}(f) = f_u$. It is clear that $R_{C^*}J_{C^*} = id_{\mathcal{U}\mathcal{C}omod(C^*)}$. Let $\eta : id_{\mathcal{U}\mathcal{C}omod(C^*)} \to R_{C^*}J_{C^*}$ be the identity natural transformation. We denote by $\varepsilon : J_{C^*}R_{C^*} \to id_{\mathcal{C}omod(C^*)}$ the natural transformation defined from the inclusion maps $M_u^* \to M^*$. R_{C^*} is a right adjoint of J_{C^*} whose unit and counit are η and ε, respectively. □

Proposition 15.3.25 *For a right C^*-comodule M^* with structure map $\lambda : M^* \to M^* \widehat{\otimes}_{K^*} C^*$, define a subspace $\tilde{M}^*$ of M^* by $\tilde{M}^* = \sum\limits_{n\in \boldsymbol{Z}} \lambda^{-1}(M^* \widehat{\otimes}_{K^*} F_{-n}C^*) \cap M^n$. Assume that C^* and M^* are 1st countable Hausdorff spaces. If $(F_iC^*)_{i\in \boldsymbol{Z}}$ satisfies $(f6^*)$, $\tilde{M}^*$ is the largest unstable subcomodule of M^*.*

Proof. Take $x \in \lambda^{-1}(M^* \widehat{\otimes}_{K^*} F_{-n}C^*) \cap M^n$. Since $(\pi_{C^*,\,m-n+1} \widehat{\otimes}_{K^*} u_m)\delta(F_{-n}C^*) = \{0\}$ by $(f6^*)$ for $m \in \boldsymbol{Z}$ and $\lambda(x) \in M^* \widehat{\otimes}_{K^*} F_{-n}C^*$, we have the following equality.

$$(id_{M^*} \widehat{\otimes}_{K^*} \pi_{C^*,\,m-n+1} \widehat{\otimes}_{K^*} u_m)(\lambda \widehat{\otimes}_{K^*} id_{C^*})\lambda(x) = (id_{M^*} \widehat{\otimes}_{K^*} \pi_{C^*,\,m-n+1} \widehat{\otimes}_{K^*} u_m)(id_{M^*} \widehat{\otimes}_{K^*} \delta)\lambda(x) = 0$$

We can put $\lambda(x) = \sum\limits_{i\in \boldsymbol{N}} x_i \widehat{\otimes} c_i$ and $\lambda(x_i) = \sum\limits_{j\in \boldsymbol{N}} x_{ij} \widehat{\otimes} c_{ij}$ by (2.3.17) and (1.3.21). We may assume that $c_1, c_2, \dots, c_i, \dots$ and $x_{i1}, x_{i2}, \dots, x_{ij}, \dots$ are linearly independent. For $k \in \boldsymbol{Z}$, we put $I(k) = \{i \in \boldsymbol{N} \,|\, \deg x_i = k\}$ and $J_i(k) = \{j \in \boldsymbol{N} \,|\, \deg x_{ij} = k\}$. Then, the following equality holds for any $m \in \boldsymbol{Z}$.

$$\begin{aligned}(id_{M^*} \widehat{\otimes}_{K^*} \pi_{C^*,\,m-n+1} \widehat{\otimes}_{K^*} u_m)(\lambda \widehat{\otimes}_{K^*} id_{C^*})\lambda(x) &= \sum_{i\in \boldsymbol{N}} (id_{M^*} \widehat{\otimes}_{K^*} \pi_{C^*,\,m-n+1} \widehat{\otimes}_{K^*} u_m)(\lambda(x_i) \widehat{\otimes} c_i) \\ = \sum_{i\in I(n-m)} (id_{M^*} \widehat{\otimes}_{K^*} \pi_{C^*,\,m-n+1})(\lambda(x_i)) \widehat{\otimes}_{K^*} c_i &= \sum_{i\in I(n-m)} \sum_{j\in \boldsymbol{N}} x_{ij} \widehat{\otimes} \pi_{C^*,\,m-n+1}(c_{ij}) \widehat{\otimes} c_i\end{aligned}$$

Hence we have $\sum\limits_{i\in I(n-m)} \sum\limits_{j\in \boldsymbol{Z}} x_{ij} \widehat{\otimes} \pi_{C^*,\,m-n+1}(c_{ij}) \widehat{\otimes} c_i = 0$ for any $k \in \boldsymbol{Z}$. Since we assume that c_i's are linearly independent, it follows from (2.3.17) that $\sum\limits_{j\in \boldsymbol{N}} x_{ij} \widehat{\otimes} \pi_{C^*,\,m-n+1}(c_{ij}) \widehat{\otimes} c_i = 0$ holds for each $i \in I(n-m)$. It follows from (2.3.16) that we have $\sum\limits_{j\in \boldsymbol{N}} x_{ij} \widehat{\otimes} \pi_{C^*,\,m-n+1}(c_{ij}) = 0$ which implies $\pi_{C^*,\,m-n+1}(c_{ij}) = 0$ by (2.3.17). Therefore $c_{ij} \in F_{m-n}C^*$ for $i \in I(n-m)$ and $j \in \boldsymbol{N}$ and this means $x_i \in \lambda^{-1}(M^* \widehat{\otimes}_{K^*} F_{m-n}C^*) \cap M^{n-m}$. Hence $\tilde{M}^* = \sum\limits_{n\in \boldsymbol{Z}} \lambda^{-1}(M^* \widehat{\otimes}_{K^*} F_{-n}C^*) \cap M^n$ is a subcomodule of M^*. It is clear from the definition of unstable comodules that $\tilde{M}^*$ is the largest unstable subcomodule of M^*. □

Proposition 15.3.26 *Let C^* be a coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with filtration $(F_iC^*)_{i\in \boldsymbol{Z}}$ which satisfies $(f5^*)$ and $(f6^*)$ of (15.3.3) and M^* a right C^*-comodule with structure map $\lambda : M^* \to M^* \widehat{\otimes}_{K^*} C^*$. Assume that C^* and M^* are 1st countable spaces. Let $\check{M}^*$ be a subspace of M^* spanned by $\bigcup\limits_{n\in \boldsymbol{Z}} \{x \in M^n \,|\, \lambda(x) \notin M^* \widehat{\otimes}_{K^*} F_{-n}C^*\}$. Then, $\check{M}^*$ is a subcomodule of M^*.*

Let M^* be a left C^*-comodule with structure map $\lambda : M^* \to C^* \widehat{\otimes}_{K^*} M^*$ and N^* an object of $\mathcal{T}op\mathcal{M}od_{K^*}$ which is complete. Let us denote by $\tilde{\mu}_{N^*} : K^* \otimes_{K^*} N^* \to N^*$ the isomorphism induced by the left K^*-module structure of N^*. Define a map $\lambda^\sharp : C^{**} \otimes_{K^*} \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(M^*, N^*)$ to be the following composition.

$$\begin{aligned} C^{**} \otimes_{K^*} \mathcal{H}om^*(M^*, N^*) &\xrightarrow{\phi} \mathcal{H}om^*(C^* \otimes_{K^*} M^*, K^* \otimes_{K^*} N^*) \xrightarrow{\tilde{\mu}_{N^*}{}_*} \mathcal{H}om^*(C^* \otimes_{K^*} M^*, N^*) \\ &\xrightarrow{(\eta^*_{C^* \otimes_{K^*} M^*})^{-1}} \mathcal{H}om^*(C^* \widehat{\otimes}_{K^*} M^*, N^*) \xrightarrow{\lambda^*} \mathcal{H}om^*(M^*, N^*) \end{aligned}$$

Then, $\mathcal{H}om^*(M^*, N^*)$ is a left C^{**}-module.

Let $H : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(C^{**}, \mathcal{H}om^*(M^*, N^*) \otimes_{K^*} C^{**})$ be the image of the identity map of $\mathcal{H}om^*(M^*, N^*) \otimes_{K^*} C^{**}$ by the adjoint map Φ from $\mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(M^*, N^*) \otimes_{K^*} C^{**}, \mathcal{H}om^*(M^*, N^*) \otimes_{K^*} C^{**})$ to $\mathrm{Hom}^c_{K^*}(\mathcal{H}om^*(M^*, N^*), \mathcal{H}om^*(C^{**}, \mathcal{H}om^*(M^*, N^*) \otimes_{K^*} C^{**}))$. If

$$\hat{\varphi}^{C^{**}}_{\mathcal{H}om^*(M^*, N^*)} : \mathcal{H}om^*(C^{**}, K^*) \widehat{\otimes}_{K^*} \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(C^{**}, \mathcal{H}om^*(M^*, N^*))^\wedge$$

is an isomorphism, $\Lambda(\lambda^\sharp) : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(M^*, N^*) \widehat{\otimes}_{K^*} \mathcal{H}om^*(C^{**}, K^*)$ is the following composition.

$$\begin{aligned} &\mathcal{H}om^*(M^*, N^*) \xrightarrow{H} \mathcal{H}om^*(C^{**}, \mathcal{H}om^*(M^*, N^*) \otimes_{K^*} C^{**}) \xrightarrow{(T_{\mathcal{H}om^*(M^*,N^*), C^{**}})_*} \\ &\quad \mathcal{H}om^*(C^{**}, C^{**} \otimes_{K^*} \mathcal{H}om^*(M^*, N^*)) \xrightarrow{\phi_*} \mathcal{H}om^*(C^{**}, \mathcal{H}om^*(C^* \otimes_{K^*} M^*, K^* \otimes_{K^*} N^*)) \xrightarrow{(\tilde{\mu}_{N^*}{}_*)_*} \\ &\quad \mathcal{H}om^*(C^{**}, \mathcal{H}om^*(C^* \otimes_{K^*} M^*, N^*)) \xrightarrow{(\eta^*_{C^* \otimes_{K^*} M^*})^{-1}_*} \mathcal{H}om^*(C^{**}, \mathcal{H}om^*(C^* \widehat{\otimes}_{K^*} M^*, N^*)) \xrightarrow{(\lambda^*)_*} \\ &\quad \mathcal{H}om^*(C^{**}, \mathcal{H}om^*(M^*, N^*)) \xrightarrow{\eta_{\mathcal{H}om^*(C^{**}, \mathcal{H}om^*(M^*, N^*))}} \mathcal{H}om^*(C^{**}, \mathcal{H}om^*(M^*, N^*))^\wedge \xrightarrow{(\hat{\varphi}^{C^{**}}_{\mathcal{H}om^*(M^*,N^*)})^{-1}} \\ &\quad \mathcal{H}om^*(C^{**}, K^*) \widehat{\otimes}_{K^*} \mathcal{H}om^*(M^*, N^*) \xrightarrow{\widehat{T}_{\mathcal{H}om^*(C^{**},K^*), \mathcal{H}om^*(M^*,N^*)}} \mathcal{H}om^*(M^*, N^*) \widehat{\otimes}_{K^*} \mathcal{H}om^*(C^{**}, K^*) \end{aligned}$$

If C^* is finite type and has skeletal topology, $\chi_{C^*} : C^* \to \mathcal{H}om^*(C^{**}, K^*)$ is an isomorphism by (3.3.6). Then

$$(id_{\mathcal{H}om^*(M^*,N^*)} \widehat{\otimes}_{K^*} \chi_{C^*}^{-1})\Lambda(\lambda^\sharp) : \mathcal{H}om^*(M^*, N^*) \to \mathcal{H}om^*(M^*, N^*) \widehat{\otimes}_{K^*} C^*$$

gives a right C^*-comodule structure on $\mathcal{H}om^*(M^*, N^*)$.

Proposition 15.3.27 *Suppose that C^* is a coalgebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with a filtration $(F_iC^*)_{i\in\boldsymbol{Z}}$. We consider the dual filtration $(F_iC^{**})_{i\in\boldsymbol{Z}}$ of $(F_iC^*)_{i\in\boldsymbol{Z}}$ on C^{**}. Let M^* be a left C^*-comodule with structure map $\lambda : M^* \to C^* \widehat{\otimes}_{K^*} M^*$. Then M^{**} is an unstable C^{**}-module if and only if the image of $\lambda : M^* \to C^* \widehat{\otimes}_{K^*} M^*$ is contained in the kernel of $\pi_{C^*, j+1} \widehat{\otimes}_{K^*} u_j : C^* \widehat{\otimes}_{K^*} M^* \to C^*/F_jC^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(M^*)$ for each $j \in \boldsymbol{Z}$.*

Proof. Suppose that M^{**} is an unstable C^{**}-module. For $D^* \in \mathcal{V}_{C^*}$ and $U^* \in \mathcal{V}_{M^*}$, let $p : C^* \to C^*/D^*$, $q : M^* \to M^*/U^*$, $p' : C^*/D^* \to C^*/(D^* + F_jC^*)$, $p'' : C^*/F_jC^* \to C^*/(D^* + F_jC^*)$ be the quotient maps. Since the following diagram commutes, it suffices to show that the image of $(p \otimes_{K^*} q)\lambda$ is contained in the kernel of $p' \otimes_{K^*} u_j$.

$$\begin{tikzcd} M^* \arrow[r, "\lambda"] & C^* \widehat{\otimes}_{K^*} M^* \arrow[r, "\pi_{C^*, j+1} \widehat{\otimes}_{K^*} u_j"] \arrow[d, "p \otimes_{K^*} q"] & C^*/F_jC^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(M^*) \arrow[d, "p'' \otimes_{K^*} \iota_j\epsilon_j(q)"] \\ & C^*/D^* \otimes_{K^*} M^*/U^* \arrow[r, "p' \otimes_{K^*} u_j"] & C^*/(D^* + F_jC^*) \otimes_{K^*} \iota_j\epsilon_j(M^*/U^*) \end{tikzcd}$$

Assume that there exists $x \in M^n$ such that $(p \otimes_{K^*} q)\lambda(x) \notin \mathrm{Ker}(p' \otimes_{K^*} u_j)$. Put $(p \otimes_{K^*} q)\lambda(x) = \sum\limits_{k=1}^{m} c_k \otimes x_k$ such that $c_1, c_2, \dots, c_m$ are linearly independent elements of C^*/D^* and $x_1, x_2, \dots, x_m$ are non zero elements of M^*/U^*. Since $\sum\limits_{k=1}^{m} p'(c_k) \otimes_{K^*} u_j(x_k) \neq 0$, there exists k_0 such that $c_{k_0} \notin (D^* + F_jC^*)/D^*$ and $x_{k_0} \in (M^*/U^*)^j$. There exist $\alpha \in \mathcal{H}om^{j-n}(C^*/D^*, K^*)$ and $\beta \in \mathcal{H}om^{-j}(M^*/U^*, K^*)$ such that $\alpha([j-n], c_{k_0}) = 1$, $\alpha([j-n], c_k) = 0$ for $k \neq k_0$, $\alpha(\Sigma^{j-n}(D^* + F_jC^*)/D^*) = 0$ and $\beta([-j], x_{k_0}) = 1$. Since we have $\alpha\Sigma^{j-n}p \in F_{-j-1}C^{**}$ and $\beta\Sigma^{-j}q \in (M^{**})^{-j}$, $\lambda^\sharp(\alpha\Sigma^{j-n}p \otimes_{K^*} \beta\Sigma^{-j}q) = 0$ holds by the assumption. On the other hand, we have

$$\begin{aligned} (\lambda^\sharp(\alpha\Sigma^{j-n}p \otimes \beta\Sigma^{-j}q))(x) &= \tilde{\mu}_{N^*{}_*}(\alpha \otimes_{K^*} \beta)(\Sigma^{j-n}p \otimes_{K^*} \Sigma^{-j}q)(\tau^{j-n,-j}_{C^*, M^*})^{-1}\Sigma^{-n}\lambda([-n], x) \\ &= (-1)^{j(j-n)}\alpha([j-n], c_{k_0})\beta([-j], x_{k_0}) = (-1)^{j(j-n)}. \end{aligned}$$

This contradicts the assumption.

Conversely, suppose that the image of $\lambda : M^* \to C^* \widehat{\otimes}_{K^*} M^*$ is contained in the kernel of $\pi_{C^*,\,j+1} \widehat{\otimes}_{K^*} u_j$ for any $j \in \boldsymbol{Z}$. For $\alpha \in (F_{-j-1}C^{**})^i$ and $\beta \in (M^{**})^{-j}$, there exist unique $\bar{\alpha} \in \mathcal{H}om^i(C^*/F_jC^*, K^*)$ and $\bar{\beta} \in \mathcal{H}om^{-j}(\iota_j\epsilon_j(M^*), K^*)$ such that $\alpha = \pi^*_{C^*,\,j+1}(\bar{\alpha})$ and $\beta = u_j^*(\bar{\beta})$. Hence we have

$$\begin{aligned}\lambda^\sharp(\alpha \otimes_{K^*} \beta) &= \tilde{\mu}_{N^**}(\bar{\alpha} \otimes_{K^*} \bar{\beta})(\Sigma^i\pi_{C^*,\,j+1} \otimes_{K^*} \Sigma^{-j}u_j)(\tau^{j-n,-j}_{C^*,\,M^*})^{-1}\Sigma^{i-j}\lambda \\ &= \tilde{\mu}_{N^**}(\bar{\alpha} \otimes_{K^*} \bar{\beta})(\tau^{j-n,-j}_{C^*/F_jC^*,\iota_j\epsilon_j(M^*)})^{-1}\Sigma^{i-j}((\pi_{C^*,\,j+1} \widehat{\otimes}_{K^*} u_j)\lambda) = 0\end{aligned}$$

which shows that M^{**} is an unstable C^{**}-module. □

Remark 15.3.28 *Suppose that $(F_iC^*)_{i\in\boldsymbol{Z}}$ satisfies $(f5^*)$. Let $\delta_i : F_iC^* \to C^* \widehat{\otimes}_{K^*} F_iC^*$ the map obtained by restricting the domain of the coproduct δ of C^*. We regard $\Sigma^nF_iC^*$ as a left C^* comodule with comultiplication defined to be a composition $\Sigma^nF_iC^* \xrightarrow{\Sigma^n\delta_i} \Sigma^n(C^* \widehat{\otimes}_{K^*} F_iC^*) \xrightarrow{(\hat{\tau}^{0,n}_{C^*,\,F_iC^*})^{-1}} C^* \widehat{\otimes}_{K^*} \Sigma^nF_iC^*$. (15.3.27) implies that $\mathcal{H}om^*(\Sigma^nF_iC^*, K^*)$ is an unstable C^{**}-module if and only if the image of the above composition is contained in the kernel of $\pi_{C^*,\,j+1} \widehat{\otimes}_{K^*} u_j : C^* \widehat{\otimes}_{K^*} \Sigma^nF_iC^* \to C^*/F_jC^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(\Sigma^nF_iC^*)$ for any $j \in \boldsymbol{Z}$. The following diagram is commutative by (1.2.6).*

$$\begin{tikzcd}
\Sigma^n(C^* \widehat{\otimes}_{K^*} F_iC^*) \arrow[r, "(\hat{\tau}^{0,n}_{C^*,\,F_iC^*})^{-1}"] \arrow[d, "\Sigma^n(\pi_{C^*,\,j+1}\widehat{\otimes}_{K^*}u_{j-n})"'] & C^* \widehat{\otimes}_{K^*} \Sigma^nF_iC^* \arrow[r, "\pi_{C^*,\,j+1}\widehat{\otimes}_{K^*}u_j"] \arrow[dr, "\pi_{C^*,\,j+1}\widehat{\otimes}_{K^*}\Sigma^nu_{j-n}"] & C^*/F_jC^* \widehat{\otimes}_{K^*} \iota_j\epsilon_j(\Sigma^nF_iC^*) \arrow[d, equal] \\
\Sigma^n(C^*/F_jC^* \widehat{\otimes}_{K^*} \iota_{j-n}\epsilon_{j-n}(F_iC^*)) \arrow[rr, "(\hat{\tau}^{0,n}_{C^*/F_jC^*,\,\iota_{j-n}\epsilon_{j-n}(F_iC^*)})^{-1}"] & & C^*/F_jC^* \widehat{\otimes}_{K^*} \Sigma^n\iota_{j-n}\epsilon_{j-n}(F_iC^*)
\end{tikzcd}$$

Therefore $\mathcal{H}om^*(\Sigma^nF_iC^*, K^*)$ is an unstable C^{**}-module if and only if the image of δ_i is contained in the kernel of $\pi_{C^*,\,j+1} \widehat{\otimes}_{K^*} u_{j-n} : C^* \widehat{\otimes}_{K^*} F_iC^* \to C^*/F_jC^* \widehat{\otimes}_{K^*} \iota_{j-n}\epsilon_{j-n}(F_iC^*)$ for any $j \in \boldsymbol{Z}$. In particular, $\mathcal{H}om^*(\Sigma^iF_iC^*, K^*)$ is an unstable C^{**}-module if and only if the image of δ_i is contained in the kernel of $\pi_{C^*,\,j+1} \widehat{\otimes}_{K^*} u_{j-i} : C^* \widehat{\otimes}_{K^*} F_iC^* \to C^*/F_jC^* \widehat{\otimes}_{K^*} \iota_{j-i}\epsilon_{j-i}(F_iC^*)$ for any $j \in \boldsymbol{Z}$. Thus we conclude that $\mathcal{H}om^*(\Sigma^iF_iC^*, K^*)$ is an unstable C^{**}-module for any $i \in \boldsymbol{Z}$ if and only if $(F_iC^*)_{i\in\boldsymbol{Z}}$ satisfies $(f6^*)$.

