Contents

Chapter 1

Γ-graded schemes

1.1 Notations and terminology

We fix two universes $\mathcal{U}$ and $\mathcal{V}$ such that $\boldsymbol{N} \in \mathcal{U}$ and $\mathcal{U} \in \mathcal{V}$. We call an element of $\mathcal{V}$ a set. A set is said to be small if it has the same cardinality as some element of $\mathcal{U}$.

If $\mathcal{C}$ is a category, we denote by $\mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Mor}\,\mathcal{C}$ the class of objects and the class of morphisms of $\mathcal{C}$, respectively. We often write $a \in \mathcal{C}$ instead of $a \in \mathrm{Ob}\,\mathcal{C}$. For $a, b \in \mathrm{Ob}\,\mathcal{C}$, we denote by $\mathcal{C}(a,b)$ the class of morphisms from a to b. When $\mathcal{C}$ is an abelian category, we use the familiar notation $\mathrm{Hom}_{\mathcal{C}}(a,b)$ instead of $\mathcal{C}(a,b)$.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, we denote by $\mathcal{C}\mathcal{D}$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$ whose morphisms are natural transformations. For an object S of a category $\mathcal{C}$, $\mathcal{C}/S$ denotes the category with $\mathrm{Ob}\,\mathcal{C}/S = \{$*morphisms of* $\mathcal{C}$ *whose targets are* $S\}$, and if $p : X \to S$, $q : Y \to S \in \mathrm{Ob}\,\mathcal{C}/S$, $\mathcal{C}/S(p,q) = \{f \in \mathcal{C}(X,Y)|\, qf = p\}$. The opposite category of $\mathcal{C}$ is denoted by $\mathcal{C}^{op}$.

We denote by $\mathbf{Ens}, \mathbf{Mon}, \mathbf{Gr}, \mathbf{Ab}, \mathcal{A}n, \mathcal{T}op, \ldots$ the categories of sets, monoids, groups, abelian groups, commutative unital rings, topological spaces,... belonging to $\mathcal{V}$.

$\mathcal{E}$ denotes one of categories $\mathbf{Ens}$, $\mathbf{Ab}$ or $\mathcal{T}op$.

Let F and G be functors from a category $\mathcal{C}$ to $\mathcal{E}$. If $\mathrm{G}(X)$ is a subset of $\mathrm{F}(X)$ for each object X of $\mathcal{C}$ and inclusion maps $\mathrm{G}(X) \hookrightarrow \mathrm{F}(X)$ give a natural transformation, we say that G is a subfunctor of F.

Let $\mathrm{S} : \mathcal{C} \to \mathcal{E}$ be a functor. The product exists in the category $\mathcal{C}\mathcal{E}/\mathrm{S}$. In fact, if $f : \mathrm{F} \to \mathrm{S}$ and $g : \mathrm{G} \to \mathrm{S}$ are objects of $\mathcal{C}\mathcal{E}/\mathrm{S}$, then $(f : \mathrm{F} \to \mathrm{S}) \times (g : \mathrm{G} \to \mathrm{S})$ is given by $f \times g : \mathrm{F} \times_{\mathrm{S}} \mathrm{G} \to \mathrm{S}$, where $(\mathrm{F} \times_{\mathrm{S}} \mathrm{G})(A) = \{(x,y) \in \mathrm{F}(A) \times \mathrm{G}(A)|\, f_A(x) = g_A(y)\}$ and $(f \times g)_A(x,y) = f_A(x)$. We denote by $\mathrm{pr}_{\mathrm{F}} : \mathrm{F} \times_{\mathrm{S}} \mathrm{G} \to \mathrm{F}$, $\mathrm{pr}_{\mathrm{G}} : \mathrm{F} \times_{\mathrm{S}} \mathrm{G} \to \mathrm{G}$ the projections.

Let $\mathcal{D}$ be a full subcategory of a category $\mathcal{C}$, and A an object of $\mathcal{C}$. We write PA for the functor $\mathcal{D} \to \mathcal{E}$ represented by A, that is, $PA(R) = \mathcal{C}(A,R)$ for $R \in \mathcal{D}$. If $f : A \to B$ is a morphism of $\mathcal{C}$, $Pf : PB \to PA$ is the natural transformation that maps $\varphi \in PB$ to φf. Thus we have a functor $P : \mathcal{C}^{op} \to \mathcal{D}\mathcal{E}$.

Proposition 1.1.1 *If $R \in \mathcal{D}$, $\mathrm{X} \in \mathcal{D}\mathcal{E}$ and $\rho \in \mathrm{X}(R)$, we define $\rho^\sharp : PR \to \mathrm{X}$ by $\rho^\sharp_S(f) = \mathrm{X}(f)(\rho)$ for $S \in \mathcal{D}$, $f \in PR(S)$. Then the correspondence $\rho \mapsto \rho^\sharp$ gives a natural bijection $\mathrm{X}(R) \to \mathcal{D}\mathcal{E}(PR, \mathrm{X})$, whose inverse is given by $\sigma \mapsto \sigma_R(id_R)$.*

For a functor $\mathrm{F} : \mathcal{C} \to \mathcal{E}$, we define the category of F-models $\mathcal{C}_{\mathrm{F}}$ by $\mathrm{Ob}\,\mathcal{C}_{\mathrm{F}} = \{(X,\rho)|\, X \in \mathrm{Ob}\,\mathcal{C}, \rho \in \mathrm{F}(X)\}$, $\mathcal{C}_{\mathrm{F}}((X,\rho),(Y,\sigma)) = \{\varphi|\, \varphi \in \mathcal{C}(X,Y),\, \mathrm{F}(\varphi)(\rho) = \sigma\}$. A functor $\mathrm{T} : \mathcal{C}_{\mathrm{F}} \to \mathcal{E}$ is called an F-functor.

Let $\mathrm{S} : \mathcal{C} \to \mathcal{E}$ be a functor. We define functors $i_{\mathrm{S}} : \mathcal{C}\mathcal{E}/\mathrm{S} \to \mathcal{C}_{\mathrm{S}}\mathcal{E}$ and $j_{\mathrm{S}} : \mathcal{C}_{\mathrm{S}}\mathcal{E} \to \mathcal{C}\mathcal{E}/\mathrm{S}$ as follows. For a functor $p : \mathrm{X} \to \mathrm{S}$ over S and an S-model (R,ρ), we set $i_{\mathrm{S}}(p)(R,\rho) = (\mathcal{C}\mathcal{E}/\mathrm{S})(\rho^\sharp, p)$. If $f : (p : \mathrm{X} \to \mathrm{S}) \to (q : \mathrm{Y} \to \mathrm{S})$ is a morphism of $\mathcal{C}\mathcal{E}/\mathrm{S}$, we define $i_{\mathrm{S}}(f) : i_{\mathrm{S}}(p) \to i_{\mathrm{S}}(q)$ by $i_{\mathrm{S}}(f)_{(R,\rho)}(\varphi) = f\varphi$ for $\varphi \in (\mathcal{C}\mathcal{E}/\mathrm{S})(\rho^\sharp, p)$. Let T be an S-functor. We define a functor ${}_{\boldsymbol{Z}}\mathrm{T} : \mathcal{C} \to \mathcal{E}$ by ${}_{\boldsymbol{Z}}\mathrm{T}(R) = \coprod_{\rho \in \mathrm{S}(R)} \mathrm{T}(R,\rho)$ and also define $p_{\mathrm{T}} : {}_{\boldsymbol{Z}}\mathrm{T} \to \mathrm{S}$ by $(p_{\mathrm{T}})_R(x) = \rho$ if $x \in \mathrm{T}(R,\rho)$. We set $j_{\mathrm{S}}(\mathrm{T}) = (p_{\mathrm{T}} : {}_{\boldsymbol{Z}}\mathrm{T} \to \mathrm{S})$. If $\psi : \mathrm{T} \to \mathrm{U}$ is a morphism of S-functors, define $j_{\mathrm{S}}(\psi) : j_{\mathrm{S}}(\mathrm{T}) \to j_{\mathrm{S}}(\mathrm{U})$ by $j_{\mathrm{S}}(\psi)(x) = \psi_{(R,\rho)}(x)$ for $x \in \mathrm{T}(R,\rho)$. Since $\coprod_{\rho \in \mathrm{S}(R)} (\mathcal{C}\mathcal{E}/\mathrm{S})(\rho^\sharp, p) = \mathcal{C}\mathcal{E}(PR, \mathrm{X}) \cong \mathrm{X}(R)$, we have the following.

Proposition 1.1.2 *The functor $i_{\mathrm{S}} : \mathcal{C}\mathcal{E}/\mathrm{S} \to \mathcal{C}_{\mathrm{S}}\mathcal{E}$ is an equivalence of categories, whose inverse is j_{S}.*

If T is an S-functor, we call $_{\boldsymbol{Z}}\mathrm{T}$ the underlying functor of T and $p : {}_{\boldsymbol{Z}}\mathrm{T} \to \mathrm{S}$ the structural projection. We note that, since $\mathcal{CE}/\mathrm{S}$ has a terminal object $id_{\mathrm{S}} : \mathrm{S} \to \mathrm{S}$, $i_{\mathrm{S}}(id_{\mathrm{S}})$ is a terminal object of $\mathcal{C}_{\mathrm{S}}\mathcal{E}$. We denote this by S., then, for an S-model (A, ρ), $\mathrm{S}.(A, \rho)$ consists of a single element (A, ρ).

Let $f : \mathrm{S} \to \mathrm{T}$ be a morphism of $\mathcal{CE}$. We define functors $f^* : \mathcal{C}_{\mathrm{T}}\mathcal{E} \to \mathcal{C}_{\mathrm{S}}\mathcal{E}$, $f_!, f_* : \mathcal{C}_{\mathrm{S}}\mathcal{E} \to \mathcal{C}_{\mathrm{T}}\mathcal{E}$ as follows. If Y is a T-functor and (R, ρ) is an S-model, define $f^*\mathrm{Y}$ by $f^*\mathrm{Y}(R, \rho) = \mathrm{Y}(R, f_R(\rho))$ and for a morphism $\alpha : \mathrm{Y}_1 \to \mathrm{Y}_2$ of T-functors, define $f^*\alpha : f^*\mathrm{Y}_1 \to f^*\mathrm{Y}_2$ by $(f^*\alpha)_{(R,\rho)} = \alpha_{(R,f_R(\rho))}$. If X is an S-functor and (R, ρ) is a T-model, define $f_!\mathrm{X}$ and $f_*\mathrm{X}$ by $f_!\mathrm{X}(R, \rho) = \coprod_{\sigma \in f_R^{-1}(\rho)} \mathrm{X}(R, \sigma)$ and $f_*\mathrm{X}(R, \rho) = \mathcal{C}_{\mathrm{S}}\mathcal{E}(f^*P(R, \rho), \mathrm{X})$, respectively. If $\alpha : \mathrm{X}_1 \to \mathrm{X}_2$ is a morphism of S-functors, define $f_!\alpha : f_!\mathrm{X}_1 \to f_!\mathrm{X}_2$ and $f_*\alpha : f_*\mathrm{X}_1 \to f_*\mathrm{X}_2$ by $(f_!\alpha)_{(R,\rho)} = \coprod_{\sigma \in f_R^{-1}(\rho)} \alpha_{(R,\sigma)}$ and $(f_*\alpha)_{(R,\rho)}(\theta) = \alpha\theta$, respectively. f^*, $f_!$ and f_* are called a base extension, a base restriction and a Weil restriction, respectively.

For f as above, we define functors $f^* : \mathcal{CE}/\mathrm{T} \to \mathcal{CE}/\mathrm{S}$ and $f_! : \mathcal{CE}/\mathrm{S} \to \mathcal{CE}/\mathrm{T}$ as follows. For a functor $p : \mathrm{X} \to \mathrm{T}$ over T, we set $f^*(p : \mathrm{X} \to \mathrm{T}) = (\mathrm{pr}_{\mathrm{S}} : \mathrm{X} \times_{\mathrm{T}} \mathrm{S} \to \mathrm{S})$. If $\alpha : (p : \mathrm{X} \to \mathrm{T}) \to (q : \mathrm{Y} \to \mathrm{T})$ is a morphism, we set $f^*\alpha = \alpha \times_{\mathrm{T}} id_{\mathrm{S}}$. For a functor $u : \mathrm{X} \to \mathrm{S}$ over S, we set $f_!(u : \mathrm{X} \to \mathrm{S}) = (fu : \mathrm{X} \to \mathrm{T})$. If $\alpha : (u : \mathrm{X} \to \mathrm{S}) \to (v : \mathrm{Y} \to \mathrm{S})$ is a morphism, we set $f_!\alpha = \alpha$.

Proposition 1.1.3 *1) We have $f_!j_{\mathrm{S}} = j_{\mathrm{T}}f_!$ and there are natural equivalences of functors $i_{\mathrm{S}}f^* \xrightarrow{\cong} f^*i_{\mathrm{T}}$ and $f^*j_{\mathrm{T}} \xrightarrow{\cong} j_{\mathrm{S}}f^*$.*

2) *Let X be an S-functor and Y a T-functor. There are natural equivalences $\chi(\mathrm{X}, \mathrm{Y}) : \mathcal{C}_{\mathrm{T}}\mathcal{E}(f_!\mathrm{X}, \mathrm{Y}) \to \mathcal{C}_{\mathrm{S}}\mathcal{E}(\mathrm{X}, f^*\mathrm{Y})$ and $\xi(\mathrm{Y}, \mathrm{X}) : \mathcal{C}_{\mathrm{S}}\mathcal{E}(f^*\mathrm{Y}, \mathrm{X}) \to \mathcal{C}_{\mathrm{T}}\mathcal{E}(\mathrm{Y}, f_*\mathrm{X})$ defined as follows. For $g : f_!\mathrm{X} \to \mathrm{Y}$ and $h : f^*\mathrm{Y} \to \mathrm{X}$, we set*

$$\chi(\mathrm{X}, \mathrm{Y})(g)_{(R,\rho)}(x) = g_{(R,f_R(\rho))}(x) \quad \text{if } (R, \rho) \in \mathcal{C}_{\mathrm{S}};\,; \text{and } x \in \mathrm{T}(R, \rho),$$

$$\xi(\mathrm{Y}, \mathrm{X})(h)_{(R,\rho)}(x) = h(f^*x^\sharp) \quad \text{if } (R, \rho) \in \mathcal{C}_{\mathrm{T}} \text{ and } x \in \mathrm{Y}(R, \rho).$$

Here $x^\sharp : P(R, \rho) \to \mathrm{Y}$ is a morphism defined by $x^\sharp_{(A,\sigma)}(\varphi) = \mathrm{Y}(\varphi)(x)$ for $(A, \sigma) \in \mathcal{C}_{\mathrm{T}}$ and $\varphi \in (P(R, \rho))(A, \sigma)$.

The category of Γ-rings is denoted by $\mathcal{A}n^\Gamma$. If $A \in \mathcal{A}n^\Gamma$, $\mathbf{Mod}_A^\Gamma$ represents the category of Γ-graded A-modules belonging to $\mathcal{V}$. A monoid, group, ring, module, topological space,... is called small if the underlying set is small. We fix a full subcategory $\boldsymbol{M}^\Gamma$ of $\mathcal{A}n^\Gamma$ such that $\mathrm{Ob}\boldsymbol{M}^\Gamma$ consists of small Γ-rings and every small Γ-ring is isomorphic to some object of $\boldsymbol{M}^\Gamma$. We call an object of $\boldsymbol{M}^\Gamma$ a Γ-model. If $k \in \mathcal{A}n^\Gamma$, we write $\mathcal{A}n_k^\Gamma$ for the category of commutative Γ-graded k-algebras. Similarly, if $k \in \boldsymbol{M}^\Gamma$, $\boldsymbol{M}_k^\Gamma$ represents full subcategory of $\mathcal{A}n_k^\Gamma$ formed by the k-algebra having a Γ-model as underlying Γ-ring. An object of $\boldsymbol{M}_k^\Gamma$ is called a Γ-k-model or k-model for short.

1.2 Graded additive category

Let Γ be an abelian group.

Definition 1.2.1 *A category $\mathcal{A}$ is a Γ-graded preadditive category if it satisfies the following axioms (A1), (A2) and (A3).*

(A1) If A and B are objects of $\mathcal{A}$, the set of morphisms $\mathrm{Hom}_{\mathcal{A}}(A, B)$ has a structure of Γ-graded abelian group. We write $\mathrm{Hom}_{\mathcal{A}}(A, B) = \sum_{g \in \Gamma} \mathrm{Hom}^g_{\mathcal{A}}(A, B)$ and we call an element of $\mathrm{Hom}^g_{\mathcal{A}}(A, B)$ a morphism of degree g.

(A2) Composition of morphisms is biadditive and the composition of a morphism of degree g and a morphism of degree h is a morphism of degree $g + h$.

(A3) $\mathcal{A}$ has a null object, that is, there is an object 0 such that $\mathrm{Hom}_{\mathcal{A}}(0, 0)$ is the zero group.

Moreover, if $\mathcal{A}$ satisfies (A4) below, it is called a Γ-graded additive category.

(A4) For $A, B \in \mathcal{A}$, there exists an object $C \in \mathcal{A}$ and morphisms $i_1 : A \to C$, $i_2 : B \to C$, $p_1 : C \to A$ and $p_2 : C \to B$ of degree zero which satisfy the identities $p_1i_1 = 1_A$, $p_2i_2 = 1_B$ and $i_1p_1 + i_2p_2 = 1_C$.

We denote by $\mathcal{A}_0$ a subcategory of $\mathcal{A}$ consisting of the same objects and morphisms of degree zero.

A functor $F : \mathcal{A} \to \mathcal{B}$ of Γ-graded preadditive categories is said to be additive if $F : \mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(B))$ is a homomorphism of Γ-graded abelian groups of degree zero for any objects A, B of $\mathcal{A}$.

An element of $\bigcup_{g \in \Gamma} \mathrm{Hom}^g_{\mathcal{A}}(A, B)$ is called a homogeneous morphism ("homomorphism" for short).

For a morphism $f : A \to B$, an equalizer (resp. coequalizer) of f and zero morphism is called a kernel (resp. cokernel). We denote by $\ker f : \mathrm{Ker} f \to A$ (resp. $\mathrm{coker} f : B \to \mathrm{Coker} f$) a kernel (resp. cokernel) of f.

Definition 1.2.2 *If a Γ-graded additive category $\mathcal{A}$ satisfies the following axioms, it is called a Γ-graded abelian category.*

(A5) For any homogeneous morphism $f : A \to B$, a kernel and a cokernel of f exist.

(A6) Every homogeneous monomorphism is a kernel of a homomorphism and every homogeneous epimorphism is a cokernel of a homomorphism.

For a Γ-graded abelian group M and $g \in \Gamma$, we denote by $\Sigma^g M$ a Γ-graded abelian group defined by $(\Sigma^g M)_h = M_{h-g}$. We call $\Sigma^g M$ the suspension of M by degree g.

Definition 1.2.3 *Let $\mathcal{A}$ be a Γ-graded preadditive category.*

1) For an object A of $\mathcal{A}$ and $g \in \Gamma$, consider a functor $B \mapsto \Sigma^g \mathrm{Hom}_{\mathcal{A}}(B, A)$ from $\mathcal{A}^{op}$ to the category of Γ-graded abelian groups. If there exists an object X of $\mathcal{A}$ and a natural equivalence $S^g_A : \Sigma^g \mathrm{Hom}_{\mathcal{A}}(B, A) \to \mathrm{Hom}_{\mathcal{A}}(B, X)$ of degree zero, we call X a suspension of A by degree g and denote this by $\Sigma^g A$. We put $s^g_A = S^g_A(id_A)$ then $s^g : A \to \Sigma^g A$ is an isomorphism of degree g.

2) If $\mathcal{A}$ satisfies the following axiom (A7), $\mathcal{A}$ is said to be stable.

(A7) $\mathcal{A}$ has a suspension of each object by any degree.

Proposition 1.2.4 *1) If a suspension of A by degree g exists, $s^g_{A*} : \Sigma^g \mathrm{Hom}_{\mathcal{A}}(B, A) \to \mathrm{Hom}_{\mathcal{A}}(B, \Sigma^g A)$ and $s^{g*}_A : \mathrm{Hom}_{\mathcal{A}}(\Sigma^g A, B) \to \Sigma^{-g} \mathrm{Hom}_{\mathcal{A}}(A, B)$ are isomorphisms of Γ-graded abelian groups of degree zero.*

2) If $\Sigma^g A$ and $\Sigma^h(\Sigma^g A)$ exist, then $\Sigma^{g+h} A$ exists and for any choice of $\Sigma^{g+h} A$ there is a unique isomorphism $\kappa : \Sigma^{g+h} A \to \Sigma^h(\Sigma^g A)$ such that $\kappa s^{g+h}_A = s^h_{\Sigma^g A} s^g_A$.

If $\mathcal{A}$ has a suspension by degree g for any object, $A \mapsto \Sigma^g A$ defines a functor $\Sigma^g : \mathcal{A} \to \mathcal{A}$ such that $\Sigma^g : \mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{A}}(\Sigma^g A, \Sigma^g B)$ is an isomorphism of degree zero. In fact, Σ is the composition of isomorphisms $(s^{g*}_A)^{-1}$ and s^g_{B*}. Moreover, correspondence $A \mapsto s^g_A$ gives a natural transformation $s_g : id_{\mathcal{A}} \to \Sigma^g$.

Let $\mathcal{A}$ be an additive category. We can form a Γ-graded additive category $\mathcal{A}^\Gamma$ as follows.

Construction 1.2.5 *An object of $\mathcal{A}^\Gamma$ is a Γ-indexed family of objects of $\mathcal{A}$. A morphism $f = (f_h)_{h\in\Gamma} : (A_h)_{h\in\Gamma} \to (B_h)_{h\in\Gamma}$ of degree g is a Γ-indexed family of morphisms $f_h : A_h \to B_{g+h}$ of $\mathcal{A}$. Thus we have*

$$\mathrm{Hom}^g_{\mathcal{A}^\Gamma}((A_h)_{h\in\Gamma}, (B_h)_{h\in\Gamma}) = \prod_{h\in\Gamma} \mathrm{Hom}_{\mathcal{A}}(A_h, B_{g+h})$$

and $\mathrm{Hom}_{\mathcal{A}^\Gamma}((A_h)_{h\in\Gamma}, (B_h)_{h\in\Gamma})$ is defined to be the direct sum of $\mathrm{Hom}^g_{\mathcal{A}^\Gamma}((A_h)_{h\in\Gamma}, (B_h)_{h\in\Gamma})$ for $g \in \Gamma$.

There is a faithful functor $I : \mathcal{A} \to \mathcal{A}^\Gamma$ given by $I(A) = (A_g)_{g\in\Gamma}$ and $I(f) = (f_g)_{g\in\Gamma}$, where $A_0 = A$, $A_g = 0$ if $g \neq 0$ and $f_0 = f$, $f_g = 0$ if $g \neq 0$.

If $\mathcal{A}$ has a direct sum of any Γ-indexed family of objects, $\mathcal{A}^\Gamma$ is identified with a subcategory of $\mathcal{A}$ consisting of objects of the form $\sum_{g\in\Gamma} A_g$ for $A_g \in \mathcal{A}$ and morphisms of the form $f = f_1 + f_2 + \cdots + f_n$ where $f_i : \sum_{h\in\Gamma} A_h \to \sum_{h\in\Gamma} B_h$ maps A_h to B_{h+g_i}.

Proposition 1.2.6 *1) $\mathcal{A}^\Gamma$ is stable.*

2) If $\mathcal{A}$ is an abelian category, $\mathcal{A}^\Gamma$ is a Γ-graded abelian category.

Proof. In fact, for an object $A = (A_h)_{h\in\Gamma}$ and $g \in \Gamma$, $\Sigma^g A$ is given by $\Sigma^g A = ((\Sigma^g A)_h)_{h\in\Gamma}$ where $(\Sigma^g A)_h = A_{h-g}$. □

Let $\sigma : \Gamma \to \boldsymbol{Z}/2 = \{0, 1\}$ be a homomorphism of abelian groups. We call σ a signature of Γ. From now on, we fix a signature σ of Γ.

Let $\mathcal{A}$ and $\mathcal{B}$ be Γ-graded preadditive categories.

Construction 1.2.7 *Let us denote by $\mathcal{A} \times_\Gamma \mathcal{B}$ a Γ-graded preadditive category defined as follows. An object of $\mathcal{A} \times_\Gamma \mathcal{B}$ is a pair (A, B) of an object A of $\mathcal{A}$ and an object B of $\mathcal{B}$. For $g \in \Gamma$, $\mathrm{Hom}^g_{\mathcal{A}\times_\Gamma\mathcal{B}}((A_1, B_1), (A_2, B_2))$ is defined to be $\sum_{h+k=g} \mathrm{Hom}^h_{\mathcal{A}}(A_1, A_2) \otimes \mathrm{Hom}^k_{\mathcal{B}}(B_1, B_2)$. The composition law $\mathrm{Hom}_{\mathcal{A}\times_\Gamma\mathcal{B}}((A_2, B_2), (A_3, B_3)) \times \mathrm{Hom}_{\mathcal{A}\times_\Gamma\mathcal{B}}((A_1, B_1), (A_2, B_2)) \to \mathrm{Hom}_{\mathcal{A}\times_\Gamma\mathcal{B}}((A_1, B_1), (A_3, B_3))$ is given by $(\varphi' \otimes \psi', \varphi \otimes \psi) \mapsto (-1)^{\sigma(h)\sigma(k)} \varphi'\varphi \otimes$*

$\psi'\psi$, where $\varphi \in \mathrm{Hom}^h(A_1, A_2)$, $\varphi' \in \mathrm{Hom}^j(A_2, A_3)$, $\psi \in \mathrm{Hom}^i(B_1, B_2)$, $\psi' \in \mathrm{Hom}^k(B_2, B_3)$. We call $\mathcal{A} \times_\Gamma \mathcal{B}$ the Γ-graded product (Γ-product for short) of $\mathcal{A}$ and $\mathcal{B}$.

More generally, for Γ-graded preadditive categories $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ we define the Γ-graded product $\mathcal{A}_1 \times_\Gamma \cdots \times_\Gamma \mathcal{A}_n$ as follows. An object of $\mathcal{A}_1 \times_\Gamma \cdots \times_\Gamma \mathcal{A}_n$ is an ordered n-tuple $(A_1, A_2, \ldots, A_n)$ of objects A_i of $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$. For $g \in \Gamma$, we define $\mathrm{Hom}^g_{\mathcal{A}_1 \times_\Gamma \cdots \times_\Gamma \mathcal{A}_n}((A_1, \ldots, A_n), (B_1, \ldots, B_n))$ to be the direct sum of abelian groups $\mathrm{Hom}^{h_1}_{\mathcal{A}_1}(A_1, B_1) \otimes \cdots \otimes \mathrm{Hom}^{h_n}_{\mathcal{A}_n}(A_n, B_n)$ for $h_1 + \cdots + h_n = g$. The composition law in $\mathcal{A}_1 \times_\Gamma \cdots \times_\Gamma \mathcal{A}_n$ is given by $(\varphi'_1 \otimes \cdots \otimes \varphi'_n, \varphi_1 \otimes \cdots \otimes \varphi_n) \mapsto (-1)^\varepsilon \varphi'_1\varphi_1 \otimes \cdots \otimes \varphi'_n\varphi_n$, where $\varphi_i \in \mathrm{Hom}^{h_i}_{\mathcal{A}_i}(A_i, B_i)$, $\varphi'_i \in \mathrm{Hom}^{k_i}_{\mathcal{A}_i}(B_i, C_i)$ and $\varepsilon = \sum\limits_{i<j} \sigma(h_i)\sigma(k_j)$.

Proposition 1.2.8 *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be Γ-graded preadditive categories. There is an isomorphism of categories $\Psi : (\mathcal{A}_1 \times_\Gamma \cdots \times_\Gamma \mathcal{A}_m) \times_\Gamma (\mathcal{A}_{m+1} \times_\Gamma \cdots \times_\Gamma \mathcal{A}_n) \to \mathcal{A}_1 \times_\Gamma \cdots \times_\Gamma \mathcal{A}_n$ defined by*

$$\Psi((A_1, \ldots, A_m), (A_{m+1}, \ldots, A_n)) = (A_1, \ldots, A_n), \quad \Psi((\varphi_1 \otimes \cdots \otimes \varphi_m) \otimes (\varphi_{m+1} \otimes \cdots \otimes \varphi_n)) = \varphi_1 \otimes \cdots \otimes \varphi_n).$$

From now on, we identify categories $(\mathcal{A}_1 \times_\Gamma \cdots \times_\Gamma \mathcal{A}_m) \times_\Gamma (\mathcal{A}_{m+1} \times_\Gamma \cdots \times_\Gamma \mathcal{A}_n)$ and $\mathcal{A}_1 \times_\Gamma \cdots \times_\Gamma \mathcal{A}_n$. Consequently, $\mathcal{A} \times_\Gamma (\mathcal{B} \times_\Gamma (\mathcal{C} \times_\Gamma \mathcal{D}))$, $(\mathcal{A} \times_\Gamma \mathcal{B}) \times_\Gamma (\mathcal{C} \times_\Gamma \mathcal{D})$, $((\mathcal{A} \times_\Gamma \mathcal{B}) \times_\Gamma \mathcal{C}) \times_\Gamma \mathcal{D}$, $\mathcal{A} \times_\Gamma ((\mathcal{B} \times_\Gamma \mathcal{C}) \times_\Gamma \mathcal{D})$ and $(\mathcal{A} \times_\Gamma (\mathcal{B} \times_\Gamma \mathcal{C})) \times_\Gamma \mathcal{D}$ are all identified with $\mathcal{A} \times_\Gamma \mathcal{B} \times_\Gamma \mathcal{C} \times_\Gamma \mathcal{D}$.

If $F_i : \mathcal{A}_i \to \mathcal{B}_i$ $(1 \leqq i \leqq n)$ are additive functors of preadditive categories, we can form the (Γ-graded) product of these functors. That is, $F_1 \times_\Gamma \cdots \times_\Gamma F_n : \mathcal{A}_1 \times_\Gamma \cdots \times_\Gamma \mathcal{A}_n \to \mathcal{B}_1 \times_\Gamma \cdots \times_\Gamma \mathcal{B}_n$ is defined by $F_1 \times_\Gamma \cdots \times_\Gamma F_n(A_1, \ldots A_n) = (F_1(A_1), \ldots, F_n(A_n))$ and $F_1 \times_\Gamma \cdots \times_\Gamma F_n(\varphi_1 \otimes \cdots \otimes \varphi_n) = F_1(\varphi_1) \otimes \cdots \otimes F_n(\varphi_n)$.

Define a functor $T : \mathcal{A} \times_\Gamma \mathcal{B} \to \mathcal{B} \times_\Gamma \mathcal{A}$ by $T(A, B) = (B, A)$ and $T(\varphi \otimes \psi) = (-1)^{\sigma(h)\sigma(k)}\psi \otimes \varphi$ for homomorphisms φ, ψ of degree h, k, respectively.

Definition 1.2.9 *Let $\mathcal{A}$ be an Γ-graded preadditive category. If an additive functor $\mathcal{T} : \mathcal{A} \times_\Gamma \mathcal{A} \to \mathcal{A}$ satisfies the following conditions (T1) and (T2), we call $(\mathcal{A}, \mathcal{T})$ a Γ-graded preadditive symmetric monoidal category.*

(T1) There are an object E of $\mathcal{A}$ and three natural equivalences of functors $\alpha : \mathcal{T}(1_\mathcal{A} \times_\Gamma \mathcal{T}) \to \mathcal{T}(\mathcal{T} \times_\Gamma 1_\mathcal{A})$, $\lambda : \mathcal{T}(E, -) \to 1_\mathcal{A}$ and $\gamma : \mathcal{T} \to T\mathcal{T}$ such that the following diagrams commute.

$$\begin{tikzcd}
\mathcal{T}(A, \mathcal{T}(B, \mathcal{T}(C, D))) \arrow[r, "\alpha"] \arrow[d, "\mathcal{T}(1,\alpha)"] & \mathcal{T}(\mathcal{T}(A, B), \mathcal{T}(C, D)) \arrow[r, "\alpha"] & \mathcal{T}(\mathcal{T}(\mathcal{T}(A, B), C), D) \\
\mathcal{T}(A, \mathcal{T}(\mathcal{T}(B, C), D)) \arrow[r, "\alpha"] & \mathcal{T}(\mathcal{T}(A, \mathcal{T}(B, C)), D) \arrow[ur, "\mathcal{T}(\alpha,1)"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{T}(A, \mathcal{T}(B, C)) \arrow[r, "\alpha"] \arrow[d, "\mathcal{T}(1,\gamma)"] & \mathcal{T}(\mathcal{T}(A, B), C) \arrow[r, "\gamma"] & \mathcal{T}(C, \mathcal{T}(A, B)) \arrow[d, "\alpha"] \\
\mathcal{T}(A, \mathcal{T}(C, B)) \arrow[r, "\alpha"] & \mathcal{T}(\mathcal{T}(A, C), B) \arrow[r, "\mathcal{T}(\gamma,1)"] & \mathcal{T}(\mathcal{T}(C, A), B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T}(A, \mathcal{T}(E, C)) \arrow[r, "\alpha"] \arrow[d, "\mathcal{T}(1,\lambda)"] & \mathcal{T}(\mathcal{T}(A, E), C) \arrow[d, "\mathcal{T}(\gamma,1)"] \\
\mathcal{T}(A, C) & \mathcal{T}(\mathcal{T}(E, A), C) \arrow[l, "\mathcal{T}(\lambda,1)"']
\end{tikzcd}$$

Moreover, compositions $\lambda\gamma : \mathcal{T}(E, E) \to \mathcal{T}(E, E) \to E$, $\gamma_T\gamma : \mathcal{T} \to \mathcal{T}T \to \mathcal{T}TT = \mathcal{T}$ coincides with λ, $1_\mathcal{T}$, respectively.

(T2) $\mathcal{T}$ commutes with sums. Namely, if a sum of a family $(A_i)_{i \in I}$ of objects of $\mathcal{A}$ exists, sums of $(\mathcal{T}(A_i, B))_{i \in I}$, $(\mathcal{T}(B, A_i))_{i \in I}$ exists for any object B and the canonical maps $\mathcal{T}(\iota_i, 1_B) : \mathcal{T}(A_i, B) \to \mathcal{T}\Big(\sum\limits_{i \in I} A_i, B\Big)$, $\mathcal{T}(1_B, \iota_i) : \mathcal{T}(B, A_i) \to \mathcal{T}\Big(B, \sum\limits_{i \in I} A_i\Big)$ induces isomorphisms $\sum\limits_{i \in I} \mathcal{T}(A_i, B) \to \mathcal{T}(\sum\limits_{i \in I} A_i, B)$, $\sum\limits_{i \in I} \mathcal{T}(B, A_i) \to \mathcal{T}(B, \sum\limits_{i \in I} A_i)$, respectively.

We often denote $\mathcal{T}(A, B)$ by $A \otimes B$.

Suppose that an additive category $\mathcal{A}$ has sums and that it is a symmetric monoidal category by an additive functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ with a unital object E and natural equivalences $\alpha : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, $\gamma : A \otimes B \to B \otimes A$, $\lambda : E \otimes A \to A$. If $\otimes$ commutes with sums, we can give $\mathcal{A}^\Gamma$ a structure of a Γ-graded symmetric monoidal category as follows. Define $\otimes : \mathcal{A}^\Gamma \times \mathcal{A}^\Gamma \to \mathcal{A}^\Gamma$ by $(\sum\limits_{g \in \Gamma} A_g) \otimes (\sum\limits_{g \in \Gamma} B_g) = \sum\limits_{g \in \Gamma} (\sum\limits_{h+k=g} A_h \otimes B_k)$. For objects $A = \sum\limits_{g \in \Gamma} A_g$, $B = \sum\limits_{g \in \Gamma} B_g$ and $C = \sum\limits_{g \in \Gamma} C_g$, let $\rho_{h,k} : \sum\limits_{i+j=k} A_h \otimes (B_i \otimes C_j) \to A_h \otimes (\sum\limits_{i+j=k} B_i \otimes C_j)$ and $\nu_{k,j} : \sum\limits_{h+i=k} (A_h \otimes B_i) \otimes C_j \to (\sum\limits_{h+i=k} A_h \otimes B_i) \otimes C_j$ be natural isomorphisms. We define $\alpha : A \otimes (B \otimes C) \to$

$(A\otimes B)\otimes C$ to be the composition of the following morphisms. $\sum\limits_{g\in\Gamma}(\sum\limits_{h+k=g}\rho^{-1}_{h,k}) : A\otimes(B\otimes C) = \sum\limits_{g\in\Gamma}(\sum\limits_{h+k=g} A_h\otimes(\sum\limits_{i+j=k} B_i\otimes C_j)) \to \sum\limits_{g\in\Gamma}(\sum\limits_{h+k=g}\sum\limits_{i+j=k} A_h\otimes(B_i\otimes C_j)) = \sum\limits_{g\in\Gamma}(\sum\limits_{h+i+j=g} A_h\otimes(B_i\otimes C_j))$,
$\sum\limits_{g\in\Gamma}(\sum\limits_{h+i+j=g}\alpha) : \sum\limits_{g\in\Gamma}(\sum\limits_{h+i+j=g} A_h\otimes(B_i\otimes C_j)) \to \sum\limits_{g\in\Gamma}(\sum\limits_{h+i+j=g}(A_h\otimes B_i)\otimes C_j)$, $\sum\limits_{g\in\Gamma}(\sum\limits_{k+j=g}\nu_{k,j}) : \sum\limits_{g\in\Gamma}(\sum\limits_{h+i+j=g}(A_h\otimes B_i)\otimes C_j) = \sum\limits_{g\in\Gamma}(\sum\limits_{k+j=g}\sum\limits_{h+i=k}(A_h\otimes B_i)\otimes C_j) \to \sum\limits_{g\in\Gamma}(\sum\limits_{k+j=g}(\sum\limits_{h+i=k}A_h\otimes B_i)\otimes C_j) = (A\otimes B)\otimes C$.
Define $\gamma : A\otimes B\to B\otimes A$ to be the sum of morphisms $(-1)^{\sigma(i)\sigma(j)}\gamma : A_i\otimes B_j\to B_j\otimes A_i$. The unital object E of $\mathcal{A}^\Gamma$ is given by $E=\sum\limits_{g\in\Gamma}E_g$, where E_0 is the unital object of $\mathcal{A}$ and $E_g=0$ if $g\neq 0$. Since $0\otimes A_k$ is a null object of $\mathcal{A}$, there is a natural isomorphism $\eta_g : \sum\limits_{h+k=g}E_h\otimes A_k\to E_0\otimes A_g$. We define $\lambda : E\otimes A\to A$ to be the composition of $\sum\limits_{g\in\Gamma}\eta_g : E\otimes A=\sum\limits_{g\in\Gamma}(\sum\limits_{h+k=g}E_h\otimes A_k)\to\sum\limits_{g\in\Gamma}E_0\otimes A_g$ and $\sum\limits_{g\in\Gamma}\lambda : \sum\limits_{g\in\Gamma}E_0\otimes A_g\to\sum\limits_{g\in\Gamma}A_g=A$.
It is easy to verify that $(\mathcal{A}^\Gamma,\otimes)$ is a Γ-graded additive symmetric monoidal category.

Let $(\mathcal{A},\otimes)$ be a Γ-graded preadditive symmetric monoidal category.

Definition 1.2.10 *A Γ-graded ring in $(\mathcal{A},\otimes)$ is an object A of $\mathcal{A}$ with homomorphisms $\mu : A\otimes A\to A$ and $\eta : E\to A$ of degree zero such that the following diagrams commutes.*

$$\begin{tikzcd}
A\otimes(A\otimes A) \arrow[r, "1\otimes\mu"] \arrow[d, "\alpha"] & A\otimes A \arrow[r, "\mu"] & A \\
(A\otimes A)\otimes A \arrow[r, "\mu\otimes 1"] & A\otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
E\otimes A \arrow[r, "\eta\otimes 1"] \arrow[dr, "\gamma"'] & A\otimes A \arrow[d, "\mu"] & A\otimes E \arrow[l, "1\otimes\eta"'] \arrow[d, "\gamma"] \\
 & A & E\otimes A \arrow[l, "\lambda"']
\end{tikzcd}$$

Moreover, if the composite $A\otimes A\xrightarrow{\gamma}A\otimes A\xrightarrow{\mu}A$ coincides with $\mu : A\otimes A\to A$, (A,μ,η) is called a Γ-graded commutative ring (Γ-ring for short).

A morphism $f : (A,\mu,\eta)\to(A',\mu',\eta')$ of Γ-graded rings is a homomorphism $f : A\to A'$ of degree zero such that $f\mu=\mu'(f\otimes f) : A\otimes A\to A'$, $f\eta=\eta' : E\to A$.

Definition 1.2.11 *Let (A,μ,η) be a Γ-graded ring in $(\mathcal{A},\otimes)$. A left A-module is an object M of $\mathcal{A}$ with a homomorphism $\nu : A\otimes M\to M$ of degree zero such that the following diagram commutes.*

$$\begin{tikzcd}
A\otimes(A\otimes M) \arrow[r, "1\otimes\nu"] \arrow[d, "\alpha"] & A\otimes M \arrow[r, "\nu"] & M \\
(A\otimes A)\otimes M \arrow[r, "\mu\otimes 1"] & A\otimes M \arrow[ur, "\nu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
E\otimes M \arrow[r, "\lambda"] \arrow[d, "\eta\otimes 1"] & M \\
A\otimes M \arrow[ur, "\nu"'] &
\end{tikzcd}$$

A morphism $f : (M,\nu)\to(M',\nu')$ of A-modules is a morphism $f : M\to M'$ of $\mathcal{A}$ such that $f\nu=\nu'(1\otimes f) : A\otimes M\to M'$.

A right A-module is defined similarly, that is, an object M of $\mathcal{A}$ with a homomorphism $\nu : M\otimes A\to M$ of degree zero such that the following diagram commutes.

$$\begin{tikzcd}
M\otimes(A\otimes A) \arrow[r, "1\otimes\mu"] \arrow[d, "\alpha"] & M\otimes A \arrow[r, "\nu"] & M \\
(M\otimes A)\otimes A \arrow[r, "\nu\otimes 1"] & M\otimes A \arrow[ur, "\nu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M\otimes E \arrow[r, "\gamma"] \arrow[d, "1\otimes\eta"] & E\otimes M \arrow[d, "\lambda"] \\
M\otimes A \arrow[r, "\nu"] & M
\end{tikzcd}$$

If a Γ-graded ring (A,μ,η) is commutative, we can regard a left (resp. right) A-module (M,ν) as a right (resp. left) A-module (M,ν^*) by defining $\nu^* : M\otimes A\to M$ (resp. $\nu^* : A\otimes M\to M$) to be the composite $M\otimes A\xrightarrow{\gamma}A\otimes M\xrightarrow{\nu}M$ (resp. $A\otimes M\xrightarrow{\gamma}M\otimes A\xrightarrow{\nu}M$).

1.3 Internal graded abelian groups

Throughout this section, $\mathcal{E}$ is a category which has a terminal object and finite products.

Definition 1.3.1 *(1) An internal monoid in $\mathcal{E}$ consists of an object M and morphisms $e : 1\to M$, $m : M\times M\to M$ making following diagrams commute.*

$$\begin{tikzcd} M \times M \times M \arrow[r, "m\times id_M"] \arrow[d, "id_M\times m"] & M \times M \arrow[d, "m"] \\ M \times M \arrow[r, "m"] & M \end{tikzcd} \qquad \begin{tikzcd} M \times 1 \arrow[r, "id_M\times e"] \arrow[dr, "\mathrm{pr}_1"'] & M \times M \arrow[d, "m"] & 1 \times M \arrow[l, "e\times id_M"'] \arrow[dl, "\mathrm{pr}_2"] \\ & M & \end{tikzcd}$$

A morphism $f : (M_1, e_1, m_1) \to (M_2, e_2, m_2)$ of internal monoids is a morphism $f : M_1 \to M_2$ of $\mathcal{E}$ such that $fe_1 = e_2$ and $fm_1 = m_2(f \times f)$. We denote by $\mathbf{mon}(\mathcal{E})$ *the category of internal monoids in $\mathcal{E}$.*

(2) An internal group in $\mathcal{E}$ consists of an object G and morphisms $e : 1 \to G$, $m : G \times G \to G$ $i : G \to G$ such that (G, e, m) is an internal monoid and the following diagrams commutes.

$$\begin{tikzcd} G \arrow[r, "(id_G,i)"] \arrow[d] & G \times G \arrow[d, "\mu"] & G \arrow[l, "(i,id_G)"'] \arrow[d] \\ 1 \arrow[r, "\varepsilon"] & G & 1 \arrow[l, "\varepsilon"'] \end{tikzcd}$$

(3) An internal monoid (M, e, m) is said to be commutative if $Tm = m$ holds, where $T = (\mathrm{pr}_2, \mathrm{pr}_1) : M \times M \to M \times M$. A commutative internal monoid (resp. group) is called an internal abelian monoid (resp. group).

Proposition 1.3.2 *Let (M, m, e) be an internal monoid in $\mathcal{E}$. If morphisms $f, g, h : N \to M$ make the following diagram commute, then $f = h$.*

$$\begin{tikzcd} N \arrow[r, "(g,h)"] \arrow[d] & M \times M \arrow[d, "m"] & N \arrow[l, "(f,g)"'] \arrow[d] \\ 1 \arrow[r, "e"] & M & 1 \arrow[l, "e"'] \end{tikzcd}$$

In particular, if $i_1, i_2 : M \to M$ are morphisms making

$$\begin{tikzcd} M \arrow[r, "(id_M,i_2)"] \arrow[d] & M \times M \arrow[d, "\mu"] & M \arrow[l, "(i_1,id_M)"] \arrow[d] \\ 1 \arrow[r, "e"] & M & 1 \arrow[l, "e"] \end{tikzcd}$$

commute, we have $i_1 = i_2$.

This implies the uniqueness of the morphism $i : G \to G$ making the diagram in (2) of (1.3.1) and that $i^2 = id_G$ holds. Moreover, if $f : G \to H$ is a morphism of internal monoids and both G and H are internal groups, it is easily verified that f commutes with i by applying the above proposition. Hence we can regard the category of internal groups in $\mathcal{E}$ as a full subcategory of $\mathbf{mon}(\mathcal{E})$. We denote by $\mathbf{grp}(\mathcal{E})$ (resp. $\mathbf{cmon}(\mathcal{E})$, $\mathbf{ab}(\mathcal{E})$) the category of internal groups (resp. abelian monoids, abelian groups) in $\mathcal{E}$. Note that there are forgetful functors $\mathbf{mon}(\mathcal{E}) \to \mathcal{E}$, $\mathbf{grp}(\mathcal{E}) \to \mathcal{E}$.

In particular, if $\mathcal{E} = \mathcal{S}$ the category of sets, $\mathbf{mon}(\mathcal{S})$, $\mathbf{grp}(\mathcal{S})$, $\mathbf{cmon}(\mathcal{S})$ and $\mathbf{ab}(\mathcal{S})$ are the categories of monoids, groups, abelian monoids and abelian groups respectively. We denote these categories **mon**, **grp**, **cmon** and **ab** for short.

Proposition 1.3.3 *Categories* $\mathbf{mon}(\mathcal{E})$, $\mathbf{grp}(\mathcal{E})$, $\mathbf{cmon}(\mathcal{E})$ *and* $\mathbf{ab}(\mathcal{E})$ *have products and forgetful functors preserves them.*

In fact, a product of internal monoids (M_i, m_i, e_i) $(i = 1, 2)$ is given by $(M_1 \times M_2, (m_1 \times m_2)(id_{M_1} \times T \times id_{M_2}), (e_1, e_2))$.

Definition 1.3.4 *Let $\mathcal{C}$ be a category.*

1) An epimorphism is said to be regular if it is a coequalizer of some pair of morphisms.

2) A pair of morphisms $R \underset{\tau}{\overset{\sigma}{\rightrightarrows}} X$ is called an equivalence relation on X if $(\sigma_*, \tau_*) : \mathrm{Hom}_{\mathcal{C}}(Y, R) \to \mathrm{Hom}_{\mathcal{C}}(Y, X) \times \mathrm{Hom}_{\mathcal{C}}(Y, X)$ is injective and its image is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ for any object Y of $\mathcal{C}$.

3) A kernel pair of a morphism $f : X \to Y$ is a pair of morphisms such that

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "h"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a pull-back square.

Remark 1.3.5 *If $\mathcal{C}$ has finite limits, an equivalence relation is a pair of morphisms* $R \overset{\sigma}{\underset{\tau}{\rightrightarrows}} X$ *satisfying the following conditions.*
1) $(\sigma, \tau) : R \to X \times X$ is a monomorphism.
2) There is a morphism $\varepsilon : X \to R$ such that $\sigma\varepsilon = \tau\varepsilon = id_X$.
3) There is a morphism $\iota : R \to R$ such that $\sigma\iota = \tau$ and $\tau\iota = \sigma$.
4) If

$$\begin{tikzcd} T \arrow[r, "p_2"] \arrow[d, "p_1"] & R \arrow[d, "\sigma"] \\ R \arrow[r, "\tau"] & X \end{tikzcd}$$

is a pull-back square, there is a morphism $\mu : T \to R$ such that $\sigma\mu = \sigma p_1$ and $\tau\mu = \tau p_2$.

Definition 1.3.6 *A category $\mathcal{C}$ is said to be exact if it satisfies the following axioms.*
E1) Each morphism of $\mathcal{C}$ has a kernel pair.
E2) Every kernel pair has a coequalizer.
E3) If $f : Y' \to Y$ is a morphism of $\mathcal{C}$ and $p : X \to Y$ is a regular epimorphism, then a pull-back $p' : X' \to Y'$ of p by f exists and p' is a regular epimorphism.
E4) Every equivalence relation is a kernel pair of a certain morphism.

Proposition 1.3.7 $\mathbf{ab}(\mathcal{E})$ *is an additive category.*

This follows from (1.3.3).

Definition 1.3.8 *Let (A_i, m_i, e_i) $(i = 1, 2, 3)$ be internal abelian groups in $\mathcal{E}$. A morphism $\mu : A_1 \times A_2 \to A_3$ in $\mathcal{E}$ is said to be biadditive if the following diagram commutes.*

$$\begin{tikzcd} A_1 \times A_1 \times A_2 \times A_2 \arrow[r, "m_1 \times m_2"] \arrow[d, "(\mathrm{pr}_1, \mathrm{pr}_3, \mathrm{pr}_2, \mathrm{pr}_3, \mathrm{pr}_1, \mathrm{pr}_4, \mathrm{pr}_2, \mathrm{pr}_4)"] & A_1 \times A_2 \arrow[d, "\mu"] \\ A_1 \times A_2 \times A_1 \times A_2 \times A_1 \times A_2 \times A_1 \times A_2 \arrow[d, "\mu \times \mu \times \mu \times \mu"] & A_3 \\ A_3 \times A_3 \times A_3 \times A_3 \arrow[r, "m_3 \times m_3"] & A_3 \times A_3 \arrow[u, "m_3"] \end{tikzcd}$$

Let (A_i, m_i, e_i) $(i = 1, 2, 3)$ be internal abelian groups in $\mathcal{E}$. We denote by $\mathrm{Biad}(A_1, A_2; A_3)$ *the set of biadditive morphisms $A_1 \times A_2 \to A_3$ in $\mathcal{E}$.*

Proposition 1.3.9 *Let $\mathcal{E}$ be an exact category with finite products and A_1, A_2 internal abelian groups in $\mathcal{E}$. Suppose that the forgetful functor $U : \mathbf{ab}(\mathcal{E}) \to \mathcal{E}$ has a left adjoint. Then a functor $\mathbf{ab}(\mathcal{E}) \to \mathbf{ab}$ given by $B \mapsto \mathrm{Biad}(A_1, A_2 : B)$ is representable.*

Proof. Let $F : \mathcal{E} \to \mathbf{ab}(\mathcal{E})$ be a left adjoint of U. Define morphisms $\varphi, \psi : A_1 \times A_1 \times A_2 \times A_2 \to F(A_1 \times A_2)$ as follows. φ is a composite $A_1 \times A_1 \times A_2 \times A_2 \xrightarrow{m_1 \times m_2} A_1 \times A_2 \xrightarrow{\eta} F(A_1 \times A_2)$, where the latter map η is the unit of the adjunction. ψ is the sum of $\eta(\mathrm{pr}_1, \mathrm{pr}_3)$, $\eta(\mathrm{pr}_2, \mathrm{pr}_3)$, $\eta(\mathrm{pr}_1, \mathrm{pr}_4)$ and $\eta(\mathrm{pr}_2, \mathrm{pr}_4)$ in $F(A_1 \times A_2)$. Then φ, ψ induces $\bar{\varphi}, \bar{\psi} : F(A_1 \times A_1 \times A_2 \times A_2) \to F(A_1 \times A_2)$ and let $\rho : F(A_1 \times A_2) \to A_1 \otimes A_2$ be the coequalizer of them. It is easy to verify that $A_1 \otimes A_2$ represents the above functor. □

Note that a functor $\otimes : \mathbf{ab}(\mathcal{E}) \times \mathbf{ab}(\mathcal{E}) \to \mathbf{ab}(\mathcal{E})$ is additive and we have a symmetric monoidal category $(\mathbf{ab}(\mathcal{E}), \otimes)$ with a unital object $E = F(1)$.

Definition 1.3.10 *A category J is said to be filtered if it is nonempty and has the following properties.*

(1) For any pair of morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ *of J, there is a morphism $h : Y \to Z$ such that $hf = hg$.*

(2) For any objects X, Y of J, there are an object Z and morphisms $X \to Z$, $Y \to Z$.

In order to define a notion of graded objects in $\mathbf{ab}(\mathcal{E})$, we require the following condition on the underlying category $\mathcal{E}$.

Condition 1.3.11 *Products in $\mathcal{E}$ commute with filtered colimits. That is, if the colimit $\varinjlim_J D(j)$ exists for a diagram $D : J \to \mathcal{E}$ with J a filtered category, then for any object A of $\mathcal{E}$, the canonical morphisms $\varinjlim_J (D(j) \times A) \to (\varinjlim_J D(j)) \times A$, $\varinjlim_J (A \times D(j)) \to A \times (\varinjlim_J D(j))$ are isomorphisms.*

Proposition 1.3.12 *If $\mathcal{E}$ satisfies the above condition, the forgetful functor $U : \mathbf{mon}(\mathcal{E}) \to \mathcal{E}$ reflects filtered colimits. Similarly, forgetful functors $\boldsymbol{gr}(\mathcal{E}) \to \mathcal{E}$, $\mathbf{cmon}(\mathcal{E}) \to \mathcal{E}$ and $\mathbf{ab}(\mathcal{E}) \to \mathcal{E}$ reflect filtered colimits.*

Proof. Let J be a filtered category and $D : J \to \mathbf{mon}(\mathcal{E})$ a diagram such that $\varinjlim_J UD(j)$ exists. Put $D(j) = (D_j, m_j, e_j)$ and $C = \varinjlim_J D_j$. Define $m : C \times C \to C$ to be the composition of following morphisms. By the assumption, there is an isomorphism $C \times C \to \varinjlim_{i \in J} \varinjlim_{j \in J} D_i \times D_j$. There also is an isomorphism $\varinjlim_{i \in J} \varinjlim_{j \in J} D_i \times D_j \to \varinjlim_{(i,j) \in J \times J} D_i \times D_j$. Since J is filtered, the diagonal functor $\Delta : J \to J \times J$ induces an isomorphism $\varinjlim_{(i,j) \in J \times J} D_i \times D_j \to \varinjlim_J D_i \times D_i$. Finally, m_j's induce a morphism $\varinjlim_J D_i \times D_i \to \varinjlim_J D_j$. $e : 1 \to C$ is given by a composition $1 \xrightarrow{e_j} D_j \to C$. A routine verification shows that (C, m, e) is an internal monoid and $\varinjlim_J D(j) = (C, m, e)$. The rest of the assertion is now obvious. □

Let I be a set and I^f be the set of finite subsets of I. Then I^f can be regarded as a filtered category whose morphisms are inclusion maps and I can be regarded as a full subcategory of I^f.

Let $(A_i)_{i \in I}$ be an I-indexed family of internal abelian groups. For each $j \in I^f$, put $A(j) = \Pi_{i \in j} A_i$ and $\mathrm{pr}_i : A(j) \to A_i$ denotes the projection. Suppose $j \subset k$ in I^f, define $D_{j,k} : A(j) \to A(k)$ by $\mathrm{pr}_i D_{j,k} = \mathrm{pr}_i$ if $i \in j$ and $\mathrm{pr}_i D_{j,k} = 0$ if $i \notin j$. Thus we have a filtered diagram $D : I^f \to \mathbf{ab}(\mathcal{E})$.

Proposition 1.3.13 *If the colimit of $UD : I^f \to \mathcal{E}$ exists, the colimit of D exists and $\varinjlim_{I^f} D(j)$ is the sum (coproduct) of $(A_i)_{i \in I}$ in $\mathbf{ab}(\mathcal{E})$.*

We denote $\varinjlim_{I^f} D(j)$ by $\sum\limits_{i \in I} A_i$.

Definition 1.3.14 *Let $\mathcal{E}$ be a category satisfying (1.3.11). An I-graded structure of an internal abelian group A is a family of monomorphisms $(s_i : A_i \to A)_{i \in I}$ inducing an isomorphism $\sum\limits_{i \in I} A_i \to A$. An internal abelian group with an I-graded structure is called an I-graded internal abelian group.*

Let Γ be a abelian group.

1.4 Internal graded rings and modules

As in the previous section, $\mathcal{E}$ is a category which has a terminal object and finite products.

Definition 1.4.1 *An internal ring in $\mathcal{E}$ is an object (A, m, e) of $\mathbf{ab}(\mathcal{E})$ with a biadditive morphism $\mu : A \times A \to A$ and a morphism $u : 1 \to A$ such that (A, μ, u) is an internal monoid. A morphism $f : (A_1, m_1, e_1; \mu_1, u_1) \to (A_2, m_2, e_2; \mu_2, u_2)$ of internal rings is a morphism $f : A_1 \to A_2$ such that $f : (A_1, m_1, e_1) \to (A_2, m_2, e_2)$ and $f : (A_1, \mu_1, u_1) \to (A_2, \mu_2, u_2)$ are morphisms of $\mathbf{ab}(\mathcal{E})$ and $\mathbf{mon}(\mathcal{E})$, respectively.*

Let $\sigma : \Gamma \to \boldsymbol{Z}/2 = \{0, 1\}$ be a fixed homomorphism of abelian groups.

Definition 1.4.2 *A Γ-graded structure on an internal ring $(A, m, e; \mu, u)$ is a Γ-graded structure $(s_g : A_g \to A)$ of an internal abelian group (A, m, e) such that there is a (unique) morphism $\mu_{g,h} : A_g \times A_h \to A_{g+h}$ satisfying $\mu(s_g \times s_h) = s_{g+h} \mu_{g,h}$ for each $g, h \in \Gamma$. We call an internal ring with a Γ-graded structure a Γ-graded internal ring. A morphism of Γ-graded internal rings is a morphism of internal rings which is also a morphism of Γ-graded internal abelian groups of degree zero.*
We say $(A, m, e; \mu, u)$ is commutative if $\mu_{h,g} T = (-1)^{\sigma(g)\sigma(h)} \mu_{g,h} : A_g \times A_h \to A_{g+h}$ holds for any $g, h \in \Gamma$, where $T = (\mathrm{pr}_2, \mathrm{pr}_1)$ and $-1 : A_{g+h} \to A_{g+h}$ is the inverse.

Definition 1.4.3 *Let $(A, m, e; \mu, u)$ be a Γ-graded internal ring. A Γ-graded structure on an internal left A-module $(M, n, \varepsilon; \lambda)$ is a Γ-graded structure $(t_g : M_g \to M)$ of an internal abelian group (M, n, ε) such that there is a (unique) morphism $\lambda_{g,h} : A_g \times M_h \to M_{g+h}$ satisfying $\lambda(s_g \times t_h) = t_{g+h}\lambda_{g,h}$ for each $g, h \in I$. We call an internal A-module with a Γ-graded structure a Γ-graded internal A-module. A morphism $f : (s_g : M_g \to M)_{g\in\Gamma} \to (t_g : N_g \to N)_{g\in\Gamma}$ of Γ-graded internal left A-modules of degree $g \in \Gamma$ is a morphism of internal abelian groups such that there is a (unique) morphism $f_h : M_h \to N_{g+h}$ of $\mathbf{ab}(\mathcal{E})$ satisfying $fs_h = t_{g+h}f_h$ and $\lambda_{h,g+k}(id_{A_h} \times f_k) = i^{\sigma(g)\sigma(h)} f_{h+k}\lambda_{h,k}$ for each $h, k \in \Gamma$.*
We denote by $\mathrm{Hom}^g_A(M, N)$ the set of morphisms of degree g. The set of morphisms $\mathrm{Hom}_A(M, N)$ is defined to be a Γ-graded abelian group $\sum_{g\in\Gamma} \mathrm{Hom}^g_A(M, N)$. The category of Γ-graded internal (left) A-modules in $\mathcal{E}$ is denoted by $\mathcal{M}od_A(\mathcal{E})$.

Proposition 1.4.4 *(1) $\mathcal{M}od_A(\mathcal{E})$ is a stable Γ-graded additive category.*

$$\begin{tikzcd}
A_1 \times A_1 \times A_2 \arrow[r, "m_1 \times id_{A_2}"] \arrow[d, "id_{A_1} \times id_{A_1} \times \Delta"] & A_1 \times A_2 \arrow[d, "\mu"] & A_1 \times A_2 \times A_2 \arrow[l, "id_{A_1} \times m_2"'] \arrow[d, "\Delta \times id_{A_2} \times id_{A_2}"] \\
A_1 \times A_1 \times A_2 \times A_2 \arrow[d, "id_{A_1} \times T \times id_{A_2}"] & A_3 & A_1 \times A_1 \times A_2 \times A_2 \arrow[d, "id_{A_1} \times T \times id_{A_2}"] \\
A_1 \times A_2 \times A_1 \times A_2 \arrow[r, "\mu \times \mu"] & A_3 \times A_3 \arrow[u, "m_3"] & A_1 \times A_2 \times A_1 \times A_2 \arrow[l, "\mu \times \mu"']
\end{tikzcd}$$

(2) Let $(A, m, e; \mu, u)$ be an internal ring. An internal left A-module in $\mathcal{E}$ is an object (M, n, ε) of $\mathbf{ab}(\mathcal{E})$ with a biadditive morphism $\lambda : A \times M \to M$ such that the following diagrams commute.

$$\begin{tikzcd}
A \times A \times M \arrow[r, "\mu \times id_M"] \arrow[d, "id_A \times \lambda"] & A \times M \arrow[d, "\lambda"] \\
A \times M \arrow[r, "\lambda"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \times M \arrow[r, "u \times id_M"] \arrow[dr, "\mathrm{pr}_2"'] & M \times M \arrow[d, "\lambda"] \\
 & M
\end{tikzcd}$$

A morphism $f : (M_1, m_1, e_1; \lambda_1) \to (M_2, m_2, e_2; \lambda_2)$ of internal A-modules is a morphism $f : M_1 \to M_2$ such that $f : (M_1, m_1, e_1) \to (M_2, m_2, e_2)$ is a morphism of $\mathbf{ab}(\mathcal{E})$ and $\lambda_2(id_A \times f) = f\lambda_1 : A \times M_1 \to M_2$. A morphism $f : (s_i : A_i \to A)_{i\in I} \to (t_i : B_i \to B)_{i\in I}$ of I-graded internal abelian groups is a morphism $f : A \to B$ of $\mathbf{ab}(\mathcal{E})$ such that there is a (unique) morphism $f_i : A_i \to B_i$ of $\mathbf{ab}(\mathcal{E})$ satisfying $fs_i = t_if_i$ for each $i \in I$.

1.5 Γ-graded rings

Let Γ be an abelian group with a homomorphism $\sigma : \Gamma \to \boldsymbol{Z}/2 = \{0, 1\}$. We call σ the signature of Γ.

Definition 1.5.1 *A Γ-graded commutative ring $R = \sum_{g\in\Gamma} R_g$ is a Γ-graded ring with unit satisfying*

$$xy = (-1)^{\sigma(g)\sigma(h)}yx \ \text{ for } \ g, h \in \Gamma, \ \ x \in R_g, \ y \in R_h.$$

A map $f : R \to S$ is a homomorphism of Γ-graded rings if f is a degree preserving ring homomorphism with $f(1) = 1$. A Γ-graded ring is said to be strictly commutative if $xy = yx$ holds for any homogeneous elements x, y (hence for any elements). If S is a subset of a Γ-ring R, we put $S^\natural = S \cap (\bigcup_{g\in\Gamma} R_g)$ and we call $S^\natural$ the homogeneous part of S. Moreover, $\deg : R^\natural \to \Gamma$ denotes a function defined by $\deg(R_g - \{0\}) = \{g\}$, $\deg 0 = 0$ and we put $|x| = \sigma(\deg x)$.

Let Γ and Δ be abelian groups with signatures σ and τ, respectively, and let $\varphi : \Gamma \to \Delta$ be a homomorphism of abelian groups such that $\tau\varphi = \sigma$. For a Γ-graded commutative ring R, we denote by R_φ a Δ-graded commutative ring defined by $R_\varphi = \sum_{h\in\Delta} R'_h$, where $R'_h = \sum_{g\in\varphi^{-1}(h)} R_g$. In the case $\tau = id_{\boldsymbol{Z}/2}$, $\varphi = \sigma$, we put $R_{ev} = R'_0$, $R_{od} = R'_1$.

From now on, we only consider Γ-graded commutative rings, so we simply call a Γ-graded commutative ring a *Γ-ring*. An ideal of a Γ-ring is said to be *homogeneous* if it is generated by homogeneous elements. Since we mainly deal with homogeneous ideals, *ideal* means homogeneous ideal unless otherwise stated.

Definition 1.5.2 *An ideal $\mathfrak{p}$ of a Γ-ring R is prime if $\mathfrak{p} \neq R$ and $xy \in \mathfrak{p}$ ($x, y \in R^\natural$) implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. A Γ-ring R with no zero divisors in $R^\natural - \{0\}$ is called a Γ-integral domain, or Γ-domain for short, and if each element of $R^\natural - \{0\}$ is a unit, R is called a Γ-field.*

The following facts are easily verified.

Proposition 1.5.3 *1) If $\mathfrak{p}$ is a prime ideal of R, then $\mathfrak{p} \ni 2$ or $\mathfrak{p} \supseteq R_{od}$.*

2) *If $x \in R_g$ is not a zero divisor, $|x| = 0$ or char $R = 2$. Hence if R is aΓ-domain, $R_{od} = 0$ or char $R = 2$, and a Γ-domain is always strictly commutative.*

3) *A unit $u \in R_g$ defines isomorphisms $R_h \to R_{h+g}$, $x \mapsto ux$ ($h \in \Gamma$) of R_0-modules. Thus if R is a Γ-field, R_0 is a field and $dim_{R_0} R_g \leq 1$.*

4) *$\mathfrak{p}$ is a prime ideal of R if and only if $R/\mathfrak{p}$ is a Γ-domain. $\mathfrak{m}$ is a maximal ideal of R if and only if $R/\mathfrak{m}$ is a Γ-field.*

Definition 1.5.4 *A Γ-ring R with exactly one maximal ideal $\mathfrak{m}$ is called a Γ-local ring. The Γ-field $k = R/\mathfrak{m}$ is called the residue field of R.*

Proposition 1.5.5 *Let $\mathfrak{m} \neq R$ be an ideal of R. Then the following three conditions are equivalent.*

(1) *R is a Γ-local ring with maximal ideal $\mathfrak{m}$.*

(2) *Each element of $(R - \mathfrak{m})^\natural$ is a unit.*

(3) *$\mathfrak{m}$ is a maximal ideal and every element of $1 + \mathfrak{m}_0$ is a unit of R_0.*

Let R be a Γ-ring. For a family $(\mathfrak{a}_\lambda)_{\lambda \in I}$ of ideals of R, the sum $\sum\limits_{\lambda \in I} \mathfrak{a}_\lambda$ and the intersection $\bigcap_{\lambda \in I} \mathfrak{a}_\lambda$ are both homogeneous. The product of ideals is also homogeneous. The radical of an ideal $\mathfrak{a}$ is an ideal generated by $\{x \in R^\natural | x^n \in \mathfrak{a} \text{ for some } n \in \boldsymbol{N}\}$, which we denote by $\sqrt{\mathfrak{a}}$. The radical of zero ideal is called the nilradical. It is generally different from the set of nilpotent elements which are not necessarily homogeneous. For example, in the case $\Gamma = \boldsymbol{Z}/2$, $\sigma = id_{\boldsymbol{Z}/2}$ and $R = \boldsymbol{Z}[x, y]/(x^2 + y^2, 2xy, 2y^2)$, where $\deg x = 0$, $\deg y = 1$, then $(x + y)^2 = 0$ but $x + y \notin \sqrt{0}$.

We state several propositions without proofs.

Proposition 1.5.6 *$\sqrt{0} =$ (the set of nilpotent elements) holds in the following cases.*

(1) *R is strictly commutative and the characteristic is a prime number p. Moreover, Γ is p-torsion free.*

(2) *Γ is a totally ordered abelian group.*

Proposition 1.5.7 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a Γ-ring R. Then the following conditions are equivalent.*

(1) *$\mathfrak{p}$ is a prime ideal.*

(2) *$\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ implies $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$.*

(3) *$\mathfrak{p} \subsetneq \mathfrak{a}$ and $\mathfrak{p} \subsetneq \mathfrak{b}$ imply $\mathfrak{a}\mathfrak{b} \not\subseteq \mathfrak{p}$.*

Proposition 1.5.8 *1) If a multiplicatively closed subset S of $R^\natural$ does not intersect with an ideal $\mathfrak{b}$, then a set of ideals $\{\mathfrak{a} | S \cap \mathfrak{a} = \phi, \mathfrak{a} \supseteq \mathfrak{b}\}$ has a maximal element with respect to inclusion, which is a prime ideal.*

2) *Let $\mathfrak{a}$ be an ideal of R, then $\{\mathfrak{p} | \mathfrak{p}$ is a prime ideal containing $\mathfrak{a}\}$ has a minimal element with respect to inclusion.*

3) *The radical of $\mathfrak{a}$ is the intersection of all prime ideals containing $\mathfrak{a}$, and $R/\sqrt{\mathfrak{a}}$ has no nilpotent element except for 0.*

The Jacobson radical $\mathfrak{R}$ of R is defined to be the intersection of all the maximal ideals of R.

Proposition 1.5.9 *For $g \in \Gamma$, $x \in \mathfrak{R}_g$ if and only if $1 - xy$ is a unit of R_0 for any $y \in R_{-g}$.*

Let S be a multiplicatively closed subset of $R^\natural$. Define a relation of $R \times S$ by "$(x, s) \equiv (y, t) \Leftrightarrow (x, s) = (y, t)$ *or* $(xt - (-1)^{|s||t|} ys)u = 0$ *for some* $u \in S$". Then, $\equiv$ is an equivalence relation, and we define the ring of fractions $S^{-1}R$ of R with respect to R to be the quotient set $R \times S/\equiv$. We denote by x/s the class of (x, s). Define a Γ-ring structure on $S^{-1}R$ by $x/s + y/t = (xt + ys)/st$, $x/s \cdot y/t = xy/st$, $\deg(x/s) = \deg x - \deg s$. Since $s/1$ is a unit of $S^{-1}R$, $S \cap R_{od} \neq \phi$ implies $char S^{-1}R = 2$. $\rho = \rho_S : R \to S^{-1}R$ denotes the canonical homomorphism $\rho(x) = x/1$.

Proposition 1.5.10 *Let $f : R \to A$ be a ring homomorphism such that $f(s)$ is a unit of A for any $s \in S$. Then there exists a unique homomorphism $g : S^{-1}R \to A$ such that $f = g\rho$*

Proposition 1.5.11 *Consider the following conditions on a ring homomorphism $f : R \to A$.*
(1) *$f(s)$ is a unit if $s \in S$.*
(2) *$xs = 0$ for some s if $f(x) = 0$.*
(3) *Every element of A is of the form $f(x)f(s)^{-1}$ for $x \in R$, $s \in S$.*
$\rho : R \to S^{-1}R$ satisfies these three conditions. If $f : R \to A$ satisfies them, there is a unique isomorphism $g : S^{-1}R \to A$ such that $f = g\rho$.

Example 1.5.12 *1) Let $\mathfrak{p}$ is a prime ideal, then $S_\mathfrak{p} = (R - \mathfrak{p})^\mathfrak{h}$ is multiplicatively closed and we write $R_\mathfrak{p}$ for $S_\mathfrak{p}^{-1}$. $R_\mathfrak{p}$ is a Γ-local ring with maximal ideal generated by $\rho(\mathfrak{p})$. $R_\mathfrak{p}$ is called the localization of R at $\mathfrak{p}$. If R is a Γ-domain and $\mathfrak{p} = 0$, $R_\mathfrak{p}$ is the field of fractions of R and we denote this by* Frac*R.*
2) If $f \in R^\mathfrak{h}$, R_f denotes the ring of fractions with respect to $\{f^n | n \geqq 0\}$.

Let $f : A \to B$ be a homomorphism of Γ-rings and $\mathfrak{a}$, $\mathfrak{b}$ ideals of A, B, respectively. We denote by $\mathfrak{a}^e$ the ideal of B generated by $f(\mathfrak{a})$, and $\mathfrak{b}^c$ denotes $f^{-1}(\mathfrak{b})$. $\mathfrak{a}^e$ is called the extension of $\mathfrak{a}$ and $\mathfrak{b}^c$ is called the contraction of $\mathfrak{b}$.

Proposition 1.5.13 *1) Let $\mathfrak{a}$ and $\mathfrak{b}$ be as above, then $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$, $\mathfrak{b} \supseteq \mathfrak{b}^{ce}$,$\mathfrak{a}^e = \mathfrak{a}^{ece}$, $\mathfrak{b}^c = \mathfrak{b}^{cec}$. If $\mathfrak{b}$ is a prime ideal, so is $\mathfrak{b}^c$.*
2) If C is the set of contracted ideals in A and E is the set of extended ideals in B. Then $C = \{\mathfrak{a} | \mathfrak{a}^{ec} = \mathfrak{a}\}$, $E = \{\mathfrak{b} | \mathfrak{b}^{ce} = \mathfrak{b}\}$. $\mathfrak{a} \mapsto \mathfrak{a}^e$ is a bijection from C onto E, whose inverse is $\mathfrak{b} \mapsto \mathfrak{b}^c$. Moreover, a prime ideal $\mathfrak{p}$ of A is the contraction of a prime ideal of B if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$.
3) If $f : A \to B$ is surjective, then the above C coincides with the set of ideals containing $\ker f$ *and E is the set of all ideals of B. The map $\mathfrak{b} \mapsto \mathfrak{b}^c$ gives a bijection between the set of prime ideals of B and the set of prime ideals containing* $\ker f$.

Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of A. We put $(\mathfrak{a} : \mathfrak{b}) = \{x \in A | x\mathfrak{b} \subset A\}$, $(\mathfrak{a} : x) = (\mathfrak{a} : \mathfrak{b})$ if $\mathfrak{b} = (x)$. Let S be a multiplicatively closed subset of $A^\mathfrak{h}$. For an ideal $\mathfrak{a}$ of A, we write $S^{-1}\mathfrak{a}$ for the extended ideal $\mathfrak{a}^e$ of $\mathfrak{a}$ by $\rho : A \to S^{-1}A$.

Proposition 1.5.14 *1) Every ideal of $S^{-1}A$ is an extended ideal.*
2) If $\mathfrak{a}$ is an ideal of A, $\mathfrak{a}^{ec} = \bigcup_{s \in S}(\mathfrak{a} : s)$. Hence $\mathfrak{a}^e = (1)$ if and only if $\mathfrak{a} \cap S \neq \phi$.
3) $\mathfrak{a}$ is a contracted ideal in A if and only if no element of S is a zero divisor in $A/\mathfrak{a}$.
4) The correspondence $\mathfrak{p} \to S^{-1}\mathfrak{p}$ gives a bijection from $\{\mathfrak{p} | \mathfrak{p}$ is a prime ideal such that $\mathfrak{p} \cap S = \phi\}$ to the set of all prime ideals of $S^{-1}A$.
5) The following equalities hold for ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A. $S^{-1}(\mathfrak{a} + \mathfrak{b}) = S^{-1}\mathfrak{a} + S^{-1}\mathfrak{b}$, $S^{-1}\mathfrak{a}\mathfrak{b} = S^{-1}\mathfrak{a}S^{-1}\mathfrak{b}$, $S^{-1}(\mathfrak{a} \cap \mathfrak{b}) = S^{-1}\mathfrak{a} \cap S^{-1}\mathfrak{b}$, $S^{-1}\sqrt{\mathfrak{a}} = \sqrt{S^{-1}\mathfrak{a}}$.

Remark 1.5.15 *If $f : A \to B$ is a homomorphism of* Γ*-rings and $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}$ are ideals of A, $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{b}$ are ideals of B, then $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = \mathfrak{a}_1^e + \mathfrak{a}_2^e$, $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subseteq \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$, $(\mathfrak{a}_1\mathfrak{a}_2)^e = \mathfrak{a}_1^e\mathfrak{a}_2^e$, $(\mathfrak{a}_1 : \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1^e : \mathfrak{a}_2^e)$, $(\sqrt{\mathfrak{a}})^e \subseteq \sqrt{\mathfrak{a}^e}$, $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c + \mathfrak{b}_2^c$, $(\mathfrak{b}_1 \cap \mathfrak{b}_2)^e = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c$, $(\mathfrak{b}_1\mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c\mathfrak{b}_2^c$, $(\mathfrak{b}_1 : \mathfrak{b}_2)^c \subseteq (\mathfrak{b}_1^c : \mathfrak{b}_2^c)$, $(\sqrt{\mathfrak{b}})^c = \sqrt{\mathfrak{b}^c}$.*

Definition 1.5.16 *A multiplicatively closed subset S of $R^\mathfrak{h}$ is said to be saturated if $xy \in S$ ($x, y \in R^\mathfrak{h}$) implies $x \in S$ and $y \in S$.*

Proposition 1.5.17 *1) A subset S of $R^\mathfrak{h}$ is a saturated multiplicatively closed subset if and only if $R^\mathfrak{h} - S$ is the homogeneous part of a union of prime ideals.*
2) If S is a multiplicatively closed subset of $R^\mathfrak{h}$, then $\bar{S} = R^\mathfrak{h} - \bigcup\{\mathfrak{p} | \mathfrak{p}$ is a prime ideal such that $\mathfrak{p} \cap S = \phi\}$ is the smallest saturated multiplicatively closed subset of $R^\mathfrak{h}$ containing S and $\bar{S} = \rho_S^{-1}(\{u \in (S^{-1}R)^\mathfrak{h} | u$ is a unit $\})$ holds.
3) Let S and T be multiplicatively closed subsets of $R^\mathfrak{h}$. Assume that each element of $\rho_T(S)$ is a unit of $T^{-1}R$. Let $\varphi : S^{-1}R \to T^{-1}R$ be the homomorphism satisfying $\varphi\rho_S = \rho_T$. Then the following conditions are equivalent. (1) *φ is bijective.* (2) *$T \subseteq \bar{S}$.* (3) *Each element of $\rho_S(T)$ is a unit of $S^{-1}R$.* (4) *For each $t \in T$, there exists $x \in R^\mathfrak{h}$ such that $xt \in S$.* (5) *Every prime ideal which meets T also meets S.*

Example 1.5.18 *If $\mathfrak{p}$ is a prime ideal of R, $S_\mathfrak{p}$ is saturated. For an ideal $\mathfrak{a}$, we set $S(\mathfrak{a}) = \bigcap\{S_\mathfrak{p} | \mathfrak{p}$ is a prime ideal not containing $\mathfrak{a}\}$. Then it is saturated. In fact, S is a saturated multiplicatively closed subset of R if and only if S is an intersection of the sets of the form $S_\mathfrak{p}$. We call $\bar{S}$ in the above proposition the saturation of S. The saturation of $\{1, s, s^2, \dots\}$ is $S((s))$.*

1.6 Γ-graded modules

Let R be a Γ-ring. A Γ-graded left (resp. right) R-module $M = \sum_{g \in \Gamma} M_g$ is a Γ-graded abelian group with unitary and associative multiplication of R on the left (resp. right) such that $R_g M_h \subseteq M_{g+h}$ (resp. $M_h R_g \subseteq M_{g+h}$). As in the previous section, we set $S^{\natural} = S \cap (\bigcup_{g \in \Gamma} M_g)$ for the subset S of M, and define a function $\deg : M^{\natural} \to \Gamma$ by $\deg(M_g - \{0\}) = \{g\}$, $\deg 0 = 0$. We also put $|x| = \sigma(\deg x)$, if $x \in M^{\natural}$.

Let M, N be Γ-graded left R-modules. For $g \in \Gamma$, a homomorphism of degree g is a map $f : M \to N$ such that f is a homomorphism of abelian groups which maps M_j into N_{j+g} for any $j \in \Gamma$ and that $f(ax) = (-1)^{\sigma(g)|a|} a f(x)$ holds for any $a \in R^{\natural}$, $x \in M$. We denote by $\mathrm{Hom}_R^g(M, N)$ the set of homomorphism from M to N of degree g. $\mathrm{Hom}_R^g(M, N)$ has a structure of an abelian group by the addition of homomorphisms. We put $\mathrm{Hom}_R(M, N) = \sum_{g \in \Gamma} \mathrm{Hom}_R^g(M, N)$ and give this a structure of a Γ-graded R-module by $(af)(x) = af(x)$ for $a \in R$, $f \in \mathrm{Hom}_R(M, N)$.

We often regard a Γ-graded left R-module M as a Γ-graded right R-module by $xa = (-1)^{|a||x|} ax$, for $a \in R^{\natural}$, $x \in M^{\natural}$. Then, $\mathrm{Hom}_R^g(M, N)$ can be regarded as the set of right R-module homomorphisms in the usual sense, and the right R-module structure is given by $(fa)(x) = f(ax) = (-1)^{|a||x|} f(xa)$.

We denote $\mathrm{Hom}_R(M, M)$ by $\mathrm{End}_R(M)$, and $\mathrm{Hom}_R(M, R)$ by M^* which is called the dual of M.

For an R-module homomorphism $f : M \to L$ of degree g, we define $f^* : \mathrm{Hom}_R(L, N) \to \mathrm{Hom}_R(M, N)$, $f_* : \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, L)$ by $f^*(\varphi) = (-1)^{\sigma(g)|\varphi|} \varphi f$, $f_*(\psi) = f\psi$. Then, both of them are R-module homomorphisms of degree g. We define $\theta_M : M \to M^{**}$ by $\theta_M(x)(\varphi) = (-1)^{|x||\varphi|} \varphi(x)$.

Proposition 1.6.1 *1) θ_M is an R-module homomorphism of degree zero and $\theta_N f = f^{**} \theta_M$ holds for $f \in \mathrm{Hom}_R(M, N)$.*

2) θ_M is an isomorphism if M is a finitely generated projective R-module.

Let M, N be Γ-graded R-modules. Define a Γ-graded tensor product $M \otimes_R N$ as follows. Let $F(M, N)$ be the free R-module generated by $M \times N$, $D(M, N)$ a submodule of $F(M, N)$ generated by elements of the forms $(x_1 + x_2, y) - (x_1, y) - (x_2, y)$, $(x, y_1 + y_2) - (x, y_1) - (x, y_2)$ for $x, x_1, x_2 \in M$, $y, y_1, y_2 \in N$ and $r(x, y) - (rx, y)$, $r(x, y) - (-1)^{|r||x|}(x, ry)$ for $r \in R^{\natural}$, $x \in M^{\natural}$, $y \in N^{\natural}$. $M \otimes_R N$ is defined to be the quotient $F(M, N)/D(M, N)$ and $x \otimes y$ denotes the class represented by (x, y). We assign degree $g + h$ to $x \otimes y$ if $x \in M_g$, $y \in N_h$, so that $M \otimes_R N$ is a Γ-graded R-module.

Let $f_i : M_i \to N_i$ $(i = 1, 2)$ be R-module homomorphisms of degree g_i. Define a homomorphism $f_1 \otimes f_2 : M_1 \otimes_R M_2 \to N_1 \otimes_R N_2$ of degree $g_1 + g_2$ by $(f_1 \otimes f_2)(x_1 \otimes x_2) = (-1)^{\sigma(g_2)|x_1|} f_1(x_1) \otimes f_2(x_2)$.

Proposition 1.6.2 *There is a natural isomorphism of Γ-graded R-modules*

$$\mathrm{Hom}_R(M \otimes_R N, L) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, L)).$$

For a Γ-graded R-module M and $g \in \Gamma$, we define a Γ-graded R-module $\Sigma^g M$ as follows; $(\Sigma^g M)_h$ is M_{h-g} as an abelian group. The R-module structure of $\Sigma^g M$ is given by $r \cdot m = (-1)^{\sigma(g)|r|} rm$ for $r \in R^{\natural}$ and $m \in M$, where the multiplication on the right hand side is the original one in M. We call $\Sigma^g M$ the suspension of M by degree g. Note that the identity map of M defines an isomorphism $id_M^g : M \to \Sigma^g M$ of degree g, and that a homomorphism $f : M \to N$ induces a homomorphism $\Sigma^g f : \Sigma^g M \to \Sigma^g N$.

Proposition 1.6.3 *There are natural isomorphisms of Γ-graded R-modules of degree zero; $\Sigma^g(\Sigma^h M) \cong \Sigma^{g+h} M$, $\mathrm{Hom}_R(\Sigma^{-g} M, N) \cong \Sigma^g \mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_R(M, \Sigma^g N)$, $(\Sigma^g M) \otimes_R (\Sigma^h N) \cong \Sigma^{g+h}(M \otimes_R N)$.*

Let $\alpha : R \to S$ be a homomorphism of Γ-rings, then correspondence $M \mapsto M \otimes_R N$, $(f : M \to N) \mapsto (f \otimes 1 : M \otimes_R S \to N \otimes_R S)$ gives a functor from the category of Γ-graded R-modules to the category of Γ-graded S-modules. We denote this functor by $\alpha_\sharp$. There is a natural equivalence $\alpha_\sharp(M \otimes_R N) \cong \alpha_\sharp M \otimes_S \alpha_\sharp N$, and also there is a natural homomorphism $\alpha_\sharp \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(\alpha_\sharp M, \alpha_\sharp N)$.

Let R be a Γ-ring. A Γ-graded R-algebra A is a Γ-graded R-module with homomorphisms $\mu_A : A \otimes_R A \to A$ and $\eta_A : R \to A$ of degree zero which satisfy $\mu_A(\mu_A \otimes 1) = \mu_A(1 \otimes \mu_A)$ and $\mu(1 \otimes \eta_A) = \iota_1$, $\mu(\eta_A \otimes 1) = \iota_2$, where $\iota_1 : R \otimes_R A \to A$ and $\iota_2 : A \otimes_R R \to A$ are isomorphisms given by $\iota_1(r \otimes_R a) = ra$, $\iota_2(a \otimes_R r) = (-1)^{|a||r|} ra$.

A homomorphism $f : A \to B$ of Γ-graded R-algebras is an R-module homomorphism of degree zero which satisfies $f\mu_A = \mu_B(f \otimes f)$ and $f\eta_A = \eta_B$. For Γ-graded R-modules M and N, define the switching map $T : M \otimes_R N \to N \otimes_R M$ by $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$ for $x \in M^{\natural}$, $y \in N^{\natural}$. T is an isomorphism of R-modules. A Γ-graded R-algebra A is said to be commutative if $\mu_A T = \mu_A$.

A homomorphism $\varepsilon : A \to R$ of R-algebras is called an augmentation if $\varepsilon\eta = id_R$. A Γ-graded R-algebra with an augmentation is called augmented Γ-graded R-algebra. $I(A)$ denotes the kernel of the augmentation. An augmented R-algebra A is said to be anti-commutative if $\mu_A T(x \otimes y) = -\mu_A(x \otimes y)$, $\mu(x \otimes x) = 0$ for $x, y \in I(A)^{\natural}$.

We construct functors T, S, E from the category of Γ-graded R-modules and homomorphisms of degree zero to the categories of Γ-graded R-algebras, commutative R-algebras and anti-commutative R-algebras, respectively.

For a Γ-graded R-module M, we put $T(M)_i = 0$ if $i < 0$, $T(M)_0 = R$, $T(M)_i = M \otimes_R M \otimes_R \cdots \otimes_R M$ (i factors) and $T(M) = \sum_{i \in \boldsymbol{Z}} T(M)_i$. Define $\mu_T : T(M) \otimes_R T(M) \to T(M)$, $\eta_T : R \to T(M)$ by $\mu_T((x_1 \otimes \cdots \otimes x_m) \otimes (y_1 \otimes \cdots \otimes y_n)) = x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n \in T(M)_{m+n}$ for $x_1 \otimes \cdots \otimes x_m \in T(M)_m$, $y_1 \otimes \cdots \otimes y_n \in T(M)_n$, $\eta_T(r) = r \in T(M)_0$.

Let $J_S(M)$ and $J_E(M)$ be two sided ideals generated by $\{x \otimes y - T(x \otimes y) \in T(M)_2 | \, x, y \in M^{\natural}\}$ and $\{x \otimes y + T(x \otimes y) \in T(M)_2 | \, x, y \in M^{\natural}\} \cup \{x \otimes x \in T(M)_2 | \, x \in M^{\natural}\}$ respectively. Define $S(M)$ and $E(M)$ by $S(M) = T(M)/J_S(M)$ and $E(M) = T(M)/J_E(M)$. The canonical projections $T(M) \to S(M)$ and $T(M) \to E(M)$ are denoted by π_S and π_E. We put $\pi_S(x_1 \otimes \cdots \otimes x_m) = x_1 \cdots x_m$, $\pi_E(x_1 \otimes \cdots \otimes x_m) = x_1 \wedge \cdots \wedge x_m$, $S(M)_i = \pi_S(T(M)_i)$, $E(M)_i = \pi_E(T(M)_i)$. Then we have $S(M) = \sum_{i \in \boldsymbol{Z}} S(M)_i$, $E(M) = \sum_{i \in \boldsymbol{Z}} E(M)_i$. The multiplications μ_S, μ_E of $S(M)$, $E(M)$ are homomorphisms induced by μ_T, and the units η_S, η_E are defined to be $\pi_S\eta_T$, $\pi_E\eta_T$ respectively. We also define augmentations $\varepsilon_T : T(M) \to R$, $\varepsilon_S : S(M) \to R$, $\varepsilon_E : E(M) \to R$ by $\varepsilon_T(r) = \varepsilon_S(r) = \varepsilon_E(r) = r$ for $r \in T(M) = S(M) = E(M) = R$, $\varepsilon_T(T(M)_i) = \varepsilon_S(S(M)_i) = \varepsilon_E(E(M)_i) = \{0\}$ if $i \neq 0$.

We call $T(M)$, $S(M)$ and $E(M)$ tensor algebra, symmetric algebra and exterior algebra, respectively.

Proposition 1.6.4 *Let M_i and N_i ($i = 1, 2$) be Γ-graded R-modules. There is a natural homomorphism $\Theta : \mathrm{Hom}_R(M_1, N_1) \otimes_R \mathrm{Hom}_R(M_2, N_2) \to \mathrm{Hom}_R(M_1 \otimes_R M_2, N_1 \otimes_R N_2)$ defined by $\Theta(f_1 \otimes f_2) = f_1 \otimes f_2$. Θ induces $\Theta_T : T(M^*)_i \to (T(M)_i)^*$ for each $i \in \boldsymbol{Z}$. If M, M_1 and M_2 are finitely generated projective R-modules, Θ and Θ_T are isomorphisms.*

Proposition 1.6.5 *Let $\alpha : R \to S$ be a homomorphism of Γ-rings and M a Γ-graded R-modules. There are natural isomorphisms $T(\alpha_\sharp M) \cong \alpha_\sharp T(M)$, $S(\alpha_\sharp M) \cong \alpha_\sharp S(M)$ of Γ-graded S-algebras.*

Let $\iota_T : M \to T(M)$, $\iota_S : M \to S(M)$, $\iota_E : M \to E(M)$ be natural inclusions into $T(M)_1 = S(M)_1 = E(M)_1 = M$. For an R-module homomorphism $f : M \to N$ of degree zero, $f \otimes \cdots \otimes f : T(M)_i \to T(N)_i$ ($i \in \boldsymbol{Z}$) induce an R-module homomorphism $T(f) : T(M) \to T(N)$. $S(f) : S(M) \to S(N)$ and $E(f) : E(M) \to E(N)$ are the homomorphisms induced by $T(f)$.

Proposition 1.6.6 *Let M be a Γ-graded R-module and A a Γ-graded R-algebra. For an R-module homomorphism $f : M \to A$ of degree zero, there exists a unique homomorphism $\bar{f} : T(M) \to A$ of Γ-graded R-algebras such that $\bar{f}\iota_T = f$. If A is augmented by $\varepsilon : A \to R$ and $\mathrm{Im}\, f \subseteq I(A)$, $\bar{f}$ satisfies $\varepsilon\bar{f} = \varepsilon_T$. If A is commutative (resp. anti-commutative and $\mathrm{Im}\, f \subseteq I(A)$), there exists a unique homomorphism $\tilde{f} : S(M) \to A$ (resp. $\hat{f} : E(M) \to A$) such that $\tilde{f}\iota_S = f$ (resp. $\hat{f}\iota_E = f$ and $\varepsilon\hat{f} = \varepsilon_E$).*

Remark 1.6.7 *1) Let A be an anti-commutative Γ-graded R-algebra. If $x, y \in A_g$ with $\sigma(g) = 1$ and $x + y \in I(A)$, then $2xy = 0$ and $xy = yx$.*

2) If $\mathrm{char}\, R = 2$ or $\sigma(g) = 0$, an R-module homomorphism $f : M \to N$ of degree g induces R-module homomorphisms $f^{\otimes i} : T(M)_i \to T(M)_i$, $f^{(i)} : S(M)_i \to S(M)_i$ and $f^{\wedge i} : E(M)_i \to E(M)_i$ of degree ig defined by $f^{\otimes i} = f \otimes \cdots \otimes f$ (i factors), $f^{(i)}\pi_S = \pi_S f^{\otimes i}$ and $f^{\wedge i}\pi_E = \pi_E f^{\otimes i}$, respectively.

Let A and B be Γ-graded R-algebras with products μ_A, μ_B, units η_A, η_B, respectively. Define $\mu : (A \otimes_R B) \otimes_R (A \otimes_R B) \to A \otimes_R B$ by $\mu((x \otimes y) \otimes (z \otimes w)) = (-1)^{|y||z|}\mu_A(x \otimes z) \otimes \mu_B(y \otimes w)$ for $x, z \in A$, $y, w \in B$, and define $\eta : R \to A \otimes_R B$ by $\eta(r) = \eta_A(r) \otimes 1$. We also define $\iota_A : A \to A \otimes_R B$, $\iota_B : B \to A \otimes_R B$ by $\iota_A(x) = x \otimes 1$, $\iota_B(y) = 1 \otimes y$.

Proposition 1.6.8 *If $f : A \to C$ and $f' : B \to C$ are homomorphisms of Γ-graded R-algebras such that $xy = (-1)^{|x||y|}yx$ holds for any $x \in f(A)^{\natural}$, $y \in f'(B)^{\natural}$, then there is a unique homomorphism $F : A \otimes_R B \to C$ satisfying $F\iota_A = f$, $F\iota_B = f'$. Hence the sum of two Γ-graded commutative R-algebras A and B is given by $A \otimes_R B$ in the category of Γ-graded commutative R-algebras.*

Proposition 1.6.9 *There is a natural isomorphism of* Γ*-graded commutative* R*-algebras* $S(M\oplus N)\cong S(M)\otimes_R S(N)$.

As in the ungraded case, the following fact holds.

Proposition 1.6.10 *If* R *is a* Γ*-field, a* Γ*-graded* R*-module is free and it has a homogeneous basis.*

Let M be a Γ-graded free R-module with basis $S=\{x_1,\cdots,x_l,y_1,\cdots,y_m\}\subseteq M^\natural$, where $|x_i|=0$, $|y_j|=1$. Consider the following subsets of $S(M)$.
$A_S=\{x_1^{i_1}\cdots x_l^{i_l}y_1^{\varepsilon_1}\cdots y_m^{\varepsilon_m}\,|\,i_k\geq 0,\ \varepsilon_k\geq 0,\varepsilon_j\geq 2 \textit{ for some } j\}$
$B_S=\{x_1^{i_1}\cdots x_l^{i_l}y_1^{\varepsilon_1}\cdots y_m^{\varepsilon_m}\,|\,i_k\geq 0,\ \varepsilon_k=0 \textit{ or } 1\}$
Let $F_S(M)$ be an R-submodule of $S(M)$ generated by B_S if *char* $R\neq 2$, $A_S\cup B_S$ if *char* $R=2$ and let $T_S(M)$ be an R-submodule of $S(M)$ generated by A_S if *char* $R\neq 2$, $T_S(M)=0$ if *char* $R=2$

Proposition 1.6.11 $S(M)$ *is a direct sum of* $F_S(M)$ *and* $T_S(M)$ *as a* Γ*-graded* R*-module.* $F_S(M)$ *is a free* R*-module with basis* B_S *if char* $R\neq 2$, $A_S\cup B_S$ *if char* $R=2$*, and* $T_S(M)$ *is a free* $R/(2)$*-module with basis* A_S *if char* $R\neq 2$*. Therefore* $S(M)$ *is a free* R*-module if* 2 *is invertible in* R *or char* $R=2$.

Definition 1.6.12 *A finitely generated* Γ*-graded* R*-module* M *is said to be strongly free if there exists* $m\in\boldsymbol{N}$ *such that* $E(M)_m$ *is a free* R*-module of rank one and* M *is generated by* m *elements.*

It is easily follows from the definition of exterior algebra that $x_1\wedge\cdots\wedge x_m=0$ in $E(M)$ if $x_1,\ldots x_m$ are linearly dependent elements of M. Hence if M is generated by m elements, then $E(M)_i=0$ for $i>m$.
If M is a strongly free Γ-graded R-module generated by $\{x_1,\ldots,x_m\}$, then $x_1\wedge\cdots\wedge x_m$ generates $E(M)_m$ and $x_1,\ldots,x_m$ are linearly independent. Hence M is a free R-module with basis $\{x_1,\ldots,x_m\}$. For example, a free Γ-graded R-module is strongly free if *char* $R=2$, or R is strictly commutative and M has a homogeneous basis $\{x_1,\ldots,x_m\}$ such that $|x_i|=0$ for any i.

Proposition 1.6.13 *1) If* M *is a strongly free* Γ*-graded* R*-module with basis* $\{x_1,\ldots,x_m\}$*. Then* $\{x_{i_1}\wedge\cdots\wedge x_{i_r}|1\leq i_1<i_2<\cdots<i_r\leq m\}$ *is a basis of* $E(M)_r$ *and the pairing* $\mu_E: E(M)_r\otimes_R E(M)_{m-r}\to E(M)_m\cong R$ *is non-degenerate. Therefore* $E(M)_r$ *is a free* R*-module of rank* $\binom{m}{r}$ *and* $E(M)_r$ *is isomorphic to* $\mathrm{Hom}_R(E(M)_{m-r},E(M)_m)$.

2) *If* M *and* N *are* Γ*-graded free* R*-modules and* $M\oplus N$ *is strongly free, so are* M *and* N.

3) *If* M *is strongly free, so is* M^*.