Let us denote by $\mathcal{UC}omod_s(C^*)$ the full subcategory of $\mathcal{UC}omod(C^*)$ consisting of objects whose underlying topological vector spaces have the skeletal topology.

Suppose that C^* satisfies $(f5^*)$ and $(f6^*)$ of (15.3.3). For $M^* \in \operatorname{Ob}\mathcal{T}op\mathcal{M}od_{K^*}$ and $f \in \operatorname{Mor}\mathcal{T}op\mathcal{M}od_{K^*}$, we put

$$\mathcal{F}(M^*) = \sum_{n\in\boldsymbol{Z}} \mathcal{H}om^*(F_{-n}C^*, K^*) \otimes_{K^*} \iota_n\epsilon_n(M^*) \quad \text{and} \quad \mathcal{F}(f) = \sum_{n\in\boldsymbol{Z}} id_{\mathcal{H}om^*(F_{-n}C^*,K^*)} \otimes_{K^*} \iota_n\epsilon_n(f).$$

Since the coproduct $\delta : C^* \to C^* \otimes_{K^*} C^*$ induces $\delta_n : F_{-n}C^* \to C^* \otimes_{K^*} F_{-n}C^*$, each summand of $\mathcal{F}(M^*)$ is a left C^{**}-module with structure map

$$\delta_n^\sharp \otimes_{K^*} id_{\iota_n\epsilon_n(M^*)} : C^{**} \otimes_{K^*} \mathcal{H}om^*(F_{-n}C^*, K^*) \otimes_{K^*} \iota_n\epsilon_n(M^*) \to \mathcal{H}om^*(F_{-n}C^*, K^*) \otimes_{K^*} \iota_n\epsilon_n(M^*).$$

It follows from (15.3.27) that $\Sigma^n\mathcal{H}om^*(F_{-n}C^*, K^*) = \mathcal{H}om^*(\Sigma^{-n}F_{-n}C^*, K^*)$ is an unstable C^{**}-module. Hence $\Sigma^n\mathcal{H}om^*(F_{-n}C^*, K^*) \otimes_{K^*} \iota_0\epsilon_n(M^*) \cong \mathcal{H}om^*(F_{-n}C^*, K^*) \otimes_{K^*} \iota_n\epsilon_n(M^*)$ is an unstable C^{**}-module.

Assume that C^* is finite type and it has the skeletal topology. Consider the followiing map.

$$\begin{aligned}&(id_{\mathcal{H}om^*(F_{-n}C^*,K^*)} \widehat{\otimes}_{K^*} \chi^{-1}_{C^*,\,K^*})\Lambda(\delta_n^\sharp \otimes_{K^*} id_{\iota_n\epsilon_n(M^*)}) : \\ &\qquad\qquad \mathcal{H}om^*(F_{-n}C^*, K^*) \otimes_{K^*} \iota_n\epsilon_n(M^*) \to \mathcal{H}om^*(F_{-n}C^*, K^*) \otimes_{K^*} \iota_n\epsilon_n(M^*) \widehat{\otimes}_{K^*} C^*\end{aligned}$$

Then, $\mathcal{H}om^*(F_{-n}C^*, K^*) \otimes_{K^*} \iota_n\epsilon_n(M^*)$ is an unstable C^*-comodule by (15.3.21). Clearly, if f is a morphism in $\mathcal{T}op\mathcal{M}od_{K^*}$, $\mathcal{F}(f)$ is a homomorphism of C^*-comodules. Thus we have a functor $\mathcal{F} : \mathcal{T}op\mathcal{M}od_{K^*} \to \mathcal{UC}omod(C^*)$.

Proposition 15.3.29 *Assume that C^* is finite type and has the skeletal topology. If $\mathfrak{F}$ satisfies $(f5^*)$ and $(f6^*)$ of (15.3.3), $\mathcal{F} : \mathcal{T}op\mathcal{M}od_{K^*} \to \mathcal{UC}omod(C^*)$ is a left adjoint of the forgetful functor $\mathcal{O} : \mathcal{UC}omod(C^*) \to \mathcal{T}op\mathcal{M}od_{K^*}$.*

Proof. Let $\varepsilon_n \in \mathcal{H}om^0(F_{-n}C^*, K^*)$ be the restriction of the counit of C^*. For $M^* \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od_{K^*}$ define $\eta_{M^*} : M^* \to \mathcal{OF}(M^*)$ by $\eta_{M^*}(x) = \varepsilon_n \otimes_{K^*} x$ if $x \in M^n$.

□

For right C^*-comodules M^* and N^* with structure maps $\lambda : M^* \to M^* \widehat{\otimes}_{K^*} C^*$ and $\nu : N^* \to N^* \widehat{\otimes}_{K^*} C^*$, define a map $\gamma : M^* \widehat{\otimes}_{K^*} N^* \to M^* \widehat{\otimes}_{K^*} N^* \widehat{\otimes}_{K^*} C^*$ to be the following composition.

$$M^* \widehat{\otimes}_{K^*} N^* \xrightarrow{\lambda \widehat{\otimes}_{K^*} \nu} M^* \widehat{\otimes}_{K^*} C^* \widehat{\otimes}_{K^*} N^* \widehat{\otimes}_{K^*} C^* \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} \widehat{T}_{C^*,\,N^*} \widehat{\otimes}_{K^*} id_{N^*}}$$
$$M^* \widehat{\otimes}_{K^*} N^* \widehat{\otimes}_{K^*} C^* \widehat{\otimes}_{K^*} C^* \xrightarrow{id_{M^*} \widehat{\otimes}_{K^*} id_{N^*} \widehat{\otimes}_{K^*} \hat{\mu}} M^* \widehat{\otimes}_{K^*} N^* \widehat{\otimes}_{K^*} C^*$$

Proposition 15.3.30 *Let C^* be a Hopf algebra in $\mathcal{T}op\mathcal{M}od_{K^*}$ with a filtration $(F_iC^*)_{i\in \boldsymbol{Z}}$. Suppose that $(F_iC^*)_{i\in \boldsymbol{Z}}$ satisfies $(f9^*)$. If M^* and N^* are unstable C^*-comodules, so is $M^* \widehat{\otimes}_{K^*} N^*$.*

Proof. Let $\lambda : M^* \to M^* \widehat{\otimes}_{K^*} C^*$ and $\nu : N^* \to N^* \widehat{\otimes}_{K^*} C^*$ be the structure maps of right comodules. Since $\lambda(M^{-j}) \subset M^* \widehat{\otimes}_{K^*} F_jC^*$ and $\nu(N^{j-n}) \subset N^* \widehat{\otimes}_{K^*} F_{n-j}C^*$ for any $j, n \in \boldsymbol{Z}$, we have the following.

$$\begin{aligned}
\gamma(M^{-j} \widehat{\otimes}_{K^*} N^{j-n}) &= (id_{M^*} \widehat{\otimes}_{K^*} id_{N^*} \widehat{\otimes}_{K^*} \hat{\mu})(id_{M^*} \widehat{\otimes}_{K^*} \widehat{T}_{C^*,\,N^*} \widehat{\otimes}_{K^*} id_{N^*})(\lambda \widehat{\otimes}_{K^*} \nu)(M^{-j} \widehat{\otimes}_{K^*} N^{j-n}) \\
&\subset (id_{M^*} \widehat{\otimes}_{K^*} id_{N^*} \widehat{\otimes}_{K^*} \hat{\mu})(id_{M^*} \widehat{\otimes}_{K^*} \widehat{T}_{C^*,\,N^*} \widehat{\otimes}_{K^*} id_{N^*})(M^* \widehat{\otimes}_{K^*} F_jC^* \widehat{\otimes}_{K^*} N^* \widehat{\otimes}_{K^*} F_{n-j}C^*) \\
&= (id_{M^*} \widehat{\otimes}_{K^*} id_{N^*} \widehat{\otimes}_{K^*} \hat{\mu})(M^* \widehat{\otimes}_{K^*} N^* \widehat{\otimes}_{K^*} F_jC^* \widehat{\otimes}_{K^*} F_{n-j}C^*) \\
&\subset M^* \widehat{\otimes}_{K^*} N^* \widehat{\otimes}_{K^*} F_nC^*
\end{aligned}$$

Therefore $\gamma((M^* \widehat{\otimes}_{K^*} N^*)^{-n}) \subset M^* \widehat{\otimes}_{K^*} N^* \widehat{\otimes}_{K^*} F_nC^*$ holds for any $n \in \boldsymbol{Z}$. □

15.4 Examples

We denote by $\mathcal{A}_p^*$ the mod p Steenrod algebra as before. Let Seq^o be a subset of Seq consisting of sequences $(i_1, i_2, \dots, i_n, \dots)$ such that $i_k = 0, 1$ if k is odd.

Definition 15.4.1 *([18]) For $I = (\varepsilon_0, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n) \in \mathrm{Seq}^o$ and an odd prime p, we put*

$$d_p(I) = -2(p-1)\sum_{s=1}^{n} i_s - \sum_{s=0}^{n} \varepsilon_s, \quad e_p(I) = -\sum_{s=0}^{n} \varepsilon_s - 2\sum_{s=1}^{n}(i_s - pi_{s+1} - \varepsilon_s).$$

For $J = (j_1, j_2, \dots, j_n) \in \mathrm{Seq}$, we put

$$d_2(J) = |J| = -\sum_{s=1}^{n} j_s, \quad e_2(J) = -\sum_{s=1}^{n}(j_s - 2j_{s+1}).$$

Then $\wp^I = \beta^{\varepsilon_0}\wp^{i_1}\beta^{\varepsilon_1}\wp^{i_2}\beta^{\varepsilon_2}\dots\wp^{i_n}\beta^{\varepsilon_n} \in \mathcal{A}_p^{d_p(I)}$ for $I \in \mathrm{Seq}^o$ and $Sq^J = Sq^{j_1}Sq^{j_2}\dots Sq^{j_n} \in \mathcal{A}_2^{d_2(J)}$ for $J \in \mathrm{Seq}$. We call $d_p(I)$ the degree of I and $e_p(I)$ the excess of I.

Definition 15.4.2 *([18]) We say that $I = (\varepsilon_0, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n, \dots) \in \mathrm{Seq}^o$ is (p-)admissible if p is an odd prime and $i_s \geq pi_{s+1} + \varepsilon_s$ for $s = 1, 2, \dots$. For $p = 2$, we say that $I = (i_1, i_2, \dots, i_n, \dots) \in \mathrm{Seq}$ is (2-)admissible if $i_s \geq 2i_{s+1}$ for $s = 1, 2, \dots$. We denote by Seq_p the subset of Seq consisting of p-admissible sequences.*

Definition 15.4.3 *Let $F_i\mathcal{A}_p^*$ be a subspace of $\mathcal{A}_p^*$ spanned by*

$$\left\{\wp^I \,\middle|\, I \in \mathrm{Seq}_p,\ e_p(I) \leqq i\right\} \text{ if } p \neq 2, \quad \left\{Sq^I \,\middle|\, I \in \mathrm{Seq}_2,\ e_2(I) \leqq i\right\} \text{ if } p = 2.$$

Thus we have an increasing filtration $\mathfrak{F}_p = (F_i\mathcal{A}_p^*)_{i\in \boldsymbol{Z}}$ on $\mathcal{A}_p^*$. We call $\mathfrak{F}_p$ the excess filtration.

Proposition 15.4.4 *([12]) (1) $Q(E)\wp(R) \in F_{|E|+2|R|}\mathcal{A}_p^* - F_{|E|+2|R|-1}\mathcal{A}_p^*$ for $R \in \mathrm{Seq}$ and $E \in \mathrm{Seq}^b$ if p is an odd prime. $Sq(R) \in F_{|R|}\mathcal{A}_2^* - F_{|R|-1}\mathcal{A}_2^*$ for $R \in \mathrm{Seq}$.*

(2) $\{Q(E)\wp(R) \mid E \in \mathrm{Seq}^b, R \in \mathrm{Seq}, |E| + 2|R| \leqq i\}$ is a basis of $F_i\mathcal{A}_p^*$ for an odd prime p. $\{Sq(R) \mid R \in \mathrm{Seq}, |R| \leqq i\}$ is a basis of $F_i\mathcal{A}_2^*$.

Clearly, $\mathfrak{F}_p$ satisfies $(f1)$ and $(f2)$. It is shown in [22] that $\mathfrak{F}_p$ satisfies $(f5)$, $(f6)$ and $(f9)$. The following result is a direct consequence of (1.16) of [22].

Proposition 15.4.5 *Let i be a non-positive integer and $\varepsilon = 0$ or 1.*
(1) $E^j_{2i-\varepsilon}\mathcal{A}^*_p = \{0\}$ if $j > 2i(p-1) - \varepsilon$ or $2i - \varepsilon + j \not\equiv 0, -2$ modulo $2p$.
(2) If p is an odd prime, $E^{2i(p-1)-\varepsilon}_{2i-\varepsilon}\mathcal{A}^*_p$ is a one dimensional vector space spanned by $\rho_{\mathcal{A}^*_p, 2i-\varepsilon}(\beta^\varepsilon \wp^{-i})$. $E^i_i\mathcal{A}^*_2$ is a one dimensional vector space spanned by $\rho_{\mathcal{A}^*_2, i}(Sq^{-i})$.

It follows from (15.4.5) that $S(\mathfrak{F}_p)$ is the set of all non-positive integers and that $c_{\mathfrak{F}_p} : S(\mathfrak{F}_p) \to \boldsymbol{Z}$ is given by $c_{\mathfrak{F}_p}(2i-\varepsilon) = 2i(p-1) - \varepsilon$ $(\varepsilon = 0, 1)$. Hence $\mathfrak{F}_p$ satisfies $(f4)$ and $(f3)$ by (1) of (15.4.5). It is shown in (2.2) of [22] that $\mathfrak{F}_p$ satisfies $(f7)$ and $(f8)$. Thus we see the following.

Proposition 15.4.6 *The excess filtration $\mathfrak{F}_p$ of the Steenrod algebra $\mathcal{A}^*_p$ satisfies conditions $(f1) \sim (f9)$.*

For $E = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n) \in \mathrm{Seq}^b$ and $R = (r_1, r_2, \dots, r_n) \in \mathrm{Seq}$, we define monomials $\tau(E)$ and $\xi(R)$ of the dual Steenrod algebra $\mathcal{A}_{p*}$ by $\tau(E) = \tau^{\varepsilon_0}\tau^{\varepsilon_1}\cdots\tau^{\varepsilon_n}$, $\xi(R) = \xi_1^{r_1}\xi_2^{r_2}\cdots\xi_n^{r_n}$ for an odd prime p. We also define a monomial $\zeta(R)$ of $\mathcal{A}_{2*}$ by $\zeta(R) = \zeta_1^{r_1}\zeta_2^{r_2}\cdots\zeta_n^{r_n}$. We put $\|R\| = -|R| = \sum\limits_{i\geqq 1} r_i$. The dual filtration $\mathfrak{F}^*_p = (F_i\mathcal{A}_{p*})$ of $\mathfrak{F}_p$ is given as follows.

Proposition 15.4.7 *([22]) $\{\tau(E)\xi(R) \,|\, E \in \mathrm{Seq}^b,\ R \in \mathrm{Seq},\ \|E\| + 2\|R\| \leqq i\}$ is a basis of $F_i\mathcal{A}_{p*}$ if p is an odd prime and $\{\zeta(R) \,|\, R \in \mathrm{Seq},\ \|R\| \leqq i\}$ is a basis of $F_i\mathcal{A}_{2*}$.*

We call $\mathfrak{F}^*_p$ the dual excess filtration of $\mathcal{A}_{p*}$. It follows from (15.1.27) that $S(\mathfrak{F}^*_p)$ is the set of all non-negative integers and that $c_{\mathfrak{F}^*_p}(2i+\varepsilon) = -c_{\mathfrak{F}_p}(-2i-\varepsilon) = 2i(p-1)+\varepsilon$ $(\varepsilon = 0, 1)$. Hence the results (15.1.16), (15.1.22), (15.1.30), (15.1.31), (15.3.5), (15.3.7), (15.3.13) and (15.3.19) imply the following result.

Proposition 15.4.8 *The dual excess filtration $\mathfrak{F}^*_p$ of the dual Steenrod algebra $\mathcal{A}_{p*}$ satisfies conditions $(f1^*) \sim (f9^*)$.*

We denote by $\mathcal{A}^{ev}_{p*}$ the polynomial part $\boldsymbol{F}_p[\xi_1, \xi_2, \dots]$ of $\mathcal{A}_{p*}$, in other words, $\mathcal{A}^{ev}_{p*} = \mathcal{A}_{p*}/(\tau_0, \tau_1, \dots, \tau_n, \dots)$. For an odd prime p, we give a filtration $\mathfrak{F}^{ev*}_p = (F_i\mathcal{A}^{ev}_{p*})_{i\in\boldsymbol{Z}}$ by $F_i\mathcal{A}^{ev}_{p*} = \mathcal{A}^{ev}_{p*} \cap F_i\mathcal{A}_{p*}$. It follows from (15.1.15), (15.1.21), (15.3.4) and (15.3.18) that $\mathfrak{F}^{ev*}_p$ satisfies $(f1^*)$, $(f2^*)$, $(f5^*)$, $(f6^*)$ and $(f9^*)$. The following fact follows from (15.4.7).

Proposition 15.4.9 *For an odd prime p, $\{\xi(R) \,|\, R \in \mathrm{Seq},\ 2\|R\| \leqq i\}$ is a basis of $F_i\mathcal{A}^{ev}_{p*}$*

For $E = (\varepsilon_0, \varepsilon_1, \dots,) \in \mathrm{Seq}^b$ and $R = (r_1, r_2, \dots) \in \mathrm{Seq}$, we put $d^*_p(E, R) = \sum\limits_{i\geqq 0} \varepsilon_i(2p^i - 1) + \sum\limits_{i\geqq 1} 2r_i(p^i - 1)$ if p is an odd prime and $d^*_2(R) = \sum\limits_{i\geqq 1} r_i(2^i - 1)$. We denote $d^*_p(\boldsymbol{0}, R)$ by $d^*_p(R)$. Then $\tau(E)\xi(R) \in \mathcal{A}_{p\, d^*_p(E,R)}$ if p is an odd prime and $\zeta(R) \in \mathcal{A}_{2\, d^*_2(R)}$.

Proposition 15.4.10 *(1) $S(\mathfrak{F}^{ev*}_p)$ is the set of all non-negative even integers and $c^*_{\mathfrak{F}^{ev*}_p} : S(\mathfrak{F}^{ev*}_p) \to \boldsymbol{Z}$ is given by $c^*_{\mathfrak{F}^{ev*}_p}(2i) = 2i(p-1)$.*
(2) $\mathfrak{F}^{ev*}_p$ satisfies $(f3^*)$ and $(f4^*)$.

Proof. (1) By (15.4.9), $F_{2i}\mathcal{A}^{ev}_{p*} = F_{2i+1}\mathcal{A}^{ev}_{p*}$ holds, which shows $E^*_{2i+1}\mathcal{A}^{ev}_{p*} = \{0\}$ for any $i \in \boldsymbol{Z}$. Since ξ_1^i is an element of $F_{2i}\mathcal{A}^{ev}_{p*} - F_{2i-1}\mathcal{A}^{ev}_{p*}$, we have $E^{2i(p-1)}_{2i}\mathcal{A}^{ev}_{p*} \neq \{0\}$. Hence $S(\mathfrak{F}^{ev*}_p)$ is the set of all non-negative even integers. For $R = (r_1, r_2, \dots) \in \mathrm{Seq}$, since

$$d^*_p(R) = 2(p-1)\sum_{j\geqq 1} r_j(p^{j-1} + p^{j-2} + \cdots + 1) = 2(p-1)\|R\| + 2p\sum_{j\geqq 2} r_j(p^{j-1} - 1) \cdots (*)$$

holds, we have $d^*_p(R) \geqq 2(p-1)\|R\| = 2i(p-1)$ if $\xi(R) \in F_{2i}\mathcal{A}^{ev}_{p*} - F_{2i-1}\mathcal{A}^{ev}_{p*}$. It follows that $E^j_{2i}\mathcal{A}^{ev}_{p*} = \{0\}$ if $j < 2i(p-1)$. Thus $c^*_{\mathfrak{F}^{ev*}_p}(2i) = 2i(p-1)$.

(2) Since $k + c^*_{\mathfrak{F}^{ev*}_p}(k) = kp$ for $k \in S(\mathfrak{F}^{ev*}_p)$, $\mathfrak{F}^{ev*}_p$ satisfies $(f4^*)$. It follows from $(*)$ that $d^*_p(R) + 2\|R\| = 2p\|R\| + 2p\sum\limits_{i\geqq 2} r_i(p^{i-1} - 1) \equiv 0$ modulo $2p$. Therefore (15.4.9) implies $E^j_i\mathcal{A}^{ev}_{p*} = \{0\}$ if $i + j \not\equiv 0$ modulo $2p$, equivalently, $E^j_i\mathcal{A}^{ev}_{p*} = \{0\}$ if $i + j \neq k + c^*_{\mathfrak{F}^{ev*}_p}(k)$ for any $k \in S(\mathfrak{F}^{ev*}_p)$. Hence $\mathfrak{F}^{ev*}_p$ satisfies $(f3^*)$. $\square$

Lemma 15.4.11 *$E_{2i}^{2i(p-1)}\mathcal{A}_{p*}^{ev}$ is a one dimensional vector space spanned by $\rho_{\mathcal{A}_{p*}^{ev},2i}(\xi_1^i)$.*

Proof. It follows from (15.4.9) that $\{\rho_{\mathcal{A}_{p*}^{ev},2i}(\xi(R))\,|\,\|R\|=i,\ d_p^*(R)=2i(p-1)\}$ is a basis of $E_{2i}^{*}\mathcal{A}_{p*}^{ev}$. For $R=(r_1,r_2,\dots)\in\mathrm{Seq}$, it follows from $(*)$ in the proof of (15.4.10) that $d_p^*(R)=2i(p-1)+2p\sum_{j\geqq 2}r_j(p^{j-1}-1)$ if $\|R\|=i$. Hence both $\|R\|=i$ and $d_p^*(R)=2i(p-1)$ hold if and only if $r_1=i$ and $r_j=0$ for $j\geqq 2$. $\square$

We define maps $\rho:\mathrm{Seq}\to\boldsymbol{Z}$ and $\tau:\mathrm{Seq}\to\mathrm{Seq}$ as follows.

$$\rho((r_0,r_1,\dots,r_n,\dots))=r_0!r_1!\cdots r_n!\cdots,\qquad \tau((r_0,r_1,\dots,r_n,\dots))=(r_1,r_2,\dots,r_n,\dots)$$

For a positive integer k, we put $\mathrm{Seq}[k]=\{(r_1,r_2,\dots)\in\mathrm{Seq}\,|\,r_i=0\text{ if }i\geqq k+1\}$. For positive integers k and p, we also define a map $\sigma_{p,k}:\mathrm{Seq}[k+1]\to\mathrm{Seq}[k]$ by $\sigma_{p,k}((r_0,r_1,\dots,r_k))=(r_{k-1}p^{k-1},r_{k-2}p^{k-2},\dots,r_0)$.

Lemma 15.4.12 *For $R=(r_1,r_2,\dots)\in\mathrm{Seq}$, the following equalities hold.*

$$\delta(\xi(R))=\sum_{(L_1,L_2,\dots)\in\prod_{k\geqq1}\mathrm{Seq}[k+1],(\|L_1\|,\|L_2\|,\dots)=R}\frac{\rho(R)}{\rho(L_1)\rho(L_2)\cdots}\xi\Big(\sum_{k\geqq1}\sigma_{p,k}(L_k)\Big)\otimes\xi\Big(\tau\Big(\sum_{k\geqq1}L_k\Big)\Big)$$

Proof. In fact, we have the following equality.

$$\begin{aligned}\delta(\xi(R))&=\prod_{k\geqq1}\delta(\xi_k)^{r_k}=\prod_{k\geqq1}\Big(\sum_{l=0}^{k}\xi_{k-l}^{p^l}\otimes\xi_l\Big)^{r_k}\\&=\prod_{k\geqq1}\Big(\sum_{l_{k0}+l_{k1}+\cdots+l_{kk}=r_k}\frac{r_k!}{l_{k0}!l_{k1}!\cdots l_{kk}!}\xi_k^{l_{k0}}\xi_{k-1}^{l_{k1}p}\cdots\xi_1^{l_{k\,k-1}p^{k-1}}\otimes\xi_1^{l_{k1}}\xi_2^{l_{k2}}\cdots\xi_k^{l_{kk}}\Big)\\&=\prod_{k\geqq1}\Big(\sum_{L_k\in\mathrm{Seq}[k+1],\|L_k\|=r_k}\frac{r_k!}{\rho(L_k)}\xi(\sigma_{p,k}(L_k))\otimes\xi(\tau(L_k))\Big)\\&=\sum_{(L_1,L_2,\dots)\in\prod_{k\geqq1}\mathrm{Seq}[k+1],(\|L_1\|,\|L_2\|,\dots)=R}\frac{\rho(R)}{\rho(L_1)\rho(L_2)\cdots}\xi\Big(\sum_{k\geqq1}\sigma_{p,k}(L_k)\Big)\otimes\xi\Big(\tau\Big(\sum_{k\geqq1}L_k\Big)\Big)\end{aligned}$$ $\square$

Proposition 15.4.13 *$\tilde{\delta}_{i,j}^{c_{\mathfrak{F}_p^{ev*}}(i)+j}:E_{i-j}^{c^*_{\mathfrak{F}_p^{ev*}}(i)+j}\mathcal{A}_{p*}^{ev}\to(E_i^*\mathcal{A}_{p*}^{ev}\otimes_{K^*}\iota_j\epsilon_j(F_{i-j}\mathcal{A}_{p*}^{ev}))^{c^*_{\mathfrak{F}_p^{ev*}}(i)+j}$ is an isomorphism for any $i\in S(\mathfrak{F}_p^{ev*})$, $j\in\boldsymbol{Z}$. Hence $\mathfrak{F}_p^{ev*}$ satisfies $(f7^*)$ and $(f8^*)$.*

Proof. Suppose that $R=(r_1,r_2,\dots)\in\mathrm{Seq}$ satisfies $\|R\|=i-j$ and $d_p^*(R)=2i(p-1)+2j$, namely $\sum_{k\geqq1}r_k=i-j$ and $\sum_{k\geqq1}2r_k(p^k-1)=2i(p-1)+2j$. Then, we have $i=\sum_{k\geqq1}r_kp^{k-1}$ and $j=\sum_{k\geqq2}r_k(p^{k-1}-1)$. Assume that, for $k\geqq1$, $L_k=(l_{k0},l_{k1},\dots,l_{kk})\in\mathrm{Seq}[k+1]$ satisfy $(\|L_1\|,\|L_2\|,\dots)=R$, $\Big\|\sum_{k\geqq1}\sigma_{p,k}(L_k)\Big\|\geqq i$ and $d_p^*\Big(\tau\Big(\sum_{k\geqq1}L_k\Big)\Big)=2j$. It follows from the calculation below, we have $l_{kk}=0$ for any $k\geqq1$.

$$\begin{aligned}2\|R\|=2i-2j\leqq 2\Big\|\sum_{k\geqq1}\sigma_{p,k}(L_k)\Big\|-d_p^*\Big(\tau\Big(\sum_{k\geqq1}L_k\Big)\Big)&=\sum_{k\geqq1}\sum_{s=0}^{k-1}2l_{ks}p^s-\sum_{k\geqq1}\sum_{s=1}^{k}2l_{ks}(p^s-1)\\&=2\sum_{k\geqq1}\Big(\sum_{s=0}^{k-1}l_{ks}p^s-\sum_{s=0}^{k}l_{ks}(p^s-1)\Big)=2\sum_{k\geqq1}\Big(\sum_{s=0}^{k}l_{ks}-l_{kk}p^k\Big)=2\|R\|-2\sum_{k\geqq1}l_{kk}p^k\end{aligned}$$

Hence $d_p^*\Big(\tau\Big(\sum_{k\geqq1}L_k\Big)\Big)=\sum_{k\geqq2}\sum_{s=0}^{k-1}2l_{ks}(p^s-1)$ holds. On the other hand, since

$$d_p^*\Big(\tau\Big(\sum_{k\geqq1}L_k\Big)\Big)=2j=\sum_{k\geqq2}2r_k(p^{k-1}-1)=\sum_{k\geqq2}2\|L_k\|(p^{k-1}-1)=\sum_{k\geqq2}\sum_{s=0}^{k-1}2l_{ks}(p^{k-1}-1),$$

it follows that we have $\sum\limits_{k\geqq 2}\sum\limits_{s=0}^{k-1}2l_{ks}(p^{k-1}-p^s)=0$ which implies that $l_{ks}=0$ for $k\geqq 2$ and $s=0,1,\dots,k-2$ and that $r_k=l_{k\,k-1}$ for $k\geqq 1$, in other words $L_k=r_kE_k$. We note that $\sum\limits_{k\geqq 1}\sigma_{p,k}(r_kE_k)=\Big(\sum\limits_{k\geqq 1}r_kp^{k-1}\Big)E_1=iE_1$ and $\sum\limits_{k\geqq 1}r_kE_k=R$ hold. We have the following equality for $R=(r_1,r_2,\dots)\in\mathrm{Seq}$ which satisfies $\|R\|=i-j$ and $d_p^*(R)=2i(p-1)+2j$ by (15.4.12) and the above argument.

$$\begin{aligned}(\pi_{\mathcal{A}^{ev}_{p*},2i}\otimes_{\boldsymbol{F}_p}u_{2j})\delta(\xi(R))&=\pi_{\mathcal{A}^{ev}_{p*},2i}\left(\xi\left(\sum_{k\geqq 1}\sigma_{p,k}(r_kE_k)\right)\right)\otimes u_{2j}\left(\xi\left(\tau\left(\sum_{k\geqq 1}r_kE_k\right)\right)\right)\\&=\pi_{\mathcal{A}^{ev}_{p*},2i}(\xi_1^i)\otimes\xi(\tau(R))\end{aligned}\qquad(*)$$

Suppose that $R=(r_1,r_2,\dots)\in\mathrm{Seq}$ satisfies $\|R\|=i-j$ and $d_p^*(R)=2i(p-1)+2j$. Then we have $\|\tau(R)\|=\|R\|-r_1=i-j-r_1\leqq i-j$ and $d_p^*(\tau(R))=\sum\limits_{k\geqq 2}2r_k(p^{k-1}-1)=2j$. Hence τ maps a subset $\{R\in\mathrm{Seq}\,|\,\|R\|=i-j,\,d_p^*(R)=2i(p-1)+2j\}$ of Seq maps into $\{R\in\mathrm{Seq}\,|\,\|R\|\leqq i-j,\,d_p^*(R)=2j\}$. It is clear that, if $R=(r_1,r_2,\dots),S=(s_1,s_2,\dots)\in\mathrm{Seq}$ satisfy $\tau(R)=\tau(S)$ and $\|R\|=\|S\|=i-j$, then $R=S$. For $T=(t_1,t_2,\dots)\in\{R\in\mathrm{Seq}\,|\,\|R\|\leqq i-j,\,d_p^*(R)=2j\}$, put $R=(i-j-\|T\|,t_1,t_2,\dots)$. Then, $\tau(R)=T$ and $\|R\|=i-j$. Since $2j=\sum\limits_{k\geqq 1}2t_k(p^k-1)$, we have the following equality.

$$\begin{aligned}d_p^*(R)-(2i(p-1)+2j)&=2(i-j-\|T\|)(p-1)+\sum_{k\geqq 1}2t_k(p^{k+1}-1)-(2i(p-1)+2j)\\&=\sum_{k\geqq 1}2t_k(p^{k+1}-1)-\sum_{k\geqq 1}2t_k(p^{k+1}-p)-\sum_{k\geqq 1}2t_k(p-1)=0\end{aligned}$$

Therefore τ maps a subset $\{R\in\mathrm{Seq}\,|\,\|R\|=i-j,\,d_p^*(R)=2i(p-1)+2j\}$ of Seq maps bijectively onto $\{R\in\mathrm{Seq}\,|\,\|R\|\leqq i-j,\,d_p^*(R)=2j\}$. Since $\{\rho_{\mathcal{A}^{ev}_{p*},2i-2j}(\xi(R))\,|\,R\in\mathrm{Seq},\,\|R\|=i-j,\,d_p^*(R)=2i(p-1)+2j\}$ is a basis of $E_{2i-2j}^{2i(p-1)+2j}\mathcal{A}^{ev}_{p*}$ and $\{\xi(R)\,|\,R\in\mathrm{Seq},\,\|R\|\leqq i-j,\,d_p^*(R)=2j\}$ is a basis of $(F_{2i-2j}\mathcal{A}^{ev}_{p*})^{2j}$ by (15.4.9), it follows from (15.4.11) and $(*)$ that $\tilde{\delta}_{2i,2j}^{2i(p-1)+2j}:E_{2i-2j}^{2i(p-1)+2j}\mathcal{A}^{ev}_{p*}\to E_{2i}^{2i(p-1)}\mathcal{A}^{ev}_{p*}\otimes_{K^*}(F_{2i-2j}\mathcal{A}^{ev}_{p*})^{2j}$ is an isomorphism. We note that $E_{i-j}^{c^*_{\mathfrak{F}^{ev*}_p}(i)+j}\mathcal{A}^{ev}_{p*}=\iota_j\epsilon_j(F_{i-j}\mathcal{A}^{ev}_{p*})=\{0\}$ if j is odd. $\square$

Let $I_{2,n}$ be an ideal of $\mathcal{A}_{2*}$ generated by $\zeta_1^{2^n},\zeta_2^{2^{n-1}},\dots,\zeta_n^2$ and ζ_i for $i\geqq n+1$. For an odd prime p, let $I_{p,n}$ be an ideal of $\mathcal{A}_{p*}$ generated by $\xi_1^{p^n},\xi_2^{p^{n-1}},\dots,\xi_n^p$ and τ_i, ξ_i for $i\geqq n+1$ and $I^{ev}_{p,n}$ an ideal of $\mathcal{A}^{ev}_{p*}$ generated by $\xi_1^{p^n},\xi_2^{p^{n-1}},\dots,\xi_n^p$ and ξ_i for $i\geqq n+1$. We put

$$\begin{aligned}\mathcal{A}_2(n)_*&=\mathcal{A}_{2*}/I_{2,n}=\boldsymbol{F}_2[\zeta_1,\zeta_2,\dots,\zeta_n]/(\zeta_1^{2^n},\zeta_2^{2^{n-1}},\dots,\zeta_n^2)\\\mathcal{A}_p(n)_*&=\mathcal{A}_{p*}/I_{p,n}=E(\tau_0,\tau_1,\dots,\tau_n)\otimes_{\boldsymbol{F}_p}\boldsymbol{F}_p[\xi_1,\xi_2,\dots,\xi_n]/(\xi_1^{p^n},\xi_2^{p^{n-1}},\dots,\xi_n^p)\\\mathcal{A}^{ev}_p(n)_*&=\mathcal{A}^{ev}_{p*}/I^{ev}_{p,n}=\boldsymbol{F}_p[\xi_1,\xi_2,\dots,\xi_n]/(\xi_1^{p^n},\xi_2^{p^{n-1}},\dots,\xi_n^p).\end{aligned}$$

We give $\mathcal{A}_p(n)_*$ the quotient filtration $\mathfrak{F}^*_{p,n}=(F_i\mathcal{A}_p(n)_*)_{i\in\boldsymbol{Z}}$ of $\mathfrak{F}^*_p=(F_i\mathcal{A}_{p*})_{i\in\boldsymbol{Z}}$ and give $\mathcal{A}^{ev}_p(n)_*$ the quotient filtration $\mathfrak{F}^{ev*}_{p,n}=(F_i\mathcal{A}^{ev}_p(n)_*)_{i\in\boldsymbol{Z}}$ of $\mathfrak{F}^{ev*}_p=(F_i\mathcal{A}^{ev}_{p*})_{i\in\boldsymbol{Z}}$. It follows from (15.1.15), (15.1.21), (15.3.4) and (15.3.18) that $\mathfrak{F}^*_{p,n}$ and $\mathfrak{F}^{ev*}_{p,n}$ satisfy $(f1^*)$, $(f2^*)$, $(f5^*)$, $(f6^*)$ and $(f9^*)$.

We define a relation $\leqq$ of Seq by

"$(r_1,r_2,\dots)\leqq(s_1,s_2,\dots)$ if and only if $r_i\leqq s_i$ for all $i=1,2,\dots$".

We put $N_{p,n}=(p^n-1,p^{n-1}-1,\dots,p-1,0,0,\dots)$ and $B_n=(\overbrace{1,1,\dots,1}^{n+1},0,0,\dots)$. The following result is clear from (15.4.7).