Let S be a multiplicatively closed subset of $R^\natural$ and M a Γ-graded R-module. Define a relation of $M\times S$ by "$(m,s)\equiv(n,t)\Leftrightarrow(m,s)=(n,t)$ or $u(tm-(-1)^{|s||t|}sn)=0$ *for some* $u\in S$". Then $\equiv$ is an equivalence relation, and we define the module of fractions $S^{-1}M$ of R with respect to$M$ to be the quotient set $M\times S/\equiv$. We denote by m/s the class of (m,s). Define a Γ-graded $S^{-1}R$-module structure on $S^{-1}M$ by $m/s+n/t=(tm+sn)/st$, $x/s\cdot m/t=xm/st$, $\deg(m/s)=\deg m-\deg s$. An R-module homomorphism $f:M\to N$ induces an $S^{-1}R$-module homomorphism $S^{-1}f:S^{-1}M\to S^{-1}N$ by $S^{-1}f(m/s)=f(m)/s$. Thus S^{-1} is a functor from the category of Γ-graded R-modules to that of Γ-graded $S^{-1}R$-modules. For a prime ideal $\mathfrak{p}$ of R, we denote $S_{\mathfrak{p}}^{-1}M$ by $M_{\mathfrak{p}}$.

Proposition 1.6.14 *1) A homomorphism* $\varphi:S^{-1}R\otimes_R M\to S^{-1}M$ *defined by* $\varphi((x/s)\otimes m)=xm/s$ *is an isomorphism.*

2) S^{-1} *is an exact functor. Hence* $S^{-1}R$ *is a flat* R*-module.*

3) *If* N *and* P *are submodules of* M, $S^{-1}(N+P)=S^{-1}N+S^{-1}P$ *and* $S^{-1}(N\cap P)=S^{-1}N\cap S^{-1}P$ *hold.*

4) *A homomorphism* $\psi:S^{-1}M\otimes_R S^{-1}N\to S^{-1}(M\otimes_R N)$ *defined by* $\psi((m/s)\otimes(n/t))=(m\otimes n)/st$ *is an isomorphism.*

Proposition 1.6.15 *Let* M *and* N *be* Γ*-graded* R*-modules and* $f:M\to N$ *a homomorphism.*

1) $M=0$ *if and only if* $M_{\mathfrak{m}}=0$ *for all maximal ideals* $\mathfrak{m}$ *of* R.

2) $f:M\to N$ *is injective* (*resp. surjective*) *if and only if* $f_{\mathfrak{m}}:M_{\mathfrak{m}}\to N_{\mathfrak{m}}$ *is injective* (*resp. surjective*) *for all maximal ideals* $\mathfrak{m}$ *of* R.

3) M *is flat* (*resp. torsion free*) *if and only if* $M_{\mathfrak{m}}$ *is flat* (*resp. torsion free*) *for all maximal ideals* $\mathfrak{m}$ *of* R.

Proposition 1.6.16 *Let M be a finitely generated Γ-graded R-module.*

1) For an ideal $\mathfrak{a}$ of R, we set $S = 1 + \mathfrak{a}_0$. Then, $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of R.

2) *If $\mathfrak{a}$ is an ideal of R such that $\mathfrak{a}M = M$, then there exists $x \in R_0$ satisfying $x - 1 \in \mathfrak{a}$ and $xM = 0$.*

3) *Let $\mathfrak{a}$ be an ideal of R contained in the Jacobson radical of R. If $M = \mathfrak{a}M + N$ for a submodule N, then $M = N$.*

4) *Let R be a Γ-local ring, $\mathfrak{m}$ its maximal ideal, $k = R/\mathfrak{m}$ its residue field. If $x_1, \ldots, x_n$ are elements of M whose images in $M/\mathfrak{m}M$ form a basis of this Γ-graded vector space, then $x_1, \ldots, x_n$ generate M.*

Let M be a Γ-graded R-module and N, P submodules of M. We put $\mathrm{Ann}(M) = \{x \in R |\, xM = 0\}$, $(N : P) = \{x \in R |\, xP \subseteq N\}$, then both of them are ideals of R. $\mathrm{Ann}(M)$ is called the annihilator of M. M is said to be faithful if $\mathrm{Ann}(M) = 0$. M is always faithful $R/\mathrm{Ann}(M)$-module.

Proposition 1.6.17 *1)* $\mathrm{Ann}(M) = (0 : M)$, $(N : P) = \mathrm{Ann}((N + P)/N)$, $\mathrm{Ann}(N + P) = \mathrm{Ann}(M) \cap \mathrm{Ann}(N)$.

2) *If M is finitely generated, $S^{-1}(\mathrm{Ann}(M)) = \mathrm{Ann}(S^{-1}M)$. Hence $S^{-1}(N : P) = (S^{-1}N : S^{-1}P)$ if P is finitely generated.*

Proposition 1.6.18 *Let $f : A \to B$ be a flat homomorphism of Γ-rings. Then, the following conditions are equivalent.*

(1) *$\mathfrak{a}^{ec} = \mathfrak{a}$ for any ideal $\mathfrak{a}$ of A.*

(2) *All prime ideals of A are contractions of prime ideals of B.*

(3) *For each maximal ideal $\mathfrak{m}$ of A, we have $\mathfrak{m}^e \neq (1)$.*

(4) *If M is a non-zero Γ-graded A-module, then $B \otimes_R M \neq 0$.*

(5) *For every Γ-graded A-module M, the mapping $x \mapsto 1 \otimes x$ of M into $B \otimes_R M$ is injective.*

A flat homomorphism satisfying the above conditions is called faithfully flat.

Proposition 1.6.19 *Let $f_1, f_2, \ldots, f_n$ be elements of R. A homomorphism $\varphi : A \to \prod_{i=1}^n A_{f_i}$ defined by $\varphi(x) = (\rho_1(x), \ldots, \rho_n(x))$ is faithfully flat if and only if $(f_1, \ldots, f_n) = (1)$, where $\rho_i : A \to A_{f_i}$ is the canonical homomorphism.*

Proposition 1.6.20 *Let $f : A \to B$ be a faithfully flat homomorphism of Γ-rings, M a Γ-graded A-module. Then, the following complex of A-modules is acyclic.*

$$0 \to M \xrightarrow{\partial_0} M \otimes_A B \to \cdots \to M \otimes_A \overbrace{B \otimes_A \cdots \otimes_A B}^{i \text{ times}} \xrightarrow{\partial_i} M \otimes_A \overbrace{B \otimes_A \cdots \otimes_A B}^{i+1 \text{ times}} \to \cdots$$

where $\partial_0(m) = m \otimes 1$, $\partial_i(m \otimes b_1 \otimes \cdots \otimes b_i) = \sum\limits_{j=0}^{i} (-1)^j m \otimes b_1 \otimes \cdots \otimes b_{i-j} \otimes 1 \otimes b_{i-j+1} \otimes \cdots \otimes b_i$.

1.7 Matrices and determinants

Let R be a Γ-ring. For $g_1, \ldots, g_m, h_1, \ldots, h_n, g \in \Gamma$, we call a $n \times m$ matrix whose ij-component belongs to $R_{g_j - h_i + g}$ a matrix of type $(g_1, \ldots, g_m; h_1, \ldots, h_n; g)$. We denote by $M(g_1, \ldots, g_m; h_1, \ldots, h_n; g)$ the set of matrices of type $(g_1, \ldots, g_m; h_1, \ldots, h_n; g)$. We define componentwise addition in $M(g_1, \ldots, g_m; h_1, \ldots, h_n; g)$. Put $M(g_1, \ldots, g_m; h_1, \ldots, h_n) = \sum\limits_{g \in \Gamma} M(g_1, \ldots, g_m; h_1, \ldots, h_n; g)$, which is a Γ-graded abelian group. A scalar multiplication in $M(g_1, \ldots, g_m; h_1, \ldots, h_n)$ is defined by

$$(a_{ij})r = ((-1)^{\sigma(g_j)|r|} a_{ij} r) \in M(g_1, \ldots, g_m; h_1, \ldots, h_n; g + h) \text{ for } r \in R^{\mathfrak{h}},\ (a_{ij}) \in M(g_1, \ldots, g_m; h_1, \ldots, h_n; g).$$

Then, $M(g_1, \ldots, g_m; h_1, \ldots, h_n)$ is a Γ-graded right R-module. We also define a product

$$\mu : M(h_1, \ldots, h_n; k_1, \ldots, k_l; h) \times M(g_1, \ldots, g_m; h_1, \ldots, h_n; g) \to M(g_1, \ldots, g_m; k_1, \ldots, k_l; g + h)$$

by $\mu((b_{ij}), (a_{ij})) = (c_{ij})$, where $c_{ij} = \sum\limits_{k=1}^{n} b_{ik} a_{kj}$. We usually denote $\mu((b_{ij}), (a_{ij}))$ by $(b_{ij})(a_{ij})$. Obviously, this product is biadditive and induces an R-module homomorphism

$$M(h_1, \ldots, h_n; k_1, \ldots, k_l) \otimes_R M(g_1, \ldots, g_m; h_1, \ldots, h_n) \to M(g_1, \ldots, g_m; k_1, \ldots, k_l),$$

which we also denote by μ.
In the case $m = n$ and $g_i = h_i$, we set $M(g_1, \ldots, g_m; g) = M(g_1, \ldots, g_m; g_1, \ldots, g_m; g)$ and $M(g_1, \ldots, g_m) = M(g_1, \ldots, g_m; g_1, \ldots, g_m)$ for short. Define $\eta : R \to M(g_1, \ldots, g_m)$ by $\eta(r) = ((-1)^{\sigma(g_j)|r|} r\delta_{ij})$ for $r \in R^{\mathfrak{h}}$, where $\delta_{ii} = 1$, $\delta_{ij} = 0$ if $i \neq j$. Thus $M(g_1, \ldots, g_m)$ is a Γ-graded R-algebra with product μ and unit η.

Let M and N be Γ-graded free (right) R-modules with basis $S = \{x_1, \ldots, x_m\}$ and $T = \{y_1, \ldots, y_n\}$. We put $\deg x_i = g_i$, $\deg y_j = h_j$.

Proposition 1.7.1 *Let $\Phi_{S,T} : \mathrm{Hom}_R(M, N) \to M(g_1, \ldots, g_m; h_1, \ldots, h_n)$ be a map defined by $\Phi_{S,T}(f) = (a_{ij})$ if $f(x_j) = \sum_{i=1}^{n} y_i a_{ij}$. Then, $\Phi_{S,T}$ is an isomorphism of Γ-graded R-modules. If L is a free R-module with basis U and $f_1 : M \to N$, $f_2 : N \to L$ are homomorphisms, we have $\Phi_{S,U}(f_2 f_1) = \Phi_{T,U}(f_2)\Phi_{S,T}(f_1)$. In particular, in the case $M = N$ and $S = T$, $\Phi_{S,S} : \mathrm{End}_R(M) \to M(g_1, \ldots, g_m)$ is an isomorphism of Γ-graded R-algebras.*

Proposition 1.7.2 *Let M and N be as above, and let $S^* = \{x_1^*, \ldots, x_m^*\}$ and $T^* = \{y_1^*, \ldots, y_n^*\}$ the dual basis to S and T, respectively. For an R-linear map $f : M \to N$, we put $\Phi_{S,T}(f) = (a_{ij})$ and ${}^t a_{ij} = (-1)^{\sigma(g_i+g)\sigma(g_i+h_j)} a_{ji}$. Then $\Phi_{T^*,S^*} : \mathrm{Hom}_R(N^*, M^*) \to M(-h_1, \ldots, -h_n; -g_1, \ldots, -g_m)$ maps $f^* : N^* \to M^*$ to $({}^t a_{ij})$.*

We call the matrix $({}^t a_{ij})$ the transpose of (a_{ij}) and denote this by ${}^t(a_{ij})$.

Let Σ_m be the symmetric group on $\{1, 2, \ldots, m\}$ and $sgn : \Sigma_m \to \{-1, 1\}$ the signature of Σ_m. Let Σ_m act on $(\boldsymbol{Z}/2)^m$ by $(\gamma_1, \ldots, \gamma_m)\tau = (\gamma_{\tau(1)}, \ldots, \gamma_{\tau(m)})$. We define a function $\varepsilon : (\boldsymbol{Z}/2)^m \times \Sigma_m \to \boldsymbol{Z}/2$ as follows. Regard $\boldsymbol{Z}$ as a $\boldsymbol{Z}/2$-graded ring by $\boldsymbol{Z}_0 = \boldsymbol{Z}$, $\boldsymbol{Z}_1 = 0$ with signature $\sigma = id_{\boldsymbol{Z}/2}$. For $(\gamma_1, \ldots, \gamma_m) \in (\boldsymbol{Z}/2)^m$, let $V(\boldsymbol{Z}; \gamma_1, \ldots, \gamma_m)$ be a $\boldsymbol{Z}/2$-graded free $\boldsymbol{Z}$-module generated by $X_1, \ldots, X_m$ with $\deg X_i = \gamma_i$. Define $\varepsilon((\gamma_1, \ldots, \gamma_m), \tau)$ by $X_{\tau(1)} \cdots X_{\tau(m)} = (-1)^{\varepsilon((\gamma_1,\ldots,\gamma_m),\tau)} X_1 \cdots X_m$ in $S(V(\boldsymbol{Z}; \gamma_1, \ldots, \gamma_m))_m$.

Lemma 1.7.3 *1) For $\gamma \in (\boldsymbol{Z}/2)^m$ and $\tau, \tau' \in \Sigma_m$, equalities $\varepsilon(\gamma, 1) = 0$, $\varepsilon(\gamma, \tau\tau') = \varepsilon(\gamma, \tau) + \varepsilon(\gamma\tau, \tau')$ and $\varepsilon(\gamma, \tau^{-1}) = \varepsilon(\gamma\tau^{-1}, \tau)$ hold.*

2) Suppose $\tau = \tau_1\tau_2 \cdots \tau_s$ where τ_t is a transposition of α_t and β_t. If there exist $k_t \in \{1, 2, \ldots, m-1\}$ such that $\alpha_t = \tau_1 \cdots \tau_{t-1}(k_t)$ and $\beta_t = \tau_1 \cdots \tau_{t-1}(k_t + 1)$ for each t, then $\varepsilon((\gamma_1, \ldots, \gamma_m), \tau) = \sum_{t=1}^{s} \gamma_{\alpha_t}\gamma_{\beta_t}$.

3) $\varepsilon((\gamma_1, \ldots, \gamma_m), \tau) = \sum_{s<t, \tau^{-1}(s) > \tau^{-1}(t)} \gamma_s\gamma_t$.

We define the determinant $\det : M(g_1, \ldots, g_m; h_1, \ldots, h_m; g) \to R_{mg + \sum_{i=1}^{m}(g_i - h_i)}$ by

$$\det A = \sum_{\tau \in \Sigma_m} sgn\,\tau (-1)^{\sum_{s<t}(\eta_s\eta_t + \gamma_s\eta_{\tau(t)}) + \gamma \sum_{s=1}^{m-1} s\eta_{\tau(s+1)} + \varepsilon((\eta_1,\ldots,\eta_m),\tau)} a_{\tau(1)1} \cdots a_{\tau(m)m}$$

where we put $A = (a_{ij})$, $\gamma_i = \sigma(g_i)$, $\eta_i = \sigma(h_i)$, $\gamma = \sigma(g)$.

Let N be a Γ-graded R-module and $y_1, \ldots, y_m$ elements of N with $\deg y_i = h_i$. For $A = (a_{ij}) \in M(g_1, \ldots, g_m; h_1, \ldots, h_m; g)$, $\det A$ is an element of R satisfying $(\sum_{i=1}^{m} y_i a_{i1}) \wedge \cdots \wedge (\sum_{i=1}^{m} y_i a_{im}) = (y_1 \wedge \cdots \wedge y_m) \det A$ in $E(N)_m$. Hence if $E(N)_m$ is torsion free, $\det A$ is uniquely determined by this equality.

Let $V = V(R; h_1, \ldots, h_m)$ be a Γ-graded free R-module with basis $e_1, \ldots, e_m$ such that $\deg e_j = h_j$. Define a map $\Psi : V_{g_1+g} \times \cdots \times V_{g_m+g} \to M(g_1, \ldots, g_m; h_1, \ldots, h_m; g)$ by $\Psi(\sum_{i=1}^{m} e_i a_{i1}, \ldots, \sum_{i=1}^{m} e_i a_{im}) = (a_{ij})$, then Ψ is bijective and we denote $\det \Psi$ also by $\det$.

Proposition 1.7.4 *For $a_i, a_i' \in V_{g_i+g}$ and $r \in R^{\mathfrak{h}}$, the following equalities hold.*

1) $\det(a_1, \ldots, a_j + a_j', \ldots, a_m) = \det(a_1, \ldots, a_j, \ldots, a_m) + \det(a_1, \ldots, a_j', \ldots, a_m)$.

2) $\det(a_1, \ldots, a_j r, \ldots, a_m) = (-1)^{|r|(\gamma_{j+1}+\cdots+\gamma_m+(m-j)\gamma)} \det(a_1, \ldots, a_j, \ldots, a_m)r$.

3) $\det(e_1, \ldots, e_m) = 1$.

4) *Assume that V is strongly free. Then,* $\det(a_1, \ldots, a_m) = 0$ *if $a_i = a_j$ for some $i \neq j$, and for $\tau \in \Sigma_m$,* $\det(a_{\tau(1)}, \ldots, a_{\tau(m)}) = sgn\,\tau(-1)^{\varepsilon((\gamma_1,\ldots,\gamma_m),\tau)} \det(a_1, \ldots, a_m)$. *Hence* det *induces an isomorphism of R-modules* $\det : E(M)_m \to R$ *of degree $-(h_1 + \cdots + h_m)$ which maps $e_1 \wedge \cdots \wedge e_m$ to 1.*

Proposition 1.7.5 *1) If $V(R;k_1,\dots,k_m)$ is strongly free, then for $A\in M(g_1,\dots,g_m;h_1,\dots,h_m;g)$ and $B\in M(h_1,\dots,h_m;k_1,\dots,k_m;h)$, $\det BA=\det B\det A$ holds .*

2) Assume that $V(R;h_1,\dots,h_m)$ is strongly free. If $C\in M(g_1,\dots,g_m;h_1,\dots,h_m;g)$ is a matrix of the form $\left(\begin{smallmatrix}A&*\\0&B\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}A&0\\ *&B\end{smallmatrix}\right)$ for some $A\in M(g_1,\dots,g_s;h_1,\dots,h_s;g)$ and $B\in M(g_{s+1},\dots,g_m;h_{s+1},\dots,h_m;g)$, $\det C=(-1)^{(s\gamma+\sum_{i=1}^{s}(\gamma_i+\eta_i))(\sum_{i=s+1}^{m}\eta_i)}\det A\det B$.

3) For $A\in M(g_1,\dots,g_m;g)$, $\det{}^tA=\det A$ if $char\,R=2$ or $\sigma(g)=0$.

Let M be a strongly free Γ-graded R-module of rank m. We define the determinant $\det:\mathrm{End}_R^g(M)\to R_{mg}$ when $char\,R=2$ or $\sigma(g)=0$ by $f^{\wedge m}(z)=z(\det f)$ for $f\in\mathrm{End}_R^g(M)$, $z\in E(M)_m$. Choosing a basis $S=\{x_1,\dots,x_m\}$ of M with $\deg x_i=g_i$, $\det f$ is also defined by $\det f=\det\Phi_{S,S}(f)$.

Proposition 1.7.6 *Let M be as above.*

1) For $f,f'\in\mathrm{End}_R(M)^{\mathfrak{h}}$, $\det f'f=(\det f')(\det f)$, $\det id_M=1$ and $\det f^*=\det f$ hold, where $f^*:M^*\to M^*$ is the dual of f.

2) If $M=N\oplus L$ for free submodule N, L and $f\in\mathrm{End}_R^g(M)$ maps N into N, then $\det f=(\det f|_N)(\det\tilde f)$, where $f|_N:N\to N$ is the restriction of f, and $\tilde f:M/N\to M/N$ is the homomorphism induced by f.

3) If N and $M\otimes_R N$ are also strongly free, then $\det(f\otimes f')=(\det f)^m(\det f')^n$ for $f\in\mathrm{End}_R(M)^{\mathfrak{h}}$, $f'\in\mathrm{End}_R(N)^{\mathfrak{h}}$ where $n=rank\,N$.

Let M be as above, then a homomorphism $\zeta:M\to\mathrm{Hom}_R(E(M)_{m-1},E(M)_m)$ defined by $\zeta(x)(y)=x\wedge y$ is an isomorphism. In the case $char\,R=2$ or $\sigma(g)=0$, for $f\in\mathrm{End}_R^g(M)$, we define a homomorphism $f^{ad}:M\to M$ of degree $(m-1)g$ by $f^{ad}=\zeta^{-1}(f^{\wedge(m-1)})^*\zeta$.

Choose a basis $S=\{x_1,\dots,x_m\}$ of M. For $1\le i_1<\cdots<i_r\le m$, $1\le j_1<\cdots<j_r\le m$ and $f\in\mathrm{End}_R^g(M)$, define $P_S(f)\binom{i_1\cdots i_r}{j_1\cdots j_r}\in R_{rg+g_{j_1}+\cdots+g_{j_r}-(g_{i_1}+\cdots+g_{i_r})}$ by $f(x_{j_1})\wedge\cdots\wedge f(x_{j_r})=\sum_{1\le i_1<\cdots<i_r\le m}x_{i_1}\wedge\cdots\wedge x_{i_r}P_S(f)\binom{i_1\cdots i_r}{j_1\cdots j_r}$.
For a matrix $A=(a_{ij})\in M(g_1,\dots,g_m;g)$, we denote by $A\binom{i_1\cdots i_r}{j_1\cdots j_r}$ the matrix of type $(g_{j_1},\dots,g_{j_r};g_{i_1},\dots,g_{i_r};g)$ whose st-component is $a_{i_sj_t}$. Then, it is easy to verify that $P_S(f)\binom{i_1\cdots i_r}{j_1\cdots j_r}=\det A\binom{i_1\cdots i_r}{j_1\cdots j_r}$, if $A=\Phi_{S,S}(f)$.
For A as above, we define the adjoint matrix $A^{ad}=(a^*_{ij})\in M(g_1,\dots,g_m;(m-1)g)$ of A by

$$a^*_{ij}=(-1)^{\varepsilon_{ij}}\det A\begin{pmatrix}1\cdots j-1\,j+1\cdots m\\1\cdots i-1\,i+1\cdots m\end{pmatrix}$$

where $\varepsilon_{ij}=i+j+\gamma_i\gamma_j+\gamma_i\sum_{s=i+1}^{m}\gamma_s+\gamma_j\sum_{s=j}^{m}\gamma_s$, $\gamma_i=\sigma(g_i)$.

Proposition 1.7.7 *1) For M, S and f as above, $\Phi_{S,S}(f^{ad})=\Phi_{S,S}(f)^{ad}$.*

2) We put $\delta(i_1,\dots,i_r;j_1,\dots,j_r)=\sum_{s<t}\gamma_{i_s}\gamma_{i_t}+\sum_{s\le t}\gamma_{j_s}\gamma_{j_t}+\sum_{s,t=1}^{r}\gamma_{i_s}\gamma_{j_t}$, then if $char\,R=2$ or $\sigma(g)=0$,

$$\det({}^tA)\begin{pmatrix}j_1\cdots j_r\\i_1\cdots i_r\end{pmatrix}=(-1)^{\delta(i_1,\dots,i_r;j_1,\dots,j_r)}\det A\begin{pmatrix}i_1\cdots i_r\\j_1\cdots j_r\end{pmatrix}.$$

3) $({}^tA)^{ad}={}^t(A^{ad})$ if $char\,R=2$ or $\sigma(g)=0$.

Theorem 1.7.8 *1) Let M be a strongly free Γ-graded R-module. Suppose that $char\,R=2$ or $\sigma(g)=0$, then $(f^*)^{ad}=(f^{ad})^*\in\mathrm{End}_R^{(m-1)g}(M^*)$ for $f\in\mathrm{End}_R^g(M)$.*

2) Under the above assumption, $f^{ad}f=ff^{ad}=id_M(\det f)$. Therefore, $A^{ad}A=AA^{ad}=I_m(\det A)$ for $A\in M(g_1,\dots,g_m;g)$ if $V(R;g_1,\dots,g_m)$ is strongly free, and $char\,R=2$ or $\sigma(g)=0$.

Corollary 1.7.9 *$f\in\mathrm{End}_R^g(M)$ is an automorphism if and only if $\det f\in R_{mg}$ is a unit.*

Lemma 1.7.10 *Let $x_1,\dots,x_m$ be homogeneous elements of a Γ-graded R-module M. Suppose $\{i_1,\dots,i_r\}\cup\{j_1,\dots,j_{m-r}\}=\{1,2,\dots,m\}$, $i_1<\cdots<i_r$, $j_1<\cdots<j_{m-r}$, then we have $x_{i_1}\wedge\cdots\wedge x_{i_r}=(-1)^{e(i_1,\dots,i_r)}x_1\wedge\cdots\wedge x_m$ in $E(M)_m$, where we set $e(i_1,\dots,i_m)=r(r+1)/2+\sum_{s=1}^{r}i_s+\sum_{s=1}^{m-r}\sum_{t=j_s-s+1}^{r}\gamma_{j_s}\gamma_{i_s}$, $\gamma_s=|x_s|$.*

Let M be a strongly free Γ-graded R-module with basis $S = \{x_1, \dots, x_m\} \subseteq M^\natural$ and let $i_1, \dots, i_r, j_1, \dots, j_{m-r}$ be as in the above lemma.

Proposition 1.7.11 *Assume that char $R = 2$ or $\sigma(g) = 0$. For $f \in \mathrm{End}_R^g(M)$, we have*

$$\det f = \sum (-1)^{e(i_1,\dots,i_r)+e(k_1,\dots,k_r)} P_S(f)\begin{pmatrix} k_1 \cdots k_r \\ i_1 \cdots i_r \end{pmatrix} P_S(f)\begin{pmatrix} l_1 \cdots l_{m-r} \\ j_1 \cdots j_{m-r} \end{pmatrix}$$

where the summation is taken over $k_1, \dots, k_r, l_1, \dots, l_{m-r}$ such that $\{k_1, \dots, k_r\} \cup \{l_1, \dots, l_{m-r}\} = \{1, 2, \dots, m\}$, $k_1 < \dots < k_r$, $l_1 < \dots < l_{m-r}$. Hence if $V(R; g_1, \dots, g_m)$ is strongly free and $A \in M(g_1, \dots, g_m; g)$, we have

$$\det A = \sum (-1)^{e(i_1,\dots,i_r)+e(k_1,\dots,k_r)} \det A\begin{pmatrix} k_1 \cdots k_r \\ i_1 \cdots i_r \end{pmatrix} \det A\begin{pmatrix} l_1 \cdots l_{m-r} \\ j_1 \cdots j_{m-r} \end{pmatrix}.$$

Here we put $\gamma_i = \sigma(g_i)$ in the definition of the function e.

Let M be a Γ-graded R-module and f an endomorphism of M of degree g. We denote by $R[f]$ a subalgebra of $\mathrm{End}_R(M)$ generated by f. Assume that *char* $R = 2$ or $\sigma(g) = 0$, then $R[f]$ is a commutative Γ-graded R-algebra. We regard M as a right $R[f]$-module. On the other hand, $R[X]$ denotes a symmetric algebra generated by a free R-module spanned by a single element X of degree g. For $A \in M(g_1, \dots, g_m; g)$, we put $\varphi_A(X) = \det(XI_m - A) \in R[X]_{mg}$. If $M \otimes_R R[X]$ is a strongly free $R[X]$-module, we put $\varphi_f(X) = \det(X id_{M\otimes_R R[X]} - f \otimes id_{R[X]}) \in R[X]_{mg}$. Then $\varphi_f(X) = \varphi_{\Phi_{S,S}(f)}(X)$ for any homogeneous basis S of M over R. We call $\varphi_A(X)$ and $\varphi_f(X)$ characteristic polynomials of A and f, respectively.

Theorem 1.7.12 *If $V(R[X]; g_1, \dots, g_m)$ is strongly free, $\varphi_A(A) = 0$ in $M(g_1, \dots, g_m; mg)$. Hence if $M \otimes_R R[X]$ is strongly free, $\varphi_f(f) = 0$ in $\mathrm{End}_R^{mg}(M)$.*

Corollary 1.7.13 *Let M be a finitely generated R-module, f an endomorphism of M of degree g whose image is contained in $\mathfrak{a}M$ for an ideal $\mathfrak{a}$ of R. Suppose that M is generated by elements $x_1, \dots, x_m$ such that $f(x_j) = \sum_{i=1}^{m} x_i a_{ij}$ with $a_{ij} \in \mathfrak{a}$ and $V(R[x]; \deg x_1, \dots, \deg x_m)$ is strongly free. If char $R = 2$ or $\sigma(g) = 0$, then f satisfies an equation in $\mathrm{End}_R^{mg}(M)$ of the form $f^m + a_1 f^{m-1} + \dots + a_m = 0$ for some $a_i \in \mathfrak{a}$.*

1.8 Sheaves of Γ-graded abelian groups

First we recall several basic facts on sheaves of abelian groups. Let us denote by $\mathbf{Ab}_X$ (resp. $\widetilde{\mathbf{Ab}}_X$) the category of sheaves (resp. presheaves) of abelian groups over a topological space X. $\mathbf{Ab}_X$ is a full subcategory of $\widetilde{\mathbf{Ab}}_X$.

Definition 1.8.1 *Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$, $\mathcal{G}$ presheaves of abelian groups on X, Y, respectively.*

1) The direct image $f_*\mathcal{F}$ of $\mathcal{F}$ by f is a presheaf on Y defined by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$ for an open set V of Y. The restriction $\rho_{V'V} : f_*\mathcal{F}(V) \to f_*\mathcal{F}(V')$ ($V' \subseteq V$) is the restriction $\rho_{f^{-1}(V'),f^{-1}(V)} : \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(V'))$. Then, homomorphisms $f_*\mathcal{F}(V) \to \mathcal{F}(U)$ for $U \subseteq f^{-1}(V)$ induce a homomorphism of stalks $i_x : (f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x$. If $\varphi : \mathcal{F} \to \mathcal{F}'$ is a morphism of presheaves on X, define $f_*\varphi : f_*\mathcal{F} \to f_*\mathcal{F}'$ by $(f_*\varphi)_V = \varphi_{f^{-1}(V)}$. If $\mathcal{F}$ is a sheaf, so is $f_*\mathcal{F}$. Thus f_* is a functor $\widetilde{\mathbf{Ab}}_X \to \widetilde{\mathbf{Ab}}_Y$ and this restricts to a functor $\mathbf{Ab}_X \to \mathbf{Ab}_Y$.

2) The inverse image $f^{-1}\mathcal{G}$ of $\mathcal{G}$ by f is a presheaf on X defined by $f^{-1}\mathcal{G}(U) = \{(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{G}_{f(x)} |$ For any $x \in U$, there exists a neighborhood V of $f(x)$, $t \in \mathcal{G}(V)$ and a neighborhood W of x contained in $f^{-1}(V) \cap U$ such that $t_{f(w)} = s_w$ for any $w \in W\}$. The restriction $\rho_{U'U} : f^{-1}\mathcal{G}(U) \to f^{-1}\mathcal{G}(U')$ ($U' \subseteq U$) maps $(s_x)_{x\in U} \to (s_x)_{x\in U'}$. The restriction of the projection $f^{-1}\mathcal{G}(U) \to \mathcal{G}_{f(x)}$ gives an isomorphism of stalks $(f^{-1}\mathcal{G})_x \to \mathcal{G}_{f(x)}$ for each $x \in X$. It is easily verified that $f^{-1}\mathcal{G}$ is a sheaf. We call $id_Y^{-1}\mathcal{G}$ the sheaf associated with $\mathcal{G}$ and denote this by $\mathcal{G}\hat{}$. If $\varphi : \mathcal{G} \to \mathcal{G}'$ is a morphism of presheaves on Y, define $f^{-1}\varphi : f^{-1}\mathcal{G} \to f^{-1}\mathcal{G}'$ by $(f^{-1}\varphi)_U((s_x)_{x\in U}) = (\varphi_x(s_x))_{x\in U}$. Thus f^{-1} is a functor $\widetilde{\mathbf{Ab}}_Y \to \mathbf{Ab}_X$. We also denote by $f^{-1} : \mathbf{Ab}_Y \to \mathbf{Ab}_X$ the restriction of $f^{-1} : \widetilde{\mathbf{Ab}}_Y \to \mathbf{Ab}_X$ to $\mathbf{Ab}_X$.

Let Z be a subspace of X and $i : Z \to X$ the inclusion map. We denote by $\mathcal{F}|_Z$ the inverse image $i^{-1}\mathcal{F}$ and call this the the restriction of a sheaf $\mathcal{F}$ to Z. Note that $(\mathcal{F}|_Z)_x = \mathcal{F}_x$ for any $x \in Z$.

Proposition 1.8.2 *$f^{-1} : \widetilde{\mathbf{Ab}}_Y \to \mathbf{Ab}_X$ is a left adjoint to $f_* : \mathbf{Ab}_X \to \widetilde{\mathbf{Ab}}_Y$ with unit $\eta : id_{\widetilde{\mathbf{Ab}}_Y} \to f_*f^{-1}$ and counit $\epsilon : f^{-1}f_* \to id_{\mathbf{Ab}_X}$ given as follows.*
For a presheaf $\mathcal{G}$ of abelian groups on Y and $s \in \mathcal{G}(V)$, $(\eta_{\mathcal{G}})_V(s) = (s_{f(x)})_{x\in f^{-1}(V)}$.
For a sheaf $\mathcal{F}$ of abelian groups on X and $(s_x)_{x\in U} \in f^{-1}f_*\mathcal{F}(U) \subseteq \prod_{x\in U}(f_*\mathcal{F})_{f(x)}$, there is a unique $t \in \mathcal{F}(U)$ such that $t_x = i_x(s_x)$ for any $x \in U$. Set $(\epsilon_{\mathcal{F}})_U((s_x)_{x\in U}) = t$.
In particular, the restriction $f^{-1} : \mathbf{Ab}_Y \to \mathbf{Ab}_X$ is a left adjoint to $f_* : \mathbf{Ab}_X \to \mathbf{Ab}_Y$. Thus f_* preserves limits and f^{-1} preserves colimits.

Definition 1.8.3 *Let $(\mathcal{F}_j)_{j\in J}$ be a family of presheaves of abelian groups over X.*

1) The product $\prod_{j\in J}\mathcal{F}_j$ is defined by $(\prod_{i\in J}\mathcal{F}_j)(U) = \prod_{j\in J}\mathcal{F}_j(U)$. The set theoretical projection $\prod_{j\in J}\mathcal{F}_j(U) \to \mathcal{F}_j(U)$ gives the projection $p_j : \prod_{j\in J}\mathcal{F}_j \to \mathcal{F}_j$. It is a sheaf if each $\mathcal{F}_j$ is a sheaf.

2) The presheaf sum $\sum'_{j\in J}\mathcal{F}_j$ is defined by $(\sum'_{j\in J}\mathcal{F}_j)(U) = \sum_{j\in J}\mathcal{F}_j(U)$. The set theoretical inclusion $\mathcal{F}_j(U) \to \sum_{j\in J}\mathcal{F}_j(U)$ gives the inclusion $i'_j : \mathcal{F}_j \to \sum'_{j\in J}\mathcal{F}_j$. If each $\mathcal{F}_j$ is a sheaf, the (sheaf) sum $\sum_{j\in J}\mathcal{F}_j$ is defined by $(\sum_{j\in J}\mathcal{F}_j)(U) = \{(s_j)_{j\in J} \in \prod_{j\in J}\mathcal{F}_j(U)|$ For any $x \in U$, there exists a neighborhood V of x such that $(s_j|_V)_{j\in J} \in \sum_{j\in J}\mathcal{F}_j(V)\}$. Then, since $\sum_{j\in J}\mathcal{F}_j(U) \subseteq (\sum_{j\in J}\mathcal{F}_j)(U) \subseteq \prod_{j\in J}\mathcal{F}_j(U)$ for an open set U of X, there are morphisms of presheaves $\sum'_{j\in J}\mathcal{F}_j \to \sum_{j\in J}\mathcal{F}_j$ and $\sum_{j\in J}\mathcal{F}_j \to \prod_{j\in J}\mathcal{F}_j$. The inclusion i_j is the composition $\mathcal{F}_j \xrightarrow{i'_j} \sum'_{j\in J}\mathcal{F}_j \to \sum_{j\in J}\mathcal{F}_j$.

Remark 1.8.4 *1) We note that the sheaf sum $\sum_{j\in J}\mathcal{F}_j$ is nothing but the sheaf associated with the presheaf sum $\sum'_{j\in J}\mathcal{F}_j$. If J is a finite set, $\sum'_{j\in J}\mathcal{F}_j = \sum_{j\in J}\mathcal{F}_j$.*

2) Let $f : X \to Y$ be a continuous map, then $\sum_{j\in J} f_*\mathcal{F}_j$ is a subsheaf of $f_*(\sum_{j\in J}\mathcal{F}_j)$. If J is a finite set, $\sum_{j\in J} f_*\mathcal{F}_j = f_*(\sum_{j\in J}\mathcal{F}_j)$.

Definition 1.8.5 *Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of abelian groups over X. Define the presheaf tensor product $\mathcal{F} \otimes' \mathcal{G}$ by $(\mathcal{F} \otimes' \mathcal{G})(U) = \mathcal{F}(U) \otimes \mathcal{G}(U)$ for an open set U. For morphisms $\varphi : \mathcal{F} \to \mathcal{F}'$, $\psi : \mathcal{G} \to \mathcal{G}'$, define $\varphi \otimes' \psi : \mathcal{F} \otimes' \mathcal{G} \to \mathcal{F}' \otimes' \mathcal{G}'$ by $(\varphi \otimes' \psi)_U = \varphi_U \otimes \psi_U$. For $x \in X$ and an open set U containing x, the canonical maps $\mathcal{F}(U) \to \mathcal{F}_x$ and $\mathcal{G}(U) \to \mathcal{G}_x$ induce $(\mathcal{F}\otimes'\mathcal{G})(U) \to \mathcal{F}_x \otimes \mathcal{G}_x$. This factors through $(\mathcal{F}\otimes'\mathcal{G})(U) \to (\mathcal{F}\otimes'\mathcal{G})_x$ and we have an isomorphism $(\mathcal{F} \otimes' \mathcal{G})_x \to \mathcal{F}_x \otimes \mathcal{G}_x$.*

If $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups, define the tensor product $\mathcal{F} \otimes \mathcal{G}$ by $\mathcal{F} \otimes \mathcal{G} = (\mathcal{F} \otimes' \mathcal{G})\hat{}$. Morphisms $\varphi : \mathcal{F} \to \mathcal{F}'$, $\psi : \mathcal{G} \to \mathcal{G}'$ induce $\varphi \otimes \psi : \mathcal{F} \otimes \mathcal{G} \to \mathcal{F}' \otimes \mathcal{G}'$ by $\varphi \otimes \psi = (\varphi \otimes' \psi)\hat{}$. For $x \in X$, the isomorphism $(\mathcal{F} \otimes' \mathcal{G})_x \to \mathcal{F}_x \otimes \mathcal{G}_x$ induces an isomorphism $(\mathcal{F} \otimes \mathcal{G})_x \to \mathcal{F}_x \otimes \mathcal{G}_x$. Hence $(\mathcal{F} \otimes \mathcal{G})(U)$ is identified with a subset of $\prod_{x\in U}\mathcal{F}_x \otimes \mathcal{G}_x$.

Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be presheaves of abelian groups over X. A morphism $\rho : \mathcal{F} \times \mathcal{G} \to \mathcal{H}$ of presheaves of sets is said to be biadditive if it satisfies $\rho_U(x + y, z) = \rho_U(x, z) + \rho_U(y, z)$ and $\rho_U(x, z + w) = \rho_U(x, z) + \rho_U(x, w)$ for each open set U and $x, y \in \mathcal{F}(U)$, $z, w \in \mathcal{G}$. The set of biadditive morphisms $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$ is denoted by $\mathrm{Biad}(\mathcal{F}, \mathcal{G}; \mathcal{H})$. Note that the addition of biadditive morphisms makes $\mathrm{Biad}(\mathcal{F}, \mathcal{G}; \mathcal{H})$ an abelian group.

Proposition 1.8.6 *Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be presheaves of abelian groups over X. There is a natural isomorphism of abelian groups $\mathrm{Biad}(\mathcal{F}, \mathcal{G}; \mathcal{H}) \cong \mathrm{Hom}_{\widetilde{\mathbf{Ab}}_X}(\mathcal{F} \otimes' \mathcal{G}, \mathcal{H})$. If $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are sheaves of abelian groups, there is a natural isomorphism of abelian groups $\mathrm{Biad}(\mathcal{F}, \mathcal{G}; \mathcal{H}) \cong \mathrm{Hom}_{\mathbf{Ab}_X}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H})$.*

For an abelian group Γ, a sheaf of Γ-graded abelian groups is the sheaf sum of a Γ-indexed family of sheaves of abelian groups. Hence it is not a presheaf of Γ-graded abelian groups in general.

Let $\mathcal{F} = \sum_{g\in\Gamma}\mathcal{F}_g$ and $\mathcal{G} = \sum_{g\in\Gamma}\mathcal{G}_g$ (resp. $\mathcal{F} = \sum'_{g\in\Gamma}\mathcal{F}_g$ and $\mathcal{G} = \sum'_{g\in\Gamma}\mathcal{G}_g$) be sheaves (resp. presheaf) of Γ-graded abelian groups over X. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of degree $g \in \Gamma$ is a morphism of sheaves (resp. presheaves) of abelian groups such that φ maps each summand $\mathcal{F}_h$ into $\mathcal{G}_{h+g}$. The set of morphisms of degree g is denoted by $\mathrm{Hom}^g_X(\mathcal{F},\mathcal{G})$. The addition of morphisms makes $\mathrm{Hom}^g_X(\mathcal{F},\mathcal{G})$ an abelian group. Put $\mathrm{Hom}_X(\mathcal{F},\mathcal{G}) = \sum_{g\in\Gamma} \mathrm{Hom}^g_X(\mathcal{F},\mathcal{G})$ and we call an element of this set a morphism from $\mathcal{F}$ to $\mathcal{G}$. We define the composition of elements $\varphi = \sum_{g\in\Gamma} \varphi_g \in \mathrm{Hom}_X(\mathcal{F},\mathcal{G})$ and $\psi = \sum_{g\in\Gamma} \psi_g \in \mathrm{Hom}_X(\mathcal{G},\mathcal{H})$ by $\psi\varphi = \sum_{g\in\Gamma}(\sum_{k+l=g} \psi_k\varphi_l)$. Obviously, the composition $\mathrm{Hom}_X(\mathcal{G},\mathcal{H}) \times \mathrm{Hom}_X(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_X(\mathcal{F},\mathcal{H})$ is biadditive. An element of $\bigcup_{g\in\Gamma} \mathrm{Hom}^g_X(\mathcal{F},\mathcal{G})$ is called a homogeneous morphism ("homomorphism" for short). We denote by $\mathbf{Ab}^\Gamma_X$ (resp. $\widetilde{\mathbf{Ab}}^\Gamma_X$) the category of sheaves (resp. presheaves) of Γ-graded abelian groups over X. We note that $\mathbf{Ab}^\Gamma_X$ (resp. $\widetilde{\mathbf{Ab}}^\Gamma_X$) has a zero object 0 given by $U \mapsto 0$ for any open set U of X.

Note that $\mathbf{Ab}^\Gamma_X$ is not a subcategory of $\widetilde{\mathbf{Ab}}^\Gamma_X$.

Let $\mathcal{F}$ be a (pre)sheaf of Γ-graded abelian groups over X and U an open set of X. If S is a subset of $\mathcal{F}(U)$, we put $S^\mathfrak{h} = S \cap (\bigcup_{g\in\Gamma} \mathcal{F}_g(U))$ and call this the homogeneous part of S.

Since the inverse image commutes with sum by (1.8.2), a continuous map $f : X \to Y$ gives functors $f^{-1} : \widetilde{\mathbf{Ab}}^\Gamma_Y \to \mathbf{Ab}^\Gamma_X$ and $f^{-1} : \mathbf{Ab}^\Gamma_Y \to \mathbf{Ab}^\Gamma_X$. Namely, $f^{-1}(\sum'_{g\in\Gamma} \mathcal{G}_g) = \sum_{g\in\Gamma} f^{-1}(\mathcal{G}_g)$, for a presheaf $\sum'_{g\in\Gamma} \mathcal{G}_g$ of Γ-graded abelian groups and $f^{-1}(\sum_{g\in\Gamma} \mathcal{G}_g) = \sum_{g\in\Gamma} f^{-1}(\mathcal{G}_g)$ for a sheaf $\sum_{g\in\Gamma} \mathcal{G}_g$ of Γ-graded abelian groups.

There is a natural isomorphism $(f^{-1}\mathcal{G})_x \to \mathcal{G}_{f(x)}$ of Γ-graded abelian groups of degree zero for an (pre)sheaf $\mathcal{G}$ and $x \in X$. We denote the functor id_X^{-1} by $\hat{}: \widetilde{\mathbf{Ab}}^\Gamma_X \to \mathbf{Ab}^\Gamma_X$.

Define the direct image functors $f_* : \widetilde{\mathbf{Ab}}^\Gamma_X \to \widetilde{\mathbf{Ab}}^\Gamma_Y$ and $f_* : \mathbf{Ab}^\Gamma_X \to \mathbf{Ab}^\Gamma_Y$ by $f_*\mathcal{F} = \sum'_{g\in\Gamma} f_*\mathcal{F}_g$ for $\mathcal{F} \in \widetilde{\mathbf{Ab}}^\Gamma_X$ and $f_*\mathcal{F} = \sum_{g\in\Gamma} f_*\mathcal{F}_g$ for $\mathcal{F} \in \mathbf{Ab}^\Gamma_X$, respectively.

For $x \in X$, there is a natural homomorphism $i_x : (f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x$ of Γ-graded abelian groups of degree zero.

For $\mathcal{F} \in \mathbf{Ab}^\Gamma_X$, we denote by $\mathcal{F}\check{}$ the presheaf sum of $\mathcal{F}_g$'s $\sum'_{g\in\Gamma} \mathcal{F}_g$ and call this the presheaf associated with $\mathcal{F}$. This defines a functor $\check{}: \mathbf{Ab}^\Gamma_X \to \widetilde{\mathbf{Ab}}^\Gamma_X$. Let us denote by $f\check{}_* : \mathbf{Ab}^\Gamma_X \to \widetilde{\mathbf{Ab}}^\Gamma_Y$ the composite functor $f_* : \mathbf{Ab}^\Gamma_X \to \mathbf{Ab}^\Gamma_Y$ and $\check{}: \mathbf{Ab}^\Gamma_Y \to \widetilde{\mathbf{Ab}}^\Gamma_Y$. We have an analog of (1.8.2).

Proposition 1.8.7 *The functor $f^{-1} : \widetilde{\mathbf{Ab}}^\Gamma_Y \to \mathbf{Ab}^\Gamma_X$ is a left adjoint to the functor $f\check{}_* : \mathbf{Ab}^\Gamma_X \to \widetilde{\mathbf{Ab}}^\Gamma_Y$. Similarly, the functor $f^{-1} : \mathbf{Ab}^\Gamma_Y \to \mathbf{Ab}^\Gamma_X$ is a left adjoint to the functor $f_* : \mathbf{Ab}^\Gamma_X \to \mathbf{Ab}^\Gamma_Y$.*

Proposition 1.8.8 *1) For a sheaf $\mathcal{F}$ of Γ-graded abelian groups over X, the canonical inclusion $\mathcal{F}\check{} \hookrightarrow \mathcal{F}$ induces a natural isomorphism $(\mathcal{F}\check{})\hat{} \to \mathcal{F}$. In fact, this isomorphism is the counit of the adjoints.*

2) If $\mathcal{F}$ is a presheaf of Γ-graded abelian groups over X, then $\mathcal{F}\hat{} = \sum_{g\in\Gamma} \mathcal{F}_g\hat{}$. Hence the unit $\mathcal{F} \to (\mathcal{F}\hat{})\check{}$ is an isomorphism if $\mathcal{F}_g$ is a sheaf for any $g \in \Gamma$.

3) The functor $\check{}: \mathbf{Ab}^\Gamma_X \to \widetilde{\mathbf{Ab}}^\Gamma_X$ is fully faithful.

By virtue of 3) above, $\mathbf{Ab}^\Gamma$ is equivalent to a full subcategory of $\widetilde{\mathbf{Ab}}^\Gamma$.

Definition 1.8.9 *Let $(\mathcal{F}_j)_{j\in J}$ be a family of sheaves of Γ-graded abelian groups over X.*

1) The product $\prod_{j\in J} \mathcal{F}_j$ is defined to be $\sum_{g\in\Gamma}(\prod_{j\in J} \mathcal{F}_{j\,g})$. The canonical projection $\prod_{j\in J} \mathcal{F}_{j\,g} \to \mathcal{F}_{j,g}$ induces the projection $p_j : \prod_{j\in J} \mathcal{F}_j \to \mathcal{F}_j$.

2) The sum $\sum_{j\in J} \mathcal{F}_j$ is defined by $\sum_{g\in\Gamma}(\sum_{j\in J} \mathcal{F}_{j\,g})$. The canonical inclusion $\mathcal{F}_{j\,g} \to \sum_{j\in J} \mathcal{F}_{j\,g}$ induces the inclusion $i_j : \mathcal{F}_j \to \sum_{j\in J} \mathcal{F}_j$.

Remark 1.8.10 *If J is a finite set, $\prod_{j\in J} \mathcal{F}_j = \sum_{j\in J} \mathcal{F}_j$. Hence $\mathbf{Ab}^\Gamma_X$ is a graded additive category. In fact $\mathbf{Ab}^\Gamma$ is a graded abelian category as we will see below.*

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of Γ-graded abelian groups over X. If $\mathcal{G}$ is a subpresheaf of $\mathcal{F}$, we define the quotient presheaf $\mathcal{F}/\mathcal{G}$ by $\mathcal{F}/\mathcal{G}(U) = \mathcal{F}(U)/\mathcal{G}(U)$. If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of $\widetilde{\mathbf{Ab}}^\Gamma_X$, define the kernel and the cokernel of f by $(\mathrm{Ker} f)(U) = \mathrm{Ker}(f_U : \mathcal{F}(U) \to \mathcal{G}(U))$, $(\mathrm{Coker} f)(U) = \mathrm{Coker}(f_U : \mathcal{F}(U) \to \mathcal{G}(U))$.

Also define the image and the coimage of f by $(\mathrm{Im} f)(U) = \mathrm{Im}(f_U : \mathcal{F}(U) \to \mathcal{G}(U))$, $(\mathrm{Coim} f)(U) = \mathrm{Coim}(f_U : \mathcal{F}(U) \to \mathcal{G}(U))$. It is easy to verify that $\widetilde{\mathbf{Ab}}_X^{\Gamma}$ is an abelian category.

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of Γ-graded abelian groups over X. If $f : \mathcal{F} \to \mathcal{G}$ is a homomorphism of $\mathbf{Ab}_X^{\Gamma}$, define the kernel and the cokernel of f by $\mathrm{Ker} f = (\mathrm{Ker}(f\check{} : \mathcal{F}\check{} \to \mathcal{G}\check{}))\hat{}$, $\mathrm{Coker} f = (\mathrm{Coker}(f\check{} : \mathcal{F}\check{} \to \mathcal{G}\check{}))\hat{}$. Also define the image and the coimage of f by $\mathrm{Im} f = \mathrm{Ker}(\mathcal{G} \to \mathrm{Coker} f)$, $\mathrm{Coim} f = \mathrm{Coker}(\mathrm{Ker} f \to \mathcal{F})$.

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of Γ-graded abelian groups. We define the presheaf tensor product $\mathcal{F} \otimes' \mathcal{G}$ by $(\mathcal{F} \otimes' \mathcal{G})_g = \sum'_{h+k=g} \mathcal{F}_h \otimes' \mathcal{G}_k$. If $f_i : \mathcal{F}_i \to \mathcal{F}'_i$ $(i = 1, 2)$ are morphisms of $\widetilde{\mathbf{Ab}}_X^{\Gamma}$ of degree g_i, a morphism $f_1 \otimes' f_2 : \mathcal{F}_1 \otimes' \mathcal{F}_2 \to \mathcal{F}'_1 \otimes' \mathcal{F}'_2$ of degree $g_1 + g_2$ is defined by $(f_1 \otimes' f_2)_U(x_1 \otimes x_2) = (-1)^{\sigma(g_2)|x_1|} f_{1U}(x_1) \otimes f_{2U}(x_2)$ for $x_i \in \mathcal{F}_i(U)^{\mathfrak{h}}$.

If $\mathcal{F}$ and $\mathcal{G}$ be sheaves of Γ-graded abelian groups, define the tensor product $\mathcal{F} \otimes \mathcal{G}$ by $\mathcal{F} \otimes \mathcal{G} = (\mathcal{F}\check{} \otimes' \mathcal{G}\check{})\hat{}$. Then, $(\mathcal{F} \otimes \mathcal{G})_g = \sum\limits_{h+k=g} \mathcal{F}_h \otimes \mathcal{G}_k$ for $g \in \Gamma$. If $f_i : \mathcal{F}_i \to \mathcal{F}'_i$ $(i = 1, 2)$ are morphisms of $\mathbf{Ab}_X^{\Gamma}$ of degree g_i, we define a morphism $f_1 \otimes f_2 : \mathcal{F}_1 \otimes \mathcal{F}_2 \to \mathcal{F}'_1 \otimes \mathcal{F}'_2$ of degree $g_1 + g_2$ by $f_1 \otimes f_2 = (f_1\check{} \otimes' f_2\check{})\hat{}$.

Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be (pre)sheaves of Γ-graded abelian groups over X. We put

$$\mathrm{Biad}^g(\mathcal{F}, \mathcal{G}; \mathcal{H}) = \{(\rho_{k,l})_{k,l \in \Gamma} |\, \rho_{k,l} \in \mathrm{Biad}(\mathcal{F}_k, \mathcal{G}_l; \mathcal{H}_{g+k+l}) \textit{ for } k, l \in \Gamma\}.$$

Note that the componentwise addition of biadditive morphisms makes $\mathrm{Biad}^g(\mathcal{F}, \mathcal{G}; \mathcal{H})$ an abelian group. If $f_i : \mathcal{F}_i \to \mathcal{F}'_i$ $(i = 1, 2)$ and $h : \mathcal{H}' \to \mathcal{H}$ are morphisms of $\mathbf{Ab}_X^{\Gamma}$ (resp. $\widetilde{\mathbf{Ab}}_X^{\Gamma}$) of degree g_i and h, we define $\mathrm{Biad}(f_1, f_2; h) : \mathrm{Biad}^g(\mathcal{F}'_1, \mathcal{F}'_2; \mathcal{H}') \to \mathrm{Biad}^{g+g_1+g_2+h}(\mathcal{F}_1, \mathcal{F}_2; \mathcal{H})$ by $\mathrm{Biad}(f_1, f_2; h)((\rho_{k,l})_{k,l \in \Gamma}) = (\rho'_{k,l})_{k,l \in \Gamma}$ where $\rho'_{k,l} \in \mathrm{Biad}(\mathcal{F}_{1,k}, \mathcal{F}_{2,l}; \mathcal{H}_{g+k+l})$ is given by

$$\rho'_{k,l}(x_1, x_2) = (-1)^{\sigma(g_2)\sigma(k)+\sigma(g)\sigma(g_1+g_2)} h\rho_{k+g_1, l+g_2}(f_1(x_1), f_2(x_2)) \;\text{ for }\; x_1 \in \mathcal{F}_{1,k}(U), ; x_2 \in \mathcal{F}_{2,l}(U).$$

Proposition 1.8.11 *If $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are presheaves of Γ-graded abelian groups over X, then $\mathrm{Biad}^g(\mathcal{F}, \mathcal{G}; \mathcal{H})$ is naturally isomorphic to $\mathrm{Hom}_X^g(\mathcal{F} \otimes' \mathcal{G}, \mathcal{H})$. If $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are sheaves of Γ-graded abelian groups over X, then $\mathrm{Biad}^g(\mathcal{F}, \mathcal{G}; \mathcal{H})$ is naturally isomorphic to $\mathrm{Hom}_X^g(\mathcal{F} \otimes \mathcal{G}, \mathcal{H})$.*

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of Γ-graded abelian groups over X and $f : X \to Y$ a continuous map. Define $\rho^f_{k,l} \in \mathrm{Biad}(f_*\mathcal{F}_k, f_*\mathcal{G}_l; f_*(\mathcal{F} \otimes \mathcal{G})_{g+k})$ by $(\rho^f_{k,l})_U(s,t) = \eta(s \otimes t)$ for $s \in \mathcal{F}(f^{-1}(U))_k$ and $t \in \mathcal{G}(f^{-1}(U))_l$, where $\eta : (\mathcal{F} \otimes' \mathcal{G})_{g+k} \to (\mathcal{F} \otimes \mathcal{G})_{g+k}$ is the unit ((1.8.2)). Then we have a natural morphism $f_*\mathcal{F} \otimes f_*\mathcal{G} \to f_*(\mathcal{F} \otimes \mathcal{G})$ of degree zero by (1.8.11).

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of Γ-graded abelian groups over Y and $f : X \to Y$ a continuous map. Define $\rho_{f\,k,l} \in \mathrm{Biad}(f^{-1}\mathcal{F}_k, f^{-1}\mathcal{G}_l; f^{-1}(\mathcal{F} \otimes \mathcal{G})_{g+k})$ by $(\rho_{f\,k,l})_U((s_x)_{x \in U}, (t_x)_{x \in U}) = (s_x \otimes t_x)_{x \in U}$. Then the natural morphism $f^{-1}\mathcal{F} \otimes f^{-1}\mathcal{G} \to f^{-1}(\mathcal{F} \otimes \mathcal{G})$ of degree zero associated with $(\rho_{f\,k,l})_{k,l \in \Gamma}$ is an isomorphism.

Proposition 1.8.12 *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of Γ-graded abelian groups. φ is an isomorphism if and only if $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for any $x \in X$.*

1.9 Γ-geometric spaces

Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves of Γ-graded abelian groups over X. We denote by $T' : \mathcal{F} \otimes' \mathcal{G} \to \mathcal{G} \otimes' \mathcal{F}$ the switching morphism $T'_U(x \otimes y) = (-1)^{|x||y|}(y \otimes x)$ $(x \in \mathcal{F}(U)^{\mathfrak{h}}, y \in \mathcal{G}(U)^{\mathfrak{h}})$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of Γ-graded abelian groups over X, the switching morphism $T : \mathcal{F} \otimes \mathcal{G} \to \mathcal{G} \otimes \mathcal{F}$ is the morphism induced by $T' : \mathcal{F}\check{} \otimes' \mathcal{G}\check{} \to \mathcal{G}\check{} \otimes' \mathcal{F}\check{}$.

Let us denote by $\boldsymbol{Z}_X$ the constant sheaf over X associated with a Γ-graded abelian group $\boldsymbol{Z}$ ($\boldsymbol{Z}_0 = \boldsymbol{Z}$, $\boldsymbol{Z}_g = 0$ if $g \neq 0$). We note that an evaluation map $e : \mathrm{Hom}_X^g(\boldsymbol{Z}_X, \mathcal{F}) \to \mathcal{F}_g(X)$, $e(\varphi) = \varphi(1)$ is an isomorphism of abelian groups. There are natural isomorphisms $R : \mathcal{F} \otimes \boldsymbol{Z}_X \to \mathcal{F}$ and $L : \boldsymbol{Z}_X \otimes \mathcal{F} \to \mathcal{F}$.

Definition 1.9.1 *A sheaf $\mathcal{F} = \sum\limits_{g \in \Gamma} \mathcal{F}_g$ of Γ-graded abelian groups over X with morphisms $\mu : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ (multiplication) and $\eta : \boldsymbol{Z}_X \to \mathcal{F}$ (unit) of degree zero is called a sheaf of Γ-ring if μ and η satisfies $\mu(id_{\mathcal{F}} \otimes \mu) = \mu(\mu \otimes id_{\mathcal{F}})$, $\mu T = \mu$ and $\mu(id_{\mathcal{F}} \otimes \eta) = R$, $\mu(\eta \otimes id_{\mathcal{F}}) = L$. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves of Γ-rings over X is a morphism of sheaves of Γ-graded abelian groups which commutes with the multiplications and the units.*

If $\mathcal{F} = \sum_{g\in\Gamma} \mathcal{F}_g$ is a sheaf of Γ-rings, $\mathcal{F}(U) = \sum_{g\in\Gamma} \mathcal{F}_g(U)$ is a Γ-ring for each open set U of X, although $\mathcal{F}(U)$ is not in general. Hence the presheaf associated with $\mathcal{F}$ is a presheaf of Γ-rings over X, which is a subpresheaf of $\mathcal{F}$.

We define the notions of Γ-ringed space and Γ-geometric space (locally Γ-ringed space).

Definition 1.9.2 *A Γ-ringed space $E = (X, \mathcal{O}_X)$ is a pair of a topological space X and a sheaf of Γ-rings $\mathcal{O}_X = \sum_{g\in\Gamma} \mathcal{O}_{X,g}$.*
A Γ-geometric space (locally Γ-ringed space) is a Γ- ringed space such that the stalk $\mathcal{O}_{X,x}$ at x is a Γ-local ring for each $x \in X$.
We denote by $\mathfrak{m}_x$ the unique maximal ideal of $\mathcal{O}_{X,x}$ and set $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$. For a neighborhood U of x and $s \in \mathcal{O}_X(U)$, we denote by s_x and $s(x)$ the canonical image of s in $\mathcal{O}_{X,x}$ and $\kappa(x)$, respectively. s_x and $s(x)$ are called the germ of s at x and the value of s at x, respectively.

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of Γ-rings over X and Y with multiplications $\mu_{\mathcal{F}} : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$, $\mu_{\mathcal{G}} : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ and units $\eta_{\mathcal{F}} : \boldsymbol{Z}_X \to \mathcal{F}$, $\eta_{\mathcal{G}} : \boldsymbol{Z}_Y \to \mathcal{G}$, respectively. For a continuous map $f : X \to Y$, compositions $f_*\mathcal{F} \otimes f_*\mathcal{F} \to f_*(\mathcal{F} \otimes \mathcal{F}) \xrightarrow{f_*\mu_{\mathcal{F}}} f_*\mathcal{F}$ and $f^{-1}\mathcal{G} \otimes f^{-1}\mathcal{G} \to f^{-1}(\mathcal{G} \otimes \mathcal{G}) \xrightarrow{f^{-1}\mu_{\mathcal{G}}} f^{-1}\mathcal{G}$ define multiplications of $f_*\mathcal{F}$ and $f^{-1}\mathcal{G}$. Thus $f_*\mathcal{F}$ and $f^{-1}\mathcal{G}$ have structures of sheaves of Γ-rings with units $\boldsymbol{Z}_Y \to f_*\boldsymbol{Z}_X \xrightarrow{f_*\eta} f_*\mathcal{F}$ and $\boldsymbol{Z}_X \to f^{-1}\boldsymbol{Z}_Y \xrightarrow{f^{-1}\eta} f^{-1}\mathcal{G}$, where $\boldsymbol{Z}_Y \to f_*\boldsymbol{Z}_X$ and $\boldsymbol{Z}_X \to f^{-1}\boldsymbol{Z}_Y$ are morphisms determined by $(1_x)_{x\in X} \in f^{-1}\boldsymbol{Z}_Y(X)$ and $1 \in f_*\boldsymbol{Z}_X(Y) = \boldsymbol{Z}_X(X)$, respectively.

Definition 1.9.3 *A morphism of Γ-ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $f^{\mathfrak{c}} : X \to Y$ and a morphism of sheaves of Γ-rings $f^{\mathfrak{f}} : \mathcal{O}_Y \to f_*\mathcal{O}_X$.*
A morphism of Γ-geometric spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of Γ-ringed space such that, for each $x \in X$, $f^{\mathfrak{f}}_x : \mathcal{O}_{Y,f^{\mathfrak{c}}(x)} \to \mathcal{O}_{X,x}$ is local, that is, $f^{\mathfrak{f}}_x(\mathfrak{m}_{f^{\mathfrak{c}}(x)}) \subseteq \mathfrak{m}_x$. The category of Γ-ringed spaces is denoted by $\mathbf{Esa}^{\Gamma}$ *and the category of Γ-geometric spaces is denoted by* $\mathbf{Esg}^{\Gamma}$ *. If U is an open set of X and V is an open set of Y which contains $f(U)$, we denote by $f^V_U : \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ the homomorphism induced by $f^{\mathfrak{f}}$.*

Let $(X, \mathcal{O}_X)$ be a Γ-ringed space and $g : Y \to X$ a continuous map. Then, there is a morphism of Γ-ringed space $\tilde{g} : (Y, g^{-1}\mathcal{O}_X) \to (X, \mathcal{O}_X)$ such that $\tilde{g}^{\mathfrak{c}} = g$ and $\tilde{g}^{\mathfrak{f}} : \mathcal{O}_X \to g_*g^{-1}\mathcal{O}_X$ is the adjoint of $id_{g^{-1}\mathcal{O}_X}$.

Proposition 1.9.4 *Let $g : Y \to X$ be a continuous map.*

1) If $(X, \mathcal{O}_X)$ is a Γ-geometric space, so is $(Y, g^{-1}\mathcal{O}_X)$ and $\tilde{g} : (Y, g^{-1}\mathcal{O}_X) \to (X, \mathcal{O}_X)$ is a morphism of Γ-geometric spaces.

2) *If $f : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a morphism of Γ-ringed spaces and there is a continuous map $\bar{g} : Z \to Y$ satisfying $g\bar{g} = f^{\mathfrak{c}}$, then there is a unique morphism $f' : (Z, \mathcal{O}_Z) \to (Y, g^{-1}\mathcal{O}_X)$ of Γ-ringed spaces such that $f = \tilde{g}f'$. Moreover, if f is a morphism of Γ-geometric spaces, so is f'.*

Let $(X, \mathcal{O}_X)$ be a Γ-ringed space and P a subspace of X. A Γ-ringed space $(P, \mathcal{O}_X|_P)$ is called a subspace of $(X, \mathcal{O}_X)$. If P is open in X, it is called an open subspace. A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of Γ-ringed spaces is called an open embedding if f induces an isomorphism onto an open subspace of $(Y, \mathcal{O}_Y)$.

Let $(X, \mathcal{O}_X)$ be a Γ-geometric space. For $s \in \mathcal{O}_X(U)^{\mathfrak{h}}$, if $s(x) \neq 0$ for $x \in U$, then there exist a neighborhood V of x contained in U and $t \in \mathcal{O}_X(V)$ such that $s_xt_x = 1$. Hence there is a neighborhood W of x such that $s_yt_y = 1$ for each $y \in W$. Therefore a set $\{x \in U | s(x) \neq 0\}$ is open in U. Such an open set is called a special open set of U and denoted by U_s. Note that U_s is the maximum element of $\{V \subseteq U | V$ *is an open set*, $s|_V \in \mathcal{O}_X(V)$ *is a unit.*$\}$.

Lemma 1.9.5 *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of Γ-geometric spaces and V an open set of Y. For $s \in \mathcal{O}_Y(V)^{\mathfrak{h}}$, ${f^{\mathfrak{c}}}^{-1}(V_s) = U_t$ hold, where we put $U = {f^{\mathfrak{c}}}^{-1}(V)$ and $t = f^V_U(s)$.*

Proposition 1.9.6 *If $\boldsymbol{T}$ is a category such that* $\mathrm{Ob}\,\boldsymbol{T}$ *and* $\mathrm{Mor}\,\boldsymbol{T}$ *are in $\mathcal{V}$, each functor $D : \boldsymbol{T} \to \mathbf{Esg}^{\Gamma}$ has a direct limit.*

Proof. It suffices to show that, for a family $(X_j)_{j\in J}$ of Γ-geometric spaces, a coproduct $\coprod_{j\in J}(X_j)$ and a coequalizer of any pair of morphisms of $\mathbf{Esg}^{\Gamma}$ exist in $\mathbf{Esg}^{\Gamma}$.

Let C be the disjoint union of topological spaces $(X_j)_{j\in J}$ and $i_j^{\mathfrak{e}} : X_j \to C$ the canonical inclusion. Put $\mathcal{O}_C = \prod_{j\in J} i^{\mathfrak{e}}_{j*}\mathcal{O}_{X_j}$ then $\mathcal{O}_{C,g}(U) = \prod_{j\in J} \mathcal{O}_{X_j,g}(U\cap X_j)$ and multiplications of $\mathcal{O}_{X_j}(U\cap X_j)$ define a multiplication of $\mathcal{O}_C$. Define $i_j^{\mathfrak{f}} : \mathcal{O}_C \to i^{\mathfrak{e}}_{j*}\mathcal{O}_{X_j}$ to be the canonical projection. It is easy to verify that $\{i_j : (C, \mathcal{O}_C) \to (X_j, \mathcal{O}_{X_j})\}$ is a product diagram of $\mathbf{Esg}^{\Gamma}$.

For morphisms $f, g : X \to Y$ of Γ-geometric spaces, let $p^{\mathfrak{e}} : Y \to Z$ be the coequalizer of $f^{\mathfrak{e}}$ and $g^{\mathfrak{e}}$ in the category of topological spaces. For an open set U of Z, put $V = p^{\mathfrak{e}\,-1}(U)$, $W = f^{\mathfrak{e}\,-1}(V) = g^{\mathfrak{e}\,-1}(V)$ and $\mathcal{O}_Z(U) = \{s \in p^{\mathfrak{e}}_*\mathcal{O}_Y(U)|\, f^V_W(s) = g^V_W(s)\}$. This defines a sheaf of Γ-rings $\mathcal{O}_Z$ and it is a subsheaf of $p^{\mathfrak{e}}_*\mathcal{O}_Y$. Let $p^{\mathfrak{f}} : \mathcal{O}_Z \to p^{\mathfrak{e}}_*\mathcal{O}_Y$ be the inclusion map. We have to show that the stalk $\mathcal{O}_{Z,z}$ is a Γ-local ring for any $z \in Z$ and that $p^{\mathfrak{f}}_x : \mathcal{O}_{Z,p^{\mathfrak{e}}(y)} \to \mathcal{O}_{Y,y}$ is local for any $y \in Y$. For U, V and W as above, let $s \in \mathcal{O}_Z(U)^{\mathfrak{h}}$ and put $t = p^U_V(s)$, $u = f^V_W(t) = g^V_W(t)$. By (1.9.5), we have $f^{\mathfrak{e}\,-1}(V_t) = g^{\mathfrak{e}\,-1}(V_t) = W_u$ and this implies that V_t is closed under the equivalence relation of Y. Put $p^{\mathfrak{e}}(V_t) = U'$, then $p^{\mathfrak{e}\,-1}(U') = V_t$ and U' is an open set of Z contained in U. We claim that $s_z \in \mathcal{O}_{Z,z}$ is invertible if $z \in U'$. In fact, if $z \in U'$, since $t|_{V_t} \in \mathcal{O}_Y(V_t)$ is invertible and its inverse is contained in $\mathcal{O}_Z(U')$, $s|_{U'} = t|_{V_t}$ is invertible. If $z \in U - U'$, then $p^{\mathfrak{e}\,-1}(z) \cap V_t$ is empty and $t_y \in \mathfrak{m}_y$ for any $y \in p^{\mathfrak{e}\,-1}(z)$. Suppose that $s, s' \in \mathcal{O}_Z(U)^{\mathfrak{h}}$ have non-invertible germs at z, then for any $y \in p^{\mathfrak{e}\,-1}(z)$, the germs of $p^U_V(s)$ and $p^U_V(s')$ at y belong to $\mathfrak{m}_y$. Thus $p^U_V(s+s')_y \in \mathfrak{m}_y$ if $y \in p^{\mathfrak{e}\,-1}(z)$ and this implies that $s + s'$ has non-invertible germ at z. Hence $\mathcal{O}_{Z,z}$ is a Γ-local ring and $p^{\mathfrak{f}}_x : \mathcal{O}_{Z,p^{\mathfrak{e}}(y)} \to \mathcal{O}_{Y,y}$ is local for any $y \in Y$. It can be verified that $p : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ is a coequalizer of f and g. □

1.10 The prime spectrum of a Γ-ring

We write $\mathcal{O} : \mathbf{Esa}^{\Gamma} \to \mathbf{An}^{\Gamma\,op}$ for the functor defined by $\mathcal{O}(X) = \sum_{g\in\Gamma} \mathcal{O}_X(X)_g$, $\mathcal{O}(f) = \sum_{g\in\Gamma} (f^Y_X)_g : \mathcal{O}(Y) \to \mathcal{O}(X)$ for a Γ-ringed space $(X, \mathcal{O}_X)$ and a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. The restriction of $\mathcal{O}$ to the subcategory $\mathbf{Esg}^{\Gamma}$ is also denoted by $\mathcal{O}$.

Theorem 1.10.1 *For each Γ-ring A, there is a Γ-geometric space $(\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})$ and a homomorphism $\varphi_A : A \to \mathcal{O}(\mathrm{Spec}A)$ having the following properties.*
If X is a Γ-geometric space and $\varphi : A \to \mathcal{O}(X)$ is a homomorphism of Γ-rings, there is a unique morphism $f : (X, \mathcal{O}_X) \to (\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})$ such that $\varphi = \mathcal{O}(f)\varphi_A$. In other words, correspondence $f \mapsto \mathcal{O}(f)\varphi_A$ gives a bijection $\Phi : \mathbf{Esg}^{\Gamma}(X, \mathrm{Spec}A) \to \mathcal{A}n^{\Gamma}(A, \mathcal{O}(X))$. Moreover, φ_A is natural in A, hence $\mathrm{Spec} : \mathbf{An}^{\Gamma\,op} \to \mathbf{Esg}^{\Gamma}$ is a right adjoint to $\mathcal{O} : \mathbf{Esg}^{\Gamma} \to \mathbf{An}^{\Gamma\,op}$ and such a pair $((\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A}), \varphi_A)$ is uniquely determined up to natural isomorphism.

Construction of $(\mathrm{Spec}A, \varphi_A)$: Let $\mathrm{Spec}A$ be the set of prime ideals of A and for an ideal $\mathfrak{a}$ of A, put $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}A|\, \mathfrak{p} \supseteq \mathfrak{a}\}$, $D(\mathfrak{a}) = \mathrm{Spec}A - V(\mathfrak{a})$. Then the following facts are easily verified. (1) $\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{b}}$ if and only if $V(\mathfrak{a}) \supseteq V(\mathfrak{b})$. (2) $V(0) = \mathrm{Spec}A$, $V(A) = \phi$. (3) $V(\sum_{\lambda} \mathfrak{a}_{\lambda}) = \bigcap_{\lambda} V(\mathfrak{a}_{\lambda})$. (4) $V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$. (5) $D(\mathfrak{a}) \subseteq D(\mathfrak{b})$ implies $S(\mathfrak{a}) \supseteq S(\mathfrak{b})$ (See (1.5.18)).
We give a topology on $\mathrm{Spec}A$ such that $\{D(\mathfrak{a})|\, \mathfrak{a}$ *is an ideal of* $A\}$ is the set of open sets. Consider a presheaf $\mathcal{F}_A$ of Γ-rings on $\mathrm{Spec}A$ given by $\mathcal{F}_A(D(\mathfrak{a})) = S(\mathfrak{a})^{-1}A$ and define the structure sheaf $\mathcal{O}_{\mathrm{Spec}A}$ to be the sheaf associated with $\mathcal{F}_A$. It follows from (1.5.17) and (1.5.18) that, for $s \in A^{\mathfrak{h}}$, the canonical map $s^{-1}A \to S((s))^{-1}A$ is bijective. Thus we have $\mathcal{F}_A(D((s))) = s^{-1}A$, in particular, $\mathcal{F}_A(\mathrm{Spec}A) = A$. We define $\varphi_A : A = \mathcal{F}_A(\mathrm{Spec}A) \to \mathcal{O}(\mathrm{Spec}A)$, using the canonical morphism $\mathcal{F}_A \to \mathcal{O}_{\mathrm{Spec}A}$ of presheaves. Since $D(\mathfrak{a})$ contains $\mathfrak{p}$ if and only if $\mathfrak{a} \not\subseteq \mathfrak{p}$, and $\bigcup_{\mathfrak{a}\not\subseteq\mathfrak{p}} S(\mathfrak{a}) = S_{\mathfrak{p}}$, the stalk at $\mathfrak{p}$ is $\mathcal{O}_{\mathrm{Spec}A,\mathfrak{p}} = \mathcal{F}_{A,\mathfrak{p}} = A_{\mathfrak{p}}$.
For a homomorphism $\varphi : A \to B$ of Γ-rings, define $\mathrm{Spec}\,\varphi = ((\mathrm{Spec}\,\varphi)^{\mathfrak{e}}, (\mathrm{Spec}\,\varphi)^{\mathfrak{f}}) : \mathrm{Spec}A \to \mathrm{Spec}B$ as follows. Set $(\mathrm{Spec}\,\varphi)^{\mathfrak{e}}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$ for $\mathfrak{p} \in \mathrm{Spec}B$ then we have $((\mathrm{Spec}\,\varphi)^{\mathfrak{e}})^{-1}(D(\mathfrak{a})) = D(B\varphi(\mathfrak{a}))$ for any ideal $\mathfrak{a}$ of A. Therefore $(\mathrm{Spec}\,\varphi)^{\mathfrak{e}}$ is continuous. Since $\varphi(S(\mathfrak{a})) \subseteq S(B\varphi(\mathfrak{a}))$, composition $A \xrightarrow{\varphi} B = \mathcal{F}_B(\mathrm{Spec}B) \xrightarrow{res} \mathcal{F}_B(D(B\varphi(\mathfrak{a}))) = (\mathrm{Spec}\,\varphi)^{\mathfrak{e}}_*\mathcal{F}_B(D(\mathfrak{a}))$ factors through $A = \mathcal{F}_A(\mathrm{Spec}A) \xrightarrow{res} \mathcal{F}_A(D(\mathfrak{a}))$. Thus we have a morphism of presheaves $\mathcal{F}_A \to (\mathrm{Spec}\,\varphi)^{\mathfrak{e}}_*\mathcal{F}_B$ and this induces a morphism of sheaves $(\mathrm{Spec}\,\varphi)^{\mathfrak{f}} : \mathcal{O}_{\mathrm{Spec}A} \to (\mathrm{Spec}\,\varphi)^{\mathfrak{e}}_*\mathcal{O}_{\mathrm{Spec}B}$. We note that $\mathcal{O}(\mathrm{Spec}\,\varphi)\varphi_A = \varphi_B\varphi$ holds.

Construction of $\Psi : \mathcal{A}n^{\Gamma}(A, \mathcal{O}(X)) \to \mathbf{Esg}^{\Gamma}(X, \mathrm{Spec}A)$: Let $\varphi : A \to \mathcal{O}(X)$ be a homomorphism of Γ-rings. We define $g^{\mathfrak{e}}_{\varphi} : X \to \mathrm{Spec}A$ by $g^{\mathfrak{e}}_{\varphi}(x) = ($*the inverse image of* $\mathfrak{m}_x$ *by composition* $A \xrightarrow{\varphi} \mathcal{O}(X) \hookrightarrow \mathcal{O}_X(X) \xrightarrow{can} \mathcal{O}_{X,x})$, for each $x \in X$. Then we have $(g^{\mathfrak{e}}_{\varphi})^{-1}(D(\mathfrak{a})) = \bigcup_{f\in\varphi(\mathfrak{a})^{\mathfrak{h}}} X_f$ and this implies that $g^{\mathfrak{e}}_{\varphi}$ is continuous. Since $\varphi(y)(x) = 0$ in $\kappa(x)$ for $y \in A^{\mathfrak{h}}$ and $x \in X$ if and only if $y \notin g^{\mathfrak{e}}_{\varphi}(x)$, it follows that

composition $A \xrightarrow{\varphi} \mathcal{O}(X) \xrightarrow{res} \sum_{\Gamma} \mathcal{O}_X((g^{\mathfrak{c}}_{\varphi})^{-1}(D(\mathfrak{a}))_g$ maps each element of $S(\mathfrak{a})$ to a unit. Hence this factors through $A \to S(\mathfrak{a})^{-1}A = \mathcal{F}_A(D(\mathfrak{a}))$ and defines $\mathcal{F}_A(D(\mathfrak{a})) \to g^{\mathfrak{c}}_{\varphi *}\mathcal{O}_X(D(\mathfrak{a}))$. Thus we have a morphism of sheaves $g^{\mathfrak{f}}_{\varphi} : \mathcal{O}_{\mathrm{Spec}A} \to g^{\mathfrak{c}}_{\varphi *}\mathcal{O}_X$. Define $\Psi : \mathcal{A}n^{\Gamma}(A, \mathcal{O}(X)) \to \mathbf{Esg}^{\Gamma}(X, \mathrm{Spec}A)$ by $\Psi(\varphi) = (g^{\mathfrak{c}}_{\varphi}, g^{\mathfrak{f}}_{\varphi})$. It is easy to verify that Ψ is the inverse of $\Phi : \mathbf{Esg}^{\Gamma}(X, \mathrm{Spec}A) \to \mathcal{A}n^{\Gamma}(A, \mathcal{O}(X))$.
We remark that $\mathrm{Spec}\,\varphi : \mathrm{Spec}A \to \mathrm{Spec}B$ coincides with the image of composition $A \xrightarrow{\varphi} B \xrightarrow{\varphi_B} \mathcal{O}(\mathrm{Spec}B)$ by $\Psi : \mathcal{A}n^{\Gamma}(A, \mathcal{O}(X)) \to \mathbf{Esg}^{\Gamma}(X, \mathrm{Spec}A)$. In fact, since composition $B \xrightarrow{\varphi_B} \mathcal{O}(\mathrm{Spec}B) \hookrightarrow \mathcal{O}_{\mathrm{Spec}B}(\mathrm{Spec}B) \to \mathcal{O}_{\mathrm{Spec}B,\mathfrak{p}} = B_{\mathfrak{p}}$ is the localization map, the inverse image of $\mathfrak{m}_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ by this map is $\mathfrak{p}$.
Finally, we note that $\Phi(\mathrm{Spec}\,\varphi)_* = \varphi^*\Phi$ holds for any homomorphism $\varphi : A \to B$ of Γ-rings and $\Phi f^* = \mathcal{O}(f)_*\Phi$ holds for any morphism $f : X \to Y$ of Γ-geometric spaces.

Definition 1.10.2 *For each Γ-ring A,* Spec$A$ *is called the prime spectrum of A.*

Proposition 1.10.3 *1) Let P be a subspace of* Spec$A$*, then $\bar{P} = V(\bigcap_{\mathfrak{p}\in P} \mathfrak{p})$.*

2) *For a homomorphism $\varphi : A \to B$ and an ideal $\mathfrak{b}$ of B, we have $\overline{(\mathrm{Spec}\,\varphi)^{\mathfrak{c}}(V(\mathfrak{b}))} = V(\varphi^{-1}(\mathfrak{b}))$. In particular, $\overline{\mathrm{Im}(\mathrm{Spec}\,\varphi)^{\mathfrak{c}}} = V(\mathrm{Ker}\,\varphi)$. Therefore $\mathrm{Im}(\mathrm{Spec}\,\varphi)^{\mathfrak{c}}$ is dense in* Spec$A$ *if and only if $\mathrm{Ker}\,\varphi \subseteq \sqrt{0}$.*

For $g \in \Gamma$, we denote by $P(g)$ a symmetric algebra $S(V(\boldsymbol{Z}; g))$ over $\boldsymbol{Z}$ generated by a single element T of degree g. $P(g)$ is isomorphic to $\boldsymbol{Z}[T]$ if $\sigma(g) = 0$, to $\boldsymbol{Z}[T]/(2T^2)$ if $\sigma(g) = 1$. It follows from (1.10.1) that $\mathbf{Esg}^{\Gamma}(X, \mathrm{Spec}P(g))$ is isomorphic to $\mathcal{O}(X)_g$.

Definition 1.10.4 *A morphism $\varphi : X \to \mathrm{Spec}P(g)$ of Γ-geometric spaces is called a function of degree g over X. The Γ-ring $\mathcal{O}(X)$ is called the ring of functions over X.*

Proposition 1.10.5 *Let A be a Γ-ring. The presheaf $\mathcal{F}_A$ over* Spec$A$ *takes the same values as its associated sheaf $\mathcal{O}_{\mathrm{Spec}A}$ over the special open sets $(\mathrm{Spec}A)_f = D((f))$ if $f \in \mathcal{O}_{\mathrm{Spec}A}(\mathrm{Spec}A)^{\mathfrak{h}}$, that is, $\mathcal{O}_{\mathrm{Spec}A}(D((f))) = \mathcal{F}_A(D((f))) = f^{-1}A$. In particular, $\varphi_A : A \to \mathcal{O}(\mathrm{Spec}A)$ is an isomorphism and $\mathcal{O}(\mathrm{Spec}A) = \mathcal{O}_{\mathrm{Spec}A}(\mathrm{Spec}A)$.*

Corollary 1.10.6 *The functor* $\mathrm{Spec} : \mathbf{An}^{\Gamma\, op} \to \mathbf{Esg}^{\Gamma}$ *is fully faithful.*

For a Γ-geometric space X, there is a natural morphism $\psi_X = \Psi(id_{\mathcal{O}(X)}) : X \to \mathrm{Spec}\mathcal{O}(X)$. Note that $\psi^{\mathfrak{c}}_X$ maps $x \in X$ to a prime ideal $\{s \in \mathcal{O}(X) |\, s(x) = 0\}$.

Definition 1.10.7 *A Γ-geometric space X is called a Γ-prime spectrum (Γ-spectrum, for short) if $\psi_X : X \to \mathrm{Spec}\,\mathcal{O}(X)$ is an isomorphism. X is called a Γ-spectral space if X has an open covering by Γ-spectra.*

Proposition 1.10.8 *A Γ-geometric space X is a Γ-prime spectrum if and only if X is isomorphic to* Spec$A$ *for some Γ-ring A.*

Proof. In fact, if $f : X \to \mathrm{Spec}A$ is an isomorphism, $\psi_X f^{-1} = \mathrm{Spec}\,\varphi_A^{-1}\mathcal{O}(f^{-1})$ is an isomorphism. □

Proposition 1.10.9 *If X is a Γ-spectral space, the map $x \mapsto \overline{\{x\}}$ is a bijection of X onto the set of irreducible closed subsets of X.*

Special open subsets of Spec$A$ are Γ-prime spectrum and form an open base for Spec$A$. Hence each open Γ-spectral space has an open base consisting of Γ-prime spectra. It follows that each open subspace of a Γ-spectral space is also a Γ-spectral space.

1.11 Z-functors

Definition 1.11.1 *We call a functor from $\boldsymbol{M}^{\Gamma}$ into $\mathcal{E}$ a $\boldsymbol{Z}$-functor and $\boldsymbol{M}^{\Gamma}\mathcal{E}$ denotes the category of $\boldsymbol{Z}$-functors.*

For $A \in \mathcal{A}n^{\Gamma}$, $PA : \boldsymbol{M}^{\Gamma} \to \mathcal{E}$ is the functor represented by A. If A is a Γ-model, we say that PA is an affine scheme of the Γ-ring A. The following is a special case of (1.1.1).

Proposition 1.11.2 *If $R \in \boldsymbol{M}^{\Gamma}$, $\mathrm{X} \in \boldsymbol{M}^{\Gamma}\mathcal{E}$ and $\rho \in \mathrm{X}(R)$, we define $\rho^{\sharp} : PR \to \mathrm{X}$ by $\rho^{\sharp}_S(f) = \mathrm{X}(f)(\rho)$ for $S \in \boldsymbol{M}^{\Gamma}$, $f \in PR(S)$. Then the correspondence $\rho \mapsto \rho^{\sharp}$ gives a natural bijection $\mathrm{X}(R) \to \boldsymbol{M}^{\Gamma}\mathcal{E}(PR, \mathrm{X})$, whose inverse is given by $\sigma \mapsto \sigma_R(id_R)$.*

Example 1.11.3 *For $g \in \Gamma$, let O_g be a $\boldsymbol{Z}$-functor defined by $\mathrm{O}_g(R) = R_g$ (the underlying set of R_g) for a Γ-model R. Then, O_g is represented by $P(g)$. If X is a $\boldsymbol{Z}$-functor, $\boldsymbol{M}^\Gamma\mathcal{E}(\mathrm{X}, \mathrm{O}_g)$ has a natural structure of abelian group and $\sum_{g\in\Gamma} \boldsymbol{M}^\Gamma\mathcal{E}(\mathrm{X}, \mathrm{O}_g)$ has a natural structure of Γ-ring, namely, for natural transformations $\varphi : \mathrm{X} \to \mathrm{O}_g$ and $\psi : \mathrm{X} \to \mathrm{O}_h$, define $\varphi+\psi$ and $\varphi\psi$ by $(\varphi+\psi)_R(x) = \varphi_R(x)+\psi_R(x)$ and $(\varphi\psi)_R(x) = \varphi_R(x)\psi_R(x)$ $(R \in \boldsymbol{M}^\Gamma$, $x \in \mathrm{X}(R))$, respectively. Each element of $\boldsymbol{M}^\Gamma\mathcal{E}(\mathrm{X}, \mathrm{O}_g)$ is called a function of degree g on X and $\sum_{g\in\Gamma} \boldsymbol{M}^\Gamma\mathcal{E}(\mathrm{X}, \mathrm{O}_g)$ is called the ring of functions on X, which we denote by $\mathrm{O}(\mathrm{X})$. Thus we have a functor $\mathrm{O} : \boldsymbol{M}^\Gamma\mathcal{E} \to \mathbf{An}^{\Gamma\,op}$.*

Proposition 1.11.4 *$P : \mathbf{An}^{\Gamma\,op} \to \boldsymbol{M}^\Gamma\mathcal{E}$ is the right adjoint to $\mathrm{O} : \boldsymbol{M}^\Gamma\mathcal{E} \to \mathbf{An}^{\Gamma\,op}$.*

Proof. In fact, for a Γ-ring A and a $\boldsymbol{Z}$-functor X, a map $\Phi : \boldsymbol{M}^\Gamma\mathcal{E}(\mathrm{X}, PA) \to \mathcal{A}n^\Gamma(A, \mathrm{O}(X))$ defined by $(\Phi(\varphi)(a))_R(\rho) = (\varphi(\rho)_R)(a)$ for $\varphi : \mathrm{X} \to PA$, $a \in A$ and $\rho \in \mathrm{X}(R)$ is a natural equivalence. The inverse Φ^{-1} is given by $(\Phi^{-1}(\psi)_R(\rho))(a) = \psi(a)_R(\rho)$ for $\psi : A \to \mathrm{O}(X)$, $\rho \in \mathrm{X}(R)$ and $a \in A$. □

Example 1.11.5 *For a Γ-geometric space $(X, \mathcal{O}_X)$, we define a $\boldsymbol{Z}$-functor $\mathrm{S}X$ by $\mathrm{S}X(R) = \mathbf{Esg}^\Gamma(\mathrm{Spec}R, X)$. Then $\mathrm{S}(\mathrm{Spec}A) = PA$. Note that if Y is a subspace of X, $\mathrm{S}Y$ is a subfunctor of $\mathrm{S}X$.*

Let $\mathfrak{a}$ be an ideal of a Γ-ring A. If $\varphi \in PA(R)$, it follows from $(\mathrm{Spec}\varphi)^{-1}(D(\mathfrak{a})) = D(R\varphi(\mathfrak{a}))$ that $\mathrm{Spec}\,\varphi : \mathrm{Spec}R \to \mathrm{Spec}A$ factors through $D(\mathfrak{a}) \hookrightarrow \mathrm{Spec}A$ if and only if $R\varphi(\mathfrak{a}) = R$. Therefore $\mathrm{S}(D(\mathfrak{a}))(R)$ can be identified with a subset $\{\varphi \in PA(R) |\, R\varphi(\mathfrak{a}) = R\}$ of $PA(R)$ and $\mathrm{S}(D(\mathfrak{a}))$ is regarded as a subfunctor of PA in this way. We denote $\mathrm{S}(D(\mathfrak{a}))$ by $(PA)_\mathfrak{a}$ and call this the subfunctor of PA defined by $\mathfrak{a}$. Note that $(PA)_\mathfrak{a} \subseteq (PA)_\mathfrak{b}$ if and only if $D(\mathfrak{a}) \subseteq D(\mathfrak{a})$.

If $f : \mathrm{X} \to \mathrm{Y}$ is a morphism of $\boldsymbol{M}^\Gamma\mathcal{E}$ and Z is a subfunctor of Y, $f^{-1}(\mathrm{Z})$ denotes a subfunctor of X defined by $f^{-1}(\mathrm{Z})(R) = \{x \in \mathrm{X}(R) |\, f_R(x) \in \mathrm{Z}(R)\}$ for each $R \in \boldsymbol{M}^\Gamma$. The image functor of f is a subfunctor of Y defined by $\mathrm{Im}f(R) = \{y \in \mathrm{Y}(R) |\, f_R(x) = y$ for some $x \in \mathrm{X}(R)\}$.

Definition 1.11.6 *Let X be a $\boldsymbol{Z}$-functor and U a subfunctor of X. We say U is open in X if, for each Γ-model A and each morphism $f : PA \to \mathrm{X}$, $f^{-1}(\mathrm{U})$ is a subfunctor of PA defined by an ideal of A. A morphism of $\boldsymbol{M}^\Gamma\mathcal{E}$ is said to be an open embedding if it is a monomorphism and the image functor is open in X.*

Let U be a subfunctor of X and $f : PA \to \mathrm{X}$ a morphism. Then we have $f^{-1}(\mathrm{U})(R) = \{\varphi A \to R |\, \mathrm{X}(\varphi)(f_A(id_A)) \in \mathrm{U}(R)\}$. This implies that U is open in X if and only if the following condition $(*)$ is satisfied.

$(*)$ *For any $A \in \boldsymbol{M}^\Gamma$ and $\alpha \in \mathrm{X}(A)$, there exists an ideal $\mathfrak{a}$ of A satisfying a condition that the image of $\mathfrak{a}$ by a homomorphism $\varphi : A \to R$ generates the unit ideal of R if and only if $\mathrm{X}(\varphi)(\alpha) \in \mathrm{U}(R)$.*

Proposition 1.11.7 *1) Let X be a Γ-geometric space and Y an open subspace of X, then $\mathrm{S}Y$ is an open subfunctor of $\mathrm{S}X$.*

2) If A is a Γ-model, a subfunctor U of PA is open if and only if U is of the form $(PA)_\mathfrak{a}$ for some ideal $\mathfrak{a}$ of A.

3) Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of $\boldsymbol{M}^\Gamma\mathcal{E}$ and Z is an open subfunctor of Y, then $f^{-1}(\mathrm{Z})$ is an open subfunctor of X.

Proof. For 1), let A be a Γ-model and α an element of $\mathrm{S}X(A)$. Then, there exists an ideal $\mathfrak{a}$ of A such that $\alpha^{-1}(Y) = D(\mathfrak{a})$. If $\varphi : A \to R$ is a homomorphism, $R\varphi(\mathfrak{a}) = R$ holds if and only if $\mathrm{Spec}\,\varphi(\mathrm{Spec}R) \subseteq D(\mathfrak{a})$. On the other hand, $\mathrm{S}X(\varphi)(\alpha) = \alpha\mathrm{Spec}\,\varphi \in \mathrm{S}Y(R)$ if and only if $\mathrm{Spec}\,\varphi(\mathrm{Spec}R) \subseteq \alpha^{-1}(Y)$. For 2), suppose $\mathrm{U} = (PA)_\mathfrak{a}$ for some ideal $\mathfrak{a}$. If B is a Γ-model and $\alpha \in PA(B)$, $R\varphi(B\alpha(\mathfrak{a})) = R$ holds for a homomorphism $\varphi : B \to R$ if and only if $P\varphi(\alpha) = \varphi\alpha \in (PA)_\mathfrak{a} = \mathrm{U}(R)$. The converse implication of 2) and 3) are obvious. □

Example 1.11.8 *If $R \in \boldsymbol{M}^\Gamma$ and $f \in R^\mathfrak{h}$, $\rho_f : R \to R_f$ denotes the canonical homomorphism. Then, $P\rho_f : PR_f \to PR$ is an open embedding whose image functor is $(PR)_{(f)}$. In the case $R = P(g)$ and $f = T$, $PP(g)_T$ is identified with a subfunctor μ_g of O_g which assigns to each $R \in \boldsymbol{M}^\Gamma$ its set of units of degree g. If X is a $\boldsymbol{Z}$-functor and f is a function on X of degree g, we write X_f for the inverse image $f^{-1}(\mu_g)$. We say that X_f is the subfunctor of X where f does not vanish.*

Proposition 1.11.9 *1) Let $\mathfrak{a}$ be an ideal of a Γ-ring A. For $f \in A^\mathfrak{h}$, $(PA_f)_{\mathfrak{a}A_f} = (PA)_{f\mathfrak{a}}$ holds in PA.*

2) If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of A, $(PA)_\mathfrak{a} \cap (PA)_\mathfrak{b} = (PA)_{\mathfrak{a}\mathfrak{b}}$ holds in PA. Hence if Y and Z are open subfunctors of a $\boldsymbol{Z}$-functor X, so is $\mathrm{Y} \cap \mathrm{Z}$.

Definition 1.11.10 *Let X be a $\boldsymbol{Z}$-functor. A family $(\mathrm{X}_i)_{i\in I}$ of subfunctors of X is said to cover X if $\mathrm{X}(K) = \bigcup_{i\in I}\mathrm{X}_i(K)$holds for every Γ-field $K \in \boldsymbol{M}^\Gamma$.*

Proposition 1.11.11 *1) If X is a Γ-geometric space and $(X_i)_{i\in I}$ is an open covering of X, then $(\mathrm{S}X_i)_{i\in I}$ is an open covering of $\mathrm{S}X$, namely, a covering consisting of open subfunctors.*

2) Let $(\mathfrak{a}_i)_{i\in I}$ be ideals of a Γ-ring A, then $((PA)_{\mathfrak{a}_i})_{i\in I}$ covers $(PA)_{\Sigma_i \mathfrak{a}_i}$. Therefore $((PA)_{\mathfrak{a}_i})_{i\in I}$ covers PA if and only if $\sum\limits_i \mathfrak{a}_i = A$.

Definition 1.11.12 *Let $f_1, f_2, \dots, f_m$ be homogeneous elements of a Γ-ring A such that $(f_1,\dots,f_m) = A$ and let $\rho_i : A \to A_{f_i}$, $\rho_{i,j} : A \to A_{f_if_j}$ be canonical homomorphisms. For a $\boldsymbol{Z}$-functor X, define $u : \mathrm{X}(A) \to \prod\limits_i \mathrm{X}(A_{f_i})$ and $v, w : \prod\limits_i \mathrm{X}(A_{f_i}) \to \prod\limits_{i,j}\mathrm{X}(A_{f_if_j})$ by $\mathrm{pr}_i u = \mathrm{X}(\rho_i)$, $\mathrm{pr}_{i,j} v = \mathrm{X}(\rho_{i,j})\mathrm{pr}_i$, $\mathrm{pr}_i w = \mathrm{X}(\rho_{j,i})\mathrm{pr}_j$.*
1) X is said to be local if, for any Γ-model A and any $f_1,\dots,f_m \in A^\natural$ such that $(f_1,\dots,f_m) = A$,

$$\mathrm{X}(A) \xrightarrow{u} \prod_i \mathrm{X}(A_{f_i}) \underset{w}{\overset{v}{\rightrightarrows}} \prod_{i,j}\mathrm{X}(A_{f_if_j})$$

is exact.
2) X is called a Γ-scheme if X is local and has an open covering $(\mathrm{X}_i)_{i\in I}$ such that each X_i is isomorphic to an affine scheme and I is small. The full subcategory of $\boldsymbol{M}^\Gamma\mathcal{E}$ formed by Γ-schemes is denoted by $\mathbf{Sch}^\Gamma$.

Proposition 1.11.13 *1) For a Γ-geometric space X, $\mathrm{S}X$ is local.*

2) If Y is an open subfunctor of a local $\boldsymbol{Z}$-functor X, then Y is also local.

3) If Y is an open subfunctor of a Γ-scheme X, then Y is also a Γ-scheme. We call such Y an open subscheme of X.

4) If X is a Γ-spectral space such that the underlying topological space $X^{\mathfrak{e}}$ and $\mathcal{O}_X(U)$ are small for any open set U of $X^{\mathfrak{e}}$, then $\mathrm{S}X$ is a Γ-scheme.

Proof. The sequence $\mathrm{S}X(A) \xrightarrow{u} \prod\limits_i \mathrm{S}X(A_{f_i}) \underset{w}{\overset{v}{\rightrightarrows}} \prod\limits_{i,j}\mathrm{S}X(A_{f_if_j})$ is identified with an exact sequence

$$\mathbf{Esg}^\Gamma(\mathrm{Spec}A, X) \xrightarrow{u} \mathbf{Esg}^\Gamma\Big(\coprod_i \mathrm{Spec}A_{f_i}, X\Big) \underset{w}{\overset{v}{\rightrightarrows}} \mathbf{Esg}^\Gamma\Big(\coprod_{i,j}(\mathrm{Spec}A_{f_i}\cap \mathrm{Spec}A_{f_j}), X\Big)\ .$$

Thus $\mathrm{S}X$ is local and the assertion 1) follows.
For 2), let A be a Γ-model and $f_1,\dots,f_m$ elements of $A^\natural$. Suppose that $(\varphi_1,\dots,\varphi_m) \in \prod\limits_{i=1}^m \mathrm{Y}(A_{f_i})$ satisfies $v(\varphi_1,\dots,\varphi_m) = w(\varphi_1,\dots,\varphi_m)$, then there exists $\varphi \in \mathrm{X}(A)$ such that $\mathrm{X}(\rho_i)(\varphi) = \varphi_i$. Let $\mathfrak{a}$ be an ideal of A such that $\varphi^{\sharp-1}(\mathrm{Y}) = (PA)_{\mathfrak{a}}$. Since composition $PA_{f_i} \xrightarrow{\varphi_i^\sharp} \mathrm{Y} \hookrightarrow \mathrm{X}$ coincides with composition $PA_{f_i} \xrightarrow{P\rho_i} PA \xrightarrow{\varphi^\sharp} \mathrm{X}$, PA_{f_i} is a subfunctor of $\mathrm{X}(\rho_i)(\varphi) = \varphi_i$. This implies that $D((f_i)) \subseteq D(\mathfrak{a})$ for $i = 1, 2, \dots, m$, hence $D(\mathfrak{a}) \supseteq \bigcup_{i=1}^m D((f_i)) = D((f_1,\dots,f_m)) = PA$. Then we have $\mathfrak{a} = A$ and this means that $\varphi^\sharp : PA \to \mathrm{X}$ factors through $\mathrm{Y} \hookrightarrow \mathrm{X}$. Therefore $\varphi = \varphi^\sharp(id_A) \in \mathrm{Y}(A)$.
For 3), let $(\mathrm{X}_i)_{i\in I}$ be an affine open cover of X and $u_i : PA_i \to \mathrm{X}_i$ isomorphisms. For each $i \in I$, there is an ideal $\mathfrak{a}_i$ of A_i such that $u_i^{-1}(\mathrm{X}_i \cap \mathrm{Y}) = (PA_i)_{\mathfrak{a}_i}$ which is covered by open subfunctors $(PA_{i\,f})_{f\in\mathfrak{a}_i^\natural}$. Note that the restriction $u_i|_{PA_{i\,f}} : PA_{i\,f} \to \mathrm{Y}$ is an isomorphism onto an open subfunctor $\mathrm{Y}_{i,f}$ of Y. Since $(\mathrm{Y}_{i,f})_{f\in\mathfrak{a}_i^\natural}$ covers $\mathrm{X}_i \cap \mathrm{Y}$, $(\mathrm{Y}_{i,f})_{f\in\mathfrak{a}_i^\natural, i\in I}$ covers Y.
Now 4) is obvious. □

1.12 The geometric realization of Z-functors

For a $\boldsymbol{Z}$-functor F, we consider the category of F-models $\boldsymbol{M}^\Gamma_{\mathrm{F}}$ (See §0.). In the case $\mathrm{F} = Pk$ for a Γ-ring k, we remark that the category of F-models is nothing but the the category of k-models $\boldsymbol{M}^\Gamma_k$.

Theorem 1.12.1 *The functor $\mathrm{S} : \mathbf{Esg}^\Gamma \to \boldsymbol{M}^\Gamma\mathcal{E}$ has left adjoint functor $|?| : \boldsymbol{M}^\Gamma\mathcal{E} \to \mathbf{Esg}^\Gamma$.*

Proof. Construction of $|?| : \boldsymbol{M}^{\Gamma}\mathcal{E} \to \mathbf{Esg}^{\Gamma}$; For a $\boldsymbol{Z}$-functor F, define a functor $d_{\mathrm{F}} : \boldsymbol{M}_{\mathrm{F}}^{\Gamma\, op} \to \mathbf{Esg}^{\Gamma}$ by $d_{\mathrm{F}}(R, \rho) = \mathrm{Spec}R$, $d_{\mathrm{F}}(\varphi) = \mathrm{Spec}\,\varphi$. We put $|\mathrm{F}| = \varinjlim d_{\mathrm{F}}$. For a morphism $f : \mathrm{E} \to \mathrm{F}$ of $\boldsymbol{M}^{\Gamma}\mathcal{E}$, $M_f : \boldsymbol{M}_{\mathrm{E}}^{\Gamma} \to \boldsymbol{M}_{\mathrm{F}}^{\Gamma}$ denotes a functor $M_f(R, \rho) = (R, f_R(\rho))$, $M_f(\varphi) = \varphi$. Since $d_{\mathrm{F}}M_f = d_{\mathrm{E}}$, M_f induces $|f| : |\mathrm{E}| = \varinjlim d_{\mathrm{E}} \to \varinjlim d_{\mathrm{F}} = |\mathrm{F}|$.

Construction of $\varphi(\mathrm{F}, X) : \mathbf{Esg}^{\Gamma}(|\mathrm{F}|, X) \to \boldsymbol{M}^{\Gamma}\mathcal{E}(\mathrm{F}, \mathrm{S}X)$; For a morphism $f : |\mathrm{F}| \to X$ of Γ-geometric spaces, define $\varphi(\mathrm{F}, X)(f)_R : \mathrm{F}(R) \to \mathrm{S}X(R)$ by $\varphi(\mathrm{F}, X)(f)_R(\rho) = f\, i(\rho)$, where $i(\rho) : \mathrm{Spec}R = d_{\mathrm{F}}(R, \rho) \to \varinjlim d_{\mathrm{F}} = |\mathrm{F}|$ is the canonical morphism. It is easy to verify the naturality of $\varphi(\mathrm{F}, X)$, that is, for any morphism $f : \mathrm{E} \to \mathrm{F}$ of $\boldsymbol{M}^{\Gamma}\mathcal{E}$ and morphism $\varphi : X \to Y$ of $\mathbf{Esg}^{\Gamma}$, $f^*\varphi(\mathrm{F}, X) = \varphi(\mathrm{E}, X)(f)|f|^*$ and $\mathrm{S}\varphi_*\varphi(\mathrm{F}, X) = \varphi(\mathrm{F}, Y)\varphi_*$ hold. Define $\psi(\mathrm{F}, X) : \boldsymbol{M}^{\Gamma}\mathcal{E}(\mathrm{F}, \mathrm{S}X) \to \mathbf{Esg}^{\Gamma}(|\mathrm{F}|, X)$ as follows. For a morphism $f : \mathrm{E} \to \mathrm{F}$ of $\boldsymbol{M}^{\Gamma}\mathcal{E}$ and a morphism $\theta : (R, \rho) \to (S, \sigma)$ of $\boldsymbol{M}_{\mathrm{F}}^{\Gamma}$, since $f_S(\sigma) = \mathrm{S}X(\theta)f_R(\rho) = f_R(\rho)\mathrm{Spec}\,\theta$ holds, $\psi(\mathrm{F}, X)(f) : |\mathrm{F}| \to X$ is the morphism induced by $f_R(\rho) : d_{\mathrm{F}}(R, \rho) = \mathrm{Spec}R \to X$. Then it can be shown that $\psi(\mathrm{F}, X) = \varphi(\mathrm{F}, X)^{-1}$. □

We call $|?| : \boldsymbol{M}^{\Gamma}\mathcal{E} \to \mathbf{Esg}^{\Gamma}$ the geometric realization functor. In the case $\mathrm{F} = PA$, since $\boldsymbol{M}_{\mathrm{F}}^{\Gamma}$ has an initial object (A, id_A), we have $|PA| = \mathrm{Spec}A$. For a Γ-geometric space X, define $\Phi(X) : |\mathrm{S}X| \to X$ to be $\psi(\mathrm{S}X, X)(id_{\mathrm{S}X})$ and for a $\boldsymbol{Z}$-functor F, define $\Psi(\mathrm{F}) : \mathrm{F} \to \mathrm{S}X$ to be $\varphi(\mathrm{F}, |\mathrm{F}|)(id_{|\mathrm{F}|})$. We note that $\Phi(\mathrm{Spec}A)$ and $\Psi(PA)$ are isomorphisms if A is a Γ-model.

We write $x \in \mathrm{F}$ for $x \in |\mathrm{F}|$ and $P \subseteq \mathrm{F}$ for $P \subseteq |\mathrm{F}|$, and call$x$, P a point of F, a subset of F, respectively. We shall say a morphism $f : \mathrm{F} \to \mathrm{E}$ is surjective (resp. injective, open, closed) if the continuous map $|f|^{\mathfrak{c}}$ is surjective (resp. injective, open, closed). We also call $|f|^{\mathfrak{c}}$ the map underlying f and denote it by $f^{\mathfrak{c}}$. Finally, we write $\mathcal{O}_{\mathrm{F}}$ for $\mathcal{O}_{|\mathrm{F}|}$ and call $\mathcal{O}_{\mathrm{F}}$ the structure sheaf of the $\boldsymbol{Z}$-functor F.

Let $\boldsymbol{K}^{\Gamma}$ be the full subcategory of Γ-fields of $\boldsymbol{M}^{\Gamma}$. For a functor $\mathrm{H} : \boldsymbol{K}^{\Gamma} \to \mathcal{E}$, $\varinjlim \mathrm{H}$ is the quotient set of $\coprod_{K \in \boldsymbol{K}^{\Gamma}} \mathrm{H}(K)$ by the smallest equivalence relation containing all pairs (a, b) for $a \in \mathrm{H}(K)$, $b \in \mathrm{H}(L)$ such that there is a homomorphism $\varphi : K \to L$ satisfying $\mathrm{H}(\varphi)(a) = b$. If $x \in \varinjlim \mathrm{H}$, H_x denotes a subfunctor of H defined by $\mathrm{H}_x(K) = x \cap \mathrm{H}(K)$. Then $\mathrm{H}(K)$ is the disjoint union of $\mathrm{H}_x(K)$ for $x \in \varinjlim \mathrm{H}$.

Definition 1.12.2 *Let $\mathcal{C}$ be a category. A functor $\mathrm{F} : \mathcal{C} \to \mathcal{E}$ is said to be indecomposable if it is not the disjoint sum of two non-empty subfunctors.*

Proposition 1.12.3 *1) Let $\mathrm{H} : \boldsymbol{K}^{\Gamma} \to \mathcal{E}$ be a functor. H_x is indecomposable for any $x \in \varinjlim \mathrm{H}$.*

2) If $\mathrm{H} : \boldsymbol{K}^{\Gamma} \to \mathcal{E}$ is a functor represented by $K \in \boldsymbol{K}^{\Gamma}$, then it is indecomposable.

3) Let X be a geometric space and H the restriction of $\mathrm{S}X : \boldsymbol{M}^{\Gamma} \to \mathcal{E}$ to $\boldsymbol{K}^{\Gamma}$. Then H is the disjoint sum of the indecomposable subfunctors represented by $\kappa(x)$, where x is a point of X such that $\kappa(x)$ is isomorphic to a Γ-model. Hence $\varinjlim \mathrm{H} = \{x \in X \,|\, \kappa(x)$ is isomorphic to a Γ-model.$\}$.

4) Let A be a Γ-model and H the restriction of PA to $\boldsymbol{K}^{\Gamma}$. For $x \in \mathrm{H}(K)$, $\bar{x} : \mathrm{Frac}(A/\mathrm{Ker}\,x) \to K$ denotes the homomorphism induced by x. Then, the correspondence $x \mapsto (\mathrm{Ker}\,x, \bar{x})$ gives a bijection $\mathrm{H}(K) \to \coprod_{\mathfrak{p} \in (\mathrm{Spec}A)^{\mathfrak{c}}} \boldsymbol{K}^{\Gamma}(\mathrm{Frac}(A/\mathfrak{p}), K)$. Therefore the indecomposable components of H are represented by residue fields $\mathrm{Frac}(A/\mathfrak{p})$ and $\varinjlim \mathrm{H} = (\mathrm{Spec}A)^{\mathfrak{c}}$ holds.

Proof. Suppose that $\mathrm{H} = \mathrm{H}_1 \coprod \mathrm{H}_2$ (disjoint) and $\mathrm{H}_1(K) \neq \phi$, $\mathrm{H}_2(L) \neq \phi$ for some $K, L \in \boldsymbol{K}^{\Gamma}$. Take $a \in \mathrm{H}_1(K)$, $b \in \mathrm{H}_2(L)$, then there exist morphisms $f_i : K_{2i} \to K_{2i+1}$, $g_i : K_{2i+2} \to K_{2i+1}$ $(i = 0, 1, \ldots, n-1)$ of $\boldsymbol{K}^{\Gamma}$ and $a_i \in \mathrm{H}(K_i)$ such that $K = K_0$, $L = K_{2n}$, $a = a_0$, $b = a_{2n}$ $\mathrm{H}(f_i)(a_{2i}) = a_{2i+1}$, $\mathrm{H}(g_i)(a_{2i+2}) = a_{2i+1}$. It follows that $a_i \in \mathrm{H}_1(K_i)$ inductively, then we have $b \in \mathrm{H}_1(L)$. This contradicts the assumption and 1) follows. For 2), suppose that $\mathrm{H} = \mathrm{H}_1 \coprod \mathrm{H}_2$ (disjoint) and $\mathrm{H}_1(L_i) \neq \phi$ $(i = 1, 2)$ for some $L_i \in \boldsymbol{K}^{\Gamma}$. Take $(\varphi_i : K \to L_i) \in \mathrm{H}_i(L_i)$ and form a tensor product $L_1 \otimes_K L_2$. Let L be the residue field of $L_1 \otimes_K L_2$ by a maximal ideal and let $\psi_i : L_i \to L$ be the composition of the canonical injection $L_i \to L_1 \otimes_K L_2$ and the projection $L_1 \otimes_K L_2 \twoheadrightarrow L$. Then we have $\mathrm{H}(\psi_1)(\varphi_1) = \mathrm{H}(\psi_2)(\varphi_2) \in \mathrm{H}_1(L) \cap \mathrm{H}_2(L)$, which contradicts the assumption. 3) and 4) follow from 1) and 2). □

Proposition 1.12.4 *For a $\boldsymbol{Z}$-functor F, the underlying set of the geometric realization $|\mathrm{F}|$ of F is naturally isomorphic to $\varinjlim(\mathrm{F}|_{\boldsymbol{K}^{\Gamma}})$. We identify each point of F with an equivalence class of $\coprod_{K \in \boldsymbol{K}^{\Gamma}} \mathrm{F}(K)$.*

Proof. For each F-model (R, ρ), define $\Psi_\rho : (\mathrm{Spec}R)^{\mathfrak{c}} \to \varinjlim(\mathrm{F}|_{\boldsymbol{K}^{\Gamma}})$ as follows. Let $\psi_{\mathfrak{p}} : R \to \mathrm{Frac}(R/\mathfrak{p})$ denote the composition of the projection $R \twoheadrightarrow R/\mathfrak{p}$ and the inclusion $R/\mathfrak{p} \hookrightarrow \mathrm{Frac}(R/\mathfrak{p})$. $\Psi_\rho(\mathfrak{p})$ is defined to be the equivalence class of $\mathrm{F}(\psi_{\mathfrak{p}})(\rho) \in \mathrm{F}(\mathrm{Frac}(R/\mathfrak{p}))$. If $\theta : (S, \sigma) \to (R, \rho)$ is a morphism of $\boldsymbol{M}_{\mathrm{F}}^{\Gamma}$, for each $\mathfrak{p} \in (\mathrm{Spec}R)^{\mathfrak{c}}$, θ induces $\bar{\theta} : \mathrm{Frac}(S/\theta^{-1}(\mathfrak{p})) \to \mathrm{Frac}(R/\mathfrak{p})$ satisfying $\bar{\theta}\psi_{\theta^{-1}(\mathfrak{p})} = \psi_{\mathfrak{p}}\theta$. Since $\rho = \mathrm{F}(\theta)(\sigma)$, we have

$\mathrm{F}(\psi_{\mathfrak{p}})(\rho) = \mathrm{F}(\bar{\theta})(\mathrm{F}(\psi_{\theta^{-1}(\mathfrak{p})})(\sigma))$ and thus $\Psi_\rho(\mathfrak{p}) = \Psi_\sigma((\mathrm{Spec}\,\theta)^{\mathfrak{c}}(\mathfrak{p}))$. This implies that $(\Psi_\rho)_{(R,\rho)\in \boldsymbol{M}^\Gamma_{\mathrm{F}}}$ induces $\Psi : |\mathrm{F}|^{\mathfrak{c}} = \varinjlim(\mathrm{Spec}R)^{\mathfrak{c}} \to \varinjlim(\mathrm{F}|_{\boldsymbol{K}^\Gamma})$. On the other hand, For $K \in \boldsymbol{K}^\Gamma$, we define $\varphi_K : \mathrm{F}(K) \to \varinjlim(\mathrm{Spec}R)^{\mathfrak{c}} = |\mathrm{F}|^{\mathfrak{c}}$ as follows. For $a \in \mathrm{F}(K)$, $\varphi_K(a)$ is the equivalence class of $0 \in (\mathrm{Spec}K)^{\mathfrak{c}} = (d_{\mathrm{F}}(K,a))^{\mathfrak{c}}$. Since $(\mathrm{Spec}K)^{\mathfrak{c}}$ consists of a single element 0 if $K \in \boldsymbol{K}^\Gamma$, $(\varphi_K)_{K\in\boldsymbol{K}^\Gamma}$ induces $\Psi^{-1} : \varinjlim(\mathrm{F}|_{\boldsymbol{K}^\Gamma}) \to \varinjlim(\mathrm{Spec}R)^{\mathfrak{c}} = |\mathrm{F}|^{\mathfrak{c}}$. It is easy to verify that Ψ^{-1} is the inverse of Ψ. If $f : \mathrm{G} \to \mathrm{F}$ is a morphism of $\boldsymbol{Z}$-functors, $f_{\boldsymbol{K}^\Gamma} : \varinjlim(\mathrm{G}|_{\boldsymbol{K}^\Gamma}) \to \varinjlim(\mathrm{F}|_{\boldsymbol{K}^\Gamma})$ denotes the map induced by f. We verify that $\Psi|f|^{\mathfrak{c}} = f_{\boldsymbol{K}^\Gamma}\Psi$. This completes the proof of the assertion. □

It follows from the above proposition that if G is a subfunctor of a $\boldsymbol{Z}$-functor F, $|\mathrm{G}|^{\mathfrak{c}}$ can be regarded as a subset of $|\mathrm{F}|^{\mathfrak{c}}$.

Corollary 1.12.5 *Let $(\mathrm{F}_i)_{i\in I}$ be a family of subfunctors of a $\boldsymbol{Z}$-functor* F*. If $(\mathrm{F}_i)_{i\in I}$ covers* F*, then $(|\mathrm{F}_i|^{\mathfrak{c}})_{i\in I}$ covers $|\mathrm{F}|^{\mathfrak{c}}$.*

Consider a $\boldsymbol{Z}$-functor F and a subset P of F. Define a subfunctor F_P of F by $\mathrm{F}_P(R) = \{\rho \in \mathrm{F}(R)|\, \mathrm{F}(\varphi)(\rho) \in \bigcup_{x\in P}(\mathrm{F}|_{\boldsymbol{K}^\Gamma})_x(K)$ *for each* $K \in \boldsymbol{K}^\Gamma$ *and* $\varphi \in \boldsymbol{M}^\Gamma(R,K)\}$.

Proposition 1.12.6 *1) For* F *and P as above, $\mathrm{F}_P|_{\boldsymbol{K}^\Gamma} = \bigcup_{x\in P}(\mathrm{F}|_{\boldsymbol{K}^\Gamma})_x$ and $P = \varinjlim(\mathrm{F}_P|_{\boldsymbol{K}^\Gamma})$ hold.*

2) If $f : \mathrm{G} \to \mathrm{F}$ is a morphism of $\boldsymbol{M}^\Gamma\mathcal{E}$, $\mathrm{G}_{|f|^{\mathfrak{c}\,-1}(P)} = f^{-1}(\mathrm{F}_P)$ holds for P as above.

3) If $\mathfrak{a}$ is an ideal of a Γ-ring A, we have $(PA)_{D(\mathfrak{a})} = (PA)_{\mathfrak{a}}$.

Proposition 1.12.7 *1) Let X be a Γ-geometric space, then $\Phi(X)^{\mathfrak{c}} : |\mathrm{S}X|^{\mathfrak{c}} \to X^{\mathfrak{c}}$ is an injection onto the subspace $\{x \in X^{\mathfrak{c}}|\, \kappa(x)$ is isomorphic to a Γ-model$\}$.*

2) For $P \subseteq |\mathrm{S}X|^{\mathfrak{c}}$, we regard P as a Γ-geometric space $(P, \mathcal{O}_X|_P)$. Then we have $(\mathrm{S}X)_P = \mathrm{S}P$. Hence if P is an open subset of X, $(\mathrm{S}X)_P$ is an open subfunctor of S*X.*

Proof. Consider the composition $\varinjlim(\mathrm{S}X|_{\boldsymbol{K}^\Gamma}) \xrightarrow{\Psi^{-1}} |\mathrm{S}X|^{\mathfrak{c}} \xrightarrow{\Phi(X)^{\mathfrak{c}}} X^{\mathfrak{c}}$. Suppose $\Phi(X)^{\mathfrak{c}}\Psi^{-1}(\alpha_1) = \Phi(X)^{\mathfrak{c}}\Psi^{-1}(\alpha_2)$, where α_i is represented by $\rho_i \in \mathrm{S}X(K_i)$ for $K_i \in \boldsymbol{K}^\Gamma$. Then $\rho_1^{\mathfrak{c}}(0) = \rho_2^{\mathfrak{c}}(0) = x \in X^{\mathfrak{c}}$ and there are homomorphisms $\rho^{\mathfrak{f}}_{i,0} : \kappa(x) \to K_i$. Let L be an extension of $\kappa(x)$ containing both K_1 and K_2 and $\iota_i : K_i \to L$ the inclusions. We have $\mathrm{S}X(\iota_1)(\rho_1) = \mathrm{S}X(\iota_2)(\rho_2)$, hence $\alpha_1 = \alpha_2$. Since Ψ^{-1} is bijective, $\Phi(X)^{\mathfrak{c}}$ is injective. This shows 1). For a Γ-model R, we have $\mathrm{S}X(R) = \{\rho \in \mathbf{Esg}^\Gamma(\mathrm{Spec}R, X)|$ *For any* $K \in \boldsymbol{K}^\Gamma$ *and* $\varphi \in \boldsymbol{M}^\Gamma(R,K)$, $\rho^{\mathfrak{c}}(\mathrm{Spec}\,\varphi)^{\mathfrak{c}}(\mathrm{Spec}K)^{\mathfrak{c}} \subseteq P\} = \{\rho \in \mathbf{Esg}^\Gamma(\mathrm{Spec}R, X)|\rho(\mathrm{Spec}R)^{\mathfrak{c}} \subseteq P\} = \mathrm{S}P(R)$. □

Lemma 1.12.8 *If* U *is an open subfunctor of a $\boldsymbol{Z}$-functor* F*, then $\mathrm{U}(R) = \{\rho \in \mathrm{F}(R)|$ For any $K \in \boldsymbol{K}^\Gamma$ and $\varphi \in \boldsymbol{M}^\Gamma(R,K)$, $\mathrm{F}(\varphi)(\rho) \in \mathrm{U}(K)\,\}$ for each $R \in \boldsymbol{M}^\Gamma$.*

Proof. Obviously, $\mathrm{U}(R)$ is contained in the right hand side. Suppose that ρ belongs to the right hand side. Since U is open, $\rho^{\sharp\,-1}(\mathrm{U}) = (PR)_{\mathfrak{a}}$ for some ideal $\mathfrak{a}$ of R. If $\mathfrak{a} \neq R$, take a maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$, then $\rho^\sharp(\pi) = \mathrm{F}(\pi)(\rho) \in \mathrm{U}(R/\mathfrak{m})$, where $\pi : R \to R/\mathfrak{m}$ denotes the projection. Hence $\pi \in \rho^{\sharp\,-1}(\mathrm{U})(R/\mathfrak{m}) = (PR)_{\mathfrak{a}}(R/\mathfrak{m})$. But this implies $\pi(\mathfrak{a}) \neq 0$ which contradicts $\mathfrak{a} \subseteq \mathfrak{m}$. Therefore $\mathfrak{a} = R$, namely, $\rho^{\sharp\,-1}(\mathrm{U}) = PR$. Then, $\rho = \rho^\sharp(id_R) \in \mathrm{U}(R)$. □

Proposition 1.12.9 *Let* F *be a $\boldsymbol{Z}$-functor. Then the correspondence $P \mapsto \mathrm{F}_P$ defines a bijection between the open subsets of $|\mathrm{F}|$ and the open subfunctors of* F*. The inverse map is given by $\mathrm{U} \mapsto \varinjlim(\mathrm{U}|_{\boldsymbol{K}^\Gamma})$.*

Proof. If $P \subseteq |\mathrm{F}|^{\mathfrak{c}}$, P is open if and only if $i(\rho)^{-1}(P)$ is open in $(\mathrm{Spec}R)^{\mathfrak{c}}$ for any $(R,\rho) \in \boldsymbol{M}^\Gamma_{\mathrm{F}}$, and F_P is an open subfunctor if and only if $\rho^{\sharp\,-1}(\mathrm{F}_R)$ is an open subfunctor of PR for any $(R,\rho) \in \boldsymbol{M}^\Gamma$. Since $|\rho^\sharp| : \mathrm{Spec}R = |PR| \to |\mathrm{F}|$ is identified with the canonical morphism $i(\rho)$, it follows from 2) of (1.12.6) that $\rho^{\sharp\,-1}(\mathrm{F}_P) = (PR)_{i(\rho)^{-1}(P)}$. Thus P is open if and only if F_P is open. For a subfunctor U of F, we put $P = \varinjlim(\mathrm{U}|_{\boldsymbol{K}^\Gamma})$. Then $P = |\mathrm{U}|^{\mathfrak{c}} \subseteq |\mathrm{F}|^{\mathfrak{c}}$ by (1.12.4),and it can be verified that U is a subfunctor of F_P. If U is open, $(\mathrm{U}|_{\boldsymbol{K}^\Gamma})_x = (\mathrm{F}|_{\boldsymbol{K}^\Gamma})_x$ holds for any $x \in P$. In fact, if $\rho \in (\mathrm{F}|_{\boldsymbol{K}^\Gamma})_x(K)$, then $\rho \in x \in P$ and there exists a homomorphism $\varphi : K \to L$ of $\boldsymbol{K}^\Gamma$ such that $\mathrm{F}(\varphi)(\rho) \in \mathrm{U}(L)$. Consider $\rho^\sharp : PK \to \mathrm{F}$, then $\rho^\sharp_L(\varphi) = \mathrm{F}(\varphi)(\rho) \in \mathrm{U}(L)$. Since U is an open subfunctor, it follows that $\rho^{\sharp\,-1}(\mathrm{U})$ is a non-empty open subfunctor of PK. Therefore we have $\rho^{\sharp\,-1}(\mathrm{U}) = PK$, hence $\rho = \rho^\sharp_K(id_K) \in \mathrm{U}(K)$. This shows $(\mathrm{U}|_{\boldsymbol{K}^\Gamma})_x \supseteq (\mathrm{F}|_{\boldsymbol{K}^\Gamma})_x$. The reverse inclusion is obvious. Now $\mathrm{U} = \mathrm{F}_P$ follows from (1.12.8). □

Let F and G be $\boldsymbol{Z}$-functors. We denote by $\mathcal{O}(\mathrm{G},\mathrm{F})$ a presheaf of sets on $|\mathrm{G}|$ defined by $\mathcal{O}(\mathrm{G},\mathrm{F})(U) = \boldsymbol{M}^\Gamma\mathcal{E}(\mathrm{G}_U,\mathrm{F})$.

Proposition 1.12.10 *The following conditions on a $\boldsymbol{Z}$-functor* F *is equivalent.*

(1) F *is local.*

(2) *For each* Γ*-model* A*, the presheaf* $\mathcal{O}(PA, \mathrm{F})$ *is a sheaf of open sets over* $\mathrm{Spec}A$*.*

(3) *For each* $\boldsymbol{Z}$*-functor* G*, the presheaf* $\mathcal{O}(\mathrm{G}, \mathrm{F})$ *is a sheaf over* $|\mathrm{G}|$*.*

Proof. (3)⇒(1); For a Γ-model R, suppose $(f_1, \ldots, f_m) = R$ and put $U_i = (\mathrm{Spec}R)_{f_i}$. Since $U_i \cap U_j = (\mathrm{Spec}R)_{f_i f_j}$, the diagram $\mathrm{F}(A) \xrightarrow{u} \prod_i \mathrm{F}(A_{f_i}) \underset{w}{\overset{v}{\rightrightarrows}} \prod_{i,j} \mathrm{F}(A_{f_i f_j})$ is naturally isomorphic to

$$\boldsymbol{M}^{\Gamma}\mathcal{E}(PR, \mathrm{F}) \to \prod_i \boldsymbol{M}^{\Gamma}\mathcal{E}((PR)_{U_i}, \mathrm{F}) \rightrightarrows \prod_{i,j} \boldsymbol{M}^{\Gamma}\mathcal{E}((PR)_{U_i \cap U_j}, \mathrm{F}),$$

which is exact by the assumption.
(2)⇒(3); We first note that, for each $(R, \rho) \in \boldsymbol{M}^{\Gamma}_{\mathrm{G}}$, $\rho^{\sharp} : PR \to \mathrm{G}$ induces an isomorphism by (1.12.2). If U is an open set of $|\mathrm{G}|$, since $(PR)i(\rho)^{-1}(U) = \rho^{\sharp\,-1}(\mathrm{G}_U)$, $\rho^{\sharp}$ restricts to $\rho^{\sharp}_U : (PR)i(\rho)^{-1}(U) :\to \mathrm{G}_U$ and this induces an isomorphism $\varinjlim (PR)i(\rho)^{-1}(U) \to \mathrm{G}_U$. Then, it is easy to check the sheaf property of $\mathcal{O}(\mathrm{G}, \mathrm{F})$.
(1)⇒(2); Let U be an open set of $\mathrm{Spec}A$ and $(U_i)_{i \in I}$ an open covering of U. For each $i \in I$, there is an ideal $\mathfrak{a}_i$ of A such that $U_i = D(\mathfrak{a})$. Then, we have $U_i = \bigcup_{f \in \mathfrak{a}_i^{\mathfrak{h}}} \mathrm{Spec}A_f$ and $U = D(\sum_{i \in I} \mathfrak{a}_i)$. Suppose that for each $i \in I$, a morphism $\varphi_i : \mathrm{S}U_i = (PA)_{U_i} \to \mathrm{F}$ of $\boldsymbol{M}^{\Gamma}\mathcal{E}$ is given such that composition $\mathrm{S}(U_i \cap U_j) \hookrightarrow \mathrm{S}U_i \xrightarrow{\varphi_i} \mathrm{F}$ coincides with $\mathrm{S}(U_i \cap U_j) \hookrightarrow \mathrm{S}U_j \xrightarrow{\varphi_j} \mathrm{F}$ for $i, j \in I$. Let us denote by α_f ($f \in \mathfrak{a}_{i(f)}^{\mathfrak{h}}$, $i(f) \in I$) the element of $\mathrm{F}(A_f)$ such that $\alpha_f^{\sharp} : PA_f \to \mathrm{F}$ is the composite $PA_f \hookrightarrow \mathrm{S}U_{i(f)} \xrightarrow{\varphi_{i(f)}} \mathrm{F}$. Then α_f and α_g are mapped to the same element of $\mathrm{F}(A_{fg})$ by $\mathrm{F}(A_f) \to \mathrm{F}(A_{fg})$ and $\mathrm{F}(A_g) \to \mathrm{F}(A_{fg})$, respectively. We define $\varphi : \mathrm{S}U \to \mathrm{F}$ as follows. For a Γ-model R and $\rho \in \mathrm{S}U(R)$, there exist $f_1, \ldots, f_m \in \bigcup_{i \in I} \mathfrak{a}_i^{\mathfrak{h}}$ such that $(\rho(f_1), \ldots, \rho(f_m)) = R$. We can apply (1) to have a unique $\bar{\rho} \in \mathrm{F}(R)$ that maps to $\mathrm{F}(\rho_{f_j})(\alpha_{f_j}) \in \mathrm{F}(R_{\rho(f_j)})$. Note that $\bar{\rho}$ does not depend on the choice of $f_1, \ldots, f_m$ and that $\bar{\rho}$ maps to $\mathrm{F}(\rho_f)(\alpha_f) \in \mathrm{F}(R_{\rho(f)})$ for any $f \in \bigcup_{i \in I} \mathfrak{a}_i^{\mathfrak{h}}$. We set $\varphi_R(\rho) = \bar{\rho}$. The uniqueness of $\bar{\rho}$ implies the naturality of φ_R in R. The composite $PA_f \hookrightarrow \mathrm{S}U_{i(f)} \hookrightarrow \mathrm{S}U \xrightarrow{\varphi} \mathrm{F}$ coincides with $\alpha_f^{\sharp}$ for $f \in \bigcup_{i \in I} \mathfrak{a}_i^{\mathfrak{h}}$. This implies that φ restricts to φ_i for each i, and the uniqueness of φ also follows. □

Let R be a Γ-model and V an open set of $\mathrm{Spec}R$. We put $R_V = S(\mathfrak{a})^{-1}R$ if $V = D(\mathfrak{a})$. Then $(PR_V)(A)$ can be identified with $\{\varphi \in (PR)(A) | \, \varphi(x)$ *is a unit if* $x \in S(\mathfrak{a})\,\}$. Suppose that there exists $\varphi \in (PR)_V(A)$ such that $\varphi \notin (PR_V)(A)$, then there exists $x \in S(\mathfrak{a})$ such that $\varphi(x)$ is not a unit. Let $\mathfrak{m}$ be a maximal ideal of A containing $\varphi(x)$ and put $\mathfrak{p} = \varphi^{-1}(\mathfrak{m})$. Then $x \in \mathfrak{p} \in \mathrm{Spec}A$ and $A\varphi(\mathfrak{p}) \subseteq \mathfrak{m}$. Since $x \in S(\mathfrak{a})$, $x \in \mathfrak{p}$ implies $\mathfrak{p} \supseteq \mathfrak{a}$. Therefore $A\varphi(\mathfrak{a}) \subseteq \mathfrak{m} \subsetneq A$, which contradicts $\varphi \in (PR)_V(A)$.

Lemma 1.12.11 $(PR)_V$ *is a subfunctor of* PR_V *in* PR*, and the inclusion* $(PR)_V \hookrightarrow PR_V$ *is natural with respect to* V *and* R*. That is, if* $\varphi : R \to S$ *is a homomorphism and* V*,* W *are open sets of* $\mathrm{Spec}R$*,* $\mathrm{Spec}S$ *such that* $(\mathrm{Spec}\,\varphi)(W) \subseteq V$*, then the composite* $(PS)_W \hookrightarrow PS_W \xrightarrow{P\varphi} PR_V$ *coincides with* $(PS)_W \xrightarrow{P\varphi} (PR)_V \hookrightarrow PR_V$*.*

Proposition 1.12.12 *Let* F *be a* $\boldsymbol{Z}$*-functor.* $\mathcal{O}_{|\mathrm{F}|,g}$ *is canonically isomorphic to* $\mathcal{O}(\mathrm{F}, \mathcal{O}_g)$ *for* $g \in \Gamma$*.*

Proof. First note that $\mathcal{O}(\mathrm{F}, \mathcal{O}_g)$ is a sheaf since $\mathcal{O}_g$ is local. In the case $\mathrm{F} = PR$, the inclusion map $(PR)_V \hookrightarrow PR_V$ induces a map $\mathcal{F}_R(V)_g = \mathcal{A}n^{\Gamma}(P(g), R_V) = \boldsymbol{M}^{\Gamma}\mathcal{E}(PR_V, \mathcal{O}_g) \to \boldsymbol{M}^{\Gamma}\mathcal{E}((PR)_V, \mathcal{O}_g) = \mathcal{O}(PR, \mathcal{O}_g)(V)$, which gives a morphism of presheaves $\mathcal{F}_{R\,g} \to \mathcal{O}(PR, \mathcal{O}_g)$. Hence a morphism of sheaves $\psi_R : \mathcal{O}_{\mathrm{Spec}R,g} \to \mathcal{O}(PR, \mathcal{O}_g)$ is induced. Since $(PR)_V = PR_V$ if V is a special open set, $\psi_{R\,V}$ is an isomorphism. It follows that ψ_R is an isomorphism of sheaves. For a homomorphism $\varphi : R \to S$ of Γ-models, the preceding lemma implies that composition $((\mathrm{Spec}\,\varphi)_*\psi_S)(\mathrm{Spec}\,\varphi)^{\mathfrak{f}} : \mathcal{O}_{\mathrm{Spec}R,g} \to (\mathrm{Spec}\,\varphi)_*\mathcal{O}_{\mathrm{Spec}S,g} \to (\mathrm{Spec}\,\varphi)_*\mathcal{O}(PS, \mathcal{O}_g)$ coincides with composition $(P\varphi)^*\psi_R : \mathcal{O}_{\mathrm{Spec}R,g} \to \mathcal{O}(PR, \mathcal{O}_g) \to (\mathrm{Spec}\,\varphi)_*\mathcal{O}(PS, \mathcal{O}_g)$. Thus, for a $\boldsymbol{Z}$-functor F, ψ_R induces an isomorphism $\varprojlim \mathcal{O}_{\mathrm{Spec}R,g} \to \varprojlim \mathcal{O}(PR, \mathcal{O}_g)$, where the limits are taken over the category of F-models. On the other hand, for $(R, \rho) \in \boldsymbol{M}^{\Gamma}_{\mathrm{F}}$ and an open set U of $\mathrm{Spec}R$, $i(\rho)^{\mathfrak{f}} : \mathcal{O}_{|\mathrm{F}|,g} \to i(\rho)_*\mathcal{O}_{\mathrm{Spec}R,g}$ and $\rho^{\sharp}_U : (PR)_{i(\rho)^{-1}(U)} \to \mathrm{F}_U$ induce isomorphisms $\mathcal{O}_{|\mathrm{F}|,g} \to \varprojlim \mathcal{O}_{\mathrm{Spec}R,g}$ and $\varinjlim (PR)_{i(\rho)^{-1}(U)} \to \mathrm{F}_U$, respectively. Hence $\rho^{\sharp *}_U : \mathcal{O}(\mathrm{F}, \mathcal{O}_g)(U) = \boldsymbol{M}^{\Gamma}\mathcal{E}(\mathrm{F}_U, \mathcal{O}_g) \to \boldsymbol{M}^{\Gamma}\mathcal{E}((PR)_{i(\rho)^{-1}(U)}, \mathcal{O}_g) = i(\rho)_*\mathcal{O}(PR, \mathcal{O}_g)(U)$ induces an isomorphism $\mathcal{O}(\mathrm{F}, \mathcal{O}_g) \to \varprojlim \mathcal{O}(PR, \mathcal{O}_g)$.

□

There is a morphism $\mu_{g,h} : \mathcal{O}_g \times \mathcal{O}_h \to \mathcal{O}_{g+h}$ induced by a homomorphism $\mu^*_{g,h} : P(g+h) \to P(g) \otimes_{\boldsymbol{Z}} P(h)$ defined by $\mu^*_{g,h}(T_{g+h}) = T_g \otimes T_h$. Then we have a pairing of sheaves $\mathcal{O}(\mathrm{F}, \mathcal{O}_g) \times \mathcal{O}(\mathrm{F}, \mathcal{O}_h) \to \mathcal{O}(\mathrm{F}, \mathcal{O}_{g+h})$ and $\sum_{g\in\Gamma} \mathcal{O}(\mathrm{F}, \mathcal{O}_g)$ has a structure of a sheaf of Γ-rings on $|\mathrm{F}|$. It is easy to verify that the isomorphism given in the preceding proposition is compatible with this pairing. Thus we have the following result.

Corollary 1.12.13 *$\mathcal{O}_{|\mathrm{F}|}$ is canonically isomorphic to $\sum_{g\in\Gamma} \mathcal{O}(\mathrm{F}, \mathcal{O}_g)$.*

Corollary 1.12.14 *If Y is an open subfunctor of X, $|\mathrm{Y}|$ is an open subspace of $|\mathrm{X}|$.*

Theorem 1.12.15 *(Comparison theorem) 1) Let X be a Γ-geometric space. Then $\Phi(X) : |\mathrm{S}X| \to X$ is an isomorphism if there exists an open covering $(X_i)_{i\in I}$ of X by prime spectra such that $I \in \mathcal{U}$ and $\mathcal{O}(X_i)$ is isomorphic to a model for each $i \in I$.*

2) Let F be a $\boldsymbol{Z}$-functor. In order for $|\mathrm{F}|$ to satisfy the above condition and for $\Psi(\mathrm{F}) : \mathrm{F} \to \mathrm{S}|\mathrm{F}|$ to be invertible, it is necessary and sufficient that F be a scheme.

Proof. Suppose that $(X_i)_{i\in I}$ is an open covering of X satisfying the condition of 1). It follows from (1.12.7) that $\Phi^{\mathfrak{e}}$ is bijective. For each $i \in I$, $\Phi(X_i) : |\mathrm{S}X_i| \to X_i$ is an isomorphism and $\Phi(X_i)$ is a restriction of $\Phi(X)$ to an open subspace of $|\mathrm{S}X|$ by (1.11.6) and (1.12.14). This implies the assertion of 1). For a $\boldsymbol{Z}$-functor F, since $\mathrm{S}|\mathrm{F}|$ is local, so is F if $\Psi(\mathrm{F})$ is invertible. Assume that $|\mathrm{F}|$ has an open covering $(U_i)_{i\in I}$ satisfying the condition 1). Then $(\Psi(\mathrm{F})^{-1}(\mathrm{S}U_i))_{i\in I}$ is a covering of F by affine open subfunctors. Thus F is a Γ-scheme. Conversely, Suppose that F is a Γ-scheme. Let $(\mathrm{U}_i)_{i\in I}$ be an affine open covering of F such that $I \in \mathcal{U}$ and $(\mathrm{U}_{i,j,k})_{k\in I(i,j)}$ an affine open covering of $\mathrm{U}_i \cap \mathrm{U}_j$. Then $(|\mathrm{U}_i|)_{i\in I}$ is an affine open covering of $|\mathrm{F}|$ consisting of prime spectra. Since $\Psi(\mathrm{U}_i) : \mathrm{U}_i \to \mathrm{S}|\mathrm{U}_i|$ is invertible, we define $\psi_i : \mathrm{S}|\mathrm{U}_i| \to \mathrm{F}$ to be the composition $\mathrm{S}|\mathrm{U}_i| \xrightarrow{\Psi(\mathrm{U}_i)^{-1}} \mathrm{U}_i \hookrightarrow \mathrm{F}$. It can be verified that the restriction of ψ_i to $\mathrm{S}|\mathrm{U}_{i,j,k}|$ coincides with the restriction of ψ_j to $\mathrm{S}|\mathrm{U}_{i,j,k}|$ for any i, j, k. Hence we can define $\psi : \mathrm{S}|\mathrm{F}| \to \mathrm{F}$ by $\psi|_{\mathrm{S}|\mathrm{U}_i|} = \psi_i$ and this is the inverse of $\Psi(\mathrm{F})$. □

1.13 Fibred product of Γ-schemes

Definition 1.13.1 *Let $f : \mathrm{X} \to \mathrm{Z}$ and $g : \mathrm{Y} \to \mathrm{Z}$ be morphisms of $\boldsymbol{Z}$-functors. We define the fibred product functor $\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}$ by $(\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y})(R) = \mathrm{X}(R) \times_{\mathrm{Z}(R)} \mathrm{Y}(R) = \{(x, y) \in \mathrm{X}(R) \times \mathrm{Y}(R) | \, \mathrm{X}_R(x) = \mathrm{Y}_R(y)\,\}$ for each Γ-model R.*

Proposition 1.13.2 *1) If Y is a subfunctor of Z and $g : \mathrm{Y} \to \mathrm{Z}$ is the inclusion, then $\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}$ is identified with the inverse image of Y by $f : \mathrm{X} \to \mathrm{Z}$.*

2) If $\mathrm{Y} \to \mathrm{Z}$ is an open embedding, so is the projection $\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y} \to \mathrm{X}$.

3) If X, Y and Z are local, so is $\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}$.

4) If X, Y and Z are Γ-schemes, so is $\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}$.

5) If X, Y and Z are Γ-schemes, $|\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}|$ is a fibred product of $|\mathrm{X}| \xrightarrow{|f|} |\mathrm{Z}| \xleftarrow{|g|} |\mathrm{Y}|$ in $\mathbf{Esg}^{\Gamma}$.

Proof. For 4), let (Z_i) be an open covering of Z, and put $\mathrm{X}_i = \mathrm{X} \times_{\mathrm{Z}} \mathrm{Z}_i$, $\mathrm{Y} = \mathrm{Z}_i \times_{\mathrm{Z}} \mathrm{Y}$. Let $(\mathrm{X}_{i,j})$ and $(\mathrm{Y}_{i,k})$ be affine open coverings of X_i and Y_i respectively. Then, $\mathrm{X}_{i,j} \times_{\mathrm{Z}_i} \mathrm{Y}_{i,k} = (\mathrm{X}_{i,j} \times_{\mathrm{Z}} \mathrm{Y}) \cap (\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}_{i,k})$ and it is an open subfunctor of $\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}$. $(\mathrm{X}_{i,j} \times_{\mathrm{Z}_i} \mathrm{Y}_{i,k})_{i,j,k}$ covers $\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}$ and if $\mathrm{X}_{i,j} \cong PA$, $\mathrm{Y}_{i,k} \cong PB$ and $\mathrm{Z}_i \cong PR$, then $\mathrm{X}_{i,j} \times_{\mathrm{Z}_i} \mathrm{Y}_{i,k} \cong P(A \otimes_R B)$. Hence $(\mathrm{X}_{i,j} \times_{\mathrm{Z}_i} \mathrm{Y}_{i,k})_{i,j,k}$ is an affine open covering of $\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}$. For 5), let $d : T \to |\mathrm{X}|$ and $e : T \to |\mathrm{Y}|$ be morphisms of $\mathbf{Esg}^{\Gamma}$ such that $|f|d = |g|e$. First we consider the case that X, Y and Z are affine. Suppose that $\mathrm{X} = PA$, $\mathrm{Y} = PB$ and $\mathrm{Z} = PR$, then $|\mathrm{X}| = \mathrm{Spec}A$, $|\mathrm{Y}| = \mathrm{Spec}B$, $|\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}| = \mathrm{Spec}(A \otimes_R B)$. By (1.10.1) and (1.10.6), there are homomorphisms $\Phi(d) : A \to \mathcal{O}(T)$, $\Phi(e) : B \to \mathcal{O}(T)$, $f^* : R \to A$ and $g^* : R \to B$ such that $|f| = \mathrm{Spec}f^*$, $|g| = \mathrm{Spec}g^*$ and $\Phi(d)f^* = \Phi(e)g^*$. Hence there is a unique homomorphism $\varphi : A \otimes_R B \to \mathcal{O}(T)$ such that $\Phi(d) = \varphi\iota_A$ and $\Phi(e) = \varphi\iota_B$ where ι_A and ι_B are canonical homomorphisms (See (1.3.8)). We set $\rho = \Phi^{-1}(\varphi) : T \to \mathrm{Spec}(A \otimes_R B) = |\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}|$ satisfying $|\mathrm{pr}_X|\rho = d$ and $|\mathrm{pr}_Y|\rho = e$. The uniqueness of ρ is obvious. Therefore $|\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}|$ is the fiber product of $|f|$ and $|g|$ in $\mathbf{Esg}^{\Gamma}$. In the general case, we put $U_{i,j} = d^{-1}|\mathrm{X}_{i,j}|$, $V_{i,k} = e^{-1}|\mathrm{Y}_{i,k}|$. Then, $(U_{i,j} \cap V_{i,k})_{i,j,k}$ is an open cover of T. The restrictions $d_{i,j,k} : U_{i,j} \cap V_{i,k} \to |\mathrm{X}_{i,j}|$, $e_{i,j,k} : U_{i,j} \cap V_{i,k} \to |\mathrm{Y}|$ of d, e induce the unique morphisms $\rho_{i,j,k} : U_{i,j} \cap V_{i,k} \to |\mathrm{X}_{i,j} \times_{\mathrm{Z}_i} \mathrm{Y}_{i,k}|$ such that $|\mathrm{pr}_{\mathrm{X}_{i,j}}|\rho_{i,j,k} = d_{i,j,k}$, $|\mathrm{pr}_{\mathrm{Y}_{i,k}}|\rho_{i,j,k} = e_{i,j,k}$. Then we have a morphism $\rho : T \to |\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}|$ whose restriction on $U_{i,j} \cap V_{i,k}$ coincides with $\rho_{i,j,k}$. It is easy to verify that ρ is the unique morphism satisfying $|\mathrm{pr}_{\mathrm{X}}|\rho = d$, $|\mathrm{pr}_{\mathrm{Y}}|\rho = e$. □

Corollary 1.13.3 *1) Finite inverse limits exist in $\mathcal{S}ch^{\Gamma}$.*

2) *The restriction of the geometric realization functor to $\mathcal{S}ch^{\Gamma}$ commutes with finite inverse limits.*

Let X be a Γ-spectral space. For $x \in X$, we define morphisms $\varepsilon_x : \mathrm{Spec}\mathcal{O}_{X,x} \to X$ and $\varepsilon(x) : \mathrm{Spec}\,\kappa(x) \to X$ as follows. Choose an open neighborhood V of x which is a prime spectrum and let $\rho_x : \mathcal{O}(V) = \mathcal{O}_X(V) \to \mathcal{O}_{X,x}$ denote the canonical homomorphism. ε_x is defined to be the composition $\mathrm{Spec}\mathcal{O}_{X,x} \xrightarrow{\mathrm{Spec}\,\rho_x} \mathrm{Spec}\mathcal{O}(V) \xrightarrow{\psi_V^{-1}} V \hookrightarrow X$. Note that ε_x does not depend on the choice of V. We define $\varepsilon(x)$ to be the composition $\mathrm{Spec}\,\kappa(x) \xrightarrow{\mathrm{Spec}\pi_x} \mathrm{Spec}\mathcal{O}_{X,x} \xrightarrow{\varepsilon_x} X$, where $\pi_x : \mathcal{O}_{X,x} \to \kappa(x)$ is the canonical projection. If X is a Γ-scheme and x is a point of $|\mathrm{X}|$, we also denote by $\varepsilon_x : P\mathcal{O}_{|\mathrm{X}|,x} \to \mathrm{X}$, $\varepsilon(x) : P\kappa(x) \to \mathrm{X}$ the morphisms whose geometric realizations $|\varepsilon_x| : \mathrm{Spec}\mathcal{O}_{|\mathrm{X}|,x} \to |rx|$, $|\varepsilon(x)| : \mathrm{Spec}\,\kappa(x) \to |\mathrm{X}|$ coincide with ε_x, $\varepsilon(x)$ defined above. We note that $\varepsilon(x) : \mathrm{Spec}\,\kappa(x) \to X$ is a monomorphism in $\mathbf{Esg}^{\Gamma}$, hence $\varepsilon(x) : P\kappa(x) \to \mathrm{X}$ is a monomorphism in $\mathcal{S}ch^{\Gamma}$.

Proposition 1.13.4 *Let X be a Γ-spectral space and x a point of X. We put $P_x = \{y \in X \,|\, x \in \overline{\{y\}}\}$, then $\varepsilon_x : \mathrm{Spec}\mathcal{O}_{X,x} \to X$ is an isomorphism onto the Γ-geometric space $(P_x, \mathcal{O}_X|_{P_x})$.*

Lemma 1.13.5 *If $f : \mathrm{X} \to \mathrm{Y}$ is a monomorphism of Γ-schemes, $|f| : |\mathrm{X}| \to |\mathrm{Y}|$ is injective.*

Proof. Suppose that $x_1, x_2 \in |\mathrm{X}|$ satisfy $|f|(x_1) = |f|(x_2)$. There are homomorphisms $\iota_j : \kappa(y) \to \kappa(x_j)$ $(j = 1, 2)$ such that $\varepsilon(y)\mathrm{Spec}\,\iota_j = |f|\varepsilon(x_j)$. Hence $\varepsilon(y)P\,\iota_j = f\varepsilon(x_j) : P\kappa(x_j) \to \mathrm{Y}$. Let $p_j : P(\kappa(x_1) \otimes_{\kappa(y)} \kappa(x_2)) \to P\kappa(x_j)$ induced by the canonical inclusions $\kappa(x_j) \hookrightarrow \kappa(x_1) \otimes_{\kappa(y)} \kappa(x_2)$. It follows that $f\varepsilon(x_1)p_1 = \varepsilon(y)(P\,\iota_1)p_1 = \varepsilon(y)(P\,\iota_2)p_2 = f\varepsilon(x_2)p_2$. Since f is a monomorphism, we have $\varepsilon(x_1)p_1 = \varepsilon(x_2)p_2$. Therefore $x_1 = x_2$. □

Proposition 1.13.6 *1) Let $f : \mathrm{X} \to \mathrm{Z}$ and $g : \mathrm{Y} \to \mathrm{Z}$ be morphisms of Γ-schemes and x, y, z points of X, Y, Z satisfying $|f|(x) = |g|(y) = z$. Then, the morphism $\varepsilon(x) \times_{\varepsilon(z)} \varepsilon(y) : P\kappa(x) \times_{P\kappa(z)} P\kappa(y) \to \mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}$ induces a bijection of the underlying set of $\mathrm{Spec}(\kappa(x) \otimes_{\kappa(z)} \kappa(y))$ onto $\{t \in |\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}| \,|\, \mathrm{pr}_{\mathrm{X}}(t) = x,\ \mathrm{pr}_{\mathrm{Y}}(t) = y\}$.*

2) *Let f, g, x, y, z be as above. Put $Q = \{t \in |\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}| \,|\, x \in \overline{\{\mathrm{pr}_{\mathrm{X}}(t)\}},\ y \in \overline{\{\mathrm{pr}_{\mathrm{Y}}(t)\}}\}$. Then, $\varepsilon_x \times_{\varepsilon_z} \varepsilon_y : P\mathcal{O}_{X,x} \times_{P\mathcal{O}_{Z,z}} P\mathcal{O}_{Y,y} \to \mathrm{X} \times_{\mathrm{Z}} \mathrm{Y}$ induces an isomorphism of $\mathrm{Spec}(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y})$ onto the Γ-geometric space $(Q, \mathcal{O}_{\mathrm{X}\times_{\mathrm{Z}}\mathrm{Y}}|_Q)$.*

Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of Γ-schemes and y a point of Y. Since $\varepsilon(y) : P\kappa(y) \to \mathrm{Y}$ is a monomorphism, so is the projection $\mathrm{pr}_{\mathrm{X}} : P\kappa(y) \times_{\mathrm{Y}} \mathrm{X} \to \mathrm{X}$. We denote the image functor of pr_{X} by $f^{-1}(y)$ which we call the fiber of f over y. The set of points of $f^{-1}(y)$ is a subset of X since pr_{X} is a monomorphism.

Proposition 1.13.7 *The topology of the space of points of $f^{-1}(y)$ is induced by the topology of $|\mathrm{X}|$. If $x \in |\mathrm{X}|$ and $f(x) = y$, then the local ring of $f^{-1}(y)$ at x is canonically isomorphic to $\mathcal{O}_{X,x}/\mathfrak{m}\mathcal{O}_{X,x}$.*

1.14 Relativization

Let S be a $\boldsymbol{Z}$-functor. We call a functor from $\boldsymbol{M}_{\mathrm{S}}^{\Gamma}$ to $\mathcal{E}$ an S-functor. For $A \in \boldsymbol{M}_{\mathrm{S}}^{\Gamma}$, $P_{\mathrm{S}}A$ denotes the S-functor represented by A. This is called an affine S-scheme. If $\mathrm{S} = Pk$ for a Γ-ring k, we denote $P_{Pk}A$ by P_kA. For a k-model R, the set $P_kA(R)$ consists of k-algebra homomorphisms. In this case, we call a S-functor a k-functor.

The following is a special case of (1.1.2).

Proposition 1.14.1 *There is an equivalence of categories $i_{\mathrm{S}} : \boldsymbol{M}^{\Gamma}\mathcal{E}/\mathrm{S} \xrightarrow{\cong} \boldsymbol{M}_{\mathrm{S}}^{\Gamma}\mathcal{E}$ with inverse $j_{\mathrm{S}} : \boldsymbol{M}_{\mathrm{S}}^{\Gamma}\mathcal{E} \to \boldsymbol{M}^{\Gamma}\mathcal{E}/\mathrm{S}$.*

If T is an S-functor, we call ${}_{\boldsymbol{Z}}\mathrm{T}$ the underlying $\boldsymbol{Z}$-functor of T and $p : {}_{\boldsymbol{Z}}\mathrm{T} \to \mathrm{S}$ the structural projection.

In the case $\mathrm{S} = Pk$, $\mathrm{T} = P_{\mathrm{S}}A = P_kA$ for $k \in \boldsymbol{M}^{\Gamma}$, $A \in \boldsymbol{M}_k^{\Gamma}$, we have ${}_{\boldsymbol{Z}}(P_kA)(R) = \coprod_{\rho \in \boldsymbol{M}^{\Gamma}(k,R)} \boldsymbol{M}^{\Gamma}(A, {}_{\rho}R)$ where $R \in \boldsymbol{M}^{\Gamma}$ and ${}_{\rho}R$ denotes the Γ-graded k-algebra with underlying Γ-ring R and structure map $\rho : k \to R$. Let us denote by ${}_{\boldsymbol{Z}}A$ the underlying Γ-ring of A then ${}_{\boldsymbol{Z}}(P_kA)$ is identified with $P_{\boldsymbol{Z}}A$.

For a given S-functor T, we say that T is local if ${}_{\boldsymbol{Z}}\mathrm{T}$ is local, that T is an S-scheme if ${}_{\boldsymbol{Z}}\mathrm{T}$ is a Γ-scheme and that a subfunctor U is open if ${}_{\boldsymbol{Z}}\mathrm{U}$ is open in ${}_{\boldsymbol{Z}}\mathrm{T}$. We set $|\mathrm{T}| = |{}_{\boldsymbol{Z}}\mathrm{T}|$ and call $|\mathrm{T}|$ the geometric realization of T.

If X and Y are S-schemes, ${}_{\boldsymbol{Z}}\mathrm{pr}_{\mathrm{X}} : {}_{\boldsymbol{Z}}(\mathrm{X} \times \mathrm{Y}) \to {}_{\boldsymbol{Z}}\mathrm{X}$ and ${}_{\boldsymbol{Z}}\mathrm{pr}_{\mathrm{Y}} : {}_{\boldsymbol{Z}}(\mathrm{X} \times \mathrm{Y}) \to {}_{\boldsymbol{Z}}\mathrm{Y}$ induce an isomorphism of $\boldsymbol{M}^{\Gamma}\mathcal{E}/\mathrm{S}$ from ${}_{\boldsymbol{Z}}(\mathrm{X} \times \mathrm{Y}) \to \mathrm{S}$ to $({}_{\boldsymbol{Z}}\mathrm{X}) \times_{\mathrm{S}} ({}_{\boldsymbol{Z}}\mathrm{Y}) \to \mathrm{S}$.

Let $f : \mathrm{S} \to \mathrm{T}$ be a morphism of $\boldsymbol{Z}$-functors. Recall the base extension $f^* : \boldsymbol{M}_{\mathrm{T}}^{\Gamma}\mathcal{E} \to \boldsymbol{M}_{\mathrm{S}}^{\Gamma}\mathcal{E}$, the base restriction $f_! : \boldsymbol{M}_{\mathrm{S}}^{\Gamma}\mathcal{E} \to \boldsymbol{M}_{\mathrm{T}}^{\Gamma}\mathcal{E}$ and the Weil restriction $f_* : \boldsymbol{M}_{\mathrm{S}}^{\Gamma}\mathcal{E} \to \boldsymbol{M}_{\mathrm{T}}^{\Gamma}\mathcal{E}$ as defined in section 0. We also recall the functor $f^* : \boldsymbol{M}^{\Gamma}\mathcal{E}/\mathrm{T} \to \boldsymbol{M}^{\Gamma}\mathcal{E}/\mathrm{S}$ and $f_! : \boldsymbol{M}^{\Gamma}\mathcal{E}/\mathrm{S} \to \boldsymbol{M}^{\Gamma}\mathcal{E}/\mathrm{T}$, then (1.1.3) implies

Proposition 1.14.2 *1) We have $f_! j_{\mathrm{S}} = j_{\mathrm{T}} f_!$ and there are natural equivalences of functors $i_{\mathrm{S}} f^* \xrightarrow{\cong} f^* i_{\mathrm{T}}$ and $f^* j_{\mathrm{T}} \xrightarrow{\cong} j_{\mathrm{S}} f^*$. Hence if Y is a T-scheme, $f^*\mathrm{Y}$ is an S-scheme, and if X is an S-scheme, $f_!\mathrm{X}$ is a T-scheme.*

2) Let X be an S-functor and Y a T-functor. There are natural equivalences $\chi(\mathrm{X},\mathrm{Y}) : \boldsymbol{M}^{\Gamma}_{\mathrm{T}}\mathcal{E}(f_!\mathrm{X},\mathrm{Y}) \to \boldsymbol{M}^{\Gamma}_{\mathrm{S}}\mathcal{E}(\mathrm{X}, f^*\mathrm{Y})$ and $\xi(\mathrm{Y},\mathrm{X}) : \boldsymbol{M}^{\Gamma}_{\mathrm{S}}\mathcal{E}(f^*\mathrm{Y},\mathrm{X}) \to \boldsymbol{M}^{\Gamma}_{\mathrm{T}}\mathcal{E}(\mathrm{Y}, f_*\mathrm{X})$ defined as follows. For $g : f_!\mathrm{X} \to \mathrm{Y}$ and $h : f^*\mathrm{Y} \to \mathrm{X}$,

$$\chi(\mathrm{X},\mathrm{Y})(g)_{(R,\rho)}(x) = g_{(R,f_R(\rho))}(x) \quad \text{if } (R,\rho) \in \boldsymbol{M}^{\Gamma}_{\mathrm{S}} \text{ and } x \in \mathrm{T}(R,\rho),$$

$$\xi(\mathrm{Y},\mathrm{X})(h)_{(R,\rho)}(x) = h(f^* x^\sharp) \quad \text{if } (R,\rho) \in \boldsymbol{M}^{\Gamma}_{\mathrm{T}} \text{ and } x \in \mathrm{Y}(R,\rho).$$

Here $x^\sharp : P_{\mathrm{T}}(R,\rho) \to \mathrm{Y}$ is a morphism defined by

$$x^\sharp_{(A,\sigma)}(\varphi) = \mathrm{Y}(\varphi)(x) \quad \text{for } (A,\sigma) \in \boldsymbol{M}^{\Gamma}_{\mathrm{T}} \text{ and } \varphi \in (P_{\mathrm{T}}(R,\rho))(A,\sigma).$$

Remark 1.14.3 *Let $\varphi : k' \to k$ be a homomorphism of Γ-models. Consider the case $\mathrm{S} = Pk$, $\mathrm{T} = Pk'$ and $f = P\varphi : \mathrm{S} \to \mathrm{T}$. If $\mathrm{X} = P_k A$ and $\mathrm{Y} = P_{k'} B$ for a k-model A and a k'-model B, then we have $f_!\mathrm{X} = P_{k'\,\varphi}A$ and $f^*\mathrm{Y} = P_k(B \otimes_{k'\,\varphi} k)$, where ${}_{\varphi}A$ denotes a k'-model with underlying Γ-ring A and the k'-algebra structure map given by $k' \xrightarrow{\varphi} k \to A$. If X is a k-functor, $f_*\mathrm{X}(R) = \mathrm{X}(R \otimes_{k'\,\varphi} k)$ holds for any k'-model R.*

Let T be a Γ-geometric space. We denote by $\mathbf{Esg}^{\Gamma}_T$ the category of geometric spaces over T defined below. An object is a morphism of $\mathbf{Esg}^{\Gamma}$ with target T and a morphism $f : (p : X \to T) \to (q : Y \to T)$ is a morphism $f : X \to Y$ of $\mathbf{Esg}^{\Gamma}$ satisfying $p = qf$. We define functors $\mathrm{S}_T : \mathbf{Esg}^{\Gamma}_T \to \boldsymbol{M}^{\Gamma}_{\mathrm{S}T}\mathcal{E}$ and $|?|_T : \boldsymbol{M}^{\Gamma}_{\mathrm{S}T}\mathcal{E} \to \mathbf{Esg}^{\Gamma}_T$ by $\mathrm{S}_T(p : X \to T) = (\mathrm{S}(p) : \mathrm{S}X \to \mathrm{S}T)$ and $|\mathrm{F}|_T = \varphi(_{\boldsymbol{Z}}\mathrm{F}, T)^{-1} j_{\mathrm{S}T}(\mathrm{F})$, respectively. Moreover, we construct a map $\varphi_T(\mathrm{F}, p) : \mathbf{Esg}^{\Gamma}_T(|\mathrm{F}|_T, p) \to \boldsymbol{M}^{\Gamma}_{\mathrm{S}T}\mathcal{E}(\mathrm{F}, \mathrm{S}_T(p))$ as follows. If $f : (|\mathrm{F}|_T : |\mathrm{F}| \to T) \to (p : X \to T)$ is a morphism of $\mathbf{Esg}^{\Gamma}_T$, $\varphi(_{\boldsymbol{Z}}\mathrm{F}, X)(f) : {}_{\boldsymbol{Z}}\mathrm{F} \to \mathrm{S}X$ satisfies $\mathrm{S}(p)\varphi(_{\boldsymbol{Z}}\mathrm{F}, X)(f) = p_{\mathrm{F}}$, namely, $\varphi(_{\boldsymbol{Z}}\mathrm{F}, X)(f)$ is a morphism of $\boldsymbol{M}^{\Gamma}\mathcal{E}/\mathrm{S}T$. Hence there is a morphism $g : \mathrm{F} \to \mathrm{S}_T(p)$ such that ${}_{\boldsymbol{Z}}g = \varphi(_{\boldsymbol{Z}}\mathrm{F}, X)(f)$. We set $\varphi_T(\mathrm{F}, p) = g$.

Proposition 1.14.4 $\varphi_T(\mathrm{F}, p) : \mathbf{Esg}^{\Gamma}_T(|\mathrm{F}|_T, p) \to \boldsymbol{M}^{\Gamma}_{\mathrm{S}T}\mathcal{E}(\mathrm{F}, \mathrm{S}_T(p))$ *is a natural equivalence.*

1.15 Quasi-coherent modules over S-functors

Let $\varphi : R \to S$ be a homomorphism of Γ-rings. For a left (resp. right) S-module N, we denote by ${}_{\varphi}N$ (resp. N_{φ}) a left R-module defined as follows. ${}_{\varphi}N = N_{\varphi} = N$ as abelian groups and the multiplication by R is given by $rn = \varphi(r)n$ (resp. $nr = n\varphi(r)$) for $r \in R$, $n \in N$.

For a $\boldsymbol{Z}$-functor S, we define a category $\mathcal{MOD}^{\Gamma}_{\mathrm{S}}$ as follows. An object of $\mathcal{MOD}^{\Gamma}_{\mathrm{S}}$ is a pair $((R,\rho), M)$ of an S-model (R,ρ) and a Γ-graded R-module M. A morphism $(\varphi,\alpha) : ((R,\rho), M) \to ((S,\sigma), N)$ of degree g of $\mathcal{MOD}^{\Gamma}_{\mathrm{S}}$ consists of a homomorphism $\varphi : (R,\rho) \to (S,\sigma)$ of S-models and a homomorphism $\alpha : M \to {}_{\varphi}N$ of Γ-graded R-modules of degree g. We denote by $\mathcal{MOD}^{\Gamma}_{\mathrm{S}0}$ a subcategory of $\mathcal{MOD}^{\Gamma}_{\mathrm{S}}$ such that $\mathrm{Ob}\mathcal{MOD}^{\Gamma}_{\mathrm{S}0} = \mathrm{Ob}\mathcal{MOD}^{\Gamma}_{\mathrm{S}}$ and morphisms of $\mathcal{MOD}^{\Gamma}_{\mathrm{S}0}$ are morphisms of $\mathcal{MOD}^{\Gamma}_{\mathrm{S}}$ of degree 0. Let $\mathfrak{G} : \mathcal{MOD}^{\Gamma}_{\mathrm{S}} \to \boldsymbol{M}^{\Gamma}_{\mathrm{S}}$ denote a functor which assign (R,ρ) to $((R,\rho), M)$, $\varphi : (R,\rho) \to (S,\sigma)$ to $(\varphi,\alpha) : ((R,\rho), M) \to ((S,\sigma), N)$ and $\mathfrak{F} : \mathcal{MOD}^{\Gamma}_{\mathrm{S}} \to \mathcal{E}$, $\mathfrak{F}_g : \mathcal{MOD}^{\Gamma}_{\mathrm{S}0} \to \mathcal{E}$ $(g \in \Gamma)$ denote forgetful functors which assign $((R,\rho), M)$ to the underlying set of M and M_g respectively.

Definition 1.15.1 *1) Let S be a $\boldsymbol{Z}$-functor and F an S-functor. A functor $\mathrm{M} : (\boldsymbol{M}^{\Gamma}_{\mathrm{S}})_{\mathrm{F}} \to \mathcal{MOD}^{\Gamma}_{\mathrm{S}}$ is called an F-module if composition $\mathfrak{G}\mathrm{M} : (\boldsymbol{M}^{\Gamma}_{\mathrm{S}})_{\mathrm{F}} \to \boldsymbol{M}^{\Gamma}_{\mathrm{S}}$ maps each F-model $((R,\rho),\tau)$ to (R,ρ), $\varphi : ((R,\rho),\tau) \to ((S,\sigma),\kappa)$ to $\varphi : (R,\rho) \to (S,\sigma)$ and $\mathrm{M}(\varphi)$ is a homomorphism of degree zero. A natural transformation $\alpha : \mathrm{M} \to \mathrm{N}$ of F-modules is called a homomorphism of degree g if, for each F-model $((R,\rho),\tau)$, $\alpha_{((R,\rho),\tau)} = (id_{(R,\rho)}, \theta)$ where $\theta : \mathrm{M}((R,\rho),\tau) \to \mathrm{N}((R,\rho),\tau)$ is a homomorphism of degree g. We denote the category of F-modules by $\mathcal{M}od^{\Gamma}_{\mathrm{F}}$ and put $\mathrm{Hom}^{g}_{\mathrm{F}}(\mathrm{M},\mathrm{N}) = \{\alpha : \mathrm{M} \to \mathrm{N} \mid \alpha \text{ is a homomorphism of degree } g\}$. Then* $\mathrm{Hom}_{\mathrm{F}}(\mathrm{M},\mathrm{N}) = \sum_{g\in\Gamma} \mathrm{Hom}^{g}_{\mathrm{F}}(\mathrm{M},\mathrm{N})$ *has a structure of a Γ-graded abelian group.*

2) An F-module M is said to be quasi-coherent if, for any morphism $\varphi : ((R,\rho),\tau) \to ((S,\sigma),\kappa)$ of F-models, the homomorphism $S_{\varphi} \otimes_R \mathrm{M}((R,\rho),\tau) \to \mathrm{M}((S,\sigma),\kappa)$ induced by φ is an isomorphism. $\mathcal{Q}mod^{\Gamma}_{\mathrm{F}}$ denotes the full subcategory of $\mathcal{M}od^{\Gamma}_{\mathrm{F}}$ consisting of quasi-coherent F-modules.

3) A quasi-coherent F-module M is called a vector bundle if $\mathrm{M}((R,\rho),\tau)$ is a projective R-module of finite rank for each F-model $((R,\rho),\tau)$.

If $\mathrm{S} = PA$, we can give $\mathrm{Hom}_{\mathrm{F}}(\mathrm{M},\mathrm{N})$ a structure of a Γ-graded A-module as follows. For $\alpha \in \mathrm{Hom}^g_{\mathrm{F}}(\mathrm{M},\mathrm{N})$ and $a \in A_h$, define $a\alpha \in \mathrm{Hom}^{g+h}_{\mathrm{F}}(\mathrm{M},\mathrm{N})$ by $(a\alpha)_{((R,\rho),\tau)}(x) = \rho(a)\alpha_{((R,\rho),\tau)}(x)$ for $((R,\rho),\tau) \in (\boldsymbol{M}^\Gamma_{\mathrm{S}})_{\mathrm{F}}$, $x \in \mathrm{M}((R,\rho),\tau)$.

Definition 1.15.2 *Let $f : \mathrm{F} \to \mathrm{E}$ be a morphism of* S*-functors. We define the base extension $f^* : \mathcal{M}od^\Gamma_{\mathrm{E}} \to \mathcal{M}od^\Gamma_{\mathrm{F}}$ and the Weil restriction $f_* : \mathcal{M}od^\Gamma_{\mathrm{F}} \to \mathcal{M}od^\Gamma_{\mathrm{E}}$ as follows. For an E-module N and an F-model $((R,\rho),\tau)$, $(f^*\mathrm{N})((R,\rho),\tau) = \mathrm{N}((R,\rho), f_{(R,\rho)}(\tau))$, and for a homomorphism $\alpha : \mathrm{N}_1 \to \mathrm{N}_2$ of E-modules, $(f^*\alpha)_{((R,\rho),\tau)} = \alpha_{((R,\rho),f_{(R,\rho)}(\tau))}$. We set $(f_*\mathrm{M})((R,\rho),\tau)_g = (\boldsymbol{M}^\Gamma_{\mathrm{S}})_{\mathrm{F}}\mathcal{E}(f^*P_{\mathrm{E}}((R,\rho),\tau), \mathfrak{F}_g\mathrm{M})$ for an F-module M and an E-model $((R,\rho),\tau)$. The abelian group structure of $(f_*\mathrm{M})((R,\rho),\tau)_g$ is given by $(\varphi+\psi)_{((A,\kappa),\sigma)}(\theta) = \varphi_{((A,\kappa),\sigma)}(\theta) + \psi_{((A,\kappa),\sigma)}(\theta)$ for $\varphi,\psi \in (f_*\mathrm{M})((R,\rho),\tau)_g$, $((A,\kappa),\sigma) \in (\boldsymbol{M}^\Gamma_{\mathrm{S}})_{\mathrm{F}}$ and $\theta \in (f^*P_{\mathrm{E}}((R,\rho),\tau)((A,\kappa),\sigma)$. We give $\sum_{g\in\Gamma}(f_*\mathrm{M})((R,\rho),\tau)_g$ an R-module structure by $(r\varphi)_{((A,\kappa),\sigma)}(\theta) = \theta(r)\varphi_{((A,\kappa),\sigma)}(\theta)$ for φ, $((A,\kappa),\sigma)$ and θ as above. Put $(f_*\mathrm{M})((R,\rho),\tau) = ((R,\rho), \sum_{g\in\Gamma}(f_*\mathrm{M})((R,\rho),\tau)_g)$. If $\alpha : \mathrm{M} \to \mathrm{N}$ is a homomorphism of F-modules, define $f_*\alpha$ by $(f_*\alpha)_{((R,\rho),\tau)}(\varphi) = \mathfrak{F}_g(\alpha)\varphi$ for $\varphi \in (f_*\mathrm{M})((R,\rho),\tau)_g$.*

Example 1.15.3 *1) Let k be a Γ-ring, A a k-model and M a Γ-graded A-module. We define a P_kA-module $\widetilde{M} : (\boldsymbol{M}^\Gamma_k)_{P_kA} \to \mathcal{MOD}^\Gamma_k$ by $\widetilde{M}(R,\rho) = R_\rho \otimes_A M$. Then $\widetilde{M}$ is a quasi-coherent P_kA-module. If $\varphi : M \to N$ is a homomorphism of Γ-graded A-modules, let $\tilde{\varphi} : \widetilde{M} \to \widetilde{N}$ be a homomorphism of P_kA-module given by $\tilde{\varphi}_{(R,\rho)} = id_R \otimes \varphi$. Thus we have a functor $\widetilde{?} : \boldsymbol{M}^\Gamma_A \to \mathcal{M}od^\Gamma_{P_kA}$ which takes values in $\mathcal{Q}mod^\Gamma_{P_kA}$.*

2) Let $\varphi : B \to A$ be a homomorphism of k-models and put $f = P_k\varphi : P_kA \to P_kB$. Then, for an A-module M and a B-module N, $f^*\widetilde{N} = \widetilde{A_\varphi \otimes_B N}$ and $f_*\widetilde{M} = \widetilde{{}_\varphi M}$ hold.

3) Let X be an S*-functor. We define an X-module $\mathrm{E_X}$ by $\mathrm{E_X}((R,\rho),\tau) = ((R,\rho),$ (the free R-module generated by $((R,\rho),\tau)))$. Then $\mathrm{E_X}$ is a vector bundle over X of rank one. We call this the trivial line bundle over X.*

Proposition 1.15.4 *Let $f : \mathrm{F} \to \mathrm{E}$ be a morphism of* S*-functors and* M, N *an* F*-module,* E*-module respectively. Then, there is a natural equivalence $\xi(\mathrm{N},\mathrm{M}) : \mathrm{Hom}_{\mathrm{F}}(f^*\mathrm{N},\mathrm{M}) \to \mathrm{Hom}_{\mathrm{E}}(\mathrm{N}, f_*\mathrm{M})$ of Γ-graded abelian groups of degree zero.*

Proof. In fact, for $\alpha \in \mathrm{Hom}_{\mathrm{F}}(f^*\mathrm{N},\mathrm{M})$, define $\xi(\mathrm{N},\mathrm{M})(\alpha) : \mathrm{N} \to f_*\mathrm{M}$ by $\xi(\mathrm{N},\mathrm{M})(\alpha)_{((R,\rho),\tau)}(x) = \alpha f^*(x^\sharp)$ for $x \in \mathrm{N}((R,\rho),\tau)_h$ and $x^\sharp : P_{\mathrm{F}}((R,\rho),\tau) \to \mathfrak{F}_h\mathrm{N}$ is defined by $x^\sharp_{((A,\kappa),\sigma)}(\varphi) = \mathrm{N}(\varphi)(x)$. □

Proposition 1.15.5 *1) Let A be a k-model and M a Γ-graded A-module. For a P_kA-module N, a map Φ :* $\mathrm{Hom}_{P_kA}(\widetilde{M},\mathrm{N}) \to \mathrm{Hom}_A(M, \mathrm{N}(A, id_A))$ *given by $\Phi(\alpha) = \alpha_{(A,id_A)}$ is an isomorphism of Γ-graded A-modules. Hence if $\mathrm{I} : \mathcal{M}od^\Gamma_{P_kA} \to \mathcal{M}od^\Gamma_A$ denotes a functor defined by $\mathrm{I}(\mathrm{M}) = \mathrm{M}(A, id_A)$, $\widetilde{?}$ is the right adjoint to* I.

2) *The functor $\widetilde{?} : \boldsymbol{M}^\Gamma_A \to \mathcal{Q}mod^\Gamma_{P_kA}$ is an equivalence of categories with inverse* I.

3) *Let $f : \mathrm{F} \to \mathrm{E}$ be a morphism of* S*-functors. If* N *is a quasi-coherent* E*-module, $f^*\mathrm{N}$ is a quasi-coherent* F*-module. Moreover, if* N *is a vector bundle over* E*, $f^*\mathrm{N}$ is a vector bundle over* F.

Definition 1.15.6 *If $(\mathrm{M}_i)_{i\in I}$ is a family of F-modules, define the direct sum $\sum_{i\in I}\mathrm{M}_i$ and the product $\prod_{i\in I}\mathrm{M}_i$ of F-modules by $(\sum_{i\in I}\mathrm{M}_i)((R,\rho),\tau) = \sum_{i\in I}\mathrm{M}_i((R,\rho),\tau)$ and $(\prod_{i\in I}\mathrm{M}_i)((R,\rho),\tau) = \prod_{i\in I}\mathrm{M}_i((R,\rho),\tau)$. Let M and N be F-modules. Define F-modules $\mathrm{M}\otimes_{\mathrm{F}}\mathrm{N}$ and $\mathcal{H}om_{\mathrm{F}}(\mathrm{M},\mathrm{N})$ by $(\mathrm{M}\otimes_{\mathrm{F}}\mathrm{N})((R,\rho),\tau) = \mathrm{M}((R,\rho),\tau)\otimes_R\mathrm{N}((R,\rho),\tau)$ and $\mathcal{H}om_{\mathrm{F}}(\mathrm{M},\mathrm{N})((R,\rho),\tau) = \mathrm{Hom}_{P_{\mathrm{S}}(R,\rho)}(\tau^{\sharp*}\mathrm{M}, \tau^{\sharp*}\mathrm{N})$ for $((R,\rho),\tau) \in \boldsymbol{M}^\Gamma_{\mathrm{F}}$, respectively.*

Proposition 1.15.7 *1) There are natural isomorphisms of Γ-graded abelian groups*

$$\mathrm{Hom}_{\mathrm{F}}(\sum_{i\in I}\mathrm{M}_i,\mathrm{N}) \xrightarrow{\cong} \prod_{i\in I}\mathrm{Hom}_{\mathrm{F}}(\mathrm{M}_i,\mathrm{N}) \quad \textit{and} \quad \mathrm{Hom}_{\mathrm{F}}(\mathrm{N},\prod_{i\in I}\mathrm{M}_i) \xrightarrow{\cong} \prod_{i\in I}\mathrm{Hom}_{\mathrm{F}}(\mathrm{N},\mathrm{M}_i).$$

2) *There is a natural equivalence $\Lambda : \mathrm{Hom}_{\mathrm{F}}(\mathrm{M}\otimes_{\mathrm{F}}\mathrm{N},\mathrm{L}) \to \mathrm{Hom}_{\mathrm{F}}(\mathrm{M}, \mathcal{H}om_{\mathrm{F}}(\mathrm{N},\mathrm{L}))$ defined by*

$$(\Lambda(\alpha)_{((R,\rho),\tau)}(x))_{((A,\sigma),\mu)}(y) = \alpha_{((A,\sigma),\mathrm{F}(\mu)(\tau)}(\mathrm{M}(\mu)(x)\otimes y)$$

for $x \in \mathrm{M}((R,\rho),\tau)$, $((A,\sigma),\mu) \in (\boldsymbol{M}^\Gamma_{\mathrm{S}})_{P_{\mathrm{S}}(R,\rho)}$, $y \in \mathrm{N}((A,\sigma),\mathrm{F}(\mu)(\tau))$. The inverse equivalence is given by $\Lambda^{-1}(\beta)_{((R,\rho),\tau)}(x\otimes y) = (\beta_{((R,\rho),\tau)}(x))_{((R,\rho),id_{(R,\rho)})}(y)$, for $x,y \in \mathrm{M}(R,\rho)$.

Proposition 1.15.8 *1) If* M *and* N *are quasi-coherent* F*-modules, so is* $\mathrm{M}\otimes_{\mathrm{F}}\mathrm{N}$.

2) *If* M *is a vector bundle over* F *and* N *is a quasi-coherent* F*-module, $\mathcal{H}om_{\mathrm{F}}(\mathrm{M},\mathrm{N})$ is a quasi-coherent* F*-module.*

1.16 Modules over Γ-ringed spaces

Definition 1.16.1 *Let $(X, \mathcal{O}_X)$ be a Γ-ringed space. An $\mathcal{O}_X$-module is a sheaf $\mathcal{M} = \sum_{g\in\Gamma} \mathcal{M}_g$ of Γ-graded abelian groups over X with a morphism $\lambda : \mathcal{O}_X \otimes \mathcal{M} \to \mathcal{M}$ of degree zero satisfying $\lambda(id_{\mathcal{O}_X} \otimes \lambda) = \lambda(\mu \otimes id_{\mathcal{M}})$ and $\lambda(\eta \otimes id_{\mathcal{M}}) = L$. We simply write $\lambda_U(s \otimes m) = sm$ for $s \in \mathcal{O}_X(U)$ and $m \in \mathcal{M}(U)$.*

A morphism $f : \mathcal{M} \to \mathcal{N}$ of degree $g \in \Gamma$ of $\mathcal{O}_X$-modules is a morphism of sheaves of Γ-graded abelian groups of degree g such that $\lambda_{\mathcal{N}}(id_{\mathcal{O}_X} \otimes f) = f\lambda_{\mathcal{M}}$. The set of morphisms of degree g is denoted by $\mathrm{Hom}^g_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$, which has a structure of an abelian group by the addition of morphisms.

We put $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) = \sum_{g\in\Gamma} \mathrm{Hom}^g_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$. Note that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ has a structure of $\mathcal{O}(X)$-module, given by $(af)(s) = a|_U f(s)$ for $a \in \mathcal{O}_{X,g}(X)$, $f \in \mathrm{Hom}^h_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$, $s \in \mathcal{M}(U)$.

We denote by $\mathcal{M}od^{\Gamma}_{\mathcal{O}_X}$ the category of $\mathcal{O}_X$-modules. We can show that $\mathcal{M}od^{\Gamma}_{\mathcal{O}_X}$ is an abelian category.

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_x$-modules. We define the tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ to be the sheaf associated with a presheaf $U \mapsto \check{\mathcal{M}}(U) \otimes_{\check{\mathcal{O}}_X(U)} \check{\mathcal{N}}(U)$. We also define the sheaf of local morphisms $\mathcal{H}om^g_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ of degree $g \in \Gamma$ to be the sheaf $U \mapsto \mathrm{Hom}^g_{\mathcal{O}_X|_U}(\mathcal{M}|_U, \mathcal{N}|_U)$. Set $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) = \sum_{g\in\Gamma} \mathcal{H}om^g_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$. We give $\mathcal{O}_X$-module structures to $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ as follows.

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Chapter 2

An introduction to Grothendieck topos

2.1 Grothendieck topology

We fix a universe $\mathcal{U}$.

Definition 2.1.1 *Let $\mathcal{C}$ be a $\mathcal{U}$-category.*

(1) *A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is called a sieve if it satisfies the following condition.*

$$\text{If } U \in \operatorname{Ob}\mathcal{C} \text{ and } \mathcal{C}(U,V) \neq \emptyset \text{ for some } V \in \operatorname{Ob}\mathcal{D}, \text{ then } U \in \operatorname{Ob}\mathcal{D}.$$

(2) *For $X \in \operatorname{Ob}\mathcal{C}$, sieves of $\mathcal{C}/X$ is called sieves on X.*

Remark 2.1.2 *For a sieve R on X, $\operatorname{Ob} R$ is a set of morphisms in $\mathcal{C}$ whose codomains are X. If we put $R(Y) = \{f : Y \to X \,|\, f \in \operatorname{Ob} R\}$ for $Y \in \operatorname{Ob}\mathcal{C}$, then R is a subfunctor of the presheaf $h_X : \mathcal{C}^{op} \to \mathcal{U}$-**Ens** represented by X. Namely, $R \mapsto R(-)$ gives a bijective correspondence between the set of sieves on X and the set of subfunctors of h_X. Thus we identify a sieve on X with a subfunctor of h_X.*

Definition 2.1.3 *Let $\mathcal{C}$ be a $\mathcal{U}$-category. For each $X \in \operatorname{Ob}\mathcal{C}$, a set $J(X)$ of sieves on X is given. If the following conditions are satisfied, this correspondence J is called a (Grothendieck) topology on $\mathcal{C}$. A category with a topology is called a site.*

($T1$) *For any $X \in \operatorname{Ob}\mathcal{C}$, $h_X \in J(X)$.*

($T2$) *For any $X \in \operatorname{Ob}\mathcal{C}$, $R \in J(X)$ and morphism $f : Y \to X$ in $\mathcal{C}$, a subfunctor $h_f^{-1}(R)$ of h_Y defined by $h_f^{-1}(R)(Z) = \{g : Z \to Y \,|\, fg \in R(Z)\}$ belongs to $J(Y)$.*

($T3$) *A sieve S on X belongs to $J(X)$, if there exists $R \in J(X)$ such that $h_f^{-1}(S) \in J(\operatorname{dom}(f))$ for any $f \in \operatorname{Ob} R$.*

Proposition 2.1.4 *Consider the following conditions on J.*

($T3'$) *A sieve S on X belongs to $J(X)$, if there exists $R \in J(X)$ such that S is a subfunctor of R and $h_f^{-1}(S) \in J(\operatorname{dom}(f))$ for any $f \in \operatorname{Ob} R$.*

($T4$) *A sieve S on X belongs to $J(X)$ if S has a subfunctor which belongs to $J(X)$.*

($T5$) *If $R \in J(X)$ and $R_f \in J(\operatorname{dom}(f))$ is given for each $f \in \operatorname{Ob} R$, then $\{fg \,|\, f \in \operatorname{Ob} R,\, g \in \operatorname{Ob} R_f\} \in J(X)$.*

(1) *($T2$) and ($T3$) imply ($T4$). ($T1$) and ($T3$) imply ($T5$).*

(2) *($T4$) and ($T5$) imply ($T3$). ($T3'$) and ($T4$) imply ($T3$).*

Proof. (1) Let S be a sieve on X which has a subfunctor $R \in J(X)$. Then, $h_f^{-1}(S) \supset h_f^{-1}(R)$ holds for any $f \in \operatorname{Ob}\mathcal{C}/X$. Suppose that $f \in \operatorname{Ob} R$ and $g \in h_f^{-1}(S)(Z)$ for $Z \in \operatorname{Ob}\mathcal{C}$. Since R is a sieve, we have $fg = R(g)(f) \in \operatorname{Ob} R$, which shows that $g \in h_f^{-1}(R)(Z)$. Thus we also have $h_f^{-1}(S) \subset h_f^{-1}(R)$. It follows that $h_f^{-1}(S) = h_f^{-1}(R) \in J(\operatorname{dom}(f))$ by ($T2$). Hence ($T3$) implies $S \in J(X)$.

Put $T = \{fg \,|\, f \in \operatorname{Ob} R, g \in \operatorname{Ob} R_f\}$. Since R_f is a sieve, so is T. For any $f \in \operatorname{Ob} R,$, $Z \in \operatorname{Ob}\mathcal{C}$ and $g \in R_f(Z)$, since $fgh \in \operatorname{Ob} T$ for any $h \in \operatorname{Ob} h_Z$, we see that $h_g^{-1}(h_f^{-1}(T)) = h_{fg}^{-1}(T) = h_Z \in J(Z)$ by ($T1$). Thus we have $h_f^{-1}(T) \in J(\operatorname{dom}(f))$ by ($T3$). Hence $T \in J(X)$ follows from ($T3$).

(2) For a sieve S on X, suppose that there exists $R \in J(X)$ such that $h_f^{-1}(S) \in J(\operatorname{dom}(f))$ for any $f \in \operatorname{Ob} R(Y)$. Put $T = \{fg \,|\, f \in \operatorname{Ob} R, g \in \operatorname{Ob} h_f^{-1}(S)\}$. Since $h_f^{-1}(S)$ is a sieve, T is a subfunctor of S.

Hence if we show that T belongs to $J(X)$, S belongs to $J(X)$ under the assumption $(T4)$. Clearly $(T5)$ implies that T belongs to $J(X)$. We assume $(T3')$. Since T is a subfunctor of S, $\mathrm{Ob}\, h_f^{-1}(T) \subset \mathrm{Ob}\, h_f^{-1}(S)$ for any $f \in \mathrm{Ob}\, R(Y)$. Since $fg \in T(\mathrm{dom}(g))$ if $g \in \mathrm{Ob}\, h_f^{-1}(S)$, it follows that $g \in \mathrm{Ob}\, h_f^{-1}(T)$. Thus we also have $\mathrm{Ob}\, h_f^{-1}(T) \supset \mathrm{Ob}\, h_f^{-1}(S)$. It follows that $h_f^{-1}(T) = h_f^{-1}(S) \in J(\mathrm{dom}(f))$. Since T is a subfunctor of R, $(T3')$ implies $T \in J(X)$. □

For subfunctors G and H of a presheaf F on $\mathcal{C}$, let us denote by $G \cap H$ a subfunctor of F defined by $(G \cap H)(X) = G(X) \cap H(X)$.

Proposition 2.1.5 *If $R, S \in J(X)$, then $R \cap S \in J(X)$.*

Proof. For any $f \in \mathrm{Ob}\, R$, $h_f^{-1}(S) \in J(\mathrm{dom}(f))$ by $(T2)$. Then, $(T5)$ implies $T = \{fg \,|\, f \in \mathrm{Ob}\, R, g \in \mathrm{Ob}\, h_f^{-1}(S)\} \in J(X)$. T is a subfunctor of both R and S, that is, $T \subset R \cap S$. Hence $R \cap S \in J(X)$ by $(T4)$. □

Definition 2.1.6 *Let J, J' be topologies on $\mathcal{C}$. If $J(X) \subset J'(X)$ for any $X \in \mathrm{Ob}\,\mathcal{C}$, J' is said to be finer than J, or J be coarser than J'. Hence the set of all topologies on $\mathcal{C}$ is an ordered set.*

Let $(J_i)_{i \in I}$ be a family of topologies on $\mathcal{C}$. We set $J(X) = \bigcap_{i \in I} J_i(X)$ for each $X \in \mathrm{Ob}\,\mathcal{C}$, then J is a topology on $\mathcal{C}$ and $J = \inf\{J_i |\, i \in I\}$. If T is the set of all topologies on $\mathcal{C}$ that are finer than every J_i, then $\sup\{J_i |\, i \in I\} = \inf T$.

A topology J on $\mathcal{C}$ given by $J(X) =$ (the set of all sieves on X) is the finest topology on $\mathcal{C}$. On the other hand, a topology J given by $J(X) = \{h_X\}$ is the coarsest topology.

Proposition 2.1.7 *For a set R of morphisms in $\mathcal{C}$ with codomain X, we put*

$$\bar{R} = \bigcup_{f \in R} \mathrm{Im}(h_f : h_{\mathrm{dom}(f)} \to h_X) = \{u \,|\, u = fg \textit{ for some } f \in R,\ g \in \mathrm{Mor}\,\mathcal{C} \textit{ such that } \mathrm{codom}(g) = \mathrm{dom}(f)\}.$$

Then, $\bar{R}$ is the smallest sieve containing R.

Proof. Let $j : Y \to Z$ be a morphism in $\mathcal{C}$. For $u \in \bar{R}(Z) = \{u \in \bar{R} \,|\, \mathrm{dom}(u) = Z\}$, there exist $f \in R$ and $g \in \mathrm{Mor}\,\mathcal{C}$ such that $\mathrm{dom}(g) = Z$ and $u = fg$. Then we have $h_X(j)(u) = fgj \in \bar{R}(Y)$, hence $\bar{R}$ is a sieve. Let S be a sieve on X containing R. For $Z \in \mathrm{Ob}\,\mathcal{C}$ and $u \in \bar{R}(Z)$, there exist $f \in R$ and $g \in \mathrm{Mor}\,\mathcal{C}$ such that $\mathrm{dom}(g) = Z$ and $u = fg$. Since $f \in R \subset S$ and $u = h_X(g)(f)$, it follows that $u \in S$ which shows $\bar{R} \subset S$. □

Definition 2.1.8 *Let $(\mathcal{C}, J)$ be a site.*

(1) *For a set R of morphisms in $\mathcal{C}$ with codomain X, we call $\bar{R}$ the sieve generated by R.*

(2) *A family of morphisms $(f_i : X_i \to X)_{i \in I}$ is called a covering of X if the sieve generated by f_i's belongs to $J(X)$.*

Let $\mathcal{C}$ be a category. Suppose that, for each object X, a set $P(X)$ of families of morphisms of $\mathcal{C}$ with codomain X is given. Then, there is the coarsest topology J_P on $\mathcal{C}$ such that for each object X, every element of $P(X)$ is a covering. In fact, J_P is the intersection of all topologies satisfying the above condition. We call J_P the topology generated by P.

Definition 2.1.9 *Let $\mathcal{C}$ be a $\mathcal{U}$-category. For each $X \in \mathrm{Ob}\,\mathcal{C}$, a set $P(X)$ of families of morphisms of $\mathcal{C}$ with codomain X is given. Consider the following conditions.*

- $(P0)$ *For any object X of $\mathcal{C}$ and $S \in P(X)$, each member of S has a pull-back along arbitrary morphism with codomain X.*
- $(P1)$ *For any $X \in \mathrm{Ob}\,\mathcal{C}$, $\{id_X\} \in P(X)$.*
- $(P2)$ *If $(f_i : X_i \to X)_{i \in I} \in P(X)$, then for any morphism $f : Y \to X$ in $\mathcal{C}$, there exists $(g_j : Y_j \to Y)_{j \in I'} \in P(Y)$ such that for each $j \in I'$, fg_j factors through some f_i.*
- $(P2')$ *For any $X \in \mathrm{Ob}\,\mathcal{C}$, $(f_i : X_i \to X)_{i \in I} \in P(X)$ and morphism $f : Y \to X$ in $\mathcal{C}$, the family $(X_i \times_X Y \to Y)_{i \in I}$ of pull-backs along f belongs to $P(Y)$.*
- $(P3)$ *If $(f_i : X_i \to X)_{i \in I} \in P(X)$ and $(g_{ij} : X_{ij} \to X_i)_{j \in I_i} \in P(X_i)$ for each $i \in I$ are given, then $(f_i g_{ij} : X_{ij} \to X)_{(i,j) \in K} \in P(X)$, where $K = \{(i, j) |\, i \in I, j \in I_i\}$.*

If conditions $(P1)$, $(P2)$ and $(P3)$ are satisfied, this correspondence P is called a basis for a (Grothendieck) topology on $\mathcal{C}$. If P satisfies $(P0)$, $(P1)$, $(P2')$ and $(P3)$, P is called a pretopology on $\mathcal{C}$.

We note that $(P2')$ implies $(P2)$ if P satisfies $(P0)$ hence a pretopology is a basis for a topology.