Proposition 15.4.14 *For an odd prime p, $\{\tau(E)\xi(R)\,|\,E\in\mathrm{Seq}^b,R\in\mathrm{Seq},\|E\|+2\|R\|\leqq i,E\leqq B_n,R\leqq N_{p,n}\}$ is a basis of $F_i\mathcal{A}_p(n)_*$ and $\{\xi(R)\,|\,R\in\mathrm{Seq},2\|R\|\leqq i,R\leqq N_{p,n}\}$ is a basis of $F_i\mathcal{A}^{ev}_p(n)_*$. Similarly, a basis of $F_i\mathcal{A}_2(n)_*$ is given by $\{\zeta(R)\,|\,R\in\mathrm{Seq},\,\|R\|\leqq i,\,R\leqq N_{2,n}\}$.*

Put $\mathcal{A}_p(n;k)_* = E(\tau_k) \otimes_{\boldsymbol{F}_p} \boldsymbol{F}_p[\xi_{k+1}]/(\xi_{k+1}^{p^{n-k}})$, $\mathcal{A}_p^{ev}(n;k)_* = \boldsymbol{F}_p[\xi_{k+1}]/(\xi_{k+1}^{p^{n-k}})$ for $k = 0, 1, \dots, n-1$ and $\mathcal{A}_p(n;n)_* = E(\tau_n)$ if p is an odd prime. We also put $\mathcal{A}_2(n;k)_* = \boldsymbol{F}_2[\zeta_{k+1}]/(\zeta_{k+1}^{2^{n-k}})$ for $k = 0, 1, \dots, n-1$. Then,

$$\begin{aligned}
\mathcal{A}_p(n)_* &= \mathcal{A}_p(n;0)_* \otimes_{\boldsymbol{F}_p} \mathcal{A}_p(n;1)_* \otimes_{\boldsymbol{F}_p} \cdots \otimes_{\boldsymbol{F}_p} \mathcal{A}_p(n;n-1)_* \otimes_{\boldsymbol{F}_p} \mathcal{A}_p(n;n)_* \\
\mathcal{A}_p^{ev}(n)_* &= \mathcal{A}_p^{ev}(n;0)_* \otimes_{\boldsymbol{F}_p} \mathcal{A}_p^{ev}(n;1)_* \otimes_{\boldsymbol{F}_p} \cdots \otimes_{\boldsymbol{F}_p} \mathcal{A}_p^{ev}(n;n-1)_* \\
\mathcal{A}_2(n)_* &= \mathcal{A}_2(n;0)_* \otimes_{\boldsymbol{F}_p} \mathcal{A}_2(n;1)_* \otimes_{\boldsymbol{F}_p} \cdots \otimes_{\boldsymbol{F}_p} \mathcal{A}_2(n;n-1)_*
\end{aligned}$$

Let $F_i\mathcal{A}_p(n;k)_*$ be the subspace of $\mathcal{A}_p(n;k)_*$ spanned by $\{\tau_k^{\varepsilon}\xi_{k+1}^r \,|\, \varepsilon + 2r \leqq i\}$ if $k = 0, 1, \dots, n-1$ and $F_i\mathcal{A}_p(n;n)_*$ the subspace of $\mathcal{A}_p(n;n)_*$ spanned by $\{\tau_n^{\varepsilon} \,|\, \varepsilon \leqq i\}$, which defines a filtration $\mathfrak{F}^*_{p,n,k} = (F_i\mathcal{A}_p(n;k)_*)_{i\in\boldsymbol{Z}}$ of $F_i\mathcal{A}_p(n;k)_*$. It is clear that $E^*_{\varepsilon+2r}\mathcal{A}_p(n;k)_*$ is one dimensional vector space spanned by the class of $\tau_k^{\varepsilon}\xi_{k+1}^r$ if $k = 0, 1, \dots, n-1$, $\varepsilon = 0, 1$ and $r = 0, 1, \dots, p^{n-k}-1$.

For $k = 0, 1, \dots, n-1$, let $F_i\mathcal{A}_p^{ev}(n;k)_*$ and $F_i\mathcal{A}_2(n;k)_*$ be the subspaces of $\mathcal{A}_p^{ev}(n;k)_*$ and $\mathcal{A}_2(n;k)_*$ spanned by $\{\xi_{k+1}^r \,|\, 2r \leqq i\}$ and $\{\zeta_{k+1}^r \,|\, r \leqq i\}$, respectively. Thus we have a filtration $\mathfrak{F}^{ev*}_{p,n,k} = (F_i\mathcal{A}_p^{ev}(n;k)_*)_{i\in\boldsymbol{Z}}$ of $F_i\mathcal{A}_p^{ev}(n;k)_*$ and a filtration $\mathfrak{F}^*_{2,n,k} = (F_i\mathcal{A}_2(n;k)_*)_{i\in\boldsymbol{Z}}$ of $F_i\mathcal{A}_2(n;k)_*$. It is clear that $E^*_{2r}\mathcal{A}_p^{ev}(n;k)_*$ is one dimensional vector space spanned by the class of ξ_{k+1}^r for $r = 0, 1, \dots, p^{n-k}-1$ and that $E^*_r\mathcal{A}_2(n;k)_*$ is one dimensional vector space spanned by the class of ζ_{k+1}^r for $r = 0, 1, \dots, 2^{n-k}-1$.

Lemma 15.4.15 *The following assertions hold.*

(1) $S(\mathfrak{F}^*_{p,n,k}) = \{0, 1, 2, \dots, 2p^{n-k}-1\}$, $S(\mathfrak{F}^{ev*}_{p,n,k}) = \{0, 2, 4, \dots, 2(p^{n-k}-1)\}$, $S(\mathfrak{F}^*_{2,n,k}) = \{0, 1, 2, \dots, 2^{n-k}-1\}$.

(2) $c^*_{\mathfrak{F}_{p,n,k}} : S(\mathfrak{F}^*_{p,n,k}) \to \boldsymbol{Z}$ *is given by* $c^*_{\mathfrak{F}^*_{p,n,k}}(2i) = 2i(p^{k+1}-1)$, $c^*_{\mathfrak{F}^*_{p,n,k}}(2i+1) = 2i(p^{k+1}-1) + 2p^k - 1$ *if* $k = 0, 1, \dots, n-1$ *and* $c^*_{\mathfrak{F}^*_{p,n,n}}(0) = 0$, $c^*_{\mathfrak{F}^*_{p,n,n}}(1) = 2p^n - 1$. $c^*_{\mathfrak{F}^{ev*}_{p,n,k}} : S(\mathfrak{F}^{ev*}_{p,n,k}) \to \boldsymbol{Z}$ *and* $c^*_{\mathfrak{F}^*_{2,n,k}} : S(\mathfrak{F}^*_{2,n,k}) \to \boldsymbol{Z}$ *are given by* $c^*_{\mathfrak{F}^{ev*}_{p,n,k}}(2i) = 2i(p^{k+1}-1)$ *and* $c^*_{\mathfrak{F}^*_{2,n,k}}(i) = i(2^{k+1}-1)$, *respectively.*

Remark 15.4.16 $c^*_{\mathfrak{F}^*_{p,n,k}} : S(\mathfrak{F}^*_{p,n,k}) \to \boldsymbol{Z}$ *is also given by* $c^*_{\mathfrak{F}^*_{p,n,k}}(i) = i(p^{k+1}-1) - \dfrac{1-(-1)^i}{2}p^k(p-2)$ *for* $k = 0, 1, \dots, n-1$.

Proposition 15.4.17 *Let p be an odd prime. The following assertiond hold.*

(1) We have $S(\mathfrak{F}^*_{p,n}) = \{0, 1, 2, \dots, 2(p + \cdots + p^n) - n + 1\}$, $S(\mathfrak{F}^{ev*}_{p,n}) = \{0, 2, 4, \dots, 2(p + p^2 + \cdots + p^n - n)\}$ *and* $S(\mathfrak{F}^*_{2,n}) = \{0, 1, 2, \dots, 2^{n+1} - n - 2\}$.

(2) $c^*_{\mathfrak{F}^*_{p,n}} : S(\mathfrak{F}^*_{p,n}) \to \boldsymbol{Z}$, $c^*_{\mathfrak{F}^{ev*}_{p,n}} : S(\mathfrak{F}^{ev*}_{p,n}) \to \boldsymbol{Z}$ *and* $c^*_{\mathfrak{F}^*_{2,n}} : S(\mathfrak{F}^*_{2,n}) \to \boldsymbol{Z}$ *are given as follows.*

$$\begin{aligned}
c^*_{\mathfrak{F}^*_{p,n}}(i) &= \sum_{k=0}^{s-1} c^*_{\mathfrak{F}^*_{p,n,k}}(2p^{n-k}-1) + c^*_{\mathfrak{F}^*_{p,n,s}}\left(i - \sum_{k=0}^{s-1}(2p^{n-k}-1)\right) \quad \textit{if} \quad \sum_{k=0}^{s-1}(2p^{n-k}-1) \leqq i \leqq \sum_{k=0}^{s}(2p^{n-k}-1) \\
c^*_{\mathfrak{F}^{ev*}_{p,n}}(i) &= \sum_{k=0}^{s-1} c^*_{\mathfrak{F}^{ev*}_{p,n,k}}(2(p^{n-k}-1)) + c^*_{\mathfrak{F}^{ev*}_{p,n,s}}\left(i - \sum_{k=0}^{s-1}2(p^{n-k}-1)\right) \quad \textit{if} \quad \sum_{k=0}^{s-1}2(p^{n-k}-1) \leqq i \leqq \sum_{k=0}^{s}2(p^{n-k}-1) \\
c^*_{\mathfrak{F}^*_{2,n}}(i) &= \sum_{k=0}^{s-1} c^*_{\mathfrak{F}^*_{2,n,k}}(2^{n-k}-1) + c^*_{\mathfrak{F}^*_{2,n,s}}\left(i - \sum_{k=0}^{s-1}(2^{n-k}-1)\right) \quad \textit{if} \quad \sum_{k=0}^{s-1}(2^{n-k}-1) \leqq i \leqq \sum_{k=0}^{s}(2^{n-k}-1)
\end{aligned}$$

Proof. (1) By (15.4.14), the following equalities hold.

$$\begin{aligned}
F_i\mathcal{A}_p(n)_* &= \sum_{j_0+j_1+\cdots+j_n=i} F_{j_0}\mathcal{A}_p(n;0)_* \otimes_{\boldsymbol{F}_p} F_{j_1}\mathcal{A}_p(n;1)_* \otimes_{\boldsymbol{F}_p} \cdots \otimes_{\boldsymbol{F}_p} F_{j_{n-1}}\mathcal{A}_p(n;n-1)_* \otimes_{\boldsymbol{F}_p} F_{j_n}\mathcal{A}_p(n;n)_* \\
F_i\mathcal{A}_p^{ev}(n)_* &= \sum_{j_0+j_1+\cdots+j_{n-1}=i} F_{j_0}\mathcal{A}_p^{ev}(n;0)_* \otimes_{\boldsymbol{F}_p} F_{j_1}\mathcal{A}_p^{ev}(n;1)_* \otimes_{\boldsymbol{F}_p} \cdots \otimes_{\boldsymbol{F}_p} F_{j_{n-1}}\mathcal{A}_p^{ev}(n;n-1)_* \\
F_i\mathcal{A}_2(n)_* &= \sum_{j_0+j_1+\cdots+j_{n-1}=i} F_{j_0}\mathcal{A}_2(n;0)_* \otimes_{\boldsymbol{F}_p} F_{j_1}\mathcal{A}_2(n;1)_* \otimes_{\boldsymbol{F}_p} \cdots \otimes_{\boldsymbol{F}_p} F_{j_{n-1}}\mathcal{A}_2(n;n-1)_*
\end{aligned}$$

Hence we have the following equalities by (15.1.32) and (15.4.15).

$$S(\mathfrak{F}^*_{p,n}) = \left\{ i \in \boldsymbol{Z} \,\middle|\, i = \sum_{k=0}^{n} j_k \text{ for } j_k \in S(\mathfrak{F}^*_{p,n,k}) \ (k = 0,1,2,\ldots,n) \right\} = \{0,1,2,\ldots,2(p+p^2+\cdots+p^n)-n+1\}$$

$$S(\mathfrak{F}^{ev*}_{p,n}) = \left\{ i \in \boldsymbol{Z} \,\middle|\, i = \sum_{k=0}^{n-1} j_k \text{ for } j_k \in S(\mathfrak{F}^{ev*}_{p,n,k}) \ (k = 0,1,2,\ldots,n-1) \right\} = \{0,2,4,\ldots,2(p+p^2+\cdots+p^n-n)\}$$

$$S(\mathfrak{F}^*_{2,n}) = \left\{ i \in \boldsymbol{Z} \,\middle|\, i = \sum_{k=0}^{n-1} j_k \text{ for } j_k \in S(\mathfrak{F}^*_{2,n,k}) \ (k = 0,1,2,\ldots,n-1) \right\} = \{0,1,2,\ldots,2^{n+1}-n-2\}$$

(2) We have the following equalities by (15.1.32).

$$c^*_{\mathfrak{F}^*_{p,n}}(i) = \min\left\{ m \in \boldsymbol{Z} \,\middle|\, m = \sum_{k=0}^{n} c^*_{\mathfrak{F}^*_{p,n,k}}(j_k) \text{ for } j_k \in S(\mathfrak{F}^*_{p,n,k}) \ (k=0,1,2,\ldots,n) \text{ satisfying } \sum_{k=0}^{n} j_k = i \right\}$$

$$c^*_{\mathfrak{F}^{ev*}_{p,n}}(i) = \min\left\{ m \in \boldsymbol{Z} \,\middle|\, m = \sum_{k=0}^{n-1} c^*_{\mathfrak{F}^{ev*}_{p,n,k}}(j_k) \text{ for } j_k \in S(\mathfrak{F}^{ev*}_{p,n,k}) \ (k=0,1,2,\ldots,n-1) \text{ satisfying } \sum_{k=0}^{n-1} j_k = i \right\}$$

$$c^*_{\mathfrak{F}^*_{2,n}}(i) = \min\left\{ m \in \boldsymbol{Z} \,\middle|\, m = \sum_{k=0}^{n-1} c^*_{\mathfrak{F}^*_{2,n,k}}(j_k) \text{ for } j_k \in S(\mathfrak{F}^*_{2,n,k}) \ (k=0,1,2,\ldots,n-1) \text{ satisfying } \sum_{k=0}^{n-1} j_k = i \right\}$$

Suppose that $j_k, l_k \in S(\mathfrak{F}^*_{p,n,k})$ $(k = 0,1,\ldots,n)$ satisfy $\sum\limits_{k=0}^{n} j_k = \sum\limits_{k=0}^{n} l_k = i$ and $j_k = l_k$ if $k \neq a,b$ for some $0 \leqq a < b \leqq n$ and $j_a > l_a$. Then, $l_b - j_b = j_a - l_a$ and we have the following relations by (15.4.15).

$$\begin{aligned}
\sum_{k=0}^{n} c^*_{\mathfrak{F}^*_{p,n,k}}(l_k) - \sum_{k=0}^{n} c^*_{\mathfrak{F}^*_{p,n,k}}(j_k) &= c^*_{\mathfrak{F}^*_{p,n,a}}(l_a) + c^*_{\mathfrak{F}^*_{p,n,b}}(l_b) - c^*_{\mathfrak{F}^*_{p,n,a}}(j_a) - c^*_{\mathfrak{F}^*_{p,n,b}}(j_b) \\
&= (l_b - j_b)(p^{b+1}-1) + (l_a - j_a)(p^{a+1}-1) - \frac{1}{2}(-1)^{j_a}(1-(-1)^{l_a-j_a})p^a(p-2) \\
&\quad - \frac{1}{2}(-1)^{j_b}(1-(-1)^{l_b-j_b})p^b(p-2) = \\
&= (j_a - l_a)(p^{b+1} - p^{a+1}) - \frac{1}{2}(1-(-1)^{j_a-l_a})(p-2)((-1)^{j_a}p^a + (-1)^{j_b}p^b) \\
&\geqq p^{b+1} - p^{a+1} - (p-2)(p^a+p^b) = 2(p^b - p^a(p-1)) > 2(p-1)(p^{b-1}-p^a) \geqq 0
\end{aligned}$$

Similarly, if $j_k, l_k \in S(\mathfrak{F}^{ev*}_{p,n,k})$ or $j_k, l_k \in S(\mathfrak{F}^*_{2,n,k})$ $(k = 0,1,\ldots,n-1)$ satisfy $\sum\limits_{k=0}^{n-1} j_k = \sum\limits_{k=0}^{n-1} l_k = i$ and $j_k = l_k$ if $k \neq a,b$ for some $0 \leqq a < b \leqq n$ and $j_a > l_a$, the following relations hold.

$$\begin{aligned}
\sum_{k=0}^{n-1} c^*_{\mathfrak{F}^{ev*}_{p,n,k}}(l_k) - \sum_{k=0}^{n-1} c^*_{\mathfrak{F}^{ev*}_{p,n,k}}(j_k) &= c^*_{\mathfrak{F}^{ev*}_{p,n,a}}(l_a) + c^*_{\mathfrak{F}^{ev*}_{p,n,b}}(l_b) - c^*_{\mathfrak{F}^{ev*}_{p,n,a}}(j_a) - c^*_{\mathfrak{F}^{ev*}_{p,n,b}}(j_b) \\
&= (l_b - j_b)(p^{b+1}-1) + (l_a - j_a)(p^{a+1}-1) = (j_a - l_a)(p^{b+1} - p^{a+1}) > 0 \\
\sum_{k=0}^{n-1} c^*_{\mathfrak{F}^*_{2,n,k}}(l_k) - \sum_{k=0}^{n-1} c^*_{\mathfrak{F}^*_{2,n,k}}(j_k) &= c^*_{\mathfrak{F}^*_{2,n,a}}(l_a) + c^*_{\mathfrak{F}^*_{2,n,b}}(l_b) - c^*_{\mathfrak{F}^*_{2,n,a}}(j_a) - c^*_{\mathfrak{F}^*_{2,n,b}}(j_b) \\
&= (l_b - j_b)(2^{b+1}-1) + (l_a - j_a)(2^{a+1}-1) = (j_a - l_a)(2^{b+1} - 2^{a+1}) > 0
\end{aligned}$$

Assume that $i \in S(\mathfrak{F}^*_{p,n})$ satisfies $\sum\limits_{k=0}^{s-1}(2p^{n-k}-1) \leqq i \leqq \sum\limits_{k=0}^{s}(2p^{n-k}-1)$ for some $0 \leqq s \leqq n$ and that $(l_0, l_1, \ldots, l_n) \in \prod\limits_{k=0}^{n} S(\mathfrak{F}^*_{p,n,k})$ satisfies $\sum\limits_{k=0}^{n} l_k = i$ and $c^*_{\mathfrak{F}^*_{p,n}}(i) = \sum\limits_{k=0}^{n} c^*_{\mathfrak{F}^*_{p,n,k}}(l_k)$. If $l_b > 0$ for some $b \geqq s+1$, then $l_a < 2p^{n-a}-1$ for some $0 \leqq a \leqq s-1$ or $l_a < i - \sum\limits_{k=0}^{s-1}(2p^{n-k}-1)$ for $a = s$. We put $j_a = l_a + 1$, $j_b = l_b - 1$ and $j_k = l_k$ for $k \neq a,b$. It follows from the above argument that $\sum\limits_{k=0}^{n} c^*_{\mathfrak{F}^*_{p,n,k}}(l_k) > \sum\limits_{k=0}^{n} c^*_{\mathfrak{F}^*_{p,n,k}}(j_k)$ holds, which contradicts $c^*_{\mathfrak{F}^*_{p,n}}(i) = \sum_{k=0}^{n} c^*_{\mathfrak{F}^*_{p,n,k}}(l_k)$. Hence $l_k = 0$ for $s+1 \leqq k \leqq n$. Similarly, if $l_s > i - \sum_{k=0}^{s-1}(2p^{n-k}-1)$, then $l_a < 2p^{n-a}-1$ for some $0 \leqq a \leqq s-1$. We put $j_a = l_a + 1$, $j_s = l_s - 1$ and $j_k = l_k$ for $k \neq a, s$. Then, $\sum_{k=0}^{n} c^*_{\mathfrak{F}^*_{p,n,k}}(l_k) > \sum_{k=0}^{n} c^*_{\mathfrak{F}^*_{p,n,k}}(j_k)$ holds and we have $l_s \leqq i - \sum_{k=0}^{s-1}(2p^{n-k}-1)$. Since $l_k \leqq 2p^{n-k}-1$ for $0 \leqq k \leqq s-1$ and $l_k = 0$ for $s+1 \leqq k \leqq n$, we have $\sum_{k=0}^{n} l_k < i$ if $l_k < 2p^{n-k}-1$ for some $0 \leqq k \leqq s-1$ or $l_s < i - \sum_{k=0}^{s-1}(2p^{n-k}-1)$. Therefore $l_k = 2p^{n-k}-1$ for $0 \leqq k \leqq s-1$, $l_s = i - \sum_{k=0}^{s-1}(2p^{n-k}-1)$ and $l_k = 0$ for $s+1 \leqq k \leqq n$. We can show $c^*_{\mathfrak{F}^{ev*}_{p,n}}(i) = \sum_{k=0}^{s-1} c^*_{\mathfrak{F}^{ev*}_{p,n,k}}(2(p^{n-k}-1)) + c^*_{\mathfrak{F}^{ev*}_{p,n,s}}\Big(i - \sum_{k=0}^{s-1} 2(p^{n-k}-1)\Big)$ if $\sum_{k=0}^{s-1} 2(p^{n-k}-1) \leqq i \leqq \sum_{k=0}^{s} 2(p^{n-k}-1)$ and $c^*_{\mathfrak{F}^*_{2,n}}(i) = \sum_{k=0}^{s-1} c^*_{\mathfrak{F}^*_{2,n,k}}(2^{n-k}-1) + c^*_{\mathfrak{F}^*_{2,n,s}}\Big(i - \sum_{k=0}^{s-1}(2^{n-k}-1)\Big)$ if $\sum_{k=0}^{s-1}(2^{n-k}-1) \leqq i \leqq \sum_{k=0}^{s}(2^{n-k}-1)$ by the similar argument. $\square$

By (15.4.15) and the proof of (2) of (15.4.17), we see the following result.

Proposition 15.4.18 *The following assertions hold.*

(1) For $i \in S(\mathfrak{F}^*_{p,n}) = \{0,1,2,\dots,2(p+\dots+p^n)-n+1\}$, suppose that $\sum_{l=0}^{s-1}(2p^{n-l}-1) \leqq i \leqq \sum_{l=0}^{s}(2p^{n-l}-1)$ for some $s = 0,1,\dots,n$. We put $\varepsilon(i) = \dfrac{1-(-1)^{i+s}}{2}$ and $r(i) = \dfrac{1}{2}\Big(i - \varepsilon(i) - \sum_{l=0}^{s-1}(2p^{n-l}-1)\Big)$. Then, $E_i^{c^*_{\mathfrak{F}^*_{p,n}}(i)}\mathcal{A}_p(n)_*$ is a one dimensional vector space spanned by $\rho_{\mathcal{A}_p(n)_*,i}\big(\tau_0\tau_1\cdots\tau_{s-1}\tau_s^{\varepsilon(i)}\xi_1^{p^n-1}\xi_2^{p^{n-1}-1}\cdots\xi_s^{p^{n-s+1}-1}\xi_{s+1}^{r(i)}\big)$.

(2) For $i \in S(\mathfrak{F}^{ev*}_{p,n}) = \{0,2,4,\dots,2(p+p^2+\dots+p^n-n)\}$, suppose that $\sum_{l=0}^{s-1} 2(p^{n-l}-1) \leqq i \leqq \sum_{l=0}^{s} 2(p^{n-l}-1)$ for some $s = 0,1,\dots,n-1$. We put $r(i) = \dfrac{1}{2}\Big(i - \sum_{l=0}^{s-1} 2(p^{n-l}-1)\Big)$. Then, $E_i^{c^*_{\mathfrak{F}^{ev*}_{p,n}}(i)}\mathcal{A}^{ev}_p(n)_*$ is a one dimensional vector space spanned by $\rho_{\mathcal{A}^{ev}_p(n)_*,i}\big(\xi_1^{p^n-1}\xi_2^{p^{n-1}-1}\cdots\xi_s^{p^{n-s+1}-1}\xi_{s+1}^{r(i)}\big)$.

(3) For $i \in S(\mathfrak{F}^*_{2,n}) = \{0,1,2,\dots,2^{n+1}-n-2\}$, suppose that $\sum_{l=0}^{s-1}(2^{n-l}-1) \leqq i \leqq \sum_{l=0}^{s}(2^{n-l}-1)$ for some $s = 0,1,\dots,n-1$. We put $r(i) = i - \sum_{l=0}^{s-1}(2^{n-l}-1)$. Then, $E_i^{c^*_{\mathfrak{F}^*_{2,n}}(i)}\mathcal{A}_2(n)_*$ is a one dimensional vector space spanned by $\rho_{\mathcal{A}_2(n)_*,i}\big(\zeta_1^{2^n-1}\zeta_2^{2^{n-1}-1}\cdots\zeta_s^{2^{n-s+1}-1}\zeta_{s+1}^{r(i)}\big)$.

Proposition 15.4.19 *$\mathfrak{F}^*_{p,n}$, $\mathfrak{F}^{ev*}_{p,n}$ and $\mathfrak{F}^*_{2,n}$ satisfy $(f4^*)$.*

Proof. For $i, i+1 \in S(\mathfrak{F}^*_{p,n})$, there exits $0 \leqq s \leqq n$ such that $\sum_{k=0}^{s-1}(2p^{n-k}-1) \leqq i < \sum_{k=0}^{s}(2p^{n-k}-1)$. Then we have the following equality by (15.4.17) and (15.4.15).

$$\begin{aligned} c^*_{\mathfrak{F}^*_{p,n}}(i+1) - c^*_{\mathfrak{F}^*_{p,n}}(i) &= c^*_{\mathfrak{F}^*_{p,n,s}}\Big(i+1-\sum_{k=0}^{s-1}(2p^{n-k}-1)\Big) - c^*_{\mathfrak{F}^*_{p,n,s}}\Big(i-\sum_{k=0}^{s-1}(2p^{n-k}-1)\Big) \\ &= p^{s+1}-1-(-1)^{i+s}p^s(p-2) = \begin{cases} 2p^s-1 & i+s \text{ is even} \\ 2p^s(p-1)-1 & i+s \text{ is odd} \end{cases} \end{aligned}$$

Hence $c^*_{\mathfrak{F}^*_{p,n}}(i+1) - c^*_{\mathfrak{F}^*_{p,n}}(i) > 0$ and $\mathfrak{F}^*_{p,n}$ satisfies $(f4^*)$.

For $i, i+2 \in S(\mathfrak{F}^{ev*}_{p,n})$, there exits $0 \leqq s \leqq n-1$ such that $\sum_{k=0}^{s-1} 2(p^{n-k}-1) \leqq i < \sum_{k=0}^{s} 2(p^{n-k}-1)$. Then,

$$c^*_{\mathfrak{F}^{ev*}_{p,n}}(i+2) - c^*_{\mathfrak{F}^{ev*}_{p,n}}(i) = c^*_{\mathfrak{F}^{ev*}_{p,n,s}}\Big(i+2-\sum_{k=0}^{s-1}2(p^{n-k}-1)\Big) - c^*_{\mathfrak{F}^{ev*}_{p,n,s}}\Big(i-\sum_{k=0}^{s-1}2(p^{n-k}-1)\Big) = 2(p^{s+1}-1) > 0.$$

For $i, i+1 \in S(\mathfrak{F}^*_{2,n})$, there exits $0 \leqq s \leqq n-1$ such that $\sum_{k=0}^{s-1}(2^{n-k}-1) \leqq i < \sum_{k=0}^{s}(2^{n-k}-1)$. Then,

$$c^*_{\mathfrak{F}^*_{2,n}}(i+1) - c^*_{\mathfrak{F}^*_{2,n}}(i) = c^*_{\mathfrak{F}^*_{2,n,s}}\left(i+2-\sum_{k=0}^{s-1}(2^{n-k}-1)\right) - c^*_{\mathfrak{F}^*_{2,n,s}}\left(i-\sum_{k=0}^{s-1}(2^{n-k}-1)\right) = 2^{s+1}-1 > 0.$$

Therefore $\mathfrak{F}^{ev*}_{p,n}$ and $\mathfrak{F}^*_{2,n}$ also satisfy $(f4^*)$. □

Remark 15.4.20 *(1) For $k \in S(\mathfrak{F}^*_{p,n})$, suppose that $\sum_{l=0}^{s-1}(2p^{n-l}-1) \leqq k \leqq \sum_{l=0}^{s}(2p^{n-l}-1)$ for some $s = 0, 1, \ldots, n$. We put $m = k - \sum_{l=0}^{s-1}(2p^{n-l}-1)$, then $0 \leqq m \leqq 2p^{n-s}-1$ and we have the following equality.*

$$\begin{aligned}
k + c^*_{\mathfrak{F}^*_{p,n}}(k) &= \sum_{l=0}^{s-1}(2p^{n-l}-1+c^*_{\mathfrak{F}_{p,n,l}}(2p^{n-l}-1)) + k - \sum_{l=0}^{s-1}(2p^{n-l}-1) + c^*_{\mathfrak{F}^*_{p,n,s}}\left(k-\sum_{l=0}^{s-1}(2p^{n-l}-1)\right) \\
&= \sum_{l=0}^{s-1}(2p^{n+1}-2p^l(p-1)) + m + m(p^{s+1}-1) - \frac{1-(-1)^m}{2}p^s(p-2) \\
&= 2sp^{n+1} - 2(p^s-1) + mp^{s+1} - \frac{1-(-1)^m}{2}p^s(p-2) = \begin{cases} 2sp^{n+1}+(mp-2)p^s+2 & m \text{ is even} \\ 2sp^{n+1}+(mp-p)p^s+2 & m \text{ is odd} \end{cases}
\end{aligned}$$

For $k \in S(\mathfrak{F}^{ev*}_{p,n})$, suppose that $\sum_{l=0}^{s-1}2(p^{n-l}-1) \leqq k \leqq \sum_{l=0}^{s}2(p^{n-l}-1)$ for some $s = 0, 1, \ldots, n-1$. We put $m = k - \sum_{l=0}^{s-1}2(p^{n-l}-1)$, then $0 \leqq m \leqq 2(p^{n-s}-1)$ and we have the following equality.

$$\begin{aligned}
k + c^*_{\mathfrak{F}^{ev*}_{p,n}}(k) &= \sum_{l=0}^{s-1}(2(p^{n-l}-1)+c^*_{\mathfrak{F}^{ev}_{p,n,l}}(2(p^{n-l}-1))) + k - \sum_{l=0}^{s-1}2(p^{n-l}-1) + c^*_{\mathfrak{F}^{ev*}_{p,n,s}}\left(k-\sum_{l=0}^{s-1}2(p^{n-l}-1)\right) \\
&= \sum_{l=0}^{s-1}(2(p^{n-l}-1)+2(p^{n-l}-1)(p^{l+1}-1)) + m + m(p^{s+1}-1) \\
&= \sum_{l=0}^{s-1}2(p^{n+1}-p^{l+1}) + mp^{s+1} = 2sp^{n+1} + mp^{s+1} - \sum_{l=0}^{s-1}2p^{l+1}
\end{aligned}$$

For $k \in S(\mathfrak{F}^*_{2,n})$, suppose that $\sum_{l=0}^{s-1}(2^{n-l}-1) \leqq k \leqq \sum_{l=0}^{s}(2^{n-l}-1)$ for some $s = 0, 1, \ldots, n-1$. We put $m = k - \sum_{l=0}^{s-1}(2^{n-l}-1)$, then $0 \leqq m \leqq 2^{n-s}-1$ and we have the following equality.

$$\begin{aligned}
k + c^*_{\mathfrak{F}^*_{2,n}}(k) &= \sum_{l=0}^{s-1}(2^{n-l}-1+c^*_{\mathfrak{F}_{2,n,l}}(2^{n-l}-1)) + k - \sum_{l=0}^{s-1}(2^{n-l}-1) + c^*_{\mathfrak{F}^*_{2,n,s}}\left(k-\sum_{l=0}^{s-1}(2^{n-l}-1)\right) \\
&= \sum_{l=0}^{s-1}(2^{n-l}-1+(2^{n-l}-1)(2^{l+1}-1)) + m + m(2^{s+1}-1) \\
&= \sum_{l=0}^{s-1}(2^{n+1}-2^{l+1}) + m2^{s+1} = s2^{n+1} + (m-1)2^{s+1} + 2
\end{aligned}$$

(2) $\{\rho_{\mathcal{A}_p(n)_*,i}(\tau(E)\xi(R)) \,|\, E \in \mathrm{Seq}^b, R \in \mathrm{Seq}, \|E\|+2\|R\| = i, d^*_p(E,R) = j, E \leqq B_n, R \leqq N_{p,n}\}$ is a basis of $E^j_i\mathcal{A}_p(n)_*$. Suppose that $E \in \mathrm{Seq}^b$ and $R \in \mathrm{Seq}$ satisfy $\|E\|+2\|R\| = i$, $d^*_p(E,R) = j$, $E \leqq B_n$ and $R \leqq N_{p,n}$. Put $E = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$, $R = (r_1, r_2, \ldots, r_n)$ and $t_l = \varepsilon_l + r_l$. Then, we have the following equality.

$$i + j = \|E\| + 2\|R\| + d^*_p(E,R) = \sum_{l=0}^{n}2\varepsilon_l p^l + \sum_{l=1}^{n}2r_l p^l = 2\varepsilon_0 + 2\sum_{l=1}^{n}t_l p^l \;\cdots\; (i)$$

Note that t_l takes values in $\{0,1,2,\ldots,p^{n-l+1}\}$ for $l=1,2,\ldots,n$. For a non-negative integer k which is less than np^n, we assume that there exist integers $t_1,t_2,\ldots,t_n$ such that $k=\sum_{l=1}^{n} t_l p^{l-1}$ and $t_l \leqq p^{n-l+1}$ for $l=1,2,\ldots,n$. Since $\sum_{l=1}^{n} t_l p^{l-1} = np^n$ if and only if $t_l = p^{n-l+1}$ for $l=1,2,\ldots,n$, there exists an integer $1 \leqq m \leqq n$ such that $t_m < p^{n-m+1}$ and $t_l = p^{n-l+1}$ for $l=1,2,\ldots,m-1$. We put $s_l = t_l - p + 1$ for $l=1,2,\ldots,m-1$, $s_m = t_m+1$ and $s_l = t_l$ for $l=m+1,m+2,\ldots,n$. Then,

$$\sum_{l=1}^{n} s_l p^{l-1} = \sum_{l=1}^{m-1}(t_l-p+1)p^{l-1} + (t_m+1)p^{m-1} + \sum_{l=m+1}^{n} t_l p^{l-1} = \sum_{l=1}^{n} t_l p^{l-1} + p^{m-1} - (p-1)\sum_{l=1}^{m-1} p^{l-1} = k+1.$$

Hence $\sum_{l=1}^{n} t_l p^{l-1}$ takes every integers between 0 and np^n. It follows from (i) that $E_i^j \mathcal{A}_p(n)_* \neq \{0\}$ if and only if $i+j \equiv 0$ or 2 modulo $2p$ and $0 \leqq i+j \leqq 2np^{n+1}+2$.

$\{\rho_{\mathcal{A}_p^{ev}(n)_*,i}(\xi(R)) \mid R \in \mathrm{Seq}, 2\|R\| = i, d_p^*(R) = j, R \leqq N_{p,n}\}$ is a basis of $E_i^j \mathcal{A}_p^{ev}(n)_*$. Suppose that $R \in \mathrm{Seq}$ satisfy $2\|R\| = i$, $d_p^*(R) = j$ and $R \leqq N_{p,n}$. Put $R=(r_1,r_2,\ldots,r_n)$. Then, we have the following equality.

$$i+j = 2\|R\| + d_p^*(R) = \sum_{l=1}^{n} 2r_l p^l \ \cdots\ (ii).$$

Note that r_l takes values in $\{0,1,2,\ldots,p^{n-l+1}-1\}$ for $l=1,2,\ldots,n$. For a non-negative integer k which is less than $np^n - \sum_{l=1}^{n} p^{l-1}$, we assume that there exist integers $r_1,r_2,\ldots,r_n$ such that $k=\sum_{l=1}^{n} r_l p^{l-1}$ and $r_l \leqq p^{n-l+1}-1$ for $l=1,2,\ldots,n$. Since $\sum_{l=1}^{n} r_l p^{l-1} = np^n - \sum_{l=1}^{n} p^{l-1}$ if and only if $r_l = p^{n-l+1}-1$ for $l=1,2,\ldots,n$, there exists an integer $1 \leqq m \leqq n$ such that $r_m < p^{n-m+1}-1$ and $r_l = p^{n-l+1}-1$ for $l=1,2,\ldots,m-1$. We put $s_l = r_l - p + 1$ for $l=1,2,\ldots,m-1$, $s_m = r_m+1$ and $s_l = r_l$ for $l=m+1,m+2,\ldots,n$. Then,

$$\sum_{l=1}^{n} s_l p^{l-1} = \sum_{l=1}^{m-1}(r_l-p+1)p^{l-1} + (r_m+1)p^{m-1} + \sum_{l=m+1}^{n} r_l p^{l-1} = \sum_{l=1}^{n} r_l p^{l-1} + p^{m-1} - (p-1)\sum_{l=1}^{m-1} p^{l-1} = k+1.$$

Hence $\sum_{l=1}^{n} r_l p^{l-1}$ takes every integers between 0 and $np^{n-1} - \sum_{l=1}^{n} p^{l-1}$. It follows from (ii) that $E_i^j \mathcal{A}_p^{ev}(n)_* \neq \{0\}$ if and only if $i+j \equiv 0$ modulo $2p$ and $0 \leqq i+j \leqq 2np^n - \sum_{l=1}^{n} 2p^l$.

$\{\rho_{\mathcal{A}_2(n)_*,i}(\zeta(R)) \mid R \in \mathrm{Seq}, 2\|R\| = i, d_2^*(R) = j, R \leqq N_{2,n}\}$ is a basis of $E_i^j \mathcal{A}_2(n)_*$. Suppose that $R \in \mathrm{Seq}$ satisfy $\|R\| = i$, $d_2^*(R) = j$ and $R \leqq N_{2,n}$. Put $R=(r_1,r_2,\ldots,r_n)$. Then, we have the following equality.

$$i+j = \|R\| + d_2^*(R) = \sum_{l=1}^{n} r_l 2^l \ \cdots\ (ii).$$

Note that r_l takes values in $\{0,1,2,\ldots,2^{n-l+1}-1\}$ for $l=1,2,\ldots,n$. For a non-negative integer k which is less than $n2^n - \sum_{l=1}^{n} 2^{l-1}$, we assume that there exist integers $r_1,r_2,\ldots,r_n$ such that $k=\sum_{l=1}^{n} r_l 2^{l-1}$ and $r_l \leqq 2^{n-l+1}-1$ for $l=1,2,\ldots,n$. Since $\sum_{l=1}^{n} r_l 2^{l-1} = (n-1)2^n+1$ if and only if $r_l = 2^{n-l+1}-1$ for $l=1,2,\ldots,n$, there exists an integer $1 \leqq m \leqq n$ such that $r_m < 2^{n-m+1}-1$ and $r_l = 2^{n-l+1}-1$ for $l=1,2,\ldots,m-1$. We put $s_l = r_l - 2 + 1$ for $l=1,2,\ldots,m-1$, $s_m = r_m+1$ and $s_l = r_l$ for $l=m+1,m+2,\ldots,n$. Then,

$$\sum_{l=1}^{n} s_l 2^{l-1} = \sum_{l=1}^{m-1}(r_l-1)2^{l-1} + (r_m+1)2^{m-1} + \sum_{l=m+1}^{n} r_l 2^{l-1} = \sum_{l=1}^{n} r_l 2^{l-1} + 2^{m-1} - \sum_{l=1}^{m-1} 2^{l-1} = k+1.$$

Hence $\sum_{l=1}^{n} r_l 2^{l-1}$ takes every integers between 0 and $(n-1)2^n+1$. It follows from (ii) that $E_i^j \mathcal{A}_2(n)_* \neq \{0\}$ if and only if $i+j$ is even and $0 \leqq i+j \leqq (n-1)2^{n+1}+2$.