Lemma 2.1.10 *Let $f : Y \to X$ be a morphism and S a set of morphisms of $\mathcal{C}$ with codomain X. Suppose that each element of S has a pull-back along f. S_f denotes the set of pull-backs of elements of S along f. Then we have $h_f^{-1}(\bar{S}) = \bar{S}_f$.*

Proof. For each $p : Z \to X \in S$, we have a pull-back diagram

$$\begin{tikzcd} Z \times_X Y \arrow[r, "p_f"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ Z \arrow[r, "p"] & X. \end{tikzcd}$$

If $g : V \to Y \in \bar{S}_f$, there exist $p : Z \to X \in S$ and $h : V \to Z \times_X Y$ such that $g = p_f h$. Hence $fg = fp_f h = pf'h \in \bar{S}$ and $g \in h_f^{-1}(\bar{S})$. If $g : W \to Y \in h_f^{-1}(\bar{S})$, there exist $p : Z \to X \in S$ and $q : W \to Z$ such that $fg = pq$. Hence there exists $h : W \to Z \times_X Y$ such that $g = p_f h$. Since $p_f \in S_f$, $g \in \bar{S}_f$. $\square$

Proposition 2.1.11 *Let $\mathcal{C}$ be a category and J a topology on $\mathcal{C}$. For each $X \in \mathrm{Ob}\,\mathcal{C}$, let $P(X)$ be the set of all coverings of X. Then P is a basis for a topology. If $\mathcal{C}$ has pull-backs, P is a pretopology.*

Proof. Since the sieve generated by $\{id_X\}$ is $h_X \in J(X)$, $\{id_X\}$ is a covering, thus $\{id_X\} \in P(X)$.
Suppose that $(f_i : X_i \to X)_{i \in I} \in P(X)$ and let $f : Y \to X$ be a morphism in $\mathcal{C}$. Put $S = \{f_i | \, i \in I\}$, then $\bar{S} \in J(X)$, hence $h_f^{-1}(\bar{S}) \in J(Y)$ by $(T2)$. Thus $h_f^{-1}(\bar{S}) \in P(Y)$ and for each $g \in h_f^{-1}(\bar{S})$, $fg = h_f(g) \in \bar{S}$ factors through some f_i.
Suppose that $(f_i : X_i \to X)_{i \in I} \in P(X)$ and, for each $i \in I$, $(g_{ij} : X_{ij} \to X)_{j \in I_i} \in P(X_i)$ is given. Put $S = \{f_i | \, i \in I\}$, $S_i = \{g_{ij} | \, j \in I_i\}$ and $T = \{f_i g_{ij} | \, i \in I, j \in I_i\}$. For each $f : Y \to X \in \bar{S}$, choose $i \in I$ and $\bar{f} : Y \to X_i$ such that $f = f_i \bar{f}$ and $\bar{f} = id_{X_i}$ if $f = f_i$ for some i. Put $R_f = h_{\bar{f}}^{-1}(\bar{S}_i)$ and $T' = \{fg | \, f \in \bar{S}, g \in R_f\}$, then $(T5)$ implies $T' \in J(X)$ and it is obvious that $\bar{T} \supset T'$. Since $R_{f_i} = \bar{S}_i$, $\bar{T} \subset T'$. Thus $\bar{T} \in J(X)$ and $(P3)$ holds.

Assume that $\mathcal{C}$ has pull-backs, then $(P0)$ is obviously satisfied. Suppose $(f_i : X_i \to X)_{i \in I} \in P(X)$ and put $S = \{f_i | \, i \in I\}$, then $\bar{S} \in J(X)$. For any morphism $f : Y \to X$ of $\mathcal{C}$, $\bar{S}_f = h_f^{-1}(\bar{S}) \in J(Y)$ by $(T2)$ and the above result. Hence $(X_i \times_X Y \to Y)_{i \in I} \in P(Y)$. $\square$

Proposition 2.1.12 *(1) Let P be a basis for a topology on $\mathcal{C}$ and J_P the topology generated by P. Then, $J_P(X) = \{R \subset h_X \,|\, R \supset S \text{ for some } S \in P(X)\}$.*

(2) Let J be a topology on $\mathcal{C}$ and P be as in (2.1.11). Then the topology generated by P coincides with J.

Proof. (1) We put $J(X) = \{R \subset h_X | \, R \supset S \text{ for some } S \in P(X)\}$, then $J(X) \subset J_P(X)$. It suffices to show that J is a topology. Since $h_X \supset \{id_X\} \in P(X)$, J satisfies $(T1)$. Let $f : Y \to X$ be a morphism of $\mathcal{C}$. If $R \in J(X)$, then $R \supset S$ for some $S \in P(X)$ thus $R \supset \bar{S}$ since R is a sieve. By $(P2)$, there exists $T \in P(Y)$ such that $h_f^{-1}(\bar{S}) \supset T$. Thus $h_f^{-1}(R) \supset h_f^{-1}(\bar{S}) \supset T \in P(Y)$. Hence $h_f^{-1}(R) \in J(Y)$. It is clear that J satisfies $(T4)$. Suppose that $R \in J(X)$ and, for each $f \in R$, $R_f \in J(\mathrm{dom}(f))$ is given. Then, there exist $S \in P(X)$ and $S_f \in P(\mathrm{dom}(f))$ such that $R \supset S$, $R_f \supset S_f$. By $(P3)$, we have $\{fg | \, f \in S, g \in S_f\} \in P(X)$, hence $\{fg | \, f \in S, g \in S_f\} \subset \{fg | \, f \in R, g \in R_f\} \in J(X)$.

(2) Let us denote by J_P the topology generated by P. For an object X of $\mathcal{C}$, it follows from the above result and $(T4)$ that $R \in J_P(X)$ implies $R \in J(X)$. It is obvious that $R \in J(X)$ implies $R \in J_P(X)$. $\square$

Proposition 2.1.13 *Let $S = (f_i : X_i \to X)_{i \in I}$ be a family of morphisms in $\mathcal{C}$. We denote by $f_i^\sharp : h_{X_i} \to \bar{S}$ the unique morphism satisfying $\iota f_i^\sharp = h_{f_i}$, where $\iota : \bar{S} \to h_X$ is the inclusion morphism.*

(1) For a presheaf F on $\mathcal{C}$, define a map $\Phi : \widehat{\mathcal{C}}(\bar{S}, F) \to \prod_{i \in I} \widehat{\mathcal{C}}(h_{X_i}, F)$ by $\Phi(\varphi) = (\varphi f_i^\sharp)_{i \in I}$. Then, Φ is injective and its image consists of families $(g_i : h_{X_i} \to F)_{i \in I}$ which satisfy a condition "If $f_i u = f_j v$ for $u : Z \to X_i$ and $v : Z \to X_j$, then $g_i h_u = g_j h_v$." for any $i, j \in I$ and any object Z of $\mathcal{C}$.

(2) Define maps $p, q : \prod_{i \in I} \widehat{\mathcal{C}}(h_{X_i}, F) \to \prod_{i,j \in I} \widehat{\mathcal{C}}(h_{X_i} \times_{h_X} h_{X_j}, F)$ by $\mathrm{pr}_{ij} p = p_{ij}^* \mathrm{pr}_i$, $\mathrm{pr}_{ij} q = q_{ij}^* \mathrm{pr}_j$, where p_{ij} and q_{ij} are given by the following pull-back diagram.

$$\begin{tikzcd} h_{X_i} \times_{h_X} h_{X_j} \arrow[r, "q_{ij}"] \arrow[d, "p_{ij}"] & h_{X_j} \arrow[d, "h_{f_j}"] \\ h_{X_i} \arrow[r, "h_{f_i}"] & h_X \end{tikzcd}$$

Then, the following diagram is an equalizer.

$$\widehat{\mathcal{C}}(\bar{S}, F) \xrightarrow{\Phi} \prod_{i \in I} \widehat{\mathcal{C}}(h_{X_i}, F) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j \in I} \widehat{\mathcal{C}}(h_{X_i} \times_{h_X} h_{X_j}, F)$$

Proof. (1) We set $\psi_i = (\theta_F^{-1})_{X_i} : \widehat{\mathcal{C}}(h_{X_i}, F) \to F(X_i)$ (A.1.6) and $\Psi = (\prod_{i \in I} \psi_i)\Phi : \widehat{\mathcal{C}}(\bar{S}, F) \to \prod_{i \in I} F(X_i)$. Then, $\prod_{i \in I} \psi_i$ is a bijection and we have $\Psi(\varphi) = (\varphi_{X_i}(f_i))_{i \in I}$. For any object Z and $g \in \bar{S}(Z)$, there exist $i \in I$ and a morphism $u : Z \to X_i$ such that $g = f_i u = \bar{S}(u)(f_i)$. Then, $\varphi_Z(g) = \varphi_Z \bar{S}(u)(f_i) = F(u)\varphi_{X_i}(f_i)$ and this implies that φ is determined by $\Psi(\varphi)$. Hence Ψ is a monomorphism and so is Φ.

If $g = f_i u = f_j v$, then $\varphi_Z(g) = F(u)\varphi_{X_i}(f_i) = F(v)\varphi_{X_j}(f_j)$. On the other hand, since $F(u)\varphi_{X_i}(f_i) = \varphi_Z \bar{S}(u)(f_i) = \varphi_Z(f_i u) = \varphi_Z f_i^\sharp(u) = \varphi_Z f_i^\sharp h_u(id_Z)$, we have $\varphi_Z f_i^\sharp h_u = \varphi_Z f_j^\sharp h_j$, hence each element of the image of Φ satisfies the above condition. Conversely, if $(g_i : h_{X_i} \to F)_{i \in I}$ satisfies the above condition, define φ by $\varphi_Z(g) = (g_i)_Z(u)$ for $g = f_i u \in \bar{S}(Z)$. It is easy to verify that this definition does not depend on the choice of i and $u : Z \to X_i$ and that φ is natural.

(2) It is obvious that each element $(g_i)_{i \in I}$ in the image of Φ satisfies $g_i p_{ij} = g_j q_{ij}$ for any $i, j \in I$. Conversely, assume that $(g_i)_{i \in I} \in \prod_{i \in I} \widehat{\mathcal{C}}(h_{X_i}, F)$ satisfies $g_i p_{ij} = g_j q_{ij}$ for any $i, j \in I$. For an object Y of $\mathcal{C}$ and $\alpha \in R(X)$, we choose $i \in I$ and a morphism $\beta : Y \to X_i$ such that $\alpha = f_i \beta$. Define $\varphi : R \to F$ by $\varphi_Y(\alpha) = g_{iY}(\beta)$. It is easy to verify that $\varphi_Y(\alpha)$ does not depend on the choice of i and β and that φ is a natural transformation such that $\Phi(\varphi) = (g_i)_{i \in I}$. □

Remark 2.1.14 *For a family $S = (f_i : X_i \to X)_{i \in I}$ of morphisms in $\mathcal{C}$, we regard f_i as an element of $\bar{S}(X_i)$ for each $i \in I$. Since there is a natural bijection $\theta_F : F(X_i) \to \widehat{\mathcal{C}}(h_{X_i}, F)$ by (A.1.6), it follows from (1) of (2.1.13) that a map $\Psi : \widehat{\mathcal{C}}(\bar{S}, F) \to \prod_{i \in I} F(X_i)$ defined by $\Psi(\varphi) = (\varphi_{X_i}(f_i))_{i \in I}$ is injective and its image consists of families $(x_i)_{i \in I}$ which satisfy a condition "If $f_i u = f_j v$ for $u : Z \to X_i$ and $v : Z \to X_j$, then $F(u)(x_i) = F(v)(x_j)$." for any $i, j \in I$ and any object Z of $\mathcal{C}$.*

Corollary 2.1.15 *Let $S = (f_i : X_i \to X)_{i \in I}$ be a family of morphisms in $\mathcal{C}$. For an object Y of $\mathcal{C}$, define a map $\Phi : \widehat{\mathcal{C}}(\bar{S}, h_Y) \to \prod_{i \in I} \mathcal{C}(X_i, Y)$ by $\Phi(\varphi) = (\varphi_{X_i}(f_i))_{i \in I}$. Then, Φ is injective and its image consists of families $(g_i : X_i \to Y)_{i \in I}$ satisfying the following condition for any $i, j \in I$ and any object Z.*

If $f_i u = f_j v$ for $u : Z \to X_i$ and $v : Z \to X_j$, then $g_i u = g_j v$.

Proof. Consider the case $F = h_Y$ in (2.1.13) and apply (A.1.7). □

2.2 Sheaves on a site

We fix a universe $\mathcal{U}$ and let $\mathcal{C}$ be a $\mathcal{U}$-category. The category of $\mathcal{U}$-presheaves $\widehat{\mathcal{C}}_{\mathcal{U}}$ on $\mathcal{C}$ is denoted by $\widehat{\mathcal{C}}$ for short.

Definition 2.2.1 *Let $(\mathcal{C}, J)$ be a site. A presheaf F on $\mathcal{C}$ is said to be separated if, for any object X and $R \in J(X)$, the map $\widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(R, F)$ induced by $R \hookrightarrow h_X$ is injective. A presheaf F on $\mathcal{C}$ is called a sheaf if, for any object X and $R \in J(X)$, the above map is bijective. We denote by $\widetilde{\mathcal{C}}_{\mathcal{U}}$ the full subcategory of $\widehat{\mathcal{C}}_{\mathcal{U}}$ consisting of sheaves.*

Proposition 2.2.2 *Let $\mathcal{C}$ be a category, F a presheaf on $\mathcal{C}$, $(f_i : X_i \to X)_{i \in I}$ a family of morphisms in $\mathcal{C}$ and R the sieve on X generated by $(f_i : X_i \to X)_{i \in I}$. We denote by $\iota : R \to h_X$ the inclusion morphism.*

1) $\iota^* : \widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(R, F)$ is injective if and only if $(F(f_i))_{i \in I} : F(X) \to \prod_{i \in I} F(X_i)$ is injective.

2) We set

$$M = \left\{ (x_i)_{i \in I} \in \prod_{i \in I} F(X_i) \,\middle|\, F(g)(x_i) = F(h)(x_j) \text{ if } f_i g = f_j h \text{ for } i, j \in I \text{ and } g : Z \to X_i,\ h : Z \to X_j \right\}.$$

$\iota^* : \widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(R, F)$ is surjective if and only if the image of $(F(f_i))_{i \in I} : F(X) \to \prod_{i \in I} F(X_i)$ is M.

3) Suppose that, for any $i, j \in I$, the following pull-back exists.

$$\begin{array}{ccc} X_i \times_X X_j & \xrightarrow{f'_{ij}} & X_j \\ \downarrow{\scriptstyle f_{ij}} & & \downarrow{\scriptstyle f_j} \\ X_i & \xrightarrow{f_i} & X \end{array}$$

Then $M = \{(x_i)_{i\in I} \in \prod_{i\in I} F(X_i)|$ *For any* $i, j \in I$, $F(f_{ij})(x_i) = F(f'_{ij})(x_j)\}$.

Proof. 1) Suppose that $\iota^* : \widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(R, F)$ is injective. If $x, y \in F(X)$ satisfy $F(f_i)(x) = F(f_i)(y)$ for any $i \in I$, it is clear that $F(f)(x) = F(f)(y)$ for any $f \in R$. x, y define natural transformations $x^\sharp, y^\sharp : h_X \to F$ by $x^\sharp(g) = F(g)(x)$, $y^\sharp(g) = F(g)(y)$. Then the above equality implies that the restriction of $x^\sharp$ to R coincides with the restriction of $y^\sharp$ to R. Hence $x^\sharp = y^\sharp$ by assumption and we have $x = x^\sharp(id_X) = y^\sharp(id_X) = y$.

Suppose that $F(X) \to \prod_{i\in I} F(X_i)$ is injective. If φ and ψ are natural transformations $h_X \to F$ whose restrictions to R coincide, $F(f_i)(\varphi_X(id_X)) = \varphi_{X_i}(f_i) = \psi_{X_i}(f_i) = F(f_i)(\psi_X(id_X))$ for any $i \in I$ since $f_i \in R$. Then, the assumption implies $\varphi_X(id_X) = \psi_X(id_X)$. Hence we have $\varphi = \psi$ and this shows that $\iota^* : \widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(R, F)$ is injective.

2) Suppose that $\iota^* : \widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(R, F)$ is surjective. It is clear that $(F(f_i)(x))_{i\in I}$ $(x \in F(X))$ belongs to M. For $(x_i)_{i\in I} \in M$, define a natural transformation $\varphi : R \to F$ by $\varphi_Y(f) = F(g)(x_i)$ if $f = f_i g$ for some $i \in I$ and $g : Z \to X_i$. If $f = f_i g = f_j h$ for $g : Z \to X_i$ and $h : Z \to X_j$, we have $F(g)(x_i) = F(h)(x_j)$ by the assumption and this shows that φ is well-defined. For $\alpha : W \to Z$ and $f = f_i g \in R(Z)$, $\varphi_W R(\alpha)(f) = \varphi_W(f\alpha) = F(g\alpha)(x_i) = F(\alpha)\varphi_Z(f)$. Thus φ is natural and there exists a natural transformation $\bar{\varphi} : h_X \to F$ such that $\bar{\varphi}$ restricts to φ. Put $x = \bar{\varphi}_X(id_X)$ then $F(f_i)(x) = F(f_i)(\bar{\varphi}_X(id_X)) = \bar{\varphi}_{X_i} R(f_i)(id_X) = \bar{\varphi}_{X_i}(f_i) = \varphi_{X_i}(f_i) = F(id_{X_i})(x_i) = x_i$ for any $i \in I$.

Conversely, suppose that the image of $(F(f_i))_{i\in I} : F(X) \to \prod_{i\in I} F(X_i)$ is M. Let $\varphi : R \to F$ be a natural transformation. If $g : Z \to X_i$ and $h : Z \to X_j$ satisfy $f_i g = f_j h$ then, $F(g)(\varphi_{X_i}(f_i)) = \varphi_Z R(g)(f_i) = \varphi_Z(f_i g) = \varphi_Z(f_j h) = \varphi_Z R(h)(f_j) = F(h)(\varphi_{X_j}(f_j))$. Hence $(\varphi_{X_i}(f_i))_{i\in I}$ is contained in the image of $(F(f_i))_{i\in I} : F(X) \to \prod_{i\in I} F(X_i)$ and there exists $x \in F(X)$ such that $F(f_i)(x) = \varphi_{X_i}(f_i)$ for any $i \in I$. Define a natural transformation $\bar{\varphi} : h_X \to F$ by $\bar{\varphi}_Y(f) = F(f)(x)$. For any $f \in R(Y)$, choose $i \in I$ and $g : Y \to X_i$ such that $f = f_i g$. Then, $\bar{\varphi}_Y(f) = F(f)(x) = F(g)F(f_i)(x) = F(g)(\varphi_{X_i}(f_i)) = \varphi_Y(R(g)(f_i)) = \varphi_Y(f_i g) = \varphi_Y(f)$. Therefore $\bar{\varphi}_Y(f) = \varphi_Y(f)$, that is, $\bar{\varphi}$ restricts to φ.

3) Since there is a natural bijection

$$\mathcal{C}(Z, X_i \times_X X_j) \to \{(g, h) \in \mathcal{C}(Z, X_i) \times \mathcal{C}(Z, X_j)|\, f_i g = f_j h\} \qquad \theta \mapsto (f_{ij}\theta, f'_{ij}\theta),$$

$(x_i)_{i\in I} \in M$ if and only if it satisfies $F(f_{ij})(x_i) = F(f'_{ij})(x_j)$ for any $i, j \in I$. □

Corollary 2.2.3 *Let* $\mathcal{C}$ *be a category with a basis* P *for a topology,* J_P *the topology generated by* P *and* F *a presheaf on* F.

1) F *is separated if and only if for any object* X *of* $\mathcal{C}$ *and* $(f_i : X_i \to X)_{i\in I} \in P(X)$, $(F(f_i))_{i\in I} : F(X) \to \prod_{i\in I} F(X_i)$ *is injective.*

2) The following conditions are equivalent.

(1) A presheaf F *is a sheaf on* $(\mathcal{C}, J_P)$.

(2) For any object X *and* $(f_i : X_i \to X)_{i\in I} \in P(X)$, $(F(f_i))_{i\in I} : F(X) \to \prod_{i\in I} F(X_i)$ *is injective and* $(x_i)_{i\in I} \in \prod_{i\in I} F(X_i)$ *belongs to the image of this map if and only if for any* $i, j \in I$ *and morphisms* $g : Z \to X_i$, $h : Z \to X_j$ *such that* $f_i g = f_j h$, $F(g)(x_i) = F(h)(x_j)$ *holds.*

If P *is a pretopology, the above condition (2) is equivalent to the following.*

(2′) For any object X *and* $(f_i : X_i \to X)_{i\in I} \in P(X)$, *let*

$$\begin{array}{ccc} X_i \times_X X_j & \xrightarrow{f'_{ij}} & X_j \\ \downarrow{\scriptstyle f_{ij}} & & \downarrow{\scriptstyle f_j} \\ X_i & \xrightarrow{f_i} & X \end{array}$$

be a pull-back and $\beta, \beta' : \prod_{i\in I} F(X_i) \to \prod_{i,j\in I} F(X_i \times_X X_j)$ *maps satisfying* $\mathrm{pr}_{ij}\beta = F(f_{ij})\mathrm{pr}_i$, $\mathrm{pr}_{ij}\beta' = F(f'_{ij})\mathrm{pr}_j$, *then the following diagram is an equalizer.*

$$F(X) \xrightarrow{\alpha} \prod_{i\in I} F(X_i) \overset{\beta}{\underset{\beta'}{\rightrightarrows}} \prod_{i,j\in I} F(X_i \times_X X_j)$$

Here we set $\alpha = (F(f_i))_{i\in I}$.

Proof. 1) Suppose that F is separated. For any $X \in \mathrm{Ob}\,\mathcal{C}$ and $S = (f_i : X_i \to X)_{i\in I} \in P(X)$, $\bar{S} \in J_P(X)$ and the map $\widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(\bar{S}, F)$ induced by the inclusion morphism $\bar{S} \to h_X$ is injective. Hence $(F(f_i))_{i\in I} : F(X) \to \prod_{i\in I} F(X_i)$ is injective by (2.2.2).

Conversely, suppose that, for any object X of $\mathcal{C}$ and $(f_i : X_i \to X)_{i\in I} \in P(X)$, $(F(f_i))_{i\in I} : F(X) \to \prod_{i\in I} F(X_i)$ is injective. For $R \in J_P(X)$, there exists $S = (f_i : X_i \to X)_{i\in I} \in P(X)$ such that $R \supset S$ by (2.1.12). Let $\iota : R \to h_X$ and $\kappa : \bar{S} \to R$ be inclusion morphisms. It follows from (2.2.2) that the composition $\widehat{\mathcal{C}}(h_X, F) \xrightarrow{\iota^*} \widehat{\mathcal{C}}(R, F) \xrightarrow{\kappa} \widehat{\mathcal{C}}(\bar{S}, F)$ is injective. Hence ι^* is injective.

2) Suppose that F is a sheaf and $S = (f_i : X_i \to X)_{i\in I} \in P(X)$. Then, $\bar{S} \in J_P(X)$ and since the inclusion morphism $\bar{S} \to h_X$ induces a bijection $\widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(\bar{S}, F)$, it follows from (2.2.2) that $(F(f_i))_{i\in I} : F(X) \to \prod_{i\in I} F(X_i)$ is injective and $(x_i)_{i\in I} \in \prod_{i\in I} F(X_i)$ belongs to the image of this map if and only if for any $i, j \in I$ and morphisms $g : Z \to X_i$, $h : Z \to X_j$ such that $f_i g = f_j h$, $F(g)(x_i) = F(h)(x_j)$ holds.

Conversely, suppose that the condition (2) holds. Then, F is separated by 1). For $R \in J_P(X)$, there exists $S = (f_i : X_i \to X)_{i\in I} \in P(X)$ such that $R \supset S$ by (2.1.12). Let $\iota : R \to h_X$ and $\kappa : \bar{S} \to R$ be inclusion morphisms. It follows from (2.2.2) that $\widehat{\mathcal{C}}(h_X, F) \xrightarrow{\kappa^*\iota^*} \widehat{\mathcal{C}}(\bar{S}, F)$ is surjective, hence, for $\varphi \in \widehat{\mathcal{C}}(R, F)$, there exists $\bar{\varphi} \in \widehat{\mathcal{C}}(h_X, F)$ such that $\bar{\varphi}\iota\kappa = \varphi\kappa$. For any $Y \in \mathrm{Ob}\,\mathcal{C}$ and $f \in R(Y)$, there exists $(g_k : Y_k \to Y)_{k\in K} \in P(Y)$ such that $fg_k \in \bar{S}$ by $(P2)$. Then, for any $k \in K$, $F(g_k)(\bar{\varphi}_Y(f)) = \bar{\varphi}_{Y_k} h_X(g_k)(f) = \bar{\varphi}_{Y_k}(fg_k) = \varphi_{Y_k}(fg_k) = \varphi_{Y_k} R(g_k)(f) = F(g_k)(\varphi_Y(f))$. Since $(F(g_k))_{k\in K} : F(Y) \to \prod_{k\in K} F(Y_k)$ is injective, we have $\bar{\varphi}_Y(f) = \varphi_Y(f)$. Therefore, $\bar{\varphi}\iota = \varphi$ and ι^* is surjective.

If P is a pretopology, the assertion easily follows from (2.2.2). $\square$

Proposition 2.2.4 *Let* $\mathcal{C}$ *be a category and* $\mathcal{F} = (F_i)_{i\in I}$ *a family of presheaves on* $\mathcal{C}$. *For each object* X *of* $\mathcal{C}$, *we denote by* $J_{\mathcal{F}}(X)$ *the set of sieves* R *on* X *such that for any morphism* $f : Y \to X$ *with codomain* X, *the map* $\widehat{\mathcal{C}}(h_Y, F_i) \to \widehat{\mathcal{C}}(h_f^{-1}(R), F_i)$ *induced by* $h_f^{-1}(R) \hookrightarrow h_Y$ *is bijective (resp. injective) for all* $i \in I$. *Then,* $J_{\mathcal{F}}(X)$ *defines the finest topology on* $\mathcal{C}$ *such that each* F_i *is a sheaf (resp. separated presheaf).*

Proof. It is obvious that $J_{\mathcal{F}}$ satisfies axioms $(T1)$ and $(T2)$. It suffices to show T3$'$ and $(T4)$ by (2.1.4). In both cases, the assumption implies $h_f^{-1}(S) \in J_{\mathcal{F}}(Y)$ for any $Y \in \mathrm{Ob}\,\mathcal{C}$ and $f \in R(Y)$. Hence the inclusion map $\iota_f : h_f^{-1}(S) \hookrightarrow h_Y$ induces a bijection (resp. injection) $\iota_f^* : \widehat{\mathcal{C}}(h_Y, F_i) \to \widehat{\mathcal{C}}(h_f^{-1}(S), F_i)$.

By (A.4.2), $(hP\langle Y, \alpha\rangle \xrightarrow{\alpha} R)_{\langle Y,\alpha\rangle\in\mathrm{Ob}(h\downarrow R)}$ is a colimiting cone of a functor $hP : (h\downarrow R) \to \widehat{\mathcal{C}}$. Hence $(\widehat{\mathcal{C}}(R, F_i) \xrightarrow{\alpha^*} \widehat{\mathcal{C}}(hP\langle Y, \alpha\rangle, F_i))_{\langle Y,\alpha\rangle\in\mathrm{Ob}(h\downarrow R)}$ is a limiting cone. Define a functor $D : (h\downarrow R) \to \widehat{\mathcal{C}}$ by $D\langle Y, \alpha\rangle = \alpha^{-1}(S)$, $D(f) = ($*the restriction of* h_f *to* $\alpha^{-1}(S))$ for $f : \langle Y, \alpha\rangle \to \langle Z, \beta\rangle$. Since colimits in $\widehat{\mathcal{C}}$ are universal by (A.4.1) and (A.4.3), we have a colimiting cone $(D\langle Y, \alpha\rangle \xrightarrow{\bar{\alpha}} S)_{\langle Y,\alpha\rangle\in\mathrm{Ob}(h\downarrow R)}$, where $\bar{\alpha} : D\langle Y, \alpha\rangle = \alpha^{-1}(S) \to S$ is the restriction of $\alpha : h_Y \to R$. Then, $(\widehat{\mathcal{C}}(S, F_i) \xrightarrow{\bar{\alpha}} \widehat{\mathcal{C}}(D\langle Y, \alpha\rangle, F_i))_{\langle Y,\alpha\rangle\in\mathrm{Ob}(h\downarrow R)}$ is a limiting cone. Note that the inclusion morphism $\iota_\alpha : \alpha^{-1}(S) \hookrightarrow h_Y$ defines a natural transformation $\varphi : D \to hP$ and that the following diagram commutes.

$$\begin{tikzcd}
\widehat{\mathcal{C}}(R, F_i) \arrow[r, "\alpha^*"] \arrow[d, "\iota^*"] & \widehat{\mathcal{C}}(hP\langle Y, \alpha\rangle, F_i) \arrow[d, "\iota_\alpha^*"] \\
\widehat{\mathcal{C}}(S, F_i) \arrow[r, "\bar{\alpha}"] & \widehat{\mathcal{C}}(D\langle Y, \alpha\rangle, F_i)
\end{tikzcd}$$

It follows that $\iota^* : \widehat{\mathcal{C}}(R, F_i) \to \widehat{\mathcal{C}}(S, F_i)$ $(\iota : S \hookrightarrow R)$ is bijective (resp. injective) for any $i \in I$. Therefore the assumptions of T3$'$ and $(T4)$ imply that $\widehat{\mathcal{C}}(h_X, F_i) \to \widehat{\mathcal{C}}(S, F_i)$ and $\widehat{\mathcal{C}}(h_X, F_i) \to \widehat{\mathcal{C}}(R, F_i)$ are bijective (resp.

injective), respectively. For any morphism $f : Y \to X$ of $\mathcal{C}$, replacing X by Y, S by $h_f^{-1}(S)$ and R by $h_f^{-1}(R)$ in the above result, we have bijections (resp. injections) $\widehat{\mathcal{C}}(h_Y, F_i) \to \widehat{\mathcal{C}}(h_f^{-1}(S), F_i)$ and $\widehat{\mathcal{C}}(h_Y, F_i) \to \widehat{\mathcal{C}}(h_f^{-1}(R), F_i)$, respectively. This shows T3$'$ and $(T4)$. □

Corollary 2.2.5 *Let $\mathcal{C}$ be a category. For each object X of $\mathcal{C}$, let $K(X)$ be a set of sieves on X satisfying $(T2)$. A presheaf F on $\mathcal{C}$ is a sheaf (resp. separated presheaf) with respect to the topology generated by $K(X)$ if and only if for any object X and $R \in K(X)$, the map $\widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(R, F)$ induced by $R \hookrightarrow h_X$ is bijective (resp. injective).*

Proof. Let F be a presheaf satisfying the condition above and put $\mathcal{F} = (F)$. Then F is a sheaf (resp. separated presheaf) with respect to the topology generated by $K(X)$ if and only if $K(X) \subset J_{\mathcal{F}}(X)$. But it follows from the assumption and the previous result that $R \in K(X)$ implies $R \in J_{\mathcal{F}}(X)$. □

Definition 2.2.6 *Let $\mathcal{C}$ be a category.*

1) The canonical topology on $\mathcal{C}$ is the finest topology for which all the representable functors are sheaves. We say that a topology is sub-canonical if it is coarser than the canonical topology.

2) Let J be the canonical topology on $\mathcal{C}$. An element of $J(X)$ is called a universal strict epimorphic sieve on X and a covering of X is called a universal strict epimorphic family.

Proposition 2.2.7 *Let $R = (f_i : X_i \to X)_{i\in I}$ be a family of morphisms of $\mathcal{C}$ such that, for any morphism $g : W \to X$ and $i \in I$, a pull-back $f'_i : X_i \times_X W \to W$ of f_i along g exists. Then R is a universal strict epimorphic family if and only if for any morphism $g : W \to X$ and $i \in I$, $(f'_i : X_i \times_X W \to W)_{i\in I}$ is a strict epimorphic family in the sense of (A.1.11).*

Proof. Let J be the canonical topology on $\mathcal{C}$. By (2.1.10) and (2.2.4), $\bar{R} \in J(X)$ if and only if for any object Y and morphism $g : W \to X$, the map $\iota^* : \widehat{\mathcal{C}}(h_W, h_Y) \to \widehat{\mathcal{C}}(\bar{R}_g, h_Y)$ induced by the inclusion $\iota : \bar{R}_g \hookrightarrow h_W$ is bijective. Since $e : \mathcal{C}(W,Y) \to \prod\limits_{i\in I} \mathcal{C}(X_i \times_X W, Y)$ defined for R_g in (A.1.11) is the composition of $h : \mathcal{C}(W,Y) \xrightarrow{\cong} \widehat{\mathcal{C}}(h_W, h_Y)$, $\iota^* : \widehat{\mathcal{C}}(h_W, h_Y) \to \widehat{\mathcal{C}}(\bar{R}_g, h_Y)$ and $\Phi : \widehat{\mathcal{C}}(\bar{R}_g, h_Y) \to \prod\limits_{i\in I} \mathcal{C}(X_i \times_X W, Y)$ in (2.1.15), it follows from (2.1.15) that ι^* is bijective if and only if $R_g = (f'_i : X_i \times_X W \to W)_{i\in I}$ is a strict epimorphic family. □

Proposition 2.2.8 *Let $\mathcal{C}$ be a category and $j : Y \to X$ a monomorphism in $\mathcal{C}$. If $(f_i : X_i \to Y)_{i\in I}$ is a family of morphism in $\mathcal{C}$ such that $(jf_i : X_i \to X)_{i\in I}$ is a universal strict epimorphic family, then j is an isomorphism and $(f_i : X_i \to Y)_{i\in I}$ is a universal strict epimorphic family.*

Proof. We denote by R the sieve on X generated by $(jf_i : X_i \to X)_{i\in I}$. Define a morphism $\varphi : R \to h_Y$ in $\widehat{\mathcal{C}}$ by $\varphi_Z(\alpha) = f_i\beta$ if $\alpha = jf_i\beta$ for some $i \in I$ and $\beta \in \mathcal{C}(Z, X_i)$. Since j is a monomorphism, φ is well-defined and natural. R is a universal strict epimorphic sieve and the map $\widehat{\mathcal{C}}(h_X, h_Y) \to \widehat{\mathcal{C}}(R, h_Y)$ induced by the inclusion morphism $\iota : R \to h_X$ is bijective. Hence we have a morphism $s : X \to Y$ such that $h_s\iota = \varphi$. Then, for each $i \in I$, $f_i = \varphi_{X_i}(jf_i) = (h_s\iota)_{X_i}(jf_i) = sjf_i$ and this implies that $\iota^* : \widehat{\mathcal{C}}(h_X, h_X) \to \widehat{\mathcal{C}}(R, h_X)$ maps h_{js} to ι. In fact, for any $\alpha \in R(Z)$, $\alpha = jf_i\beta$ for some $i \in I$ and $\beta \in \mathcal{C}(Z, X_i)$, thus we have $\iota^*(h_{js})_Z(\alpha) = jsjf_i\beta = jf_i\beta = \alpha = \iota_Z(\alpha)$. Since ι^* is injective and this also maps id_{h_X} to ι, we have $js = id_X$. Therefore $jsj = j$ and $sj = id_Y$, for j is a monomorphism. □

Proposition 2.2.9 *Let $\mathcal{C}$ be a $\mathcal{U}$-category with a strict initial object 0 and J a topology on $\mathcal{C}$ finer than the canonical topology.*

1) The empty sieve $\emptyset$ is a covering sieve on 0 for J.

2) If F is a sheaf on $\mathcal{C}$ for J, $F(0)$ consists of a single element.

3) Let F be a sheaf on $\mathcal{C}$ for J and $(X_j)_{j\in I}$ a family of objects of $\mathcal{C}$. If there exists a disjoint coproduct $\coprod\limits_{j\in I} X_j$ such that the family $(\iota_k : X_k \to \coprod\limits_{j\in I} X_j)_{j\in I}$ of the canonical morphisms is a covering for J, $F(\iota_k) : F(\coprod\limits_{j\in I} X_j) \to F(X_k)$ induces a bijection $\Phi : F(\coprod\limits_{j\in I} X_j) \to \prod\limits_{j\in I} F(X_j)$.

4) Suppose that J is the canonical topology on $\mathcal{C}$. Then, the Yoneda embedding $h : \mathcal{C} \to \widehat{\mathcal{C}}$ takes values in $\widetilde{\mathcal{C}}_J$. Let $\tilde{h} : \mathcal{C} \to \widetilde{\mathcal{C}}_J$ be the functor such that $h = i\tilde{h}$, where $i : \widetilde{\mathcal{C}}_J \to \widehat{\mathcal{C}}$ is the inclusion functor. If a family $(X_j)_{j\in I}$ of objects of $\mathcal{C}$ satisfies the condition of 3), the morphism $\coprod\limits_{j\in I} \tilde{h}(X_j) \to \tilde{h}(\coprod\limits_{j\in I} X_j)$ induced by $\tilde{h}(\iota_k)$ $(k \in I)$ is an isomorphism.

Proof. 1) By (2.2.4), it suffices to show that, for any morphism $f : Y \to 0$ with codomain 0 and any object Z, the map $\widehat{\mathcal{C}}(h_Y, h_Z) \to \widehat{\mathcal{C}}(h_f^{-1}(\emptyset), h_Z)$ induced by $h_f^{-1}(\emptyset) \hookrightarrow h_Y$ is bijective. Since every morphism $f : Y \to 0$ is an isomorphism, Y is an initial object of $\mathcal{C}$ and $\widehat{\mathcal{C}}(h_Y, h_Z) \cong \mathcal{C}(Y, Z)$ consists of a single element. On the other hand, since $h_f^{-1}(\emptyset)$ is the empty sieve on Y, $\widehat{\mathcal{C}}(h_f^{-1}(\emptyset), h_Z)$ also consists of a single element.

2) Since the empty sieve $\emptyset$ is a covering sieve on 0 for a topology finer than the canonical topology by 1), we have a bijection $F(0) \cong \widehat{\mathcal{C}}(h_0, F) \cong \widehat{\mathcal{C}}(\emptyset, F)$ and $\widehat{\mathcal{C}}(\emptyset, F)$ consists of a single element.

3) By the assumption, the map $\Phi : F(\coprod_{j\in I} X_j) \to \prod_{j\in I} F(X_j)$ induced by $F(\iota_k)$ is injective. For $(x_j)_{j\in I} \in \prod_{j\in I} F(X_j)$ and $j, k \in I$, let $f : Y \to X_j$ and $g : Y \to X_k$ be morphisms in $\mathcal{C}$ satisfying $\iota_j f = \iota_k g$. If $j \neq k$, there exists $z : Y \to 0$ such that $f = f'z$ and $g = g'z$, where $f' : 0 \to X_j$ and $g' : 0 \to X_k$ are the unique morphisms. Since $F(0)$ consists of a single element, $F(f')(x_j) = F(g')(x_k)$. Hence, $F(f)(x_j) = F(z)F(f')(x_j) = F(z)F(g')(x_k) = F(g)(x_k)$. If $j = k$, we have $f = g$ and obviously, $F(f)(x_j) = F(g)(x_k)$. It follows that Φ is surjective.

4) Let F be a sheaf on $\mathcal{C}$ for the canonical topology and $\Psi : \widetilde{\mathcal{C}}_J(\tilde{h}(\coprod_{j\in I} X_j), F) \to \prod_{j\in I} \widetilde{\mathcal{C}}_J(\tilde{h}(X_j), F)$ the map whose k-th component is $\tilde{h}(\iota_k)^* : \widetilde{\mathcal{C}}_J(\tilde{h}(\coprod_{j\in I} X_j), F) \to \widetilde{\mathcal{C}}_J(\tilde{h}(X_k), F)$. Then the assertion follows from 3) and the following commutative diagram, where the horizontal maps are bijections and Ψ' is a map whose k-th component is $h(\iota_k)^* : \widehat{\mathcal{C}}(h(\coprod_{k\in I} X_k), iF) \to \widehat{\mathcal{C}}(h(X_k), iF)$.

$$\begin{tikzcd}
\widetilde{\mathcal{C}}_J(\tilde{h}(\coprod_{j\in I} X_j), F) \arrow[r, "i"] \arrow[d, "\Psi"] & \widehat{\mathcal{C}}(h(\coprod_{j\in I} X_j), iF) \arrow[r, "\cong"] \arrow[d, "\Psi'"] & F(\coprod_{j\in I} X_j) \arrow[d, "\Phi"] \\
\prod_{j\in I} \widetilde{\mathcal{C}}_J(\tilde{h}(X_j), F) \arrow[r, "\prod i"] & \prod_{j\in I} \widehat{\mathcal{C}}(h(X_j), iF) \arrow[r, "\cong"] & \prod_{j\in I} F(X_j)
\end{tikzcd}$$

□

We note that if the coproduct $\coprod_{i\in I} X_i$ is universal, the family $(\iota_j : X_j \to \coprod_{i\in I} X_i)_{j\in I}$ of the canonical morphisms is a universal strict epimorphic family by (2.2.7). In this case $(X_i)_{i\in I}$ satisfies the condition of (2.2.9), 3).

Proposition 2.2.10 *Let $\mathcal{C}$ be a $\mathcal{U}$-category and $(\mathcal{C}, J)$ a site, then the category of sheaves $\widetilde{\mathcal{C}}$ is $\mathcal{U}$-complete and the inclusion functor $i : \widetilde{\mathcal{C}} \to \widehat{\mathcal{C}}$ creates limits.*

Proof. For a $\mathcal{U}$-small category I and functor $D : I \to \widetilde{\mathcal{C}}$, $\varprojlim_I iD$ exists in $\widehat{\mathcal{C}}$ and $\varprojlim_I iD$ is a sheaf. In fact, the canonical projections $\varprojlim_I iD \to iD(k)$ induces an bijection $\widehat{\mathcal{C}}(G, \varprojlim_I iD) \to \varprojlim_I \widehat{\mathcal{C}}(G, iD(k))$ which is natural in $G \in \widehat{\mathcal{C}}$. Since $D(k)$ is a sheaf for each $k \in I$, the restriction map $\widehat{\mathcal{C}}(h_X, iD(k)) \to \widehat{\mathcal{C}}(R, iD(k))$ is bijective for any $x \in \mathcal{C}$ and $R \in J(X)$. Hence $\widehat{\mathcal{C}}(h_X, \varprojlim_I iD) \to \widehat{\mathcal{C}}(R, \varprojlim_I iD)$ is bijective. □

Definition 2.2.11 *Let $\mathcal{C}$ be a regular category (A.8.1). For each object X of $\mathcal{C}$, we set $P(X) = \{(f : Y \to X) | f$ is a regular epimorphism$\}$. Then P is a pretopology by R3 and (A.8.7). We call the topology generated by P the regular epimorphism topology on $\mathcal{C}$.*

Proposition 2.2.12 *1) Let $\mathcal{C}$ be a regular category. Then, the regular epimorphism topology on $\mathcal{C}$ is coarser than the canonical topology.*

2) Let J be a topology on a category $\mathcal{C}$ coarser than the canonical topology. If every regular epimorphism in $\mathcal{C}$ is a covering for J, the Yoneda embedding $h : \mathcal{C} \to \widehat{\mathcal{C}}$ defines a fully faithful exact functor $\tilde{h} : \mathcal{C} \to \widetilde{\mathcal{C}}_J$ (A.8.15). In particular, if every regular epimorphism in $\mathcal{C}$ has a kernel pair, $\tilde{h}$ preserves regular epimorphisms.

Proof. 1) For $Y \in \operatorname{Ob}\mathcal{C}$ and a regular epimorphism $p : X \to Y$, a kernel pair $Z \rightrightarrows X$ exists and $Z \rightrightarrows X \xrightarrow{p} Y$ is exact by (A.8.14). Hence for any $W \in \operatorname{Ob}\mathcal{C}$, $h_W(Y) \xrightarrow{h_W(p)} h_W(X) \rightrightarrows h_W(Z)$ is an equalizer. It follows from (2.2.3) that each representable functor is a sheaf for the regular epimorphism topology and the regular epimorphism topology on $\mathcal{C}$ is coarser than the canonical topology.

2) Since J is coarser than the canonical topology, $h : \mathcal{C} \to \widehat{\mathcal{C}}$ takes values in $\widetilde{\mathcal{C}}_J$ and $h = i\tilde{h}$ for a unique functor $\tilde{h} : \mathcal{C} \to \widetilde{\mathcal{C}}_J$, where $i : \widetilde{\mathcal{C}}_J \to \widehat{\mathcal{C}}$ is the inclusion functtor. Since h is fully faithful and $\widetilde{\mathcal{C}}_J$ is a full subcategory of $\widehat{\mathcal{C}}$, $\tilde{h}$ is fully faithful. Since h preserves limits, so does $\tilde{h}$ by (2.2.10).

Suppose that $Z \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{p} Y$ is exact in $\mathcal{C}$. $\tilde{h}(Z) \overset{\tilde{h}(f)}{\underset{\tilde{h}(p)}{\rightrightarrows}} \tilde{h}(Y)$ is exact in $\widetilde{\mathcal{C}}_J$ if and only if for any sheaf F,

$$\widetilde{\mathcal{C}}_J(\tilde{h}(Y), F) \xrightarrow{\tilde{h}(p)^*} \widetilde{\mathcal{C}}_J(\tilde{h}(X), F) \overset{\tilde{h}(f)^*}{\underset{\tilde{h}(g)^*}{\rightrightarrows}} \widetilde{\mathcal{C}}_J(\tilde{h}(Z), F)$$

is an equalizer. But this diagram is isomorphic to $F(Y) \xrightarrow{F(p)} F(X) \overset{F(f)}{\underset{F(g)}{\rightrightarrows}} F(Z)$, which is an equalizer, for p tis a covering. □

2.3 The sheaf associated with a presheaf

Definition 2.3.1 *Let $(\mathcal{C}, J)$ be a site. A set G of objects of $\mathcal{C}$ is said to be a topologically generating family if for any object X of $\mathcal{C}$, there exists a covering of X consisting of morphisms with domain belonging to G.*

Definition 2.3.2 *Let $\mathcal{U}$ be a universe.*

1) A site $(\mathcal{C}, J)$ is called a $\mathcal{U}$-site if $\mathcal{C}$ is a $\mathcal{U}$-category and there is a ($\mathcal{U}$-)small topologically generating family.

2) A topology J on a $\mathcal{U}$-category $\mathcal{C}$ is called a $\mathcal{U}$-topology if $(\mathcal{C}, J)$ is a $\mathcal{U}$-site.

3) We say that a site $(\mathcal{C}, J)$ is $\mathcal{U}$-small if $\mathcal{C}$ is $\mathcal{U}$-small.

Proposition 2.3.3 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site and G a $\mathcal{U}$-small topologically generating family of $\mathcal{C}$. For $X \in \mathrm{Ob}\,\mathcal{C}$, we set $J_G(X) = \{R \in J(X) | R = \bar{S}$ for some covering $S = (X_i \to X)_{i \in I}$ with $X_i \in G$ for any $i \in I\}$.*

1) $J_G(X)$ is $\mathcal{U}$-small.

2) $J_G(X)$ is cofinal in $J(X)$, that is, for any $R \in J(X)$, there exists $S \in J_G(X)$ such that $S \subset R$.

3) For any $R \in J_G(X)$, there exists a $\mathcal{U}$-small epimorphic family $(u_i : h_{Y_i} \to R)_{i \in I}$ with $Y_i \in G$. Hence $\widehat{\mathcal{C}}(R, F)$ is $\mathcal{U}$-small for any presheaf F.

Proof. 1) Put $A(X) = \coprod_{Y \in G} \mathcal{C}(Y, X)$. Then, $A(X)$ is $\mathcal{U}$-small and $\mathrm{card}(J_G(X)) \leqq 2^{\mathrm{card}(A(X))}$. Thus $J_G(X)$ is $\mathcal{U}$-small.

2) For $R \in J(X)$, put $A(R) = \coprod_{Y \in G} \widehat{\mathcal{C}}(h_Y, R)$ and $S = \{u(id_Y) : Y \to X | u : h_Y \to R \in A(R)\}$. Then $\bar{S} \subset R$ and it suffices to show $\bar{S} \in J(X)$. For any object Z and $f \in R(Z)$, we show that $h_f^{-1}(\bar{S}) \in J(Z)$. By assumption, there exists a covering $T = (Z_i \to Z)_{i \in I}$ such that $Z_i \in G$. Then $\bar{T} \subset h_f^{-1}(\bar{S})$ and this implies $h_f^{-1}(\bar{S}) \in J(Z)$ by $(T4)$.

3) If $R \in J_G(X)$, $A(R) = (u : h_Y \to R)$ is an epimorphic family. Since $A(R) \subset A(X)$ and $A(X)$ is small, $A(R)$ is small. □

Let $(\mathcal{C}, J)$ be a site where $\mathcal{C}$ is a $\mathcal{U}$-category, $\mathcal{V}$ a universe such that $\mathcal{C} \in \mathcal{V}$ and $\mathcal{U} \subset \mathcal{V}$. Then the category $\widehat{\mathcal{C}}_{\mathcal{U}}$ of presheaves of $\mathcal{U}$-sets on $\mathcal{C}$ is a $\mathcal{V}$-category and $J(X)$ is $\mathcal{V}$-small for each object X.

Let F be a presheaf on $\mathcal{C}$. For an object X of $\mathcal{C}$, $(J(X), \supset)$ is a directed set and if $R \supset S$ for $R, S \in J(X)$, the restriction map $\rho_S^R : \widehat{\mathcal{C}}_{\mathcal{U}}(R, F) \to \widehat{\mathcal{C}}_{\mathcal{U}}(S, F)$ defines an inductive system of $\mathcal{V}$-small sets. We put $LF(X) = \varinjlim_{R \in J(X)} \widehat{\mathcal{C}}_{\mathcal{U}}(R, F)$ and $\rho^R : \widehat{\mathcal{C}}_{\mathcal{U}}(R, F) \to LF(X)$ denotes the canonical map. Note that $LF(X)$ is a $\mathcal{V}$-set and that if $(\mathcal{C}, J)$ is a $\mathcal{U}$-site, it follows from (2.3.3) that $LF(X)$ is $\mathcal{U}$-set.

If $f : Y \to X$ is a morphism of $\mathcal{C}$ and $R \in J(X)$, the restriction $f_\sharp : h_f^{-1}(R) \to R$ of $h_f : h_Y \to h_X$ to $h_f^{-1}(R)$ induces $LF(f) : LF(X) \to LF(Y)$ such that

$$\begin{tikzcd}
\widehat{\mathcal{C}}_{\mathcal{U}}(R, F) \arrow[r, "(f_\sharp)^*"] \arrow[d, "\rho^R"] & \widehat{\mathcal{C}}_{\mathcal{U}}(h_f^{-1}(R), F) \arrow[d, "\rho^{h_f^{-1}(R)}"] \\
LF(X) \arrow[r, "LF(f)"] & LF(Y)
\end{tikzcd}$$

commutes. Thus we have a presheaf LF of $\mathcal{V}$-set and $LF \in \widehat{\mathcal{C}}_{\mathcal{U}}$ if $(\mathcal{C}, J)$ is a $\mathcal{U}$-site.

If $\alpha : F \to G$ is a morphism of $\widehat{\mathcal{C}}_{\mathcal{U}}$, $\alpha_* : \widehat{\mathcal{C}}_{\mathcal{U}}(R, F) \to \widehat{\mathcal{C}}_{\mathcal{U}}(R, G)$ defines a morphism of inductive systems. Hence this defines a morphism of presheaves $L(\alpha) : LF \to LG$ and $F \mapsto LF$ gives a functor $L : \widehat{\mathcal{C}}_{\mathcal{U}} \to \widehat{\mathcal{C}}_{\mathcal{V}}$. If $(\mathcal{C}, J)$ is a $\mathcal{U}$-site, L is regarded as a functor $L : \widehat{\mathcal{C}}_{\mathcal{U}} \to \widehat{\mathcal{C}}_{\mathcal{U}}$.

Let $\theta_F : F(X) \to \widehat{\mathcal{C}}_{\mathcal{U}}(h_X, F)$ denote the natural bijection in (A.1.6). Since $h_X \in J(X)$, the composite $F(X) \xrightarrow{\theta_F} \widehat{\mathcal{C}}_{\mathcal{U}}(h_X, F) \xrightarrow{\rho^{h_X}} LF(X)$ defines a morphism $\ell(F) : F \to LF$ of $\widehat{\mathcal{C}}_{\mathcal{V}}$. Clearly, $\ell(F)$ is natural in F, that is, we have a natural transformation $\ell : I \to L$, where $I : \widehat{\mathcal{C}}_{\mathcal{U}} \to \widehat{\mathcal{C}}_{\mathcal{V}}$ denotes the inclusion functor. If $(\mathcal{C}, J)$ is a $\mathcal{U}$-site, ℓ is regarded as a natural transformation $\ell : id_{\widehat{\mathcal{C}}_{\mathcal{U}}} \to L$

Define a map $Z_R : \widehat{\mathcal{C}}_{\mathcal{U}}(R, F) \to \widehat{\mathcal{C}}_{\mathcal{V}}(h_X, LF)$ for $R \in J(X)$ to be the composite $\widehat{\mathcal{C}}_{\mathcal{U}}(R, F) \xrightarrow{\rho^R} LF(X) \xrightarrow{\theta_{LF}} \widehat{\mathcal{C}}_{\mathcal{V}}(h_X, LF)$. Then, for a morphism $f : Y \to X$ of $\mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd}
\widehat{\mathcal{C}}_{\mathcal{U}}(R, F) \arrow[r, "Z_R"] \arrow[d, "(f_\sharp)^*"] & \widehat{\mathcal{C}}_{\mathcal{V}}(h_X, LF) \arrow[d, "h_f^*"] \\
\widehat{\mathcal{C}}_{\mathcal{U}}(h_f^{-1}(R), F) \arrow[r, "Z_{h_f^{-1}(R)}"] & \widehat{\mathcal{C}}_{\mathcal{V}}(h_Y, LF)
\end{tikzcd}$$

Lemma 2.3.4 *1) For any $R \in J(X)$ and $u \in \widehat{\mathcal{C}}_{\mathcal{U}}(R, F)$, the diagram*

$$\begin{tikzcd}
R \arrow[r, "\iota_R"] \arrow[d, "u"] & h_X \arrow[d, "Z_R(u)"] \\
F \arrow[r, "\ell(F)"] & LF
\end{tikzcd}$$

is commutative, where $\iota_R : R \to h_X$ is the inclusion morphism.

2) For any morphism $v : h_X \to LF$, there exist $R \in J(X)$ and a morphism $u : R \to F$ such that $Z_R(u) = v$.

3) For $R \in J(X)$ and $u, v : R \to F$ morphisms satisfying $\ell(F)u = \ell(F)v$, the equalizer of u and v belongs to $J(X)$.

4) Suppose that $R, S \in J(X)$ and $u : R \to F$, $v : S \to F$ are morphisms. Then $Z_R(u) = Z_S(v)$ if and only if $\rho^R_T(u) = \rho^S_T(v)$ for some $T \subset R \cap S$.

Proof. 1) For $f \in \mathcal{C}(Y, X)$, $Z_R(u)_Y(f) = LF(f)(\rho^R(u)) = \rho^{h_f^{-1}(R)}(uf_\sharp)$ by definition. If $f \in R(Y)$, then $h_f^{-1}(R) = h_Y$ hence $\rho^{h_f^{-1}(R)}(uf_\sharp) = \rho^{h_Y}\theta_F(u_Y f_{\sharp Y}(id_Y)) = \ell(F)_Y(u_Y f_{\sharp Y}(id_Y)) = \ell(F)_Y u_Y(f)$.

2) There exist $R \in J(X)$ and $u \in \widehat{\mathcal{C}}_{\mathcal{U}}(R, F)$ such that $\rho^R(u) = v(id_X)$, then $Z_R(u) = v$.

3) For any $Y \in \mathrm{Ob}\mathcal{C}$ and $f \in R(Y)$, since $\rho^{h_Y}\theta_F u_Y(f) = \rho^{h_Y}\theta_F v_Y(f)$ by the assumption, there exists $R_f \in J(Y)$ such that $\theta_F(u_Y(f))\iota_{R_f} = \theta_F(v_Y(f))\iota_{R_f} : R_f \to F$, namely, $u\theta_R(f)\iota_{R_f} = v\theta_R(f)\iota_{R_f}$ where $\iota_{R_f} : R_f \to h_Y$ is the inclusion functor. Define a subfunctor S of h_X by $S(Z) = \{fg | f \in R(Y),\ g \in R_f(Z)\}$, then $S \in J(X)$ by (2.1.4) and S is a subfunctor of R. It follows that $\rho^R_S(u) = \rho^R_S(v)$, which means that S is a subfunctor of the equalizer of u and v. Then the result follows from $(T4)$.

4) Since $Z_R(u) = Z_S(v)$ if and only if $\rho^R(u) = \rho^S(v)$, the assertion is obvious. □

Proposition 2.3.5 *1) $L : \widehat{\mathcal{C}}_{\mathcal{U}} \to \widehat{\mathcal{C}}_{\mathcal{V}}$ is left exact.*

2) For any presheaf F, LF is a separated presheaf.

3) A presheaf F is separated if and only if $\ell(F) : F \to LF$ is a monomorphism. In this case LF is a sheaf.

4) F is a sheaf if and only if $\ell(F) : F \to LF$ is an isomorphism.

5) Let F, G be presheaves and $f : F \to G$ a morphism. If $g : LF \to LG$ is a morphism satisfying $g\ell(F) = \ell(G)f$, then $g = L(f)$. Hence, if G is a sheaf, there exists a unique morphism $\bar{f} : LF \to G$ such that $\bar{f}\ell(F) = f$.

Proof. 1) It suffices to show that for each object X, a functor $F \mapsto LF(X)$ commutes with finite limits. Fixing $R \in J(X)$, a functor $F \mapsto \widehat{\mathcal{C}}_{\mathcal{U}}(R, F)$ commutes with limits. Since filtered colimits commutes with finite limits in the category of sets, the functor mentioned above commutes with finite limits.

2) Let $f, g : h_X \to LF$ be morphisms such that $f\iota_R = g\iota_R$ for some $R \in J(X)$. We can choose $S \in J(X)$ and $u, v \in \widehat{\mathcal{C}}_{\mathcal{U}}(S, F)$ such that $S \subset R$ and $Z_S(u) = f$, $Z_S(v) = g$ by (2.3.4), 2). By (2.3.4), 1) and the assumption, $\ell(F)u = \ell(F)v$. It follows from (2.3.4), 3) that there exists $T \in J(X)$ contained in S such that the restriction w of u to T coincides with that of v. Hence $f = Z_S(u) = Z_T(w) = Z_S(v) = g$.

3) If F is separated, the restriction map $\rho^{h_X}_R : \widehat{\mathcal{C}}_{\mathcal{U}}(h_X, F) \to \widehat{\mathcal{C}}_{\mathcal{U}}(R, F)$ is injective for any $R \in J(X)$. Hence $\rho^{h_X} : \widehat{\mathcal{C}}_{\mathcal{U}}(h_X, F) \to LF(X)$ is injective, that is, $\ell(F)_X : F(X) \to LF(X)$ is injective for any object X. Conversely, if $\ell(F)$ is a monomorphism, F is a sub-presheaf of a separated presheaf LF. Therefore F is separated.

Suppose that F is separated. For $R \in J(X)$, let $u : R \to LF$ be a morphism. Consider the pull-back $R' = F \times_{LF} R$ of u along $\ell(F)$. Since $\ell(F)$ is a monomorphism, R' is regarded as a subfunctor of R and we show that $R' \in J(X)$. For $(f : Y \to X) \in R$, put $\bar{f} = \theta_X(f) : h_Y \to R$, then there exist $S \in J(Y)$ and $g : S \to F$ such that $Z_S(g) = u\bar{f}$ by (2.3.4), 2). The both squares of the diagram below are pull-back squares, it follows from (2.3.4), 1) that there exists $k : S \to h_f^{-1}(R')$ satisfying $jk = \iota_S$ and $u'\tilde{f}k = g$.

$$\begin{tikzcd}
h_f^{-1}(R') \arrow[r, "\tilde{f}"] \arrow[d, "j"] & R' \arrow[r, "u'"] \arrow[d, "i"] & F \arrow[d, "\ell(F)"] \\
h_Y \arrow[r, "\bar{f}"] & R \arrow[r, "u"] & LF
\end{tikzcd}$$

Since $j : h_f^{-1}(R') \to h_Y$ and $\iota_S : S \to h_Y$ are monomorphisms, $h_f^{-1}(R')$ is regarded as a subfunctor of h_Y containing S. Thus we have $h_f^{-1}(R') \in J(Y)$ by $(T4)$ and $R' \in J(X)$ by $(T3)$. We put $v = Z_{R'}(u') : h_X \to LF$, then it suffices to show that $u = v\iota_R$. For $f \in R(Y)$ as above, then $v\iota_R i = v\iota_{R'} = \ell(F)u' = ui$ by (2.3.4), 1). Hence we have $v\iota_R\bar{f}j = v\iota_R i\tilde{f} = ui\tilde{f} = u\bar{f}j$. Since j is a monomorphism and LF is separated, $v\iota_R\bar{f} = u\bar{f}$. Therefore $v\iota_R(f) = u(f)$ for any $f \in R$.

4) If F is a sheaf, the restriction map $\rho_R^{h_X} : \widehat{\mathcal{C}}_{\mathcal{U}}(h_X, F) \to \widehat{\mathcal{C}}_{\mathcal{U}}(R, F)$ is bijective for any object X and $R \in J(X)$. Hence $\ell(F) : F \to LF$ is an isomorphism. Conversely, if $\ell(F) : F \to LF$ is an isomorphism, F is separated by 2). Then, F is a sheaf by 3).

5) It suffices to show that $gv = L(f)v$ for any morphism $v : h_X \to LF$. By 2) of (2.3.4), there exist $R \in J(X)$ and a morphism $u : R \to F$ such that $v = Z_R(u)$. Then, $gv\iota_R = g\ell(F)u = \ell(G)fu = L(f)\ell(F)u = L(f)v\iota_R$ by (2.3.4), 1). Since LG is separated, we have $gv = L(f)v$. □

Corollary 2.3.6 *Let $\mathcal{C}$ be a $\mathcal{U}$-category and $(\mathcal{C}, J)$ a site.*

1) A presheaf F on $\mathcal{C}$ is a sheaf if and only if, for each object X of $\mathcal{C}$, there exists a cofinal subset $J'(X)$ of $J(X)$ such that the map $\widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(R, F)$ induced by the inclusion morphism $R \to h_X$ is bijective for any $R \in J'(X)$.

2) If G is a topologically generating family, a presheaf F on $\mathcal{C}$ is a sheaf if and only if, for each object X of $\mathcal{C}$ and $R \in J_G(X)$, the map $\widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(R, F)$ induced by the inclusion morphism $R \to h_X$ is bijective.

3) If G is a topologically generating family, a presheaf F on $\mathcal{C}$ is a sheaf if and only if, for each object X of $\mathcal{C}$ and a covering $(f_i : X_i \to X)_{i \in I}$ such that $X_i \in G$, $(F(f_i))_{i \in I} : F(X) \to \prod_{i \in I} F(X_i)$ is injective and its image is $\{(x_i)_{i \in I} \in \prod_{i \in I} F(X_i) |$ For any $i, j \in I$ and morphisms $g : Z \to X_i$, $h : Z \to X_j$ such that $f_i g = f_j h$, $F(g)(x_i) = F(h)(x_j)$ holds$\}$.

Proof. 1) is a direct consequence of 4) of the previous result. 2) follows from 1) and (2.3.3), and 3) follows from 2) and (2.2.2). □

Theorem 2.3.7 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site. The inclusion functor $i : \widetilde{\mathcal{C}}_{\mathcal{U}} \to \widehat{\mathcal{C}}_{\mathcal{U}}$ has a left adjoint $a : \widehat{\mathcal{C}}_{\mathcal{U}} \to \widetilde{\mathcal{C}}_{\mathcal{U}}$ which is left exact. The functor $ia : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ is canonically isomorphic to LL and the unit of the adjunction $\eta : id_{\widehat{\mathcal{C}}_{\mathcal{U}}} \to ia$ is given by the composition $\ell(LF)\ell(F) : F \to LLF$. In particular, $\widetilde{\mathcal{C}}_{\mathcal{U}}$ is a reflexive subcategory of $\widehat{\mathcal{C}}_{\mathcal{U}}$.*

Proof. Since LLF is a sheaf for any presheaf F by the previous result, define $a : \widehat{\mathcal{C}}_{\mathcal{U}} \to \widetilde{\mathcal{C}}_{\mathcal{U}}$ by $aF = LLF$. Then, η_F is an isomorphism if and only if F is a sheaf. If F is a sheaf and $f : G \to F$ is a morphism of presheaves, $\bar{f} : aG \to F$ defined by $i(\bar{f}) = \eta_F^{-1} ia(f)$ is the unique morphism satisfying $i(\bar{f})\eta_G = f$. In fact, the uniqueness follows from (2.3.5), 5). Hence a is a left adjoint of i and the unit is η defined above. Since L is left exact, so is a. □

If J is not necessarily a $\mathcal{U}$-topology on a $\mathcal{U}$-category $\mathcal{C}$, take a universe $\mathcal{V}$ such that $\mathcal{C} \in \mathcal{V}$ and $\mathcal{U} \subset \mathcal{V}$. Then, $LF \in \operatorname{Ob} \widehat{\mathcal{C}}_{\mathcal{V}}$ and since $J(X)$ is $\mathcal{V}$-small, $LLF \in \operatorname{Ob} \widetilde{\mathcal{C}}_{\mathcal{V}}$. Thus we have a left exact functor $a : \widehat{\mathcal{C}}_{\mathcal{U}} \to \widetilde{\mathcal{C}}_{\mathcal{V}}$ defined by $aF = LLF$.

Definition 2.3.8 *The sheaf $aF \in \widetilde{\mathcal{C}}_{\mathcal{V}}$ is called the sheaf associated with F and we call $a : \widehat{\mathcal{C}}_{\mathcal{U}} \to \widetilde{\mathcal{C}}_{\mathcal{V}}$ the associated sheaf functor. If $(\mathcal{C}, J)$ is a $\mathcal{U}$-site, the associated sheaf functor is regarded as a functor $a : \widehat{\mathcal{C}}_{\mathcal{U}} \to \widetilde{\mathcal{C}}_{\mathcal{U}}$.*

The next result follows from the construction of the associated sheaf functor.

Proposition 2.3.9 *Let $\mathcal{U}$, $\mathcal{V}$ be universes such that $\mathcal{U} \subset \mathcal{V}$ and $(\mathcal{C}, J)$ a $\mathcal{U}$-site. Regard $(\mathcal{C}, J)$ as a $\mathcal{V}$-site. We denote by $a_{\mathcal{U}} : \widehat{\mathcal{C}}_{\mathcal{U}} \to \widetilde{\mathcal{C}}_{\mathcal{U}}$ and $a_{\mathcal{V}} : \widehat{\mathcal{C}}_{\mathcal{V}} \to \widetilde{\mathcal{C}}_{\mathcal{V}}$ the associated sheaf functors. Then the following diagram commutes up to natural equivalence, where the vertical arrows are the inclusion functors.*

$$\begin{tikzcd} \widehat{\mathcal{C}}_{\mathcal{U}} \arrow[r, "a_{\mathcal{U}}"] \arrow[d] & \widetilde{\mathcal{C}}_{\mathcal{U}} \arrow[d] \\ \widehat{\mathcal{C}}_{\mathcal{V}} \arrow[r, "a_{\mathcal{V}}"] & \widetilde{\mathcal{C}}_{\mathcal{V}} \end{tikzcd}$$

Proposition 2.3.10 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site. Then, J is coarser than the canonical topology if and only if $\mathcal{C} \xrightarrow{h} \widehat{\mathcal{C}} \xrightarrow{a} \widetilde{\mathcal{C}}$ is fully faithful.*

Proof. Suppose that J is sub-canonical. Then, h_X is a sheaf for any $X \in \operatorname{Ob}\mathcal{C}$ and h induces a functor $\tilde{h} : \mathcal{C} \to \widetilde{\mathcal{C}}$ such that $h = i\tilde{h}$. Since i and h are fully faithful, so is $\tilde{h}$. The counit $\varepsilon : ai \to id_{\widetilde{\mathcal{C}}}$ is an equivalence and this gives an equivalence $ah = ai\tilde{h} \xrightarrow{\cong} \tilde{h}$. Thus ah is fully faithful.

Conversely, suppose that ah is fully faithful. Let X and Y be objects of $\mathcal{C}$. Since the Yoneda embedding h is fully faithful, the assumption implies that $a : \widehat{\mathcal{C}}(h_Y, h_X) \to \widetilde{\mathcal{C}}(ah_Y, ah_X)$ is bijective. The composite $\widehat{\mathcal{C}}(h_Y, h_X) \xrightarrow{a} \widetilde{\mathcal{C}}(ah_Y, ah_X) \xrightarrow{\cong} \widehat{\mathcal{C}}(h_Y, iah_X)$ is induced by the unit $\eta_{h_X} : h_X \to iah_X$ and it is bijective for any $Y \in \operatorname{Ob}\mathcal{C}$, fixing $X \in \operatorname{Ob}\mathcal{C}$. Regarding $\mathcal{C}$ as a full subcategory of $\widehat{\mathcal{C}}$ by h, it follows from (A.4.2) that $\mathcal{C}$ is a generating subcategory by strict epimorphisms. Hence η_{h_X} is an isomorphism by (A.4.10) and h_X is a sheaf for J. $\square$

2.4 Properties of the category of sheaves

Theorem 2.4.1 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site, $\widetilde{\mathcal{C}}$ the category of sheaves and $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}$ the associated sheaf functor.*

1) The associated sheaf functor preserves colimits and it is left exact.

2) $\widetilde{\mathcal{C}}$ is $\mathcal{U}$-cocomplete. In fact, for any $\mathcal{U}$-small category I and functor $D : I \to \widetilde{\mathcal{C}}$, if $(iD(j) \xrightarrow{\iota_j} \varinjlim_I iD)_{j \in \operatorname{Ob} I}$ is a colimiting cone of iD in $\widehat{\mathcal{C}}$, $(D(j) \xrightarrow{a(\iota_j)\varepsilon^{-1}} a(\varinjlim_I iD))_{j \in \operatorname{Ob} I}$ is a colimiting cone of D in $\widetilde{\mathcal{C}}$, where $\varepsilon : ai \to id_{\widetilde{\mathcal{C}}}$ is the counit of the adjunction.

3) $\mathcal{U}$-small filtered colimits in $\widetilde{\mathcal{C}}$ commute with finite limits.

Proof. 1) Since a has a right adjoint i, a preserves colimits. We have already shown in (2.3.7) that a is left exact.

2) Since $\widehat{\mathcal{C}}$ is $\mathcal{U}$-cocomplete, so is $\widetilde{\mathcal{C}}$ by (2.4.2) below.

3) Let $\mathcal{M}$ be a filtered category and $\mathcal{N}$ a finite category. It follows from (A.4.4) and (A.4.1) that filtered colimits in $\widehat{\mathcal{C}}$ indexed by $\mathcal{U}$-set commute with finite limits. Hence, for a functor $D : \mathcal{M} \times \mathcal{N} \to \widetilde{\mathcal{C}}$, the canonical morphism $\kappa : \varinjlim_m \varprojlim_n iD(m,n) \to \varprojlim_n \varinjlim_m iD(m,n)$ is an isomorphism in $\widehat{\mathcal{C}}$. By 2), we have natural isomorphisms

$$a(\varinjlim_m \varprojlim_n iD(m,n)) = a(\varinjlim_m i(\varprojlim_n D(m,n))) \cong \varinjlim_m \varprojlim_n D(m,n),$$
$$a(\varprojlim_n \varinjlim_m iD(m,n)) \cong \varprojlim_n a(\varinjlim_m iD(m,n)) \cong \varprojlim_n \varinjlim_m D(m,n).$$

Apply the associated functor to κ, the canonical morphism $\kappa : \varinjlim_m \varprojlim_n D(m,n) \to \varprojlim_n \varinjlim_m D(m,n)$ is an isomorphism in $\widetilde{\mathcal{C}}$. $\square$

Proposition 2.4.2 *Let $\mathcal{C}$ be a category and $\mathcal{D}$ a reflexive full subcategory of $\mathcal{C}$, namely, the inclusion functor $i : \mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint $L : \mathcal{C} \to \mathcal{D}$.*

1) If $\mathcal{C}$ is $\mathcal{U}$-cocomplete, so is $\mathcal{D}$.

2) If $\mathcal{C}$ is finitely complete and L is left exact, $\mathcal{D}$ is finitely complete.

Proof. By the assumption, the counit $\varepsilon : Li \to id_{\mathcal{D}}$ is a natural equivalence. For any $\mathcal{U}$-small (resp. finite) category I and functor $D : I \to \mathcal{D}$, let $(iD(k) \xrightarrow{\iota_k} \varinjlim_I iD)_{k \in I}$ (resp. $(\varprojlim_I iD \xrightarrow{\pi_k} iD(k))_{k \in I}$) be the colimiting (resp. limiting) cone in $\mathcal{C}$, then it is easy to verify that $(D(k) \xrightarrow{L(\iota_k)\varepsilon_{D(k)}^{-1}} L(\varinjlim_I iD))_{k \in I}$ (resp. $(L(\varprojlim_I iD) \xrightarrow{L(\pi_k)\varepsilon_{D(k)}} D(k))_{k \in I}$) is a colimiting (resp. limiting) cone in $\mathcal{D}$. $\square$

Corollary 2.4.3 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site and F a sheaf on $\mathcal{C}$. For a morphism $f : h_X \to iF$ in $\widehat{\mathcal{C}}$, $\tilde{f} : ah_X \to F$ denotes the adjoint of f. $(ahP\langle X, f\rangle \xrightarrow{\tilde{f}} F)_{\langle X,f\rangle \in \mathrm{Ob}(h\downarrow iF)}$ is a colimiting cone.*

Proof. Applying $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}$ to the colimiting cone $(hP\langle X, f\rangle \xrightarrow{f} iF)_{\langle X,f\rangle \in \mathrm{Ob}(h\downarrow iF)}$ (A.4.2), we have a colimiting cone $(ahP\langle X, f\rangle \xrightarrow{a(f)} aiF)_{\langle X,f\rangle \in \mathrm{Ob}(h\downarrow iF)}$ by (2.4.1). Since the counit $\varepsilon_F : aiF \to F$ is an isomorphism and $\tilde{f} = \varepsilon_F a(f)$, the result follows. □

Lemma 2.4.4 *1) Let $f : A \to X$ and $g : A \to Y$ be maps of sets satisfying that $f(x) = f(y)$ implies $g(x) = g(y)$. Let Z be the quotient set of the disjoint union of X and Y modulo an equivalence relation $\sim$ generated by $f(x) \sim g(x)$ for $x \in A$. Then the composition of the inclusion $Y \hookrightarrow X \coprod Y$ and the quotient map $X \coprod Y \to Z$ is injective.*

2) Let

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle k} \\ H & \xrightarrow{j} & K \end{array}$$

be a cocartesian square in $\widehat{\mathcal{C}}$. If f is a monomorphism, so is j. Moreover, if both f and g are monomorphisms, the above square is cartesian.

Proof. By the assumption, g factors through $f : A \to f(A)$. Hence we may assume that f is the inclusion map $A \hookrightarrow X$. Then, for $y \in Y$, $y \sim x$ if and only if $y = x$ or $y = g(x)$ $(x \in A)$. Hence the restriction of the equivalence relation to Y is trivial. Therefore the composition is injective. Now, the first assertion of 2) follows from 1). If f and g are monomorphisms, the above square is isomorphic to the following square which is cartesian.

$$\begin{array}{ccc} k(G) \cap j(H) & \longrightarrow & k(G) \\ \downarrow & & \downarrow \\ j(H) & \longrightarrow & K \end{array}$$

□

Proposition 2.4.5 *Let $\mathcal{D}$ be a reflexive full subcategory of $\widehat{\mathcal{C}}$ with a left exact reflection $L : \widehat{\mathcal{C}} \to \mathcal{D}$. Then, a morphism in $\mathcal{D}$ which is a monomorphism and an epimorphism is an isomorphism. In particular, if $(\mathcal{C}, J)$ is a $\mathcal{U}$-site, a morphism in $\widetilde{\mathcal{C}}$ which is a monomorphism and an epimorphism is an isomorphism.*

Proof. Let $f : F \to G$ be a monomorphism and an epimorphism in $\mathcal{D}$. Since the inclusion functor $i : \mathcal{D} \hookrightarrow \widehat{\mathcal{C}}$ is left exact, f is a monomorphism in $\widehat{\mathcal{C}}$. Consider the cocartesian square

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle k} \\ G & \xrightarrow{j} & H \end{array}$$

in $\widehat{\mathcal{C}}$. Since f is a monomorphism in $\widehat{\mathcal{C}}$, the left square is cartesian by (2.4.4). Applying the reflection to the above square, we have a cartesian and cocartesian square

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Lk} \\ G & \xrightarrow{Lj} & L(H) \end{array}$$

in $\mathcal{D}$. Since f is an epimorphism in $\mathcal{D}$, $f^* : \mathcal{D}(G, K) \to \mathcal{D}(F, K)$ is injective for any sheaf K. It follows from the following cartesian square that $(Lj)^*$ is bijective (A.3.2).

$$\begin{array}{ccc} \mathcal{D}(L(H), K) & \xrightarrow{(Lj)^*} & \mathcal{D}(G, K) \\ \downarrow{\scriptstyle (Lk)^*} & & \downarrow{\scriptstyle f^*} \\ \mathcal{D}(G, K) & \xrightarrow{f^*} & \mathcal{D}(F, K) \end{array}$$

Hence Lj is an isomorphism, and since f is a pull-back of Lj, f is also an isomorphism. $\square$

Proposition 2.4.6 *Let $\mathcal{D}$ be a reflexive full subcategory of $\widehat{\mathcal{C}}$ with a left exact reflection $L : \widehat{\mathcal{C}} \to \mathcal{D}$.*

1) Colimits indexed by $\mathcal{U}$-set exist in $\mathcal{D}$, which are universal.

2) An epimorphic family of morphisms of $\mathcal{D}$ is universal effective.

3) An equivalence relation in $\mathcal{D}$ is effective.

In particular, if $(\mathcal{C}, J)$ be a $\mathcal{U}$-site, the category of sheaves has the above properties.

Proof. 1) In (2.4.2), we have already shown the existence of colimits in $\mathcal{D}$ indexed by $\mathcal{U}$-set. Let $D : I \to \mathcal{D}$ be a functor. Suppose that a cone $(D(j) \xrightarrow{f_j} F)_{j \in \mathrm{Ob}\, I}$ and a morphism $f : G \to F$ are given and define a functor $D_F : I \to \mathcal{D}$ by $D_F(j) = D(j) \times_F G$. If $(iD(j) \xrightarrow{\iota_j} C)_{j \in \mathrm{Ob}\, I}$ is a colimiting cone of $iD : I \to \widehat{\mathcal{C}}$, $(iD(j)(X) \xrightarrow{\iota_j X} C(X))_{j \in \mathrm{Ob}\, I}$ is a colimiting cone of $E_X iD : I \to \mathcal{U}$-**Ens** by (A.4.1). Then, by (A.4.3), $(iD(j)(X) \times_{F(X)} G(X) \xrightarrow{\iota_j X \times id_{G(X)}} C(X) \times_{F(X)} G(X))_{j \in \mathrm{Ob}\, I}$ is a colimiting cone in $\mathcal{U}$-**Ens**. Again, by (A.4.1), $(iD(j) \times_F G \xrightarrow{\iota_j \times id_G} C \times_F G)_{j \in \mathrm{Ob}\, I}$ is a colimiting cone of $iD_F : I \to \widehat{\mathcal{C}}$. Since the reflection is left exact and the counit $Li \to id_{\mathcal{D}}$ is a natural equivalence, $(D(j) \times_F G \xrightarrow{\iota_j \times id_G} C \times_F G)_{j \in \mathrm{Ob}\, I}$ is a colimiting cone of $D_F : I \to \mathcal{D}$.

2) Recall that $\mathcal{D}$ is finitely complete (2.4.2). Let $(f_i : F_i \to F)_{i \in I}$ be an epimorphic family in $\mathcal{D}$. It suffices to show that the following assertions.

(i) For any morphism $g : G \to F$ in $\mathcal{D}$, the family of morphisms $(\bar{f}_i : F_i \times_F G \to G)_{i \in I}$ is an epimorphic family of $\mathcal{D}$, where $\bar{f}_i$ is the pull-back of f_i along g.

(ii) For any $H \in \mathrm{Ob}\, \mathcal{D}$, $\mathcal{D}(F, H) \to \prod\limits_{i \in I} \mathcal{D}(F_i, H) \rightrightarrows \prod\limits_{i,k \in I} \mathcal{D}(F_i \times_F F_k, H)$ is an equalizer.

Let $F' \in \widehat{\mathcal{C}}$ be the union of the images of f_i's, that is, F' is defined by $F'(X) = \bigcup_{i \in I} f_{iX}(F_i(X))$ for each $X \in \mathrm{Ob}\, \mathcal{C}$. We denote by $j : F' \hookrightarrow F$ the inclusion morphism. Then, we have morphisms $f'_i : F_i \to F'$ $(i \in I)$ such that $f_i = jf'_i$ and $(f'_i : F_i \to F')_{i \in I}$ is an epimorphic family in $\widehat{\mathcal{C}}$. Since j is a monomorphism in $\widehat{\mathcal{C}}$ and L is left exact, $L(j) : L(F') \to F$ is a monomorphism in $\mathcal{D}$. On the other hand, since $f_i = L(j)L(f'_i)$ for any $i \in I$ and $(f_i : F_i \to F)_{i \in I}$ is an epimorphic family in $\mathcal{D}$, $L(j)$ is an epimorphism, hence it is an isomorphism by (2.4.5). Consider the following cartesian diagrams.

$$\begin{tikzcd}
F_i \times_F G \arrow[r, "\bar{f}'_i"] \arrow[d] & F' \times_F G \arrow[r, "\bar{j}"] \arrow[d] & G \arrow[d, "g"] \\
F_i \arrow[r, "f'_i"] & F' \arrow[r, "j"] & F
\end{tikzcd}$$

Since $L : \widehat{\mathcal{C}} \to \mathcal{D}$ is left exact, $L(\bar{j})$ is an isomorphism. We claim that $(\bar{f}'_i : F_i \times_F G \to F' \times_F G)_{i \in I}$ is an epimorphic family in $\widehat{\mathcal{C}}$. In fact, for each $X \in \mathrm{Ob}\, \mathcal{C}$, since $F'(X) = \bigcup_{i \in I} f_{iX}(F_i(X))$ and $(x, y) \in F'(X) \times G(X)$ belongs to $(F' \times_F G)(X)$ if and only if $j(x) = g(y)$, we have $(F' \times_F G)(X) = \bigcup_{i \in I} \bar{f}'_i(F_i \times_F G)(X)$. Hence $(L(\bar{f}'_i) : F_i \times_F G \to L(F' \times_F G))_{i \in I}$ is an epimorphic family in $\mathcal{D}$ by (A.3.13) and this shows (i).

For $H \in \mathrm{Ob}\, \mathcal{D}$, $j^* : \mathcal{D}(F, H) = \widehat{\mathcal{C}}(F, H) \to \widehat{\mathcal{C}}(F', H)$ is bijective. Since $(f'_i : F_i \to F')_{i \in I}$ is an epimorphic family in $\widehat{\mathcal{C}}$,

$$\widehat{\mathcal{C}}(F', H) \to \prod_{i \in I} \widehat{\mathcal{C}}(F_i, H) \rightrightarrows \prod_{i,k \in I} \widehat{\mathcal{C}}(F_i \times_{F'} F_k, H)$$

is an equalizer. By (A.3.6), a monomorphism $j : F' \to F$ induces an isomorphism $F_i \times_{F'} F_k \to F_i \times_F F_k$ which commutes with the projections. Thus $\mathcal{D}(F, H) \to \prod\limits_{i \in I} \mathcal{D}(F_i, H) \rightrightarrows \prod\limits_{i,k \in I} \mathcal{D}(F_i \times F_k, H)$ is an equalizer.

3) Let $R \overset{p_1}{\underset{p_2}{\rightrightarrows}} F$ be an equivalence relation in $\mathcal{D}$. Regarding this as an equivalence relation in $\widehat{\mathcal{C}}$, let G be a presheaf defined by $G(X) = F(X)/R(X)$ (the quotient set of $F(X)$ by $R(X)$). Then, the following square is a cartesian and cocartesian square in $\widehat{\mathcal{C}}$.

$$\begin{tikzcd}
R \arrow[r, "p_1"] \arrow[d, "p_2"] & F \arrow[d, "q"] \\
F \arrow[r, "q"] & G
\end{tikzcd}$$

Apply the reflection L to this square, we have a cartesian and cocartesian square in $\mathcal{D}$. Therefore $R \underset{p_2}{\overset{p_1}{\rightrightarrows}} F$ is an effective equivalence relation. $\square$

Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site. We denote by $\epsilon_J : \mathcal{C} \to \widetilde{\mathcal{C}}$ the composition of functors $h : \mathcal{C} \to \widehat{\mathcal{C}}$ and $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}$. We call ϵ_J the canonical functor of $(\mathcal{C}, J)$.

Theorem 2.4.7 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site and $(f_i : X_i \to X)_{i\in I}$ be a family of morphisms in $\mathcal{C}$. Then, the following conditions are equivalent.*
i) $(\epsilon_J(f_i) : \epsilon_J(X_i) \to \epsilon_J(X))_{i\in I}$ is an epimorphic family in $\widetilde{\mathcal{C}}$.
ii) $(f_i : X_i \to X)_{i\in I}$ is a covering of X (2.1.8).

Proof. *ii)* $\Rightarrow$ *i)*: Let R be a sieve generated by $(f_i : X_i \to X)_{i\in I}$ and $f_i' : h_{X_i} \to R$ the morphism induced by $h_{f_i} : h_{X_i} \to h_X$. Then, $R \in J(X)$ and $(f_i' : h_{X_i} \to R)_{i\in I}$ is an epimorphic family in $\widehat{\mathcal{C}}$. Hence, for any sheaf F, the map $\widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(R, F)$ is bijective and $\widehat{\mathcal{C}}(R, F) \to \prod_{i\in I} \widehat{\mathcal{C}}(h_{X_i}, F)$ induced by f_i''s is injective. It follows that $\widehat{\mathcal{C}}(h_X, F) \to \prod_{i\in I} \widehat{\mathcal{C}}(h_{X_i}, F)$ induced by h_{f_i}'s is injective, thus $\widetilde{\mathcal{C}}(ah_X, F) \to \prod_{i\in I} \widetilde{\mathcal{C}}(ah_{X_i}, F)$ induced by $\epsilon_J(f_i)$'s is injective.

i) $\Rightarrow$ *ii)*: With the above notations, let $\iota_R : R \hookrightarrow h_X$ be the inclusion morphism. Since the upper horizontal map in the following commutative diagram is injective for any sheaf F, so is the left vertical arrow. Thus $a(\iota_R) : aR \to ah_X$ is an epimorphism in $\widetilde{\mathcal{C}}$.

$$\begin{tikzcd}
\widetilde{\mathcal{C}}(ah_X, F) \arrow[r, "(\epsilon_J(f_i)^*)"] \arrow[d, "a(\iota_R)^*"] & \prod_{i\in I} \widetilde{\mathcal{C}}(ah_{X_i}, F) \\
\widetilde{\mathcal{C}}(aR, F) \arrow[ur, "(a(f_i')^*)"'] &
\end{tikzcd}$$

On the other hand, since $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}$ is left exact hence preserves monomorphisms, $a(\iota_R)$ is a monomorphism. Therefore $a(\iota_R)$ is a isomorphism of sheaves by (2.4.5). We have the following commutative diagram.

$$\begin{tikzcd}
R \arrow[r, "\ell(R)"] \arrow[d, "\iota_R"] & LR \arrow[r, "\ell(LR)"] \arrow[d, "L(\iota_R)"] & aR \arrow[d, "a(\iota_R)"] \\
h_X \arrow[r, "\ell(h_X)"] & Lh_X \arrow[r, "\ell(Lh_X)"] & ah_X
\end{tikzcd}$$

Applying (2.3.4), 2) to $v = a(\iota_R)^{-1}\ell(Lh_X)\ell(h_X) : h_X \to aR = LLR$, there exist $S \in J(X)$ and a morphism $u : S \to LR$ such that
$$\ell(Lh_X)\ell(h_X)\iota_S = a(\iota_R)\ell(LR)u = \ell(Lh_X)L(\iota_R)u.$$
Let $e : T \to S$ be the equalizer of $\ell(h_X)\iota_S, L(\iota_R)u : S \to Lh_X$. Then, $T \in J(X)$ by (2.3.4), 3).

Let Z be an object of $\mathcal{C}$ and $\beta \in T(Z)$. Applying (2.3.4), 2) to $v = ueh_\beta : h_Z \to LR$, there exist $Q \in J(Z)$ and a morphism $w : Q \to R$ such that $ueh_\beta\iota_Q = \ell(R)w$. Let $f : P \to Q$ be the equalizer of $\iota_T h_\beta \iota_Q, \iota_R w : Q \to h_X$. Then, we have $\ell(h_X)\iota_T h_\beta\iota_Q = \ell(h_X)\iota_S e h_\beta \iota_Q = L(\iota_R)ueh_\beta\iota_Q = L(\iota_R)\ell(R)w = \ell(h_X)\iota_R w$ and by 3) of (2.3.4), $h_\alpha^{-1}(P) \in J(Y)$ and by $(T3)$, $P \in J(Z)$.

We claim that $P \subset h_\beta^{-1}(T\cap R)$ in h_Z. In fact, since $\beta \in T(Z)$, $P \subset h_\beta^{-1}(T)$ is obvious, and for any $\alpha \in P(Y)$, $h_\beta(\alpha) = w(\alpha) \in R(Y)$ by the construction of P. Therefore $h_\beta^{-1}(T \cap R) \in J(Z)$ by $(T4)$, hence $T \cap R \in J(X)$. Again, by $(T4)$, we have $R \in J(X)$. $\square$

Proposition 2.4.8 *Let $\mathcal{C}$ be a $\mathcal{U}$-category and $(s_i : X_i \to X)_{i\in I}$ a family of morphisms in $\mathcal{C}$. For a $\mathcal{U}$-topology J on $\mathcal{C}$, we denote by $\widetilde{\mathcal{C}}_J$ the category of sheaves associated with J and by $\epsilon_J : \mathcal{C} \to \widetilde{\mathcal{C}}_J$ the canonical functor.*
1) Let J be a $\mathcal{U}$-topology such that
$$\coprod_{i\in I} \epsilon_J(X_i) \xrightarrow{(\epsilon_J(s_i))} \epsilon_J(X)$$
is an isomorphism. If J' is a topology finer than J, then the following morphism is an isomorphism.
$$\coprod_{i\in I} \epsilon_{J'}(X_i) \xrightarrow{(\epsilon_{J'}(s_i))} \epsilon_{J'}(X)$$

2) Let J be a $\mathcal{U}$-topology on $\mathcal{C}$. The following conditions (i) and (ii) are equivalent.

(i) (a) $(s_i : X_i \to X)_{i\in I}$ *is a covering for* J.

(b) *For any* $i \in I$, *the morphism* $a(\Delta_i) : a(h_{X_i}) \to a(h_{X_i} \times_{h_X} h_{X_i})$ *induced by the diagonal morphism* $\Delta_i : h_{X_i} \to h_{X_i} \times_{h_X} h_{X_i}$ *is an isomorphism.*

(c) *If* $i, j \in I$ *and* $i \neq j$, $a(h_{X_i} \times_{h_X} h_{X_j})$ *is an initial object of* $\widetilde{\mathcal{C}}_J$.

(ii) $\epsilon_J(X)$ *is a coproduct of* $\epsilon_J(X_i)$.

Proof. 1) It suffices to show that for any object F of $\widetilde{\mathcal{C}}_{J'}$, the morphism $\widetilde{\mathcal{C}}_{J'}(\epsilon_{J'}(X), F) \to \prod_{i\in I} \widetilde{\mathcal{C}}_{J'}(\epsilon_{J'}(X_i), F)$ induced by $(\epsilon_{J'}(s_i))$ is an isomorphism. The following diagram commutes, where the vertical maps are the adjoint isomorphisms.

$$\begin{tikzcd}
\widetilde{\mathcal{C}}_{J'}(\epsilon_{J'}(X), F) \arrow[r, "(\epsilon_{J'}(s_i)^*)"] \arrow[d, "ad^{-1}"] & \prod_{i\in I} \widetilde{\mathcal{C}}_{J'}(\epsilon_{J'}(X_i), F) \arrow[d, "ad^{-1}"] \\
\widehat{\mathcal{C}}(h_X, F) \arrow[r, "(h_{s_i}^*)"] \arrow[d, "ad"] & \prod_{i\in I} \widehat{\mathcal{C}}(h_{X_i}, F) \arrow[d, "ad"] \\
\widetilde{\mathcal{C}}_J(\epsilon_J(X), F) \arrow[r, "(\epsilon_J(s_i)^*)"] & \prod_{i\in I} \widetilde{\mathcal{C}}_J(\epsilon_J(X_i), F)
\end{tikzcd}$$

Since F is an object of $\widetilde{\mathcal{C}}_J$ and the lower horizontal map is bijective by the assumption, so is the upper one.

2) Consider a morphism $\Phi = (h_{s_i})_{i\in I} : \coprod_{i\in I} h_{X_i} \to h_X$ in $\widehat{\mathcal{C}}$. Then, the condition (ii) holds if and only if $a(\Phi)$ is an isomorphism. Hence, by (2.4.5), (ii) holds if and only if $a(\Phi)$ is an epimorphism and monomorphism. It follows from (2.4.7) that $a(\Phi)$ is an epimorphism if and only if (a) holds.

Since $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}_J$ preserves finite limits, the kernel pair of $a(\Phi)$ is given by

$$a\Big(\Big(\coprod_{i\in I} h_{X_i}\Big) \times_{h_X} \Big(\coprod_{i\in I} h_{X_i}\Big)\Big) \underset{a(\mathrm{pr}_2)}{\overset{a(\mathrm{pr}_1)}{\rightrightarrows}} a\Big(\coprod_{i\in I} h_{X_i}\Big) \xrightarrow{a(\Phi)} ah_X.$$

Hence $a(\Phi)$ is a monomorphism if and only if $a(\Delta) : a\Big(\coprod_{i\in I} h_{X_i}\Big) \to a\Big(\Big(\coprod_{i\in I} h_{X_i}\Big) \times_{h_X} (\coprod_{i\in I} h_{X_i}\Big)\Big)$ is an isomorphism, where $\Delta : \coprod_{i\in I} h_{X_i} \to \Big(\coprod_{i\in I} h_{X_i}\Big) \times_{h_X} \Big(\coprod_{i\in I} h_{X_i}\Big)$ is the diagonal morphism . Δ is regarded the coproduct of $\Delta_{i,j}$ $(i, j \in I)$ defined as follows. $\Delta_{i,i} : h_{X_i} \to h_{X_i} \times_{h_X} h_{X_i}$ is the diagonal morphism. If $i \neq j$, $\Delta_{i,j} : 0_{\widehat{\mathcal{C}}} \to h_{X_i} \times_{h_X} h_{X_j}$ is the unique morphism, where $0_{\widehat{\mathcal{C}}}$ is the initial object of $\widehat{\mathcal{C}}$. Since $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}_J$ preserves coproducts, $a(\Delta)$ is an isomorphism if and only if $a(\Delta_{i,j})$ are isomorphisms and this is equivalent to the conditions (b) and (c). □

Corollary 2.4.9 *Let $\mathcal{C}$ be a $\mathcal{U}$-category.*

1) Let J be a $\mathcal{U}$-topology on $\mathcal{C}$ and X an object of $\mathcal{C}$. Then, $\epsilon_J(X)$ is an initial object of $\widetilde{\mathcal{C}}_J$ if and only if the empty sieve $\emptyset$ belongs to $J(X)$.

2) If 0 is a strict initial object of $\mathcal{C}$ and J is a $\mathcal{U}$-topology on $\mathcal{C}$ finer than the canonical topology, then, $\epsilon_J(0)$ is an initial object of $\widetilde{\mathcal{C}}_J$.

Proof. 1) The result follows from the case $I =$ (*the empty set*) in (2.4.8).

2) Since the empty sieve $\emptyset$ on 0 is a universal strict epimorphic sieve by (2.2.9), the assertion follows from 1). □

Corollary 2.4.10 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site and $(s_i : X_i \to X)_{i\in I}$ a family of morphisms in $\mathcal{C}$ having the following properties.*

(1) For any morphism $f : Y \to X$ and any $i \in I$, the pull-back of s_i along f exists.

(2) $(s_i : X_i \to X)_{i\in I}$ is a covering.

(3) s_i is a monomorphism for any $i \in I$.

(4) For any $i, j \in I$ $(i \neq j)$, the empty sieve $\emptyset$ on $X_i \times_X X_j$ belongs to $J(X_i \times_X X_j)$, that is, for any sheaf F on $\mathcal{C}$, $F(X_i \times_X X_j)$ consists of a single element.

Then, $\epsilon_J(X)$ is a coproduct of $\epsilon_J(X_i)$ $(i \in I)$.

Proof. Since s_i is a monomorphism, the diagonal morphism $\Delta_i : X_i \to X_i \times_X X_i$ is an isomorphism. By 1) of (2.4.9), $\epsilon_J(X_i \times_X X_j)$ is an initial object. Then, the result follows from (2.4.8), 2). □

Corollary 2.4.11 *Let $\mathcal{C}$ be a $\mathcal{U}$-category and $(s_i : X_i \to X)_{i\in I}$ a family of morphisms in $\mathcal{C}$ such that, for any morphism $f : Y \to X$ and any $i \in I$, the pull-back of s_i along f exists. Suppose that the canonical topology on $\mathcal{C}$ is a $\mathcal{U}$-topology. The following conditions are equivalent.*

i) There exists a $\mathcal{U}$-topology J on $\mathcal{C}$ coarser than the canonical topology such that $\epsilon_J(X)$ is a coproduct of $\epsilon_J(X_i)$ $(i \in I)$.
ii) X is a universally disjoint coproduct of X_i $(i \in I)$.

Proof. *ii)* $\Rightarrow$ *i)*: Let J be the canonical topology on $\mathcal{C}$. The assumption immediately implies the conditions (1) and (3) of (2.4.10). Since X is a universal disjoint coproduct of X_i $(i \in I)$, $(s_i : X_i \to X)_{i\in I}$ is a universal strict epimorphic family by (2.2.7), that is, it is a covering for J. Hence the condition (2) of (2.4.10) is satisfied. Moreover, by (A.3.16), $X_i \times_X X_j$ is a strict initial object if $i \neq j$. It follows from (2.4.9) that $\epsilon_J(X_i \times_X X_j)$ is an initial object of $\widetilde{\mathcal{C}}_J$ and satisfies (4) of (2.4.10).

i) $\Rightarrow$ *ii)*: Let J be a $\mathcal{U}$-topology on $\mathcal{C}$ coarser than the canonical topology such that $\epsilon_J(X)$ is a coproduct of $\epsilon_J(X_i)$ $(i \in I)$. Let $f : Y \to X$ be an arbitrary morphism in $\mathcal{C}$, then $\epsilon_J(Y)$ is a sum of $\epsilon_J(Y \times_X X_i)$ $(i \in I)$. In fact, since $\epsilon_J : \mathcal{C} \to \widetilde{\mathcal{C}}_J$ preserves pull-backs and a coproduct in $\widetilde{\mathcal{C}}_J$ is universally disjoint (see (2.4.14) below), $\epsilon_J(Y)$ is a coproduct of $\epsilon_J(Y) \times_{\epsilon_J(X)} \epsilon_J(X_i) \cong \epsilon_J(Y \times_X X_i)$ $(i \in I)$. Since every representable functor is a sheaf for J, $\widehat{\mathcal{C}}(h_Y, h_Z) \to \widehat{\mathcal{C}}\Big(\coprod_{i\in I} h_{Y\times_X X_i}, h_Z\Big)$ is bijective for any object Z of $\mathcal{C}$. It follows that Y is a coproduct of $Y \times_X X_i$ $(i \in I)$ in $\mathcal{C}$.

Similarly if $\epsilon_J(W)$ is an initial object of $\widetilde{\mathcal{C}}_J$, we have $\mathcal{C}(W, Z) \cong \widetilde{\mathcal{C}}_J(\epsilon_J(W), h_Z)$, which consists of a single element for any object Z of $\mathcal{C}$. Thus W is an initial object in $\mathcal{C}$.