(3) It follows from the above results that $\mathfrak{F}_{p,n}^*$, $\mathfrak{F}_{p,n}^{ev*}$ and $\mathfrak{F}_{2,n}^*$ do not satisfy $(f3^*)$ unless $n=1$.

We consider Hopf subalgebras $\mathcal{A}_p\langle k\rangle_*$ of $\mathcal{A}_{p*}$ introduced in [24] which are defined by

$$\mathcal{A}_p\langle k\rangle_* = \begin{cases} \boldsymbol{F}_2\big[\zeta_1^{2^k}, \zeta_2^{2^k}, \ldots, \zeta_n^{2^k}, \ldots\big] & \text{if } p=2 \\ E(\tau_0) \otimes_{\boldsymbol{F}_p} \boldsymbol{F}_p\big[\xi_1^p, \xi_2^p, \ldots, \xi_n^p, \ldots\big] & \text{if } p \text{ is an odd prime and } k=1 \\ \boldsymbol{F}_p\big[\xi_1^{p^k}, \xi_2^{p^k}, \ldots, \xi_n^{p^k}, \ldots\big] & \text{if } p \text{ is an odd prime and } k \geqq 2. \end{cases}$$

Then $\mathcal{A}_p\langle k\rangle_*$ is a Hopf subalgebra of $\mathcal{A}_{p*}$. We give a filtration $\mathfrak{F}_p\langle k\rangle^* = (F_i\mathcal{A}_p\langle k\rangle_*)_{i\in \boldsymbol{Z}}$ of $\mathcal{A}_p\langle k\rangle_*$ by restricting $\mathfrak{F}_p^*$ to $\mathcal{A}_p\langle k\rangle_*$, namely, $F_i\mathcal{A}_p\langle k\rangle_* = \mathcal{A}_p\langle k\rangle_* \cap F_i\mathcal{A}_{p*}$. It follows from (15.1.15), (15.1.21), (15.3.4) and (15.3.18) that $\mathfrak{F}_p\langle k\rangle^*$ satisfies $(f1^*)$, $(f2^*)$, $(f5^*)$, $(f6^*)$ and $(f9^*)$. We see the following result from (15.4.7).

Proposition 15.4.21 *If p is an odd prime, $\{\tau_0^\varepsilon\xi(pR)\,|\,\varepsilon = 0,1, R\in \mathrm{Seq}, \varepsilon + 2p\|R\| \leqq i\}$ is a basis of $F_i\mathcal{A}_p\langle 1\rangle_*$ and $\{\xi(p^kR)\,|\,R\in\mathrm{Seq}, 2p^k\|R\|\leqq i\}$ is a basis of $\mathcal{A}_p\langle k\rangle_*$ for $k\geqq 2$. $\{\zeta(2^kR)\,|\,R\in\mathrm{Seq}, 2^k\|R\|\leqq i\}$ is a basis of $\mathcal{A}_2\langle k\rangle_*$ for $k\geqq 1$.*

The above result result immediately implies the following.

Proposition 15.4.22 *Let p be an odd prime. Then, we have $S(\mathfrak{F}_p\langle 1\rangle^*) = \{2ip+\varepsilon\,|\,\varepsilon = 0,1, i = 0,1,2,\dots\}$ and $S(\mathfrak{F}_p\langle k\rangle^*) = \{2ip^k\,|\,i = 0,1,2,\dots\}$ if $k\geqq 2$. We also have $S(\mathfrak{F}_2\langle k\rangle^*) = \{i2^k\,|\,i=0,1,2,\dots\}$.*

Proposition 15.4.23 *The map $c^*_{\mathfrak{F}_p\langle k\rangle^*} : S(\mathfrak{F}_p\langle k\rangle^*) \to \boldsymbol{Z}$ is given as follows. $c^*_{\mathfrak{F}_p\langle 1\rangle^*}(2ip+1) = 2ip(p-1)+1$, $c^*_{\mathfrak{F}_p\langle k\rangle^*}(2ip^k) = 2ip^k(p-1)$, for $i = 0,1,2,\dots,$ $k\geqq 1$ if p is an odd prime and $c^*_{\mathfrak{F}_2\langle k\rangle^*}(i2^k) = i2^k$. Hence $\mathfrak{F}_p\langle k\rangle^*$ satisfies $(f4^*)$.*

Proof. It follows from (15.4.21) that $\{\rho_{\mathcal{A}_p\langle 1\rangle_*,2ip+\varepsilon}(\tau_0^\varepsilon\xi(pR))\,|\,\varepsilon = 0,1, R\in\mathrm{Seq}, \|R\| = i\}$ is a basis of $E^*_{2ip+\varepsilon}\mathcal{A}_p\langle 1\rangle_*$. If $R = (r_1, r_2, \dots)$ and $\|R\| = i$, we have the following equality.

$$\begin{aligned}\deg\rho_{\mathcal{A}_p\langle 1\rangle_*,2ip+\varepsilon}(\tau_0^\varepsilon\xi(pR)) &= \varepsilon + \sum_{s\geqq 1}2r_sp(p^s-1) = \varepsilon + 2p^2\left(i - \sum_{s\geqq 2}r_s\right) + \sum_{s\geqq 2}2r_sp^{s+1} - 2ip\\ &= \varepsilon + 2ip(p-1) + \sum_{s\geqq 2}2r_sp^2(p^{s-1}-1)\end{aligned}$$

Hence the degree of $\rho_{\mathcal{A}_p\langle 1\rangle_*,2ip+\varepsilon}(\tau_0^\varepsilon\xi(pR))$ takes the minimum value $\varepsilon + 2ip(p-1)$ if and only if $r_s = 0$ for $s\geqq 2$. Thus we have $c^*_{\mathfrak{F}_p\langle 1\rangle^*}(2ip+\varepsilon) = 2ip(p-1)+\varepsilon$ for $\varepsilon = 0,1$ and $i = 0,1,2,\dots$.

Assume that $k\geqq 2$. It follows from (15.4.21) that $\{\rho_{\mathcal{A}_p\langle k\rangle_*,2ip^k}(\xi(p^kR))\,|\,R\in\mathrm{Seq}, \|R\| = i\}$ is a basis of $E^*_{2ip^k}\mathcal{A}_p\langle k\rangle_*$. If $R = (r_1,r_2,\dots)$ and $\|R\| = i$, we have the following equality.

$$\begin{aligned}\deg\rho_{\mathcal{A}_p\langle k\rangle_*,2ip^k}(\xi(p^kR)) &= \sum_{s\geqq 1}2r_sp^k(p^s-1) = 2p^{k+1}\left(i - \sum_{s\geqq 2}r_s\right) + \sum_{s\geqq 2}2r_sp^{k+s} - 2ip^k\\ &= 2ip^k(p-1) + \sum_{s\geqq 2}2r_sp^{k+1}(p^{s-1}-1)\end{aligned}$$

Hence the degree of $\rho_{\mathcal{A}_p\langle k\rangle_*,2ip^k}(\xi(p^kR))$ takes the minimum value $2ip^k(p-1)$ if and only if $r_s = 0$ for $s\geqq 2$. Thus we have $c^*_{\mathfrak{F}_p\langle k\rangle^*}(2ip^k) = 2ip^k(p-1)$ for $i = 0,1,2,\dots$.

Assume that $k\geqq 1$. It follows from (15.4.21) that $\{\rho_{\mathcal{A}_2\langle k\rangle_*,i2^k}(\zeta(2^kR))\,|\,R\in\mathrm{Seq}, \|R\| = i\}$ is a basis of $E^*_{i2^k}\mathcal{A}_2\langle k\rangle_*$. If $R = (r_1,r_2,\dots)$ and $\|R\| = i$, we have the following equality.

$$\begin{aligned}\deg\rho_{\mathcal{A}_2\langle k\rangle_*,i2^k}(\zeta(2^kR)) &= \sum_{s\geqq 1}r_s2^k(2^s-1) = \sum_{s\geqq 1}r_s2^{k+s} - i2^k = 2^{k+1}\left(i - \sum_{s\geqq 2}r_s\right) + \sum_{s\geqq 2}r_s2^{k+s} - i2^k\\ &= i2^k + \sum_{s\geqq 2}r_s2^{k+1}(2^{s-1}-1)\end{aligned}$$

Hence the degree of $\rho_{\mathcal{A}_2\langle k\rangle_*,i2^k}(\zeta(2^kR))$ takes the minimum value $i2^k$ if and only if $r_s = 0$ for $s\geqq 2$. Thus we have $c^*_{\mathfrak{F}_2\langle k\rangle^*}(i2^k) = i2^k$ for $i = 0,1,2,\dots$. $\square$

We have the following result from the proof of (15.4.23).

Proposition 15.4.24 *$E_{2ip+\varepsilon}^{c^*_{\mathfrak{F}_p\langle 1\rangle^*}(2ip+\varepsilon)}\mathcal{A}_p\langle 1\rangle_*$ is a one dimensional vector space spanned by $\rho_{\mathcal{A}_p\langle 1\rangle_*,2ip+\varepsilon}(\tau_0^\varepsilon\xi_1^{ip})$. $E_{2ip^k}^{c^*_{\mathfrak{F}_p\langle k\rangle^*}(2ip^k)}\mathcal{A}_p\langle k\rangle_*$ is a one dimensional vector space spanned by $\rho_{\mathcal{A}_p\langle k\rangle_*,2ip^k}(\xi_1^{ip^k})$. $E_{i2^k}^{c^*_{\mathfrak{F}_2\langle k\rangle^*}(i2^k)}\mathcal{A}_2\langle k\rangle_*$ is a one dimensional vector space spanned by $\rho_{\mathcal{A}_2\langle k\rangle_*,i2^k}(\zeta_1^{i2^k})$.*

Proposition 15.4.25 *$\mathfrak{F}_p\langle k\rangle^*$ satisfies $(f3^*)$.*

Proof. Since $2ip+\varepsilon+c^*_{\mathfrak{F}_p\langle 1\rangle^*}(2ip+\varepsilon)=2ip^2+2\varepsilon$ by (15.4.23), the image of a map $S(\mathfrak{F}_p\langle 1\rangle^*)\to \boldsymbol{Z}$ defined by $k\mapsto k+c^*_{\mathfrak{F}_p\langle 1\rangle^*}(k)$ is $\{n\in\boldsymbol{Z}\,|\,n\geqq 0, n\equiv 0 \text{ or } 2 \text{ modulo } 2p^2\}$. It follows from (15.4.21) that

$$\left\{\rho_{\mathcal{A}_p\langle 1\rangle_*,2ip+\varepsilon}(\tau_0^\varepsilon\xi(pR))\,\middle|\, R=(r_1,r_2,\dots)\in\mathrm{Seq}, \|R\|=i, j=\varepsilon+2ip(p-1)+\sum_{s\geqq 2}2r_sp^2(p^{s-1}-1)\right\}$$

is a basis of $E^j_{2ip+\varepsilon}\mathcal{A}_p\langle 1\rangle_*$. Hence $E^j_i\mathcal{A}_p\langle 1\rangle_*=\{0\}$ if $i+j\not\equiv 0,2$ modulo $2p^2$.

Since $2ip^k+c^*_{\mathfrak{F}_p\langle k\rangle^*}(2ip^k)=2ip^{k+1}$ by (15.4.23), the image of a map $S(\mathfrak{F}_p\langle k\rangle^*)\to\boldsymbol{Z}$ defined by $k\mapsto k+c^*_{\mathfrak{F}_p\langle 1\rangle^*}(k)$ is $\{n\in\boldsymbol{Z}\,|\,n\geqq 0, n\equiv 0 \text{ modulo } 2p^{k+1}\}$. It follows from (15.4.21) that

$$\left\{\rho_{\mathcal{A}_p\langle k\rangle_*,2ip^k}(\xi(p^kR))\,\middle|\, R=(r_1,r_2,\dots)\in\mathrm{Seq}, \|R\|=i, j=2ip^k(p-1)+\sum_{s\geqq 2}2r_sp^{k+1}(p^{s-1}-1)\right\}$$

is a basis of $E^j_{2ip^k}\mathcal{A}_p\langle k\rangle_*$. Hence $E^j_i\mathcal{A}_p\langle k\rangle_*=\{0\}$ if $i+j\not\equiv 0$ modulo $2p^{k+1}$.

Since $i2^k+c^*_{\mathfrak{F}_2\langle k\rangle^*}(i2^k)=i2^{k+1}$ by (15.4.23), the image of a map $S(\mathfrak{F}_2\langle k\rangle^*)\to\boldsymbol{Z}$ defined by $k\mapsto k+c^*_{\mathfrak{F}_2\langle 1\rangle^*}(k)$ is $\{n\in\boldsymbol{Z}\,|\,n\geqq 0, n\equiv 0 \text{ modulo } 2^{k+1}\}$. It follows from (15.4.21) that

$$\left\{\rho_{\mathcal{A}_2\langle k\rangle_*,i2^k}(\zeta(2^kR))\,\middle|\, R=(r_1,r_2,\dots)\in\mathrm{Seq}, \|R\|=i, j=i2^k+\sum_{s\geqq 2}r_s2^{k+1}(2^{s-1}-1)\right\}$$

is a basis of $E^j_{i2^k}\mathcal{A}_2\langle k\rangle_*$. Hence $E^j_i\mathcal{A}_2\langle k\rangle_*=\{0\}$ if $i+j\not\equiv 0$ modulo 2^{k+1}. □

Let $F_i\tilde{A}^*_{p\infty}$ be the subspace of $\tilde{A}^*_{p\infty}$ spanned by

$$\{x_{i_1j_1}x_{i_2j_2}\cdots x_{i_nj_n}\,|\,j_1+j_2+\cdots+j_n\leq i\}.$$

By this definition and (14.2.5), (14.2.6), it is easy to verify the following assertions.

Proposition 15.4.26 *(1) The filtration $(F_i\tilde{A}^*_{p\infty})_{i\in\boldsymbol{Z}}$ on $\tilde{A}^*_{p\infty}$ satisfies the conditions $(f9)$, $(f1^*)$, $(f0)$, $(f3)$, $(f5^*)$, $(f6^*)$, $(f3^*)$.*

(2) $\rho_p(F_i\tilde{A}^*_{p\infty})=F_i\mathcal{A}_{p*}$.

Let $F_iA^*_{p\infty}$ be the subspace of $A^*_{p\infty}$ spanned by

$$\left\{x_{k_11}x_{k_21}\cdots x_{k_m1}x_{i_1j_1}x_{i_2j_2}\cdots x_{i_nj_n}\,\middle|\, m+2\sum_{s=1}^n p^{j_s-2}\leq i,\ j_1,j_2,\dots,j_n\geq 2\right\}\quad \text{if } p\neq 2$$

$$\left\{x_{i_1j_1}x_{i_2j_2}\cdots x_{i_nj_n}\,\middle|\,\sum_{s=1}^n 2^{j_s-1}\leq i\right\}\quad \text{if } p=2.$$

By this definition and (14.2.7), (14.2.8), it is easy to verify the following assertions.

Proposition 15.4.27 *(1) The filtration $(F_iA^*_{p\infty})_{i\in\boldsymbol{Z}}$ on $A^*_{p\infty}$ satisfies the conditions $(f9)$, $(f1^*)$, $(f0)$, $(f3)$, $(f5^*)$, $(f6^*)$, $(f3^*)$, $(c4)$.*

(2) $\rho_p(F_iA^*_{p\infty})=F_i\mathcal{A}_{p*}$.

Proposition 15.4.28 *If p is an odd prime, then for $\varepsilon=0,1$ and $s=0,1,2,\dots,$*

$$\left\{x_{21}^\varepsilon\prod_{j\geq 2}x_{j+1\,j}^{m_j}\,\middle|\,\sum_{j\geq 2}m_jp^{j-2}=s\right\}$$

is a basis of $E^{2s(p-1)+\varepsilon}_{2s+\varepsilon}A^*_{p\infty}$. For $s=0,1,2,\dots,$

$$\left\{\prod_{j\geq 1}x_{j+1\,j}^{m_j}\,\middle|\,\sum_{j\geq 2}m_j2^{j-1}=s\right\}$$

is a basis of $E^s_sA^*_{2\infty}$.

15.5 Division functor

We assume that K^* is a field such that $K^i = \{0\}$ if $i \neq 0$. Let us denote by $\mathcal{M}od^s_{K^*}$ the category of topological graded K^*-modules with skeletal topology and linear maps. We also denote by $\mathcal{M}od^{cs}_{K^*}$ a full subcategory of $\mathcal{M}od^s_{K^*}$ whose objects are 1-coconnected. Define a functor $\mathrm{Tr} : \mathcal{M}od^s_{K^*} \to \mathcal{M}od^{cs}_{K^*}$ as follows. For $M^* \in \mathrm{Ob}\,\mathcal{M}od^s_{K^*}$, put $\mathrm{Tr}(M^*) = M^*/\sum\limits_{n\geqq 1} M^n$. Let $\pi_{M^*} : M^* \to \mathrm{Tr}(M^*)$ be the quotient map. For a morphism $f : M^* \to N^*$ of $\mathcal{M}od^s_{K^*}$, let $\mathrm{Tr}(f) : \mathrm{Tr}(M^*) \to \mathrm{Tr}(N^*)$ be unique morphism that makes the following diagram commute.

$$\begin{array}{ccc} M^* & \xrightarrow{\quad f \quad} & N^* \\ \big\downarrow{\scriptstyle \pi_{M^*}} & & \big\downarrow{\scriptstyle \pi_{N^*}} \\ \mathrm{Tr}(M^*) & \xrightarrow{\mathrm{Tr}(f)} & \mathrm{Tr}(N^*) \end{array}$$

Let $I : \mathcal{M}od^{cs}_{K^*} \to \mathcal{M}od^s_{K^*}$ be the inclusion functor. Then, the quotient maps $\pi_{M^*} : M^* \to \mathrm{Tr}(M^*)$ define a natural transformation $\pi : id_{\mathcal{M}od^s_{K^*}} \to I\mathrm{Tr}$. We denote $\mathcal{H}om^*(M^*, K^*)$ by M^{**} below.