Applying 2) of (2.4.8), $\epsilon_J((Y \times_X X_i) \times_Y (Y \times_X X_i))$ is an initial object of $\widetilde{\mathcal{C}}_J$ if $i \neq j$, hence so is $(Y \times_X X_i) \times_Y (Y \times_X X_i)$ in $\mathcal{C}$. Moreover, since $\epsilon_J : \mathcal{C} \to \widetilde{\mathcal{C}}_J$ is fully-faithful, it reflects isomorphisms. Hence the diagonal morphism $\Delta_i : Y \times_X X_i \to (Y \times_X X_i) \times_Y (Y \times_X X_i)$ is an isomorphism namely, the canonical inclusion $Y \times_X X_i \to Y$ is a monomorphism. □

Consider the case that I is empty in (2.4.11), then we have the following result.

Corollary 2.4.12 *Let $\mathcal{C}$ be a $\mathcal{U}$-category and X an object of $\mathcal{C}$. If the canonical topology on $\mathcal{C}$ is a $\mathcal{U}$-topology, the following conditions are equivalent.*

(i) There exists a topology J on $\mathcal{C}$ coarser than the canonical topology such that $\epsilon_J(X)$ is an initial object of $\widetilde{\mathcal{C}}_J$.
(ii) X is a strict initial object of $\mathcal{C}$

Proposition 2.4.13 *Let $\mathcal{C}$ be a $\mathcal{U}$-category, $\widetilde{\mathcal{C}}$ the category of sheaves for the canonical topology J, $R \rightrightarrows X$ an equivalence relation with a coequalizer $\pi : X \to Y$. Suppose that the pull-back of π along an arbitrary morphism with codomain Y exists. Consider the following properties.*

(i) $\epsilon_J(\pi) : \epsilon_J(X) \to \epsilon_J(Y)$ is a coequalizer of the equivalence relation $\epsilon_J(R) \rightrightarrows \epsilon_J(X)$.
(ii) The equivalence relation $R \rightrightarrows X$ is effective and universal.

Then, (ii) implies (i) and the converse holds if $\mathcal{C}$ has a $\mathcal{U}$-topology coarser than the canonical topology.

Proof. *ii)* $\Rightarrow$ *i)*: Let S be the sieve generated by $\pi : X \to Y$ and $\pi^\sharp : h_X \to S$ a morphism defined by $\pi^\sharp(f) = f\pi$, then for any presheaf F, $(\pi^\sharp)^* : \widehat{\mathcal{C}}(S, F) \to \widehat{\mathcal{C}}(h_X, F)$ is injective and its image consists of morphisms $g : h_X \to F$ satisfying "$\pi u = \pi v$ for $u, v : Z \to X \Rightarrow gh_u = gh_v$" by (2.1.13). Since $R \rightrightarrows X$ is a kernel pair of π, π is an effective epimorphism hence we have an equalizer

$$\widehat{\mathcal{C}}(S, F) \xrightarrow{(\pi^\sharp)^*} \widehat{\mathcal{C}}(h_X, F) \rightrightarrows \widehat{\mathcal{C}}(h_R, F).$$

We show that the inclusion morphism $\iota : S \to h_Y$ induces a bijection $\iota^* : \widehat{\mathcal{C}}(h_Y, F) \to \widehat{\mathcal{C}}(S, F)$ for any sheaf F for the canonical topology on $\mathcal{C}$. Then, from the above equalizer, we have the following equalizer which shows *i)*.

$$\widetilde{\mathcal{C}}(\epsilon_J(Y), F) \xrightarrow{\epsilon_J(\pi)^*} \widetilde{\mathcal{C}}(\epsilon_J(X), F) \rightrightarrows \widetilde{\mathcal{C}}(\epsilon_J(R), F)$$

By the assumption, $\pi : X \to Y$ is a universal effective epimorphism. Thus $S \in J(Y)$ and this implies the assertion.

$i) \Rightarrow ii)$: By 3) of (2.4.6), $\epsilon_J(R) \rightrightarrows \epsilon_J(X)$ is a kernel pair of $\epsilon_J(\pi) : \epsilon_J(X) \to \epsilon_J(Y)$. Since the functor $\epsilon_J : \mathcal{C} \to \widetilde{\mathcal{C}}$ is fully faithful, it follows that $R \rightrightarrows X$ is the kernel pair of π by (A.3.3). Hence the equivalence relation $R \rightrightarrows X$ is effective. By the assumption and (2.4.7), $(\pi : X \to Y)$ is a covering for the canonical topology, in other words, $\pi : X \to Y$ is a universal strict epimorphism. Since the pull-back of π along an arbitrary morphism with codomain Y exists, $\pi : X \to Y$ is a universal effective epimorphism. □

Proposition 2.4.14 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site, then the category of sheaves on $\mathcal{C}$ has the following properties.*

1) $\widetilde{\mathcal{C}}$ has finite limits.

2) $\widetilde{\mathcal{C}}$ has coproducts indexed by $\mathcal{U}$-set and they are universally disjoint.

3) Every equivalence relation in $\widetilde{\mathcal{C}}$ is effective and has a coequalizer which is a universal effective epimorphism.

Proof. 1) follows from (2.2.10).

2) In the category of sets, a coproduct is universally disjoint (A.3.5). Hence a coproduct is universally disjoint in the category of presheaves of sets. It follows from (2.4.1), 2) that if $(\mathcal{C}, J)$ is a $\mathcal{U}$-site, coproducts exist in $\widetilde{\mathcal{C}}$ and they are universally disjoint.

3) follows from (2.4.6). □

Proposition 2.4.15 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site.*

1) Every epimorphic family in $\widetilde{\mathcal{C}}$ is strict.

2) Every monomorphic family in $\widetilde{\mathcal{C}}$ is strict.

3) For a family $(\varphi_k : F_k \to F)_{k\in I}$ of morphisms in $\widetilde{\mathcal{C}}$, there exist a strict epimorphic family $(\psi_k : F_k \to G)_{k\in I}$ and a strict monomorphism $j : G \to F$ such that $\varphi_k = j\psi_k$ for any $i \in I$.

Hence the conditions of 2) of (A.4.10) are all equivalent in $\widetilde{\mathcal{C}}$.

Proof. 1) By 2) of (2.4.6), an epimorphic family in $\widetilde{\mathcal{C}}$ is universal effective.

2) Since the inclusion functor $i : \widetilde{\mathcal{C}} \to \widehat{\mathcal{C}}$ has a left adjoint, i preserves monomorphic families. Since monomorphic families in the category of $\mathcal{U}$-set are strict, every monomorphic family in $\widehat{\mathcal{C}}$ is strict. Thus every monomorphic family in $\widetilde{\mathcal{C}}_{\mathcal{U}}$ is strict.

3) By 1) and 2), it suffices to show that for a family $(\varphi_k : F_k \to F)_{k\in I}$ of morphisms in $\widetilde{\mathcal{C}}$, there exist an epimorphic family $(\psi_k : F_k \to G)_{k\in I}$ and a monomorphism $j : G \to F$ such that $\varphi_k = j\psi_k$ for any $i \in I$. Define $F' \in \widehat{\mathcal{C}}$ and $\varphi'_k : F_k \to F'$ by $F'(X) = \bigcup_{k\in I}(\varphi_k)_X(F_k(X))$ and $(\varphi'_k)_X(x) = (\varphi_k)_X(x)$, then $(\varphi'_k : F_k \to F')_{k\in I}$ is an epimorphic family in $\widehat{\mathcal{C}}$ and $\varphi_k = j\varphi'_k$ holds for $i \in I$, where $j : F' \hookrightarrow F$ is the inclusion morphism. Since the associated sheaf functor has a right adjoint, it preserves epimorphic families, hence $(a(\varphi'_k)\varepsilon^{-1} : F_k \cong aF_k \to aF')_{k\in I}$ is an epimorphic family in $\widetilde{\mathcal{C}}$, where $\varepsilon : ai \to id_{\widetilde{\mathcal{C}}}$ is the counit of the adjunction. Moreover, since the associated sheaf functor is a left exact, it preserves monomorphisms, hence $\varepsilon a(j) : aF' \to aF \cong F$ is a monomorphism in $\widetilde{\mathcal{C}}$. By the naturality of ε, we have $\varepsilon a(j)a(\varphi'_k)\varepsilon^{-1} = \varepsilon a(\varphi_k)\varepsilon^{-1} = \varphi_k$. □

Proposition 2.4.16 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site with a $\mathcal{U}$-small topologically generating family G. Consider the canonical functor $\epsilon_J : \mathcal{C} \to \widetilde{\mathcal{C}}$. Then, a set $\epsilon_J(G) = \{\epsilon_J(X) | X \in G\}$ of objects of $\widetilde{\mathcal{C}}$ is a generator of $\widetilde{\mathcal{C}}$. Moreover, $\epsilon_J(G)$ is a topologically generating family for the canonical topology on $\widetilde{\mathcal{C}}$.*

Proof. It suffices to show that $\bigcup_{X\in G}\widetilde{\mathcal{C}}(\epsilon_J(X), F)$ is an epimorphic family (which becomes universal and effective by (2.4.6)) in $\widetilde{\mathcal{C}}$ for a sheaf F. Let $u, v : F \to H$ be morphisms of sheaves such that $uh = vh$ for any $h : \epsilon_J(X) \to F$ with $X \in G$. For any object Y of $\mathcal{C}$, there exists a covering $(g_i : X_i \to Y)_{i\in I}$ of Y for J such that $X_i \in G$ for any $i \in I$. Then, $(\epsilon_J(g_i) : \epsilon_J(X_i) \to \epsilon_J(Y))_{i\in I}$ is an epimorphic family in $\widetilde{\mathcal{C}}$ by (2.4.7). Hence for any morphism $k : \epsilon_J(Y) \to F$, we have $ukg_i = vkg_i$ and it follows that $uk = vk$. Thus we have $u = v$ by (2.4.3). □

Corollary 2.4.17 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site.*

1) $\widetilde{\mathcal{C}}$ is a $\mathcal{U}$-category with a $\mathcal{U}$-small set of generators and the canonical topology on $\widetilde{\mathcal{C}}$ is a $\mathcal{U}$-topology.

2) For an object X of $\widetilde{\mathcal{C}}$, the set of subobjects of X and the set of quotient objects of X are $\mathcal{U}$-small.

Proof. 1) Let G be a $\mathcal{U}$-small topologically generating family. By (2.4.16), $\{\epsilon_J(X) | X \in G\}$ is a $\mathcal{U}$-small set of generators. We set $M = \bigcup_{X\in G}\widetilde{\mathcal{C}}(\epsilon_J(X), F)$ and $\epsilon_J(X_f) = \mathrm{dom}(f)$ for $f \in M$. If $F, H \in \mathrm{Ob}\,\widetilde{\mathcal{C}}$, since M is an epimorphic family, we have a monomorphism $\widetilde{\mathcal{C}}(F,H) \to \prod_{f\in M} \widetilde{\mathcal{C}}(\epsilon_J(X_f),H)$. We note that there is a natural bijection $\widetilde{\mathcal{C}}(\epsilon_J(X),F) \to F(X)$, hence $\widetilde{\mathcal{C}}(\epsilon_J(X),F)$ is $\mathcal{U}$-small and so is M. It follows that $\widetilde{\mathcal{C}}(F,H)$ is $\mathcal{U}$-small.

2) Since every subobject and quotient object in $\widetilde{\mathcal{C}}$ are strict, the assertion follows from (A.4.12) and (A.4.14).□

Let $(\mathcal{C},J)$ be a $\mathcal{U}$-site and F a sheaf on $\mathcal{C}$ for J. We denote by $\mathrm{Sub}(F)$ the set of subobjects of F. By (2.4.17), $\mathrm{Sub}(F)$ is $\mathcal{U}$-small and by (A.9.6) and (A.9.8), we have the following result.

Proposition 2.4.18 *$(\mathrm{Sub}(F),\cap,\cup)$ is a lattice such that, for any family $(F_i)_{i\in I}$ of subobjects of F and a subobject G of F, the upper bound (coproduct) $\bigcup_{i\in I} F_i$ of $(F_i)_{i\in I}$ exists and $G\cap(\bigcup_{i\in I} F_i) = \bigcup_{i\in I}(G\cap F_i)$ holds.*

2.5 Extension of topologies

Lemma 2.5.1 *Let J be a topology on $\mathcal{C}$ and R a sieve on X. We denote by $\iota : R \to h_X$ the inclusion morphism. Then, $a(\iota) : aR \to ah_X$ is an isomorphism if and only if $R \in J(X)$.*

Proof. Suppose $R \in J(X)$, then for any sheaf F, $\iota^* : \widehat{\mathcal{C}}(h_X,F) \to \widehat{\mathcal{C}}(R,F)$ is bijective. Hence $a(\iota)^* : \widetilde{\mathcal{C}}(ah_X,F) \to \widetilde{\mathcal{C}}(aR,F)$ is bijective and this implies that $a(\iota)$ is an isomorphism. Conversely, suppose that $a(\iota)$ is an isomorphism. Let $(f_i : X_i \to X)_{i\in I}$ be a family which generates R. Then, $h_{f_i} : h_{X_i} \to h_X$ factors through ι, that is $h_{f_i} = \iota f_i^\sharp$ for $f_i^\sharp : h_{X_i} \to R$. Obviously, $(f_i^\sharp : h_{X_i} \to R)_{i\in I}$ is an epimorphic family in $\widehat{\mathcal{C}}$. Since the associated sheaf has a right adjoint, $(a(f_i^\sharp) : ah_{X_i} \to aR)_{i\in I}$ is an epimorphic family in $\widetilde{\mathcal{C}}$. Thus $(\epsilon_J(f_i) = a(\iota)a(f_i^\sharp) : \epsilon_J(X_i) \to \epsilon_J(X))_{i\in I}$ is an epimorphic family and we see $R \in J(X)$ by (2.4.7). □

Proposition 2.5.2 *Let $(\mathcal{C},J)$ be a $\mathcal{U}$-site and $f : H \to K$ a morphism of $\widehat{\mathcal{C}}$. The following conditions are equivalent.*

i) For any morphism $\alpha : h_X \to K$ with $X \in \mathrm{Ob}\,\mathcal{C}$, the image of the pull-back $\bar{f} : H \times_K h_X \to h_X$ of f along α belongs to $J(X)$.

ii) $a(f) : aH \to aK$ is an epimorphism in $\widetilde{\mathcal{C}}$.

iii) For any sheaf F on $\mathcal{C}$, $f^* : \widehat{\mathcal{C}}(K,F) \to \widehat{\mathcal{C}}(H,F)$ is injective.

Proof. It is clear that *iii)* is equivalent to *ii)*.

i) ⇒ *ii)*: We first show that $a(\bar{f}) : a(H \times_K h_X) \to ah_X$ is an epimorphism. Let R be the image of $\bar{f}$ and $H \times_K h_X \xrightarrow{p} R \xhookrightarrow{\iota} h_X$ the factorization of $\bar{f}$. Then, p is an epimorphism, hence $a(p)$ is an epimorphism. It follows from the assumption and (2.5.1) that $a(\iota)$ is an isomorphism. Hence $a(\bar{f})$ is an epimorphism. Take an epimorphic family $(\alpha_i : h_{X_i} \to K)_{i\in I}$, then $(a(\alpha_i) : ah_{X_i} \to aK)_{i\in I}$ is also an epimorphic family. Let F be an arbitrary sheaf. It follows from the following commutative diagram that $a(f) : aH \to aK$ is an epimorphism in $\widetilde{\mathcal{C}}$.

$$\begin{tikzcd}
\widetilde{\mathcal{C}}(aK,F) \arrow[r, "a(f)^*"] \arrow[d, "(a(\alpha_i)^*)_{i\in I}"] & \widetilde{\mathcal{C}}(aH,F) \arrow[d, "(a(\bar{\alpha}_i)^*)_{i\in I}"] \\
\prod_{i\in I} \widetilde{\mathcal{C}}(ah_{X_i},F) \arrow[r, "\prod_{i\in I} a(\bar{f}_i)^*"] & \prod_{i\in I} \widetilde{\mathcal{C}}(a(H \times_K h_{X_i}),F)
\end{tikzcd}$$

ii) ⇒ *i)*: Let R be the image of $\bar{f}$ and $H \times_K h_X \xrightarrow{p} R \xhookrightarrow{\iota} h_X$ the factorization of $\bar{f}$. Since the associated sheaf functor is left exact, the following diagram is a pull-back and $a(\iota)$ is a monomorphism.

$$\begin{tikzcd}
a(H \times_K h_X) \arrow[r, "a(\bar{f})"] \arrow[d] & ah_X \arrow[d, "a(\alpha)"] \\
aH \arrow[r, "a(f)"] & aK
\end{tikzcd}$$

Epimorphisms in $\widetilde{\mathcal{C}}$ are universal, hence $a(\bar{f}) = a(\iota)a(p)$ is an epimorphism. It follows that $a(\iota)$ is an epimorphism and by (2.4.5), it is an isomorphism. Therefore, $R \in J(X)$ by (2.5.1). □

Definition 2.5.3 *1) A morphism $f : H \to K$ in $\widehat{\mathcal{C}}$ satisfying the equivalent conditions of (2.5.2) is called a covering. A family of morphisms $(f_i : H_i \to K)_{i\in I}$ in $\widehat{\mathcal{C}}$ such that I is $\mathcal{U}$-small is called a covering if the morphism $\coprod_{i\in I} H_i \to K$ induced by f_i's is a covering.*

2) A morphism $f : H \to K$ in $\widehat{\mathcal{C}}$ is said to be a bicovering if it is a covering and the diagonal morphism $\Delta : H \to H \times_K H$ is a covering. A family of morphisms $(f_i : H_i \to K)_{i\in I}$ in $\widehat{\mathcal{C}}$ such that I is $\mathcal{U}$-small is called a bicovering if the morphism $\coprod_{i\in I} H_i \to K$ induced by f_i's is a bicovering.

By the condition *ii)* of (2.5.3), a family of morphisms $(f_i : H_i \to K)_{i\in I}$ in $\widehat{\mathcal{C}}$ is a covering if and only if $(f_i^*)_{i\in I} : \widehat{\mathcal{C}}(K, F) \to \prod_{i\in I} \widehat{\mathcal{C}}(H_i, F)$ is injective for any sheaf F. In other words, $(f_i : H_i \to K)_{i\in I}$ is a covering if and only if $(a(f_i) : aH_i \to aK)_{i\in I}$ is an epimorphic family in $\widetilde{\mathcal{C}}$.

Proposition 2.5.4 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site and $f : H \to K$ a morphism of $\widehat{\mathcal{C}}$. The following conditions are equivalent.*

i) f is a bicovering.

ii) f is a covering and for any object X of $\mathcal{C}$ and any pair $u, v : h_X \to H$ of morphisms in $\widehat{\mathcal{C}}$ such that $fu = fv$, the equalizer of u and v belongs to $J(X)$.

iii) $a(f) : aH \to aK$ is an isomorphism in $\widetilde{\mathcal{C}}$.

iv) For any sheaf F on $\mathcal{C}$, $f^* : \widehat{\mathcal{C}}(K, F) \to \widehat{\mathcal{C}}(H, F)$ is bijective.

Proof. For morphisms $u, v : h_X \to H$ in $\widehat{\mathcal{C}}$, then $e : R \to h_X$ is an equalizer of u and v if and only if the following left square is a pull-back. If $fu = fv$, $(u, v) : H \to H \times H$ factors through the monomorphism $H \times_K H \to H \times H$ and the following right square is also a pull-back by (A.3.6).

$$\begin{array}{ccc} R & \xrightarrow{e} & h_X \\ \downarrow & & \downarrow{\scriptstyle (u,v)} \\ H & \xrightarrow{\Delta} & H \times H \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{e} & h_X \\ \downarrow & & \downarrow{\scriptstyle (u,v)} \\ H & \xrightarrow{\Delta} & H \times_K H \end{array}$$

Hence *ii)* is equivalent to *i)* by the condition *i)* of (2.5.2).

The equivalence *iii)* $\Leftrightarrow$ *iv)* is obvious.

i) $\Rightarrow$ *iii)*: By (2.5.1), $a(f) : aH \to aK$ and $a(\Delta) : aH \to a(H \times_K H)$ are epimorphisms. Since $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}$ is left exact, $a(H \times_K H) \overset{a(\mathrm{pr}_1)}{\underset{a(\mathrm{pr}_2)}{\rightrightarrows}} aH$ is a kernel pair of $a(f)$. Moreover, $a(\mathrm{pr}_i)a(\Delta) = id_{aH}$ $(i = 1, 2)$. It follows from (A.3.2) that $a(f)$ is a monomorphism and by (2.4.5), $a(f)$ is an isomorphism.

iii) $\Rightarrow$ *i)*: Since $a(f) : aH \to aK$ is an isomorphism and $a(H \times_K H) \overset{a(\mathrm{pr}_1)}{\underset{a(\mathrm{pr}_2)}{\rightrightarrows}} aH$ is a kernel pair of $a(f)$, f is a covering and $a(\mathrm{pr}_1)$ is an isomorphism. It follows from $a(\mathrm{pr}_1)a(\Delta) = id_{aH}$ that $a(\Delta)$ is an isomorphism. □

It follows from (2.5.4) and the fact that $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}$ preserves coproducts that a family $(f_i : H_i \to K)_{i\in I}$ in $\widehat{\mathcal{C}}$ is a bicovering if and only if $(f_i^*)_{i\in I} : \widehat{\mathcal{C}}(K, F) \to \prod_{i\in I} \widehat{\mathcal{C}}(H_i, F)$ is bijective for any sheaf F.

We note that a topology T on $\widehat{\mathcal{C}}$ finer than the canonical topology if and only if every epimorphic family in $\widehat{\mathcal{C}}$ is a covering for T. In fact, since every epimorphic family in $\widehat{\mathcal{C}}$ is effective and universal, a family of morphisms with common codomain is a covering for the canonical topology on $\widehat{\mathcal{C}}$ if and only if it is an epimorphic family.

Proposition 2.5.5 *Let $\mathcal{C}$ be a $\mathcal{U}$-category and $\mathcal{D}$ a reflexive full subcategory of $\widehat{\mathcal{C}}$. Suppose that the left adjoint $L : \widehat{\mathcal{C}} \to \mathcal{D}$ of the inclusion functor $\mathcal{D} \hookrightarrow \widehat{\mathcal{C}}$ is left exact. For a presheaf K, let $T_{\mathcal{D}}(K)$ be the set of sieves on K such that each of them contains a family $(f_i : H_i \to K)_{i\in I}$ such that $(L(f_i) : L(H_i) \to L(K))_{i\in I}$ is an epimorphic family in $\mathcal{D}$. Then $T_{\mathcal{D}}$ is a topology on $\widehat{\mathcal{C}}$ finer than the canonical topology.*

Proof. Let $P(K)$ be the set of families $(f_i : H_i \to K)_{i\in I}$ such that $(L(f_i) : L(H_i) \to L(K))_{i\in I}$ is an epimorphic family in $\mathcal{D}$. By (2.1.12), it suffices to show that P is a pretopology on $\widehat{\mathcal{C}}$. $(P0)$ is obvious by the completeness of $\widehat{\mathcal{C}}$. Since $(id_K : K \to K)$ is a covering, $(P1)$ is satisfied.

For $(P2)$, suppose that $S = (f_i : H_i \to K)_{i\in I} \in P(K)$ and $g : G \to K$ is a morphism in $\widehat{\mathcal{C}}$. Let $\bar{f}_i : H_i \times_K G \to G$ be the pull-back of f_i along g. Since L is left exact, $L(\bar{f}_i) : L(H_i \times_K G) \to L(G)$ is a pull-back of

$L(f_i)$ along $L(g)$. It follows from (2.4.6) that $(L(\bar{f}_i) : L(H_i \times_K G) \to L(G))_{i\in I}$ is an epimorphic family. Hence $(\bar{f}_i : H_i \times_K G \to G)_{i\in I} \in P(G)$ and this shows $(P2)$.

Suppose $(f_i : H_i \to K)_{i\in I} \in P(K)$ and $(g_{ij} : F_{ij} \to H_i)_{j\in I_i} \in P(H_i)$, then $(L(f_i) : L(H_i) \to L(K))_{i\in I}$ and $(L(g_{ij}) : L(F_{ij}) \to L(H_i))_{i\in I}$ are epimorphic families in $\mathcal{D}$. Thus so is $(L(f_ig_{ij}) : L(F_{ij}) \to L(K))_{(i,j)\in M}$ $(M = \{(i,j)|\, i \in I,\, j \in I_i\})$. Therefore $(f_ig_{ij} : F_{ij} \to K)_{(i,j)\in M} \in P(K)$ and this shows $(P3)$.

Finally, since L preserves epimorphic families (A.3.13), every epimorphic family in $\widehat{\mathcal{C}}$ is a covering for $T_{\mathcal{D}}$. $\square$

The topology $T_{\mathcal{D}}$ does not depend on the choice of the reflection L. In fact, if L and L' are left adjoints of the inclusion functor, they are naturally equivalent.

Proposition 2.5.6 *1) Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site. For a presheaf K, let $T_J(K)$ be the set of sieves on K such that each of them contains a covering family in the sense of (2.5.3). Then, T_J is a topology on $\widehat{\mathcal{C}}$ finer than the canonical topology, in fact $T_J = T_{\widetilde{\mathcal{C}}_J}$. Moreover, if $(f_i : X_i \to X)_{i\in I}$ is a covering of $X \in \operatorname{Ob}\mathcal{C}$ for J, then $(h_{f_i} : h_{X_i} \to h_X)_{i\in I}$ is a covering for T_J in $\widehat{\mathcal{C}}$.*

2) Let T be a topology on $\widehat{\mathcal{C}}$ finer than the canonical topology. For each object X of $\mathcal{C}$, let $J^T(X)$ be the set of sieves on X such that each of them contains a family of morphisms $(f_i : X_i \to X)_{i\in I}$ such that $(h_{f_i} : h_{X_i} \to h_X)_{i\in I}$ is a covering for T of h_X. Then, J^T is a topology on $\mathcal{C}$.

Proof. 1) Since we have a left exact left adjoint $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}$ of the inclusion functor $\widetilde{\mathcal{C}} \hookrightarrow \widehat{\mathcal{C}}$ and $(s_i : H_i \to K)_{i\in I}$ is a covering if and only if $(a(s_i) : aH_i \to aK)_{i\in I}$ is an epimorphic family in $\widetilde{\mathcal{C}}$, the first assertion follows from (2.5.5).

Suppose that $(f_i : X_i \to X)_{i\in I}$ is a covering for J in $\mathcal{C}$. By (2.4.7), $(\epsilon_J(f_i) : \epsilon_J(X_i) \to \epsilon_J(X))_{i\in I}$ is an epimorphic family in $\widetilde{\mathcal{C}}$. Hence $(h_{f_i} : h_{X_i} \to h_X)_{i\in I}$ is a covering for T_J and T_J satisfies *ii*).

2) Since $id_{h_X} = h_{id_X} : h_X \to h_X$ is a covering for T, $h_X = \overline{\{id_X\}} \in J^T(X)$ and J^T satisfies $(T1)$. For $R \in J^T(X)$ and a morphism $f : Y \to X$ in $\mathcal{C}$, take $S = (f_i : X_i \to X)_{i\in I}$ such that $R \supset S$ and $(h_{f_i} : h_{X_i} \to h_X)_{i\in I}$ is a covering of h_X for T. Let $\bar{f}_i : h_{X_i} \times_{h_X} h_Y \to h_Y$ be the pull-back of h_{f_i} along h_f. By (2.1.10), $(\bar{f}_i : h_{X_i} \times_{h_X} h_Y \to h_Y)_{i\in I}$ is a covering of h_Y for T. To show that J^T satisfies $(T2)$, namely, $h_f^{-1}(R) \in J^T(Y)$, it suffices to prove the following facts. In fact, consider the case $F_i = h_{X_i} \times_{h_X} h_Y$, $s_i = \bar{f}_i$ in *ii*), then we have $h_f^{-1}(\bar{S}) \in J^T(Y)$ which is contained in $h_f^{-1}(R)$, and since J^T obviously satisfies $(T4)$, it follows that $h_f^{-1}(R) \in J^T(Y)$.

i) The union of the images of $\bar{f}_i$'s coincides with $h_f^{-1}(\bar{S})$.

ii) If $(s_i : F_i \to h_Y)_{i\in I}$ is a covering for T, the union of the images of s_i's belongs to $J^T(Y)$.

i) If $g \in h_f^{-1}(\bar{S})(Z)$, $fg \in \bar{S}(Z)$ and $fg = f_ik$ for some $i \in I$ and $k : Z \to X_i$. Then $(k, g) \in (h_{X_i} \times_{h_X} h_Y)(Z)$ and this maps to g by $\bar{f}_i$. Conversely, for $(k, g) \in (h_{X_i} \times_{h_X} h_Y)(Z)$, we have $fg = f_ik \in \bar{S}(Z)$, hence $g \in h_f^{-1}(\bar{S})(Z)$. *ii*) We take an epimorphic families $(\varphi_{i\lambda} : h_{Y_{i\lambda}} \to F_i)_{\lambda\in\Lambda_i}$, which are coverings for T by the assumption. Thus $(s_i\varphi_{i\lambda} : h_{Y_{i\lambda}} \to h_Y)_{(i,\lambda)\in M}$ $(M = \{(i,\lambda)|\, i \in I,\, \lambda \in \Lambda_i\})$ is a covering for T by (2.1.11). We set $\alpha_{i\lambda} = s_i\varphi_{i\lambda}(id_{Y_{i\lambda}})$, then the sieve Q generated by $(\alpha_{i\lambda} : Y_{i\lambda} \to Y)_{(i,\lambda)\in M}$ belongs to $J^T(Y)$. Then, for any object Z of $\mathcal{C}$ and $\beta \in h_{Y_{i\lambda}}$, we have $\alpha_{i\lambda}\beta = h_Y(\beta)s_i\varphi_{i\lambda}(id_{Y_{i\lambda}}) = s_iF_i(\beta)\varphi_{i\lambda}(id_{Y_{i\lambda}}) = s_i\varphi_{i\lambda}h_{Y_{i\lambda}}(\beta)(id_{Y_{i\lambda}}) = s_i\varphi_{i\lambda}(\beta)$ by the naturality of s_i and $\varphi_{i\lambda}$. Since $(\varphi_{i\lambda})_{\lambda\in\Lambda_i}$ are epimorphic families, it follows from the above equality that Q coincides with the the union of the images of s_i's.

Clearly, J^T satisfies $(T4)$ and we show $(T5)$. Suppose that $R \in J^T(X)$ and $R_f \in J^T(\operatorname{dom}(f))$ with $f \in \operatorname{Ob} R$ and take families of morphisms $S = (f_i : X_{f_i} \to X)_{i\in I}$, $S_f = (g_{f\lambda} : X_{f\lambda} \to X_f)_{\lambda\in\Lambda_f}$ such that $(h_{f_i} : h_{X_{f_i}} \to h_X)_{i\in I}$ and $(h_{g_{f\lambda}} : h_{X_{f\lambda}} \to h_{X_f})_{\lambda\in\Lambda_f}$ are coverings for T. It follows from (2.1.11) that $(h_{f_ig_{f_i\lambda}} : h_{X_{f_i\lambda}} \to h_X)_{(i,\lambda)\in M}$ $(M = \{(i,\lambda)|\, i \in I,\, \lambda \in \Lambda_{f_i}\})$ is a covering for T. It is obvious that $\{fg|\, f \in \operatorname{Ob} R,\, g \in \operatorname{Ob} R_f\}$ contains $(f_ig_{f_i\lambda} : X_{f_i\lambda} \to X)_{(i,\lambda)\in M}$. Thus we have verified $(T5)$. $\square$

Remark 2.5.7 *If $S = (f_i : H_i \to K)_{i\in I}$ is a covering of K for T_J, it is a covering in the sense of (2.5.3). In fact, since $\bar{S}$ contains a covering $S' = (g_j : F_j \to K)_{j\in M}$ in the sense of (2.5.3), there exist $i(j) \in I$ and $s_j : F_j \to H_{i(j)}$ such that $g_j = f_{i(j)}s_j$. Hence $(a(f_{i(j)}))_{j\in M}$ is an epimorphic family and S contains a covering in the sense of (2.5.3).*

Proposition 2.5.8 *1) Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site and T_J the topology on $\widehat{\mathcal{C}}$ defined in (2.5.6), then, $J^{T_J} = J$.*

2) Let $\mathcal{C}$ be a $\mathcal{U}$-category and $\mathcal{D}$ a reflexive full subcategory of $\widehat{\mathcal{C}}$. Suppose that the reflection $L : \widehat{\mathcal{C}} \to \mathcal{D}$ is left exact, then for each object X of $\mathcal{C}$, $J^{T_{\mathcal{D}}}(X) = \{R \subset h_X|\, L(\iota) : L(R) \to L(h_X)$ is an isomorphism$\}$ $(\iota : R \hookrightarrow h_X)$.

Proof. 1) For each object X of $\mathcal{C}$, $J^{T_J}(X) = \{R \subset h_X | \, R \supset S$ *for some* $S = (f_i : X_i \to X)_{i \in I}$ *such that* $(h_{f_i} : h_{X_i} \to h_X)_{i \in I}$ *is a covering in the sense of* (2.5.3)$\}$ by (2.5.7). For $R \in J^{T_J}(X)$, choose $S = (f_i : X_i \to X)_{i \in I}$ such that $(h_{f_i} : h_{X_i} \to h_X)_{i \in I}$ is a covering in the sense of (2.5.3). Let us denote by $f : \coprod_{i \in I} h_{X_i} \to h_X$ the morphism induced by h_{f_i}'s. The image of f is $\bar{S}$ and $\iota : \bar{S} \to h_X$ denotes the inclusion morphism. Since $a(f)$ is an epimorphism and $a(\iota)$ is a monomorphism, $a(\iota)$ is an isomorphism by (2.4.5). Hence $\bar{S} \in J(X)$ and we have $R \in J(X)$. Conversely, if $R \in J(X)$, then $(h_f : h_{\mathrm{dom}(f)} \to h_X)_{f \in \mathrm{Ob}\, R}$ is a covering for T_J by (2.5.6). Therefore $R \in J^{T_J}(X)$.

2) Let R be a sieve on X such that the inclusion morphism $\iota : R \hookrightarrow h_X$ induces an isomorphism $L(\iota) : L(R) \to L(h_X)$. Take a family of morphisms $S = (f_i : X_i \to X)_{i \in I}$ such that $R = \bar{S}$. Let $f_i^\sharp : h_{X_i} \to R$ be the unique morphism such that $\iota f_i^\sharp = h_{f_i}$. Then, $(f_i^\sharp : h_{X_i} \to R)_{i \in I}$ is an epimorphic family in $\widehat{\mathcal{C}}$, hence $(L(f_i^\sharp) : L(h_{X_i}) \to L(R))_{i \in I}$ an epimorphic family in $\mathcal{D}$. Since $L(\iota)$ is an isomorphism, it follows that $(L(h_{f_i}) = L(\iota)L(f_i^\sharp) : L(h_{X_i}) \to L(h_X))_{i \in I}$ an epimorphic family in $\mathcal{D}$. Thus we see $R \in J^{T_{\mathcal{D}}}(X)$.

Conversely, suppose $R \in J^{T_{\mathcal{D}}}(X)$. There exists a family of morphisms $S = (f_i : X_i \to X)_{i \in I}$ such that $S \subset R$ and $(L(h_{f_i}) : L(h_{X_i}) \to L(h_X))_{i \in I}$ is an epimorphic family in $\mathcal{D}$. The union of the images of h_{f_i}'s is $\bar{S}$ and let us denote by $\bar{\iota} : \bar{S} \to h_X$ the inclusion morphism. $f_i^\sharp : h_{X_i} \to \bar{S}$ denotes the unique morphism such that $\bar{\iota} f_i^\sharp = h_{f_i}$. Since $(L(\bar{\iota})L(f_i^\sharp) = L(h_{f_i})_{i \in I}$ is an epimorphic family in $\mathcal{D}$, $L(\bar{\iota})$ is an epimorphism. We denote by $\iota : R \to h_X$ the inclusion morphism, then it follows that $L(\iota)$ is an epimorphism. On the other hand, since L is left exact and ι is a monomorphism, $L(\iota)$ is a monomorphism in $\mathcal{D}$. By (2.4.5), $L(\iota)$ is an isomorphism in $\mathcal{D}$. □

Proposition 2.5.9 *Let T be a topology on $\widehat{\mathcal{C}}$ finer than the canonical topology. Then, T is the coarsest topology among the topology T' on $\widehat{\mathcal{C}}$ having the following properties.*

i) T' is finer than the canonical topology on $\widehat{\mathcal{C}}$.

ii) If $(f_i : X_i \to X)_{i \in I}$ is a covering of $X \in \mathrm{Ob}\, \mathcal{C}$ for J^T, $(h_{f_i} : h_{X_i} \to h_X)_{i \in I}$ is a covering for T' in $\widehat{\mathcal{C}}$.

Proof. Let T' be a topology on $\widehat{\mathcal{C}}$ satisfying $i)$ and $ii)$ and $S = (f_i : H_i \to K)_{i \in I}$ a covering of K for T. For a morphism $g : h_X \to K$, we denote by $\bar{f}_i : H_i \times_K h_X \to h_X$ the pull-back of f_i along g. By (2.1.11), $S_g = (\bar{f}_i : H_i \times_K h_X \to h_X)_{i \in I}$ a covering of h_X for T. We show that S_g is a covering for T'. There exists an epimorphic family $(\alpha_{i\lambda} : h_{Y_{i\lambda}} \to H_i \times_K h_X)_{\lambda \in \Lambda_i}$ for each $i \in I$. Since T is finer than the canonical topology, this is a covering for T. Then, $Q = (\bar{f}_i \alpha_{i\lambda} : h_{Y_{i\lambda}} \to h_X)_{(i,\lambda) \in M}$ $(M = \{(i, \lambda) | \, i \in I, \, \lambda \in \Lambda\})$ is a covering for T by $(P3)$. Set $s_{i\lambda} = \bar{f}_i \alpha_{i\lambda}(id_{Y_{i\lambda}})$, then $\bar{f}_i \alpha_{i\lambda} = h_{s_{i\lambda}}$ and $(s_{i\lambda} : Y_{i\lambda} \to X)_{(i,\lambda) \in M}$ is a covering for J^T. By $ii)$, Q is a covering for T'. Since the sieve generated by S_g contains the sieve generated by Q, S_g is a covering for T'.

Take an epimorphic family $(g_\lambda : h_{X_\lambda} \to K)_{\lambda \in N}$ and we denote by $\bar{f}_{i\lambda} : H_i \times_K h_{X_\lambda} \to h_{X_\lambda}$ the pull-back of f_i along g_λ. Then, $S_{g_\lambda} = (\bar{f}_{i\lambda} : H_i \times_K h_{X_\lambda} \to h_{X_\lambda})_{i \in I}$ a covering of h_{X_λ} for T'. Since T' is finer than the canonical topology, $(g_\lambda : h_{X_\lambda} \to K)_{\lambda \in N}$ is a covering for T'. Therefore $R = (g_\lambda \bar{f}_{i\lambda} : H_i \times_K h_{X_\lambda} \to K)_{(i,\lambda) \in I \times N}$ is a covering for T'. Let us denote by $\bar{g}_{i\lambda} : H_i \times_K h_{X_\lambda} \to H_i$ the pull-back of g_λ along f_i, then $g_\lambda \bar{f}_{i\lambda} = f_i \bar{g}_{i\lambda}$ and it follows that the sieve generated by S contains the sieve generated by S contains R. Thus, S is a cover for T' and T is coarser than T'. □

By (2.5.6) and (2.5.8), the above result implies the following.

Corollary 2.5.10 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site. T_J is the coarsest topology among the topology T on $\widehat{\mathcal{C}}$ having the following properties.*

i) T is finer than the canonical topology on $\widehat{\mathcal{C}}$.

ii) If $(f_i : X_i \to X)_{i \in I}$ is a covering of $X \in \mathrm{Ob}\, \mathcal{C}$ for J, $(h_{f_i} : h_{X_i} \to h_X)_{i \in I}$ is a covering for T in $\widehat{\mathcal{C}}$.

Corollary 2.5.11 *Let T be a topology on $\widehat{\mathcal{C}}$ finer than the canonical topology. If J^T is a $\mathcal{U}$-topology on $\mathcal{C}$ ($\mathcal{C}$ is $\mathcal{U}$-small, for example), then $T_{J^T} = T$.*

Proof. Since $J^{T_{J^T}} = J^T$ by (2.5.8), both T and T_{J^T} are the coarsest topologies satisfying the conditions $i)$ and $ii)$ of (2.5.9). Therefore $T_{J^T} = T$. □

Let $\mathcal{C}$ be a $\mathcal{U}$-small category. We denote by $\mathcal{C}_{ref}$ the set of reflexive strictly full subcategories of $\widehat{\mathcal{C}}$ such that the left adjoints of the inclusion functors are left exact. We also denote by $\mathcal{T}_{\mathcal{C}}$ the set of topologies on $\mathcal{C}$. Define a map $\Phi : \mathcal{T}_{\mathcal{C}} \to \mathcal{C}_{ref}$ by $\Phi(J) = \widetilde{\mathcal{C}}_J$ ((2.3.7)). For $\mathcal{D} \in \mathcal{C}_{ref}$, let $L : \widehat{\mathcal{C}} \to \mathcal{D}$ be a left adjoint of the inclusion morphism $i : \mathcal{D} \to \widehat{\mathcal{C}}$. We define a map $\Psi : \mathcal{C}_{ref} \to \mathcal{T}_{\mathcal{C}}$ by $\Psi(\mathcal{D}) = J^{T_{\mathcal{D}}}$ ((2.5.5),(2.5.6)). Note that this does not depend on the choice of the reflection L.

Theorem 2.5.12 *Φ is bijective and Ψ is its inverse.*

Proof. For a topology J on $\mathcal{C}$, we have $\Psi\Phi(J) = J^{T_{\widetilde{\mathcal{C}}_J}} = J^{T_J} = J$ by (2.5.6) and (2.5.8).

Let $\mathcal{D}$ be a reflexive full subcategory of $\widehat{\mathcal{C}}$ with a left exact reflection $L : \widehat{\mathcal{C}} \to \mathcal{D}$ and consider the topology $J = J^{T_{\mathcal{D}}}$ on $\mathcal{C}$. If $F \in \operatorname{Ob}\mathcal{D}$, then for any $X \in \operatorname{Ob}\mathcal{C}$ and $R \in J(X)$, since $L(\iota) : L(R) \to L(h_X)$ ($\iota : R \hookrightarrow h_X$) is an isomorphism by (2.5.8), the following commutative diagram implies $F \in \operatorname{Ob}\widetilde{\mathcal{C}}_J$.

$$\begin{tikzcd}
\widehat{\mathcal{C}}(h_X, F) \arrow[r, "\iota^*"] \arrow[d, "\cong"] & \widehat{\mathcal{C}}(R, F) \arrow[d, "\cong"] \\
\mathcal{D}(L(h_X), F) \arrow[r, "L(\iota)^*"] & \mathcal{D}(L(R), F)
\end{tikzcd}$$

Thus we see that $\mathcal{D}$ is a subcategory of $\widetilde{\mathcal{C}}_J$.

Let $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}_J$ be the associated sheaf functor and $f : H \to K$ a morphism in $\widehat{\mathcal{C}}$. We show that if $L(f) : L(H) \to L(K)$ is an isomorphism, so is $a(f)$. We denote by $f_1 : H \times_K H \to H$ the pull-back of f along f. Since L is left exact, $L(f_1)$ is a pull-back of $L(f)$ along $L(f)$, hence it is also an isomorphism. It follows that the morphism $L(\Delta) : L(H) \to L(H \times_K H)$ is an isomorphism. Since $T_{\mathcal{D}} = T_J$ by (2.5.11), $(f : H \to K)$ and $(\Delta : H \to H \times_K H)$ are coverings in the sense of (2.5.3). Thus f is a bicovering and $a(f)$ is an isomorphism by (2.5.4).

Suppose that F is a sheaf for $J^{T_{\mathcal{D}}}$. Let us denote by $i : \widetilde{\mathcal{C}}_J \to \widehat{\mathcal{C}}$, $j : \mathcal{D} \to \widehat{\mathcal{C}}$ the inclusion functors and $\eta : id_{\widehat{\mathcal{C}}} \to jL$, $\varepsilon : Lj \to id_{\mathcal{D}}$ the unit and counit of the adjunction $\mathcal{D} \underset{L}{\overset{j}{\rightleftarrows}} \widehat{\mathcal{C}}$ Since ε is a natural equivalence, $L(\eta_{iF}) : Li(F) \to LjLi(F)$ is an isomorphism, hence so is $a(\eta_{iF}) : aiF \to ajLi(F) = aikLi(F)$, where $k : \mathcal{D} \to \widetilde{\mathcal{C}}_J$ the inclusion functor. Then, F is isomorphic to $Li(F) \in \operatorname{Ob}\mathcal{D}$, therefore $F \in \operatorname{Ob}\mathcal{D}$. We conclude that $\mathcal{D} = \widetilde{\mathcal{C}}_J$ and $\Phi\Psi(\mathcal{D}) = \widetilde{\mathcal{C}}_J = \mathcal{D}$. □

2.6 Closed subpresheaves

Definition 2.6.1 *Let F be a presheaf on $\mathcal{C}$ and G a subpresheaf of F. We denote by $\iota : G \to F$ the inclusion morphism. Suppose that a topology J on $\mathcal{C}$ is given.*

1) We say that G is dense in F if ι is a covering (hence a bicovering).

2) We say that G is closed in F if, for $X \in \operatorname{Ob}\mathcal{C}$, $R \in J(X)$, morphisms $f : h_X \to F$ and $g : R \to G$ such that the following square commutes,

$$\begin{tikzcd}
R \arrow[r, "g"] \arrow[d, "\sigma"] & G \arrow[d, "\iota"] \\
h_X \arrow[r, "f"] & F
\end{tikzcd}$$

there is a morphism $s : h_X \to G$ such that $\iota s = f$. Here $\sigma : R \to h_X$ denotes the inclusion morphism.

Proposition 2.6.2 *Let G be a subpresheaf of F with the inclusion morphism $\iota : G \to F$. G is closed if and only if the following condition holds.*

If $\bar{\iota} : R \to h_X$ is the pull-back of ι along a morphism $f : h_X \to F$ and $R \in J(X)$, then $R = h_X$.

Proof. Suppose that G is closed and that $\bar{\iota} : R \to h_X$ is the pull-back of ι along a morphism $f : h_X \to F$ such that $R \in J(X)$. There is a morphism $s : h_X \to G$ such that $\iota s = f$. Since $f id_{h_X} = \iota s$, id_X and s induce a morphism $t : h_X \to R$ such that $\bar{\iota} t = id_{h_X}$. Hence $\bar{\iota} t \bar{\iota} = \bar{\iota}$. Since $\bar{\iota}$ is a monomorphism, we have $\bar{\iota} t = id_R$. Thus $\bar{\iota}$ is an isomorphism and it follows that $R = h_X$.

We show the converse. Let R be a covering sieve on X and $\sigma : R \to h_X$ the inclusion morphism. Suppose that the following diagram on the left commutes and the right one is a pull-back.

$$\begin{tikzcd}
R \arrow[r, "g"] \arrow[d, "\sigma"] & G \arrow[d, "\iota"] \\
h_X \arrow[r, "f"] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "s"] \arrow[d, "\bar{\iota}"] & G \arrow[d, "\iota"] \\
h_X \arrow[r, "f"] & F
\end{tikzcd}$$

There is a unique morphism $\sigma' : R \to S$ such that $\bar{\iota}\sigma' = \sigma$ and $s\sigma' = g$. Since $\bar{\iota}$ is regaded as an inclusion morphism, R is a subfunctor of S. Hence $S \in J(X)$ and it follows from the assumption that $S = h_X$ and $\bar{\iota} = id_{h_X}$. Thus we have $\iota s = f\bar{\iota} = f$. □

Proposition 2.6.3 *Let G be a closed subpresheaf of F and K is a dense subpresheaf of H with the inclusion morphisms $\iota : G \to F$ and $\kappa : K \to H$. If the following square commutes, there is a unique morphism $s : H \to G$ such that $\iota s = f$.*

$$\begin{tikzcd} K \arrow[r, "g"] \arrow[d, "\kappa"] & G \arrow[d, "\iota"] \\ H \arrow[r, "f"] & F \end{tikzcd}$$

Proof. For $X \in \mathrm{Ob}\,\mathcal{C}$ and $x \in H(X)$, let $\hat{x} : h_X \to H$ be the morphism defined by $\hat{x}_Y(\varphi) = H(\varphi)(x)$. Form a pull-back $\bar{\kappa} : R \to h_X$ of κ along $\hat{x}$. Since κ is a monomorphism and a covering, $R \in J(X)$. By the assumption, there is a morphism $t : h_X \to G$ such that $\iota t = f\hat{x}$. Then, $f_X(x) = f_X\hat{x}_X(id_X) = \iota_X t_X(id_X) \in \iota_X(G(X))$. Thus the image of f is contained in the image of ι and the result follows. □

Proposition 2.6.4 *Let G be a closed subpresheaf of F with the inclusion morphism $\iota : G \to F$.*

1) If H is a subpresheaf of F containing G and the inclusion morphism $\sigma : G \to H$ is a covering, then σ is an isomorphism.

2) For a morphism $f : H \to F$, the image of the pull-back $\bar{\iota} : K \to H$ of ι along f is closed.

3) If F is a sheaf, so is G.

Proof. 1) For $X \in \mathrm{Ob}\,\mathcal{C}$ and $x \in H(X)$, let $\hat{x} : h_X \to H$ be the morphism defined by $\hat{x}_Y(\varphi) = H(\varphi)(x)$. Form a pull-back $\bar{\kappa} : R \to h_X$ of κ along $\hat{x}$.

$$\begin{tikzcd} R \arrow[r, "\bar{x}"] \arrow[d, "\bar{\sigma}"] & G \arrow[d, "\sigma"] \\ h_X \arrow[r, "\hat{x}"] & H \end{tikzcd}$$

Since σ is a covering, $R \in J(X)$. On the other hand, let us denote by $\sigma' : H \to F$ the inclusion morphism. By (A.3.6), $\bar{\kappa}$ is also a pull-back of $\iota = \sigma'\sigma$ along $\sigma'\hat{x}$. It follows from (2.6.2) that $R = h_X$ and $\bar{\sigma} = id_{h_X}$. $x = \hat{x}_X(id_X) = \sigma_X\bar{x}_X(id_X) \in \sigma_X(G)$. Thus $\sigma_X : G(X) \to H(X)$ is bijective for all $X \in \mathrm{Ob}\,\mathcal{C}$.

2) Let $g : h_X \to H$ be a morphism and form a pull-back of $\bar{\iota}$ along g.

$$\begin{tikzcd} R \arrow[r, "\bar{f}"] \arrow[d, "\tilde{\iota}"] & K \arrow[r] \arrow[d, "\bar{\iota}"] & G \arrow[d, "\iota"] \\ h_X \arrow[r, "g"] & H \arrow[r, "f"] & F \end{tikzcd}$$

Suppose $R \in J(X)$. Since the outer rectangle is also a pull-back, we have $R = h_X$ by the assumption and (2.6.2). Thus the assertion follows from (2.6.2).

3) For $X \in \mathrm{Ob}\,\mathcal{C}$, $R \in J(X)$ and a morphism $g : R \to G$, since F is a sheaf, $\iota g : R \to F$ uniquely extends to a morphism $f : h_X \to F$, namely, f satisfies $f\sigma = \iota g$, where $\sigma : R \to h_X$ is the inclusion morphism. Since G is closed, there is a morphism $s : h_X \to G$ such that $\iota s = f$. Then, $\iota s\sigma = f\sigma = \iota g$ and it follows that $s\sigma = g$. If $s' : h_X \to G$ also satisfy $s'\sigma = g$, we have $\iota s'\sigma = \iota g$, that is, $\iota s'$ is also an extension of ιg. Since F is a sheaf, $\iota s' = \iota s$ and this implies $s = s'$. □

Proposition 2.6.5 *Let G be a subpresheaf of a separated presheaf F with the inclusion morphism $\iota : G \to F$. If G is a sheaf, G is closed.*

Proof. For $X \in \mathrm{Ob}\,\mathcal{C}$, $R \in J(X)$, $\sigma : R \to h_X$ denotes the inclusion morphism. Suppose that the following square commutes.

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, "\iota"] & G \arrow[d, "\sigma"] \\ h_X \arrow[r, "f"] & F \end{tikzcd}$$

Since G is a sheaf, there is a unique morphism $s : h_X \to G$ satisfying $s\sigma = g$. Then, $\iota s\sigma = \iota g = f\sigma$ and it follows that both ιs and f are extensions of $f\sigma$. Hence we have $\iota s = f$. □

Proposition 2.6.6 *Let G be a subpresheaf of a presheaf F with the inclusion morphism $\iota : G \to F$. Assume that J is a $\mathcal{U}$-topology on $\mathcal{C}$. There exists a unique closed subpresheaf $\overline{G}$ of F containing G such that G is dense in $\overline{G}$.*

Proof. We form a pull-back $\bar\iota : \overline{G} \to F$ of $ia(\iota) : iaG \to iaF$ along the unit $\eta_F : F \to iaF$.

$$\begin{CD} \overline{G} @>{\bar\eta}>> iaG \\ @V{\bar\iota}VV @VV{ia(\iota)}V \\ F @>{\eta_F}>> iaF \end{CD}$$

Then, by the naturality of the unit, there is a unique morphism $\iota' : G \to \overline{G}$ such that $\bar\iota\iota' = \iota$ and $\bar\eta\iota' = \eta_G$. Since ι is a monomorphism, so are $ia(\iota)$, ι' and $\bar\iota$. Hence we can regard $\overline{G}$ as a subpresheaf of F containing G. Note that both iaF and iaG are sheaves. iaG is closed in iaF by (2.6.5) and so is $\overline{G}$ in F by 2) of (2.6.4). Applying the associated sheaf functor to the above diagram, we see that $a(\bar\eta) : a\overline{G} \to aiaG$ is an isomorphism. Since $a(\eta_G)$ is also an isomorphism, it follows from $\bar\eta\iota' = \eta_G$ that $a(\bar\iota) : aG \to a\overline{G}$ is an isomorphism. Hence $\bar\iota$ is an bicovering by (2.5.4) and G is dense in $\overline{G}$. Suppose that G' is also a closed subpresheaf of F containing G such that G is dense in G'. Let $\sigma' : G \to G'$ and $\bar\sigma : G' \to F$ be the inclusion morphisms. Since $\bar\sigma\sigma' = \bar\iota\iota' = \iota$, there are morphisms $s : G' \to \overline{G}$ and $t : \overline{G} \to G'$ such that $\bar\iota s = \bar\sigma$ and $\bar\sigma t = \bar\iota$ by (2.6.3). Then, we have $\bar\iota st = \bar\iota$ and $\bar\sigma ts = \bar\sigma$. Hence $st = id_{\overline{G}}$ and $ts = id_{G'}$ and s is an isomorphism. Therefore $G' = \overline{G}$ in F. □

Definition 2.6.7 *We call $\overline{G}$ in the above proposition the closure of G in F.*

Proposition 2.6.8 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site and G a subpresheaf of a presheaf F on $\mathcal{C}$. $\iota : G \to F$ denotes the inclusion morphism.*

1) If H is a closed subpresheaf of F containing G, H also contains $\overline{G}$.

2) $\overline{\overline{G}} = \overline{G}$ and, if $G \subset H$, $\overline{G} \subset \overline{H}$.

3) Let $f : H \to F$ be a morphism of presheaves. The pull-back functor $f^* : \mathrm{Sub}(F) \to \mathrm{Sub}(H)$ preserves closures, that is, $f^*(\overline{G}) = \overline{f^*(G)}$.

Proof. 1) Let $\bar\iota : \overline{G} \to F$, $\iota' : G \to \overline{G}$, $\sigma' : G \to H$ and $\sigma : H \to F$ be the inclusion morphisms. Then, $\bar\iota\iota' = \sigma\sigma' = \iota$. By (2.6.3), there is a morphism $s : \overline{G} \to H$ such that $\sigma s = \bar\iota$. Hence H contains $\overline{G}$.

2) Since $\overline{G}$ is a closed subpresheaf of F containing $\overline{G}$, $\overline{G}$ contains $\overline{\overline{G}}$ by 1). Thus the first assertion follows. Since $\overline{H}$ is closed in F and contains G, the second assertion follows from 1).

3) Let $\hat\iota : f^*(\overline{G}) \to H$ be the pull-back of the inclusion morphism $\bar\iota : \overline{G} \to F$ along f. Then, $f^*(\overline{G})$ is closed in H by (2.6.4). Consider the pull-back $\tilde\iota : f^*(G) \to H$ of ι along f. There is a morphism $f^*(\iota') : f^*(G) \to f^*(\overline{G})$ such that $\hat\iota f^*(\iota') = \tilde\iota$ and the following diagram commutes.

$$\begin{CD} f^*(G) @>{f^*(\iota')}>> f^*(\overline{G}) @>{\hat\iota}>> H \\ @VVV @VV{\bar f}V @VV{f}V \\ G @>{\iota'}>> \overline{G} @>{\bar\iota}>> H \end{CD}$$

Since the outer rectangle and the right square are pull-backs, so is the left square. Hence $f^*(\iota')$ is a pull-back of the inclusion morphism $\iota' : G \to \overline{G}$, which is a covering. Therefore $f^*(\iota')$ is also a covering and $f^*(G)$ is dense in $f^*(\overline{G})$. By the uniqueness of the closure, we have $f^*(\overline{G}) = \overline{f^*(G)}$. □

Theorem 2.6.9 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site and $(f_j : F_j \to F)_{j\in I}$ a family of morphisms in $\widehat{\mathcal{C}}$. $(a(f_j) : aF_j \to aF)_{j\in I}$ is an epimorphic family in $\widetilde{\mathcal{C}}$ if and only if, for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $x \in F(X)$, a family of morphisms $(p : Y \to X | Y \in \mathrm{Ob}\,\mathcal{C},\ f_{jY}(y) = F(p)(x)$ for some $j \in I$, $y \in F_j(Y))$ in $\mathcal{C}$ is a covering.*

Proof. Suppose that $(a(f_j) : aF_j \to aF)_{j\in I}$ is an epimorphic family. For $X \in \mathrm{Ob}\,\mathcal{C}$, set $G(X) = \{x \in aF(X)|$ *there exists a covering $(p_k : X_k \to X)_{k\in K}$ such that for each $k \in K$, $F(p_k)(x) \in f_{jX_k}(F_j(X_k))$ for some $j \in I$*$\}$. Let $\varphi : Y \to X$ be a morphism in $\mathcal{C}$. If $x \in G(X)$, there is a covering $S = (p_k : X_k \to X)_{k\in K}$ such that for each $k \in K$, $F(p_k)(x) = f_{jX_k}(y)$ for some $j \in I$ and $y \in F_j(X_k)$. Then, $h_\varphi^{-1}(\bar S)$ is a covering sieve and, if $q : Z \to Y$ belongs to $h_\varphi^{-1}(\bar{S})$, $\varphi q = p_k r$ for some $k \in K$ and $r : Z \to X_k$. Hence $F(q)F(\varphi)(x) = F(r)F(p_k)(x) = F(r)f_{jX_k}(y) = f_{jZ}F_j(r)(y)$ and it follows that $F(\varphi)(x) \in G(Y)$. Thus we have a subpresheaf G of F. $\iota : G \to F$ denotes the inclusion morphism. We show that G is closed in F. Let $S = (p_k : X_k \to X)_{k\in K}$ be a covering and $g : \bar{S} \to G$, $\chi : h_X \to F$ morphisms in $\widehat{\mathcal{C}}$ such that $\iota g = f\sigma$, where $\sigma : \bar{S} \to h_X$ is the inclusion morphism. Put $x = \chi_X(id_X) \in F(X)$, then $F(p_k)(x) = \chi_{X_k}(p_k) = g_{X_k}(p_k) \in G(X_k)$ for any $k \in K$. By the definition of G, there exists a covering $S_k = (p_{kl} : X_{kl} \to X_k)_{l\in K_l}$ such that, for each $l \in K_l$, $F(p_k p_{kl})(x) = F(p_{kl})F(p_k)(x) = f_{j_{kl}X_{kl}}(y_{kl})$ for some $j_{kl} \in I$ and $y_{kl} \in F_{j_{kl}}(X_{kl})$. Since $(p_k p_{kl} : X_{kl} \to X)_{k\in K, l\in K_l}$ is a covering of X, it follows that $x \in G(X)$. Thus the morphism $\hat{x} : h_X \to G$ defined by $\hat{x}_Y(s) = G(\varphi)(x)$ ($\varphi \in h_X(Y)$) satisfies $\iota\hat{x} = \chi$. Therefore G is closed in F.

For any $X \in \mathrm{Ob}\,\mathcal{C}$, $j \in I$ and $x \in F_j(X)$, since $(id_X : X \to X)$ is a covering and $F(id_X)(f_{jX}(x)) = f_{jX}(x)$, $f_{jX}(x) \in G(X)$. Hence there is a unique morphism $\bar{f}_j : F_j \to G$ in $\widehat{\mathcal{C}}$ such that $f_j = \iota\bar{f}_j$. Since $(a(f_j) : aF_j \to aF)_{j\in I}$ is an epimorphic family, it follows that $a(\iota)$ is an epimorphism in $\widetilde{\mathcal{C}}$. Thus ι is a covering by (2.5.2). Hence G is dense in F and, by (2.6.4), we have $G = F$. We deduce that, for $X \in \mathrm{Ob}\,\mathcal{C}$, if $x \in F(X)$, there exists a covering $(p_k : X_k \to X)_{k\in K}$ such that for each $k \in K$, $F(p_k)(x) \in f_{jX_k}(F_j(X_k))$ for some $j \in I$. It follows that a family of morphisms $(p : Y \to X \,|\, Y \in \mathrm{Ob}\,\mathcal{C},\ f_{jY}(y) = F(p)(x)$ *for some* $j \in I$, $y \in F_j(Y))$ is a covering, for it contains the covering $(p_k : X_k \to X)_{k\in K}$.

We show the converse. For $X \in \mathrm{Ob}\,\mathcal{C}$ and $x \in F(X)$, let $\hat{x} : h_X \to iF$ be the morphism in $\widehat{\mathcal{C}}$ such that $\hat{x}_X(id_X) = x$. Consider a pull-back $\hat{f}_j : h_X \times_F F_j \to h_X$ of f_j along $\hat{x}$.

$$\begin{tikzcd} h_X \times_F F_j \arrow[r, "\hat{x}_j"] \arrow[d, "\hat{f}_j"] & F_j \arrow[d, "f_j"] \\ h_X \arrow[r, "\hat{x}"] & F \end{tikzcd}$$

We claim that a morphism $p : Y \to X$ satisfies $f_{jY}(y) = F(p)(x)$ for some $j \in I$ and $y \in F_j(Y)$ if and only if there exists a morphism $g : h_Y \to h_X \times_F F_j$ for some $j \in I$ such that $\hat{f}_j g = h_p$. In fact, if $p : Y \to X$ satisfies $f_{jY}(y) = F(p)(x)$ for some $j \in I$ and $y \in F_j(Y)$, then $f_j\hat{y} = \hat{x}h_p$, where $\hat{y} : h_Y \to F_j$ denotes the morphism in $\widehat{\mathcal{C}}$ such that $\hat{y}_Y(id_Y) = y$. There is a unique morphism $g : h_Y \to h_X \times_F F_j$ such that $\hat{f}_j g = h_p$ and $\hat{x}_j g = \hat{y}$. Conversely, suppose that exists a morphism $g : h_Y \to h_X \times_F F_j$ for some $j \in I$ such that $\hat{f}_j g = h_p$. We set $y = \hat{x}_{jY} g_Y(id_Y) \in F_j(Y)$. Then, $f_{jY}(y) = f_{jY}\hat{x}_{jY}g_Y(id_Y) = \hat{x}_Y \hat{f}_{jY} g_Y(id_Y) = \hat{x}_Y(p) = F(p)(x)$. Hence $(p : Y \to X \,|\, Y \in \mathrm{Ob}\,\mathcal{C},\ f_{jY}(y) = F(p)(x)$ *for some* $j \in I$, $y \in F_j(Y)) = (p : Y \to X \,|\, Y \in \mathrm{Ob}\,\mathcal{C}$, *there exists a morphism* $g : h_Y \to h_X \times_F F_j$ *for some* $j \in I$ *such that* $\hat{f}_j g = h_p)$.

Thus, by the assumption, $(\epsilon_J(p) : \epsilon_J(Y) \to \epsilon_J(X) \,|\, Y \in \mathrm{Ob}\,\mathcal{C}$, *there exists a morphism* $g : h_Y \to h_X \times_F F_j$ *for some* $j \in I$ *such that* $\hat{f}_j g = h_p)$ is an epimorphic family in $\widetilde{\mathcal{C}}$ by the assumption and (2.4.7). It follows that $(a(\hat{f}_j) : a(h_X \times_F F_j) \to \epsilon_J(X))_{j\in I}$ is an epimorphic family. In fact, if $\alpha a(\hat{f}_j) = \beta a(\hat{f}_j)$ for any $j \in I$, then for every $p : Y \to X$ such that there exists a morphism $g : h_Y \to h_X \times_F F_j$ for some $j \in I$ satisfying $\hat{f}_j g = h_p$, we have $\alpha\epsilon_J(p) = \alpha a(\hat{f}_j g) = \beta a(\hat{f}_j g) = \beta\epsilon_J(p)$. Thus $\alpha = \beta$.

By (A.4.2), $(\rho : hP\langle X, \rho\rangle \to F)_{\langle X,\rho\rangle \in \mathrm{Ob}(h\downarrow F)}$ is an epimorphic family. For each $\langle X, \rho\rangle \in \mathrm{Ob}\,(h\downarrow F)$ and $j \in I$, let $\hat{f}_j = \hat{f}_{j,\langle X,\rho\rangle} : hP\langle X, \rho\rangle \times_F F_j \to hP\langle X, \rho\rangle$ be the pull-back of f_j along ρ. $\rho_j : hP\langle X, \rho\rangle \times_F F_j \to F_j$ denotes the canonical morphism. We note that, if we put $x = \rho_X(id_X) \in F(X)$, $\hat{x} = \rho$. Since $(a(\hat{f}_{j,\langle X,\rho\rangle}) : a(hP\langle X, \rho\rangle \times_F F_j) \to \epsilon_J P\langle X, \rho\rangle)_{j\in I}$ is an epimorphic family for a fixed $\langle X, \rho\rangle \in \mathrm{Ob}(h\downarrow F)$ and $\rho\hat{f}_{j,\langle X,\rho\rangle} = f_j\rho_j$, $(a(f_j\rho_j) : a(hP\langle X, \rho\rangle \times_F F_j) \to aF)_{\langle X,\rho\rangle \in \mathrm{Ob}\,(h\downarrow F), j\in I}$ is an epimorphic family. Therefore $(a(f_j) : aF_j \to aF)_{j\in I}$ is an epimorphic family. □

Since the counit $\varepsilon : ai \to id_{\widetilde{\mathcal{C}}}$ is an equivalence, the above result implies the following.

Corollary 2.6.10 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site. A family of morphisms $(f_j : F_j \to F)_{j\in I}$ in $\widetilde{\mathcal{C}}$ is an epimorphic family if and only if, for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $x \in F(X)$, a family of morphisms $(p : Y \to X \,|\, Y \in \mathrm{Ob}\,\mathcal{C},\ f_{jY}(y) = F(p)(x)$ for some $j \in I$, $y \in F_j(Y))$ in $\mathcal{C}$ is a covering.*

2.7 Finitary algebraic theory in a regular category

Let $(\mathcal{T}; \omega_1, \dots, \omega_k)$, $(\mathcal{T}_0; \bar{\omega}_1, \dots, \bar{\omega}_{k_0})$ be finitary algebraic theories (A.11.1) and $T_0 : \mathcal{T}_0 \to \mathcal{T}$ a morphism of finitary algebraic theories such that $T_0\bar{\omega}_s = \omega_{\sigma(s)}$ for some map $\sigma : \{1, 2, \dots, k_0\} \to \{1, 2, \dots, k\}$. Suppose that $\mathcal{C}$ is a regular category and J is the regular epimorphism topology on $\mathcal{C}$ (2.2.11). Since $\widetilde{\mathcal{C}}_J$ is complete and the inclusion functor $i : \widetilde{\mathcal{C}}_J \to \widehat{\mathcal{C}}$ creates limits (2.2.10), embeddings $\tilde{h} : \mathcal{C} \to \widetilde{\mathcal{C}}_J$ and $i : \widetilde{\mathcal{C}}_J \to \widehat{\mathcal{C}}$ give fully faithful functors $\tilde{h}_{\mathcal{T}} : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\widetilde{\mathcal{C}}_J)$ and $i_{\mathcal{T}} : \mathcal{T}(\widetilde{\mathcal{C}}_J) \to \mathcal{T}(\widehat{\mathcal{C}})$, respectively (A.11.10). Then we have $h_{\mathcal{T}} = i_{\mathcal{T}}\tilde{h}_{\mathcal{T}} : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\widehat{\mathcal{C}})$.

Proposition 2.7.1 *Let F_0 be an object of $\mathcal{T}_0(\mathcal{C})$ and consider the category $\mathcal{T}(\mathcal{C}; \mathcal{T}_0, F_0)$. We set $U_{\mathcal{T}_0}(F_0) = (V_1, \dots, V_{k_0})$. If $f = (f_1, \dots, f_k), g = (g_1, \dots, g_k) : G \to F$ are morphisms in $\mathcal{T}(\mathcal{C}; \mathcal{T}_0, F_0)$ such that $U_{\mathcal{T}}(F) = (X_1, \dots, X_k)$, $U_{\mathcal{T}}(G) = (Y_1, \dots, Y_k)$ and $Y_s \overset{f_s}{\underset{g_s}{\rightrightarrows}} X_s \xrightarrow{p_s} Z_s$ is exact and $p_{\sigma(s)} = id_{V_s}$, then there exists a unique $\mathcal{T}$-structure H on $(Z_1, \dots, Z_k)$ such that $p = (p_1, \dots, p_k)$ is a morphism in $\mathcal{T}(\mathcal{C}; \mathcal{T}_0, F_0)$. Moreover, $G \overset{f}{\underset{g}{\rightrightarrows}} F \xrightarrow{p} H$ is exact in $\mathcal{T}(\mathcal{C})$ and $\mathcal{T}(\mathcal{C}; \mathcal{T}_0, F_0)$.*

Proof. We first consider the case that $\mathcal{C}$ has finite products. We have an exact sequence

$$\prod_{s=1}^{k} Y_s^{n_s} \overset{\prod_{s=1}^{k} f_s^{n_s}}{\underset{\prod_{s=1}^{k} g_s^{n_s}}{\rightrightarrows}} \prod_{s=1}^{k} X_s^{n_s} \xrightarrow{\prod_{s=1}^{k} p_s^{n_s}} \prod_{s=1}^{k} Z_s^{n_s}$$

for each $n_1, \dots, n_k \in \boldsymbol{N}$ by (A.8.18). For a morphism $\alpha : \coprod_{s=1}^{k} [n_s]_s \to \coprod_{s=1}^{k} [m_s]_s$ in $\mathcal{T}$, since $F(\alpha)\Bigl(\prod_{s=1}^{k} f_s^{m_s}\Bigr) = \Bigl(\prod_{s=1}^{k} f_s^{n_s}\Bigr)G(\alpha)$ and $F(\alpha)\Bigl(\prod_{s=1}^{k} g_s^{m_s}\Bigr) = \Bigl(\prod_{s=1}^{k} g_s^{n_s}\Bigr)G(\alpha)$, there is a unique morphism $H(\alpha) : \prod_{s=1}^{k} Z_s^{m_s} \to \prod_{s=1}^{k} Z_s^{n_s}$ satisfying $H(\alpha)\Bigl(\prod_{s=1}^{k} p_s^{m_s}\Bigr) = \Bigl(\prod_{s=1}^{k} p_s^{n_s}\Bigr)F(\alpha)$. By the uniqueness of $H(\alpha)$, we have a functor $H : \mathcal{T}^{op} \to \mathcal{C}$ given by $\coprod_{s=1}^{k} [n_s]_s \mapsto \prod_{s=1}^{k} Z_s^{n_s}$, $\Bigl(\alpha : \coprod_{s=1}^{k} [n_s]_s \to \coprod_{s=1}^{k} [m_s]_s\Bigr) \mapsto \Bigl(H(\alpha) : \prod_{s=1}^{k} Z_s^{m_s} \to \prod_{s=1}^{k} Z_s^{n_s}\Bigr)$. It is clear that $p = (p_1, \dots, p_k)$ is a morphism in $\mathcal{T}(\mathcal{C}; \mathcal{T}_0, F_0)$.

By (A.11.14), $G \overset{f}{\underset{g}{\rightrightarrows}} F$ is a kernel pair of $p : F \to H$, hence by (A.11.7), it suffices to show that p is a coequalizer of $G \overset{f}{\underset{g}{\rightrightarrows}} F$ in $\mathcal{T}(\mathcal{C})$. Let $q = (q_1, \dots, q_k) : F \to K$ be a morphism in $\mathcal{T}(\mathcal{C})$ satisfying $qf = qg$ and set $U_{\mathcal{T}}(K) = (W_1, \dots, W_k)$. There is a unique morphism $r = (r_1, \dots, r_k) : (Z_1, \dots, Z_k) \to (W_1, \dots, W_k)$ in $\mathcal{C}$ such that $q_s = r_s p_s$ for $s = 1, \dots, k$. For a morphism $\alpha : \coprod_{s=1}^{k} [n_s]_s \to \coprod_{s=1}^{k} [m_s]_s$ in $\mathcal{T}$, the outer rectangle and the left square of the following diagram commute.

$$\begin{tikzcd}
\prod_{s=1}^{k} X_s^{m_s} \arrow[r, "\prod_{s=1}^{k} p_s^{m_s}"] \arrow[d, "F(\alpha)"] & \prod_{s=1}^{k} Z_s^{m_s} \arrow[r, "\prod_{s=1}^{k} r_s^{m_s}"] \arrow[d, "H(\alpha)"] & \prod_{s=1}^{k} W_s^{m_s} \arrow[d, "K(\alpha)"] \\
\prod_{s=1}^{k} X_s^{n_s} \arrow[r, "\prod_{s=1}^{k} p_s^{n_s}"] & \prod_{s=1}^{k} Z_s^{n_s} \arrow[r, "\prod_{s=1}^{k} r_s^{n_s}"] & \prod_{s=1}^{k} W_s^{n_s}
\end{tikzcd}$$

Since $\prod_{s=1}^{k} p_s^{m_s} : \prod_{s=1}^{k} X_s^{m_s} \to \prod_{s=1}^{k} Z_s^{m_s}$ is an epimorphism, the right square of the above diagram also commutes. Thus r is a morphism of in $\mathcal{T}(\mathcal{C})$ and this implies that p is a coequalizer of $G \overset{f}{\underset{g}{\rightrightarrows}} F$.

In the general case, we consider the embedding $\tilde{h} : \mathcal{C} \to \widetilde{\mathcal{C}}_J$ (2.2.12). Since $\tilde{h}$ is exact,

$$\tilde{h}(Y_s) \overset{\tilde{h}(f_s)}{\underset{\tilde{h}(g_s)}{\rightrightarrows}} \tilde{h}(X_s) \xrightarrow{\tilde{h}(p_s)} \tilde{h}(Z_s)$$

is exact for $s = 1, 2, \dots, k$. Thus we have a unique $\mathcal{T}$-structure H' on $(\tilde{h}(Z_1), \dots, \tilde{h}(Z_k))$ such that

$$(\tilde{h}(p_1), \dots, \tilde{h}(p_k)) : \tilde{h}_{\mathcal{T}}(F) \to H'$$

is a morphism in $\mathcal{T}(\widetilde{\mathcal{C}}_J; T_0, \tilde{h}_{\mathcal{T}_0}(F_0))$. Since $\mathcal{T}(\mathcal{C}; T_0, F_0)$ is regarded as a full subcategory of $\mathcal{T}(\widetilde{\mathcal{C}}_J; T_0, \tilde{h}_{\mathcal{T}}(F_0))$ (A.11.10), $(Z_1, \ldots, Z_k)$ has a unique $\mathcal{T}$-structure $H \in \mathrm{Ob}\mathcal{T}(\mathcal{C}; T_0, F_0)$ such that $\tilde{h}_{\mathcal{T}}(H) = H'$. Then, H is the unique $\mathcal{T}$-structure on $(Z_1, \ldots, Z_k)$ such that p is a morphism of $\mathcal{T}$-models. The preceding argument shows that $\tilde{h}_{\mathcal{T}}(G) \overset{\tilde{h}_{\mathcal{T}}(f)}{\underset{\tilde{h}_{\mathcal{T}}(g)}{\rightrightarrows}} \tilde{h}_{\mathcal{T}}(F) \xrightarrow{\tilde{h}_{\mathcal{T}}(p)} \tilde{h}_{\mathcal{T}}(H)$ is exact in $\mathcal{T}(\widetilde{\mathcal{C}}_J)$ and $\mathcal{T}(\widetilde{\mathcal{C}}_J; T_0, \tilde{h}_{\mathcal{T}}(F_0))$. Since $\tilde{h}_{\mathcal{T}} : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\widetilde{\mathcal{C}}_J)$, $\tilde{h}_{\mathcal{T}} : \mathcal{T}(\mathcal{C}; T_0, F_0) \to \mathcal{T}(\widetilde{\mathcal{C}}_J; T_0, \tilde{h}_{\mathcal{T}}(F_0))$ are fully faithful, $G \overset{f}{\underset{g}{\rightrightarrows}} F \xrightarrow{p} H$ is exact in $\mathcal{T}(\mathcal{C})$ and $\mathcal{T}(\mathcal{C}; T_0, F_0)$ by (A.3.3). □

Corollary 2.7.2 *If $\mathcal{C}$ is a regular category, the forgetful functor $\widetilde{U}_{\mathcal{T}} : \mathcal{T}(\mathcal{C}; T_0, F_0) \to \mathcal{C}^{k-m}$ ($m = \mathrm{card}(\mathrm{Im}\,\sigma)$) preserves and reflects regular epimorphisms.*

Proof. We first note that the product category $\mathcal{C}^{k-m}$ is regular by (A.8.16). Let $q : F \to H$ be a regular epimorphism in $\mathcal{T}(\mathcal{C}; T_0, F_0)$ and $Y \overset{f'}{\underset{g'}{\rightrightarrows}} \widetilde{U}_{\mathcal{T}}(F)$ a kernel pair of $\widetilde{U}_{\mathcal{T}}(q) : \widetilde{U}_{\mathcal{T}}(F) \to \widetilde{U}_{\mathcal{T}}(H)$. By (A.11.14), there exist a unique $G \in \mathrm{Ob}\mathcal{T}(\mathcal{C}; T_0, F_0)$ and morphisms $f, g : G \to F$ in $\mathcal{T}(\mathcal{C}; T_0, F_0)$ such that $\widetilde{U}_{\mathcal{T}}(G) = Y$, $\widetilde{U}_{\mathcal{T}}(f) = f'$, $\widetilde{U}_{\mathcal{T}}(g) = g'$ and $G \overset{f}{\underset{g}{\rightrightarrows}} F$ is a kernel pair of q. Thus $G \overset{f}{\underset{g}{\rightrightarrows}} F \xrightarrow{q} H$ is exact in $\mathcal{T}(\mathcal{C}; T_0, F_0)$ by (A.8.14).

Let $p : \widetilde{U}_{\mathcal{T}}(F) \to Z$ be a coequalizer of f' and g' in $\mathcal{C}^{k-m}$. Then, $Y \overset{f'}{\underset{g'}{\rightrightarrows}} \widetilde{U}_{\mathcal{T}}(F) \xrightarrow{p} Z$ is an exact sequence in $\mathcal{C}^{k-m}$ by (A.8.14). We denote by $\{\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{k-m}\}$ ($\bar{\sigma}_1 < \bar{\sigma}_2 < \cdots < \bar{\sigma}_{k-m}$) the complement of the image of σ and set $U_{\mathcal{T}_0}(F_0) = (V_1, \ldots, V_{k_0})$. Thus we have an exact sequence $U_{\mathcal{T}}(G) \overset{\bar{f}}{\underset{\bar{g}}{\rightrightarrows}} U_{\mathcal{T}}(F) \xrightarrow{\bar{p}} \bar{Z}$ whose projection onto $(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{k-m})$-th component is the above exact sequence and projection onto $\sigma(s)$-th component is an exact sequence $V_s \overset{id_{V_s}}{\underset{id_{V_s}}{\rightrightarrows}} V_s \xrightarrow{id_{V_s}} V_s$, where $\bar{Z}$ is an object of $\mathcal{C}^k$ whose $\bar{\sigma}_s$-th component is the s-th component of Z and $\sigma(s)$-component is V_s. It follows from (2.7.1) that there exist an object K' of $\mathcal{T}(\mathcal{C}; T_0, F_0)$ and a morphism $p : F \to K'$ in $\mathcal{T}(\mathcal{C}; T_0, F_0)$ such that $U_{\mathcal{T}}(K') = \bar{Z}$, $U_{\mathcal{T}}(p) = \bar{p}$ and $G \overset{f}{\underset{g}{\rightrightarrows}} F \xrightarrow{p} K'$ is exact in $\mathcal{T}(\mathcal{C}; T_0, F_0)$. Hence there exists a unique isomorphism $k : K \to K'$ in $\mathcal{T}(\mathcal{C}; T_0, F_0)$ such that $p = kq$. Since $\bar{p}$ is a regular epimorphism in $\mathcal{C}^k$, so is $\widetilde{U}_{\mathcal{T}}(q)$.

Let $p : F \to K$ be a morphism in $\mathcal{T}(\mathcal{C}; T_0, F_0)$ such that $\widetilde{U}_{\mathcal{T}}(p)$ is a regular epimorphism. Consider a kernel pair $Y \overset{f'}{\underset{g'}{\rightrightarrows}} \widetilde{U}_{\mathcal{T}}(F)$ of $\widetilde{U}_{\mathcal{T}}(p)$, then it follows from (A.11.14) that there is a unique object G of $\mathcal{T}(\mathcal{C}, T_0, F_0)$ and morphisms $f, g : G \to F$ such that $\widetilde{U}_{\mathcal{T}}(G) = Y$, $\widetilde{U}_{\mathcal{T}}(f) = f'$, $\widetilde{U}_{\mathcal{T}}(g) = g'$ and $G \overset{f}{\underset{g}{\rightrightarrows}} F$ is a kernel pair of p. Since $Y \overset{f'}{\underset{g'}{\rightrightarrows}} \widetilde{U}_{\mathcal{T}}(F) \xrightarrow{\widetilde{U}_{\mathcal{T}}(p)} \widetilde{U}_{\mathcal{T}}(K)$ is exact in $\mathcal{C}^{k-m}$, (2.7.1) implies that p is a regular epimorphism in $\mathcal{T}(\mathcal{C}; T_0, F_0)$. □

Corollary 2.7.3 *Let $(\mathcal{T}; \omega_1, \ldots, \omega_k)$, $(\mathcal{T}_0; \bar{\omega}_1, \ldots, \bar{\omega}_{k_0})$, $(\mathcal{T}'; \omega'_1, \ldots, \omega'_l)$ and $(\mathcal{T}'_0; \bar{\omega}'_1, \ldots, \bar{\omega}'_{l_0})$ be finitary algebraic theories and $T_0 : \mathcal{T}_0 \to \mathcal{T}$, $T'_0 : \mathcal{T}'_0 \to \mathcal{T}'$, $T : \mathcal{T}' \to \mathcal{T}$ and $\overline{T} : \mathcal{T}'_0 \to \mathcal{T}_0$ morphisms of finitary algebraic theories such that $T_0\bar{\omega}_s = \omega_{\sigma(s)}$ $T'_0\bar{\omega}'_s = \omega'_{\sigma'(s)}$, $T\omega'_s = \omega_{\tau(s)}$, $\overline{T}\bar{\omega}'_s = \bar{\omega}_{\tau_0(s)}$ for each s and that $T_0\overline{T} = TT'_0$. Suppose that $\mathrm{Im}\,\sigma' = \tau^{-1}(\mathrm{Im}\,\sigma)$ and that the correspondence $s \mapsto \beta(s)$ in (A.11.7) is bijective. If $\mathcal{C}$ is a regular category, then, for $F_0 \in \mathrm{Ob}\mathcal{T}_0(\mathcal{C})$, $T^* : \mathcal{T}(\mathcal{C}; T_0, F_0) \to \mathcal{T}'(\mathcal{C}; T'_0, \overline{T}^*(F))$ reflects and preserves regular epimorphisms.*

Proof. The assertion follows from (2.7.3) and (A.11.7). □

Theorem 2.7.4 *Let $(\mathcal{T}; \omega_1, \ldots, \omega_k)$, $(\mathcal{T}_0; \bar{\omega}_1, \ldots, \bar{\omega}_{k_0})$ be finitary algebraic theories and $T_0 : \mathcal{T}_0 \to \mathcal{T}$ a morphism of finitary algebraic theories such that $T_0\bar{\omega}_s = \omega_{\sigma(s)}$ for some map $\sigma : \{1, 2, \ldots, k_0\} \to \{1, 2, \ldots, k\}$. If $\mathcal{C}$ is a regular category, then for $F_0 \in \mathrm{Ob}\mathcal{T}_0(\mathcal{C})$, $\mathcal{T}(\mathcal{C}; T_0, F_0)$ is also regular. Moreover, if $\mathcal{C}$ is exact and finitely complete, so is $\mathcal{T}(\mathcal{C}; T_0, F_0)$.*

Proof. By (A.11.14), R1 of (A.8.1) is satisfied in $\mathcal{T}(\mathcal{C};T_0,F_0)$.

If $G \overset{f}{\underset{g}{\rightrightarrows}} F$ is a kernel pair of a morphism $q : F \to M$ in $\mathcal{T}(\mathcal{C};T_0,F_0)$, $\widetilde{U}_{\mathcal{T}}(G) \overset{\widetilde{U}_{\mathcal{T}}(f)}{\underset{\widetilde{U}_{\mathcal{T}}(g)}{\rightrightarrows}} \widetilde{U}_{\mathcal{T}}(F)$ is a kernel pair of $\widetilde{U}_{\mathcal{T}}(q)$ in $\mathcal{C}^{k-m}$ by (A.11.14). Hence there exists a coequalizer $p : \widetilde{U}_{\mathcal{T}}(F) \to Z$ of $\widetilde{U}_{\mathcal{T}}(f)$ and $\widetilde{U}_{\mathcal{T}}(g)$ in $\mathcal{C}^{k-m}$ and it follows from (2.7.1) that there exists a unique object K of $\mathcal{T}(\mathcal{C};T_0,F_0)$ such that $G \overset{f}{\underset{g}{\rightrightarrows}} F \xrightarrow{p} K$ is exact in $\mathcal{T}(\mathcal{C};T_0,F_0)$. Thus R2 holds in $\mathcal{T}(\mathcal{C};T_0,F_0)$.

Let $p : F \to K$ be a regular epimorphism and $g : G \to K$ a morphism in $\mathcal{T}(\mathcal{C};T_0,F_0)$. Since $\widetilde{U}_{\mathcal{T}}(p)$ is a regular epimorphism in $\mathcal{C}^{k-m}$ by (2.7.2), a pull-back $q' : W \to \widetilde{U}_{\mathcal{T}}(K)$ of $\widetilde{U}_{\mathcal{T}}(p)$ along $\widetilde{U}_{\mathcal{T}}(g)$ exists in $\mathcal{C}^{k-m}$ and q' is a regular epimorphism. It follows from (A.11.14) that there exist a unique object M of $\mathcal{T}(\mathcal{C};T_0,F_0)$ and a morphism $q : M \to G$ in $\mathcal{T}(\mathcal{C};T_0,F_0)$ such that $\widetilde{U}_{\mathcal{T}}(M) = W$, $\widetilde{U}_{\mathcal{T}}(q) = q'$ and that q is a pull-back of p along g in $\mathcal{T}(\mathcal{C};T_0,F_0)$. Since the forgetful functor reflects regular epimorphisms by (2.7.2), $q : M \to G$ is a regular epimorphism in $\mathcal{T}(\mathcal{C};T_0,F_0)$. This shows R3.

Suppose that $\mathcal{C}$ is exact and has finite limits. Then, the forgetful functor $\widetilde{U}_{\mathcal{T}} : \mathcal{T}(\mathcal{C};T_0,F_0) \to \mathcal{C}^{k-m}$ is left exact by (A.11.14) and it preserves equivalence relations by (A.3.20). Hence, if $R \overset{a}{\underset{b}{\rightrightarrows}} F$ is an equivalence relation in $\mathcal{T}(\mathcal{C};T_0,F_0)$, $\widetilde{U}_{\mathcal{T}}(R) \overset{\widetilde{U}_{\mathcal{T}}(a)}{\underset{\widetilde{U}_{\mathcal{T}}(b)}{\rightrightarrows}} \widetilde{U}_{\mathcal{T}}(F)$ is an equivalence relation in $\mathcal{C}^{k-m}$. There exists a coequalizer $p' : \widetilde{U}_{\mathcal{T}}(F) \to Z$ of $\widetilde{U}_{\mathcal{T}}(a)$ and $\widetilde{U}_{\mathcal{T}}(b)$ so that $\widetilde{U}_{\mathcal{T}}(R) \overset{\widetilde{U}_{\mathcal{T}}(a)}{\underset{\widetilde{U}_{\mathcal{T}}(b)}{\rightrightarrows}} \widetilde{U}_{\mathcal{T}}(F) \xrightarrow{p'} Z$ is exact by (A.8.14). It follows from (2.7.1) that there exist an object K of $\mathcal{T}(\mathcal{C};T_0,F_0)$ and a morphism $p : F \to K$ in $\mathcal{T}(\mathcal{C};T_0,F_0)$ such that $\widetilde{U}_{\mathcal{T}}(K) = Z$, $\widetilde{U}_{\mathcal{T}}(p) = p'$ and $R \overset{a}{\underset{b}{\rightrightarrows}} F \xrightarrow{p} K$ is exact in $\mathcal{T}(\mathcal{C};T_0,F_0)$. Therefore equivalence relations in $\mathcal{T}(\mathcal{C};T_0,F_0)$ is effective and it follows that $\mathcal{T}(\mathcal{C};T_0,F_0)$ is exact. It follows from (A.4.7) and (A.11.14) that $\mathcal{T}(\mathcal{C};T_0,F_0)$ is finitely complete. □

Corollary 2.7.5 *Let $T_0 : \mathcal{T}_0 \to \mathcal{T}$ be a morphism of finitary algebraic theories satisfying the conditions of (A.11.6) and F_0 an object of $\mathcal{T}_0(\mathcal{C})$. Suppose that there is a morphism $T : \mathcal{T}_{ab}^{k-k_0} \to \mathcal{T}$ of finitary algebraic theories satisfying the condition of (A.11.28). If $\mathcal{C}$ is a finitely complete exact category, $\mathcal{T}(\mathcal{C};T_0,F_0)$ is an abelian category. In particular, $\mathcal{T}_{mod}(\mathcal{C};T_{an},A)$ is an abelian category for any ring A in $\mathcal{C}$.*

Proof. The result follows from (A.11.28), (2.7.4), (A.10.8), and the above result. □

Lemma 2.7.6 *1) If $\mathcal{C}$ and $\mathcal{D}$ are filtered category, so is $\mathcal{C} \times \mathcal{D}$.*

2) If $\mathcal{C}$ is a filtered category, the diagonal subcategory $\mathcal{D}$ ($\operatorname{Ob}\mathcal{D} = \{(X,X) | X \in \mathcal{C}\}$, $\mathcal{D}((X,X),(Y,Y)) = \{(f,f) | f \in \mathcal{C}(X,Y)\}$) of $\mathcal{C} \times \mathcal{C}$ is cofinal.

3) Let $\mathcal{D}$ be a $\mathcal{U}$-small filtered category and $\mathcal{C}$ a category with finite products such that $\mathcal{U}$-small filtered colimits in $\mathcal{C}$ commute with finite products. Suppose that colimits of functors $D_1, D_2 : \mathcal{D} \to \mathcal{C}$ exist. Define a functor $D : \mathcal{D} \to \mathcal{C}$ by $D(i) = D_1(i) \times D_2(i)$ and $D(\theta) = D_1(\theta) \times D_2(\theta)$. Then, $\varinjlim D$ exists and it is canonically isomorphic to $\varinjlim D_1 \times \varinjlim D_2$.

Proof. 1) and 2) are straightforward. For 3), let $(D_k(i) \xrightarrow{\rho_{ki}} L_k)_{i \in \operatorname{Ob}\mathcal{D}}$ $(k = 1,2)$ be a colimiting cone of D_k. Consider a functor $D' : \mathcal{D} \times \mathcal{D} \to \mathcal{C}$ given by $D'(i,j) = D_1(i) \times D_2(j)$ and $D'(\theta,\psi) = D_1(\theta) \times D_2(\psi)$. It follows from the assumption that $(D'(i,j) \xrightarrow{\rho_{1i} \times id_{D_{2j}}} L_1 \times D_2(j))_{i \in \operatorname{Ob}\mathcal{D}}$ and $(L_1 \times D_2(j) \xrightarrow{id_{L_1} \times \rho_{2i}} L_1 \times L_2)_{j \in \operatorname{Ob}\mathcal{D}}$ are colimiting cones of functors $i \mapsto D'(i,j)$ and $j \mapsto L_1 \times D_2(j)$, respectively. Hence $(D'(i,j) \xrightarrow{\rho_{1i} \times \rho_{2i}} L_1 \times L_2)_{(i,j) \in \operatorname{Ob}\mathcal{D} \times \mathcal{D}}$ is a colimiting cone of D'. We denote by $\Delta : \mathcal{D} \to \mathcal{D} \times \mathcal{D}$ the diagonal functor, that is, $\Delta(i) = (i,i)$ $(i \in \operatorname{Ob}\mathcal{D})$, $\Delta(\theta) = (\theta,\theta)$ $(\theta \in \operatorname{Mor}\mathcal{D})$. Then, $D = D'\Delta$ and by 2), the assertion follows. □

Proposition 2.7.7 *Let $T_0 : \mathcal{T}_0 \to \mathcal{T}$ be a morphism of finitary algebraic theories and $\mathcal{C}$ a category with finite products such that $\mathcal{U}$-small filtered colimits in $\mathcal{C}$ commute with finite products. Suppose that $\mathcal{U}$-small filtered colimits always exist in $\mathcal{C}$. Then, for $F_0 \in \operatorname{Ob}\mathcal{T}_0(\mathcal{C})$, $\mathcal{U}$-small filtered colimits always exist in $\mathcal{T}(\mathcal{C};T_0,F_0)$ and the forgetful functor $\widetilde{U}_{\mathcal{T}} : \mathcal{T}(\mathcal{C};T_0,F_0) \to \mathcal{C}^{k-m}$ preserves them. Moreover, $\widetilde{U}_{\mathcal{T}}$ "creates" $\mathcal{U}$-small filtered colimits in the following sense; For each colimiting cone $(\widetilde{U}_{\mathcal{T}}(D(i)) \xrightarrow{\alpha_i} M)_{i \in \operatorname{Ob}\mathcal{D}}$ of $\widetilde{U}_{\mathcal{T}}D : \mathcal{D} \to \mathcal{C}^{k-m}$, there exists a colimiting cone $(D(i) \xrightarrow{\rho_i} L)_{i \in \operatorname{Ob}\mathcal{D}}$ of D such that $\widetilde{U}_{\mathcal{T}}(L) = M$ and $\widetilde{U}_{\mathcal{T}}(\rho_i) = \alpha_i$ for any $i \in \operatorname{Ob}\mathcal{D}$. If*

$(D(i) \xrightarrow{\rho'_i} L')_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of D such that $\widetilde{U}_{\mathcal{T}}(L') = M$ and $\widetilde{U}_{\mathcal{T}}(\rho'_i) = \alpha_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$, the unique morphism $\varphi : L \to L'$ satisfying $\varphi\rho_i = \rho'_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$ is an isomorphism in $\mathcal{T}(\mathcal{C}; T_0, F_0)$ such that $\widetilde{U}_{\mathcal{T}}(\varphi) = id_M$.

Proof. Let $\mathcal{D}$ be a $\mathcal{U}$-small filtered category and $D : \mathcal{D} \to \mathcal{T}(\mathcal{C}; T_0, F_0)$ a functor. For an object n of $\mathcal{T}$, we denote by $E_n : \mathcal{T}(\mathcal{C}; T_0, F_0) \to \mathcal{C}$ the evaluation functor $E_n(F) = F(n)$, $E_n(f : F \to G) = (f_n : F(n) \to G(n))$. By the assumption, we have a colimiting cone $(E_nD(i) \xrightarrow{\rho^n_i} L(n))_{i\in \mathrm{Ob}\,\mathcal{D}}$ for each $n \in \mathrm{Ob}\,\mathcal{T}$. A morphism $\theta : l \to n$ in $\mathcal{T}$ defines a natural transformation $E_\theta : E_n \to E_l$ by $(E_\theta)_F = F(\theta) : F(n) \to F(l)$. Thus we have a unique morphism $L(\theta) : L(n) \to L(l)$ satisfying $L(\theta)\rho^n_i = \rho^l_i(E_\theta)_{D(i)}$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. Hence the correspondences $n \mapsto L(n)$, $\theta \mapsto L(\theta)$ give a functor $L : \mathcal{T}^{op} \to \mathcal{C}$ and $\rho^n_i : D(i)(n) \to L(n)$ defines a natural transformation $\rho_i : D(i) \to L(n)$.

Since $E_{n\coprod l}D(i) = D(i)(n \coprod l) = D(i)(n) \times D(i)(l) = E_nD(i) \times E_lD(i)$, it follows from the above lemma that $(E_{n\coprod l}D(i) \xrightarrow{\rho^l_i\times\rho^n_i} L(l) \times L(n))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $E_{n\coprod l}D$. Therefore $L : \mathcal{T}^{op} \to \mathcal{C}$ is product preserving. If $n \in \mathrm{Ob}\,\mathcal{T}_0$ and $\alpha : i \to j$ is a morphism in $\mathcal{D}$, $E_{T_0(n)}D(i) = D(i)(T_0(n)) = F_0(n)$ and $E_{T_0(n)}D(T_0(\alpha)) = id_{F_0}(n)$. Hence L is an object of $\mathcal{T}(\mathcal{C}; T_0, F_0)$. Now it is easy to verify that $(D(i) \xrightarrow{\rho_i} L)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of D.

It is obvious from the above argument that the forgetful functor preserves $\mathcal{U}$-small filtered colimits and that, for each colimiting cone $(\widetilde{U}_{\mathcal{T}}(D(i)) \xrightarrow{\alpha_i} M)_{i\in \mathrm{Ob}\,\mathcal{D}}$ of $\widetilde{U}_{\mathcal{T}}D : \mathcal{D} \to \mathcal{C}^{k-m}$, there exists a colimiting cone $(D(i) \xrightarrow{\rho_i} L)_{i\in \mathrm{Ob}\,\mathcal{D}}$ of D such that $\widetilde{U}_{\mathcal{T}}(L) = M$ and $\widetilde{U}_{\mathcal{T}}(\rho_i) = \alpha_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. Suppose that $(D(i) \xrightarrow{\rho'_i} L')_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of D such that $\widetilde{U}_{\mathcal{T}}(L') = M$ and $\widetilde{U}_{\mathcal{T}}(\rho'_i) = \alpha_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. Let $\varphi : L \to L'$ be the unique morphism satisfying $\varphi\rho_i = \rho'_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. Then, $\widetilde{U}_{\mathcal{T}}(\varphi)(\alpha_i) = \alpha_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$, and it follows that $\widetilde{U}_{\mathcal{T}}(\varphi) = id_M$. Since the forgetful functor reflects isomorphisms (A.11.7), φ is an isomorphism. □

2.8 Sheaves taking values in a category

Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{U}$-categories. A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is called a presheaf on $\mathcal{C}$ taking values in $\mathcal{D}$. For a presheaf $F : \mathcal{C}^{op} \to \mathcal{D}$ and an object S of $\mathcal{D}$, we denote by F^S a presheaf of $\mathcal{U}$-sets on $\mathcal{C}$ defined by $F^S(X) = \mathcal{D}(S, F(X))$ for $X \in \mathrm{Ob}\,\mathcal{C}$ and $F^S(f) = F(f)_*$ for $f \in \mathrm{Mor}\,\mathcal{C}$.

Definition 2.8.1 *Let $(\mathcal{C}, J)$ be a site and $\mathcal{D}$ a category. A presheaf $F : \mathcal{C}^{op} \to \mathcal{D}$ is called a sheaf on $\mathcal{C}$ taking values in $\mathcal{D}$ if for any object S of $\mathcal{D}$, F^S is a sheaf. We denote by $\mathrm{Sh}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})$ consisting of sheaves on $\mathcal{C}$ taking values in $\mathcal{D}$.*

The following assertion is straightforward from the definition and (2.2.3).

Proposition 2.8.2 *Let P be a pretopology on $\mathcal{C}$ and suppose that $\mathcal{D}$ has products. Then, a presheaf $F : \mathcal{C}^{op} \to \mathcal{D}$ is a sheaf if and only if for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $(f_i : X_i \to X)_{i\in I} \in P(X)$, the following diagram is an equalizer.*

$$F(X) \to \prod_{i\in I} F(X_i) \rightrightarrows \prod_{i,j\in I} F(X_i \times_X X_j)$$

That is, a family of morphisms $(\varphi_i : S \to F(X_i))_{i\in I}$ satisfying $F(p_{ij})\varphi_i = F(q_{ji})\varphi_j$ for any $i, j \in I$ induces a unique morphism $\varphi : S \to F(X)$ such that $F(f_i)\varphi = \varphi_i$ for any $i \in I$. Here we denote by $p_{ij} : X_i \times_X X_j \to X_i$ and $q_{ij} : X_i \times_X X_j \to X_j$ the canonical projections.

Recall that for an object X of $\mathcal{C}$ and $R \in J(X)$, R is a colimit of the functor $hP : (h{\downarrow}R) \to \widehat{\mathcal{C}}$ by (A.4.2). Let $F : \mathcal{C}^{op} \to \mathcal{D}$ be a presheaf and S an object of $\mathcal{D}$. Then, $(\widehat{\mathcal{C}}(R, F^S) \xrightarrow{f^*} \widehat{\mathcal{C}}(h_Y, F^S))_{\langle Y,f\rangle\in \mathrm{Ob}(h{\downarrow}R)}$ is a limiting cone of a functor $\langle Y, f\rangle \mapsto \widehat{\mathcal{C}}(h_Y, F^S)$, $\varphi \mapsto h^*_\varphi$. We note that the following diagram commutes for $(\varphi : \langle Z, g\rangle \to \langle Y, f\rangle) \in \mathrm{Mor}(h{\downarrow}R)$, where $\iota_R : R \to h_X$ is the inclusion morphism and we set $f_\sharp = f_Y(id_Y)$.

$$\begin{tikzcd}
 & \widehat{\mathcal{C}}(h_X, F^S) \arrow[r, "\cong"] \arrow[dl, "\iota_R^*"'] \arrow[d, "(\iota_R f)^*"] & \mathcal{D}(S, F(X)) \arrow[d, "F(f_\sharp)_*"] \\
\widehat{\mathcal{C}}(R, F^S) \arrow[r, "f^*"] \arrow[dr, "g^*"'] & \widehat{\mathcal{C}}(h_Y, F^S) \arrow[r, "\cong"] \arrow[d, "h^*_\varphi"] & \mathcal{D}(S, F(Y)) \arrow[d, "F(\varphi)_*"] \\
 & \widehat{\mathcal{C}}(h_Z, F^S) \arrow[r, "\cong"] & \mathcal{D}(S, F(Z))
\end{tikzcd}$$

Thus we have the following result.

Proposition 2.8.3 *A presheaf $F : \mathcal{C}^{op} \to \mathcal{D}$ is a sheaf if and only if, for any $S \in \mathrm{Ob}\,\mathcal{D}$, $X \in \mathrm{Ob}\,\mathcal{C}$ and $R \in J(X)$ $(\mathcal{D}(S, F(X)) \xrightarrow{F(f_\sharp)_*} \mathcal{D}(S, F(Y))))_{\langle Y,f\rangle \in \mathrm{Ob}(h\downarrow R)}$ is a limiting cone of a functor $(h\downarrow R) \to \mathcal{U}$-$\mathbf{Ens}$ defined by $\langle Y, f\rangle \mapsto \mathcal{D}(S, F(Y))$ and $\varphi \mapsto F(\varphi)_*$. In other words, F is a sheaf if and only if, for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $R \in J(X)$, $(F(X) \xrightarrow{F(f_\sharp)} F(Y))_{\langle Y,f\rangle\in \mathrm{Ob}(h\downarrow R)}$ is a limiting cone of a functor $FP : (h\downarrow R) \to \mathcal{D}$.*

Lemma 2.8.4 *Let F be a presheaf on a category $\mathcal{C}$. If there exists an epimorphic family $(f_i : h_{X_i} \to F)_{i\in I}$ in $\widehat{\mathcal{C}}$, a set of objects $\{\langle X_i, f_i\rangle | i \in I\}$ of $(h\downarrow F)$ is cofinal. That is, for any object $\langle X, f\rangle$ of $(h\downarrow F)$, there exist $i \in I$ and a morphism $\varphi : \langle X, f\rangle \to \langle X_i, f_i\rangle$ in $(h\downarrow F)$.*

Proof. By the assumption, there exist $i \in I$ and $\varphi \in h_{X_i}(X)$ such that $f_{iX}(\varphi) = f_X(id_X) \in F(X)$. Then, $f_i h_\varphi = f$ and $\varphi : \langle X, f\rangle \to \langle X_i, f_i\rangle$ is a morphism in $(h\downarrow F)$. □

Proposition 2.8.5 *Let $(\mathcal{C}, J)$ be a site, $K : \mathcal{D} \to \mathcal{A}$ a functor and F a sheaf on $\mathcal{C}$ taking values in $\mathcal{D}$. If one of the following conditions is satisfied, $KF : \mathcal{C}^{op} \to \mathcal{A}$ is a sheaf, hence we have a functor $K_* : \mathrm{Sh}(\mathcal{C}, \mathcal{D}) \to \mathrm{Sh}(\mathcal{C}, \mathcal{A})$.*
(1) K preserves arbitrary limits. (2) $(\mathcal{C}, J)$ is a $\mathcal{U}$-site and K preserves $\mathcal{U}$-limits.

Proof. (1) By the assumption, since $(F(X) \xrightarrow{F(f_\sharp)} F(Y))_{\langle Y,f\rangle\in\mathrm{Ob}(h\downarrow R)}$ is a limiting cone of a functor $FP : (h\downarrow R) \to \mathcal{D}$, applying the functor K to this cone, the result follows.

(2) We use the notations of (2.3.3). By (2.3.6), it suffices to show that for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $R \in J_G(X)$, $(KF(X) \xrightarrow{KF(f_\sharp)} KF(Y))_{\langle Y,f\rangle\in\mathrm{Ob}(h\downarrow R)}$ is a limiting cone of a functor $KFP : (h\downarrow R) \to \mathcal{D}$. If $R \in J_G(X)$, there exists an epimorphic family $(f_i : h_{X_i} \to X)_{i\in I}$ such that I is $\mathcal{U}$-small and $X_i \in G$ (2.3.3). It follows from (2.8.4) that a set $\{\langle X_i, f_i\rangle | \, i \in I\}$ of objects of $(h\downarrow R)$ is cofinal. Hence $(h\downarrow R)$ contains a $\mathcal{U}$-small cofinal full subcategory. Since K preserves $\mathcal{U}$-limits, the assertion follows. □

Let $(\mathcal{C}, J)$ be a site, $(\mathcal{D}_i)_{i\in I}$ a family of categories and $P_j : \prod_{i\in I} \mathcal{D}_i \to \mathcal{D}_j$ the projection functor. Since P_j preserves arbitrary limits (A.4.7), by (2.8.4), we have a functor $P_{j*} : \mathrm{Sh}(\mathcal{C}, \prod_{i\in I} \mathcal{D}_i) \to \mathrm{Sh}(\mathcal{C}, \mathcal{D}_j)$.

Proposition 2.8.6 *Let $\rho : \mathrm{Sh}(\mathcal{C}, \prod_{i\in I} \mathcal{D}_i) \to \prod_{i\in I} \mathrm{Sh}(\mathcal{C}, \mathcal{D}_i)$ be the functor whose j-th component is P_{j*}. Then, ρ is an isomorphism.*

Proof. For each $(F_i)_{i\in I} \in \mathrm{Ob} \prod_{i\in I} \mathrm{Sh}(\mathcal{C}, \mathcal{D}_i)$, define $F : \mathcal{C}^{op} \to \prod_{i\in I} \mathcal{D}_i$ by $F(X) = (F_i(X))_{i\in I}$ it is easy to verify that F is a sheaf on $\mathcal{C}$ and $\rho(F) = (F_i)_{i\in I}$. Since ρ is the restriction of the canonical isomorphism $\mathrm{Funct}(\mathcal{C}^{op}, \prod_{i\in I} \mathcal{D}_i) \to \prod_{i\in I} \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}_i)$ to a full subcategory $\mathrm{Sh}(\mathcal{C}, \prod_{i\in I} \mathcal{D}_i)$, it is an isomorphism. □

Let $(\mathcal{C}, J)$ be a site, $\mathcal{D}$ a category with finite products and $\mathcal{T}$ a k-fold finitary algebraic theory. Since $\mathrm{Sh}(\mathcal{C}, \mathcal{D})$ is a full subcategory of $\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})$, $\mathcal{T}(\mathrm{Sh}(\mathcal{C}, \mathcal{D}))$ is regarded as a full subcategory of $\mathcal{T}(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}))$.

Let $T : \mathcal{T}' \to \mathcal{T}$ be a morphism of finitary algebraic theories. If $(\mathcal{C}, J)$ is a $\mathcal{U}$-site and $\mathcal{D}$ is $\mathcal{U}$-complete, then $T^* : \mathcal{T}(\mathcal{D}) \to \mathcal{T}'(\mathcal{D})$ and $U_\mathcal{T} : \mathcal{T}(\mathcal{D}) \to \mathcal{D}^k$ preserves $\mathcal{U}$-limits by (A.11.14) and (A.11.15). Hence we have functors $(T^*)_* : \mathrm{Sh}(\mathcal{C}, \mathcal{T}(\mathcal{D})) \to \mathrm{Sh}(\mathcal{C}, \mathcal{T}'(\mathcal{D}))$ and $U_{\mathcal{T}*} : \mathrm{Sh}(\mathcal{C}, \mathcal{T}(\mathcal{D})) \to \mathrm{Sh}(\mathcal{C}, \mathcal{D}^k)$.

Proposition 2.8.7 *The isomorphism $\Phi : \mathcal{T}(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})) \to \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}))$ given in (A.11.11) maps $\mathcal{T}(\mathrm{Sh}(\mathcal{C}, \mathcal{D}))$ into $\mathrm{Sh}(\mathcal{C}, \mathcal{T}(\mathcal{D}))$. Moreover, if $(\mathcal{C}, J)$ is a $\mathcal{U}$-site and $\mathcal{D}$ is $\mathcal{U}$-complete, Φ maps $\mathcal{T}(\mathrm{Sh}(\mathcal{C}, \mathcal{D}))$ isomorphically onto $\mathrm{Sh}(\mathcal{C}, \mathcal{T}(\mathcal{D}))$ and the following diagrams commute.*

$$\begin{tikzcd}
\mathcal{T}(\mathrm{Sh}(\mathcal{C},\mathcal{D})) \arrow[r, "\Phi"] \arrow[d, "T^*"] & \mathrm{Sh}(\mathcal{C},\mathcal{T}(\mathcal{D})) \arrow[d, "(T^*)_*"] \\
\mathcal{T}'(\mathrm{Sh}(\mathcal{C},\mathcal{D})) \arrow[r, "\Phi"] & \mathrm{Sh}(\mathcal{C},\mathcal{T}'(\mathcal{D}))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T}(\mathrm{Sh}(\mathcal{C},\mathcal{D})) \arrow[r, "\Phi"] \arrow[d, "U_\mathcal{T}"] & \mathrm{Sh}(\mathcal{C},\mathcal{T}(\mathcal{D})) \arrow[d, "U_{\mathcal{T}*}"] \\
\mathrm{Sh}(\mathcal{C},\mathcal{D})^k & \mathrm{Sh}(\mathcal{C},\mathcal{D}^k) \arrow[l, "\rho"']
\end{tikzcd}$$

Here ρ denotes the isomorphism in (2.8.6). In particular, if $\mathcal{D}$ is the category of $\mathcal{U}$-sets, we have an isomorphism of categories $\mathcal{T}(\widetilde{\mathcal{C}}_\mathcal{U}) \cong \mathrm{Sh}(\mathcal{C}, \mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}))$.

Proof. For $n \in \operatorname{Ob}\mathcal{T}$, we denote by $E_n : \mathcal{T}(\mathcal{D}) \to \mathcal{D}$ the evaluation functor at n. Let F be an object of $\mathcal{T}(\operatorname{Sh}(\mathcal{C},\mathcal{D}))$. For $X \in \operatorname{Ob}\mathcal{C}$ and $R \in J(X)$, $(E_n\Phi(F)(X) \xrightarrow{E_n\Phi(F)(f_\sharp)} E_n\varphi(F)(Y))_{\langle Y,f\rangle\in\operatorname{Ob}(h\downarrow R)}$ is a limiting cone of a functor $E_n\Phi(F)P : (h\downarrow R) \to \mathcal{D}$. It follows from (A.4.1) that $(\Phi(F)(X) \xrightarrow{\Phi(F)(f_\sharp)} \varphi(F)(Y))_{\langle Y,f\rangle\in\operatorname{Ob}(h\downarrow R)}$ is a limiting cone of a functor $\Phi(F)P : (h\downarrow R) \to \mathcal{T}(\mathcal{D})$. Therefore $\Phi(F)$ is a sheaf.

Assume that $(\mathcal{C}, J)$ is a $\mathcal{U}$-site and $\mathcal{D}$ is $\mathcal{U}$-complete. There exists a $\mathcal{U}$-small topologically generating family G (2.3.3). Let F be an object of $\mathcal{T}(\operatorname{Funct}(\mathcal{C}^{op},\mathcal{D}))$ such that $\Phi(F)$ is an object of $\operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{D}))$. Then, for each $X \in \operatorname{Ob}\mathcal{C}$ and $R \in J_G(X)$, $(\Phi(F)(X) \xrightarrow{\Phi(F)(f_\sharp)} \varphi(F)(Y))_{\langle Y,f\rangle\in\operatorname{Ob}(h\downarrow R)}$ is a limiting cone of a functor $\Phi(F)P : (h\downarrow R) \to \mathcal{T}(\mathcal{D})$. Since $(h\downarrow R)$ contains a $\mathcal{U}$-small cofinal full subcategory, it follows from (2.2.1) that, for each $n \in \operatorname{Ob}\mathcal{T}$, $(E_n\Phi(F)(X) \xrightarrow{E_n\Phi(F)(f_\sharp)} E_n\varphi(F)(Y))_{\langle Y,f\rangle\in\operatorname{Ob}(h\downarrow R)}$ is a limiting cone of a functor $E_n\Phi(F)P : (h\downarrow R) \to \mathcal{D}$. Therefore $F(n)$ is a sheaf for each $n \in \operatorname{Ob}\mathcal{T}$, hence F is an object of $\mathcal{T}(\operatorname{Sh}(\mathcal{C},\mathcal{D}))$.

The commutativity of the diagrams follows from (A.11.11). □

Let $(\mathcal{T}_0; \bar{\omega}_1, \dots, \bar{\omega}_{k_0})$ be a finitary algebraic theory and $T_0 : \mathcal{T}_0 \to \mathcal{T}$ a morphism of finitary algebraic theories such that $T_0\bar{\omega}_s = \omega_{\sigma(s)}$. For $F_0 \in \operatorname{Ob}\mathcal{T}_0(\operatorname{Sh}(\mathcal{C},\mathcal{D}))$, define a subcategory $\operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{D}); T_0, F_0)$ of $\operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{D}))$ by

$$\operatorname{Ob}\operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{D}); T_0, F_0) = \{F : \mathcal{C} \to \mathcal{T}(\mathcal{D}) \,|\, T_0^*F = \Phi_0(F_0)\},$$

where $\Phi_0 : \mathcal{T}_0(\operatorname{Sh}(\mathcal{C},\mathcal{D})) \to \operatorname{Sh}(\mathcal{C},\mathcal{T}_0(\mathcal{D}))$ is the isomorphism in (2.8.7), and

$$\operatorname{Mor}\operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{D}); T_0, F_0) = \{(\theta : F \to G) \in \operatorname{Mor}\operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{D})) \,|\, T_0^*(\theta) = id_{\Phi_0(F_0)}\}.$$

As in (A.11.12), we have the following fact.

Proposition 2.8.8 *$\Phi : \mathcal{T}(\operatorname{Sh}(\mathcal{C},\mathcal{D})) \to \operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{D}))$ gives an isomorphism of categories $\mathcal{T}(\operatorname{Sh}(\mathcal{C},\mathcal{D}); T_0, F_0) \to \operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{D}); T_0, F_0)$. Moreover, for morphisms of finitary algebraic theories $T_0' : \mathcal{T}_0' \to \mathcal{T}'$, $T : \mathcal{T}' \to \mathcal{T}$ and $\overline{T} : \mathcal{T}_0' \to \mathcal{T}_0$ satisfying $T_0\overline{T} = TT_0'$ (A.11.4), the following diagrams commute.*

$$\begin{tikzcd}
\mathcal{T}(\operatorname{Sh}(\mathcal{C},\mathcal{D}); T_0, F_0) \arrow[r, "\Phi"] \arrow[d, "T^*"] & \operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{D}); T_0, F_0) \arrow[d, "(T^*)_*"] \\
\mathcal{T}'(\operatorname{Sh}(\mathcal{C},\mathcal{D}); T_0', \overline{T}^*(F_0)) \arrow[r, "\Phi"] & \operatorname{Sh}(\mathcal{C},\mathcal{T}'(\mathcal{D}); T_0', \overline{T}^*(F_0))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T}(\operatorname{Sh}(\mathcal{C},\mathcal{D}); T_0, F_0) \arrow[r, "\Phi"] \arrow[d, "\widetilde{U}_{\mathcal{T}}"] & \operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{D}); T_0, F_0) \arrow[d, "\widetilde{U}_{\mathcal{T}*}"] \\
\operatorname{Sh}(\mathcal{C},\mathcal{D})^{k-m} & \operatorname{Sh}(\mathcal{C},\mathcal{D}^{k-m}) \arrow[l, "\rho"']
\end{tikzcd}$$

Here ρ denotes the canonical isomorphism. In particular, if $\mathcal{D}$ is the category of $\mathcal{U}$-sets, we have an isomorphism of categories $\mathcal{T}(\widetilde{\mathcal{C}}_{\mathcal{U}}; T_0, F_0) \cong \operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}); T_0, F_0)$.

Proposition 2.8.9 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site and F_0 an object of $\mathcal{T}_0(\widetilde{\mathcal{C}})$. Suppose that $T_0 : \mathcal{T}_0 \to \mathcal{T}$ satisfies (A.11.6). Then, the inclusion functor $\iota : \operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}); T_0, F_0) \to \operatorname{Funct}(\mathcal{C}^{op},\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}); T_0, F_0)$ has a left adjoint*

$$\alpha : \operatorname{Funct}(\mathcal{C}^{op},\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}); T_0, F_0) \to \operatorname{Sh}(\mathcal{C},\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}); T_0, F_0).$$

Proof. By (2.8.8), it suffices to show that the functor $i_{\mathcal{T}} : \mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0) \to \mathcal{T}(\widehat{\mathcal{C}}; T_0, i_{\mathcal{T}_0}(F_0))$ induced by the inclusion functor $i : \widetilde{\mathcal{C}} \to \widehat{\mathcal{C}}$ has a left adjoint. Since i has a left exact left adjoint $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}$ by (2.3.7) and F_0 takes values in the category of sheaves, we can apply (A.11.20). □

The next result follows from (2.8.8) and (A.11.21).

Proposition 2.8.10 *Let $T_0 : \mathcal{T}_0 \to \mathcal{T}$, $T_0' : \mathcal{T}_0' \to \mathcal{T}'$, $T : \mathcal{T}' \to \mathcal{T}$, $\overline{T} : \mathcal{T}_0' \to \mathcal{T}_0$ be morphisms of finitary algebraic theories satisfying $T_0\overline{T} = TT_0'$ and $\operatorname{Im}\sigma' = \tau^{-1}(\operatorname{Im}\sigma)$. Suppose that $T_0 : \mathcal{T}_0 \to \mathcal{T}$ and $T_0' : \mathcal{T}_0' \to \mathcal{T}'$ satisfy (A.11.6). Then, the following diagrams commute, where $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}$ denotes the associated sheaf functor.*

$$\begin{tikzcd}
\operatorname{Funct}(\mathcal{C}^{op},\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}); T_0, F_0) \arrow[r, "\alpha"] \arrow[d, "\operatorname{Funct}(id_{\mathcal{C}^{op}}, T^*)"] & \operatorname{Sh}(\mathcal{C}^{op},\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}); T_0, F_0) \arrow[d, "(T^*)_*"] \\
\operatorname{Funct}(\mathcal{C}^{op},\mathcal{T}'(\mathcal{U}\text{-}\mathbf{Ens}); T_0', \overline{T}^*(F_0)) \arrow[r, "\alpha"] & \operatorname{Sh}(\mathcal{C}^{op},\mathcal{T}'(\mathcal{U}\text{-}\mathbf{Ens}); T_0', \overline{T}^*(F_0))
\end{tikzcd}$$

$$\begin{tikzcd}
\operatorname{Funct}(\mathcal{C}^{op},\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}); T_0, F_0) \arrow[r, "\alpha"] \arrow[d, "\operatorname{Funct}(id_{\mathcal{C}^{op}}, \widetilde{U}_{\mathcal{T}})"] & \operatorname{Sh}(\mathcal{C}^{op},\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}); T_0, F_0) \arrow[d, "\widetilde{U}_{\mathcal{T}*}"] \\
\widehat{\mathcal{C}}^{k-m} \arrow[r, "a^{k-m}"] & \widetilde{\mathcal{C}}^{k-m}
\end{tikzcd}$$

Proposition 2.8.11 *Assume that* $\operatorname{Im}\sigma' = \tau^{-1}(\operatorname{Im}\sigma)$ *and the correspondence in (A.11.7)* $s \mapsto \beta(s)$ *is bijective and that* $T_0 : \mathcal{T}_0 \to \mathcal{T}$ *satisfies (A.11.6). If* $\mathcal{T}$ *is a* $\mathcal{U}$*-category,* $(\mathcal{C}, J)$ *is a* $\mathcal{U}$*-site and* F_0 *is an object of* $\mathcal{T}_0(\widetilde{\mathcal{C}})$*,* $T^* : \mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0) \to \mathcal{T}'(\widetilde{\mathcal{C}}; T'_0, \overline{T}^*(F_0))$ *has a left adjoint. In particular, the forgetful functor* $\widetilde{U}_{\mathcal{T}} : \mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0) \to \widetilde{\mathcal{C}}^{k-m}$ *has a left adjoint.*

Proof. Let us denote by $a_{\mathcal{T}} : \mathcal{T}(\widehat{\mathcal{C}}; T_0, i_{\mathcal{T}_0}(F_0)) \to \mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$ the left adjoint of $i_{\mathcal{T}} : \mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0) \to \mathcal{T}(\widehat{\mathcal{C}}; T_0, i_{\mathcal{T}_0}(F_0))$ ((2.8.9)). By (A.11.17), $T^* : \mathcal{T}(\widehat{\mathcal{C}}; T_0, i_{\mathcal{T}}(F_0)) \to \mathcal{T}'(\widehat{\mathcal{C}}; T'_0, \overline{T}^*(i_{\mathcal{T}}(F_0)))$ has a left adjoint $L : \mathcal{T}'(\widehat{\mathcal{C}}; T'_0, \overline{T}^*(i_{\mathcal{T}}(F_0))) \to \mathcal{T}(\widehat{\mathcal{C}}; T_0, i_{\mathcal{T}}(F_0))$. Define $\tilde{L} : \mathcal{T}'(\widetilde{\mathcal{C}}; T'_0, \overline{T}^*(F_0)) \to \mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$ to be the composition $\mathcal{T}'(\widetilde{\mathcal{C}}; T'_0, \overline{T}^*(F_0)) \xrightarrow{i_{\mathcal{T}}} \mathcal{T}'(\widehat{\mathcal{C}}; T'_0, \overline{T}^*(i_{\mathcal{T}}(F_0))) \xrightarrow{L} \mathcal{T}(\widehat{\mathcal{C}}; T_0, i_{\mathcal{T}}(F_0)) \xrightarrow{a_{\mathcal{T}}} \mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$. Then, $\tilde{L}$ is a left adjoint of $T^* : \mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0) \to \mathcal{T}'(\widetilde{\mathcal{C}}; T'_0, \overline{T}^*(F_0))$. □

Proposition 2.8.12 *Let* $(\mathcal{C}, J)$ *be a* $\mathcal{U}$*-site,* $T_0 : \mathcal{T}_0 \to \mathcal{T}$ *a morphism of finitary algebraic theories and* F_0 *is an object of* $\mathcal{T}_0(\widetilde{\mathcal{C}})$*.* $\mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$ *is a* $\mathcal{U}$*-complete exact category with* $\mathcal{U}$*-small set of generators.*

Proof. The result follows from (A.11.23), (2.4.17) and the above result. □

Proposition 2.8.13 *Let* $(\mathcal{C}, J)$ *be a* $\mathcal{U}$*-site,* $T_0 : \mathcal{T}_0 \to \mathcal{T}$ *a morphism of finitary algebraic theories satisfying the conditions of (A.11.6) and* F_0 *an object of* $\mathcal{T}_0(\widetilde{\mathcal{C}})$*. Suppose that there is a morphism* $T : \mathcal{T}_{ab}^{k-k_0} \to \mathcal{T}$ *of finitary algebraic theories satisfying the condition of (A.11.28). Then,* $\mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$ *is a* $\mathcal{U}$*-complete abelian category with* $\mathcal{U}$*-small set of generators having the following properties.*

i) For any family of objects $(M_i)_{i\in I}$ *of* $\mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$ *such that* I *is* $\mathcal{U}$*-small, a direct sum* $\bigoplus_{i\in I} M_i$ *exists.*

ii) Let M *be an object of* $\mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$ *and* $(M_i)_{i\in I}$ *a directed family of subobjects of* M*. For any subobject* N *of* M*, we have* $(\sum_{i\in I} M_i) \cap N = \sum_{i\in I}(M_i \cap N)$*.*

Proof. It follows from (2.7.5) and (2.8.12) that $\mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$ is a $\mathcal{U}$-complete abelian category with $\mathcal{U}$-small set of generators. Let $(M_i)_{i\in I}$ of $\mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$ be a family of objects of $\mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$ such that I is $\mathcal{U}$-small. Let us denote by $\mathcal{F}$ the category of finite subsets of I. Consider a functor $\Sigma : \mathcal{F} \to \mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$ given as follows. Set $\Sigma(J) = \bigoplus_{i\in J} M_i$ and let $\iota_j^J : M_j \to \bigoplus_{i\in J} M_i$ be the canonical morphism into the j-th summand. For an inclusion $\theta : J \to K$, $\Sigma(\theta) : \Sigma(J) \to \Sigma(K)$ is the morphism satisfying $\Sigma(\theta)\iota_j^J = \iota_j^K$. Since $\mathcal{F}$ is a $\mathcal{U}$-small filtered category, it follows from (2.4.1) and (2.7.7) that the colimit of Σ exists, which is a direct sum of $(M_i)_{i\in I}$.

Let M be an object of $\mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$, N a subobject of M and $(M_i)_{i\in I}$ a directed family of subobjects of M. It follows from (A.4.12) that we may assume that I is $\mathcal{U}$-small. Embed $\mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$ into $\mathcal{T}(\widehat{\mathcal{C}}; T_0, F_0)$, then it is easy to verify that $(\sum_{i\in I} M_i) \cap N = \sum_{i\in I}(M_i \cap N)$ holds in $\mathcal{T}(\widehat{\mathcal{C}}; T_0, F_0)$. Since the embedding $i_{\mathcal{T}} : \mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0) \to \mathcal{T}(\widehat{\mathcal{C}}; T_0, F_0)$ induced by $i : \widetilde{\mathcal{C}} \to \widehat{\mathcal{C}}$ has a left exact left adjoint $a_{\mathcal{T}} : \mathcal{T}(\widehat{\mathcal{C}}; T_0, F_0) \to \mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$. We note that both $i_{\mathcal{T}}$ and $a_{\mathcal{T}}$ preserves subobjects, hence we have a pair of functors $i_{\mathcal{T}*} : \operatorname{Sub}_{\widetilde{\mathcal{C}}}(M) \to \operatorname{Sub}_{\widehat{\mathcal{C}}}(i_{\mathcal{T}}(M))$, $a_{\mathcal{T}*} : \operatorname{Sub}_{\widehat{\mathcal{C}}}(i_{\mathcal{T}}(M)) \to \operatorname{Sub}_{\widetilde{\mathcal{C}}}(M)$ such that $a_{\mathcal{T}*}$ is a left adjoint of $i_{\mathcal{T}}$. By applying $a_{\mathcal{T}}$ to the above equality, we have $(\sum_{i\in I} M_i) \cap N = \sum_{i\in I}(M_i \cap N)$ holds in $\mathcal{T}(\widetilde{\mathcal{C}}; T_0, F_0)$. □

2.9 Filtering functor

Definition 2.9.1 *Let* $\mathcal{E}$ *be a* $\mathcal{U}$*-category and* J *a topology on* $\mathcal{E}$*. A functor* $K : \mathcal{C} \to \mathcal{E}$ *is said to be* J*-filtering if it satisfies the following conditions.*

(1) For any object U *of* $\mathcal{E}$*, there exist a covering* $(p_i : U_i \to U)_{i\in I}$ *in* $\mathcal{E}$ *and a family of morphisms* $(q_i : U_i \to K(X_i))_{i\in I}$ *in* $\mathcal{E}$*.*

(2) For any two objects Y *and* Z *of* $\mathcal{C}$ *and any diagram* $K(Y) \xleftarrow{f} U \xrightarrow{g} K(Z)$ *in* $\mathcal{E}$*, there exist a covering* $(p_i : U_i \to U)_{i\in I}$ *in* $\mathcal{E}$*, a family of diagrams* $(Y \xleftarrow{u_i} X_i \xrightarrow{v_i} Z)_{i\in I}$ *in* $\mathcal{C}$ *and a family of morphisms* $(q_i : U_i \to K(X_i))_{i\in I}$ *in* $\mathcal{E}$ *such that* $fp_i = K(u_i)q_i$ *and* $gp_i = K(v_i)q_i$ *hold for every* $i \in I$*.*

(3) For any parallel arrows $Y \underset{t}{\overset{s}{\rightrightarrows}} Z$ *in* $\mathcal{C}$ *and any morphism* $f : U \to K(Y)$ *satisfying* $K(s)f = K(t)f$*, there exist a covering* $(p_i : U_i \to U)_{i\in I}$ *in* $\mathcal{E}$*, a family of morphisms* $(w_i : X_i \to Y)_{i\in I}$ *in* $\mathcal{C}$ *and* $(q_i : U_i \to K(X_i))_{i\in I}$ *in* $\mathcal{C}$ *such that* $fp_i = K(w_i)q_i$ *and* $sw_i = tw_i$ *hold for every* $i \in I$*.*

If J *is the canonical topology on* $\mathcal{E}$*, a functor satisfying the above conditions is simply called a filtering functor.*

Proposition 2.9.2 *Let $(\mathcal{E}, J)$ be a site and $K : \mathcal{C} \to \mathcal{E}$ a functor.*

1) If $K : \mathcal{C} \to \mathcal{E}$ is a fully faithful functor such that, for any object U of $\mathcal{E}$, there exist a family of objects $(X_i)_{i\in I}$ of $\mathcal{C}$ and a covering $(p_i : K(X_i) \to U)_{i\in I}$, K is a J-filtering functor.

2) Suppose that J is a $\mathcal{U}$-topology and let $\epsilon_J : \mathcal{E} \to \widetilde{\mathcal{E}}$ be the canonical functor. If K is J-filtering, $\epsilon_J K : \mathcal{C} \to \widetilde{\mathcal{E}}$ is filtering.

Proof. 1) The condition (1) of (2.9.1) is obviously satisfied ($q_i = id_{K(X_i)}$). For $Y, Z \in \mathrm{Ob}\,\mathcal{C}$ and a diagram $K(Y) \xleftarrow{f} U \xrightarrow{g} K(Z)$ in $\mathcal{E}$, there exists a covering $(p_i : K(X_i) \to U)_{i\in I}$. Since K is fully faithful, there exist unique morphisms $u_i : X_i \to Y$ and $v_i : X_i \to Z$ for each $i \in I$ such that $K(u_i) = fp_i$ and $K(v_i) = gp_i$. Putting $q_i = id_{K(X_i)}$, we see that the condition (2) of (2.9.1) is satisfied. For a parallel arrows $Y \underset{t}{\overset{s}{\rightrightarrows}} Z$ in $\mathcal{C}$ and a morphism $f : U \to K(Y)$ satisfying $K(s)f = K(t)f$, there exist a covering $(p_i : K(X_i) \to U)_{i\in I}$ in $\mathcal{E}$. There exist a unique morphism $w_i : X_i \to Y$ for each $i \in I$ such that $fp_i = K(w_i)$. Then, $K(sw_i) = K(s)K(w_i) = K(s)fp_i = K(t)fp_i = K(t)K(w_i) = K(tw_i)$. Since K is fully faithful, we have $sw_i = tw_i$ for every $i \in I$. Hence the condition (3) of (2.9.1) is satisfied for $q_i = id_{K(X_i)}$.

2) This is a direct consequence of (2.4.6) and (2.4.7). □

Proposition 2.9.3 *Let $(\mathcal{E}, J)$ be a site. A functor $K : \mathcal{C} \to \mathcal{E}$ is J-filtering if and only if for each object U of $\mathcal{E}$ and each finite diagram $(\langle f_m, X_m\rangle \xrightarrow{\theta_{m,n}} \langle f_n, X_n\rangle)$ in $(U{\downarrow}K)$, there exists a covering $(p_i : U_i \to U)_{i\in I}$ such that for each $i \in I$, there is a cone $(\langle c_i, C_i\rangle \xrightarrow{\pi_m} \langle f_m p_i, X_m\rangle)$ of $(\langle f_m p_i, X_m\rangle \xrightarrow{\theta_{m,n}} \langle f_n p_i, X_n\rangle)$ in $(U_i{\downarrow}K)$.*

Proof. Suppose that a functor $K : \mathcal{C} \to \mathcal{E}$ is J-filtering. Let $(\langle f_m, X_m\rangle \xrightarrow{\theta_{m,n}} \langle f_n, X_n\rangle)$ a finite diagram in $(U{\downarrow}K)$. We show the assertion by induction on the number of non-identity morphisms in the given diagram. First assume that the diagram contains only identity morphisms. If the diagram is empty, by (1), there exist a covering $(p_i : U_i \to U)_{i\in I}$ in $\mathcal{E}$ and a family of morphisms $(q_i : U_i \to K(X_i))_{i\in I}$ in $\mathcal{E}$. Then $\langle q_i, X_i\rangle$ is a cone of the empty diagram in $(U_i{\downarrow}K)$. If the diagram consists of a single object, the assertion is trivial. Inductively, assume that the assertion holds if a diagram has less than n objects. Let $\mathcal{D}$ be a diagram consisting of n objects $\langle f_1, X_1\rangle, \ldots, \langle f_n, X_n\rangle$. Then, there is a covering $(p_i : U_i \to U)_{i\in I}$ such that for each $i \in I$, there is a cone $(\langle c_i, C_i\rangle \xrightarrow{\pi_m} \langle f_m p_i, X_m\rangle)$ of the diagram $(\langle f_m p_i, X_m\rangle \xrightarrow{id_{X_m}} \langle f_m p_i, X_m\rangle)_{m=1,\ldots,n-1}$. Applying the condition (2) of (2.9.1) to a diagram $K(X_n) \xleftarrow{f_n p_i} U_i \xrightarrow{c_i} K(C)$ in $\mathcal{E}$, there exist a covering $(p_{ij} : U_{ij} \to U_i)_{j\in J_i}$ in $\mathcal{E}$, a family of diagrams $(X_n \xleftarrow{u_{ij}} C_{ij} \xrightarrow{v_{ij}} C_i)_{j\in J_i}$ in $\mathcal{C}$ and a family of morphisms $(c_{ij} : U_{ij} \to K(C_{ij}))_{j\in J_i}$ in $\mathcal{E}$ such that $f_n p_i p_{ij} = K(u_{ij})c_{ij}$ and $c_i p_{ij} = K(v_{ij})c_{ij}$ hold for every $j \in J_i$. Note that $(p_i p_{ij} : U_{ij} \to U)_{(i,j)\in M}$ ($M = \bigcup_{i\in I}(\{i\} \times J_i)$) is a covering by ($P3$) of (2.1.9). Set $\pi'_m = \pi_m v_{ij}$ for $1 \leqq m \leqq n-1$ and $\pi'_n = u_{ij}$. Then $(\langle c_{ij}, C_{ij}\rangle \xrightarrow{\pi'_m} \langle f_m p_i p_{ij}, X_m\rangle)$ is a cone in $(U_{ij}{\downarrow}K)$. Therefore the assertion holds if the diagram has no non-identity morphism.

Let $\mathcal{D}$ be a finite diagram in $(U{\downarrow}K)$ such that there exist a covering $(p_i : U_i \to U)_{i\in I}$ and a cone $(\langle c_i, C_i\rangle \xrightarrow{\pi_m} \langle f_m p_i, X_m\rangle)$ of the diagram obtained by applying the functor $p_i^\sharp : (U{\downarrow}K) \to (U_i{\downarrow}K)$ to $\mathcal{D}$. We add a new morphism $\alpha : \langle f_m, X_m\rangle \to \langle f_l, X_l\rangle$ to $\mathcal{D}$. Since $f_m p_i = K(\pi_m)c_i$, $f_l p_i = K(\pi_l)c_i$ and $K(\alpha)f_m = f_l$, we can apply the condition (3) of (2.9.1) to a parallel arrows $C_i \underset{\pi_l}{\overset{\alpha\pi_m}{\rightrightarrows}} X_l$ and a morphism $c_i : U_i \to K(C_i)$. There exist a covering $(p_{ij} : U_{ij} \to U_i)_{j\in J_i}$ in $\mathcal{E}$, a family of morphisms $(c_{ij} : C_{ij} \to C_i)_{j\in J_i}$ in $\mathcal{C}$ and $(q_{ij} : U_{ij} \to K(C_{ij}))_{j\in J_i}$ in $\mathcal{E}$ such that $c_i p_{ij} = K(c_{ij})q_{ij}$ and $\alpha\pi_m c_{ij} = \pi_l c_{ij}$ hold for every $j \in J_i$. Set $\pi'_m = \pi_m c_{ij}$, then for each $(i,j) \in \bigcup_{i\in I}(\{i\} \times J_i)$, $(\langle c_{ij}, C_{ij}\rangle \xrightarrow{\pi'_m} \langle f_m p_i p_{ij}, X_m\rangle)$ is a cone of the diagram obtained by applying the functor $(p_i p_{ij})^\sharp : (U{\downarrow}K) \to (U_{ij}{\downarrow}K)$ to the new diagram.

We show the converse. For any object U of $\mathcal{E}$, consider the empty diagram in $(U{\downarrow}K)$. There exist a covering $(p_i : U_i \to U)_{i\in I}$ in $\mathcal{E}$ and a cone $\langle q_i, X_i\rangle$ of the empty diagram in $(U_i{\downarrow}K)$. Thus (1) holds.

Let Y and Z be objects of $\mathcal{C}$ and $K(Y) \xleftarrow{f} U \xrightarrow{g} K(Z)$ a diagram in $\mathcal{E}$, there exist a covering $(p_i : U_i \to U)_{i\in I}$ in $\mathcal{E}$ and morphisms $u_i : \langle q_i, X_i\rangle \to \langle fp_i, Y\rangle$ and $v_i : \langle q_i, X_i\rangle \to \langle gp_i, Z\rangle$ in $(U_i{\downarrow}K)$ for each $i \in I$. Hence $u_i : U_i \to Y$, $v_i : U_i \to Z$ and $q_i : U_i \to K(X_i)$ satisfy $fp_i = K(u_i)q_i$ and $gp_i = K(v_i)q_i$. This shows (2).

Let $Y \underset{t}{\overset{s}{\rightrightarrows}} Z$ be parallel arrows in $\mathcal{C}$ and $f : U \to K(Y)$ a morphism satisfying $K(s)f = K(t)f$. Put $g = K(s)f : U \to K(Z)$, then $\langle f, Y\rangle \underset{t}{\overset{s}{\rightrightarrows}} \langle g, Z\rangle$ are parallel arrows in $(U_i{\downarrow}K)$. There exist a covering $(p_i : U_i \to U)_{i\in I}$ in $\mathcal{E}$, for each $i \in I$, a morphism $w_i : \langle q_i, X_i\rangle \to \langle fp_i, Y\rangle$ in $(U_i{\downarrow}K)$ satisfying $sw_i = tw_i$. Then, we have $fp_i = K(w_i)q_i$ for every $i \in I$ and (3) follows. □

If $\mathcal{C}$ is a category with finite limits and $K : \mathcal{C} \to \mathcal{E}$ is left exact, it follows from (A.5.5) and the above result that K is J-filtering for any topology J on $\mathcal{E}$.

Proposition 2.9.4 *The conditions (2) and (3) of (2.9.1) imply the following condition (4). Clearly, (3) is a special case of (4). If $\mathcal{C}$ has a terminal object, (4) implies (2).*

(4) For any diagram $Y \xrightarrow{s} W \xleftarrow{t} Z$ in $\mathcal{C}$ and any diagram $K(Y) \xleftarrow{f} U \xrightarrow{g} K(Z)$ in $\mathcal{E}$ such that $K(s)f = K(t)g$, there exist a covering $(p_i : U_i \to U)_{i \in I}$ in $\mathcal{E}$, a family of diagrams $(Y \xleftarrow{u_i} X_i \xrightarrow{v_i} Z)_{i \in I}$ in $\mathcal{C}$ and a family of morphisms $(q_i : U_i \to K(X_i))_{i \in I}$ in $\mathcal{E}$ such that $fp_i = K(u_i)q_i$, $gp_i = K(v_i)q_i$ and $su_i = tv_i$ hold for every $i \in I$.

Proof. Suppose that the conditions (2) and (3) of (2.9.1) hold. Let $Y \xrightarrow{s} W \xleftarrow{t} Z$ be a diagram in $\mathcal{C}$ and $K(Y) \xleftarrow{f} U \xrightarrow{g} K(Z)$ a diagram in $\mathcal{E}$ such that $K(s)f = K(t)g$. By (2), there exist a covering $(p_i : U_i \to U)_{i \in I}$ in $\mathcal{E}$, a family of diagrams $(Y \xleftarrow{u_i} X_i \xrightarrow{v_i} Z)_{i \in I}$ in $\mathcal{C}$ and a family of morphisms $(q_i : U_i \to K(X_i))_{i \in I}$ in $\mathcal{E}$ such that $fp_i = K(u_i)q_i$ and $gp_i = K(v_i)q_i$ hold for every $i \in I$. Then, by (3), there exist a covering $(p_{ij} : U_{ij} \to U_i)_{j \in J_i}$ in $\mathcal{E}$, a family of morphisms $(w_{ij} : X_{ij} \to X_i)_{j \in J_i}$ in $\mathcal{C}$ and $(q_{ij} : U_{ij} \to K(X_{ij}))_{j \in J_i}$ in $\mathcal{E}$ such that $q_i p_{ij} = K(w_{ij})q_{ij}$ and $su_i w_{ij} = tv_i w_{ij}$ hold for every $j \in J_i$. Put $M = \bigcup_{i \in I}(\{i\} \times J_i)$, then $(p_i p_{ij} : U_{ij} \to U)_{(i,j) \in M}$ is a covering. A family of diagrams $(Y \xleftarrow{u_i w_{ij}} X_{ij} \xrightarrow{v_i w_{ij}} Z)_{(i,j) \in M}$ in $\mathcal{C}$ and a family of morphisms $(q_i q_{ij} : U_{ij} \to K(X_{ij}))_{(i,j) \in M}$ in $\mathcal{E}$ satisfies the conditions of (4).

Suppose that $\mathcal{C}$ has a terminal object and satisfies (4). Then, (2) is a special case of (4) when W is the terminal object of $\mathcal{C}$. □

Let us denote by $1_{\mathcal{E}}$ the terminal object of $\mathcal{E}$.

Proposition 2.9.5 *For a site $(\mathcal{E}, J)$ and a functor $K : \mathcal{C} \to \mathcal{E}$, consider the following conditions.*

(1) $(K(X) \to 1_{\mathcal{E}})_{X \in \mathrm{Ob}\,\mathcal{C}}$ is a covering.

(2) $\{(K(f), K(g)) : K(X) \to K(Y) \times K(Z) |\, X \in \mathrm{Ob}\,\mathcal{C},\ f \in \mathcal{C}(X, Y),\ g \in \mathcal{C}(X, Z)\}$ is a covering for each $Y, Z \in \mathrm{Ob}\,\mathcal{C}$.

(3) For morphisms $Y \underset{t}{\overset{s}{\rightrightarrows}} Z$ in $\mathcal{C}$, let $E \xrightarrow{e} K(Y)$ be an equalizer of $K(Y) \underset{K(t)}{\overset{K(s)}{\rightrightarrows}} K(Z)$ in $\mathcal{E}$. Then, $\{f : K(X) \to E |\, X \in \mathrm{Ob}\,\mathcal{C}, {}^{\exists}v \in \mathcal{C}(X, Y)$ such that $sv = tv,\ ef = K(v)\}$ is a covering.

(4) For morphisms $s : Y \to Z$ and $t : W \to Z$ in $\mathcal{C}$, let

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] & K(Y) \arrow[d, "K(s)"] \\ K(W) \arrow[r, "K(t)"] & K(Z) \end{tikzcd}$$

be a pull-back square in $\mathcal{E}$. Then, $\{f : K(X) \to P |\, X \in \mathrm{Ob}\,\mathcal{C}, {}^{\exists}u \in \mathcal{C}(X, Y),\ {}^{\exists}v \in \mathcal{C}(X, W)$ such that $su = tv,\ pf = K(u),\ qf = K(v)\}$ is a covering.

If $\mathcal{E}$ is a category with finite limits, the condition (i) of (2.9.1) is equivalent to the above (i) for $i = 1, 2, 3$ and (4) of (2.9.4) is equivalent to the above (4).

Proof. Suppose that (1) of (2.9.1) holds. There exist a covering $(p_i : U_i \to 1_{\mathcal{E}})_{i \in I}$ in $\mathcal{E}$ and a family of morphisms $(q_i : U_i \to K(X_i))_{i \in I}$ in $\mathcal{E}$. Let $r_i : K(X_i) \to 1_{\mathcal{E}}$ the unique morphism in $\mathcal{E}$. Then $r_i q_i = p_i$ for any $i \in I$ and this implies that $(r_i : K(X_i) \to 1_{\mathcal{E}})_{i \in I}$ is a covering. Hence $(K(X) \to 1_{\mathcal{E}})_{X \in \mathrm{Ob}\,\mathcal{C}}$ is a covering.

Conversely, suppose that the above (1) holds and let U be an object of $\mathcal{E}$. Set $(r_X : K(X) \to 1_{\mathcal{E}})_{X \in \mathrm{Ob}\,\mathcal{C}}$ and let $p_X : U_X \to U$ be the pull-back of r_X along the unique morphism $U \to 1_{\mathcal{E}}$. Then we have a covering $(p_X : U_X \to U)_{X \in \mathrm{Ob}\,\mathcal{C}}$ in $\mathcal{E}$ and a family of morphisms $(q_X : U_X \to K(X))_{X \in \mathrm{Ob}\,\mathcal{C}}$ in $\mathcal{E}$.

Suppose that (2) of (2.9.1) holds. Let Y and Z be objects of $\mathcal{C}$ and consider a diagram $K(Y) \xleftarrow{\mathrm{pr}_1} K(Y) \times K(Z) \xrightarrow{\mathrm{pr}_2} K(Z)$ in $\mathcal{E}$. There exist a covering $(p_i : U_i \to K(Y) \times K(Z))_{i \in I}$ in $\mathcal{E}$, a family of diagrams $(Y \xleftarrow{u_i} X_i \xrightarrow{v_i} Z)_{i \in I}$ in $\mathcal{C}$ and a family of morphisms $(q_i : U_i \to K(X_i))_{i \in I}$ in $\mathcal{E}$ such that $\mathrm{pr}_1 p_i = K(u_i)q_i$ and $\mathrm{pr}_2 p_i = K(v_i)q_i$ hold for every $i \in I$. Hence $(K(u_i), K(v_i))q_i = p_i$ and this implies that $((K(u_i), K(v_i)) : K(X_i) \to K(Y) \times K(Z))_{i \in I}$ is a covering. Therefore so is $\{(K(f), K(g)) : K(X) \to K(Y) \times K(Z) |\, X \in \mathrm{Ob}\,\mathcal{C},\ f \in \mathcal{C}(X, Y),\ g \in \mathcal{C}(X, Z)\}$.

Conversely, suppose that the above (2) holds. Let Y and Z be objects of $\mathcal{C}$ and $K(Y) \xleftarrow{f} U \xrightarrow{g} K(Z)$ a diagram in $\mathcal{E}$. Set $\{(K(u), K(v)) : K(X) \to K(Y) \times K(Z) |\, X \in \mathrm{Ob}\,\mathcal{C},\ u \in \mathcal{C}(X,Y),\ v \in \mathcal{C}(X,Z)\} = ((K(u_i), K(v_i)) : K(X_i) \to K(Y) \times K(Z))_{i\in I}$ and let $p_i : U_i \to U$ be a pull-back of $(K(u_i), K(v_i))$ along $(f, g) : U \to K(Y) \times K(Z)$. Then $(p_i : U_i \to U)_{i\in I}$ is a covering and there is a morphism $q_i : U_i \to K(X_i)$ such that $(f, g)p_i = (K(u_i), K(v_i))q_i$. Thus we have a family of diagrams $(Y \xleftarrow{u_i} X_i \xrightarrow{v_i} Z)_{i\in I}$ in $\mathcal{C}$ such that $fp_i = K(u_i)q_i$ and $gp_i = K(v_i)q_i$.

Suppose that (3) of (2.9.1) holds. For $Y \underset{t}{\overset{s}{\rightrightarrows}} Z$ in $\mathcal{C}$, let $E \xrightarrow{e} K(Y)$ be an equalizer of $K(Y) \underset{K(t)}{\overset{K(s)}{\rightrightarrows}} K(Z)$ in $\mathcal{E}$. There exist a covering $(p_i : U_i \to E)_{i\in I}$ in $\mathcal{E}$, a family of morphisms $(w_i : X_i \to Y)_{i\in I}$ in $\mathcal{C}$ and $(q_i : U_i \to K(X_i))_{i\in I}$ in $\mathcal{E}$ such that $ep_i = K(w_i)q_i$ and $sw_i = tw_i$ hold for every $i \in I$. Let $u_i : K(X_i) \to E$ be the unique morphism satisfying $eu_i = K(w_i)$. Then $eu_iq_i = K(w_i)q_i = ep_i$ and, since e is a monomorphism, we have $u_iq_i = p_i$. It follows that $(u_i : K(X_i) \to E)_{i\in I}$ is a covering and this implies that so is $\{f : K(X) \to E |\, X \in \mathrm{Ob}\,\mathcal{C}, {}^\exists v \in \mathcal{C}(X,Y)\ \textit{such that}\ sv = tv,\ ef = K(v)\}$.

Conversely, suppose that the above (3) holds. Let $Y \underset{t}{\overset{s}{\rightrightarrows}} Z$ be parallel arrows in $\mathcal{C}$ and $f : U \to K(Y)$ a morphism in $\mathcal{E}$ satisfying $K(s)f = K(t)f$. Form an equalizer $E \xrightarrow{e} K(Y)$ of $K(Y) \underset{K(t)}{\overset{K(s)}{\rightrightarrows}} K(Z)$ in $\mathcal{E}$. There exists a unique morphism $g : U \to E$ in $\mathcal{E}$ such that $eg = f$. Set $\{f : K(X) \to E |\, X \in \mathrm{Ob}\,\mathcal{C}, {}^\exists v \in \mathcal{C}(X,Y)\ \textit{such that}\ sv = tv,\ ef = K(v)\} = (r_i : K(X_i) \to E)_{i\in I}$ and let $p_i : U_i \to U$ be a pull-back of r_i along g. Then, $(p_i : U_i \to U)_{i\in I}$ is a covering and we have a morphism $q_i : U_i \to K(X_i)$ satisfying $r_iq_i = gp_i$. Moreover, there is a morphism $w_i : X_i \to Y$ such that $sw_i = tw_i$ and $er_i = K(w_i)$ for each $i \in I$. Thus we have $fp_i = egp_i = er_iq_i = K(w_i)q_i$.

Suppose that (4) of (2.9.4) holds. For morphisms $s : Y \to Z$ and $t : W \to Z$ in $\mathcal{C}$, let

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"] & K(Y) \arrow[d, "K(s)"] \\ K(W) \arrow[r, "K(t)"] & K(Z) \end{tikzcd}$$

be a pull-back square in $\mathcal{E}$. There exist a covering $(p_i : U_i \to P)_{i\in I}$ in $\mathcal{E}$, a family of diagrams $(Y \xleftarrow{u_i} X_i \xrightarrow{v_i} W)_{i\in I}$ in $\mathcal{C}$ and a family of morphisms $(q_i : U_i \to K(X_i))_{i\in I}$ in $\mathcal{E}$ such that $pp_i = K(u_i)q_i$, $qp_i = K(v_i)q_i$ and $su_i = tv_i$ hold for every $i \in I$. Hence there is a unique morphism $r_i : K(X_i) \to P$ satisfying $\mathrm{pr}_i = K(u_i)$ and $qr_i = K(v_i)$. Then, we have $\mathrm{pr}_iq_i = K(u_i)q_i = pp_i$ and $qr_iq_i = K(v_i)q_i = qp_i$ which imply $r_iq_i = p_i$. It follows that $(r_i : K(X_i) \to P)_{i\in I}$ is a covering. Since $\{f : K(X) \to P |\, X \in \mathrm{Ob}\,\mathcal{C}, {}^\exists u \in \mathcal{C}(X,Y), {}^\exists v \in \mathcal{C}(X,W)\ \textit{such that}\ su = tv,\ pf = K(u),\ qf = K(v)\}$ contains every r_i, it is a covering.

Conversely, suppose that the above (4) holds. Let $Y \xrightarrow{s} W \xleftarrow{t} Z$ be a diagram in $\mathcal{C}$ and $K(Y) \xleftarrow{f} U \xrightarrow{g} K(Z)$ a diagram in $\mathcal{E}$ such that $K(s)f = K(t)g$. Consider a pull-back

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"] & K(Y) \arrow[d, "K(s)"] \\ K(Z) \arrow[r, "K(t)"] & K(W) \end{tikzcd}$$

in $\mathcal{C}$. There is a unique morphism $h : U \to P$ such that $ph = f$ and $qh = g$. Set $\{w : K(X) \to P |\, X \in \mathrm{Ob}\,\mathcal{C}, {}^\exists u \in \mathcal{C}(X,Y), {}^\exists v \in \mathcal{C}(X,Z)\ \textit{such that}\ su = tv,\ pw = K(u),\ qw = K(v)\} = (w_i : K(X_i) \to P)_{i\in I}$ and let $p_i : U_i \to U$ be a pull-back of w_i along h. Then we have a morphism $q_i : U_i \to K(X_i)$ satisfying $w_iq_i = hp_i$. Note that $(p_i : U_i \to U)_{i\in I}$ is a covering in $\mathcal{E}$. For each $i \in I$, there are morphisms $u_i : X_i \to Y$ and $v_i : X_i \to Z$ such that $su_i = tv_i, pw_i = K(u_i)$, $qw_i = K(v_i)$. Therefore the condition (4) of (2.9.4) holds. □

We fix a universe $\mathcal{U}$. Let $\mathcal{C}$ be a $\mathcal{U}$-small category, $\mathcal{E}$ a $\mathcal{U}$-category and $K : \mathcal{C} \to \mathcal{E}$ a functor. We denote by $h^{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ and $h^{\mathcal{E}} : \mathcal{E} \to \widehat{\mathcal{E}}$ the Yoneda embeddings. If $\mathcal{E}$ is $\mathcal{U}$-cocomplete, the left Kan extension $L : \widehat{\mathcal{C}} \to \mathcal{E}$ of K along $h^{\mathcal{C}}$ exists, which is given by $L(F) = \varinjlim((h^{\mathcal{C}}\downarrow F) \xrightarrow{P} \mathcal{C} \xrightarrow{K} \mathcal{E})$ (A.6.5).

Proposition 2.9.6 *Put $R = K^*h^{\mathcal{E}} : \mathcal{E} \to \widehat{\mathcal{C}}$, then L is a left adjoint of R and L can be chosen such that $Lh^{\mathcal{C}} = K$.*

Proof. For any $F \in \mathrm{Ob}\,\widehat{\mathcal{C}}$, $(h^{\mathcal{C}}P\langle X,f\rangle \xrightarrow{f} F)_{\langle X,f\rangle\in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow F)}$ is a colimiting cone of $h^{\mathcal{C}}P : (h^{\mathcal{C}}\downarrow F) \to \widehat{\mathcal{C}}$ by (A.4.2). Hence $(\widehat{\mathcal{C}}(F, R(W)) \xrightarrow{f^*} \widehat{\mathcal{C}}(h^{\mathcal{C}}P\langle X,f\rangle, R(W)))_{\langle X,f\rangle\in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow F)}$ is a limiting cone of $h_{R(W)}h^{\mathcal{C}}P : (h^{\mathcal{C}}\downarrow F) \to \mathcal{U}$-**Ens** for any $W \in \mathrm{Ob}\,\mathcal{E}$. On the other hand, since there is a colimiting cone $(KP\langle X,f\rangle \xrightarrow{\lambda^F_{\langle X,f\rangle}} L(F))_{\langle X,f\rangle\in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow F)}$ of $KP : (h^{\mathcal{C}}\downarrow F) \to \mathcal{E}$, we have a limiting cone $(\mathcal{E}(L(F), W) \xrightarrow{\lambda^{F*}_{\langle X,f\rangle}} \mathcal{E}(KP\langle X,f\rangle, W))_{\langle X,f\rangle\in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow F)}$ of a functor $h_WKP : (h^{\mathcal{C}}\downarrow F) \to \mathcal{U}$-**Ens**. We claim that $h_{R(W)}h^{\mathcal{C}}$ and h_WK are naturally equivalent. In fact, for $X \in \mathrm{Ob}\,\mathcal{C}$, $h_{R(W)}h^{\mathcal{C}}(X) = \widehat{\mathcal{C}}(h_X, R(W))$ is naturally isomorphic to $R(W)(X) = h^{\mathcal{E}}(W)(K(X)) = h_WK(X)$. Thus we have a natural equivalence $\widehat{\mathcal{C}}(F, R(W)) \cong \mathcal{E}(L(F), W)$.

Since $(h^{\mathcal{C}}\downarrow h_Y)$ $(Y \in \mathrm{Ob}\,\mathcal{C})$ has a terminal object $\langle Y, id_{h_Y}\rangle$, $(KP\langle X,f\rangle \xrightarrow{K(f_X(id_X))} K(Y))_{\langle X,f\rangle\in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow h_Y)}$ is a colimiting cone of $KP : (h^{\mathcal{C}}\downarrow h_Y) \to \mathcal{E}$. Thus we can choose L such that $L(h_Y) = K(Y)$ for each $Y \in \mathrm{Ob}\,\mathcal{C}$. □

We remark that, for a morphism $g : h_Y \to F$ in $\widehat{\mathcal{C}}$, since the following left diagram commutes,

$$\begin{tikzcd} KP\langle Y, id_{h_Y}\rangle \arrow[r, "id"] \arrow[d, "\lambda^{h_Y}_{\langle Y, id_{h_Y}\rangle}"] & KP\langle Y, g\rangle \arrow[d, "\lambda^F_{\langle Y,g\rangle}"] \\ L(h_Y) \arrow[r, "L(g)"] & L(F) \end{tikzcd} \qquad \begin{tikzcd} KP\langle Y, h_\varphi\rangle \arrow[r, "K(\varphi)"] \arrow[dr, "\lambda^{h_Z}_{\langle Y, h_\varphi\rangle}"'] & KP\langle Z, id_{h_Z}\rangle \arrow[d, "\lambda^{h_Z}_{\langle Z, id_{h_Z}\rangle}"] \\ & L(h_Z) \end{tikzcd}$$

$L(g) : L(h_Y) \to L(F)$ is identified with $L(h_Y) = K(Y) = KP\langle Y, g\rangle \xrightarrow{\lambda^F_{\langle Y,g\rangle}} L(F)$. In particular, if $\varphi : Y \to Z$ is a morphism in $\mathcal{C}$, φ defines a morphism $\varphi : \langle Y, h_\varphi\rangle \to \langle Z, id_{h_Z}\rangle$ in $(h^{\widehat{\mathcal{C}}}\downarrow h_Z)$ and the above right diagram commutes. Hence $L(h_\varphi) : L(h_Y) \to L(h_Z)$ is identified with $K(\varphi) : K(Y) \to K(Z)$.

The counit $\varepsilon : LR \to id_{\mathcal{E}}$ is given as follows. Let Z be an object of $\mathcal{E}$. It is easy to verify that

$$(KP\langle X,f\rangle = K(X) \xrightarrow{f_X(id_X)} Z)_{\langle X,f\rangle\in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow h_ZK)} \quad \cdots \quad (*)$$

is a cone of $KP : (h^{\mathcal{C}}\downarrow h_ZK) \to \mathcal{E}$. The unique morphism $\varepsilon_Z : LR(Z) = L(h_ZK) \to Z$ satisfying $\varepsilon_Z\lambda^F_{\langle X,f\rangle} = f_X(id_X)$ for any $\langle X,f\rangle \in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow h_ZK)$ defines the counit. The following assertion is obvious.

Proposition 2.9.7 *The following conditions are equivalent.*

(1) R is fully faithful.
(2) $\varepsilon : LR \to id_{\mathcal{E}}$ is an equivalence.
(3) $(*)$ is a colimiting cone for every $Z \in \mathrm{Ob}\,\mathcal{E}$.

Corollary 2.9.8 *Suppose that $\mathcal{C}$ is a full subcategory of $\mathcal{E}$ and $K : \mathcal{C} \to \mathcal{E}$ is the inclusion functor. Then, R is fully faithful if and only if $\mathcal{C}$ is a generating subcategory by strict epimorphisms.*

Proof. Since $(h^{\mathcal{C}}\downarrow h_ZK)$ is isomorphic to $(K\downarrow Z)$ for any $Z \in \mathrm{Ob}\,\mathcal{E}$ by Yoneda's lemma, the result follows from (A.4.10). □

Proposition 2.9.9 *Let $\mathcal{C}$ be a $\mathcal{U}$-small category and $\mathcal{E}$ a $\mathcal{U}$-cocomplete and finitely complete regular category with universal coproducts. If the left Kan extension $L : \widehat{\mathcal{C}} \to \mathcal{E}$ of a functor $K : \mathcal{C} \to \mathcal{E}$ along the Yoneda embedding $h^{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ is left exact, then K is filtering. Moreover, if $\mathcal{C}$ is finitely complete, K is left exact.*

Proof. Since $L(1_{\widehat{\mathcal{C}}}) = 1_{\mathcal{E}}$ and $P : (h^{\mathcal{C}}\downarrow 1_{\widehat{\mathcal{C}}}) \to \mathcal{C}$ is an isomorphism of categories, the colimiting cone $(KP\langle X,f\rangle \xrightarrow{\lambda^{1_{\widehat{\mathcal{C}}}}_{\langle X,f\rangle}} L(1_{\widehat{\mathcal{C}}}))_{\langle X,f\rangle\in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow 1_{\widehat{\mathcal{C}}})}$ defining $L(1_{\widehat{\mathcal{C}}})$ gives a universal strict epimorphic family $(K(X) \to 1_{\mathcal{E}})_{X\in \mathrm{Ob}\,\mathcal{C}}$ by (A.8.24). Hence the condition (1) of (2.9.5) is satisfied.

For $Y, Z \in \mathrm{Ob}\,\mathcal{C}$, consider the colimiting cone $(KP\langle X,f\rangle \xrightarrow{\lambda^{h_Y\times h_Z}_{\langle X,f\rangle}} L(h_Y \times h_Z))_{\langle X,f\rangle\in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow h_Y\times h_Z)}$ of $KP : (h^{\mathcal{C}}\downarrow h_Y \times h_Z) \to \mathcal{E}$. Let $p_1 : h_Y \times h_Z \to h_Y$ and $p_2 : h_Y \times h_Z \to h_Z$ be projections. Set $u_f = (p_1f)_X(id_X) : X \to Y$, $v_f = (p_2f)_X(id_X) : X \to Z$, then u_f and v_f give morphisms $u_f : \langle X, p_1f\rangle \to \langle Y, id_{h_Y}\rangle$ in $(h^{\mathcal{C}}\downarrow h_Y)$ and $v_f : \langle X, p_f\rangle \to \langle Z, id_{h_Z}\rangle$ in $(h^{\mathcal{C}}\downarrow h_Z)$. Recall that $\lambda^{h_Y}_{\langle Y, id_{h_Y}\rangle} : KP\langle Y, id_{h_Y}\rangle \to L(h_Y)$ and $\lambda^{h_Z}_{\langle Z, id_{h_Z}\rangle} : KP\langle Z, id_{h_Z}\rangle \to L(h_Z)$ are isomorphisms. Hence we have $((\lambda^{h_Y}_{\langle Y, id_{h_Y}\rangle})^{-1} \times (\lambda^{h_Z}_{\langle Z, id_{h_Z}\rangle})^{-1})(L(p_1), L(p_2))\lambda^{h_Y\times h_Z}_{\langle X,f\rangle} = (\lambda^{h_Y}_{\langle Y, id_{h_Y}\rangle})^{-1} \times$

$(\lambda^{h_Z}_{\langle Z,id_{h_Z}\rangle})^{-1}(\lambda^{h_Y}_{\langle X,p_1f\rangle},\lambda^{h_Z}_{\langle X,p_2f\rangle} = (K(u_f),K(v_f))$. Since $(L(p_1),L(p_2)) : L(h_Y\times h_Z)\to L(h_Y)\times L(h_Z)$ is an isomorphism and , $(\lambda^{h_Y\times h_Z}_{\langle X,f\rangle})_{\langle X,f\rangle\in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow h_Y\times h_Z)}$ is a universal strict epimorphic family, so is

$$((K(u_f),K(v_f)) : K(X)\to K(Y)\times K(Z))_{\langle X,f\rangle\in\mathrm{Ob}(h^{\mathcal{C}}\downarrow h_Y\times h_Z)}.$$

Therefore the condition (2) of (2.9.5) is satisfied.

For morphisms $s,t : Y\to Z$ in $\mathcal{C}$, let $F\xrightarrow{e} h_Y$ be an equalizer of $h_Y \underset{h_t}{\overset{h_s}{\rightrightarrows}} h_Z$ in $\widehat{\mathcal{C}}$. Then, $L(F)\xrightarrow{L(e)} L(h_Y)$ is an equalizer of $L(h_Y) \underset{L(h_t)}{\overset{L(h_s)}{\rightrightarrows}} L(h_Z)$ in $\widehat{\mathcal{C}}$. We have a universal strict epimorphic family $(KP\langle X,f\rangle \xrightarrow{\lambda^F_{\langle X,f\rangle}} L(F))_{\langle X,f\rangle\in\mathrm{Ob}(h^{\mathcal{C}}\downarrow F)}$. Since $\lambda^{h_Y}_{\langle Y,id_{h_Y}\rangle} : KP\langle Y,id_{h_Y}\rangle\to L(h_Y)$ and $\lambda^{h_Z}_{\langle Z,id_{h_Z}\rangle} : KP\langle Z,id_{h_Z}\rangle\to L(h_Z)$ are isomorphisms and $L(h_s)\lambda^{h_Y}_{\langle Y,id_{h_Y}\rangle} = \lambda^{h_Z}_{\langle Z,id_{h_Z}\rangle}K(s)$, $L(h_t)\lambda^{h_Y}_{\langle Y,id_{h_Y}\rangle} = \lambda^{h_Z}_{\langle Z,id_{h_Z}\rangle}K(t)$ hold, $(\lambda^{h_Y}_{\langle Y,id_{h_Y}\rangle})^{-1}L(e) : L(F)\to K(Y)$ is an equalizer of $K(Y) \underset{K(t)}{\overset{K(s)}{\rightrightarrows}} K(Z)$. For any $\langle X,f\rangle\in\mathrm{Ob}\,(h^{\mathcal{C}}\downarrow F)$, put $(ef)_X(id_X) = w_f : X\to Y$. Then, $w_f : \langle X,ef\rangle\to\langle Y,id_{h_Y}\rangle$ is a morphism in $(h^{\mathcal{C}}\downarrow h_Y)$, hence $\lambda^{h_Y}_{\langle X,ef\rangle}K(w_f) = \lambda^{h_Y}_{\langle X,ef\rangle}$. It follows that $(\lambda^{h_Y}_{\langle Y,id_{h_Y}\rangle})^{-1}L(e)\lambda^F_{\langle X,f\rangle} = (\lambda^{h_Y}_{\langle Y,id_{h_Y}\rangle})^{-1}\lambda^{h_Y}_{\langle X,ef\rangle} = (\lambda^{h_Y}_{\langle Y,id_{h_Y}\rangle})^{-1}\lambda^{h_Y}_{\langle X,ef\rangle} = K(w_f)$. Moreover, $h_{sw_f} = h_sh_{w_f} = h_sef = h_tef = h_th_{w_f} = h_{tw_f}$ and $h^{\mathcal{C}}$ is faithful, it follows that $sw_f = tw_f$. Thus we see that K satisfies the condition (3) of (2.9.5). The second assertion follows from $Lh^{\mathcal{C}} = K$ (2.9.6). $\square$

We consider finite categories Δ_1 and Δ_2 defined as follows. $\mathrm{Ob}\,\Delta_1 = \{0,1,2\}$, $\Delta_1(0,j) = \{p_j\}$ $(j=1,2)$ and $\Delta_1(i,j)$ is empty if $i\neq j$ and $i\neq 0$. $\mathrm{Ob}\,\Delta_2 = \{0,1,2,3,4,5\}$, $\Delta_2(0,j) = \{p_j\}$ $(j=1,2,3,4,5)$, $\Delta_2(1,j) = \{q_j\}$ $(j=3,5)$, $\Delta_2(2,j) = \{r_j\}$ $(j=4,5)$ and $\Delta_2(i,j)$ is empty if $i>j$ or $(i,j) = (1,2),(1,4),(2,3),(3,4),(3,5),(4,5)$. Moreover, equalities $q_3p_1 = p_3$, $q_5p_1 = r_5p_2 = p_5$ and $r_4p_2 = p_4$ hold. For a category $\mathcal{D}$, we set $\mathcal{D}_i = \mathrm{Funct}(\Delta_i,\mathcal{D})$ $(i=1,2)$. Define a map $\xi : \mathrm{Mor}\,\mathcal{D}\to\mathrm{Ob}\,\mathcal{D}_1$ by $\xi(f)(0) = \xi(f)(1) = \mathrm{dom}(f)$, $\xi(f)(2) = \mathrm{codom}(f)$ and $\xi(f)(p_1) = id_{\mathrm{dom}(f)}$, $\xi(f)(p_2) = f$.

Let $\mathcal{D}$ be a $\mathcal{U}$-small category and $\mathcal{E}$ a $\mathcal{U}$-cocomplete category. For a functor $D : \mathcal{D}\to\mathcal{E}$, define morphisms $\sigma,\tau : \coprod_{f\in\mathrm{Mor}\,\mathcal{D}} D(\mathrm{dom}(f))\to\coprod_{i\in\mathrm{Ob}\,\mathcal{D}} D(i)$ so that the following diagram commutes.

$$\begin{tikzcd}
D(\mathrm{dom}(f)) \arrow[d, "\iota_{\mathrm{dom}(f)}"] & D(\mathrm{dom}(f)) \arrow[l, "id"'] \arrow[r, "D(f)"] \arrow[d, "\iota_f"] & D(\mathrm{codom}(f)) \arrow[d, "\iota_{\mathrm{codom}(f)}"] \\
\coprod_{i\in\mathrm{Ob}\,\mathcal{D}} D(i) & \coprod_{f\in\mathrm{Mor}\,\mathcal{D}} D(\mathrm{dom}(f)) \arrow[l, "\sigma"'] \arrow[r, "\tau"] & \coprod_{i\in\mathrm{Ob}\,\mathcal{D}} D(i)
\end{tikzcd}$$

Here, the vertical morphisms are the canonical morphisms. Let $\lambda : \coprod_{i\in\mathrm{Ob}\,\mathcal{D}} D(i)\to L$ be a coequalizer of σ and τ, then $(D(i)\xrightarrow{\lambda\iota_i} L)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of D.

Define morphisms $\mu,\nu : \coprod_{d\in\mathrm{Ob}\,\mathcal{D}_1} D(d(0))\to\coprod_{i\in\mathrm{Ob}\,\mathcal{D}} D(i)$ so that the following diagram commutes.

$$\begin{tikzcd}
D(d(1)) \arrow[d, "\iota_{d(1)}"] & D(d(0)) \arrow[l, "D(d(p_1))"'] \arrow[r, "D(d(p_2))"] \arrow[d, "\iota_d"] & D(d(2)) \arrow[d, "\iota_{d(2)}"] \\
\coprod_{i\in\mathrm{Ob}\,\mathcal{D}} D(i) & \coprod_{d\in\mathrm{Ob}\,\mathcal{D}_1} D(d(0)) \arrow[l, "\mu"'] \arrow[r, "\nu"] & \coprod_{i\in\mathrm{Ob}\,\mathcal{D}} D(i)
\end{tikzcd}$$

There is a unique morphism $\zeta : \coprod_{f\in\mathrm{Mor}\,\mathcal{D}} D(\mathrm{dom}(f))\to\coprod_{d\in\mathrm{Ob}\,\mathcal{D}_1} D(d(0))$ such that the following diagram commutes for any $f\in\mathrm{Mor}\,\mathcal{D}$.

$$\begin{tikzcd}
D(\mathrm{dom}(f)) \arrow[r, "id"] \arrow[d, "\iota_f"] & D(\xi(f)(0)) \arrow[d, "\iota_{\xi(f)}"] \\
\coprod_{f\in\mathrm{Mor}\,\mathcal{D}} D(\mathrm{dom}(f)) \arrow[r, "\zeta"] & \coprod_{d\in\mathrm{Ob}\,\mathcal{D}_1} D(d(0))
\end{tikzcd}$$

Since the following diagram commutes for any $f \in \operatorname{Mor}\mathcal{D}$,

$$\begin{tikzcd}
D(\mathrm{dom}(f)) \arrow[d, "\iota_{\mathrm{dom}(f)}"] & D(\mathrm{dom}(f)) \arrow[l, "id"'] \arrow[r, "D(f)"] \arrow[d, "\iota_{\xi(f)}=\zeta\iota_f"] & D(\mathrm{codom}(f)) \arrow[d, "\iota_{\mathrm{codom}(f)}"] \\
\coprod_{i\in\operatorname{Ob}\mathcal{D}} D(i) & \coprod_{d\in\operatorname{Ob}\mathcal{D}_1} D(d(0)) \arrow[l, "\mu"'] \arrow[r, "\nu"] & \coprod_{i\in\operatorname{Ob}\mathcal{D}} D(i)
\end{tikzcd}$$

we have $\sigma = \mu\zeta$ and $\tau = \nu\zeta$. It follows from the dual of (A.3.6) that $\lambda : \coprod_{i\in\operatorname{Ob}\mathcal{D}} D(i) \to L$ is a coequalizer of μ and ν.

We denote by $\kappa : \Delta_1 \to \Delta_1$ the functor given by $\kappa(0) = 0$, $\kappa(1) = 2$ and $\kappa(2) = 1$. Let $s : \coprod_{i\in\operatorname{Ob}\mathcal{D}} D(i) \to \coprod_{d\in\operatorname{Ob}\mathcal{D}_1} D(d(0))$ and $t : \coprod_{d\in\operatorname{Ob}\mathcal{D}_1} D(d(0)) \to \coprod_{d\in\operatorname{Ob}\mathcal{D}_1} D(d(0))$ be morphisms making the following diagrams commute.

$$\begin{tikzcd}
D(i) \arrow[r, "id"] \arrow[d, "\iota_i"] & D(\xi(id_i)(0)) \arrow[d, "\iota_{\xi(id_i)}"] \\
\coprod_{i\in\operatorname{Ob}\mathcal{D}} D(i) \arrow[r, "s"] & \coprod_{d\in\operatorname{Ob}\mathcal{D}_1} D(d(0))
\end{tikzcd}
\qquad
\begin{tikzcd}
D(d(0)) \arrow[r, "id"] \arrow[d, "\iota_d"] & D(d\kappa(0)) \arrow[d, "\iota_{d\kappa}"] \\
\coprod_{d\in\operatorname{Ob}\mathcal{D}_1} D(d(0)) \arrow[r, "t"] & \coprod_{d\in\operatorname{Ob}\mathcal{D}_1} D(d(0))
\end{tikzcd}$$

Then we have $\mu s = \nu s = id_{\coprod_{i\in\operatorname{Ob}\mathcal{D}} D(i)}$ and $\mu t = \nu$, $\nu t = \mu$.

Lemma 2.9.10 *Let $\mathcal{C}$ be a $\mathcal{U}$-small category and $\mathcal{E}$ a finitely complete, $\mathcal{U}$-cocomplete regular category whose coproducts are disjoint and universal. For a $\mathcal{U}$-presheaf $F \in \operatorname{Ob}\widehat{\mathcal{C}}$, we set $\mathcal{D} = (h^{\mathcal{C}}\downarrow F)$ and $D = KP : (h^{\mathcal{C}}\downarrow F) \to \mathcal{E}$. If $K : \mathcal{C} \to \mathcal{E}$ is a filtering functor, the morphisms $\mu, \nu : \coprod_{d\in\operatorname{Ob}\mathcal{D}_1} D(d(0)) \to \coprod_{i\in\operatorname{Ob}\mathcal{D}} D(i)$ defined above satisfy the conditions of (A.8.23). Hence the image of $(\mu,\nu) : \coprod_{d\in\operatorname{Ob}\mathcal{D}_1} D(d(0)) \to \coprod_{i\in\operatorname{Ob}\mathcal{D}} D(i) \times \coprod_{i\in\operatorname{Ob}\mathcal{D}} D(i)$ is an equivalence relation.*

Proof. We have already seen that μ and ν satisfy the conditions (1) and (2). Put $X = \coprod_{d\in\operatorname{Ob}\mathcal{D}_1} D(d(0))$ and $Y = \coprod_{i\in\operatorname{Ob}\mathcal{D}} D(i)$. Form a pull-back

$$\begin{tikzcd}
D(d(0)) \times_Y D(d'(0)) \arrow[r, "\varphi_{d,d'}"] \arrow[d, "\psi_{d,d'}"] & D(d'(0)) \arrow[d, "\mu\iota_{d'}"] \\
D(d(0)) \arrow[r, "\nu\iota_d"] & Y.
\end{tikzcd}$$

By the universality of coproducts in $\mathcal{E}$, the following diagram is cartesian (A.4.5).

$$\begin{tikzcd}
\coprod_{d,d'\in\operatorname{Ob}\mathcal{D}_1} D(d(0)) \times_Y D(d'(0)) \arrow[r, "\varphi"] \arrow[d, "\psi"] & X \arrow[d, "\mu"] \\
X \arrow[r, "\nu"] & Y
\end{tikzcd}$$

Here morphisms φ, ψ are induced by $\varphi_{d,d'}$, $\psi_{d,d'}$, respectively. On the other hand, since $\mu\iota_{d'} = \iota_{d'(1)}D(d'(p_1))$, $\nu\iota_d = \iota_{d(2)}D(d(p_2))$ and Y is a disjoint coproduct of $D(i)$'s, there is a unique morphism from $D(d(0))\times_Y D(d'(0))$ to the initial object of $\mathcal{E}$ induced by $D(d'(p_1))\varphi_{d,d'}$ and $D(d(p_2))\psi_{d,d'}$ if $d'(1) \neq d(2)$. Hence it follows from (A.3.16) that $D(d(0)) \times_Y D(d'(0))$ is an initial object if $d'(1) \neq d(2)$. If $d'(1) = d(2)$, since $\iota_{d(2)} : D(d(2)) \to Y$ is a monomorphism, it follows from (A.3.6) that the following lower diagram is cartesian.

$$\begin{tikzcd}
D(d(0)) \times_{D(d(2))} D(d'(0)) \arrow[r, "\varphi_{d,d'}"] \arrow[d, "\psi_{d,d'}"] & D(d'(0)) \arrow[d, "D(d'(p_1))"] \\
D(d(0)) \arrow[r, "D(d(p_2))"] & D(d(2))
\end{tikzcd}
\qquad
\begin{tikzcd}
D(d(0)) \times_{D(d(2))} D(d'(0)) \arrow[r, "\varphi_{d,d'}"] \arrow[d, "\psi_{d,d'}"] & D(d'(0)) \arrow[d, "\iota_{d'(1)}D(d'(p_1))"] \\
D(d(0)) \arrow[r, "\iota_{d(2)}D(d(p_2))"] & Y
\end{tikzcd}$$

Thus we identify $D(d(0)) \times_{D(d(2))} D(d'(0))$ with $D(d(0)) \times_Y D(d'(0))$ if $d'(1) = d(2)$. Set $I = \{(d,d') \mid d, d' \in \operatorname{Ob} \mathcal{D}_1,\ d'(1) = d(2)\}$, $T = \coprod_{(d,d') \in I} D(d(0)) \times_{D(d(2))} D(d'(0))$. By the above arguments, we have a cartesian square

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[d, "\psi"] & X \arrow[d, "\mu"] \\ X \arrow[r, "\nu"] & Y. \end{tikzcd}$$

Let us denote by $\eta_0, \eta_1, \eta_2 : \Delta_1 \to \Delta_2$ the functors defined by $\eta_0(0) = 0$, $\eta_0(1) = 3$, $\eta_0(2) = 4$, $\eta_1(0) = 1$, $\eta_1(1) = 3$, $\eta_1(2) = 5$, $\eta_2(0) = 2$, $\eta_2(1) = 5$, $\eta_2(2) = 4$. We put $T' = \coprod_{\delta \in \operatorname{Ob} \mathcal{D}_2} D(\delta(0))$, then $D(\delta(p_1)) : D(\delta(0)) \to D(\delta(1)) = D(\delta\eta_1(0))$ and $D(\delta(p_2)) : D(\delta(0)) \to D(\delta(2)) = D(\delta\eta_2(0))$ induce a morphism $\theta_\delta : D(\delta(0)) \to D(\delta\eta_1(0)) \times_{D(\delta\eta_1(2))} D(\delta\eta_2(0))$. Since $\delta \mapsto (\delta\eta_1, \delta\eta_2)$ gives a map $\operatorname{Ob} \mathcal{D}_2 \to I$, we have a morphism $\theta : T' \to T$ induced by $(\theta_\delta)_{\delta \in \operatorname{Ob} \mathcal{D}_2}$. For each $(d,d') \in I$, we claim that $\{\gamma : D\langle A, f\rangle \to D(d(0)) \times_{D(d(2))} D(d'(0)) \mid \langle A, f\rangle \in \operatorname{Ob} \mathcal{D}, {}^\exists u \in \mathcal{D}(\langle A, f\rangle, d(0)), {}^\exists v \in \mathcal{D}(\langle A, f\rangle, d'(0))$ *such that* $d(p_2)u = d'(p_1)v,\ \psi_{d,d'}\gamma = D(u),\ \varphi_{d,d'}\gamma = D(v)\}$ is a universal strict epimorphic family. Set $d(i) = \langle B_i, g_i\rangle$, $d'(i) = \langle B'_i, g'_i\rangle$. If $\gamma : K(A) \to D(d(0)) \times_{D(d(2))} D(d'(0))$ is a morphism in $\mathcal{E}$ such that $\psi_{d,d'}\gamma = K(u)$ and $\varphi_{d,d'}\gamma = K(v)$ hold for some $u \in \mathcal{C}(A, B_0)$, $v \in \mathcal{C}(A, B'_0)$ satisfying $d(p_2)u = d'(p_1)v$, put $f = g_0 h_u : h_A \to F$, then u and v define morphisms $\langle A, f\rangle \to d(0)$ and $\langle A, f\rangle \to d'(0)$ in $\mathcal{D}$. Since K is filtering, (2.9.5) implies the assertion. Hence by (A.8.24), θ is a regular epimorphism. Let $\rho : T' \to X$ be the morphism induced by $\iota_{\delta\eta_0} : D(\delta(0)) = D(\delta\eta_0(0)) \to X$ and $X \xrightarrow{\pi} R \xrightarrow{\varsigma} Y \times Y$ a factorization of (μ, ν) such that π is a regular epimorphism and ς a monomorphism. It is easy to verify that $\mu\rho = \mu\psi\theta, \nu\rho = \nu\varphi\theta : T' \to Y$. Applying (A.8.4) to the following commutative square, we see that the image of $(\mu\psi, \nu\varphi) : T \to Y \times Y$ is contained in R.

$$\begin{tikzcd} T' \arrow[r, "\pi\rho"] \arrow[d, "\theta"] & R \arrow[d, "\varsigma"] \\ T \arrow[r, "(\mu\psi, \nu\varphi)"] & Y \times Y \end{tikzcd}$$

□

Lemma 2.9.11 *Let $\mathcal{C}$ be a $\mathcal{U}$-small category and $\mathcal{E}$ a finitely complete, $\mathcal{U}$-cocomplete exact category whose coproducts are disjoint and universal. Suppose that $K : \mathcal{C} \to \mathcal{E}$ is a filtering functor. For a $\mathcal{U}$-presheaf $F \in \operatorname{Ob}\widehat{\mathcal{C}}$, objects $\langle A, f\rangle$, $\langle B, g\rangle$ of $(h^{\mathcal{C}}\downarrow F)$ and morphisms $\alpha : U \to KP\langle A, f\rangle$, $\beta : U \to KP\langle B, g\rangle$ in $\mathcal{E}$, $\lambda^F_{\langle A, f\rangle}\alpha = \lambda^F_{\langle B, g\rangle}\beta$ holds if and only if there exist a universal strict epimorphic family $(p_i : U_i \to U)_{i \in I}$ in $\mathcal{E}$, morphisms $u_i : W_i \to A$, $v_i : W_i \to B$ in $\mathcal{C}$ and a morphism $q_i : U_i \to K(W_i)$ in $\mathcal{E}$ for each $i \in I$ such that $fh_{u_i} = gh_{v_i}$, $K(u_i)q_i = \alpha p_i$, $K(v_i)q_i = \beta p_i$.*

Proof. We use the same notations as in the previous lemma. The image R of $(\mu, \nu) : X \to Y \times Y$ is an equivalence relation on Y and $\lambda : Y \to L(F)$ is a coequalizer of this equivalence relation $R \overset{\varpi}{\underset{\varrho}{\rightrightarrows}} Y$. Since $\mathcal{E}$ is exact, $R \overset{\varpi}{\underset{\varrho}{\rightrightarrows}} Y$ is a kernel pair of λ. Since the compositions $U \xrightarrow{\alpha} KP\langle A, f\rangle \xrightarrow{\iota_{\langle A, f\rangle}} Y$ and $U \xrightarrow{\beta} KP\langle B, g\rangle \xrightarrow{\iota_{\langle B, g\rangle}} Y$ are equalized by $\lambda : Y \to L(F)$, there exists a unique morphism $\chi : U \to R$ such that $\varpi\chi = \iota_{\langle A, f\rangle}\alpha$, $\varrho\chi = \iota_{\langle B, g\rangle}\beta$.

For $d \in \operatorname{Ob} \mathcal{D}_1$, consider the following diagram, where $\bar{\pi}$ and $\bar{\iota}_d$ are pull-backs of π and ι_d, respectively.

$$\begin{tikzcd} U_d \arrow[r, "\bar{\iota}_d"] \arrow[d, "q_d"] & U' \arrow[r, "\bar{\pi}"] \arrow[d] & U \arrow[d, "\chi"] \\ D(d(0)) \arrow[r, "\iota_d"] & Y \arrow[r, "\pi"] & R \end{tikzcd}$$

Since π is a regular epimorphism, so is $\bar{\pi}$. It follows from the universality of coproducts in $\mathcal{E}$, $(\bar{\iota}_d : U_d \to U')_{d \in \operatorname{Ob} \mathcal{D}_1}$ induces an isomorphism $\coprod_{d \in \operatorname{Ob} \mathcal{D}_1} U_d \to U$. Therefore $(\bar{\iota}_d : U_d \to U')_{d \in \operatorname{Ob} \mathcal{D}_1}$ is a universal strict epimorphic family.

Set $J = \{d \in \operatorname{Ob} \mathcal{D}_1 \mid d(1) = \langle A, f\rangle, d(2) = \langle B, g\rangle\}$ and $d(0) = \langle W_d, k_d\rangle$ for each $d \in J$. Recall that an initial object $0_{\mathcal{E}}$ of $\mathcal{E}$ is strict (A.3.16), thus $0_{\mathcal{E}} \times Z$ is also an initial object for any $Z \in \operatorname{Ob} \mathcal{E}$. Therefore, if $d \notin J$, the pull-back of $\iota_{d(1)} \times \iota_{d(2)} : D(d(1)) \times D(d(2)) \to Y \times Y$ along $\iota_{\langle A, f\rangle} \times \iota_{\langle B, g\rangle} : D\langle A, f\rangle \times D\langle B, g\rangle \to Y \times Y$ is the unique morphism $0_{\mathcal{E}} \to D(d(1)) \times D(d(2))$. Since the following diagram commutes, there exists a morphism $U_d \to 0_{\mathcal{E}}$ if $d \notin J$.

$$\begin{tikzcd}
U_d \arrow[r, "(\alpha\times\beta)\bar{\pi}\bar{\iota}_d"] \arrow[d, "(D(d(p_1)),D(d(p_2)))q_d"] & D\langle A,f\rangle\times D\langle B,g\rangle \arrow[d, "\iota_{\langle A,f\rangle}\times\iota_{\langle B,g\rangle}"] \\
D(d(1))\times D(d(2)) \arrow[r, "\iota_{d(1)}\times\iota_{d(2)}"] & Y\times Y
\end{tikzcd}$$

Hence U_d is an initial object if $d\notin J$ and $(\bar{\iota}_d : U_d\to U')_{d\in J}$ is a universal strict epimorphic family. Put $p_d=\bar{\pi}\bar{\iota}_d$, then $(p_d : U_d\to U)_{d\in J}$ is a universal strict epimorphic family. It is easy to verify that $d(p_1) : W_d\to A$, $d(p_2) : W_d\to B$ and $q_d : U_d\to K(W_d)$ satisfy $fh_{d(p_1)}=gh_{d(p_2)}$, $K(d(p_1))q_d=\alpha p_d$ and $K(d(p_2))q_d=\beta p_d$ for any $d\in J$.

Conversely, assume that there exist a universal strict epimorphic family $(p_i : U_i\to U)_{i\in I}$ in $\mathcal{E}$, morphisms $u_i : W_i\to A$, $v_i : W_i\to B$ in $\mathcal{C}$ and a morphism $q_i : U_i\to K(W_i)$ in $\mathcal{E}$ for each $i\in I$ such that $fh_{u_i}=gh_{v_i}$, $K(u_i)q_i=\alpha p_i$, $K(v_i)q_i=\beta p_i$. Then, for each $i\in I$, $\lambda^F_{\langle A,f\rangle}\alpha p_i=\lambda^F_{\langle A,f\rangle}K(u_i)q_i=\lambda^F_{\langle W_i,fh_{u_i}\rangle}q_i=\lambda^F_{\langle W_i,gh_{v_i}\rangle}q_i=\lambda^F_{\langle B,g\rangle}K(v_i)q_i=\lambda^F_{\langle B,g\rangle}\beta p_i$. Hence $\lambda^F_{\langle A,f\rangle}\alpha=\lambda^F_{\langle B,g\rangle}\beta$. □

Proposition 2.9.12 *Under the same assumptions as in (2.9.11), $L:\widehat{\mathcal{C}}\to\mathcal{E}$ preserves pull-backs.*

Proof. Consider the following cartesian squares in $\widehat{\mathcal{C}}$ and $\mathcal{E}$.

$$\begin{tikzcd}
F\times_H G \arrow[r, "\bar{\varphi}"] \arrow[d, "\bar{\psi}"] & G \arrow[d, "\psi"] \\
F \arrow[r, "\varphi"] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
L(F)\times_{L(H)} L(G) \arrow[r, "p_2"] \arrow[d, "p_1"] & L(G) \arrow[d, "L(\psi)"] \\
L(F) \arrow[r, "L(\varphi)"] & L(H)
\end{tikzcd}$$

There is a unique morphism $\Phi : L(F\times_H G)\to L(F)\times_{L(H)} L(G)$ satisfying $L(\bar{\varphi})\Phi=p_2$ and $L(\bar{\psi})\Phi=p_1$. We show that Φ is a both regular epimorphism and monomorphism, hence by (A.8.5), Φ is an isomorphism.