Proposition 15.5.1 *Let M^* be an object of $\mathcal{M}od^{cs}_{K^*}$. Define a functor $T_{M^*} : \mathcal{M}od^{cs}_{K^*} \to \mathcal{M}od^{cs}_{K^*}$ by $T_{M^*}(N^*) = M^* \otimes_{K^*} N^*$ and $T_{M^*}(f) = id_{M^*} \otimes_{K^*} f$ for an object N^* and a morphism f of $\mathcal{M}od^{cs}_{K^*}$. If M^* is coconnective and finite type, then T_{M^*} has a left adjoint.*

Proof. Let $\chi_{M^*,K^*} : M^* \to \mathcal{H}om^*(M^{**}, K^*)$ be the double dual isomorphism (3.3.6). We note that M^{**} has skeletal topology by (3.1.36). Since both $\mathcal{H}om^*(M^{**}, K^*)$ and N^* are coconnective and have skeletal topology, $\mathcal{H}om^*(M^{**}, K^*) \otimes_{K^*} N^*$ is complete by (2.3.3). Hence it follows from (4.1.8) that we have a natural isomorphism $\varphi^{M^{**}}_{N^*} : \mathcal{H}om^*(M^{**}, K^*) \otimes_{K^*} N^* \to \mathcal{H}om^*(M^{**}, N^*)$. Then, for an object L^* of $\mathcal{M}od^{cs}_{K^*}$, it follows from (3.2.6) that there is an adjoint isomorphism

$$\Phi_{L^*,M^{**},N^*} : \mathrm{Hom}^c_{K^*}(L^* \otimes_{K^*} M^{**}, N^*) \to \mathrm{Hom}^c_{K^*}(L^*, \mathcal{H}om^*(M^{**}, N^*)).$$

Since N^* is 1-coconnected, $\pi_{L^* \otimes_{K^*} M^{**}} : L^* \otimes_{K^*} M^{**} \to \mathrm{Tr}(L^* \otimes_{K^*} M^{**})$ induces an isomorphism

$$\pi^*_{L^* \otimes_{K^*} M^{**}} : \mathrm{Hom}^c_{K^*}(\mathrm{Tr}(L^* \otimes_{K^*} M^{**}), N^*) \to \mathrm{Hom}^c_{K^*}(L^* \otimes_{K^*} M^{**}, N^*).$$

Thus the following composition of isomorphisms shows that T_{M^*} has a left adjoint.

$$\begin{aligned} &\mathrm{Hom}^c_{K^*}(\mathrm{Tr}(L^* \otimes_{K^*} M^{**}), N^*) \xrightarrow{\pi^*_{L^* \otimes_{K^*} M^{**}}} \mathrm{Hom}^c_{K^*}(L^* \otimes_{K^*} M^{**}, N^*) \xrightarrow{\Phi_{L^*,M^{**},N^*}} \mathrm{Hom}^c_{K^*}(L^*, \mathcal{H}om^*(M^{**}, N^*)) \\ &\xrightarrow{(\varphi^{M^{**}}_{N^*})^{-1}_*} \mathrm{Hom}^c_{K^*}(L^*, \mathcal{H}om^*(M^{**}, K^*) \otimes_{K^*} N^*) \xrightarrow{(\chi_{M^*,K^*} \otimes_{K^*} id_{N^*})^{-1}_*} \mathrm{Hom}^c_{K^*}(L^*, M^* \otimes_{K^*} N^*) \end{aligned}$$

Namely, a functor $S_{M^*} : \mathcal{M}od^{cs}_{K^*} \to \mathcal{M}od^{cs}_{K^*}$ given by $S_{M^*}(L^*) = \mathrm{Tr}(L^* \otimes_{K^*} M^{**})$ gives a left adjoint of T_{M^*}. □

Lemma 15.5.2 *Suppose that $K^n = \{0\}$ if $n \neq 0$ and that M^* is a free K^*-module of finite type. Let $\{v_{i,j}\}_{1\leq j\leq d_i}$ be a basis of M^i and $\{v^*_{i,j}\}_{1\leq j\leq d_i}$ its dual basis of $(M^{**})^{-i} = \mathcal{H}om^{-i}(M^*, K^*)$ for each $i \in \boldsymbol{Z}$. Namely, $v^*_{i,j} : \Sigma^{-i}M^* \to K^*$ satisfies $v^*_{i,j}([-i], v_{i,j}) = 1$ and $v^*_{i,j}([-i], v_{k,l}) = 0$ if $k \neq i$ or $l \neq j$. The following equality holds for $g \in (M^{**})^{-i} = \mathrm{Hom}^c_{K^*}(\Sigma^{-i}M^*, K^*)$.*

$$g = \sum_{j=1}^{d_i} g([-i], v_{i,j})v^*_{i,j}$$

Proof. For $([-i], v_{k,l}) \in (\Sigma^{-i}M^*)^{k-i} = \{[-i]\} \times M^k$, we have the following equality.

$$\sum_{j=1}^{d_i} g([-i], v_{i,j})v^*_{i,j}([-i], v_{k,l}) = \begin{cases} g([-i], v_{i,l}) & k = i \\ 0 & k \neq i \end{cases}$$

On the other hand, $g([-i], v_{k,l}) = 0$ if $k \neq i$ since $K^n = \{0\}$ if $n \neq 0$. Thus the assertion follows. □

Proposition 15.5.3 *The unit $\eta : id_{\mathcal{M}od^{cs}_{K^*}} \to T_{M^*}S_{M^*}$ and the counit $\varepsilon : S_{M^*}T_{M^*} \to id_{\mathcal{M}od^{cs}_{K^*}}$ of the adjunction of (15.5.1) are given as follows. Let N^* be an object of $\mathcal{M}od^{cs}_{K^*}$.*

(1) We choose a basis $\{v_{i,j}\}_{1\leqq j\leqq d_i}$ of M^i and its dual basis $\{v^*_{i,j}\}_{1\leqq j\leqq d_i}$ of $(M^{**})^{-i} = \mathcal{H}om^{-i}(M^*, K^*)$ for each $i \leqq 0$. Then $\eta_{N^*} : N^* \to M^*\otimes_{K^*}\mathrm{Tr}(N^*\otimes_{K^*}M^{**})$ maps $x \in N^n$ to $\sum\limits_{i=n}^{0}\sum\limits_{j=1}^{d_i}(-1)^{in}v_{i,j}\otimes x\otimes v^*_{i,j}$.

(2) For $a \in M^k$, $x \in N^n$ and $g \in (M^{**})^l = \mathrm{Hom}^c_{K^*}(\Sigma^l M^*, K^*)$, $\varepsilon_{N^*} : \mathrm{Tr}(M^*\otimes_{K^*}N^*\otimes_{K^*}M^{**}) \to N^*$ maps $a\otimes x\otimes g$ to $(-1)^{k(k+n)}g([-k], a)x$.

Proof. (1) Recall that $\eta_{N^*} : N^* \to M^*\otimes_{K^*}\mathrm{Tr}(N^*\otimes_{K^*}M^{**})$ is the image of the identity map of $\mathrm{Tr}(N^*\otimes_{K^*}M^{**})$ by the following composition.

$$\begin{aligned}
\mathrm{Hom}^c_{K^*}(\mathrm{Tr}(N^*\otimes_{K^*}M^{**}), \mathrm{Tr}(N^*\otimes_{K^*}M^{**})) &\xrightarrow{\pi^*_{N^*\otimes_{K^*}M^{**}}} \mathrm{Hom}^c_{K^*}(N^*\otimes_{K^*}M^{**}, \mathrm{Tr}(N^*\otimes_{K^*}M^{**}))\\
&\xrightarrow{\Phi_{N^*,M^{**},\mathrm{Tr}(N^*\otimes_{K^*}M^{**})}} \mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(M^{**}, \mathrm{Tr}(N^*\otimes_{K^*}M^{**})))\\
&\xrightarrow{\left(\varphi^{M^{**}}_{\mathrm{Tr}(N^*\otimes_{K^*}M^{**})}\right)^{-1}_*} \mathrm{Hom}^c_{K^*}(N^*, \mathcal{H}om^*(M^{**}, K^*)\otimes_{K^*}\mathrm{Tr}(N^*\otimes_{K^*}M^{**}))\\
&\xrightarrow{(\chi_{M^*,K^*}\otimes_{K^*}id_{\mathrm{Tr}(N^*\otimes_{K^*}M^{**})})^{-1}_*} \mathrm{Hom}^c_{K^*}(N^*, M^*\otimes_{K^*}\mathrm{Tr}(N^*\otimes_{K^*}M^{**}))
\end{aligned}$$

Put $\bar{x} = \sum\limits_{i=n}^{0}\sum\limits_{j=1}^{d_i}(-1)^{in}v_{i,j}\otimes x\otimes v^*_{i,j}$. It suffices to verify that a composition

$$\begin{aligned}
M^*\otimes_{K^*}\mathrm{Tr}(N^*\otimes_{K^*}M^{**}) &\xrightarrow{\chi_{M^*,K^*}\otimes_{K^*}id_{\mathrm{Tr}(N^*\otimes_{K^*}M^{**})}} \mathcal{H}om^*(M^{**}, K^*)\otimes_{K^*}\mathrm{Tr}(N^*\otimes_{K^*}M^{**})\\
&\xrightarrow{\varphi^{M^{**}}_{\mathrm{Tr}(N^*\otimes_{K^*}M^{**})}} \mathcal{H}om^*(M^{**}, \mathrm{Tr}(N^*\otimes_{K^*}M^{**}))
\end{aligned}$$

maps $\bar{x}$ to $(\Phi_{N^*,M^{**},\mathrm{Tr}(N^*\otimes_{K^*}M^{**})}(\pi_{N^*\otimes_{K^*}M^{**}}))(x) : \Sigma^n M^{**} \to \mathrm{Tr}(N^*\otimes_{K^*}M^{**})$. $\chi_{M^*,K^*}(v_{i,j}) : \Sigma^i M^{**} \to K^*$ is given by $\chi_{M^*,K^*}(v_{i,j})([i], g) = (-1)^{ik}g([k], v_{i,j})$ for $g \in (M^{**})^k = \mathrm{Hom}^c_{K^*}(\Sigma^k M^*, K^*)$. Then the following equality holds.

$$\begin{aligned}
\varphi^{M^{**}}_{\mathrm{Tr}(N^*\otimes_{K^*}M^{**})}(\chi_{M^*,K^*}(v_{i,j})\otimes x\otimes v^*_{i,j})([n], g) &= (-1)^{(n-i)k}\chi_{M^*,K^*}(v_{i,j})([i], g)x\otimes v^*_{i,j}\\
&= (-1)^{kn}g([k], v_{i,j})x\otimes v^*_{i,j}
\end{aligned}$$

Hence it follows from (15.5.2) that we have the following equality.

$$\begin{aligned}
((\varphi^{M^{**}}_{\mathrm{Tr}(N^*\otimes_{K^*}M^{**})}(\chi_{M^*,K^*}\otimes_{K^*}id_{\mathrm{Tr}(N^*\otimes_{K^*}M^{**})}))(\bar{x}))([n], g) &= \sum_{i=n}^{0}\sum_{j=1}^{d_i}(-1)^{n(i+k)}g([k], v_{i,j})x\otimes v^*_{i,j}\\
&= \begin{cases} x\otimes\left(\sum\limits_{j=1}^{d_{-k}} g([k], v_{-k,j})v^*_{-k,j}\right) & n \leqq -k\\ 0 & n > -k\end{cases}\\
&= \begin{cases} x\otimes g & n \leqq -k\\ 0 & n > -k\end{cases}\\
&= \pi_{N^*\otimes_{K^*}M^{**}}(x\otimes g)
\end{aligned}$$

On the other hand, $(\Phi_{N^*,M^{**},\mathrm{Tr}(N^*\otimes_{K^*}M^{**})}(\pi_{N^*\otimes_{K^*}M^{**}}))(x) : \Sigma^n M^{**} \to \mathrm{Tr}(N^*\otimes_{K^*}M^{**})$ maps $([n], g)$ to $\pi_{N^*\otimes_{K^*}M^{**}}(x\otimes g)$.

(2) $\varepsilon_{N^*} : \mathrm{Tr}(M^*\otimes_{K^*}N^*\otimes_{K^*}M^{**}) \to N^*$ is the image of the identity map of $M^*\otimes_{K^*}N^*$ by the following composition.

$$\begin{aligned}
\mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}N^*, M^*\otimes_{K^*}N^*) &\xrightarrow{(\chi_{M^*,K^*}\otimes_{K^*}id_{N^*})_*} \mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}N^*, \mathcal{H}om^*(M^{**}, K^*)\otimes_{K^*}N^*)\\
&\xrightarrow{(\varphi^{M^{**}}_{N^*})_*} \mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}N^*, \mathcal{H}om^*(M^{**}, N^*)) \xrightarrow{\Phi^{-1}_{M^*\otimes_{K^*}N^*,M^{**},N^*}} \mathrm{Hom}^c_{K^*}(M^*\otimes_{K^*}N^*\otimes_{K^*}M^{**}, N^*)\\
&\xrightarrow{\pi^{*-1}_{M^*\otimes_{K^*}N^*\otimes_{K^*}M^{**}}} \mathrm{Hom}^c_{K^*}(\mathrm{Tr}(M^*\otimes_{K^*}N^*\otimes_{K^*}M^{**}), N^*)
\end{aligned}$$

For $a \in M^k$, $x \in N^n$ and $g \in (M^{**})^l = \mathrm{Hom}^c_{K^*}(\Sigma^l M^*, K^*)$, we have the following equality.

$$\begin{aligned}\big(\Phi^{-1}_{M^*\otimes_{K^*}N^*,M^{**},N^*}\big(\varphi^{M^{**}}_{N^*}(\chi_{M^*,K^*}\otimes_{K^*} id_{N^*})\big)\big)(a\otimes x\otimes g) &= \big(\varphi^{M^{**}}_{N^*}(\chi_{M^*,K^*}\otimes_{K^*} id_{N^*}\big)(a\otimes x))([k+n],g)\\ &= \big(\varphi^{M^{**}}_{N^*}(\chi_{M^*,K^*}(a)\otimes x)\big)([k+n],g) = (-1)^{ln}\chi_{M^*,K^*}(a)([k],g)x = (-1)^{l(k+n)}g([l],a)x\end{aligned}$$

Since $g([l],a) = 0$ if $l \neq -k$, the assertion follows. □

Let A^* be a -1-coconnected K^*-algebra with product $\mu_{A^*} : A^* \otimes_{K^*} A^* \to A^*$ and unit $\eta_{A^*} : K^* \to A^*$. We assume that A^* has skeletal topology. Define a functor $T_{A^*} : \mathcal{M}od^{cs}_{K^*} \to \mathcal{M}od^{cs}_{K^*}$ by $T_{A^*}(M^*) = A^* \otimes_{K^*} M^*$. Define a natural transformation $\tilde{\mu} : T^2_{A^*} \to T_{A^*}$ and by

$$\tilde{\mu}_{M^*} = \mu_{A^*} \otimes_{K^*} id_{M^*} : A^* \otimes_{K^*} A^* \otimes_{K^*} M^* \to A^* \otimes_{K^*} M^*$$

For $M^* \in \mathrm{Ob}\,\mathcal{M}od^{cs}_{K^*}$, let $i_2 : M^* \to K^* \otimes_{K^*} M^*$ be the isomorphism defined by $i_2(x) = 1 \otimes x$. We also define a natural transformation $\tilde{\eta} : id_{\mathcal{M}od^{cs}_{K^*}} \to T_{A^*}$ as follows. Define $\tilde{\eta}_{M^*} : M^* \to T_{A^*}(M^*)$ to be a composition

$$M^* \underset{\cong}{\xrightarrow{i_2}} K^* \otimes_{K^*} M^* \xrightarrow{\eta_{A^*}\otimes_{K^*} id_{A^*}} A^* \otimes_{K^*} M^*.$$

Then, $\boldsymbol{T}_{A^*} = \langle T_{A^*}, \tilde{\eta}, \tilde{\mu}\rangle$ is a monad on $\mathcal{M}od^{cs}_{K^*}$ and the category $(\mathcal{M}od^{cs}_{K^*})^{\boldsymbol{T}_{A^*}}$ of $\boldsymbol{T}_{A^*}$-algebras is identified with the category $\mathcal{M}od(A^*)$ of left A^*-modules. An object of $(\mathcal{M}od^{cs}_{K^*})^{\boldsymbol{T}_{A^*}}$ is a pair (M^*, α) of an object M^* of $\mathcal{M}od^{cs}_{K^*}$ and an A^*-module structure map $\alpha : A^* \otimes_{K^*} M^* \to M^*$ of M^*. We denote by $U_{\boldsymbol{T}_{A^*}} : (\mathcal{M}od^{cs}_{K^*})^{\boldsymbol{T}_{A^*}} \to \mathcal{M}od^{cs}_{K^*}$ the forgetful functor which maps (M^*, α) to M^*.

16 Haar measure on the Steenrod group

16.1 Invariant measures on profinite groups

Let G be a profinite group. We denote by $\mathcal{N}_G$ the set of all open normal subgroups of G and regard $\mathcal{N}_G$ as a directed set. We also denote by $\mathbf{Gr}_f$ the category of finite topological groups with discrete topology. Define a functor $D : \mathcal{N}_G \to \mathbf{Gr}_f$ by $D(H) = G/H$ and

$$D(H \hookrightarrow K) = \Big(\text{the unique map } G/H \xrightarrow{\tau_{H,K}} G/K \text{ satisfying } \tau_{H,K}p_H = p_K\Big)$$

where $p_H : G \to G/H$ is the quotient map. Then, $\Big(G \xrightarrow{p_H} G/H\Big)_{H\in\mathcal{N}_G}$ is a limiting cone of D.

Let $\varOmega$ be a complete ring satisfying the following conditions.

(i) There exists a fundamental system of neighborhood of 0 consisting of subsets of $\varOmega$ which are closed under addition.

(ii) There exists a closed subgroup $\mathcal{O}$ such that, for any open neighborhood U of 0, there exists an open neighborhood V of 0 satisfying $V\mathcal{O} \subset U$.

For example, if $\varOmega$ is a normed ring with non-archimedean norm $\nu : \varOmega \to [0, \infty)$, namely, ν satisfies $\nu(x+y) \leqq \max\{\nu(x), \nu(y)\}$ for any $x, y \in \varOmega$, then $\mathcal{V} = \{V \subset \varOmega \,|\, V = \{x \in \varOmega \,|\, \nu(x) < r\} \text{ for some } r > 0\}$ is a fundamental system of neighborhood of 0 which satisfies the condition (i) and $\mathcal{O} = \{x \in \varOmega \,|\, \nu(x) \leqq 1\}$ satisfies the condition (ii).

A subset $\mathcal{N}'_G$ of $\mathcal{N}_G$ is said to be cofinal if, for any $N \in \mathcal{N}_G$, there exits $H \in \mathcal{N}'_G$ which is contained in N.