We put $\mathcal{D}^F=(h^{\mathcal{C}}\downarrow F)$, $\mathcal{D}^G=(h^{\mathcal{C}}\downarrow G)$, $\mathcal{D}^{F\times_H G}=(h^{\mathcal{C}}\downarrow F\times_G H)$. Form the following pull-backs.

$$\begin{tikzcd}
U'_{\langle A,f\rangle} \arrow[r, "\bar{\lambda}^F_{\langle A,f\rangle}"] \arrow[d, "q_{\langle A,f\rangle}"] & L(F)\times_{L(H)}L(G) \arrow[d, "p_1"] \\
KP\langle A,f\rangle \arrow[r, "\lambda^F_{\langle A,f\rangle}"] & L(F)
\end{tikzcd}
\quad
\begin{tikzcd}
U''_{\langle B,g\rangle} \arrow[r, "r_{\langle B,g\rangle}"] \arrow[d, "\bar{\lambda}^G_{\langle B,g\rangle}"] & KP\langle B,g\rangle \arrow[d, "\lambda^G_{\langle B,g\rangle}"] \\
L(F)\times_{L(H)}L(G) \arrow[r, "p_2"] & L(G)
\end{tikzcd}
\quad
\begin{tikzcd}
U_{\langle A,f\rangle,\langle B,g\rangle} \arrow[r, "\tilde{\lambda}^{\langle A,f\rangle}_{\langle B,g\rangle}"] \arrow[d, "\hat{\lambda}^{\langle B,g\rangle}_{\langle A,f\rangle}"] & U''_{\langle B,g\rangle} \arrow[d, "\bar{\lambda}^G_{\langle B,g\rangle}"] \\
U'_{\langle A,f\rangle} \arrow[r, "\bar{\lambda}^F_{\langle A,f\rangle}"] & L(F)\times_{L(H)}L(G)
\end{tikzcd}$$

We set

$$q=\coprod_{\langle A,f\rangle\in\operatorname{Ob}\mathcal{D}^F} q_{\langle A,f\rangle} : \coprod_{\langle A,f\rangle\in\operatorname{Ob}\mathcal{D}^F} U'_{\langle A,f\rangle}\to\coprod_{\langle A,f\rangle\in\operatorname{Ob}\mathcal{D}^F} KP\langle A,f\rangle,$$

$$r=\coprod_{\langle B,g\rangle\in\operatorname{Ob}\mathcal{D}^G} q_{\langle B,g\rangle} : \coprod_{\langle B,g\rangle\in\operatorname{Ob}\mathcal{D}^G} U''_{\langle B,g\rangle}\to\coprod_{\langle B,g\rangle\in\operatorname{Ob}\mathcal{D}^G} KP\langle B,g\rangle$$

and let

$$\bar{\lambda}^F : \coprod_{\langle A,f\rangle\in\operatorname{Ob}\mathcal{D}^F} U'_{\langle A,f\rangle}\to L(F)\times_{L(H)} L(G),\quad \bar{\lambda}^G : \coprod_{\langle B,g\rangle\in\operatorname{Ob}\mathcal{D}^G} U''_{\langle B,g\rangle}\to L(F)\times_{L(H)} L(G)$$

be the morphisms induced by the following families of morphisms, respectively.

$$(\bar{\lambda}^F_{\langle A,f\rangle} : U'_{\langle A,f\rangle}\to L(F)\times_{L(H)} L(G))_{\langle A,f\rangle\in\operatorname{Ob}\mathcal{D}^F},\quad (\bar{\lambda}^G_{\langle B,g\rangle} : U''_{\langle B,g\rangle}\to L(F)\times_{L(H)} L(G))_{\langle B,g\rangle\in\operatorname{Ob}\mathcal{D}^G}$$

We denote by

$$\hat{\lambda}_{\langle A,f\rangle} : \coprod_{\langle B,g\rangle\in\operatorname{Ob}\mathcal{D}^G} U_{\langle A,f\rangle,\langle B,g\rangle}\to U'_{\langle A,f\rangle}\ \text{ and }\ \tilde{\lambda}_{\langle B,g\rangle} : \coprod_{\langle A,f\rangle\in\operatorname{Ob}\mathcal{D}^G} U_{\langle A,f\rangle,\langle B,g\rangle}\to U''_{\langle B,g\rangle}$$

the morphisms induced by

$$(\hat{\lambda}^{\langle B,g\rangle}_{\langle A,f\rangle} : U_{\langle A,f\rangle,\langle B,g\rangle}\to U'_{\langle A,f\rangle})_{\langle B,g\rangle\in\operatorname{Ob}\mathcal{D}^G}\ \text{ and }\ (\tilde{\lambda}^{\langle A,f\rangle}_{\langle B,g\rangle} : U_{\langle A,f\rangle,\langle B,g\rangle}\to U''_{\langle B,g\rangle})_{\langle A,f\rangle\in\operatorname{Ob}\mathcal{D}^G},$$

respectively. Also put

$$\hat{\lambda} = \coprod_{\langle A,f\rangle \in \mathrm{Ob}\,\mathcal{D}^F} \hat{\lambda}_{\langle A,f\rangle} : \coprod_{(\langle A,f\rangle,\langle B,g\rangle)\in \mathrm{Ob}(\mathcal{D}^F\times\mathcal{D}^G)} U_{\langle A,f\rangle,\langle B,g\rangle} \to \coprod_{\langle A,f\rangle \in \mathrm{Ob}\,\mathcal{D}^F} U'_{\langle A,f\rangle},$$

$$\tilde{\lambda} = \coprod_{\langle B,g\rangle \in \mathrm{Ob}\,\mathcal{D}^G} \tilde{\lambda}_{\langle B,g\rangle} : \coprod_{(\langle A,f\rangle,\langle B,g\rangle)\in \mathrm{Ob}(\mathcal{D}^F\times\mathcal{D}^G)} U_{\langle A,f\rangle,\langle B,g\rangle} \to \coprod_{\langle B,g\rangle \in \mathrm{Ob}\,\mathcal{D}^F} U''_{\langle B,g\rangle}.$$

By the universality of coproducts in $\mathcal{E}$, each square of the following diagram is a pull-back.

$$\begin{tikzcd}
\displaystyle\coprod_{(\langle A,f\rangle,\langle B,g\rangle)\in \mathrm{Ob}(\mathcal{D}^F\times\mathcal{D}^G)} U_{\langle A,f\rangle,\langle B,g\rangle} \arrow[r, "\tilde{\lambda}"] \arrow[d, "\hat{\lambda}"] & \displaystyle\coprod_{\langle B,g\rangle\in \mathrm{Ob}\,\mathcal{D}^F} U''_{\langle B,g\rangle} \arrow[r, "r"] \arrow[d, "\bar{\lambda}^G"] & \displaystyle\coprod_{\langle B,g\rangle\in \mathrm{Ob}\,\mathcal{D}^G} KP\langle B,g\rangle \arrow[d, "\lambda^G"] \\
\displaystyle\coprod_{\langle A,f\rangle\in \mathrm{Ob}\,\mathcal{D}^F} U'_{\langle A,f\rangle} \arrow[r, "\bar{\lambda}^F"] \arrow[d, "q"] & L(F)\times_{L(H)} L(G) \arrow[r, "p_2"] \arrow[d, "p_1"] & L(G) \arrow[d, "L(\psi)"] \\
\displaystyle\coprod_{\langle A,f\rangle\in \mathrm{Ob}\,\mathcal{D}^F} KP\langle A,f\rangle \arrow[r, "\lambda^F"] & L(F) \arrow[r, "L(\varphi)"] & L(H)
\end{tikzcd}$$

Since λ^F and λ^G are regular epimorphisms, so are $\bar{\lambda}^F$, $\bar{\lambda}^G$, $\hat{\lambda}$ and $\tilde{\lambda}$. For each object $(\langle A,f\rangle,\langle B,g\rangle)$ of $\mathcal{D}^F\times\mathcal{D}^G$, applying (2.9.9) to an equality

$$\lambda^H_{\langle A,\varphi f\rangle} q_{\langle A,f\rangle}\hat{\lambda}^{\langle B,g\rangle}_{\langle A,f\rangle} = L(\varphi)\lambda^F_{\langle A,\varphi f\rangle} q_{\langle A,f\rangle}\hat{\lambda}^{\langle B,g\rangle}_{\langle A,f\rangle} = L(\psi)\lambda^G_{\langle B,\psi g\rangle} r_{\langle B,g\rangle}\tilde{\lambda}^{\langle A,f\rangle}_{\langle B,g\rangle} = \lambda^H_{\langle B,\psi g\rangle} r_{\langle B,g\rangle}\tilde{\lambda}^{\langle A,f\rangle}_{\langle B,g\rangle},$$

we have a universal strict epimorphic family

$$(\pi^i_{\langle A,f\rangle,\langle B,g\rangle} : V^i_{\langle A,f\rangle,\langle B,g\rangle} \to U_{\langle A,f\rangle,\langle B,g\rangle})_{i\in I_{\langle A,f\rangle,\langle B,g\rangle}}$$

in $\mathcal{E}$ and morphisms

$$u^i_{\langle A,f\rangle,\langle B,g\rangle} : W^i_{\langle A,f\rangle,\langle B,g\rangle} \to A, \quad v^i_{\langle A,f\rangle,\langle B,g\rangle} : W^i_{\langle A,f\rangle,\langle B,g\rangle} \to B$$

in $\mathcal{C}$ and a morphism

$$s^i_{\langle A,f\rangle,\langle B,g\rangle} : V^i_{\langle A,f\rangle,\langle B,g\rangle} \to K(W^i_{\langle A,f\rangle,\langle B,g\rangle})$$

in $\mathcal{E}$ for each $i \in I_{\langle A,f\rangle,\langle B,g\rangle}$ such that $\varphi f h_{u^i_{\langle A,f\rangle,\langle B,g\rangle}} = \psi g h_{v^i_{\langle A,f\rangle,\langle B,g\rangle}}$, $K(u^i_{\langle A,f\rangle,\langle B,g\rangle})s^i_{\langle A,f\rangle,\langle B,g\rangle} = q_{\langle A,f\rangle}\hat{\lambda}^{\langle B,g\rangle}_{\langle A,f\rangle}\pi^i_{\langle A,f\rangle,\langle B,g\rangle}$, $K(v^i_{\langle A,f\rangle,\langle B,g\rangle})s^i_{\langle A,f\rangle,\langle B,g\rangle} = r_{\langle B,g\rangle}\tilde{\lambda}^{\langle A,f\rangle}_{\langle B,g\rangle}\pi^i_{\langle A,f\rangle,\langle B,g\rangle}$. There exists a unique morphism $k : h_{W^i_{\langle A,f\rangle,\langle B,g\rangle}} \to F\times_H G$ such that $\bar{\varphi}k = g h_{v^i_{\langle A,f\rangle,\langle B,g\rangle}}$, $\bar{\psi}k = f h_{u^i_{\langle A,f\rangle,\langle B,g\rangle}}$. Set $\rho_{\langle A,f\rangle,\langle B,g\rangle} = \bar{\lambda}^F_{\langle A,f\rangle}\hat{\lambda}^{\langle B,g\rangle}_{\langle A,f\rangle} = \bar{\lambda}^G_{\langle B,g\rangle}\tilde{\lambda}^{\langle A,f\rangle}_{\langle B,g\rangle}$. We claim that the following diagram commute.

$$\begin{tikzcd}
K(W^i_{\langle A,f\rangle,\langle B,g\rangle}) \arrow[d, equal] & V^i_{\langle A,f\rangle,\langle B,g\rangle} \arrow[l, "s^i_{\langle A,f\rangle,\langle B,g\rangle}"'] \arrow[r, "\pi^i_{\langle A,f\rangle,\langle B,g\rangle}"] & U_{\langle A,f\rangle,\langle B,g\rangle} \arrow[d, "\rho_{\langle A,f\rangle,\langle B,g\rangle}"] \\
KP\langle W^i_{\langle A,f\rangle,\langle B,g\rangle}, k\rangle \arrow[r, "\lambda^{F\times_H G}_{\langle W^i_{\langle A,f\rangle,\langle B,g\rangle},k\rangle}"] & L(F\times_H G) \arrow[r, "\Phi"] & L(F)\times_{L(H)} L(G)
\end{tikzcd}$$

We note that since

$$u^i_{\langle A,f\rangle,\langle B,g\rangle} : \langle W^i_{\langle A,f\rangle,\langle B,g\rangle}, f h_{u^i_{\langle A,f\rangle,\langle B,g\rangle}}\rangle \to \langle A,f\rangle \ \text{ and } \ v^i_{\langle A,f\rangle,\langle B,g\rangle} : \langle W^i_{\langle A,f\rangle,\langle B,g\rangle}, g h_{v^i_{\langle A,f\rangle,\langle B,g\rangle}}\rangle \to \langle B,g\rangle$$

are morphisms in $\mathcal{D}^F$ and $\mathcal{D}^G$ respectively, we see that equalities

$$\lambda^F_{\langle A,f\rangle}K(u^i_{\langle A,f\rangle,\langle B,g\rangle}) = \lambda^F_{\langle W^i_{\langle A,f\rangle,\langle B,g\rangle}, f h_{u^i_{\langle A,f\rangle,\langle B,g\rangle}}\rangle} \ \text{ and } \ \lambda^G_{\langle B,g\rangle}K(v^i_{\langle A,f\rangle,\langle B,g\rangle}) = \lambda^G_{\langle W^i_{\langle A,f\rangle,\langle B,g\rangle}, g h_{v^i_{\langle A,f\rangle,\langle B,g\rangle}}\rangle}$$

hold. Thus we have $p_1\rho_{\langle A,f\rangle,\langle B,g\rangle}\pi^i_{\langle A,f\rangle,\langle B,g\rangle} = \lambda^F_{\langle A,f\rangle}q_{\langle A,f\rangle}\hat{\lambda}^{\langle B,g\rangle}_{\langle A,f\rangle}\pi^i_{\langle A,f\rangle,\langle B,g\rangle} = \lambda^F_{\langle A,f\rangle}K(u^i_{\langle A,f\rangle,\langle B,g\rangle})s^i_{\langle A,f\rangle,\langle B,g\rangle}$ $= \lambda^F_{\langle W^i_{\langle A,f\rangle,\langle B,g\rangle}, f h_{u^i_{\langle A,f\rangle,\langle B,g\rangle}}\rangle} s^i_{\langle A,f\rangle,\langle B,g\rangle} = \lambda^F_{\langle W^i_{\langle A,f\rangle,\langle B,g\rangle}, \bar{\psi}k\rangle} s^i_{\langle A,f\rangle,\langle B,g\rangle} = L(\bar{\psi})\lambda^{F\times_H G}_{\langle W^i_{\langle A,f\rangle,\langle B,g\rangle}, k\rangle} s^i_{\langle A,f\rangle,\langle B,g\rangle}$

$= p_1\Phi\lambda^{F\times_H G}_{\langle W^i_{\langle A,f\rangle,\langle B,g\rangle},k\rangle} s^i_{\langle A,f\rangle,\langle B,g\rangle}$. Similarly, $p_2\rho_{\langle A,f\rangle,\langle B,g\rangle}\pi^i_{\langle A,f\rangle,\langle B,g\rangle} = p_2\Phi\lambda^{F\times_H G}_{\langle W^i_{\langle A,f\rangle,\langle B,g\rangle},k\rangle} s^i_{\langle A,f\rangle,\langle B,g\rangle}$. Hence the above diagram commutes. We set

$$\rho = \bar\lambda^F\hat\lambda = \bar\lambda^G\tilde\lambda : \coprod_{(\langle A,f\rangle,\langle B,g\rangle)\in \mathrm{Ob}(\mathcal{D}^F\times\mathcal{D}^G)} U_{\langle A,f\rangle,\langle B,g\rangle} \to L(F)\times_{L(H)} L(G),$$

which is the morphism induced by $\rho_{\langle A,f\rangle,\langle B,g\rangle}$'s and is a regular epimorphism by (A.8.7). Let $\pi_{\langle A,f\rangle,\langle B,g\rangle} : \coprod_{i\in I_{\langle A,f\rangle,\langle B,g\rangle}} V^i_{\langle A,f\rangle,\langle B,g\rangle} \to U_{\langle A,f\rangle,\langle B,g\rangle}$ be the morphism induced by universal strict epimorphic family $(\pi^i_{\langle A,f\rangle,\langle B,g\rangle})_{i\in I_{\langle A,f\rangle,\langle B,g\rangle}}$. Hence $\pi_{\langle A,f\rangle,\langle B,g\rangle}$ is a regular epimorphism.
Put $\pi = \coprod_{(\langle A,f\rangle,\langle B,g\rangle)\in \mathrm{Ob}(\mathcal{D}^F\times\mathcal{D}^G)} \pi_{\langle A,f\rangle,\langle B,g\rangle}$. We also denote by

$$s : \coprod_{\substack{i\in I_{\langle A,f\rangle,\langle B,g\rangle}\\(\langle A,f\rangle,\langle B,g\rangle)\in \mathrm{Ob}(\mathcal{D}^F\times\mathcal{D}^G)}} V^i_{\langle A,f\rangle,\langle B,g\rangle} \to \coprod_{\langle C,l\rangle\in \mathrm{Ob}\mathcal{D}^{F\times_G H}} KP\langle C,l\rangle$$

the morphism induced by

$$(s^i_{\langle A,f\rangle,\langle B,g\rangle} : V^i_{\langle A,f\rangle,\langle B,g\rangle}\to K(W^i_{\langle A,f\rangle,\langle B,g\rangle}) = KP\langle W^i_{\langle A,f\rangle,\langle B,g\rangle},k\rangle)_{\substack{i\in I_{\langle A,f\rangle,\langle B,g\rangle}\\(\langle A,f\rangle,\langle B,g\rangle)\in \mathrm{Ob}(\mathcal{D}^F\times\mathcal{D}^G)}}.$$

Since the following diagram commutes, Φ is a regular epimorphism by (A.8.6).

$$\begin{tikzcd}
\coprod_{\langle C,l\rangle\in \mathrm{Ob}\mathcal{D}^{F\times_G H}} KP\langle C,l\rangle \arrow[d, "\lambda^{F\times_H G}"] & & \coprod_{\substack{i\in I_{\langle A,f\rangle,\langle B,g\rangle}\\(\langle A,f\rangle,\langle B,g\rangle)\in \mathrm{Ob}(\mathcal{D}^F\times\mathcal{D}^G)}} V^i_{\langle A,f\rangle,\langle B,g\rangle} \arrow[ll, "s"'] \arrow[d, "\pi"] \\
L(F\times_H G) \arrow[r, "\Phi"] & L(F)\times_{L(H)} L(G) & \coprod_{(\langle A,f\rangle,\langle B,g\rangle)\in \mathrm{Ob}(\mathcal{D}^F\times\mathcal{D}^G)} U_{\langle A,f\rangle,\langle B,g\rangle} \arrow[l, "\rho"']
\end{tikzcd}$$

Next, we show that Φ is a monomorphism. Let $\alpha,\beta : U \to L(F\times_H G)$ be morphisms such that $\Phi\alpha = \Phi\beta$. Consider the following pull-backs for $\langle A,f\rangle,\langle B,g\rangle \in \mathrm{Ob}\,\mathcal{D}^{F\times_G H}$.

$$\begin{tikzcd}
U^\alpha_{\langle A,f\rangle} \arrow[r, "\lambda^\alpha_{\langle A,f\rangle}"] \arrow[d, "\alpha_{\langle A,f\rangle}"] & U \arrow[d, "\alpha"] \\
KP\langle A,f\rangle \arrow[r, "\lambda^{F\times_H G}_{\langle A,f\rangle}"] & L(F\times_H G)
\end{tikzcd}
\quad
\begin{tikzcd}
U^\beta_{\langle B,g\rangle} \arrow[r, "\lambda^\beta_{\langle B,g\rangle}"] \arrow[d, "\beta_{\langle B,g\rangle}"] & U \arrow[d, "\beta"] \\
KP\langle B,g\rangle \arrow[r, "\lambda^{F\times_H G}_{\langle B,g\rangle}"] & L(F\times_H G)
\end{tikzcd}
\quad
\begin{tikzcd}
U_{\langle A,f\rangle,\langle B,g\rangle} \arrow[r, "\mu^{\langle A,f\rangle}_{\langle B,g\rangle}"] \arrow[d, "\nu^{\langle B,g\rangle}_{\langle A,f\rangle}"] & U^\beta_{\langle B,g\rangle} \arrow[d, "\lambda^\beta_{\langle B,g\rangle}"] \\
U^\alpha_{\langle A,f\rangle} \arrow[r, "\lambda^\alpha_{\langle A,f\rangle}"] & U
\end{tikzcd}$$

Then, $(\lambda^\alpha_{\langle A,f\rangle} : U^\alpha_{\langle A,f\rangle}\to U)_{\langle A,f\rangle\in \mathrm{Ob}\mathcal{D}^{F\times_G H}}$ and $(\lambda^\beta_{\langle B,g\rangle} : U^\beta_{\langle B,g\rangle}\to U)_{\langle B,g\rangle\in \mathrm{Ob}\mathcal{D}^{F\times_G H}}$ are universal strict epimorphic families and so are

$$(\mu^{\langle A,f\rangle}_{\langle B,g\rangle} : U_{\langle A,f\rangle,\langle B,g\rangle} \to U^\beta_{\langle B,g\rangle})_{\langle A,f\rangle\in \mathrm{Ob}\mathcal{D}^{F\times_G H}} \text{ and } (\nu^{\langle B,g\rangle}_{\langle A,f\rangle} : U_{\langle A,f\rangle,\langle B,g\rangle} \to U^\alpha_{\langle A,f\rangle})_{\langle B,g\rangle\in \mathrm{Ob}\mathcal{D}^{F\times_G H}}.$$

Set $\rho_{\langle A,f\rangle,\langle B,g\rangle} = \lambda^\alpha_{\langle A,f\rangle}\nu^{\langle B,g\rangle}_{\langle A,f\rangle} = \lambda^\beta_{\langle A,f\rangle}\mu^{\langle A,f\rangle}_{\langle B,g\rangle}$. Then, $(\rho_{\langle A,f\rangle,\langle B,g\rangle} : U_{\langle A,f\rangle,\langle B,g\rangle} \to U)_{\langle A,f\rangle,\langle B,g\rangle\in \mathrm{Ob}\,\mathcal{D}^{F\times_H G}}$ is a universal strict epimorphic family. It follows from $L(\bar\psi)\alpha = L(\bar\psi)\beta$ that $\lambda^F_{\langle A,\bar\psi f\rangle}\alpha_{\langle A,f\rangle}\nu^{\langle B,g\rangle}_{\langle A,f\rangle} = \lambda^F_{\langle B,\bar\psi g\rangle}\beta_{\langle B,g\rangle}\mu^{\langle A,f\rangle}_{\langle B,g\rangle}$ for $\langle A,f\rangle,\langle B,g\rangle \in \mathrm{Ob}\,\mathcal{D}^{F\times_H G}$. By (2.9.11), there exist a universal strict epimorphic family $(p^i_{\langle A,f\rangle,\langle B,g\rangle} : V^i_{\langle A,f\rangle,\langle B,g\rangle} \to U_{\langle A,f\rangle,\langle B,g\rangle})_{i\in I_{\langle A,f\rangle,\langle B,g\rangle}}$ in $\mathcal{E}$, morphisms $u^i_{\langle A,f\rangle,\langle B,g\rangle} : X^i_{\langle A,f\rangle,\langle B,g\rangle} \to A$, $v^i_{\langle A,f\rangle,\langle B,g\rangle} : X^i_{\langle A,f\rangle,\langle B,g\rangle} \to B$ in $\mathcal{C}$ and morphisms $s^i_{\langle A,f\rangle,\langle B,g\rangle} : V^i_{\langle A,f\rangle,\langle B,g\rangle} \to K(X^i_{\langle A,f\rangle,\langle B,g\rangle})$ such that $\bar\psi f h_{u^i_{\langle A,f\rangle,\langle B,g\rangle}} = \bar\psi g h_{v^i_{\langle A,f\rangle,\langle B,g\rangle}}$ and the following diagram commute.

$$\begin{tikzcd}
K(X^i_{\langle A,f\rangle,\langle B,g\rangle}) \arrow[d, "K(u^i_{\langle A,f\rangle,\langle B,g\rangle})"] & V^i_{\langle A,f\rangle,\langle B,g\rangle} \arrow[l, "s^i_{\langle A,f\rangle,\langle B,g\rangle}"'] \arrow[r, "s^i_{\langle A,f\rangle,\langle B,g\rangle}"] \arrow[d, "p^i_{\langle A,f\rangle,\langle B,g\rangle}"] & K(X^i_{\langle A,f\rangle,\langle B,g\rangle}) \arrow[d, "K(v^i_{\langle A,f\rangle,\langle B,g\rangle})"] \\
K(A) & U_{\langle A,f\rangle,\langle B,g\rangle} \arrow[l, "\alpha_{\langle A,f\rangle}\nu^{\langle B,g\rangle}_{\langle A,f\rangle}"'] \arrow[r, "\beta_{\langle A,f\rangle}\mu^{\langle A,f\rangle}_{\langle B,g\rangle}"] & K(B)
\end{tikzcd}$$

It follows from $L(\bar{\varphi})\alpha = L(\bar{\varphi})\beta$ that $\lambda^F_{\langle A,\bar{\varphi}f\rangle}\alpha_{\langle A,f\rangle}\nu^{\langle B,g\rangle}_{\langle A,f\rangle} = \lambda^F_{\langle B,\bar{\varphi}g\rangle}\beta_{\langle B,g\rangle}\mu^{\langle A,f\rangle}_{\langle B,g\rangle}$ and we have
$\lambda^G_{\langle X^i_{\langle A,f\rangle,\langle B,g\rangle},\bar{\varphi}fh_{u^i_{\langle A,f\rangle,\langle B,g\rangle}}\rangle}s^i_{\langle A,f\rangle,\langle B,g\rangle} = \lambda^G_{\langle A,\bar{\varphi}f\rangle}K(u^i_{\langle A,f\rangle,\langle B,g\rangle})s^i_{\langle A,f\rangle,\langle B,g\rangle} = \lambda^G_{\langle A,\bar{\varphi}f\rangle}\alpha_{\langle A,f\rangle}\nu^{\langle B,g\rangle}_{\langle A,f\rangle}p^i_{\langle A,f\rangle,\langle B,g\rangle}$
$= \lambda^G_{\langle B,\bar{\varphi}g\rangle}\beta_{\langle B,g\rangle}\mu^{\langle A,f\rangle}_{\langle B,g\rangle}p^i_{\langle A,f\rangle,\langle B,g\rangle} = \lambda^G_{\langle B,\bar{\varphi}f\rangle}K(v^i_{\langle A,f\rangle,\langle B,g\rangle})s^i_{\langle A,f\rangle,\langle B,g\rangle} = \lambda^G_{\langle X^i_{\langle A,f\rangle,\langle B,g\rangle},\bar{\varphi}fh_{v^i_{\langle A,f\rangle,\langle B,g\rangle}}\rangle}s^i_{\langle A,f\rangle,\langle B,g\rangle}$.
By (2.9.11), we have a universal strict epimorphic family $(q^{ij}_{\langle A,f\rangle,\langle B,g\rangle} : W^{ij}_{\langle A,f\rangle,\langle B,g\rangle} \to V^i_{\langle A,f\rangle,\langle B,g\rangle})_{j\in J^i_{\langle A,f\rangle,\langle B,g\rangle}}$, morphisms $w^{ij}_{\langle A,f\rangle,\langle B,g\rangle} : Y^{ij}_{\langle A,f\rangle,\langle B,g\rangle} \to X^i_{\langle A,f\rangle,\langle B,g\rangle}$, $z^{ij}_{\langle A,f\rangle,\langle B,g\rangle} : Y^{ij}_{\langle A,f\rangle,\langle B,g\rangle} \to X^i_{\langle A,f\rangle,\langle B,g\rangle}$ in $\mathcal{C}$ and morphisms $t^{ij}_{\langle A,f\rangle,\langle B,g\rangle} : W^{ij}_{\langle A,f\rangle,\langle B,g\rangle} \to K(Y^{ij}_{\langle A,f\rangle,\langle B,g\rangle})$ in $\mathcal{E}$ such that $\bar{\varphi}fh_{u^i_{\langle A,f\rangle,\langle B,g\rangle}}h_{w^{ij}_{\langle A,f\rangle,\langle B,g\rangle}} = \bar{\varphi}gh_{v^i_{\langle A,f\rangle,\langle B,g\rangle}}h_{z^{ij}_{\langle A,f\rangle,\langle B,g\rangle}}$ and the following diagram commute.

$$\begin{tikzcd}
K(Y^{ij}_{\langle A,f\rangle,\langle B,g\rangle}) \arrow[d, "K(w^{ij}_{\langle A,f\rangle,\langle B,g\rangle})"] & W^{ij}_{\langle A,f\rangle,\langle B,g\rangle} \arrow[l, "t^{ij}_{\langle A,f\rangle,\langle B,g\rangle}"'] \arrow[r, "t^{ij}_{\langle A,f\rangle,\langle B,g\rangle}"] \arrow[d, "q^{ij}_{\langle A,f\rangle,\langle B,g\rangle}"] & K(Y^{ij}_{\langle A,f\rangle,\langle B,g\rangle}) \arrow[d, "K(z^{ij}_{\langle A,f\rangle,\langle B,g\rangle})"] \\
K(X^i_{\langle A,f\rangle,\langle B,g\rangle}) & V^i_{\langle A,f\rangle,\langle B,g\rangle} \arrow[l, "s^i_{\langle A,f\rangle,\langle B,g\rangle}"'] \arrow[r, "s^i_{\langle A,f\rangle,\langle B,g\rangle}"] & K(X^i_{\langle A,f\rangle,\langle B,g\rangle})
\end{tikzcd}$$

Since K is filtering, there exist a universal strict epimorphic family

$$(r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} : Z^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} \to W^{ij}_{\langle A,f\rangle,\langle B,g\rangle})_{k\in M^{ij}_{\langle A,f\rangle,\langle B,g\rangle}} \text{ in } \mathcal{E}, \text{ morphisms } e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} : Z^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} \to Y^{ij}_{\langle A,f\rangle,\langle B,g\rangle}$$

in $\mathcal{C}$ and morphisms $\bar{t}^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} : T^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} \to K(Z^{ijk}_{\langle A,f\rangle,\langle B,g\rangle})$ such that

$$w^{ij}_{\langle A,f\rangle,\langle B,g\rangle}e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} = z^{ij}_{\langle A,f\rangle,\langle B,g\rangle}e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} \text{ and } t^{ij}_{\langle A,f\rangle,\langle B,g\rangle}r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} = K(e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle})\bar{t}^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}.$$

Therefore,

$$\begin{aligned}
\bar{\psi}fh_{u^i_{\langle A,f\rangle,\langle B,g\rangle}w^{ij}_{\langle A,f\rangle,\langle B,g\rangle}e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}} &= \bar{\psi}fh_{v^i_{\langle A,f\rangle,\langle B,g\rangle}z^{ij}_{\langle A,f\rangle,\langle B,g\rangle}e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}}, \\
\bar{\varphi}fh_{u^i_{\langle A,f\rangle,\langle B,g\rangle}w^{ij}_{\langle A,f\rangle,\langle B,g\rangle}e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}} &= \bar{\varphi}fh_{v^i_{\langle A,f\rangle,\langle B,g\rangle}z^{ij}_{\langle A,f\rangle,\langle B,g\rangle}e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}}.
\end{aligned}$$

Hence we have $fh_{u^i_{\langle A,f\rangle,\langle B,g\rangle}w^{ij}_{\langle A,f\rangle,\langle B,g\rangle}e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}} = fh_{v^i_{\langle A,f\rangle,\langle B,g\rangle}z^{ij}_{\langle A,f\rangle,\langle B,g\rangle}e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}}$ and it follows that
$\lambda^{F\times_H G}_{\langle A,f\rangle}K(u^i_{\langle A,f\rangle,\langle B,g\rangle}w^{ij}_{\langle A,f\rangle,\langle B,g\rangle}e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}) = \lambda^{F\times_H G}_{\langle B,g\rangle}K(v^i_{\langle A,f\rangle,\langle B,g\rangle}z^{ij}_{\langle A,f\rangle,\langle B,g\rangle}e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle})$.
Finally, $\alpha\rho_{\langle A,f\rangle,\langle B,g\rangle}p^i_{\langle A,f\rangle,\langle B,g\rangle}q^{ij}_{\langle A,f\rangle,\langle B,g\rangle}r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} = \alpha\lambda^{\alpha}_{\langle A,f\rangle}\nu^{\langle B,g\rangle}_{\langle A,f\rangle}p^i_{\langle A,f\rangle,\langle B,g\rangle}q^{ij}_{\langle A,f\rangle,\langle B,g\rangle}r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}$
$= \lambda^{F\times_H G}_{\langle A,f\rangle}\alpha_{\langle A,f\rangle}\nu^{\langle B,g\rangle}_{\langle A,f\rangle}p^i_{\langle A,f\rangle,\langle B,g\rangle}q^{ij}_{\langle A,f\rangle,\langle B,g\rangle}r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} = \lambda^{F\times_H G}_{\langle A,f\rangle}K(u^i_{\langle A,f\rangle,\langle B,g\rangle})s^i_{\langle A,f\rangle,\langle B,g\rangle}q^{ij}_{\langle A,f\rangle,\langle B,g\rangle}r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}$
$= \lambda^{F\times_H G}_{\langle A,f\rangle}K(u^i_{\langle A,f\rangle,\langle B,g\rangle}w^{ij}_{\langle A,f\rangle,\langle B,g\rangle})t^{ij}_{\langle A,f\rangle,\langle B,g\rangle}r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}$
$= \lambda^{F\times_H G}_{\langle A,f\rangle}K(u^i_{\langle A,f\rangle,\langle B,g\rangle}w^{ij}_{\langle A,f\rangle,\langle B,g\rangle}e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle})\bar{t}^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}$
$= \lambda^{F\times_H G}_{\langle B,g\rangle}K(v^i_{\langle A,f\rangle,\langle B,g\rangle}z^{ij}_{\langle A,f\rangle,\langle B,g\rangle}e^{ijk}_{\langle A,f\rangle,\langle B,g\rangle})\bar{t}^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}$
$= \lambda^{F\times_H G}_{\langle B,g\rangle}K(v^i_{\langle A,f\rangle,\langle B,g\rangle}z^{ij}_{\langle A,f\rangle,\langle B,g\rangle})t^{ij}_{\langle A,f\rangle,\langle B,g\rangle}r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}$
$= \lambda^{F\times_H G}_{\langle B,g\rangle}K(v^i_{\langle A,f\rangle,\langle B,g\rangle})s^i_{\langle A,f\rangle,\langle B,g\rangle}q^{ij}_{\langle A,f\rangle,\langle B,g\rangle}r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} = \lambda^{F\times_H G}_{\langle B,g\rangle}\beta_{\langle B,g\rangle}\mu^{\langle A,f\rangle}_{\langle B,g\rangle}p^i_{\langle A,f\rangle,\langle B,g\rangle}q^{ij}_{\langle A,f\rangle,\langle B,g\rangle}r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}$
$= \beta\lambda^{\beta}_{\langle A,f\rangle}\mu^{\langle A,f\rangle}_{\langle B,g\rangle}p^i_{\langle A,f\rangle,\langle B,g\rangle}q^{ij}_{\langle A,f\rangle,\langle B,g\rangle}r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle} = \beta\rho_{\langle A,f\rangle,\langle B,g\rangle}p^i_{\langle A,f\rangle,\langle B,g\rangle}q^{ij}_{\langle A,f\rangle,\langle B,g\rangle}r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle}$.
Since $(\rho_{\langle A,f\rangle,\langle B,g\rangle})_{\langle A,f\rangle,\langle B,g\rangle\in\mathrm{Ob}\,\mathcal{D}^{F\times_H G}}$, $(p^i_{\langle A,f\rangle,\langle B,g\rangle})_{i\in I_{\langle A,f\rangle,\langle B,g\rangle}}$, $(q^{ij}_{\langle A,f\rangle,\langle B,g\rangle})_{j\in J^i_{\langle A,f\rangle,\langle B,g\rangle}}$ and $(r^{ijk}_{\langle A,f\rangle,\langle B,g\rangle})_{k\in M^{ij}_{\langle A,f\rangle,\langle B,g\rangle}}$ are universal strict epimorphic families, we have $\alpha = \beta$. □

Proposition 2.9.13 *Under the same assumptions as in (2.9.11), $L : \widehat{\mathcal{C}} \to \mathcal{E}$ preserves terminal objects.*

Proof. Let us denote by $1_{\mathcal{E}}$ and $1_{\widehat{\mathcal{C}}}$ the terminal objects of $\mathcal{E}$ and $\widehat{\mathcal{C}}$, respectively. Obviously, $P : (h^{\mathcal{C}}\downarrow 1_{\widehat{\mathcal{C}}}) \to \mathcal{C}$ is an isomorphism of categories. Hence $L(1_{\widehat{\mathcal{C}}}) = \varinjlim K$, that is, $\coprod_{f\in\mathrm{Mor}\,\mathcal{C}} K(\mathrm{dom}(f)) \overset{\sigma}{\underset{\tau}{\rightrightarrows}} \coprod_{X\in\mathrm{Ob}\,\mathcal{C}} K(X) \xrightarrow{\lambda} L(1_{\widehat{\mathcal{C}}})$ is a coequalizer, where σ and τ are the morphisms which make the following diagram commute.

$$\begin{tikzcd}
K(\mathrm{dom}(f)) \arrow[d, "\iota_{\mathrm{dom}(f)}"] & K(\mathrm{dom}(f)) \arrow[l, "id"'] \arrow[r, "K(f)"] \arrow[d, "\iota_f"] & K(\mathrm{codom}(f)) \arrow[d, "\iota_{\mathrm{codom}(f)}"] \\
\coprod_{i\in\mathrm{Ob}\,\mathcal{C}} K(i) & \coprod_{f\in\mathrm{Mor}\,\mathcal{C}} K(\mathrm{dom}(f)) \arrow[l, "\sigma"'] \arrow[r, "\tau"] & \coprod_{i\in\mathrm{Ob}\,\mathcal{C}} K(i)
\end{tikzcd}$$

Let $p, q : \coprod_{Y,Z\in \mathrm{Ob}\,\mathcal{C}} K(Y)\times K(Z) \to \coprod_{X\in \mathrm{Ob}\,\mathcal{C}} K(X)$ be morphisms induced by the projections onto the first and the second components. Since $(K(X) \to 1_{\mathcal{E}})_{X\in \mathrm{Ob}\,\mathcal{C}}$ is a universal strict epimorphic family by (2.9.5), $\coprod_{Y,Z\in \mathrm{Mor}\,\mathcal{C}} K(Y)\times K(Z) \overset{p}{\underset{q}{\rightrightarrows}} \coprod_{X\in \mathrm{Ob}\,\mathcal{C}} K(X) \xrightarrow{r} 1_{\mathcal{E}}$ is a coequalizer. Put $\mathcal{C}_1 = \mathrm{Funct}(\Delta_1, \mathcal{C})$ and let $\phi : \coprod_{d\in \mathrm{Ob}\,\mathcal{C}_1} K(d(0))$
$\to \coprod_{Y,Z\in \mathrm{Mor}\,\mathcal{C}} K(Y)\times K(Z)$ the morphism induced by $(K(d(p_1)), K(d(p_2))) : K(d(0)) \to K(d(1))\times K(d(0))$. Then, ϕ is an epimorphism by (2.9.5) and $\coprod_{d\in \mathrm{Ob}\,\mathcal{C}_1} K(d(0)) \overset{p\phi}{\underset{q\phi}{\rightrightarrows}} \coprod_{X\in \mathrm{Ob}\,\mathcal{C}} K(X) \xrightarrow{r} 1_{\mathcal{E}}$ is a coequalizer. For each $d \in \mathrm{Ob}\,\mathcal{C}_1$, $\lambda p\phi\iota_d = \lambda\iota_{d(1)}K(d(p_1)) = \lambda\tau\iota_{d(p_1)} = \lambda\sigma\iota_{d(p_1)} = \lambda\iota_{d(0)} = \lambda\sigma\iota_{d(p_2)} = \lambda\tau\iota_{d(p_2)} = \lambda\iota_{d(2)}K(d(p_2)) = \lambda q\phi\iota_d$. Hence we have $\lambda p\phi = \lambda q\phi$ and there exists a unique morphism $g : 1_{\mathcal{E}} \to W$ that factors through the unique morphism $r : \coprod_{X\in \mathrm{Ob}\,\mathcal{C}} K(X) \to 1_{\mathcal{E}}$, that is, $sr = \lambda$. Let $t : L(1_{\widehat{\mathcal{C}}}) \to 1_{\widehat{\mathcal{C}}}$ be the unique morphism. Clearly, $ts = id_{1_{\mathcal{E}}}$ and $st\lambda = sr = \lambda$. Since λ is an epimorphism, st is the identity morphism of $L(1_{\widehat{\mathcal{C}}})$. □

By (2.9.9), (2.9.12) and (2.9.13), we have the following result.

Theorem 2.9.14 *Let $\mathcal{C}$ be a $\mathcal{U}$-small category and $\mathcal{E}$ a finitely complete, $\mathcal{U}$-cocomplete exact category whose coproducts are disjoint and universal. A functor $K : \mathcal{C} \to \mathcal{E}$ is filtering if and only if the left Kan extension of K along the Yoneda embedding is left exact. Moreover, if $\mathcal{C}$ is finitely complete, K is filtering if and only if it is left exact.*

2.10 Giraud's theorem

Proposition 2.10.1 *Let (T, μ, η) be a monad on a category $\mathcal{C}$ such that $\mu : T^2 \to T$ is an equivalence and $\mathcal{D}$ a full subcategory of $\mathcal{C}$ given by $\mathrm{Ob}\,\mathcal{D} = \{X \in \mathrm{Ob}\,\mathcal{C} | \, \eta_X : X \to T(X)$ is an isomorphism$\}$. Then, $\mathcal{D}$ is a strictly full reflexive subcategory of $\mathcal{C}$. Moreover, $\mathcal{D}$ has a left exact reflection if and only if T is left exact.*

Proof. By the naturality of η, $\mathcal{D}$ is strictly full. Since μ is an equivalence, $T(X)$ is an object of $\mathcal{D}$ for any $X \in \mathrm{Ob}\,\mathcal{C}$. We define a functor $L : \mathcal{C} \to \mathcal{D}$ by $L(X) = T(X)$ and $L(f) = T(f)$ for $X \in \mathrm{Ob}\,\mathcal{C}$, $f \in \mathrm{Mor}\,\mathcal{C}$. Let us denote by $i : \mathcal{D} \to \mathcal{C}$ the inclusion functor. Then, $iL = T$ and we set $\varepsilon_Y = \eta_Y^{-1} : Li(Y) = T(Y) \to Y$ for $Y \in \mathrm{Ob}\,\mathcal{D}$. Hence we have a unit $\eta : id_{\mathcal{C}} \to T = iL$ and a counit $\varepsilon : Li \to id_{\mathcal{D}}$ which satisfy $\varepsilon_{L(X)}L(\eta_X) = id_{L(X)}$ for $X \in \mathrm{Ob}\,\mathcal{C}$ and $i(\varepsilon_Y)\eta_{i(Y)} = id_{i(Y)}$ for $Y \in \mathrm{Ob}\,\mathcal{D}$. Thus L is a left adjoint of i. It is clear that L is left exact if and only if T is so. □

Definition 2.10.2 *We call a category $\mathcal{E}$ a $\mathcal{U}$-topos if there exist a site $(\mathcal{C}, J)$ such that $\mathcal{C} \in \mathcal{U}$ and $\mathcal{E}$ is equivalent to the category $\widetilde{\mathcal{C}}$ of sheaves of $\mathcal{U}$-set on $\mathcal{C}$.*

The following theorem characterizes $\mathcal{U}$-topos.

Theorem 2.10.3 *A category $\mathcal{E}$ is a $\mathcal{U}$-topos if and only if the following conditions are satisfied.*

(0) $\mathcal{E}$ is a $\mathcal{U}$-category.
(1) $\mathcal{E}$ has finite limits.
(2) $\mathcal{E}$ has coproducts indexed by $\mathcal{U}$-small sets and they are disjoint and universal (A.1.10).
(3) Every equivalence relation in $\mathcal{E}$ is effective (A.1.9) and has a universal coequalizer (A.1.10).
(4) $\mathcal{E}$ has a $\mathcal{U}$-small set of generators for monomorphisms (A.1.14).

By (2.4.14) and (2.4.17), the above conditions are necessary. We note that the conditions (1) and (3) imply that $\mathcal{E}$ is an exact category by (A.8.14).

Proposition 2.10.4 *Let $\mathcal{E}$ be a finitely complete exact category with universal countable coproducts. Then, $\mathcal{E}$ has coequalizers. Hence if $\mathcal{E}$ also has $\mathcal{U}$-coproducts, it is $\mathcal{U}$-cocomplete.*

Proof. Let $Z \overset{k}{\underset{l}{\rightrightarrows}} Y$ be a parallel pair of morphisms in $\mathcal{E}$. Set $X_0 = Y$, $f_0 = g_0 = t_0 = id_Y$ and $X_1 = Y \coprod Z \coprod Z$. Let $f_1, g_1 : X_1 \to Y$, $t_1 : X_1 \to X_1$ and $j_1 : X_0 \to X_1$ be morphisms defined by $f_1\iota_1 = g_1\iota_1 = id_Y$,

$f_1\iota_2 = k$, $g_1\iota_2 = l$, $f_1\iota_3 = l$, $g_1\iota_3 = k$, $t_1\iota_1 = \iota_1$, $t_1\iota_2 = \iota_3$, $t_1\iota_3 = \iota_2$ and $j_1 = \iota_1$. Suppose that objects X_m $(0 \leqq m \leqq n)$ of $\mathcal{E}$ and morphisms $f_m, g_m : X_m \to Y$, $t_m : X_m \to X_m$ $(0 \leqq m \leqq n)$, monomorphisms $j_m : X_{m-1} \to X_m$ $(1 \leqq m \leqq n)$ in $\mathcal{E}$ are defined such that $f_m j_m = f_{m-1}$, $g_m j_m = g_{m-1}$, $t_m g_m = f_m$, $t_m f_m = g_m$. Form a pull-back

$$\begin{tikzcd} X_{n+1} \arrow[r, "\bar{g}_n"] \arrow[d, "\bar{f}_n"] & X_n \arrow[d, "f_n"] \\ X_n \arrow[r, "g_n"] & Y \end{tikzcd}$$

and set $f_{n+1} = f_n\bar{f}_n$, $g_{n+1} = g_n\bar{g}_n$. Since $g_n = f_n j_n j_{n-1}\cdots j_1 g_n$, there is a unique morphism $j_{n+1} : X_n \to X_{n+1}$ satisfying $\bar{f}_n j_{n+1} = id_{X_n}$ and $\bar{g}j_{n+1} = j_n j_{n-1}\cdots j_1 g_n$. Then j_{n+1} is a split monomorphism and we have $f_{n+1}j_{n+1} = f_n\bar{f}_n j_{n+1} = f_n$, $g_{n+1}j_{n+1} = g_n\bar{g}_n j_{n+1} = g_n j_n j_{n-1}\cdots j_1 g_n = g_n$. Since $f_n t_n \bar{f}_n = g_n\bar{f}_n = f_n\bar{g}_n = g_n t_n \bar{g}_n$, there is a unique morphism $t_{n+1} : X_{n+1} \to X_{n+1}$ satisfying $\bar{f}_n t_{n+1} = t_n\bar{g}_n$ and $\bar{g}_n t_{n+1} = t_n\bar{f}_n$. Then $g_{n+1}t_{n+1} = g_n\bar{g}_n t_{n+1} = g_n t_n \bar{f}_n = f_n\bar{f}_n = f_{n+1}$ and $f_{n+1}t_{n+1} = f_n\bar{f}_n t_{n+1} = f_n t_n \bar{g}_n = g_n\bar{g}_n = g_{n+1}$. We put $X = \coprod_{n\geqq 0} X_n$, $s = \iota_0 : Y = X_0 \to X$, $t = \coprod_{n\geqq 0} t_n : X \to X$ and let $f, g : X \to Y$ be morphisms defined by $f\iota_n = f_n$, $g\iota_n = g_n$. Clearly, $fs = gs = id_Y$ and $ft = g$, $gt = f$. Let

$$\begin{tikzcd} T_{mn} \arrow[r, "q_{mn}"] \arrow[d, "p_{mn}"] & X_n \arrow[d, "f_n"] \\ X_m \arrow[r, "g_m"] & Y \end{tikzcd}$$

be a pull-back. To verify the condition (3) of (A.8.23), it suffices to show that the image of $(f_m p_{mn}, g_n q_{mn}) : T_{mn} \to Y\times Y$ is contained in that of $(f_N, g_N) : X_N \to Y \times Y$ for sufficiently large N by the universality of countable coproducts. Choose an integer r such that $r \geqq m, n$. Then $f_n = f_r\iota_r\iota_{r-1}\cdots\iota_{n+1}$, $g_m = g_r\iota_r\iota_{r-1}\cdots\iota_{m+1}$ and we have a unique morphism $u : T_{mn} \to X_{r+1}$ such that $\bar{g}_r u = \iota_r\iota_{r-1}\cdots\iota_{n+1}q_{mn}$, $\bar{f}_r u = \iota_r\iota_{r-1}\cdots\iota_{m+1}p_{mn}$. Hence $(f_{r+1}, g_{r+1})u = (f_r\bar{f}_r u, g_r\bar{g}_r u) = (f_r\iota_r\iota_{r-1}\cdots\iota_{m+1}p_{mn}, g_r\iota_r\iota_{r-1}\cdots\iota_{n+1}q_{mn}) = (f_m p_{mn}, g_n q_{mn})$. Therefore the image of $(f_m p_{mn}, g_n q_{mn})$ is contained in that of (f_{r+1}, g_{r+1}) and the image of $(f, g) : X \to Y \times Y$ is an equivalence relation by (A.8.23).

Let $X \overset{\pi}{\twoheadrightarrow} R \overset{i}{\rightarrowtail} Y \times Y$ be a factorization of (f, g) and $p_n : Y \times Y \to Y$ be the projection onto the n-th component. By the assumption, $R \underset{p_2 i}{\overset{p_1 i}{\rightrightarrows}} Y$ has a coequalizer $\rho : Y \to W$. Let $h : Y \to U$ be a morphism satisfying $hk = hl$. We show that $hf_n = hg_n$ by induction on n. It is obvious from $hk = hl$ that $hf_1 = hg_1$. By the inductive assumption, $hf_{n+1} = hf_n\bar{f}_n = hg_n\bar{f}_n = hf_n\bar{g}_n = hg_n\bar{g}_n = hg_{n+1}$. Then, $hp_1 i\pi\iota_n = hp_2 i\pi\iota_n$ for any $n \geqq 1$ and we have $hp_1 i = hp_2 i$. Hence there exists a unique morphism $\varphi : W \to U$ such that $\varphi\rho = h$. This shows that ρ is a coequalizer of $Z \underset{l}{\overset{k}{\rightrightarrows}} Y$. □

Lemma 2.10.5 *Let $\mathcal{E}$ be a finitely complete regular $\mathcal{U}$-category with $\mathcal{U}$-small universal coproducts. If $\mathcal{E}$ has a $\mathcal{U}$-small set G of generators for monomorphisms, then G is a set of generators by universal strict epimorphisms. We denote by $\mathcal{C}$ the full subcategory of $\mathcal{E}$ with $\mathrm{Ob}\,\mathcal{C} = G$ and by $K : \mathcal{C} \to \mathcal{E}$ the inclusion functor. Then, K is a filtering functor.*

Proof. Since $\mathcal{E}$ is a $\mathcal{U}$-category and $\mathcal{C}$ is $\mathcal{U}$-small, $(K{\downarrow}X)$ is $\mathcal{U}$-small for $X \in \mathrm{Ob}\,\mathcal{E}$. Hence we can form a coproduct $W = \coprod_{\langle Y,g\rangle\in\mathrm{Ob}(K\downarrow X)} Y$. Then, a family of morphisms $(g : Y \to X)_{\langle Y,g\rangle\in\mathrm{Ob}(K\downarrow X)}$ induces $f : W \to X$. Let $W \overset{\pi}{\twoheadrightarrow} Z \overset{\iota}{\rightarrowtail} X$ be a factorization of f by a regular epimorphism π and a monomorphism ι. It follows from (A.8.24) that $(\pi\nu_{\langle Y,g\rangle} : Y \to Z)_{\langle Y,g\rangle\in\mathrm{Ob}(K\downarrow X)}$ is a universal strict epimorphic family, where $\nu_{\langle Y,g\rangle} : Y \to W$ denotes the canonical morphism into the $\langle Y, g\rangle$-th summand. On the other hand, for each $Y \in G$, $\iota_* : \mathcal{E}(Y, Z) \to \mathcal{E}(Y, X)$ is bijective. In fact, if $g \in \mathcal{E}(Y, X)$, then $\langle Y, g\rangle \in \mathrm{Ob}\,(K{\downarrow}X)$ and $\iota_*(\pi\nu_{\langle Y,g\rangle}) = g$. Since G is a set of generators for monomorphisms, ι is an isomorphism. It follows that $(g : Y \to X)_{\langle Y,g\rangle\in\mathrm{Ob}(K\downarrow X)}$ is a universal strict epimorphic family. The second assertion follows from (2.9.2). □

Now we can give a proof of (2.10.3).

Proof. Let $\mathcal{E}$ be a category satisfying the conditions of (2.10.3) and $\mathcal{C}$ a $\mathcal{U}$-small generating subcategory for monomorphisms of $\mathcal{E}$. Then, the left Kan extension $L : \widehat{\mathcal{C}} \to \mathcal{E}$ of the inclusion functor $K : \mathcal{C} \to \mathcal{E}$ along the Yoneda embedding $h^{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$ exists and it is left exact by (2.10.4), (2.10.5) and (2.9.14). Moreover, the functor $R : \mathcal{E} \to \widehat{\mathcal{C}}$ defined by $R = K^* h^{\mathcal{E}}$ is fully faithful by (2.10.5), (2.9.8) and is a right adjoint of L by (2.9.6). Let $\eta : id_{\widehat{\mathcal{C}}} \to RL$ and $\varepsilon : LR \to id_{\mathcal{E}}$ be the unit and the counit of this adjunction. We note that, since R is fully faithful, ε is a natural equivalence of functors. Consider the monad $(RL, R(\varepsilon_L), \eta)$ on $\widehat{\mathcal{C}}$ and a full subcategory of $\widehat{\mathcal{C}}$ consisting of objects X such that $\eta_X : X \to RL(X)$ is an isomorphism. Since L is left exact, so is RL, hence $\mathcal{D}$ is a reflexive strictly full subcategory of $\widehat{\mathcal{C}}$ with a left exact left adjoint of the inclusion functor $i : \mathcal{D} \to \widehat{\mathcal{C}}$ by (2.10.1). R induces an equivalence $\widetilde{R} : \mathcal{E} \to \mathcal{D}$ with quasi-inverse Li. In fact, since $R(\varepsilon)\eta_R = id_R$ and ε is an equivalence, $\eta_{R(X)} : R(X) \to RLR(X)$ is an isomorphism for any $X \in \operatorname{Ob}\mathcal{E}$, namely, $R(X) \in \operatorname{Ob}\mathcal{D}$. If $X \in \operatorname{Ob}\mathcal{D}$, $\eta_X : X \to RL(X) = \widetilde{R}Li(X)$ is an isomorphism, hence we have an equivalence $id_{\mathcal{D}} \to \widetilde{R}Li$. Therefore $\mathcal{E}$ is equivalent to a reflexive strictly full subcategory $\mathcal{D}$ of $\widehat{\mathcal{C}}$ which has a left exact reflection $\widetilde{R}L : \widehat{\mathcal{C}} \to \mathcal{D}$. It follows from (2.5.12) that $\mathcal{D}$ is the category of sheaves on $\mathcal{C}$ for the topology $J^{T_{\mathcal{D}}}$. Hence $\mathcal{E}$ is a Grothendieck topos and this completes the proof of (2.10.3). □

We remark that, for $X \in \operatorname{Ob}\mathcal{C}$, a sieve S on X belongs to $J^{T_{\mathcal{D}}}(X)$ if and only if S contains a family of morphisms $(f_i : X_i \to X)_{i \in I}$ such that $(K(f_i) : K(X_i) \to K(X))_{i \in I}$ is an epimorphic family in $\mathcal{E}$. In fact, $S \in J^{T_{\mathcal{D}}}(X)$ if and only if S contains a family of morphisms $(f_i : X_i \to X)_{i \in I}$ such that $(h_{f_i} : h_{X_i} \to h_X)_{i \in I}$ is a covering for $T_{\mathcal{D}}$. Since $\widetilde{R}L : \widehat{\mathcal{C}} \to \mathcal{D}$ is the reflection, $(h_{f_i} : h_{X_i} \to h_X)_{i \in I}$ is a covering for $T_{\mathcal{D}}$ if and only if $(\widetilde{R}L(h_{f_i}) : \widetilde{R}L(h_{X_i}) \to \widetilde{R}L(h_X))_{i \in I} = (\widetilde{R}K(f_i) : \widetilde{R}K(X_i) \to \widetilde{R}K(X))_{i \in I}$ is an epimorphic family in $\mathcal{D}$. Thus the assertion from the fact that $\widetilde{R} : \mathcal{E} \to \mathcal{D}$ is an equivalence.

Next we investigate the topology $J^{T_{\mathcal{D}}}$.

Proposition 2.10.6 *$J^{T_{\mathcal{D}}}$ is coarser than the canonical topology on $\mathcal{C}$. If $\mathcal{C}$ is closed under taking subobjects in $\mathcal{E}$, $J^{T_{\mathcal{D}}}$ is the canonical topology.*

Proof. By (2.9.6) and the above proof of (2.10.3), composition $\mathcal{C} \xrightarrow{h^{\mathcal{C}}} \widehat{\mathcal{C}} \xrightarrow{\widetilde{R}L} \mathcal{D}$ coincides with $\widetilde{R}K$ which is fully faithful. Since $\widetilde{R}L$ is naturally equivalent to the associated sheaf functor for the topology $J^{T_{\mathcal{D}}}$, it follows from (2.3.10) that $J^{T_{\mathcal{D}}}$ is coarser than the canonical topology.

Assume that $\mathcal{C}$ is closed under taking subobjects in $\mathcal{E}$. Let S be a universal strict epimorphic sieve on $X \in \operatorname{Ob}\mathcal{C}$ in $\mathcal{C}$. Since $\mathcal{C}$ is $\mathcal{U}$-small and $\mathcal{E}$ is a $\mathcal{U}$-category, S is $\mathcal{U}$-small and we can form a coproduct $\coprod_{f \in \operatorname{Ob} S} X_f$ $(X_f = \operatorname{dom}(f))$ in $\mathcal{E}$. Let $\varphi : \coprod_{f \in \operatorname{Ob} S} X_f \to X$ be the morphism induced by $f : X_f \to X$ for $f \in \operatorname{Ob} S$ and $\coprod_{f \in \operatorname{Ob} S} X_f \xrightarrow{p} Y \xrightarrow{i} X$ a mono-epi factorization of φ in $\mathcal{E}$. We denote by $p_f : X_f \to Y$ the composition of the canonical morphism into the f-th summand and p. Then, Y is an object of $\mathcal{C}$ by the assumption and $ip_f = f$ for each $f \in \operatorname{Ob} S$. It follows from (2.2.8) that i is an isomorphism and $(K(f) : K(X_f) \to K(X))_{f \in \operatorname{Ob} S}$ is an epimorphic family in $\mathcal{E}$. Hence, by the preceding remark, $(f : X_f \to X)_{f \in \operatorname{Ob} S}$ is a covering of X for $J^{T_{\mathcal{D}}}$, namely, $S \in J^{T_{\mathcal{D}}}(X)$. □

Corollary 2.10.7 *Let $\mathcal{E}$ be a $\mathcal{U}$-topos. If $\mathcal{C}$ is a $\mathcal{U}$-small generating subcategory of $\mathcal{E}$ for monomorphism, there exists a topology on $\mathcal{C}$ which is coarser than the canonical topology such that $\mathcal{E}$ is equivalent to the category of sheaves on $\mathcal{C}$. Moreover, if $\mathcal{C}$ is closed under taking subobjects in $\mathcal{E}$, $\mathcal{E}$ is equivalent to the category of sheaves on $\mathcal{C}$ for the canonical topology.*

Lemma 2.10.8 *Let $\mathcal{E}$ be a finitely complete $\mathcal{U}$-category and $\mathcal{C}$ a $\mathcal{U}$-small full subcategory of $\mathcal{E}$. There exists a $\mathcal{U}$-small full subcategory containing $\mathcal{C}$ which is closed under finite limits in $\mathcal{E}$. Moreover, if $\mathcal{C}$ is a $\mathcal{U}$-small generating subcategory for monomorphisms, there exists a $\mathcal{U}$-small full subcategory containing $\mathcal{C}$ which is closed under subobjects and finite limits in $\mathcal{E}$.*

Proof. Set $\mathcal{C}_1 = \mathcal{C}$ and suppose that $\mathcal{U}$-small full subcategories $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$ of $\mathcal{E}$ are constructed so that $\operatorname{Ob}\mathcal{C}_i \subset \operatorname{Ob}\mathcal{C}_{i+1}$ for $i = 1, 2, \dots, n-1$. We denote by $\mathcal{D}_n$ be the set of all finite diagrams in $\mathcal{C}_n$. For each element D of $\mathcal{D}_n$, we choose a limiting cone $(l_D \to d)_{d \in D_0}$ (D_0 is the set of vertices of D) in $\mathcal{E}$. Let $\mathcal{C}_{n+1}$ be the full subcategory of $\mathcal{E}$ generated by $\{l_D | D \in \mathcal{D}_n\}$ and $\operatorname{Ob}\mathcal{C}_n$. Since $\mathcal{C}_n$ and $\mathcal{D}_n$ are $\mathcal{U}$-small, so is $\mathcal{C}_{n+1}$. Let $\mathcal{C}_\infty$ be the full subcategory of $\mathcal{E}$ with $\operatorname{Ob}\mathcal{C}_\infty = \bigcup_{n \geqq 1} \operatorname{Ob}\mathcal{C}_n$. Then, $\mathcal{C}_\infty$ is $\mathcal{U}$-small, for $\mathcal{U}$ contains infinite set. Since each finite diagram in $\mathcal{C}_\infty$ is contained in some $\mathcal{C}_n$, $\mathcal{C}_\infty$ is closed under finite limits.

Suppose that $\mathcal{C}$ is a $\mathcal{U}$-small generating subcategory for monomorphisms. Let $\mathcal{C}'$ be the full subcategory of $\mathcal{E}$ with objects $\{X_1 \times X_2 \times \cdots \times X_n | n \geqq 0, X_i \in \operatorname{Ob}\mathcal{C}\}$. Then, $\mathcal{C}'$ is $\mathcal{U}$-small and closed under finite products.

It follows from (A.4.12) that for each object X of $\mathcal{E}$, $\mathrm{Sub}_{\mathcal{E}}(X)$ is $\mathcal{U}$-small. Choose one representative from each subobject of X for every $X \in \mathrm{Ob}\,\mathcal{C}'$ and let $\mathcal{C}''$ be the full subcategory of $\mathcal{E}$ consisting of the domains of such representatives. Then, $\mathcal{C}''$ is $\mathcal{U}$-small and closed under both subobjects and finite products in $\mathcal{E}$. Generally, if a full subcategory of $\mathcal{E}$ is closed under both subobjects and finite products in $\mathcal{E}$, it is closed under finite limits. □

By virtue of (2.10.7) and (2.10.8), we have the following result.

Theorem 2.10.9 *Let $\mathcal{E}$ be a $\mathcal{U}$-topos. There exists a $\mathcal{U}$-small full subcategory $\mathcal{C}$ of $\mathcal{E}$ which is closed under both subobjects and finite limits in $\mathcal{E}$ such that $\mathcal{E}$ is equivalent to the category of sheaves on $\mathcal{C}$ for the canonical topology.*

Lemma 2.10.10 *Let $\mathcal{E}$ be a $\mathcal{U}$-topos and $\mathcal{C}$ a $\mathcal{U}$-small generating subcategory of $\mathcal{E}$ for monomorphisms. Then, the canonical topology on $\mathcal{E}$ is a $\mathcal{U}$-topology with a topologically generating family* $\mathrm{Ob}\,\mathcal{C}$.

Proof. By (2.10.3), there exist a topology J on $\mathcal{C}$ coarser than the canonical topology and an equivalence $Li : \widetilde{\mathcal{C}} \to \mathcal{E}$ of categories. Then, Li preserves universal strict epimorphic families. Since J is coarser than the canonical topology, the composition $\mathcal{C} \xrightarrow{\epsilon_J} \widetilde{\mathcal{C}} \xrightarrow{Li} \mathcal{E}$ is naturally equivalent to the inclusion functor $K : \mathcal{C} \to \mathcal{E}$ by (2.9.6). Hence the assertion follows from (2.4.16). □

Theorem 2.10.11 *Let $\mathcal{E}$ be a $\mathcal{U}$-topos. We give $\mathcal{E}$ the canonical topology. Then, the unique functor $\tilde{h} : \mathcal{E} \to \widetilde{\mathcal{E}}$ such that $h = i\tilde{h}$ is an equivalence, where $i : \widetilde{\mathcal{E}} \to \widehat{\mathcal{E}}$ is the inclusion functor.*

Proof. Since $\tilde{h}$ is fully faithful, it suffices to show that, for each object H of $\widetilde{\mathcal{E}}$, there exists an object W of $\mathcal{E}$ such that $\tilde{h}(W)$ is isomorphic to H. Let $\mathcal{C}$ be a $\mathcal{U}$-small generating subcategory of $\mathcal{E}$ for monomorphisms and $K : \mathcal{C} \to \mathcal{E}$ the inclusion functor. By the above result and (2.4.16), there exists an epimorphic family $(f_j : \tilde{h}(X_j) \to H)_{j\in I}$ ($X_j \in \mathrm{Ob}\,\mathcal{C}$) indexed by a $\mathcal{U}$-small set I for each object H of $\widetilde{\mathcal{E}}$. Then f_j's induce an epimorphism $\coprod\limits_{j\in I} \tilde{h}(X_j) \to H$. Set $X = \coprod\limits_{j\in I} X_j$. We have an epimorphism $\rho : \tilde{h}(X) \to H$ by (2.2.9). Consider the kernel pair $R \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \tilde{h}(X)$ of ρ. Similarly, we take an epimorphism $\pi : \tilde{h}(Y) \to R$ for some $Y \in \mathrm{Ob}\,\mathcal{E}$. There exist morphisms $\varphi, \psi : Y \to X$ such that $\alpha\pi = \tilde{h}(\varphi)$, $\beta\pi = \tilde{h}(\psi)$. Let $Y \overset{p}{\rightarrowtail} Z \overset{\sigma}{\twoheadrightarrow} X \times X$ be a mono-epi factorization of $(\varphi, \psi) : Y \to X \times X$. Since $\tilde{h}$ is exact by (2.2.12), $\tilde{h}(Y) \overset{\tilde{h}(p)}{\rightarrowtail} \tilde{h}(Z) \overset{\tilde{h}(\sigma)}{\twoheadrightarrow} \tilde{h}(X \times X)$ is a mono-epi factorization of $\tilde{h}((\varphi, \psi)) : \tilde{h}(Y) \to \tilde{h}(X \times X)$. Let $\mathrm{pr}_1, \mathrm{pr}_2 : X \times X \to X$ be the projections and put $\sigma_1 = \mathrm{pr}_1\sigma$, $\sigma_2 = \mathrm{pr}_2\sigma$. Then, $\tilde{h}(\sigma_1)\tilde{h}(p) = \tilde{h}(\varphi) = \alpha\pi$, $\tilde{h}(\sigma_2)\tilde{h}(p) = \tilde{h}(\psi) = \beta\pi$, hence we have $(\tilde{h}(\sigma_1), \tilde{h}(\sigma_2))\tilde{h}(p) = (\alpha, \beta)\pi$. Since $(\tilde{h}(\mathrm{pr}_1), \tilde{h}(\mathrm{pr}_2)) : \tilde{h}(X \times X) \to \tilde{h}(X) \times \tilde{h}(X)$ is an isomorphism, $(\tilde{h}(\sigma_1), \tilde{h}(\sigma_2))$ is a monomorphism. It follows that $(\tilde{h}(\sigma_1), \tilde{h}(\sigma_2))\tilde{h}(p)$ and $(\alpha, \beta)\pi$ are mono-epi factorizations of the same morphism. Hence there exists a unique isomorphism $\theta : R \to \tilde{h}(Z)$ such that $\alpha = \tilde{h}(\sigma_1)\theta$, $\beta = \tilde{h}(\sigma_2)\theta$ and $\tilde{h}(Z) \overset{\tilde{h}(\sigma_1)}{\underset{\tilde{h}(\sigma_2)}{\rightrightarrows}} \tilde{h}(X)$ is a kernel pair of ρ. Thus $Z \overset{\sigma_1}{\underset{\sigma_2}{\rightrightarrows}} X$ is an equivalence relation. Let $g : X \to W$ be a coequalizer of this equivalence relation. Since $\tilde{h}$ is exact, $\tilde{h}(Z) \overset{\tilde{h}(\sigma_1)}{\underset{\tilde{h}(\sigma_2)}{\rightrightarrows}} \tilde{h}(X) \xrightarrow{\tilde{h}(g)} \tilde{h}(W)$ is exact, there is a unique isomorphism $\xi : \tilde{h}(W) \to H$ such that $\rho = \xi\tilde{h}(g)$. □

2.11 Continuous functor and cocontinuous functor

Definition 2.11.1 *Let $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ be sites such that $\mathcal{C}$ and $\mathcal{C}'$ are $\mathcal{U}$-categories and $u : \mathcal{C} \to \mathcal{C}'$ a functor.*

1) We say that u is $\mathcal{U}$-continuous if, for any sheaf F of $\mathcal{U}$-sets on $\mathcal{C}'$, $u^*(F) = Fu$ is a sheaf on $\mathcal{C}$.

2) u is said to be continuous if there exists a universe $\mathcal{V}$ containing $\mathcal{U}$ such that $\mathcal{C}$ is $\mathcal{V}$-small, $(\mathcal{C}', J')$ is a $\mathcal{V}$-site and that u is $\mathcal{V}$-continuous.

In other words, u is $\mathcal{U}$-continuous if and only if $u^* : \widehat{\mathcal{C}'}_{\mathcal{U}} \to \widehat{\mathcal{C}}_{\mathcal{U}}$ induces a functor $\tilde{u}^* : \widetilde{\mathcal{C}'}_{\mathcal{U}} \to \widetilde{\mathcal{C}}_{\mathcal{U}}$. Note that, if u is continuous, it is $\mathcal{U}$-continuous and that if $\mathcal{C}$ is $\mathcal{U}$-small, $(\mathcal{C}', J')$ is a $\mathcal{U}$-site and u is $\mathcal{U}$-continuous, it is continuous.

Proposition 2.11.2 *Let $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ be $\mathcal{U}$-sites and $u : \mathcal{C} \to \mathcal{C}'$ a functor such that the left adjoint $u_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ of u^* (A.6.12) exists (for example, $\mathcal{C}$ is $\mathcal{U}$-small). Then, the following conditions are equivalent.*

i) u is $\mathcal{U}$-continuous.

ii) For any $X \in \mathrm{Ob}\,\mathcal{C}$ and $R \in J(X)$, the morphism $u_!(R) \to u_!(h_X) = h_{u(X)}$ induced by the inclusion morphism $\iota : R \to h_X$ is a bicovering in $\widehat{\mathcal{C}'}$.

iii) For any bicovering family $(f_i : H_i \to K)_{i\in I}$ in $\widehat{\mathcal{C}}$, $(u_!(f_i) : u_!(H_i) \to u_!(K))_{i\in I}$ is a bicovering family in $\widehat{\mathcal{C}'}$.

iv) There exists a functor $\tilde{u}_! : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ preserving colimits such that the following diagram commutes up to natural equivalence.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "u"] \arrow[d, "\epsilon_J"] & \mathcal{C}' \arrow[d, "\epsilon_{J'}"] \\ \widetilde{\mathcal{C}} \arrow[r, "\tilde{u}_!"] & \widetilde{\mathcal{C}'} \end{tikzcd}$$

Proof. $i) \Rightarrow iii)$: Let $(f_i : H_i \to K)_{i\in I}$ be a bicovering family in $\widehat{\mathcal{C}'}$. For any sheaf F on $\mathcal{C}'$, $u^*(F)$ is a sheaf on $\mathcal{C}$ by the assumption. Let $f : \coprod_{i\in I} H_i \to K$ be the morphism induced by $(f_i : H_i \to K)_{i\in I}$. Then f is a bicovering and by (2.5.4), $f^* : \widehat{\mathcal{C}}(K, u^*(F)) \to \widehat{\mathcal{C}}\Big(\coprod_{i\in I} H_i, u^*(F)\Big)$ is bijective. Since $u_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ is a left adjoint of u^*, $u_!(f)^* : \widehat{\mathcal{C}'}(u_!(K), F) \to \widehat{\mathcal{C}'}\Big(u_!\Big(\coprod_{i\in I} H_i\Big), F\Big)$ is bijective. Again by (2.5.4), $u_!(f) : u_!\Big(\coprod_{i\in I} H_i\Big) \to u_!(K)$ is a bicovering. Since $u_!$ preserves colimits by (A.3.13), the morphism $\coprod_{i\in I} u_!(H_i) \to u_!(K)$ induced by $(u_!(f_i) : u_!(H_i) \to u_!(K))_{i\in I}$ is a bicovering.

$iii) \Rightarrow ii)$ is obvious.

$ii) \Rightarrow i)$: Let X be an object of $\mathcal{C}$ and $R \in J(X)$. For a sheaf F on $\mathcal{C}'$, it follows from the assumption and (2.5.4) that $u_!(\iota)^* : \widehat{\mathcal{C}'}(u_!(h_X), F) \to \widehat{\mathcal{C}'}(u_!(R), F)$ is bijective. Thus $\iota^* : \widehat{\mathcal{C}}(h_X, u^*(F)) \to \widehat{\mathcal{C}}(R, u^*(F))$ is bijective and $u^*(F)$ is a sheaf on $\mathcal{C}$.

$i) \Rightarrow iv)$: Let us denote by $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}$, $a' : \widehat{\mathcal{C}'} \to \widetilde{\mathcal{C}'}$ by the associated sheaf functors and by $i : \widetilde{\mathcal{C}} \to \widehat{\mathcal{C}}$, $i' : \widetilde{\mathcal{C}'} \to \widehat{\mathcal{C}'}$ the inclusion functors. Define $\tilde{u}_!$ to be the composition $\widetilde{\mathcal{C}} \xrightarrow{i} \widehat{\mathcal{C}} \xrightarrow{u_!} \widehat{\mathcal{C}'} \xrightarrow{a'} \widetilde{\mathcal{C}'}$. Then, for each sheaf F on $\mathcal{C}$ and G on $\mathcal{C}'$, we have a chain of natural bijections $\widetilde{\mathcal{C}}(F, \tilde{u}^*(G)) \cong \widehat{\mathcal{C}}(i(F), i\tilde{u}^*(G)) = \widehat{\mathcal{C}}(i(F), u^*i'(G)) \cong \widehat{\mathcal{C}'}(u_!i(F), i'(G)) \cong \widetilde{\mathcal{C}'}(\tilde{u}_!(F), G)$. Thus $\tilde{u}_!$ is a left adjoint of $\tilde{u}^*$ and in particular, $\tilde{u}_!$ preserves colimits. For any presheaf H on $\mathcal{C}$ and sheaf G on $\mathcal{C}'$, we have a chain of natural bijections $\widetilde{\mathcal{C}'}(\tilde{u}_!a(H), G) \cong \widetilde{\mathcal{C}}(a(H), \tilde{u}^*(G)) \cong \widehat{\mathcal{C}}(H, i\tilde{u}^*(G)) = \widehat{\mathcal{C}}(H, u^*i'(G)) \cong \widehat{\mathcal{C}'}(u_!(H), i'(G)) \cong \widetilde{\mathcal{C}'}(a'u_!(H), G)$. Hence $\tilde{u}_!a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ is naturally equivalent to $a'u_!$. It follows that $\tilde{u}_!\epsilon_J = \tilde{u}_!ah$ is naturally equivalent to $\epsilon_{J'}u = a'h'u = a'u_!h$, where $h : \mathcal{C} \to \widehat{\mathcal{C}}$ and $h' : \mathcal{C}' \to \widehat{\mathcal{C}'}$ are the Yoneda embeddings.

$iv) \Rightarrow ii)$: For any $H \in \mathrm{Ob}\,\widehat{\mathcal{C}}$, there is a colimiting cone $(hP\langle X, f\rangle \xrightarrow{f} H)_{\langle X,f\rangle\in\mathrm{Ob}(h\downarrow H)}$ of a functor $hP : (h\downarrow H) \to \widehat{\mathcal{C}}$ by (A.4.2). Since functors $u_!$, $\tilde{u}_!$, a, a' preserve colimits, $(\tilde{u}_!ahP\langle X, f\rangle \xrightarrow{\tilde{u}_!a(f)} \tilde{u}_!a(H))_{\langle X,f\rangle\in\mathrm{Ob}(h\downarrow H)}$ and $(a'u_!hP\langle X, f\rangle \xrightarrow{a'u_!(f)} a'u_!(H))_{\langle X,f\rangle\in\mathrm{Ob}(h\downarrow H)}$ are colimiting cones of functors $\tilde{u}_!ahP, a'u_!hP = a'h'uP : (h\downarrow H) \to \widetilde{\mathcal{C}'}$, respectively. There is a natural equivalence $\theta : \tilde{u}_!ah \to a'h'u$ by the assumption and this induces a unique isomorphism $\zeta_H : \tilde{u}_!a(H) \to a'u_!(H)$ satisfying $\zeta_H\tilde{u}_!a(f) = a'u_!(f)\theta_{P\langle X,f\rangle}$ for each $\langle X, f\rangle \in \mathrm{Ob}\,(h\downarrow H)$. Let $\varphi : H \to K$ be a morphism in $\widehat{\mathcal{C}}$. Then, for any $\langle X, f\rangle \in \mathrm{Ob}\,(h\downarrow H)$, we have $a'u_!(\varphi)\zeta_H\tilde{u}_!a(f) = a'u_!(\varphi)a'u_!(f)\theta_{P\langle X,f\rangle} = a'u_!(\varphi f)\theta_X = a'u_!(\varphi f)\theta_{P\langle X,\varphi f\rangle} = \zeta_K\tilde{u}_!a(\varphi f) = \zeta_K\tilde{u}_!a(\varphi)\tilde{u}_!a(f)$ and it follows that $a'u_!(\varphi)\zeta_H = \zeta_K\tilde{u}_!a(\varphi)$. This shows the naturality of $\zeta : \tilde{u}_!a \to a'u_!$. Let $v : H \to K$ be a bicovering in $\widehat{\mathcal{C}}$. Then, $a(v) : a(H) \to a(K)$ is an isomorphism by (2.5.4) and so is $\tilde{u}_!a(v) : \tilde{u}_!a(H) \to \tilde{u}_!a(K)$. By the above result, $a'u_!(v) : a'u_!(H) \to a'u_!(K)$ is an isomorphism and this implies that $u_!(v) : u_!(H) \to u_!(K)$ is a bicovering in $\widehat{\mathcal{C}'}$ by (2.5.4). In particular, since the inclusion morphism $R \to h_X$ is a bicovering if $R \in J(X)$ by (2.5.1) and (2.5.4), $ii)$ follows. □

Proposition 2.11.3 *Let $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ be $\mathcal{U}$-sites and $u : \mathcal{C} \to \mathcal{C}'$ a $\mathcal{U}$-continuous functor such that the left adjoint $u_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ of u^* exists.*

1) The functor $\tilde{u}_! : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ satisfying the condition iv) in (2.11.2) is a left adjoint of $\tilde{u}^*$ and there are natural equivalences $\tilde{u}_! \cong a'u_!i$, $\tilde{u}_!a \cong a'u_!$.

2) We denote by $\rho : \tilde{u}_! \to a'u_!i$ the above equivalence. Let $\eta : id_{\widehat{\mathcal{C}}} \to u^*u_!$, $\varepsilon : u_!u^* \to id_{\widehat{\mathcal{C}'}}$ be the unit, counit of the adjunction of $u_!$ and u^*, and $\eta_{J'} : id_{\widehat{\mathcal{C}'}} \to i'a'$, $\varepsilon_{J'} : a'i' \to id_{\widetilde{\mathcal{C}'}}$ the unit, counit of the adjunction of a'

and i'. Then, the the unit $\tilde{\eta} : id_{\widetilde{\mathcal{C}}} \to \tilde{u}^*\tilde{u}_!$ is the unique natural transformation such that $i(\tilde{\eta})$ is a composition $i \xrightarrow{\eta_i} u^*u_!i \xrightarrow{u^*(\eta_{J'u_!i})} u^*i'a'u_!i \xrightarrow{u^*i'(\rho^{-1})} u^*i'\tilde{u}_! = i\tilde{u}^*\tilde{u}_!$. The counit $\tilde{\varepsilon} : \tilde{u}_!\tilde{u}^* \to id_{\widetilde{\mathcal{C}'}}$ is given by a composition $\tilde{u}_!\tilde{u}^* \xrightarrow{\rho_{\tilde{u}^*}} a'u_!i\tilde{u}^* = a'u_!u^*i' \xrightarrow{a'(\varepsilon_{i'})} a'i' \xrightarrow{\varepsilon_{J'}} id_{\widetilde{\mathcal{C}'}}$.

3) If $u_!$ is left exact, so is $\tilde{u}_!$.

Proof. 1) The functor $\tilde{u}_! : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ satisfying the condition $iv)$ of (2.11.2) is uniquely determined up to natural equivalence. In fact, by the proof of (2.11.2), there is a natural equivalence $\zeta : \tilde{u}_!a \to a'u_!$. Since the counit $\varepsilon : ai \to id_{\widetilde{\mathcal{C}}}$ is an equivalence, so is $\tilde{u}_! \xrightarrow{\tilde{u}_!(\varepsilon)^{-1}} \tilde{u}_!ai \xrightarrow{\zeta_i} a'u_!i$. We have seen in the proof of (2.11.2) that $a'u_!i$ is a left adjoint of $\tilde{u}^*$.

2) We verify the equalities $\tilde{u}^*(\tilde{\varepsilon})\tilde{\eta}_{\tilde{u}^*} = id_{\tilde{u}^*}$ and $\tilde{\varepsilon}_{\tilde{u}_!}\tilde{u}_!(\tilde{\eta}) = id_{\tilde{u}_!}$; $i(\tilde{u}^*(\tilde{\varepsilon})\tilde{\eta}_{\tilde{u}^*}) = u^*i'(\tilde{\varepsilon})i(\tilde{\eta}_{\tilde{u}^*}) =$
$u^*i'(\varepsilon_{J'}a'(\varepsilon_{i'})\rho_{\tilde{u}^*})u^*i'(\rho^{-1}_{\tilde{u}^*})u^*(\eta_{J'u_!i\tilde{u}^*})\eta_{i\tilde{u}^*} = u^*i'(\varepsilon_{J'})u^*i'a'(\varepsilon_{i'})u^*(\eta_{J'u_!u^*i'})\eta_{u^*i'} = u^*i'(\varepsilon_{J'})u^*(\eta_{J'i'})u^*(\varepsilon_{i'})\eta_{u^*i'}$
$= u^*(i'(\varepsilon_{J'})\eta_{J'i'}) = u^*(\varepsilon_{i'})\eta_{u^*i'} = id_{u^*i'} = id_{i\tilde{u}^*}$ Since i is fully faithful, we have the first equality.
$\tilde{\varepsilon}_{\tilde{u}_!}\tilde{u}_!(\tilde{\eta}) = \varepsilon_{J'\tilde{u}_!}a'(\varepsilon_{i'\tilde{u}_!})\rho_{\tilde{u}^*\tilde{u}_!}\tilde{u}_!(\tilde{\eta}) = \varepsilon_{J'\tilde{u}_!}a'(\varepsilon_{i'\tilde{u}_!})a'u_!i(\tilde{\eta})\rho = \varepsilon_{J'\tilde{u}_!}a'(\varepsilon_{i'\tilde{u}_!})a'u_!u^*i'(\rho^{-1})a'u_!u^*(\eta_{J'u_!i})a'u_!(\eta_i)\rho$
$= \varepsilon_{J'\tilde{u}_!}a'i'(\rho^{-1})a'(\varepsilon_{i'a'u_!i})a'u_!u^*(\eta_{J'u_!i})a'u_!(\eta_i)\rho = \rho^{-1}\varepsilon_{J'a'u_!i}a'(\varepsilon_{i'a'u_!i})a'u_!u^*(\eta_{J'u_!i})a'u_!(\eta_i)\rho$
$= \rho^{-1}\varepsilon_{J'a'u_!i}a'(\eta_{J'u_!i})a'(\varepsilon_{u_!i})a'u_!(\eta_i)\rho = \rho^{-1}a'(\varepsilon_{u_!i}u_!(\eta_i))\rho = id_{\tilde{u}_!}$ Thus we have the second one.

3) Since both a' and i are left exact, so is $\tilde{u}_!$ by 1) if $u_!$ is so. □

Remark 2.11.4 *Since functors a, a', $\tilde{u}_!$ are only determined uniquely up to natural isomorphisms, we can choose them so that $a'h'u = \tilde{u}_!ah$ holds. In fact, it is possible to choose a and a' so that ah and $a'h'$ are injective on the sets of objects. Then we can choose $\tilde{u}_!$ so that $a'h'u = \tilde{u}_!ah$ holds.*

Proposition 2.11.5 *Let $\mathcal{U}$, $\mathcal{V}$ be universes such that $\mathcal{U} \subset \mathcal{V}$ and $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ $\mathcal{U}$-sites such that $\mathcal{C}$ is $\mathcal{U}$-small and $u : \mathcal{C} \to \mathcal{C}'$ a functor.*

1) u is $\mathcal{U}$-continuous if and only if it is $\mathcal{V}$-continuous. Hence if u is $\mathcal{U}$-continuous, it is continuous and $\mathcal{V}$-continuous for any universe $\mathcal{V}$ such that $\mathcal{U} \subset \mathcal{V}$.

2) If u is $\mathcal{U}$-continuous, we denote by $\tilde{u}_{\mathcal{U}!}$ (resp. $\tilde{u}_{\mathcal{V}!}$) the functor between the category of $\mathcal{U}$-sheaves (resp. $\mathcal{V}$-sheaves) given in (2.11.2). Then, the following diagram commutes up to natural equivalence, where the vertical arrows are the inclusion functors.

$$\begin{array}{ccc} \widetilde{\mathcal{C}}_{\mathcal{U}} & \xrightarrow{\tilde{u}_{\mathcal{U}!}} & \widetilde{\mathcal{C}'}_{\mathcal{U}} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{C}}_{\mathcal{V}} & \xrightarrow{\tilde{u}_{\mathcal{V}!}} & \widetilde{\mathcal{C}'}_{\mathcal{V}} \end{array}$$

Proof. If u^* maps $\mathcal{V}$-sheaves on $\mathcal{C}'$ to $\mathcal{V}$-sheaves on $\mathcal{C}$, it is clear that u^* maps $\mathcal{U}$-sheaves on $\mathcal{C}'$ to $\mathcal{U}$-sheaves on $\mathcal{C}$. Suppose that u^* maps $\mathcal{U}$-sheaves on $\mathcal{C}'$ to $\mathcal{U}$-sheaves on $\mathcal{C}$. Let us denote by $u_{\mathcal{U}!} : \widehat{\mathcal{C}}_{\mathcal{U}} \to \widehat{\mathcal{C}'}_{\mathcal{U}}$ (resp. $u_{\mathcal{V}!} : \widehat{\mathcal{C}}_{\mathcal{V}} \to \widehat{\mathcal{C}'}_{\mathcal{V}}$) the left adjoint of $u^* : \widehat{\mathcal{C}'}_{\mathcal{U}} \to \widehat{\mathcal{C}}_{\mathcal{U}}$ (resp. $u^* : \widehat{\mathcal{C}'}_{\mathcal{V}} \to \widehat{\mathcal{C}}_{\mathcal{V}}$) (A.6.12). Then, by the construction of $u_{\mathcal{U}!}$ and $u_{\mathcal{V}!}$, the following diagram commutes, where the vertical arrows are the inclusion functors.

$$\begin{array}{ccc} \widehat{\mathcal{C}}_{\mathcal{U}} & \xrightarrow{u_{\mathcal{U}!}} & \widehat{\mathcal{C}'}_{\mathcal{U}} \\ \downarrow & & \downarrow \\ \widehat{\mathcal{C}}_{\mathcal{V}} & \xrightarrow{u_{\mathcal{V}!}} & \widehat{\mathcal{C}'}_{\mathcal{V}} \end{array}$$

Since the inclusion functor $\widehat{\mathcal{C}}_{\mathcal{U}} \to \widehat{\mathcal{C}}_{\mathcal{V}}$ preserves limits and the images of morphisms, it follows from (2.5.2) that, if $f : H \to K$ is a bicovering in $\widehat{\mathcal{C}}_{\mathcal{U}}$, so is in $\widehat{\mathcal{C}}_{\mathcal{V}}$. For $X \in \operatorname{Ob}\mathcal{C}$ and $R \in J(X)$, the morphism $u_{\mathcal{U}!}(R) \to u_{\mathcal{U}!}(h_X)$ induced by the inclusion morphism is a bicovering in $\widehat{\mathcal{C}}_{\mathcal{U}}$ by (2.11.2). By the commutativity of the above diagram, $u_{\mathcal{V}!}(R) \to u_{\mathcal{V}!}(h_X)$ is a bicovering in $\widehat{\mathcal{C}}_{\mathcal{V}}$. Thus, by (2.11.2), $u^* : \widehat{\mathcal{C}'}_{\mathcal{V}} \to \widehat{\mathcal{C}}_{\mathcal{V}}$ maps $\mathcal{V}$-presheaves on $\mathcal{C}'$ to $\mathcal{V}$-presheaves on $\mathcal{C}$. The second assertion follows from (2.3.9) and (2.11.3). □

Proposition 2.11.6 *Let $(\mathcal{C}, J)$ and $(\mathcal{C}', J')$ be $\mathcal{U}$-sites and $u : \mathcal{C} \to \mathcal{C}'$ a functor. We choose a $\mathcal{U}$-small topologically generating set G of $(\mathcal{C}, J)$.*

1) If u is $\mathcal{U}$-continuous, then for every covering $(f_i : X_i \to X)_{i \in I}$ of $X \in \operatorname{Ob}\mathcal{C}$ for J, $(u(f_i) : u(X_i) \to u(X))_{i \in I}$ is a covering of $u(X)$ for J'.

2) If u has the following properties, u is $\mathcal{U}$-continuous.

i) For any $X \in \mathrm{Ob}\,\mathcal{C}$, covering $(f_i : X_i \to X)_{i\in I}$ of X such that I is $\mathcal{U}$-small and $X_i \in G$, and $i, j \in I$, if $\alpha : U \to u(X_i)$ and $\beta : U \to u(X_j)$ are morphisms in $\mathcal{C}'$ satisfying $u(f_i)\alpha = u(f_j)\beta$, there exist a covering $(p_\lambda : U_\lambda \to U)_{\lambda\in\Lambda}$ of U for J', a family of diagrams $(X_i \xleftarrow{s_\lambda} V_\lambda \xrightarrow{t_\lambda} X_j)_{\lambda\in\Lambda}$ in $\mathcal{C}$ and a family of morphisms $(q_\lambda : U_\lambda \to u(V_\lambda))_{\lambda\in\Lambda}$ in $\mathcal{C}'$ such that $\alpha p_\lambda = u(s_\lambda)q_\lambda$, $\beta p_\lambda = u(t_\lambda)q_\lambda$ and $f_i s_\lambda = f_j t_\lambda$ hold for every $\lambda \in \Lambda$.

ii) For every covering $(f_i : X_i \to X)_{i\in I}$ of $X \in \mathrm{Ob}\,\mathcal{C}$ for J such that I is $\mathcal{U}$-small and $X_i \in G$, $(u(f_i) : u(X_i) \to u(X))_{i\in I}$ is a covering of $u(X)$ for J'.

3) If $\mathcal{C}$ has pull-backs and u preserves them, u has the above property i).

Proof. 1) By (2.4.7), it suffices to show that for any sheaf F of $\mathcal{U}$-sets on $\mathcal{C}'$, the map $\widetilde{\mathcal{C}'}(\epsilon_{J'}u(X), F) \to \prod_{i\in I} \widetilde{\mathcal{C}'}(\epsilon_{J'}u(X_i), F)$ induced by $\epsilon_{J'}u(f_i)$'s is injective. But this follows from the following commutative diagram, where the vertical maps are bijections.

$$\begin{tikzcd}
\widetilde{\mathcal{C}'}(\epsilon_{J'}u(X), F) \arrow[r, "\cong"] \arrow[d] & \widehat{\mathcal{C}'}(h'_{u(X)}, i'(F)) \arrow[r, "\cong"] \arrow[d] & \widehat{\mathcal{C}'}(h_X, u^*i'(F)) \arrow[d] \\
\prod_{i\in I} \widetilde{\mathcal{C}'}(\epsilon_{J'}u(X_i), F) \arrow[r, "\cong"] & \prod_{i\in I} \widetilde{\mathcal{C}'}(h'_{u(X_i)}, i'(F)) \arrow[r, "\cong"] & \prod_{i\in I} \widetilde{\mathcal{C}'}(h_{X_i}, u^*i'(F))
\end{tikzcd}$$

2) Let F a sheaf of $\mathcal{U}$-sets on $\mathcal{C}'$ and $(f_i : X_i \to X)_{i\in I}$ a covering of $X \in \mathrm{Ob}\,\mathcal{C}$ for J such that I is $\mathcal{U}$-small and $X_i \in G$. Then, by *ii*), $(u(f_i) : u(X_i) \to u(X))_{i\in I}$ is a covering of $u(X)$ for J'. Since F is a sheaf, the map $Fu(X) \to \prod_{i\in I} Fu(X_i)$ induced by $(u(f_i) : u(X_i) \to u(X))_{i\in I}$ is injective. If $(x_i)_{i\in I} \in \prod_{i\in I} Fu(X_i)$ is in the image of this map, namely, $x_i = Fu(f_i)(X)$ for some $x \in Fu(X)$, then for any $i, j \in I$ and morphisms $s : V \to X_i$, $t : V \to X_j$ satisfying $f_i s = f_j t$, we have $Fu(s)(x_i) = Fu(t)(x_j)$. Conversely, suppose that $(x_i)_{i\in I}$ is an element of $\prod_{i\in I} Fu(X_i)$ such that, for any $i, j \in I$ and morphisms $s : V \to X_i$, $t : V \to X_j$ satisfying $f_i s = f_j t$, $Fu(s)(x_i) = Fu(t)(x_j)$ holds. Let $\alpha : U \to u(X_i)$ and $\beta : U \to u(X_j)$ be morphisms in $\mathcal{C}'$ satisfying $u(f_i)\alpha = u(f_j)\beta$. There exist a covering $(p_\lambda : U_\lambda \to U)_{\lambda\in\Lambda}$ of U for J', a family of diagrams $(X_i \xleftarrow{s_\lambda} V_\lambda \xrightarrow{t_\lambda} X_j)_{i\in I}$ in $\mathcal{C}$ and a family of morphisms $(q_\lambda : U_\lambda \to u(V_\lambda))_{\lambda\in\Lambda}$ in $\mathcal{C}'$ such that $\alpha p_\lambda = u(s_\lambda)q_\lambda$, $\beta p_\lambda = u(t_\lambda)q_\lambda$ and $f_i s_\lambda = f_j t_\lambda$ hold for every $\lambda \in \Lambda$. Then $Fu(s_\lambda)(x_i) = Fu(t_\lambda)(x_j)$ by the assumption on $(x_i)_{i\in I}$. Hence we have $F(p_\lambda)F(\alpha)(x_i) = F(q_\lambda)Fu(s_\lambda)(x_i) = F(q_\lambda)Fu(t_\lambda)(x_j) = F(p_\lambda)F(\beta)(x_j)$. Since $(p_\lambda : U_\lambda \to U)_{\lambda\in\Lambda}$ is a covering for J' and F is a sheaf on $\mathcal{C}'$, it follows that $F(\alpha)(x_i) = F(\beta)(x_j)$. Since $(u(f_i) : u(X_i) \to u(X))_{i\in I}$ is a covering of $u(X)$ for J' and F is a sheaf on $\mathcal{C}'$, it follows from (2.2.3) that there exists $x \in Fu(X)$ such that $Fu(f_i)(x) = x_i$ for any $i \in I$. Therefore Fu is a sheaf on $\mathcal{C}$ by (2.3.6).

3) For $X \in \mathrm{Ob}\,\mathcal{C}$, covering $(f_i : X_i \to X)_{i\in I}$ of X and $i, j \in I$, form a pull-back of f_j along f_i.

$$\begin{tikzcd}
X_i \times_X X_j \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X_j \arrow[d, "f_j"] \\
X_i \arrow[r, "f_i"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
u(X_i \times_X X_j) \arrow[r, "u(\pi_1)"] \arrow[d, "u(\pi_2)"] & u(X_j) \arrow[d, "u(f_j)"] \\
u(X_i) \arrow[r, "u(f_i)"] & u(X)
\end{tikzcd}$$

The above right diagram is also pull-back by the assumption. If $\alpha : U \to u(X_i)$ and $\beta : U \to u(X_j)$ are morphisms in $\mathcal{C}'$ satisfying $u(f_i)\alpha = u(f_j)\beta$, there is a unique morphism $q : U \to u(X_i \times_X X_j)$ such that $u(\pi_1)q = \alpha$ and $u(\pi_2)q = \beta$. Then, a covering $(id_U : U \to U)$, a diagram $X_i \xleftarrow{\pi_1} X_i \times_X X_j \xrightarrow{\pi_2} X_j$ and a morphism $q : U \to u(X_i \times_X X_j)$ satisfy the requirements. □

Definition 2.11.7 *Let $(\mathcal{C}, J)$ and $(\mathcal{C}', J')$ be sites and $u : \mathcal{C} \to \mathcal{C}'$ a functor. For $X \in \mathrm{Ob}\,\mathcal{C}$ and a sieve R on $u(X)$, we set $R^u = \{f \in h_X \mid u(f) \in R(u(\mathrm{dom}(f)))\}$. We say that u is cocontinuous if $R^u \in J(X)$ for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $R \in J'(u(X))$.*

By $(T4)$ of (2.1.4), u is cocontinuous if and only if, for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $R \in J'(u(X))$, there exists $S \in J(X)$ such that $\{u(f) \mid f \in S(Y)\} \subset R(u(Y))$ for any $Y \in \mathrm{Ob}\,\mathcal{C}$.

We remark that the above R^u is described as follows. Let $\iota : R \to h'_{u(X)} = u_!(h_X)$ be the inclusion morphism. Then, $u^*(\iota) : u^*(R) \to u^*(h'_{u(X)}) = u^*u_!(h_X)$ is regarded as an inclusion morphism and the inclusion morphism $R^u \to h_X$ is the pull-back of $u^*(\iota)$ along the unit $\eta_{h_X} : h_X \to u^*u_!(h_X)$ of the adjunction of $u_!$ and u^*.

The following fact is straightforward from the definition.

Proposition 2.11.8 *Let $(\mathcal{C}, J)$, $(\mathcal{C}', J')$, $(\mathcal{C}'', J'')$ be sites and $u : \mathcal{C} \to \mathcal{C}'$, $u' : \mathcal{C}' \to \mathcal{C}''$ functors. For $X \in \mathrm{Ob}\,\mathcal{C}$ and a sieve R on $u'u(X)$, we have $(R^{u'})^u = R^{u'u}$. Hence if u and u' are cocontinuous, so is $u'u : \mathcal{C} \to \mathcal{C}''$.*

Proposition 2.11.9 *Let $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ be $\mathcal{U}$-sites and $u : \mathcal{C} \to \mathcal{C}'$ a functor. For any covering (resp. bicovering) $p : H \to K$ in $\widehat{\mathcal{C}'}$, $u^*(p) : u^*(H) \to u^*(K)$ is a covering (resp. bicovering) in $\widehat{\mathcal{C}}$ if u is cocontinuous.*

Proof. A morphism $f : h_X \to u^*(K)$ in $\widehat{\mathcal{C}}$ factorizes $h_X \xrightarrow{\eta_{h_X}} u^*u_!(h_X) = u^*(h'_{u(X)}) \xrightarrow{u^*(f')} u^*(K)$, where $f' : h'_{u(X)} = u_!(h_X) \to K$ is the adjoint of f. Consider a pull-back of p along f' and $u^*(p)$ along f. Since u^* preserves pull-backs, the right square and the outer rectangle of the following diagram on the right are pull-backs.

$$\begin{tikzcd} H \times_K h'_{u(X)} \arrow[r, "p_1"] \arrow[d, "p_2"] & H \arrow[d, "p"] \\ h'_{u(X)} \arrow[r, "f'"] & K \end{tikzcd} \qquad \begin{tikzcd} u^*(H) \times_{u^*(K)} h_X \arrow[r] \arrow[d, "q"] & u^*(H \times_K h'_{u(X)}) \arrow[r, "u^*(p_2)"] \arrow[d, "u^*(p_2)"] & u^*(H) \arrow[d, "u^*(p)"] \\ h_X \arrow[r, "\eta_{h_X}"] & u^*(h'_{u(X)}) \arrow[r, "u^*(f')"] & u^*(K) \end{tikzcd}$$

Let S be the image of p_2. Since p is a covering, $S \in J'(u(X))$, hence $S^u \in J(X)$ by the assumption. For any $Y \in \mathrm{Ob}\,\mathcal{C}$ and $g \in S^u(Y)$, $u(g) \in S$ and there exists $x \in H(u(Y)) = u^*(H)(Y)$ such that $p_{u(Y)}(x) = f'_{u(Y)}(u(g))$, that is, $u^*(p)_Y(x) = u^*(f')_Y(\eta_{h_X})_Y(g)$. Hence $(x, g) \in (u^*(H) \times_{u^*(K)} h_X)(Y)$ and g is in the image of q. Thus the image of q contains a covering sieve S^u and it follows that $u^*(p)$ is a covering.

If $p : H \to K$ is a bicovering, then the diagonal morphism $\Delta : H \to H \times_K H$ is a covering. Since u_* preserves pull-backs by (A.6.2), it follows from the above result that the diagonal morphism $u^*(H) \to u^*(H) \times_{u^*(K)} u^*(H)$ is also a covering. Therefore $u^*(p)$ is a bicovering. □

Proposition 2.11.10 *Let $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ be $\mathcal{U}$-sites and $u : \mathcal{C} \to \mathcal{C}'$ a functor such that the right adjoint $u_* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ of u^* exists. u is cocontinuous if and only if, for any sheaf F on $\mathcal{C}$, $u_*(F)$ is a sheaf on $\mathcal{C}'$.*

Proof. Suppose that $u_*(F)$ is a sheaf for any sheaf F on $\mathcal{C}$. Let X be an object of $\mathcal{C}$ and $R \in J'(u(X))$. Then the inclusion morphism $\iota : R \to h'_{u(X)}$ induces a bijection $\iota^* : \widehat{\mathcal{C}'}(h'_{u(X)}, u_*(F)) \to \widehat{\mathcal{C}'}(R, u_*(F))$. By the adjunction, $u^*(\iota)^* : \widehat{\mathcal{C}}(u^*(h'_{u(X)}), F) \to \widehat{\mathcal{C}}(u^*(R), F)$ is bijective. Hence $u^*(\iota) : u^*(R) \to u^*(h'_{u(X)}) = u^*u_!(h_X)$ is a bicovering by (2.5.4). Note that u^* has a right adjoint, thus it preserves monomorphisms. It follows that $u^*(\iota)$ is a monomorphism. Consider the pull-back of $u^*(\iota)$ along the unit $\eta_{h_X} : h_X \to u^*u_!(h_X)$ of the adjunction.

$$\begin{tikzcd} u^*(R) \times_{u^*u_!(h_X)} h_X \arrow[r, "p_1"] \arrow[d, "p_2"] & u^*(R) \arrow[d, "u^*(\iota)"] \\ h_X \arrow[r, "\eta_{h_X}"] & u^*u_!(h_X) \end{tikzcd}$$

Since $u^*(\iota)$ is a bicovering, the image S of p_2 belongs to $J(X)$. It suffices to show that $S \subset R^u$. Suppose that $f : Y \to X$ is in the image of p_{2Y}. Recall that $(\eta_{h_X})_Y : h_X(Y) \to u^*u_!(h_X)(Y) = h'_{u(X)}(u(Y))$ is given by $f \mapsto u(f)$ (A.6.12). There exists an element $g \in u^*(R)(Y) = R(u(Y))$ such that $g = u(f)$. Hence we have $f \in R^u(X)$.

Suppose that u is cocontinuous. Note that, since u^* has a right adjoint, it is left exact. For $Z \in \mathrm{Ob}\,\mathcal{C}'$ and $R \in J'(Z)$, it follows from (2.11.9) that the morphism $u^*(\iota) : u^*(R) \to u^*(h'_Z)$ induced by the inclusion morphism $\iota : R \to h'_Z$ is a bicovering by (2.5.1) and (2.5.4). Then, for any sheaf F on $\mathcal{C}$, $\iota^* : \widehat{\mathcal{C}'}(h'_Z, u_*(F)) \to \widehat{\mathcal{C}'}(R, u_*(F))$ is bijective by the adjunction of u^* and u_*. Hence $u_*(F)$ is a sheaf on $\mathcal{C}'$. □

Proposition 2.11.11 *Let $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ be $\mathcal{U}$-sites and $u : \mathcal{C} \to \mathcal{C}'$ a functor. Define $\bar{u}^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ to be the composition $\widetilde{\mathcal{C}'} \xrightarrow{i'} \widehat{\mathcal{C}'} \xrightarrow{u^*} \widehat{\mathcal{C}} \xrightarrow{a} \widetilde{\mathcal{C}}$.*

1) $\bar{u}^*$ is left exact.