Definition 16.1.1 *Let $\mathcal{N}'_G$ be a cofinal subset of $\mathcal{N}_G$. A family of maps $\mu = (\mu_H : G/H \to \mathcal{O})_{H\in\mathcal{N}'_G}$ is called a measure of G if μ satisfies the following condition.*

(meas) *If $H, K \in \mathcal{N}'_G$ and $H \subset K$, $\mu_K(x) = \sum\limits_{y\in\tau_{H,K}^{-1}(x)} \mu_H(y)$ for any $x \in G/K$.*

Lemma 16.1.2 *Let $\mu = (\mu_H : G/H \to \mathcal{O})_{H\in\mathcal{N}'_G}$ be a measure of G and N an open normal subgroup of G. If $H, K \in \mathcal{N}'_G$ are contained in N, the following equality holds for $x \in G/N$.*

$$\sum_{y\in\tau_{H,N}^{-1}(x)} \mu_H(y) = \sum_{z\in\tau_{K,N}^{-1}(x)} \mu_K(z)$$

Proof. Since $H \cap K$ is an open normal normal subgroup of G, there exists $L \in \mathcal{N}'_G$ which is contained in $H \cap K$. Since $\tau_{H,N}\tau_{L,H} = \tau_{L,K}\tau_{K,N} = \tau_{L,N}$, the assertion follows from the following.

$$\begin{aligned}\sum_{y\in\tau_{H,N}^{-1}(x)} \mu_H(y) &= \sum_{y\in\tau_{H,N}^{-1}(x)} \sum_{w\in\tau_{L,H}^{-1}(y)} \mu_L(w) = \sum_{w\in\tau_{L,H}^{-1}(\tau_{H,N}^{-1}(x))} \mu_L(w) = \sum_{w\in\tau_{L,N}^{-1}(x)} \mu_L(w)\\ &= \sum_{w\in\tau_{L,K}^{-1}(\tau_{K,N}^{-1}(x))} \mu_L(w) = \sum_{z\in\tau_{K,N}^{-1}(x)} \sum_{w\in\tau_{L,K}^{-1}(z)} \mu_L(w) = \sum_{z\in\tau_{K,N}^{-1}(x)} \mu_K(z)\end{aligned}$$

□

Proposition 16.1.3 *Suppose that a measure $\mu = (\mu_H : G/H \to \mathcal{O})_{H\in\mathcal{N}'_G}$ of G is given. For $N \in \mathcal{N}_G$, we choose $H \in \mathcal{N}'_G$ which is contained in N and define $\mu_N : G/N \to \mathcal{O}$ by*

$$\mu_N(x) = \sum_{y\in\tau_{H,N}^{-1}(x)} \mu_H(y)$$

for $x \in G/N$. Then, $\tilde{\mu} = (\mu_H : G/H \to \mathcal{O})_{H\in\mathcal{N}_G}$ is a measure of G

Proof. First note that $\mu_N(x)$ does not depend on the choice of $H \in \mathcal{N}'_G$ by (16.1.2). Suppose that $M, N \in \mathcal{N}_G$ satisfy $M \subset N$. We choose $H \in \mathcal{N}'_G$ which satisfies $H \subset M$. Then we have

$$\mu_N(x) = \sum_{u\in\tau_{H,N}^{-1}(x)} \mu_H(u) = \sum_{y\in\tau_{M,N}^{-1}(x)} \sum_{u\in\tau_{H,M}^{-1}(y)} \mu_H(u) = \sum_{y\in\tau_{M,N}^{-1}(x)} \mu_M(y).$$

□

Definition 16.1.4 *Let $\mu = (\mu_H : G/H \to \mathcal{O})_{H\in\mathcal{N}'_G}$ be a measure of G. For a continuous map $f : G \to \Omega$, $H \in \mathcal{N}_G$ and a map $s : G/H \to G$ satisfying $p_H s = id_{G/H}$, we set*

$$R_\mu(f; H, s) = \sum_{x\in G/H} f(s(x))\mu_H(x).$$

We call $R_\mu(f; H, s)$ the Riemannian sum of f.

Proposition 16.1.5 *Let $f : G \to \Omega$ be a continuous map. For each $H \in \mathcal{N}'_G$, we choose a map $s_H : G/H \to G$ satisfying $p_H s_H = id_{G/H}$. Then, $(R_\mu(f; H, s_H))_{H\in\mathcal{N}'_G}$ is a Cauchy sequence in Ω.*

Proof. Suppose that $L \subset H$ for $H, L \in \mathcal{N}'_G$ and that maps $s_H : G/H \to G$ and $s_L : G/L \to G$ satisfy $p_H s_H = id_{G/H}$ and $p_L s_L = id_{G/L}$, respectively. Then we have

$$\begin{aligned}
R_\mu(f; L, s_L) - R_\mu(f; H, s_H) &= \sum_{y\in G/L} f(s_L(y))\mu_L(y) - \sum_{x\in G/H} f(s_H(x))\mu_H(x)\\
&= \sum_{y\in G/L} f(s_L(y))\mu_L(y) - \sum_{x\in G/H}\sum_{y\in\tau^{-1}_{L,H}(x)} f(s_H(x))\mu_L(y)\\
&= \sum_{y\in G/L} (f(s_L(y)) - f(s_H(\tau_{L,H}(y))))\mu_L(y)\\
&= \sum_{y\in G/L} (f(s_L(y)) - f(s_H(\tau_{L,H}(p_L(s_L(y))))))\mu_L(y)\\
&= \sum_{y\in G/L} (f(s_L(y)) - f(s_H(p_H(s_L(y)))))\mu_L(y)
\end{aligned}$$

Since $p_H s_H = id_{G/H}$, we have $p_H(s_L(y)s_H(p_H(s_L(y)))^{-1}) = p_H(s_L(y))p_H(s_H(p_H(s_L(y))))^{-1}) = 1$ for $y \in G/L$. It follow that $s_L(y)s_H(p_H(s_L(y)))^{-1} \in H$ for any $y \in G/L$.

Let O be an open neighborhood of 0 of Ω and choose an open neighborhood U of 0 of Ω satisfying $U - U \subset O$. We can also choose a neighborhood V of 0 contained in U and closed under addition. Then, there exists an open neighborhood W of 0 satisfying $W\mathcal{O} \subset V$. Since G is compact, f is uniformly continuous. Hence there exists $N \in \mathcal{N}'_G$ satisfying $f(z) - f(w) \in W$ for any $z, w \in G$ satisfying $zw^{-1} \in N$. Then, for any $H, K \in \mathcal{N}'_G$ contained in N and $y \in G/L$ where $L \in \mathcal{N}'_G$ and $L \subset H \cap K$, since $s_L(y)s_H(p_H(s_L(y)))^{-1} \in H \subset N$ and $s_L(y)s_K(p_K(s_L(y)))^{-1} \in K \subset N$, both $f(s_L(y)) - f(s_H(p_H(s_L(y))))$ and $f(s_L(y)) - f(s_K(p_K(s_L(y))))$ belongs to W. Therefore $(f(s_L(y)) - f(s_H(p_H(s_L(y)))))\mu_L(y)$ and $(f(s_L(y)) - f(s_K(p_K(s_L(y)))))\mu_L(y)$ are contained in V for any $y \in G/L$. Since V is closed under addition, we have

$$\begin{aligned}
R_\mu(f; L, s_L) - R_\mu(f; H, s_H) &= \sum_{y\in G/L} (f(s_L(y)) - f(s_H(p_H(s_L(y)))))\mu_L(y) \in V\\
R_\mu(f; L, s_L) - R_\mu(f; K, s_K) &= \sum_{y\in G/L} (f(s_L(y)) - f(s_K(p_K(s_L(y)))))\mu_L(y) \in V.
\end{aligned}$$

Thus $R_\mu(f; K, s_K) - R_\mu(f; H, s_H) \in V - V \subset U - U \subset O$ and this shows that $(R_\mu(f; H, s_H))_{H\in\mathcal{N}'_G}$ is a Cauchy sequence in Ω. □

Proposition 16.1.6 *For each $H \in \mathcal{N}'_G$, we choose maps $s_H, t_H : G/H \to G$ satisfying $p_H s_H = p_H t_H = id_{G/H}$. Let $f : G \to \Omega$ be a continuous map. Then, Cauchy sequences $(R_\mu(f; H, s_H))_{H\in\mathcal{N}'_G}$ and $(R_\mu(f; H, t_H))_{H\in\mathcal{N}'_G}$ converge to the same point of $\mathcal{O}$.*

Proof. For an open neighborhood O of 0 of Ω, we choose an open neighborhood U of 0 contained in O and closed under addition. We also choose an open neighborhood V of 0 satisfying $V\mathcal{O} \subset U$. Then, there exists $K \in \mathcal{N}'_G$ such that $f(z) - f(w) \in V$ for any $z, w \in G$ satisfying $zw^{-1} \in K$. If $H \in \mathcal{N}'_G$ and $H \subset K$, then $s_H(x)t_H(x)^{-1} \in H \subset K$ for any $x \in G/H$ which implies $f(s_H(x)) - f(t_H(x)) \in W$. It follows that $(f(s_H(x)) - f(t_H(x)))\mu_H(x) \in V\mathcal{O} \subset U$, hence

$$R_\mu(f; H, s_H) - R_\mu(f; H, t_H) = \sum_{x\in G/H} (f(s_H(x)) - f(t_H(x)))\mu_H(x) \in U \subset O.$$

Therefore $(R_\mu(f; H, s_H))_{H\in\mathcal{N}'_G}$ and $(R_\mu(f; H, t_H))_{H\in\mathcal{N}'_G}$ converge to same value of $\mathcal{O}$. □

Proposition 16.1.7 *Let $\mu = (\mu_H : G/H \to \mathcal{O})_{H\in\mathcal{N}_G}$ be a measure of G and $\mathcal{N}'_G$ a cofinal subset of $\mathcal{N}_G$. We choose maps $s_H : G/H \to G$ satisfying $p_H s_H = id_{G/H}$ for each $H \in \mathcal{N}'_G$. For $N \in \mathcal{N}_G$, we take $H \in \mathcal{N}'_G$ which is contained in N and choose a map $\sigma^H_N : G/N \to G/H$ such that $\tau_{H,N}\sigma^H_N = id_{G/N}$. Put $s_N = s_H\sigma^H_N$. Then, $(R_\mu(f; H, s_H))_{H\in\mathcal{N}'_G}$ and $(R_\mu(f; N, s_N))_{N\in\mathcal{N}_G}$ converge to the same value of $\mathcal{O}$.*

Proof. Since σ^N_N is the identity map of G/N for $N \in \mathcal{N}'_G$, $(R_\mu(f; H, s_H))_{H\in\mathcal{N}'_G}$ is a cofinal subsequence of $(R_\mu(f; N, s_N))_{N\in\mathcal{N}_G}$. Hence $(R_\mu(f; H, s_H))_{H\in\mathcal{N}'_G}$ and $(R_\mu(f; N, s_N))_{N\in\mathcal{N}_G}$ converge to the same value of $\mathcal{O}$. □

Definition 16.1.8 *Let $\mu = (\mu_H : G/H \to \mathcal{O})_{H\in\mathcal{N}'_G}$ be a measure of G. We choose maps $s : G/H \to G$ satisfying $p_H s = id_{G/H}$ for $H \in \mathcal{N}'_G$. For a continuous map $f : G \to \Omega$, we denote by*

$$\int_G f(x)\, d\mu(x)$$

the limit of a Cauchy sequence $(R_\mu(f; H, s_H))_{H\in\mathcal{N}'_G}$.

Example 16.1.9 *For $N \in \mathcal{N}_G$ and a map $\lambda : G/N \to \Omega$, we put $f = \lambda p_N : G \to \Omega$. Then f is continuous and, for $H \in \mathcal{N}_G$ which is contained in N, we have $f s_H = \lambda p_N s_H = \lambda\tau_{H,N} p_H s_H = \lambda\tau_{H,N}$. Hence*

$$\begin{aligned} R_\mu(f; H, s_H) &= \sum_{x\in G/H} f(s_H(x))\mu_H(x) = \sum_{x\in G/H} \lambda(\tau_{H,N}(x))\mu_H(x) = \sum_{y\in G/N}\sum_{x\in\tau^{-1}_{H,N}(y)} \lambda(y)\mu_H(x) \\ &= \sum_{y\in G/N} \lambda(y)\mu_N(y) \end{aligned}$$

which implies that $\displaystyle\int_G f(x)\, d\mu(x) = \sum_{y\in G/N} \lambda(y)\mu_N(y)$. In particular, for fixed $c \in G/N$, if λ maps c to 1 and others to 0, we have $\displaystyle\int_G f(x)\, d\mu(x) = \mu_N(c)$.

For $g \in G$, let $L_g, R_g : G \to G$ be left and right translations.

Definition 16.1.10 *Let $\mu = (\mu_H : G/H \to \mathcal{O})_{H\in\mathcal{N}'_G}$ be a measure of G. We say that μ is left (resp. right) invariant if $\displaystyle\int_G fL_g(x)\, d\mu(x) = \int_G f(x)\, d\mu(x)$ $\left(\text{resp.} \displaystyle\int_G fR_g(x)\, d\mu(x) = \int_G f(x)\, d\mu(x)\right)$ for any continuous map $f : G \to \Omega$.*

Proposition 16.1.11 *A measure $\mu = (\mu_H : G/H \to \mathcal{O})_{H\in\mathcal{N}'_G}$ of G is left or right invariant if and only of $\mu_H : G/H \to \mathcal{O}$ is a constant map for any $H \in \mathcal{N}'_G$. Hence μ is left invariant if and only if right invariant.*

Proof. For $g \in G$ and $H \in \mathcal{N}'_G$, define $\lambda_1, \lambda_2 : G/H \to \Omega$ by $\lambda_1(e) = \lambda_2(p_H(g)) = 1$ and $\lambda_1(x) = \lambda_2(y) = 0$ for $x \neq e$, $y \neq p_H(g)$. Here e denotes the unit of G/H. We put $f_1 = \lambda_1 p_H$ and $f_2 = \lambda_2 p_H$. Then it can be verified that $f_2 L_g = f_2 R_g = f_1$. Hence it follows from (16.1.9) that

$$\int_G f_2L_g(x)\, d\mu(x) = \int_G f_2R_g(x)\, d\mu(x) = \int_G f_1(x)\, d\mu(x) = \mu_H(e) \quad \text{and} \quad \int_G f_2(x)\, d\mu(x) = \mu_H(p_H(g)).$$

Therefore, if μ is left or right invariant, we have $\mu_H(p_H(g)) = \mu_H(e)$.

Suppose that $\mu_H : G/H \to \mathcal{O}$ is a constant map for any $H \in \mathcal{N}'_G$. We choose maps $s_H : G/H \to G$ satisfying $p_H s_H = id_{G/H}$ for each $H \in \mathcal{N}'_G$ and define $s^g_H : G/H \to G$ by $s^g_H(x) = gs_H(p_H(g)^{-1}x)$ for $g \in G$. Then s^g_H satisfies $p_H s^g_H = id_{G/H}$. For a continuous map $f : G \to \Omega$, we have

$$\begin{aligned} R_\mu(fL_g; H, s_H) &= \sum_{x\in G/H} f(gs_H(x))\mu_H(x) = \sum_{x\in G/H} f(s^g_H(p_H(g)x))\mu_H(x) = \sum_{x\in G/H} f(s^g_H(x))\mu_H(p_H(g)^{-1}x) \\ &= \sum_{x\in G/H} f(s^g_H(x))\mu_H(x) = R_\mu(f; H, s^g_H). \end{aligned}$$

Since $(R_\mu(f; H, s^g_H))_{H\in\mathcal{N}'_G}$ converges to $\displaystyle\int_G f(x)\, d\mu(x)$ by (16.1.6), we have $\displaystyle\int_G fL_g(x)\, d\mu(x) = \int_G f(x)\, d\mu(x)$. Proof of $\displaystyle\int_G fR_g(x)\, d\mu(x) = \int_G f(x)\, d\mu(x)$ is similar. □

Let $\mu = (\mu_H : G/H \to \mathcal{O})_{H \in \mathcal{N}'_G}$ be an invariant measure of G. Then, each $\mu_H : G/H \to \mathcal{O}$ is a constant map by (16.1.10). We denote by $m_H \in \mathcal{O}$ the image of μ_H and define a map $m_\mu : \mathcal{N}'_G \to \mathcal{O}$ by $m_\mu(H) = m_H$. If $H \subset K$ for $H, K \in \mathcal{N}'_G$, since the number of elements of $\tau_{H,K}^{-1}(x)$ for $x \in G/K$ is the index $[K : H]$ of H in K, we have $m_\mu(K) = [K : H]m_\mu(H)$.

Conversely, suppose that a map $m : \mathcal{N}'_G \to \mathcal{O}$ satisfies $m(K) = [K : H]m(H)$ if $H \subset K$. If we define $\mu_H : G/H \to \mathcal{O}$ by $\mu_H(x) = m(H)$, then $\mu = (\mu_H : G/H \to \mathcal{O})_{H \in \mathcal{N}'_G}$ is an invariant measure of G. Hence we see the following fact.

Proposition 16.1.12 *There is a bijection between the set of invariant measures of G and the set of maps $m : \mathcal{N}'_G \to \mathcal{O}$ satisfying $m(K) = [K : H]m(H)$ if $H \subset K$.*

Suppose that Ω contains the field $\boldsymbol{Q}$ of rational numbers as a subring and that a map $m : \mathcal{N}'_G \to \mathcal{O}$ satisfies $m(K) = [K : H]m(H)$ if $H \subset K$. Then we have $m(H) = [G : H]^{-1}m(G)$. Suppose moreover that Ω is a non-archimedean normed ring with norm ν and $\mathcal{O} = \{x \in \Omega \,|\, \nu(x) \leqq 1\}$. Then, there exists a prime number q such that the norm on $\boldsymbol{Q}$ obtained by restricting ν to $\boldsymbol{Q}$ is the q-adic norm. If there exists an invarinat measure $\mu = (\mu_H : G/H \to \mathcal{O})_{H \in \mathcal{N}'_G}$ of G, we may assume that $m_\mu(G) = 1$, then it follows from $\nu(m_\mu(H))\nu([G : H]) = 1$ that $\nu([G : H]) = \nu(m_\mu(H))^{-1} \leqq 1$. Hence $\nu([G : H]) = 1$, in other words, $[G : H]$ is coprime to q. Conversely, assume that $[G : H]$ is coprime to q for any $H \in \mathcal{N}'_G$. Then, $[G : H]^{-1} \in \mathcal{O}$ and define $\mu_H : G/H \to \mathcal{O}$ by $\mu_H(x) = [G : H]^{-1}$.

16.2 Haar measure on the Steenrod group

Let $\tau^k_{A^*} : G_p^{k+1}(A^*) \to G_p^k(A^*)$ the the unique map that makes the following diagram commute.

$$\begin{tikzcd} G_p(A^*) \arrow[r, "\pi^{k+1}_{A^*}"] \arrow[dr, "\pi^k_{A^*}"'] & G_p^{k+1}(A^*) \arrow[d, "\tau^k_{A^*}"] \\ & G_p^k(A^*) \end{tikzcd}$$

For $\alpha(X) = \sum\limits_{i=0}^{k} \alpha_i X^{p^i} \in G_p^k(A^*)$, we have

$$(\tau^k_{A^*})^{-1}(\alpha(X)) = \left\{ \beta(X) \in G_p^{k+1}(A^*) \,\middle|\, \beta(X) = \sum_{i=0}^{k+1} \beta_i X^{p^i}, \beta_i = \alpha_i \; (0 \leqq i \leqq k) \right\}.$$

Define a map $\tilde{\tau}_k : A^{2(p^{k+1}-1)} \times A^{2p^{k+1}-1} \to \tau_k^{-1}(\alpha(X))$ by $\tilde{\tau}_k(x, y) = \sum\limits_{i=0}^{k} \alpha_i X^{p^i} + (x + \epsilon y)X^{p^{k+1}}$. Then, $\tilde{\tau}_k$ is bijective.

Suppose that a graded commutative $\boldsymbol{F}_p$-algebra A^* is finite type (at least $A^{2(p^i-1)}$ and $A^{2p^{i-1}-1}$ are finite dimensional for $i \geqq 0$). Then, the order of $G_p^k(A^*)$ which is equal to $\left[G_p(A^*) : G_p^{(k)}(A^*)\right]$ is

$$\prod_{i=1}^{k} \Bigl(p^{\dim A^{2(p^{i+1}-1)} + \dim A^{2p^{i+1}-1}} \Bigr).$$

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