2) If u is cocontinuous and the right adjoint $u_* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ of u^* exists, $\bar{u}^*$ has a right adjoint $\tilde{u}_*$ such that the following diagram on the left commutes. The right diagram commutes up to a natural equivalence.

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "\tilde{u}_*"] \arrow[d, "i"] & \widetilde{\mathcal{C}'} \arrow[d, "i'"] \\ \widehat{\mathcal{C}} \arrow[r, "u_*"] & \widehat{\mathcal{C}'} \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathcal{C}'} \arrow[r, "u^*"] \arrow[d, "a'"] & \widehat{\mathcal{C}} \arrow[d, "a"] \\ \widetilde{\mathcal{C}'} \arrow[r, "\bar{u}^*"] & \widetilde{\mathcal{C}} \end{tikzcd}$$

3) Suppose that $\mathcal{C}$ is $\mathcal{U}$-small and u is cocontinuous. If $\mathcal{V}$ is a universe such that $\mathcal{U} \subset \mathcal{V}$, we denote by $\tilde{u}_{\mathcal{U}*} : \widetilde{\mathcal{C}}_{\mathcal{U}} \to \widetilde{\mathcal{C}'}_{\mathcal{U}}$ (resp. $\tilde{u}_{\mathcal{V}*} : \widetilde{\mathcal{C}}_{\mathcal{V}} \to \widetilde{\mathcal{C}'}_{\mathcal{V}}$) the right adjoint of $\bar{u}^*_{\mathcal{U}} : \widetilde{\mathcal{C}'}_{\mathcal{U}} \to \widetilde{\mathcal{C}}_{\mathcal{U}}$ (resp. $\bar{u}^*_{\mathcal{V}} : \widetilde{\mathcal{C}'}_{\mathcal{V}} \to \widetilde{\mathcal{C}}_{\mathcal{V}}$). Then, the following diagram commutes up to natural equivalence.

$$\begin{tikzcd}
\widetilde{\mathcal{C}}_{\mathcal{U}} \arrow[r, "\tilde{u}_{\mathcal{U}*}"] \arrow[d] & \widetilde{\mathcal{C}'}_{\mathcal{U}} \arrow[d] \\
\widetilde{\mathcal{C}}_{\mathcal{V}} \arrow[r, "\tilde{u}_{\mathcal{V}*}"] & \widetilde{\mathcal{C}'}_{\mathcal{V}}
\end{tikzcd}$$

Proof. 1) i', u^* and a are left exact.

2) It follows from (2.11.10) that by restricting $u_* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ to $\widetilde{\mathcal{C}}$, we have a functor $\tilde{u}_* : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ satisfying $i'\tilde{u}_* = u_* i$. For $F \in \operatorname{Ob}\widetilde{\mathcal{C}}$ and $G \in \operatorname{Ob}\widetilde{\mathcal{C}'}$, we have $\widetilde{\mathcal{C}}(\bar{u}^*(G), F) = \widetilde{\mathcal{C}}(au^*i'(G), F) \cong \widehat{\mathcal{C}}(u^*i'(G), i(F)) \cong \widehat{\mathcal{C}'}(i'(G), u_*i(F)) = \widehat{\mathcal{C}'}(i'(G), i'\tilde{u}_*(F)) = \widetilde{\mathcal{C}'}(G, \tilde{u}_*(F))$. Thus $\tilde{u}_*$ is a right adjoint of $\bar{u}^*$. For $F \in \operatorname{Ob}\widetilde{\mathcal{C}}$ and $G \in \operatorname{Ob}\widehat{\mathcal{C}'}$, we have a chain of natural bijections $\widetilde{\mathcal{C}}(au^*(G), F) \cong \widehat{\mathcal{C}}(u^*(G), i(F)) \cong \widehat{\mathcal{C}'}(G, u_*i(F)) = \widehat{\mathcal{C}'}(G, i'\tilde{u}_*(F)) \cong \widetilde{\mathcal{C}'}(a'(G), \tilde{u}_*(F)) \cong \widetilde{\mathcal{C}}(\bar{u}^*a'(G), F)$. Hence there is a natural equivalence $au^* \to \bar{u}^*a'$.

3) Since $\tilde{u}_{\mathcal{U}*}$, $\tilde{u}_{\mathcal{V}*}$ are restrictions of $u_{\mathcal{U}*} : \widehat{\mathcal{C}}_{\mathcal{U}} \to \widehat{\mathcal{C}'}_{\mathcal{U}}$, $u_{\mathcal{V}*} : \widehat{\mathcal{C}}_{\mathcal{V}} \to \widehat{\mathcal{C}'}_{\mathcal{V}}$ and the inclusion functor $i'_{\mathcal{V}} : \widetilde{\mathcal{C}'}_{\mathcal{V}} \to \widehat{\mathcal{C}'}_{\mathcal{V}}$ is fully faithful, it suffices to see that the following diagram commutes up to natural equivalence.

$$\begin{tikzcd}
\widehat{\mathcal{C}}_{\mathcal{U}} \arrow[r, "u_{\mathcal{U}*}"] \arrow[d] & \widehat{\mathcal{C}'}_{\mathcal{U}} \arrow[d] \\
\widehat{\mathcal{C}}_{\mathcal{V}} \arrow[r, "u_{\mathcal{V}*}"] & \widehat{\mathcal{C}'}_{\mathcal{V}}
\end{tikzcd}$$

But this is obvious from the construction of $u_{\mathcal{U}*}$, $u_{\mathcal{V}*}$ (A.6.6). □

Proposition 2.11.12 *Let $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ be $\mathcal{U}$-sites and $u : \mathcal{C} \to \mathcal{C}'$, $v : \mathcal{C}' \to \mathcal{C}$ functors such that v is a left adjoint of u.*

1) The following conditions are equivalent.

(i) u is $\mathcal{U}$-continuous.
(ii) v is cocontinuous.
(iii) If $(f_i : X_i \to X)_{i\in I}$ is a covering of $X \in \mathcal{C}$, $(u(f_i) : u(X_i) \to u(X))_{i\in I}$ is a covering of $u(X)$.

2) In the above case, there are natural equivalences $\tilde{v}_* \cong \tilde{u}^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$, $\tilde{v}^* \cong \tilde{u}_! : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$.

3) Suppose that $\mathcal{C}$ is $\mathcal{U}$-small. $h'u : \mathcal{C} \to \widehat{\mathcal{C}'}$ is filtering. Hence so is $\epsilon_{J'}u : \mathcal{C} \to \widetilde{\mathcal{C}'}$.

Proof. 1) Since v is a left adjoint of u, $v^* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ is a left adjoint of $u^* : \widehat{\mathcal{C}'} \to \widehat{\mathcal{C}}$ by (A.6.11). Hence the left adjoint $u_!$ of u^* exists even if $\mathcal{C}$ is not $\mathcal{U}$-small, and it is given by $u_! = v^*$. Similarly, the right adjoint v_* of v^* exists and it is given by $v_* = u^*$. Therefore, for $F \in \widehat{\mathcal{C}'}$, $v_*(F)$ is a sheaf if and only if $u^*(F)$ is so. Then, the equivalence $(i) \Leftrightarrow (ii)$ follows from (2.11.10). By (2.11.6), (i) implies (iii). Assume (iii). Then, the condition $ii)$ of (2.11.6) is satisfied. Hence, by (2.11.6), it suffices to verify the condition $i)$ of (2.11.6). Let $(f_i : X_i \to X)_{i\in I}$ be a covering of $X \in \operatorname{Ob}\mathcal{C}$. For $i, j \in I$, suppose that $\alpha : U \to u(X_i)$ and $\beta : U \to u(X_j)$ are morphisms in $\mathcal{C}'$ satisfying $u(f_i)\alpha = u(f_j)\beta$. We denote by $\alpha' : v(U) \to X_i$, $\beta' : v(U) \to X_j$ and $\eta : id_{\mathcal{C}'} \to uv$ the adjoint of α, β and the unit of the adjunction. Then, for the trivial covering $(id_U : U \to U)$, a diagram $X_i \xleftarrow{\alpha'} v(U) \xrightarrow{\beta'} X_j$ in $\mathcal{C}$ and a morphism $\eta_U : U \to uv(U)$ in $\mathcal{C}'$, equalities $\alpha id_U = u(\alpha')\eta_U$, $\beta id_U = u(\beta')\eta_U$ and $f_i\alpha' = f_j\beta'$ hold.

2) Since $\tilde{v}_* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ is the restriction of $v_* : \widehat{\mathcal{C}'} \to \widehat{\mathcal{C}}$ which is equivalent to u^*, $\tilde{v}_*$ is equivalent to $\tilde{u}^*$. Note that $\tilde{v}^* : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ is a left adjoint of $\tilde{v}_*$ and $\tilde{u}_! : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ is a left adjoint of $\tilde{u}^*$. Since $\tilde{v}_*$ is equivalent to $\tilde{u}^*$, $\tilde{v}^*$ is equivalent to $\tilde{u}_!$.

3) Recall from (A.6.2) that v^* is left exact. Thus $h'u$ has a left exact left Kan extension $u_! = v^*$ along $h : \mathcal{C} \to \widehat{\mathcal{C}}$. By (2.9.9), $h'u$ is filtering. Again, since v^* is a left Kan extension of $h'u$ along h, there is a colimiting cone $(h'uP\langle X, f\rangle \xrightarrow{\lambda^F_{\langle X,f\rangle}} v^*(F))_{\langle X,f\rangle\in\operatorname{Ob}(h'\downarrow F)}$ is for each $F \in \operatorname{Ob}\widehat{\mathcal{C}}$. Since the associated sheaf functor $a' : \widehat{\mathcal{C}'} \to \widetilde{\mathcal{C}'}$ preserves colimits, $(a'h'uP\langle X, f\rangle \xrightarrow{a'(\lambda^F_{\langle X,f\rangle})} a'v^*(F))_{\langle X,f\rangle\in\operatorname{Ob}(h'\downarrow F)}$ is a colimiting cone. It follows that $a'v^* : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ is a left Kan extension of $a'h'u : \mathcal{C} \to \widetilde{\mathcal{C}'}$ along $h : \mathcal{C} \to \widehat{\mathcal{C}}$. Hence $a'h'u$ also has a left exact left Kan extention along $h : \mathcal{C} \to \widehat{\mathcal{C}}$. It follows from (2.9.9) that $ah'u$ is also filtering. □

Proposition 2.11.13 *Let $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ be $\mathcal{U}$-sites and $u : \mathcal{C} \to \mathcal{C}'$ a $\mathcal{U}$-continuous and cocontinuous functor.*

1) If $u^* : \widehat{\mathcal{C}'} \to \widehat{\mathcal{C}}$ has both left and right adjoints $u_!, u_* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$, so does $\tilde{u}^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$.

2) We denote by $\tilde{u}_!$ and $\tilde{u}_*$ the left and right adjoint of $\tilde{u}^*$, respectively. $\tilde{u}_!$ is fully faithful if and only if so is $\tilde{u}_*$.

3) If u is fully faithful, so is $\tilde{u}_!$. If J and J' are coarser than the canonical topology, the converse holds.

Proof. 1) follows from (2.11.3) and (2.11.11).

2) is a general property of adjoint functors. (See the proof of (A.6.10).)

3) If u is fully faithful, so is u_* by (A.6.13). By the commutativity of the diagram of 2) of (2.11.11), we see that $\tilde{u}_*$ is fully faithful. Hence $\tilde{u}_!$ is fully faithful. Since the functors ϵ_J and $\epsilon_{J'}$ are fully faithful if J and J' are coarser than the canonical topology, the converse assertion follows from the commutativity of the diagram of iv) of (2.11.2). □

2.12 Induced topology

Definition 2.12.1 *Let $(\mathcal{C}', J')$ be a site and $u : \mathcal{C} \to \mathcal{C}'$ a functor. The finest topology on $\mathcal{C}$ such that u is continuous is called the topology induced by u.*

Proposition 2.12.2 *Let $(\mathcal{C}', J')$ be a site and $u : \mathcal{C} \to \mathcal{C}'$ a functor. We choose a universe $\mathcal{U}$ such that $\mathcal{C}$ is $\mathcal{U}$-small and $(\mathcal{C}', J')$ is a $\mathcal{U}$-site. The finest topology on $\mathcal{C}$ such that u is $\mathcal{U}$-continuous is the topology induced by u and it does not depend on the choice of $\mathcal{U}$.*

Proof. Let $\mathcal{V}$ be a universe containing $\mathcal{U}$. By (2.11.5), u is $\mathcal{U}$-continuous if and only if it is $\mathcal{V}$-continuous. For any universe $\mathcal{U}'$ such that $\mathcal{C}$ is $\mathcal{U}'$-small and $(\mathcal{C}', J')$ is a $\mathcal{U}'$-site, there exists a universe $\mathcal{V}$ which contains both $\mathcal{U}$ and $\mathcal{U}'$. Let J be the finest topology on $\mathcal{C}$ such that u is $\mathcal{U}$-continuous. Then, u is continuous. Suppose that T is a topology on $\mathcal{C}$ such that u is continuous. There exists a universe $\mathcal{V}$ such that $\mathcal{C}$ is $\mathcal{V}$-small, $(\mathcal{C}', J')$ is a $\mathcal{V}$-site and that u is $\mathcal{V}$-continuous. By (2.11.5), we may assume that $\mathcal{V}$ contains $\mathcal{U}$, hence u is also $\mathcal{U}$-continuous for T. Thus J is finer than T. □

Proposition 2.12.3 *Let $\mathcal{C}$ be a $\mathcal{U}$-small category, $(\mathcal{C}', J')$ a $\mathcal{U}$-site and $u : \mathcal{C} \to \mathcal{C}'$ a functor. We denote by J the topology on $\mathcal{C}$ induced by u. For $X \in \operatorname{Ob}\mathcal{C}$, $R \in J(X)$ if and only if for any morphism $f : Y \to X$, the morphism $u_!(h_f^{-1}(R)) \to u_!(h_Y)$ induced by the inclusion morphism $h_f^{-1}(R) \to h_Y$ is a bicovering in $\widehat{\mathcal{C}'}$.*

Proof. Let $u_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ be the left adjoint of u^*. If $R \in J(X)$, $h_f^{-1}(R) \in J(Y)$ by $(T2)$. Since u is continuous, $u_!(h_f^{-1}(R)) \to u_!(h_Y)$ is a bicovering by (2.11.2). Conversely, suppose that, for any morphism $f : Y \to X$, the morphism $u_!(\iota_f) : u_!(h_f^{-1}(R)) \to u_!(h_Y)$ induced by the inclusion morphism $\iota_f : h_f^{-1}(R) \to h_Y$ is a bicovering in $\widehat{\mathcal{C}'}$. Then, for any sheaf F on $\mathcal{C}'$, $u_!(\iota_f)^* : \widehat{\mathcal{C}'}(u_!(h_Y), F) \to \widehat{\mathcal{C}'}(u_!(h_f^{-1}(R)), F)$ is bijective. By the adjunction, $\iota_f^* : \widehat{\mathcal{C}}(h_Y, u^*(F)) \to \widehat{\mathcal{C}}(h_f^{-1}(R), u^*(F))$ is bijective. It follows from (2.2.4) that $R \in J(X)$. □

Corollary 2.12.4 *Let $(\mathcal{C}', J')$ be a site and $u : \mathcal{C} \to \mathcal{C}'$ a functor. We denote by J the topology on $\mathcal{C}$ induced by u. If a family of morphisms $(f_i : X_i \to X)_{i \in I}$ in $\mathcal{C}$ is a covering for J, then $(u(f_i) : u(X_i) \to u(X))_{i \in I}$ is a covering for J'. If each f_i has a pull-back along an arbitrary morphism and u preserves it, the converse holds.*

Proof. We choose a universe $\mathcal{U}$ such that $\mathcal{C}$ is $\mathcal{U}$-small and $(\mathcal{C}', J')$ is a $\mathcal{U}$-site. We denote by $\widehat{\mathcal{C}}$, $\widetilde{\mathcal{C}'}$ the categories of $\mathcal{U}$-presheaves Let $u_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ be the left adjoint of u^*. The first assertion follows from 1) of (2.11.6). Assume that $(u(f_i) : u(X_i) \to u(X))_{i \in I}$ is a covering for J'. R denotes the sieve on X generated by $(f_i : X_i \to X)_{i \in I}$. By (2.12.3), it suffices to show that for any morphism $g : Y \to X$, the morphism $u_!(h_g^{-1}(R)) \to u_!(h_Y)$ induced by the inclusion morphism $h_g^{-1}(R) \to h_Y$ is a bicovering in $\widehat{\mathcal{C}'}$.

We first consider the case $g = id_X$. Let $f_i^\sharp : h_{X_i} \to R$ be the morphism induced by $h_{f_i} : h_{X_i} \to h_X$. Then $(f_i^\sharp : h_{X_i} \to R)_{i \in I}$ is an epimorphic family. Since $u_!$ has a right adjoint, $(u_!(f_i^\sharp) : h_{u(X_i)} = u_!(X_i) \to u_!(R))_{i \in I}$ is an epimorphic family in $\widehat{\mathcal{C}'}$ (A.3.13). We denote by $\iota : R \to h_X$ the inclusion morphism. Then, applying $u_!$ to $h_{f_i} = \iota f_i^\sharp$, we have $h_{u(f_i)} = u_!(\iota)u_!(f_i^\sharp)$ by (A.6.12). We can form the following pull-back on the left. The middle square is also a pull-back by the assumption and so is the one on the right.

$$\begin{tikzcd} X_i\times_X X_j \arrow[r, "q_{ij}"] \arrow[d, "p_{ij}"] & X_j \arrow[d, "f_i"] \\ X_i \arrow[r, "f_i"] & X \end{tikzcd} \quad \begin{tikzcd} u(X_i\times_X X_j) \arrow[r, "u(q_{ij})"] \arrow[d, "u(p_{ij})"] & u(X_j) \arrow[d, "u(f_i)"] \\ u(X_i) \arrow[r, "u(f_i)"] & u(X) \end{tikzcd} \quad \begin{tikzcd} h_{u(X_i\times_X X_j)} \arrow[r, "h_{u(q_{ij})}"] \arrow[d, "h_{u(p_{ij})}"] & h_{u(X_j)} \arrow[d, "h_{u(f_i)}"] \\ h_{u(X_i)} \arrow[r, "h_{u(f_i)}"] & h_{u(X)} \end{tikzcd}$$

Let $\alpha,\beta:\coprod_{i,j\in I}h_{u(X_i\times_X X_j)}\to\coprod_{i\in I}h_{u(X_i)}$ be morphisms satisfying $\alpha\nu_{ij}=\nu_i h_{u(p_{ij})}$, $\beta\nu_{ij}=\nu_j h_{u(q_{ij})}$, where $\nu_{ij}:h_{u(X_i\times_X X_j)}\to\coprod_{i,j\in I}h_{u(X_i\times_X X_j)}$, $\nu_i:h_{u(X_i)}\to\coprod_{i\in I}h_{u(X_i)}$ denote the canonical morphisms. We also consider the morphisms $p:\coprod_{i\in I}h_{u(X_i)}\to u_!(R)$ and $q:\coprod_{i\in I}h_{u(X_i)}\to h_{u(X)}$ be induced by $(u_!(f_i^\sharp):h_{u(X_i)}=u_!(h_{X_i})\to u_!(R))_{i\in I}$ and $(h_{u(f_i)}:h_{u(X_i)}\to h_{u(X)})_{i\in I}$, respectively. Then, p is an epimorphism and $q=u_!(\iota)p$. Moreover, the following squares are cartesian.

$$\begin{tikzcd} \coprod_{i,j\in I}h_{u(X_i\times_X X_j)} \arrow[r, "\beta"] \arrow[d, "\alpha"] & \coprod_{i\in I}h_{u(X_i)} \arrow[d, "p"] \\ \coprod_{i\in I}h_{u(X_i)} \arrow[r, "p"] & u_!(R) \end{tikzcd} \qquad \begin{tikzcd} \coprod_{i,j\in I}h_{u(X_i\times_X X_j)} \arrow[r, "\beta"] \arrow[d, "\alpha"] & \coprod_{i\in I}h_{u(X_i)} \arrow[d, "q"] \\ \coprod_{i\in I}h_{u(X_i)} \arrow[r, "q"] & h_{u(X)} \end{tikzcd}$$

Hence by (A.3.6), $u_!(\iota)$ is a monomorphism. Since p is an epimorphism, it follows from the assumption that the image of $u_!(\iota)$ is a covering sieve generated by $(u(f_i):u(X_i)\to u(X))_{i\in I}$. Thus we see that $u_!(\iota)$ is a bicovering.

For general $g:Y\to X$, let $\bar f_i:X_i\times_X Y\to Y$ a pull-back of f_i along g. Then, $\bar f_i$'s generate $h_g^{-1}(R)$ by (2.1.10) and each $\bar f_i$ has a pull-back along an arbitrary morphism by (A.3.1). Since u preserves pull-backs of f_i, it also preserves pull-backs of $\bar f_i$ and $u(\bar f_i):u(X_i\times_X Y)\to u(Y)$ is a pull-back of $u(f_i)$ along $u(g)$. It follows from the assumption and (2.1.11) that $(u(\bar f_i):u(X_i\times_X Y)\to u(Y))_{i\in I}$ is a covering of $u(Y)$. Thus, we can apply the preceding result to a family $(\bar f_i:X_i\times_X Y\to Y)_{i\in I}$ which generates $h_f^{-1}(R)$. □

Corollary 2.12.5 *Let $(\mathcal{C}',J')$ be a $\mathcal{U}$-site and $\mathcal{C}$ a full subcategory of $\mathcal{C}'$ with the inclusion functor $u:\mathcal{C}\to\mathcal{C}'$. Suppose that pull-backs in $\mathcal{C}$ exist and u preserves them. Then the following conditions i) and ii) are equivalent.*

i) a) For any $X\in\operatorname{Ob}\mathcal{C}$ and a covering $R=(g_j:Y_j\to X)_{j\in K}$, there exists a covering $S=(f_i:X_i\to X)_{i\in I}$ such that $X_i\in\operatorname{Ob}\mathcal{C}$ for any $i\in I$ and $\bar S\subset\bar R$.
b) There exists a $\mathcal{U}$-small subset G of $\operatorname{Ob}\mathcal{C}$ such that, for any $X\in\operatorname{Ob}\mathcal{C}$, there exists a covering $(f_i:X_i\to X)_{i\in I}$ in $\mathcal{C}'$ such that $X_i\in\operatorname{Ob}G$ for any $i\in I$.
ii) The topology on $\mathcal{C}$ induced by u is a $\mathcal{U}$-topology and u is continuous and cocontinuous.

Proof. *i)* ⇒ *ii)*: For any $X\in\operatorname{Ob}\mathcal{C}$, there exists a covering $(f_i:X_i\to X)_{i\in I}$ in $\mathcal{C}'$ such that $X_i\in\operatorname{Ob}G$ for any $i\in I$. By the assumption on the inclusion functor u, we can apply (2.12.4). Hence $(f_i:X_i\to X)_{i\in I}$ is a covering in $\mathcal{C}$ for the induced topology and it follows that G is $\mathcal{U}$-small topologically generating set for the induced topology. Obviously, u is continuous. For any $X\in\operatorname{Ob}\mathcal{C}$ and $R\in J'(u(X))$, choose a family $(g_j:Y_j\to X)_{j\in K}$ generating R. There exists a covering $S=(f_i:X_i\to X)_{i\in I}$ such that $X_i\in\operatorname{Ob}\mathcal{C}$ for any $i\in I$ and $\bar S\subset R$. Hence $u(f_i)=f_i\in R$ for any $i\in I$ and we have $\bar S\subset R^u$. On the other hand, S is a covering for the induced topology by (2.12.4). Thus R^u is a covering sieve and u is cocontinuous.

ii) ⇒ *i)*: Let G be a $\mathcal{U}$-small topologically generating set for the induced topology. Then the condition *b)* is satisfied by (2.12.4). For $X\in\operatorname{Ob}\mathcal{C}$, let $R=(g_j:Y_j\to X)_{j\in K}$ be a covering for J'. Since u is cocontinuous, $\bar R^u$ is a covering sieve on X for the induced topology. If $S=(f_i:X_i\to X)_{i\in I}$ is a covering which generates $\bar R^u$, this satisfies the condition *a)*. □

Proposition 2.12.6 *Let $\mathcal{C}$ be a $\mathcal{U}$-category.*

1) Suppose that a $\mathcal{U}$-topology J on $\mathcal{C}$ is given. We give $\widetilde{\mathcal{C}}_J$ the canonical topology. Then, J coincides with the topology induced by $\epsilon_J:\mathcal{C}\to\widetilde{\mathcal{C}}_J$.

2) Let $\mathcal{D}$ be a reflexive full subcategory of $\widehat{\mathcal{C}}$ with a left exact reflection L. We give $\mathcal{D}$ the canonical topology. If $J^{T_{\mathcal{D}}}$ is a $\mathcal{U}$-topology on $\mathcal{C}$, it is the topology induced by $Lh:\mathcal{C}\to\mathcal{D}$.

Proof. 1) First, note that the canonical topology on $\widetilde{\mathcal{C}}_J$ is a $\mathcal{U}$-topology by (2.4.16). Let H be a sheaf on $\widetilde{\mathcal{C}}_J$ for the canonical topology. It follows from (2.10.11) that there exists $F\in\operatorname{Ob}\widetilde{\mathcal{C}}_J$ such that H is isomorphic to the sheaf $\tilde h(F)$ represented by F. Define a morphism $\alpha:\tilde h(F)\epsilon_J\to F$ of presheaves on $\mathcal{C}$ as follows. For $X\in\operatorname{Ob}\mathcal{C}$,

$\alpha_X : \tilde{h}(F)\epsilon_J(X) \to F(X)$ is the composition $\tilde{h}(F)\epsilon_J(X) = \widetilde{\mathcal{C}}_J(\epsilon_J(X), F) \to \widehat{\mathcal{C}}(h_X, i(F)) \to i(F)(X) = F(X)$, where the first map is the bijection given by the adjunction of the associated sheaf functor $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}_J$ and the inclusion functor $i : \widetilde{\mathcal{C}}_J \to \widehat{\mathcal{C}}$ and the second map is the bijection given by the Yoneda's lemma. Hence α is an isomorphism and it follows that $\tilde{h}(F)\epsilon_J$ is a sheaf. Therefore ϵ_J is continuous.

Let T be a topology on $\mathcal{C}$ such that ϵ_J is continuous. Suppose that $R \in T(X)$ for $X \in \mathrm{Ob}\,\mathcal{C}$. For any sheaf F on $\mathcal{C}$ for J, $\tilde{h}(F)$ is a sheaf on $\widetilde{\mathcal{C}}_J$ for the canonical topology. Hence $\tilde{h}(F)\epsilon_J$ is a sheaf on $\mathcal{C}$ for T by the assumption on T, and the map $\widehat{\mathcal{C}}(h_X, \tilde{h}(F)\epsilon_J) \to \widehat{\mathcal{C}}(R, \tilde{h}(F)\epsilon_J)$ induced by the inclusion morphism $\iota : R \to h_X$ is bijective. By the above isomorphism α, the map $\iota^* : \widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(R, F)$ is bijective. Thus $R \in J(X)$ by (2.5.1) and (2.5.4) and this implies that J is finer than T.

2) We set $J = J^{T_{\mathcal{D}}}$. It follows from (2.5.12) that $\mathcal{D} = \widetilde{\mathcal{C}}_J$ and the associated sheaf functor $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}_J$ is naturally equivalent to L. Hence $\epsilon_J : \mathcal{C} \to \widetilde{\mathcal{C}}_J$ is naturally equivalent to Lh and the assertion follows from 1). □

Lemma 2.12.7 *Let $\mathcal{C}$ be a $\mathcal{U}$-small category and $(u_i : F_i \to G_i)_{i\in I}$ a family of morphisms in $\widehat{\mathcal{C}}$. There exists the coarsest topology on $\mathcal{C}$ such that every u_i is a covering (resp. bicovering).*

Proof. Recall that u_i is a covering if and only if, for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \widehat{\mathcal{C}}(h_X, G_i)$, the image of the pull-back $f_i : h_X \times_{G_i} F_i \to h_X$ of u_i along f is a covering sieve. Set $T_i(X) = \{\mathrm{Im} f_i |\, f \in \widehat{\mathcal{C}}(h_X, G_i)\}$. The coarsest topology J such that $J(X) \supset \bigcup_{i\in I} T_i(X)$ for all $X \in \mathrm{Ob}\,\mathcal{C}$ is the coarsest topology on $\mathcal{C}$ such that every u_i is a covering. Since u_i is a bicovering if and only if both u_i and the diagonal morphism $F_i \to F_i \times_{G_i} F_i$ are coverings, the above argument shows that there exists the coarsest topology on $\mathcal{C}$ such that every u_i is a bicovering. □

Proposition 2.12.8 *Let $((\mathcal{C}_i, J_i))_{i\in I}$ be a family of sites, $\mathcal{C}$ a category, $(u_i : \mathcal{C}_i \to \mathcal{C})_{i\in I}$ a family of functors and $\mathcal{U}$ a universe such that $\mathcal{C}_i$ and $\mathcal{C}$ are $\mathcal{U}$-small. There exists the coarsest topology $J_{\mathcal{U}}$ on $\mathcal{C}$ such that every u_i is continuous. Moreover, $J_{\mathcal{U}}$ does not depend on the choice of a universe $\mathcal{U}$ such that $\mathcal{C}_i$ and $\mathcal{C}$ are $\mathcal{U}$-small.*

Proof. By (2.11.2), u_i is continuous if and only if, for any $X \in \mathrm{Ob}\,\mathcal{C}_i$ and $R \in J_i(X)$, the morphism $u_{i!}(R) \to u_{i!}(h_X)$ induced by the inclusion morphism is a bicovering. Hence the result follows from (2.12.7). The second assertion follows from (2.11.5). □

Proposition 2.12.9 *Let $((\mathcal{C}_i, J_i))_{i\in I}$ be a family of sites, $\mathcal{C}$ a category and $(u_i : \mathcal{C}_i \to \mathcal{C})_{i\in I}$ a family of functors. There exists the finest topology on $\mathcal{C}$ such that every u_i is cocontinuous.*

Proof. Let $\mathcal{U}$ be a universe such that $\mathcal{C}_i$ and $\mathcal{C}$ are $\mathcal{U}$-small. We denote by $u_{i*} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}_i$ the right adjoint of u^*. By (2.11.10), u_i is cocontinuous if and only if, for any sheaf F on $\mathcal{C}_i$, $u_{i*}(F)$ is a sheaf on $\mathcal{C}$. Hence the finest topology on $\mathcal{C}$ such that every element of $\bigcup_{i\in I}\{u_{i*}(F) \in \mathrm{Ob}\,\widehat{\mathcal{C}} |\, F \in \mathrm{Ob}\,\widetilde{\mathcal{C}}_i\}$ is a sheaf is the finest topology on $\mathcal{C}$ such that every u_i is cocontinuous. □

Theorem 2.12.10 *Let $\mathcal{C}$ be a $\mathcal{U}$-small category, $(\mathcal{C}', J')$ a site such that $\mathcal{C}'$ is a $\mathcal{U}$-category and $u : \mathcal{C} \to \mathcal{C}'$ a fully faithful functor. Give $\mathcal{C}$ the topology J induced by u. Consider the following properties.*

i) $\{u(X) |\, X \in \mathrm{Ob}\,\mathcal{C}\}$ is a topologically generating family for J'.

ii) The functor $u^* : \widehat{\mathcal{C}'} \to \widehat{\mathcal{C}}$ induces a equivalence from the category of sheaves on $\mathcal{C}'$ to the category of sheaves on $\mathcal{C}$.

Then, i) implies ii). If $(\mathcal{C}', J')$ is a $\mathcal{U}$-site and J' is coarser than the canonical topology, the converse holds.

Proof. i) ⇒ ii): Let $u_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ be the left adjoint of u^*. We show that, for any presheaf H on $\mathcal{C}'$, the counit $\varepsilon_H : u_!u^*(H) \to H$ is a bicovering. For a morphism $s : h'_Y \to H$, consider the pull-back of ε_H along s.

$$\begin{tikzcd} h'_Y \times_H u_!u^*(H) \arrow[r, "\bar{s}"] \arrow[d, "\varepsilon^s_H"] & u_!u^*(H) \arrow[d, "\varepsilon_H"] \\ h'_Y \arrow[r, "s"] & H \end{tikzcd}$$

By the assumption, $(f : uP\langle X, f\rangle \to Y)_{\langle X, f\rangle \in \mathrm{Ob}(u\downarrow Y)}$ is a covering family of Y for J'. Then, the image of ε^s_H contains this family. In fact, for $\langle X, f\rangle \in \mathrm{Ob}\,(u\downarrow Y)$, let $\bar{f} : h_X \to u^*(h'_Y)$ be the adjoint of $sh'_f : u_!(h_X) = h'_{u(X)} \to H$. Since $\varepsilon_H u_!(\bar{f}) = sh'_f$, there exists a unique morphism $\tilde{f} : h'_{u(X)} = u_!(h_X) \to h'_Y \times_H u_!u^*(H)$ such that $\varepsilon^s_H \tilde{f} = h'_f$ and $\bar{s}\tilde{f} = u_!(\bar{f})$. Hence $f = h'_f(id_{u(X)}) = (\varepsilon^s_H)_{u(X)}\tilde{f}_{u(X)}(id_{u(X)})$ is in the image of ε^s_H. Since the image of ε^s_H contains a covering family, it is a covering sieve and it follows that ε_H is a covering.

Let $s, t : h'_Y \to u_!u^*(H)$ be morphisms such that $\varepsilon_H s = \varepsilon_H t$. Recall from (A.6.10) that $u_!$ is fully faithful. Then, for any $\langle X, f\rangle \in \mathrm{Ob}\,(u\downarrow Y)$, there exist unique morphisms $s', t' : h_X \to u^*(H)$ such that $sh'_f = u_!(s'), th'_f = u_!(t') : u_!(h_X) = h'_{u(X)} \to u_!u^*(H)$. Since $\varepsilon_H u_!(s')$, $\varepsilon_H u_!(t')$ are adjoints of s', t' respectively and $\varepsilon_H u_!(s') = \varepsilon_H u_!(t')$, we have $s' = t'$, hence $sh'_f = th'_f$. It follows that the equalizer of s and t contains a covering family $(f : uP\langle X, f\rangle \to Y)_{\langle X,f\rangle\in\mathrm{Ob}(u\downarrow Y)}$ of Y for J'. By (2.5.4), $\varepsilon_H : u_!u^*(H) \to H$ is a bicovering.

Suppose that H is a sheaf on $\mathcal{C}'$. Consider the left adjoint $\tilde{u}_! : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ of $u^* : \widehat{\mathcal{C}'} \to \widehat{\mathcal{C}}$ given by $\tilde{u}_! = a'u_!i$ (2.11.2). Recall from (2.11.3) that the counit $\tilde{\varepsilon} : \tilde{u}_!\tilde{u}^* \to id_{\widetilde{\mathcal{C}'}}$ is a composition $\tilde{u}_!\tilde{u}^* = a'u_!i\tilde{u}^* = a'u_!u^*i' \xrightarrow{a'(\varepsilon_{i'})} a'i' \xrightarrow{\varepsilon_{J'}} id_{\widetilde{\mathcal{C}'}}$. Since $\varepsilon_{J'}$ is an isomorphism and $a'(\varepsilon_{i'(H)}) : a'u_!u^*i'(H) \to a'i'(H)$ is an isomorphism by the above result and (2.5.4), the counit $\tilde{\varepsilon}$ is a natural equivalence.

Next, we show that u is cocontinuous. For $X \in \mathrm{Ob}\,\mathcal{C}$ and $R \in J'(u(X))$, let $\iota : R \to h'_{u(X)} = u_!(h_X)$ be the inclusion morphism. Obviously, $u^*(\iota) : u^*(R) \to u^*(h'_{u(X)}) = u^*u_!(h_X)$ is regarded as a inclusion morphism. We have to show that the pull-back $\iota^u : R_u \to h_X$ of $u^*(\iota)$ along the unit $\eta_{h_X} : h_X \to u^*u_!(h_X)$ is a covering sieve on X for the induced topology on $\mathcal{C}$. By (2.12.2), it suffices show that, for any morphism $f : Y \to X$, the morphism $u_!(h_f^{-1}(R^u)) \to u_!(h_Y)$ induced by the inclusion morphism $h_f^{-1}(R^u) \to h_Y$ is a bicovering in $\widehat{\mathcal{C}'}$. Let $\iota_f : R_f \to h'_{u(X)} = u_!(h_X)$ be the pull-back of ι along $h'_{u(f)} = u_!(h_f)$. Then, R_f is a covering sieve on Y. Since u^* preserves limits, $u^*(\iota_f) : u^*(R_f) \to u^*u_!(h_Y)$ is a pull-back of $u^*(\iota)$ along $u^*u_!(h_f)$. Let $\iota^u_f : R^u_f \to h_Y$ be the pull-back of $u^*(\iota_f)$ along $\eta_{h_Y} : h_Y \to u^*u_!(h_Y)$. Then, it follows from (A.3.1) that ι^u_f is a pull-back of ι^u along h_f, hence the inclusion morphism $h_f^{-1}(R^u) \to h_Y$ is identified with ι^u_f. Therefore it suffices to show that $u_!(\iota^u) : u_!(R_u) \to u_!(h_X)$ is a bicovering. Let $\zeta : R^u \to u^*(R)$ be the pull-back of η_{h_X} along $u^*(\iota)$. Since u is fully faithful, the unit $\eta_{h_X} : h_X \to u^*u_!(h_X)$ is an isomorphism by (A.6.13), which is given by $\eta_{h_X}(\alpha) = u(\alpha)$. Thus ζ is an isomorphism. Taking the adjoints of both sides of $\eta_{h_X}\iota^u = u^*(\iota)\zeta$, we have $u_!(\iota^u) = \iota\varepsilon_R u^*(\zeta)$. Note that $\iota : R \to h'_{u(X)}$ and $\varepsilon_R : u_!u^*(R) \to R$ are bicoverings. In fact, since $R \in J'(u(X))$, the former is a bicovering by (2.5.1) and (2.5.4). The latter is so as we have seen. Hence $u_!(\iota^u)$ is a bicovering.

Since u is continuous and fully faithful, it follows from (2.11.13) that $\tilde{u}_! : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ is fully faithful. Thus the unit $\tilde{\eta} : id_{\widetilde{\mathcal{C}}} \to \tilde{u}^*\tilde{u}_!$ is a natural equivalence as well as the counit $\tilde{\varepsilon}$. Therefore $\tilde{u}^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ is an equivalence of categories with a quasi-inverse $\tilde{u}_!$.

ii) $\Rightarrow$ *i)*: By (2.4.16), $\{\epsilon_J(X) |\, X \in \mathrm{Ob}\,\mathcal{C}\}$ is a generator of $\widetilde{\mathcal{C}}$. For any object X of $\mathcal{C}'$, h'_X is a sheaf on $\mathcal{C}'$ by the assumption, hence $(\epsilon_J P\langle Y, f\rangle \xrightarrow{f} \tilde{u}^*(h'_X))_{\langle Y,f\rangle\in\mathrm{Ob}(\epsilon_J\downarrow\tilde{u}^*(h'_X))}$ is an epimorphic family in $\widetilde{\mathcal{C}}$. Since $\tilde{u}^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ is an equivalence and $\tilde{u}_! : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ is a left adjoint of $\tilde{u}^*$, $\tilde{u}_!$ is a quasi-inverse of $\tilde{u}^*$. Thus $(\tilde{u}_!\epsilon_J P\langle Y, f\rangle \xrightarrow{\tilde{u}_!(f)} \tilde{u}_!\tilde{u}^*(h'_X))_{\langle Y,f\rangle\in\mathrm{Ob}(\epsilon_J\downarrow\tilde{u}^*(h'_X))}$ is an epimorphic family in $\widetilde{\mathcal{C}'}$. The counit $\tilde{\varepsilon} : \tilde{u}_!\tilde{u}^* \to id_{\widetilde{\mathcal{C}'}}$ is a natural equivalence and recall that there is a natural equivalence $\xi : \epsilon_{J'}u \to \tilde{u}_!\epsilon_J$ (2.11.2). Then, we have an epimorphic family $(\epsilon_{J'}uP\langle Y, f\rangle \xrightarrow{\tilde{\varepsilon}_{h'_X}\tilde{u}_!(f)\xi_Y} h'_X)_{\langle Y,f\rangle\in\mathrm{Ob}(\epsilon_J\downarrow\tilde{u}^*(h'_X))}$ in $\widetilde{\mathcal{C}'}$. Since $h'_X = \epsilon_{J'}(X)$ and $\epsilon_{J'}$ is fully faithful, there is a morphism $f' : u(Y) \to X$ such that $\epsilon_J(f') = \tilde{\varepsilon}_{h'_X}\tilde{u}_!(f)\xi_Y$ for each $\langle Y, f\rangle \in \mathrm{Ob}(\epsilon_J\downarrow\tilde{u}^*(h'_X))$. It follows from (2.4.7) that $(f' : u(Y) \to X)_{\langle Y,f\rangle\in\mathrm{Ob}(\epsilon_J\downarrow\tilde{u}^*(h'_X))}$ is a covering of X. □

Proposition 2.12.11 *Let $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ be $\mathcal{U}$-sites and $u : \mathcal{C} \to \mathcal{C}'$ a functor.*

1) The condition i) in (2.11.2) implies iv) even if the left adjoint of u^* does not exist.

2) If u is $\mathcal{U}$-continuous, the functor $\tilde{u}_! : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ satisfying the condition iv) in (2.11.2) is a left adjoint of $\tilde{u}^*$.

3) If $\mathcal{C}$ has a finite limits and u is left exact and $\mathcal{U}$-continuous, the left adjoint $\tilde{u}_! : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ of $\tilde{u}^*$ is left exact.

Proof. 1) Let $\mathcal{G}$ be a full subcategory of $\mathcal{C}$ such that $\mathrm{Ob}\,\mathcal{G}$ is a $\mathcal{U}$-small topologically generating set for J. We denote by $j : \mathcal{G} \to \mathcal{C}$ the inclusion functor and give $\mathcal{G}$ the topology $J_{\mathcal{G}}$ induced by j. It follows from (2.11.2) that $\widetilde{(uj)}^* = \tilde{j}^*\tilde{u}^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{G}}$ has a left adjoint $\widetilde{(uj)}_! : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{C}'}$. Since $\tilde{j}^* : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{G}}$ is an equivalence of categories with quasi-inverse $\tilde{j}_!$ by (2.12.10), $\widetilde{(uj)}_!\tilde{j}^*$ is a left adjoint of $\tilde{u}^*$. In fact, we have a chain of natural bijections $\widetilde{\mathcal{C}'}(\widetilde{(uj)}_!\tilde{j}^*(F), G) \cong \widetilde{\mathcal{G}}(\tilde{j}^*(F), \widetilde{(uj)}^*(G)) \cong \widetilde{\mathcal{C}}(\tilde{j}_!\tilde{j}^*(F), \tilde{j}_!\tilde{j}^*\tilde{u}^*(G)) \cong \widetilde{\mathcal{C}}(F, \tilde{u}^*(G))$ for $F \in \mathrm{Ob}\,\widetilde{\mathcal{C}}$ and $G \in \mathrm{Ob}\,\widetilde{\mathcal{C}'}$. We set $\tilde{u}_! = \widetilde{(uj)}_!\tilde{j}^*$. Note that, since $\tilde{u}_!$ has a right adjoint $\tilde{u}^*$, it preserves colimits. For any $X \in \mathrm{Ob}\,\mathcal{C}$ and $G \in \mathrm{Ob}\,\widetilde{\mathcal{C}'}$, we have the following bijections which are natural in X and G. $\widetilde{\mathcal{C}'}(\epsilon_{J'}u(X), G) \cong \widehat{\mathcal{C}'}(h'_{u(X)}, i'(G)) \cong G(u(X)) = \tilde{u}^*(G)(X) \cong \widehat{\mathcal{C}}(h_X, i\tilde{u}^*(G)) \cong \widetilde{\mathcal{C}}(\epsilon_J(X), \tilde{u}^*(G)) \cong \widetilde{\mathcal{C}'}(\tilde{u}_!\epsilon_J(X), G)$. Hence $\tilde{u}_!\epsilon_J$ is naturally equivalent to $\epsilon_{J'}u$.

2) By (2.4.3), a functor $v : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ satisfying the condition *iv)* of (2.11.2) is uniquely determined up to natural equivalence. Since the functor $\tilde{u}_!$ considered above is a left adjoint of $\tilde{u}^*$ and satisfies the condition *iv)* of (2.11.2), the assertion follows.

3) By (2.10.8), we can choose a full subcategory $\mathcal{G}$ of $\mathcal{C}$ such that $\mathrm{Ob}\,\mathcal{G}$ is a $\mathcal{U}$-small topologically generating set for J and that $\mathcal{G}$ is closed under finite limits in $\mathcal{C}$. Then, the inclusion functor $j : \mathcal{G} \to \mathcal{C}$ is left exact and so is $uj : \mathcal{G} \to \mathcal{C}'$. It follows from (A.6.12) that the left adjoint $(uj)_! : \widehat{\mathcal{G}} \to \widehat{\mathcal{C}'}$ of $(uj)^*$ is left exact. Hence $\widetilde{(uj)}_!$ is also left exact by (2.11.3). Since $\tilde{j}^*$ is an equivalence, it is left exact. Thus $\tilde{u}_! = \widetilde{(uj)}_!\tilde{j}^*$ is left exact. □

Proposition 2.12.12 *Let $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ be $\mathcal{U}$-sites and $u : \mathcal{C} \to \mathcal{C}'$ a cocontinuous functor. Define $\bar{u}^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ to be the composition $\widetilde{\mathcal{C}'} \xrightarrow{i'} \widehat{\mathcal{C}'} \xrightarrow{u^*} \widehat{\mathcal{C}} \xrightarrow{a} \widetilde{\mathcal{C}}$.*

1) $\bar{u}^*$ has a right adjoint.

2) If $\mathcal{V}$ is a universe such that $\mathcal{U} \subset \mathcal{V}$, we denote by $\tilde{u}_{\mathcal{U}*} : \widetilde{\mathcal{C}}_{\mathcal{U}} \to \widetilde{\mathcal{C}'}_{\mathcal{U}}$ (resp. $\tilde{u}_{\mathcal{V}*} : \widetilde{\mathcal{C}}_{\mathcal{V}} \to \widetilde{\mathcal{C}'}_{\mathcal{V}}$) the right adjoint of $\bar{u}^*_{\mathcal{U}} : \widetilde{\mathcal{C}'}_{\mathcal{U}} \to \widetilde{\mathcal{C}}_{\mathcal{U}}$ (resp. $\bar{u}^*_{\mathcal{V}} : \widetilde{\mathcal{C}'}_{\mathcal{V}} \to \widetilde{\mathcal{C}}_{\mathcal{V}}$). Then, the following diagram commutes up to natural equivalence.

$$\begin{array}{ccc} \widetilde{\mathcal{C}}_{\mathcal{U}} & \xrightarrow{\tilde{u}_{\mathcal{U}*}} & \widetilde{\mathcal{C}'}_{\mathcal{U}} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{C}}_{\mathcal{V}} & \xrightarrow{\tilde{u}_{\mathcal{V}*}} & \widetilde{\mathcal{C}'}_{\mathcal{V}} \end{array}$$

Proof. 1) We take a $\mathcal{U}$-small full subcategory $\mathcal{G}$ of $\mathcal{C}$ such that $\mathrm{Ob}\,\mathcal{G}$ is a topologically generating family for J. We denote by $j : \mathcal{G} \to \mathcal{C}$ the inclusion functor. By the proof of (2.12.10), j is cocontinuous. Hence $uj : \mathcal{G} \to \mathcal{C}'$ is cocontinuous by the assumption and (2.11.8). Since $\mathcal{G}$ is $\mathcal{U}$-small, the right adjoint $(uj)_* : \widehat{\mathcal{G}} \to \widehat{\mathcal{C}'}$ of $(uj)^*$ exists. Therefore we can apply (2.11.11) to uj, and $\overline{uj}^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{G}}$ has a right adjoint $\widetilde{uj}_*$. Note that $\tilde{j}^* : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{G}}$ is an equivalence by (2.12.10). We set $\tilde{u}_* = \widetilde{uj}_*\tilde{j}^* : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$. For $F \in \mathrm{Ob}\,\widetilde{\mathcal{C}'}$ and $G \in \mathrm{Ob}\,\widetilde{\mathcal{C}}$, by (2.11.11), $\widetilde{\mathcal{C}'}(F, \tilde{u}_*(G)) = \widetilde{\mathcal{C}'}(F, \widetilde{uj}_*\tilde{j}^*G)) \cong \widetilde{\mathcal{G}}(\overline{uj}^*(F), \tilde{j}^*(G)) = \widetilde{\mathcal{G}}(a''j^*u^*i'(F), \tilde{j}^*(G)) \cong \widetilde{\mathcal{G}}(\tilde{j}^*au^*i'(F), \tilde{j}^*(G)) \cong \widetilde{\mathcal{C}}(\bar{u}^*(F), G)$. Therefore $\tilde{u}_* : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ is a right adjoint of $\bar{u}^*$.

2) We first consider the case $\tilde{u}_{\mathcal{U}*} = \widetilde{uj}_{\mathcal{U}*}\tilde{j}^*$, $\tilde{u}_{\mathcal{V}*} = \widetilde{uj}_{\mathcal{V}*}\tilde{j}^*$. By (2.11.11), the right square of the following diagram commutes up to a natural equivalence and the left one obviously commutes.

$$\begin{array}{ccccc} \widetilde{\mathcal{C}}_{\mathcal{U}} & \xrightarrow{\tilde{j}^*} & \widetilde{\mathcal{G}}_{\mathcal{U}} & \xrightarrow{\widetilde{uj}_{\mathcal{U}*}} & \widetilde{\mathcal{C}'}_{\mathcal{U}} \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{\mathcal{C}}_{\mathcal{V}} & \xrightarrow{\tilde{j}^*} & \widetilde{\mathcal{G}}_{\mathcal{V}} & \xrightarrow{\widetilde{uj}_{\mathcal{V}*}} & \widetilde{\mathcal{C}'}_{\mathcal{V}} \end{array}$$

Hence the assertion holds in this case. Since $\tilde{u}_{\mathcal{U}*}$ (resp. $\tilde{u}_{\mathcal{V}*}$) is a right adjoint of a functor $\bar{u}^*_{\mathcal{U}}$ (resp. $\bar{u}^*_{\mathcal{V}}$) which is independent of the choice of $\mathcal{G}$, $\tilde{u}_{\mathcal{U}*}$ (resp. $\tilde{u}_{\mathcal{V}*}$) is uniquely determined up to natural equivalence. Therefore the assertion holds generally. □

Proposition 2.12.13 *Let $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ be $\mathcal{U}$-sites and $u : \mathcal{C} \to \mathcal{C}'$ a $\mathcal{U}$-continuous and cocontinuous functor.*

1) The functor $\tilde{u}^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ has both left and right adjoints.

2) We denote by $\tilde{u}_!$ and $\tilde{u}_*$ the left and right adjoint of $\tilde{u}^*$, respectively. $\tilde{u}_!$ is fully faithful if and only if so is $\tilde{u}_*$.

3) If u is fully faithful, so is $\tilde{u}_!$. If J and J' are coarser than the canonical topology, the converse holds.

Proof. 1) follows from (2.12.11) and (2.12.12).

2) is a general property of adjoint functors. (See the proof of (A.6.10).)

3) Let $\mathcal{V}$ be a universe containing $\mathcal{U}$ such that $\mathcal{C}$ is $\mathcal{V}$-small. Then, the right adjoint $u_{\mathcal{V}*} : \widehat{\mathcal{C}}_{\mathcal{V}} \to \widehat{\mathcal{C}'}_{\mathcal{V}}$ exists. If u is fully faithful, so is $u_{\mathcal{V}*}$ by (A.6.13). By the commutativity of the diagram of 2) of (2.11.11), we see that $\tilde{u}_{\mathcal{V}*}$ is fully faithful. Hence by the commutativity of the diagram of 2) of (2.12.12), $\tilde{u}_* = \tilde{u}_{\mathcal{U}*}$ is fully faithful. It follows from above 2) that $\tilde{u}_!$ is fully faithful. Since the functors ϵ_J and $\epsilon_{J'}$ are fully faithful if J and J' are coarser than the canonical topology, the converse assertion follows from the commutativity of the diagram of *iv*) of (2.11.2) by (2.12.11). □

Proposition 2.12.14 *Let $\mathcal{E}$ be a $\mathcal{U}$-topos and $\mathcal{C}$ a $\mathcal{U}$-small full subcategory of $\mathcal{E}$ with the inclusion functor $K : \mathcal{C} \to \mathcal{E}$. We give $\mathcal{E}$ the canonical topology and $\mathcal{C}$ the topology induced by K. Then the functor $R : \mathcal{E} \to \widehat{\mathcal{C}}$ defined by $R(X) = h^{\mathcal{E}}_X K$ takes values in $\widetilde{\mathcal{C}}$. We denote by $\widetilde{R} : \mathcal{E} \to \widetilde{\mathcal{C}}$ the functor such that $R = i\widetilde{R}$. $\widetilde{R}$ is an equivalence if and only if $\mathcal{C}$ is a generating subcategory of $\mathcal{E}$.*

Proof. Since $K : \mathcal{C} \to \mathcal{E}$ is continuous, we have a functor $\widetilde{K}^* : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{C}}$ induced by $K^* : \widehat{\mathcal{E}} \to \widehat{\mathcal{C}}$. There is an equivalence $\tilde{h} : \mathcal{E} \to \widetilde{\mathcal{E}}$ given in (2.10.11) and the following diagram commutes.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\tilde{h}"] \arrow[dr, "h^{\mathcal{E}}"'] & \widetilde{\mathcal{E}} \arrow[r, "\widetilde{K}^*"] \arrow[d, "i"] & \widetilde{\mathcal{C}} \arrow[d, "i"] \\
& \widehat{\mathcal{E}} \arrow[r, "K^*"] & \widehat{\mathcal{C}}
\end{tikzcd}$$

Hence $R = K^* h^{\mathcal{E}}$ factors through the inclusion morphism $i : \widetilde{\mathcal{C}} \to \widehat{\mathcal{C}}$.

Suppose that $\mathcal{C}$ is a generating subcategory of $\mathcal{E}$. By (2.10.10), $\mathrm{Ob}\,\mathcal{C}$ is a topologically generating family for the canonical topology on $\mathcal{E}$ is a $\mathcal{U}$-topology. Hence it follows from (2.12.10) that $\widetilde{K}^*$ is an equivalence. Then, by (2.10.11), $\widetilde{R} = \widetilde{K}^* \tilde{h} : \mathcal{E} \to \widetilde{\mathcal{C}}$ is an equivalence.

Conversely, suppose that $\widetilde{R}$ is an equivalence. Then, $\widetilde{K}^*$ is an equivalence. Since $\mathcal{E}$ has an $\mathcal{U}$-small generating family, the canonical topology on $\mathcal{E}$ is a $\mathcal{U}$-topology by (2.10.10). Thus $\mathrm{Ob}\,\mathcal{C}$ is a topologically generating family for the canonical topology on $\mathcal{E}$ by (2.12.10). Generally, a topologically generating family for the canonical topology on a category is nothing but a generating family by universal strict epimorphisms. Hence the assertion follows from (A.4.10). □

Proposition 2.12.15 *Let $\mathcal{E}$ be a $\mathcal{U}$-topos and $\mathcal{C}$ a category (not necessarily a $\mathcal{U}$-category). A presheaf F on $\mathcal{E}$ taking values in $\mathcal{C}$ is a sheaf for the canonical topology if and only if $F : \mathcal{E}^{op} \to \mathcal{C}$ preserves $\mathcal{U}$-limits.*

Proof. Let $\mathcal{V}$ be a universe such that $\mathcal{U} \subset \mathcal{V}$ and $\mathcal{C}$ is a $\mathcal{V}$-category.

Suppose that a presheaf $F : \mathcal{E}^{op} \to \mathcal{C}$ preserves $\mathcal{U}$-limits. Recall that F is a sheaf taking values in $\mathcal{C}$ if and only if, for any $Y \in \mathrm{Ob}\,\mathcal{C}$, a presheaf given by $X \mapsto \mathcal{C}(Y, F(X))$ is a sheaf. Since the functor $\mathcal{C} \to \mathcal{V}$-**Ens** given by $Z \mapsto \mathcal{C}(Y, Z)$ preserves limits, the above presheaf $\mathcal{E}^{op} \to \mathcal{V}$-**Ens** preserves limits. Hence we may assume that $\mathcal{C} = \mathcal{V}$-**Ens**. Since $\mathcal{E}$ is a $\mathcal{U}$-topos, there exists a $\mathcal{U}$-small generating family G, which is also a topologically generating family for the canonical topology J on $\mathcal{E}$. We choose a $\mathcal{U}$-small family of morphisms $(f_i : X_i \to X)_{i \in I}$ generating R such that $X_i \in G$. Since $(f_i : X_i \to X)_{i \in I}$ is an effective epimorphic family, it is a colimiting cone of a diagram $(X_i \leftarrow X_i \times_X X_j \to X_j)_{i,j \in I}$. The assumption implies that $(F(X) \xrightarrow{F(f_i)} F(X_i))_{i \in I}$ is a limiting cone of $(F(X_i) \to F(X_i \times_X X_j) \leftarrow F(X_j))_{i,j \in I}$. It follows from (2.2.2) that $\widehat{\mathcal{E}}(h_X, F) \to \widehat{\mathcal{E}}(R, F)$ is bijective. Hence F is a sheaf by (2.3.6).

Conversely, suppose that $F : \mathcal{E}^{op} \to \mathcal{C}$ is a sheaf. Let $\mathcal{D}$ be a $\mathcal{U}$-small category and $D : \mathcal{D} \to \mathcal{E}$ a functor with a colimiting cone $(D(i) \xrightarrow{f_i} L)_{i \in \mathrm{Ob}\,\mathcal{D}}$. Then, $(F(L) \xrightarrow{F(f_i)} FD(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of $FD : \mathcal{D}^{op} \to \mathcal{C}$ if and only if, for any $Y \in \mathrm{Ob}\,\mathcal{C}$, $(\mathcal{C}(Y, F(L)) \xrightarrow{F(f_i)_*} \mathcal{C}(Y, FD(i)))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone in $\mathcal{V}$-**Ens**. Since a presheaf given by $X \mapsto \mathcal{C}(Y, F(X))$ is a sheaf, we may assume that $\mathcal{C} = \mathcal{V}$-**Ens** again. If $\mathcal{V} = \mathcal{U}$, it follows from (2.10.11) that $F = h_Z$ for some $Z \in \mathrm{Ob}\,\mathcal{E}$. Then, $(\mathcal{E}(L, Z) \xrightarrow{f_i^*} \mathcal{E}(D(i), Z))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of $h_Z D : \mathcal{D}^{op} \to \mathcal{V}$-**Ens**. Hence $F : \mathcal{E}^{op} \to \mathcal{V}$-**Ens** preserves limits. In the general case, let $\mathcal{G}$ be a $\mathcal{U}$-small generating subcategory of $\mathcal{E}$. We give $\mathcal{E}$ the canonical topology and $\mathcal{G}$ the topology induced by the inclusion functor $K : \mathcal{G} \to \mathcal{E}$. We denote by $\widetilde{\mathcal{G}}_{\mathcal{U}}$ (resp. $\widetilde{\mathcal{G}}_{\mathcal{V}}$) the category of presheaves of $\mathcal{U}$-sets (resp. $\mathcal{V}$-sets) on $\mathcal{G}$ and by $\widetilde{\mathcal{E}}_{\mathcal{V}}$ (resp. $\widehat{\mathcal{E}}_{\mathcal{V}}$) the category of sheaves (resp. presheaves) of $\mathcal{V}$-sets on $\mathcal{E}$. Then, by (2.12.14) and (2.12.10), $\widetilde{R} : \mathcal{E} \to \widetilde{\mathcal{G}}_{\mathcal{U}}$ and $\widetilde{K}^* : \widetilde{\mathcal{E}}_{\mathcal{V}} \to \widetilde{\mathcal{G}}_{\mathcal{V}}$ are equivalences. Let $\iota_{\mathcal{U}\mathcal{V}} : \widetilde{\mathcal{G}}_{\mathcal{U}} \to \widetilde{\mathcal{G}}_{\mathcal{V}}$ be the inclusion functor. By the construction of the associated sheaf functor and (2.4.1), $\iota_{\mathcal{U}\mathcal{V}}$ preserves colimits. Let $\tilde{h}_{\mathcal{V}} : \mathcal{E} \to \widetilde{\mathcal{E}}_{\mathcal{V}}$ be the composition of the equivalence $\tilde{h} : \mathcal{E} \to \widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_{\mathcal{U}}$ in (2.10.11) and the inclusion functor $\widetilde{\mathcal{E}}_{\mathcal{U}} \to \widetilde{\mathcal{E}}_{\mathcal{V}}$. Then, we have $\widetilde{K}^* \tilde{h}_{\mathcal{V}} = \iota_{\mathcal{U}\mathcal{V}} \widetilde{R}$. Since $\iota_{\mathcal{U}\mathcal{V}}$ preserves $\mathcal{U}$-colimits and $\widetilde{R}$, $\widetilde{K}^*$ are equivalences, $\tilde{h}_{\mathcal{V}}$ preserves $\mathcal{U}$-colimits. Since F and a representable functor on $\mathcal{E}$ are objects of $\widetilde{\mathcal{E}}_{\mathcal{V}}$, there are isomorphisms $F(X) \cong \widehat{\mathcal{E}}_{\mathcal{V}}(h_X, iF) \cong \widetilde{\mathcal{E}}_{\mathcal{V}}(\tilde{h}_{\mathcal{V}}(X), F)$ which are natural in $X \in \mathrm{Ob}\,\mathcal{E}$, where $i : \widetilde{\mathcal{E}}_{\mathcal{V}} \to \widehat{\mathcal{E}}_{\mathcal{V}}$ is the inclusion functor. We note that $X \mapsto \widetilde{\mathcal{E}}_{\mathcal{V}}(\tilde{h}_{\mathcal{V}}(X), F)$ is the composition of $\tilde{h}_{\mathcal{V}}$ and the functor $\widetilde{\mathcal{E}}_{\mathcal{V}}^{op} \to \mathcal{V}$-**Ens** represented by F, which preserves limits. Thus $F : \mathcal{E}^{op} \to \mathcal{V}$-**Ens** preserves limits. □

Corollary 2.12.16 *Let $\mathcal{E}$ be a $\mathcal{U}$-topos and $\mathcal{C}$ a $\mathcal{U}$-category. A functor $f : \mathcal{E} \to \mathcal{C}$ has a right adjoint if and only if f preserves $\mathcal{U}$-colimits.*

Proof. Suppose that f preserves $\mathcal{U}$-colimits. We regard f as a presheaf on $\mathcal{E}$ taking values in $\mathcal{C}^{op}$. It follows from (2.12.15) that f is a sheaf for the canonical topology on $\mathcal{E}$. Then, for any object Y of $\mathcal{C}$, a presheaf $X \mapsto \mathcal{C}^{op}(Y, f(X))$ of $\mathcal{U}$-sets on $\mathcal{E}$ is a sheaf. By (2.10.11), it is represented by an object R_Y, that is, there is a bijection $\mathcal{C}(f(X), Y) = \mathcal{C}^{op}(Y, f(X)) \to \mathcal{E}(X, R_Y)$ which is natural in X. Hence the correspondence $Y \mapsto R_Y$ defines a right adjoint of f. The converse follows from (A.3.13). □

Corollary 2.12.17 *Let $\mathcal{E}$, $\mathcal{F}$ be $\mathcal{U}$-topoi and $f : \mathcal{E} \to \mathcal{F}$ a functor. Then the following conditions are equivalent.*

i) f preserves $\mathcal{U}$-colimits.
ii) f has a right adjoint.
iii) f is continuous for the canonical topologies on $\mathcal{E}$, $\mathcal{F}$.

Proof. The equivalence of $i)$ and $ii)$ follows from (2.12.16). Choose a universe $\mathcal{V}$ such that $\mathcal{U} \in \mathcal{V}$. Then, $\mathcal{E}$ is $\mathcal{V}$-small. Let F be a sheaf of $\mathcal{V}$-sets on $\mathcal{F}$. By (2.12.15), $F : \mathcal{F}^{op} \to \mathcal{V}$-**Ens** preserves $\mathcal{U}$-limits. If we assume $i)$, $Ff : \mathcal{E}^{op} \to \mathcal{V}$-**Ens** preserves $\mathcal{U}$-limits. Hence Ff is a sheaf and $iii)$ follows. Conversely, assume that f is continuous. For any $Y \in \operatorname{Ob}\mathcal{F}$, $h_Y f : \mathcal{E}^{op} \to \mathcal{U}$-**Ens** is a sheaf. Hence it preserves $\mathcal{U}$-limits by (2.12.10). Note that $(D(i) \xrightarrow{\lambda_i} L)_{i\in \operatorname{Ob}\mathcal{D}}$ is a colimiting cone of a functor $D : \mathcal{D} \to \mathcal{F}$ if and only if, for any $Y \in \operatorname{Ob}\mathcal{F}$, $(\mathcal{F}(L, Y) \xrightarrow{\lambda_i^*} \mathcal{F}(D(i), Y))_{i\in \operatorname{Ob}\mathcal{D}}$ is a limiting cone of $h_Y D : \mathcal{D}^{op} \to \mathcal{U}$-**Ens**. Thus f preserves $\mathcal{U}$-colimits. □

Corollary 2.12.18 *Let $\mathcal{E}$, $\mathcal{F}$ be $\mathcal{U}$-topoi and $f : \mathcal{E} \to \mathcal{F}$ a left exact functor. Then, the following conditions are equivalent.*
(i) f has a right adjoint.
(ii) f preserves epimorphisms and $\mathcal{U}$-small coproducts.
(iii) f preserves epimorphic families indexed by a $\mathcal{U}$-small sets.

Proof. Recall from (2.10.6) that $\mathcal{E}$ has a $\mathcal{U}$-small topologically generating subcategory for the canonical topology (namely, a generating subcategory). If f preserves epimorphic families indexed by a $\mathcal{U}$-small sets, f is continuous by (2.11.6). Hence f has a right adjoint by (2.12.17). $(i) \Rightarrow (ii)$ follows from (A.3.13). Since $\mathcal{E}$ has $\mathcal{U}$-small coproducts, (ii) implies (iii). The above result shows that a functor between $\mathcal{U}$-topoi is the inverse image of a geomtric morphism (see §15) if and only if it is left exact and preserves epimorphic families indexed by $\mathcal{U}$-small sets. □

2.13 Localization

Let $(\mathcal{C}, J)$ be a site and X an object of $\widehat{\mathcal{C}}$. We give $(h{\downarrow}X)$ the topology J_X induced by the projection functor $P_X : (h{\downarrow}X) \to \mathcal{C}$. In (A.6.15), we explicitly constructed a left Kan extension $P_{X!} : \widehat{(h{\downarrow}X)} \to \widehat{\mathcal{C}}$ of P_X along the Yoneda embedding $h' : (h{\downarrow}X) \to \widehat{(h{\downarrow}X)}$ and an equivalence $e_X : \widehat{(h{\downarrow}X)} \to \widehat{\mathcal{C}}/X$ such that $P_{X!} = \Sigma_X e_X$ holds. We showed in (A.6.17) that $P_{X!}$ preserves monomorphic families, pull-backs and $\mathcal{U}$-colimits.

For $Z \in \operatorname{Ob}\mathcal{C}$ and $\langle Z, g\rangle \in \operatorname{Ob}(h{\downarrow}X)$, we denote by S_Z, $S_{\langle Z,g\rangle}$ the set of sieves on Z, $\langle Z, g\rangle$, respectively. Let $\bar{\chi}^X_{\langle Z,g\rangle} : h'_{\langle Z,g\rangle} \to P^*_X(h_Z)$ be the adjoint of the morphism $\chi^X_{\langle Z,g\rangle} : P_{X!}(h'_{\langle Z,g\rangle}) \to h_Z$ defined in the proof of (A.6.14). Since the adjoint of $\chi^X_{\langle Z,g\rangle} : P_{X!}(h'_{\langle Z,g\rangle}) \to h_Z$ is a composition $h'_{\langle Z,g\rangle} \xrightarrow{\eta^X_{h'_{\langle Z,g\rangle}}} P^*_X P_{X!}(h'_{\langle Z,g\rangle}) \xrightarrow{P^*_X(\chi^X_{\langle Z,g\rangle})} P^*_X(h_Z)$, it can be verified that $\bar{\chi}^X_{\langle Z,g\rangle}$ is given by $(\bar{\chi}^X_{\langle Z,g\rangle})_{\langle W,k\rangle}(\alpha) = P_X(\alpha)$ ($\alpha \in h'_{\langle Z,g\rangle}\langle W, k\rangle$).

Define a map $\Phi : S_Z \to S_{\langle Z,g\rangle}$ as follows. For $R \in S_Z$, we denote by $\iota : R \to h_Z$ the inclusion morphism. Let $\bar{\iota} : \Phi(R) \to h'_{\langle Z,g\rangle}$ be the pull-back of $P^*_X(\iota) : P^*_X(R) \to P^*_X(h_X)$ along $\bar{\chi}^X_{\langle Z,g\rangle}$. Since $P^*_X(\iota)$ is regarded as an inclusion morphism, so is $\bar{\iota}$.

Proposition 2.13.1 *Let $\langle Z, g\rangle$ be an object of $(h{\downarrow}X)$.*

1) Φ is an order preserving bijection. The inverse Φ^{-1} is given as follows. For $T \in S_{\langle Z,g\rangle}$, let $\sigma : T \to h'_{\langle Z,g\rangle}$ be the inclusion morphism. $\Phi^{-1}(T)$ is defined to be the image of the composition $P_{X!}(T) \xrightarrow{P_{X!}(\sigma)} P_{X!}(h'_{\langle Z,g\rangle}) \xrightarrow{\chi^X_{\langle Z,g\rangle}} h_Z$.

2) If a sieve R on Z is generated by $(g_i : Z_i \to Z)_{i\in I}$, $\Phi(R)$ is generated by $(g_i : \langle Z_i, fg_i\rangle \to \langle Z, g\rangle)_{i\in I}$.

Proof. 1) For $R \in S_Z$ and $T \in S_{\langle Z,g\rangle}$, we have the following equalities.

$$\Phi(R)\langle W, k\rangle = \{\alpha \in (h{\downarrow}X)(\langle W, k\rangle, \langle Z, g\rangle) |\, P_X(\alpha) \in R(W)\}$$
$$\Phi^{-1}(T)(W) = \{\alpha \in \mathcal{C}(W, Z) |\, \bar{\alpha} \in T\langle W, fh_\alpha\rangle\}$$

Then, it is easy to verify $\Phi^{-1}\Phi = id_{S_Z}$ and $\Phi\Phi^{-1} = id_{S_{\langle Z,g\rangle}}$.

2) Obviously, $g_i : \langle Z_i, fg_i\rangle \to \langle Z, g\rangle$ belongs to $\Phi(R)$. If $(\alpha : \langle W, k\rangle \to \langle Z, g\rangle) \in \Phi(R)$, $P_X(\alpha) \in R(W)$ hence $P_X(\alpha) = g_i u$ for some $i \in I$ and $u \in \mathcal{C}(W, Z_i)$. Since $k = fP_X(\alpha)$, u defines a morphism $u : \langle W, k\rangle \to \langle Z_i, g_i\rangle$ in $(h{\downarrow}X)$. Therefore $\alpha = g_i u$ in $(h{\downarrow}X)$. $\square$

Proposition 2.13.2 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site and X an object of $\widehat{\mathcal{C}}$.*

1) A sieve T on an object $\langle Z, g\rangle$ of $(h{\downarrow}X)$ is a covering for J_X if and only if the image of the composition $P_{X!}(T) \xrightarrow{P_{X!}(\iota)} P_{X!}(h'_{\langle Z,g\rangle}) \xrightarrow{\chi^X_{\langle Z,g\rangle}} h_Z$ is a covering for J, where $\iota : T \to h'_{\langle Z,g\rangle}$ is the inclusion morphism. Hence the bijection $\Phi : S_Z \to S_{\langle Z,g\rangle}$ induces a bijection $J(Z) \to J_X(\langle Z, g\rangle)$.

2) The projection functor $P_X : (h{\downarrow}X) \to \mathcal{C}$ is continuous and cocontinuous.

3) J_X is a $\mathcal{U}$-topology.

4) Let $\alpha : Y \to X$ be a morphism in $\widehat{\mathcal{C}}$ and set $[\alpha] = e_X^{-1}(Y \xrightarrow{\alpha} X)$. Then J_Y coincides with the topology induced by $P_\alpha = (h{\downarrow}\alpha) : (h{\downarrow}Y) \to (h{\downarrow}X)$. Moreover the topology $J_{[\alpha]}$ on $(h'{\downarrow}[\alpha])$ induced by the projection functor $P_{[\alpha]} : (h'{\downarrow}[\alpha]) \to (h{\downarrow}X)$ coincides with the topology induced by the isomorphism $Q_\alpha : (h'{\downarrow}[\alpha]) \to (h{\downarrow}Y)$ given in (A.6.18).

Proof. 1) Suppose that T is a covering sieve on $\langle Z, g\rangle$. Since P_X is continuous, $P_{X!}(\iota) : P_{X!}(T) \to P_{X!}(h'_{\langle Z,g\rangle})$ is a bicovering by (2.11.1). Moreover, since $P_{X!}(\iota)$ is a monomorphism and $\chi^X_{\langle Z,g\rangle}$ is an isomorphism, it follows from (2.5.1) and (2.5.4) that the image of $\chi^X_{\langle Z,g\rangle} P_{X!}(\iota)$ is a covering sieve. Conversely, suppose that the image of $\chi^X_{\langle Z,g\rangle} P_{X!}(\iota)$ is a covering sieve. Choose a universe $\mathcal{V}$ such that $\mathcal{U} \subset \mathcal{V}$ and $\mathcal{C}$ is $\mathcal{V}$-small. Since the Yoneda embedding $h^{\mathcal{V}} : \mathcal{C} \to \widehat{\mathcal{C}}_{\mathcal{V}}$ factors through the inclusion functor $\widehat{\mathcal{C}} = \widehat{\mathcal{C}}_{\mathcal{U}} \to \widehat{\mathcal{C}}_{\mathcal{V}}$, $(h^{\mathcal{V}}{\downarrow}X)$ is identified with $(h{\downarrow}X)$. Hence, by replacing $\mathcal{U}$ by $\mathcal{V}$ if necessary, we may assume that $\mathcal{C}$ is $\mathcal{U}$-small. Then, $(h{\downarrow}X)$ is also $\mathcal{U}$-small and we can apply (2.12.3) to show that T is a covering sieve on $\langle Z, g\rangle$. Let $\alpha : \langle W, k\rangle \to \langle Z, g\rangle$ be a morphism in $(h{\downarrow}X)$. Since $P_{X!}$ preserves pull-backs, the left square of the following diagram on the right is cartesian.

$$\begin{tikzcd}
{h'}^{-1}_{\alpha}(T) \arrow[r, "\bar\iota"] \arrow[d] & h'_{\langle W,k\rangle} \arrow[d, "h'_\alpha"] \\
T \arrow[r, "\iota"] & h'_{\langle Z,g\rangle}
\end{tikzcd}
\qquad
\begin{tikzcd}
P_{X!}({h'}^{-1}_{\alpha}(T)) \arrow[r, "P_{X!}(\bar\iota)"] \arrow[d] & P_{X!}(h'_{\langle W,k\rangle}) \arrow[r, "\chi^X_W", "\cong"'] \arrow[d, "P_{X!}(h'_\alpha)"] & h_W \arrow[d, "h_{P_X(\alpha)}"] \\
P_{X!}(T) \arrow[r, "P_{X!}(\iota)"] & P_{X!}(h'_{\langle Z,g\rangle}) \arrow[r, "\chi^X_{\langle Z,g\rangle}", "\cong"'] & h_Z
\end{tikzcd}$$

Hence $\chi^X_W P_{X!}(\bar\iota)$ is a monomorphism and its image is a covering sieve on W by the assumption and $(T2)$. Thus $P_{X!}(\bar\iota)$ is a bicovering and it follows from (2.12.3) that T is a covering sieve.

2) It is obvious that P_X is continuous. For $\langle Z, g\rangle \in \mathrm{Ob}\,(h{\downarrow}X)$ and $R \in J(Z)$, let $\iota : R \to h_Z$ be the inclusion morphism. Set $\iota' = (\chi^X_{\langle Z,g\rangle})^{-1}\iota : R \to P_{X!}(h'_{\langle Z,g\rangle})$. By the definition of R^{P_X} (2.11.7),

$$\begin{tikzcd}
R^{P_X} \arrow[r] \arrow[d, "\bar\iota"] & P_X^*(R) \arrow[d, "P_X^*(\iota')"] \\
h'_{\langle Z,g\rangle} \arrow[r, "\eta^X"] & P_X^* P_{X!}(h'_{\langle Z,g\rangle})
\end{tikzcd}$$

is a pull-back in $(h{\downarrow}X)$ and $P_{X!}$ preserves it, the left square of the following diagram is a pull-back.

$$\begin{tikzcd}
P_{X!}(R^{P_X}) \arrow[r] \arrow[d, "P_{X!}(\bar\iota)"] & P_{X!}P_X^*(R) \arrow[r, "\varepsilon^X"] \arrow[d, "P_{X!}P_X^*(\iota')"] & R \arrow[d, "\iota'"] \\
P_{X!}(h'_{\langle Z,g\rangle}) \arrow[r, "P_{X!}(\eta_X)"] & P_{X!}P_X^*P_{X!}(h'_{\langle Z,g\rangle}) \arrow[r, "\varepsilon^X"] & P_{X!}(h'_{\langle Z,g\rangle})
\end{tikzcd}$$

Generally, for $F \in \mathrm{Ob}\,\widehat{\mathcal{C}}$, it follows from (A.6.15) that $P_X^* P_{X!}(F)$ is naturally isomorphic to $F \times X$ and $\varepsilon^X : P_X^* P_{X!}(F) \to F$ is identified with the projection onto the first component. Hence the right square of the above diagram is also a pull-back. Since $\iota : R \to h_Z$ is a covering for J, so is $P_{X!}(\bar\iota) : P_{X!}(R^{P_X}) \to P_{X!}(h'_{\langle Z,g\rangle}) = h_Z$. By 1), we see that $\bar\iota : R^{P_X} \to h'_{\langle Z,g\rangle}$ is a covering for J_X. Therefore, P_X is cocontinuous.

3) Let G be a $\mathcal{U}$-small topologically generating family of $\mathcal{C}$. For any $\langle Y, f\rangle \in \mathrm{Ob}\,(h{\downarrow}X)$, there exists a covering $(g_i : Z_i \to Y)_{i\in I}$ for J such that $Z_i \in G$. Let R be the sieve on Y generated by $(g_i : Z_i \to Y)_{i\in I}$. It follows from 1) that $\Phi(R)$ is a covering sieve on $\langle Y, f\rangle$ and it is generated by $(g_i : \langle Z_i, fg_i\rangle \to \langle Y, f\rangle_{i\in I}$ ((2.13.1)). Hence $G_X = \{\langle Z, g\rangle \in \mathrm{Ob}\,(h{\downarrow}X) |\, Z \in G,\ g \in \widehat{\mathcal{C}}(h_Z, X)\}$ is a $\mathcal{U}$-small topologically generating family of $(h{\downarrow}X)$.

4) We first note that, since $P_{Y!} \cong P_{X!}P_{\alpha!}$ preserves monomorphic families and $P_{X!}$ is faithful (for Σ_X is so.), $P_{\alpha!}$ also preserves monomorphic families.

As in the proof of 1), we may assume that $\mathcal{C}$ is $\mathcal{U}$-small. Let J_α be the topology induced by P_α and $\langle Z, g\rangle$ an object of $(h\downarrow Y)$. By (2.12.3), a sieve T on $\langle Z, g\rangle$ is a covering sieve for J_α if and only if, for any morphism $f : \langle W, k\rangle \to \langle Z, g\rangle$ in $(h\downarrow Y)$, $P_{\alpha!}(\iota_f) : P_{\alpha!}({h'}_f^{-1}(T)) \to P_{\alpha!}(h'_{\langle W,k\rangle})$ is a bicovering, where $\iota_f : {h'}_f^{-1}(T) \to h'_{\langle W,k\rangle}$ is a pull-back of the inclusion morphism $\iota : T \to h'_{\langle Z,g\rangle}$ along $h'_f : h'_{\langle W,k\rangle} \to h'_{\langle Z,g\rangle}$. By 1), $P_{\alpha!}(\iota_f)$ is a bicovering for J_X if and only if $P_{X!}P_{\alpha!}(\iota_f)$ is a bicovering for J, namely, $P_{Y!}(\iota_f)$ is a bicovering for J. Since $P_{Y!}$ preserves pull-backs, T is a covering sieve for J_α if and only if $P_{Y!}(\iota)$ is a bicovering for J. Again by 1), T is a covering sieve for J_α if and only if it is a covering sieve for J_Y. The second assertion follows from $P_{[\alpha]} = P_\alpha Q_\alpha$ and the first assertion. □

Corollary 2.13.3 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site, X a presheaf on $\mathcal{C}$ and $\alpha : Y \to X$ a morphism in $\widehat{\mathcal{C}}$. We give $(h\downarrow X)$ and $(h\downarrow Y)$ the toplogies induced by $P_X : (h\downarrow X) \to \mathcal{C}$, $P_Y : (h\downarrow Y) \to \mathcal{C}$. Then, $P_\alpha = (h\downarrow\alpha) : (h\downarrow Y) \to (h\downarrow X)$ is continuous and cocontinuous.*

Proof. Put $[\alpha] = e_X^{-1}(Y \xrightarrow{\alpha} X) \in \mathrm{Ob}\,\widehat{(h\downarrow X)}$. The projection functor $P_{[\alpha]} : (h'\downarrow[\alpha]) \to (h\downarrow X)$ is continuous and cocontinuous by 2) and 4) of the previous result. Since the following diagram commutes, the assertion follows from 4) above.

$$\begin{tikzcd} (h'\downarrow[\alpha]) \arrow[r, "P_{[\alpha]}"] \arrow[d, "Q_\alpha"] & (h\downarrow X) \\ (h\downarrow Y) \arrow[ur, "P_\alpha"'] & \end{tikzcd}$$

□

Proposition 2.13.4 *If $f : F \to G$ is a morphism in $\widehat{(h\downarrow X)}$ such that $P_{X!}(f) : P_{X!}(F) \to P_{X!}(G)$ is a covering (resp. bicovering), then f is a covering (resp. bicovering).*

Proof. Suppose that $P_{X!}(f)$ is a covering. For a morphism $\alpha : h'_{\langle Z,g\rangle} \to G$, let $\bar{f} : F_f \to h'_{\langle Z,g\rangle}$ be a pull-back of f along α and T the image of $\bar{f}$. By (A.6.17), it follows from the assumption that $P_{X!}(\bar{f})$ is a covering and its image is identified with $P_{X!}(T)$. Hence T is a covering sieve by (2.13.2) and f is a covering. If $P_{X!}(f)$ is a bicovering, the diagonal morphism $P_{X!}(F) \to P_{X!}(F) \times_{P_{X!}(G)} P_{X!}(F)$ is a covering. Since $P_{X!}$ preserves pull-backs, the above diagonal morphism is identified with the morphism $P_{X!}(F) \to P_{X!}(F \times_G F)$ induced by the diagonal morphism $F \to F \times_G F$ in $\widehat{(h\downarrow X)}$. Then, the above argument shows that $F \to F \times_G F$ is a covering. Hence f is a bicovering. □

By (2.13.2), $P_X^* : \widehat{\mathcal{C}} \to \widetilde{(h\downarrow X)}$ induces $\widetilde{P}_X^* : \widetilde{\mathcal{C}} \to \widehat{(h\downarrow X)}$. Let $i_X : \widetilde{(h\downarrow X)} \to \widehat{(h\downarrow X)}$ be the inclusion functor. A left adjoint $\widetilde{P}_{X!} : \widetilde{(h\downarrow X)} \to \widetilde{\mathcal{C}}$ of $\widetilde{P}_X^*$ is defined to be $aP_{X!}i_X$ (2.11.2). There also exists a right adjoint $\widetilde{P}_{X*}$ of $\widetilde{P}_X^*$ (2.12.12).

The assosiated sheaf functor $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}$ induces a functor $a/X : \widehat{\mathcal{C}}/X \to \widetilde{\mathcal{C}}/aX$ $(F \xrightarrow{p} X) \mapsto (aF \xrightarrow{a(p)} aX)$. Let $i/aX : \widetilde{\mathcal{C}}/aX \to \widehat{\mathcal{C}}/iaX$ be the functor induced by the inclusion functor $i : \widetilde{\mathcal{C}} \to \widehat{\mathcal{C}}$ and $\eta_X^* : \widehat{\mathcal{C}}/iaX \to \widehat{\mathcal{C}}/X$ denotes the pull-back functor along $\eta_X : X \to iaX$. We define $\tilde{e}_X : \widetilde{(h\downarrow X)} \to \widetilde{\mathcal{C}}/aX$ to be a composition $\widetilde{(h\downarrow X)} \xrightarrow{i_X} \widehat{(h\downarrow X)} \xrightarrow{e_X} \widehat{\mathcal{C}}/X \xrightarrow{a/X} \widetilde{\mathcal{C}}/aX$.

Lemma 2.13.5 *1) For $(G \xrightarrow{p} aX) \in \mathrm{Ob}\,\widetilde{\mathcal{C}}/aX$, let $\bar{p} : \bar{G} \to X$ be a pull-back of $i(p) : iG \to iaX$ along the counit $\eta_X : X \to iaX$. Then, $e_X^{-1}(\bar{G} \xrightarrow{\bar{p}} X)$ (A.6.16) is a sheaf on $(h\downarrow X)$.*

2) For a sheaf F on $(h\downarrow X)$, the following square is cartesian.

$$\begin{tikzcd} F_X \arrow[r, "\eta_{F_X}"] \arrow[d, "p_X(F)"] & iaF_X \arrow[d, "ia(p_X(F))"] \\ X \arrow[r, "\eta_X"] & iaX \end{tikzcd}$$

Proof. 1) We denote by $\bar{\eta} : \bar{G} \to iG$ the morphism in $\widehat{\mathcal{C}}$ such that the following square is a pull-back.

$$\begin{tikzcd} \bar{G} \arrow[r, "\bar{\eta}"] \arrow[d, "\bar{p}"] & iG \arrow[d, "i(p)"] \\ X \arrow[r, "\eta_X"] & iaX \end{tikzcd}$$

For $\langle Z, g\rangle \in \mathrm{Ob}\,(h{\downarrow}X)$ and $T \in J_X(\langle Z, g\rangle)$, let $\sigma : T \to h'_{\langle Z,g\rangle}$ be the inclusion morphism and $\varphi : T \to e_X^{-1}(\bar{G} \xrightarrow{\bar{p}} X)$ a morphism in $(h{\downarrow}X)$. Then, we have a morphism $\lambda_{\bar{p}} e_X(\varphi) : e_X(T) \to (\bar{G} \xrightarrow{\bar{p}} X)$ in $\widehat{\mathcal{C}}/X$, where $\lambda : e_X e_X^{-1} \to id_{\widehat{\mathcal{C}}/X}$ is the natural equivalence constructed in (A.6.16). Let us denote by $\varphi' : P_{X!}(T) \to \bar{G}$ the morphism in $\widehat{\mathcal{C}}$ inducing $\lambda_{\bar{p}} e_X(\varphi)$. Hence $\bar{\eta}\varphi' : P_{X!}(T) \to iG$ defines a morphism $\psi : \Sigma_{\eta_X} e_X(T) \to (iG \xrightarrow{i(p)} iaX)$ in $(h{\downarrow}X)$. Since $P_{X!}(\sigma) : P_{X!}(T) \to P_{X!}(h'_{\langle Z,g\rangle})$ is a bicovering by (2.13.2), there exists a unique morphism $\xi : P_{X!}(h'_{\langle Z,g\rangle}) \to iG$ satisfying $\bar{\eta}\varphi' = \xi P_{X!}(\sigma)$. Then, $i(p)\xi P_{X!}(\sigma) = i(p)\bar{\eta}\varphi' = \eta_X \bar{p}\varphi' = \eta_X p_X(T) = \eta_X p_X(h'_{\langle Z,g\rangle}) P_{X!}(\sigma)$. Again, since $P_{X!}(\sigma)$ is a bicovering, we have $i(p)\xi = \eta_X p_X(h'_{\langle Z,g\rangle})$. Thus ξ induces a unique morphism $\xi' : P_{X!}(h'_{\langle Z,g\rangle}) \to \bar{G}$ satisfying $\bar{p}\xi' = p_X(h'_{\langle Z,g\rangle})$ and $\bar{\eta}\xi' = \xi$. Therefore we have a morphism $\bar{\xi} : e_X(h'_{\langle Z,g\rangle}) \to (\bar{G} \xrightarrow{\bar{p}} X)$ defined by ξ'. We claim that $\bar{\xi} e_X(\sigma) = \lambda_{\bar{p}} e_X(\varphi)$. In fact, since $\bar{p}\xi' P_{X!}(\sigma) = p_X(h'_{\langle Z,g\rangle}) P_{X!}(\sigma) = p_X(T) = \bar{p}\varphi'$ and $\bar{\eta}\xi' P_{X!}(\sigma) = \xi P_{X!}(\sigma) = \bar{\eta}\varphi'$, we have $\xi' P_{X!}(\sigma) = \varphi'$. Recall from (A.6.16) that $e_X^{-1} e_X = id_{\widehat{(h{\downarrow}X)}}$. It can be easily verified from the construtions of e_X^{-1} and λ that $e_X^{-1}(\lambda_q) = id_{e_X^{-1}(H \xrightarrow{q} X)}$ for any $(H \xrightarrow{q} X) \in \mathrm{Ob}\,\widehat{\mathcal{C}}/X$. Hence $e_X^{-1}(\xi')\sigma = e_X^{-1}(\xi' e_X(\sigma)) = e_X^{-1}(\lambda_{\bar{p}}) e_X^{-1} e_X(\varphi) = \varphi$. This shows that $\sigma^* : \widehat{(h{\downarrow}X)}(h'_{\langle Z,g\rangle}, e_X^{-1}(\bar{G} \xrightarrow{\bar{p}} X)) \to \widehat{(h{\downarrow}X)}(T, e_X^{-1}(\bar{G} \xrightarrow{\bar{p}} X))$ is surjective.

Suppose that $\varphi_1\sigma = \varphi_2\sigma$ for $\varphi_1, \varphi_2 \in \widehat{(h{\downarrow}X)}(h'_{\langle Z,g\rangle}, e_X^{-1}(\bar{G} \xrightarrow{\bar{p}} X))$. We denote by $\varphi'_i : P_{X!}(h'_{\langle Z,g\rangle}) \to \bar{G}$ ($i = 1, 2$) the morphism in $\widehat{\mathcal{C}}$ inducing $\lambda_{\bar{p}} e_X(\varphi_i)$. Then, we have $\varphi'_1 P_{X!}(\sigma) = \varphi'_2 P_{X!}(\sigma)$. Since $P_{X!}(\sigma)$ is a bicovering, it follows from $\bar{\eta}\varphi'_1 P_{X!}(\sigma) = \bar{\eta}\varphi'_2 P_{X!}(\sigma)$ that $\bar{\eta}\varphi'_1 = \bar{\eta}\varphi'_2$. On the other hand, $\bar{p}\varphi'_1 = p_X(h'_{\langle Z,g\rangle}) = \bar{p}\varphi'_2$. Thus we have $\varphi'_1 = \varphi'_2$.

2) Suppose that F is a sheaf on $(h{\downarrow}X)$. Consider a pull-back of $ia(p_X(F))$ along η_X.

$$\begin{tikzcd} \bar{F} \arrow[r, "\bar{\eta}"] \arrow[d, "\bar{p}"] & iaF_X \arrow[d, "ia(p_X(F))"] \\ X \arrow[r, "\eta_X"] & iaX \end{tikzcd}$$

There exists a unique morphism $\zeta : F_X \to \bar{F}$ satisfying $\bar{p}\zeta = p_X(F)$ and $\bar{\eta}\zeta = \eta_{F_X}$. Since $\bar{\eta}$ is a pull-back of a bicovering η_X, it is a bicovering. It follows from $\bar{\eta}\zeta = \eta_{F_X}$ that ζ is also a bicovering. We regard ζ as a morphism $e_X(F) \to (\bar{F} \xrightarrow{\bar{p}} X)$ in $\widehat{\mathcal{C}}/X$. By the naturality of $\lambda : e_X e_X^{-1} \to id_{\widehat{\mathcal{C}}/X}$, we have $\lambda_{\bar{p}} e_X e_X^{-1}(\zeta) = \zeta\lambda_{p_X(F)}$. It can be easily verified from the construtions of e_X^{-1} and λ that $\lambda_{p_X(F)} = id_{e_X(F)}$ for any $F \in \mathrm{Ob}\,\widehat{(h{\downarrow}X)}$. Hence $\lambda_{\bar{p}} e_X e_X^{-1}(\zeta) = \zeta$ and applying Σ_X to the both sides of this equality, we see that $P_{X!} e_X^{-1}(\zeta)$ is a bicovering in $\widehat{\mathcal{C}}$. Then $e_X^{-1}(\zeta) : F \to e_X^{-1}(\bar{F} \xrightarrow{\bar{p}} X)$ is a bicovering in $\widehat{(h{\downarrow}X)}$ by (2.13.4). Since both F and $e_X^{-1}(\bar{F} \xrightarrow{\bar{p}} X)$ are sheaves by the assumption and 1), it follows that $e_X^{-1}(\zeta)$ is an isomorphism. Therefore ζ is an isomorphism. □

Proposition 2.13.6 *$\tilde{e}_X : \widetilde{(h{\downarrow}X)} \to \widetilde{\mathcal{C}}/aX$ is an equivalence of categories and $\widetilde{P}_{X!} = \Sigma_{aX}\tilde{e}_X$ hold. Moreover, $\tilde{e}_X \widetilde{P}^*_X : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}/aX$ is naturally equivalent to $(aX)^*$.*

Proof. The equality $\widetilde{P}_{X!} = \Sigma_{aX}\tilde{e}_X$, obvious from the definition of $\tilde{e}_X$. By (A.6.15), $\tilde{e}_X \widetilde{P}^*_X = (a/X)X^* i$. Since the associated sheaf functor is left exact, $(a/X)X^* i$ is naturally equivalent to $(aX)^*$.

A quasi-inverse $\tilde{e}_X^{-1} : \widetilde{\mathcal{C}}/aX \to \widetilde{(h{\downarrow}X)}$ of $\tilde{e}_X$ is defined as follows. It follows from (2.13.5) that a composition $\widetilde{\mathcal{C}}/aX \xrightarrow{i/aX} \widehat{\mathcal{C}}/iaX \xrightarrow{\eta_X^*} \widehat{\mathcal{C}}/X \xrightarrow{e_X^{-1}} \widehat{(h{\downarrow}X)}$ takes values in $\widetilde{(h{\downarrow}X)}$. Hence $\tilde{e}_X^{-1}$ is the unique functor satisfying $i_X \tilde{e}_X^{-1} = e_X^{-1}\eta_X^*(i/aX)$. For a sheaf F on $(h{\downarrow}X)$, $e_X(F) = (F_X \xrightarrow{p_X(F)} X) \in \mathrm{Ob}\,\widehat{\mathcal{C}}/X$ is naturally isomorphic to $\eta_X^*(iaF_X \xrightarrow{ia(p_X(F))} iaX)$ by (2.13.5). Since $e_X^{-1} e_X = id_{\widehat{(h{\downarrow}X)}}$ by (A.6.16), $\tilde{e}_X^{-1}\tilde{e}_X(F)$ is naturally isomorphic to F. Thus we see that $\tilde{e}_X^{-1}\tilde{e}_X$ is naturally equivalent to the identity functor of $\widetilde{(h{\downarrow}X)}$. By (A.6.16) and the definitions of $\tilde{e}_X$ and $\tilde{e}_X^{-1}$, $\tilde{e}_X\tilde{e}_X^{-1}$ is naturally equivalent to a composition $\widetilde{\mathcal{C}}/aX \xrightarrow{i/aX} \widehat{\mathcal{C}}/iaX \xrightarrow{\eta_X^*} \widehat{\mathcal{C}}/X \xrightarrow{a/X} \widetilde{\mathcal{C}}/aX$. Since the associated sheaf functor is left exact, $(a/X)\eta_X^*$ is naturally equivalent to a composition

$\widehat{\mathcal{C}}/iaX \xrightarrow{a/iaX} \widetilde{\mathcal{C}}/aiaX \xrightarrow{a(\eta_X)^*} \widetilde{\mathcal{C}}/aX$. For $(G \xrightarrow{p} aX) \in \mathrm{Ob}\,\widetilde{\mathcal{C}}/aX$, since the counit $\varepsilon : ai \to id_{\widetilde{\mathcal{C}}}$ is an equivalence and $a(\eta_X) = \varepsilon_{aX}^{-1}$, the following square is cartesian.

$$\begin{tikzcd}
G \arrow[r, "\varepsilon_G^{-1}", "\cong"'] \arrow[d, "p"] & aiG \arrow[d, "ai(p)"] \\
aX \arrow[r, "a(\eta_X)=\varepsilon_{aX}^{-1}", "\cong"'] & aiaX
\end{tikzcd}$$

It follows that a composition $\widetilde{\mathcal{C}}/aX \xrightarrow{i/aX} \widehat{\mathcal{C}}/iaX\widehat{\mathcal{C}}/iaX \xrightarrow{a/iaX} \widetilde{\mathcal{C}}/aiaX \xrightarrow{a(\eta_X)^*} \widetilde{\mathcal{C}}/aX$ is naturally equivalent to the identity functor of $\widetilde{\mathcal{C}}/aX$. □

We denote by $a_X : \widehat{(h\downarrow X)} \to \widetilde{(h\downarrow X)}$ the associated sheaf functor.

Proposition 2.13.7 *Let $(\mathcal{C}, J)$ be a $\mathcal{U}$-site, X a presheaf on $\mathcal{C}$ and $\alpha : Y \to X$ a morphism in $\widehat{\mathcal{C}}$.*

1) The following diagrams commutes up to natural equivalences.

$$\begin{tikzcd}
\widehat{(h\downarrow X)} \arrow[r, "a_X"] \arrow[d, "e_X"] & \widetilde{(h\downarrow X)} \arrow[r, "i_X"] \arrow[d, "\tilde{e}_X"] & \widehat{(h\downarrow X)} \arrow[dr, "e_X"] & \\
\widehat{\mathcal{C}}/X \arrow[r, "a/X"] & \widetilde{\mathcal{C}}/aX \arrow[r, "i/aX"] & \widehat{\mathcal{C}}/iaX \arrow[r, "\eta_X^*"] & \widehat{\mathcal{C}}/X
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[r, "\widetilde{P}_X^*"] \arrow[dr, "(aX)^*"'] & \widetilde{(h\downarrow X)} \arrow[r, "\widetilde{P}_{X!}"] \arrow[d, "\tilde{e}_X"] & \widetilde{\mathcal{C}} \\
 & \widetilde{\mathcal{C}}/aX \arrow[ur, "\Sigma_{aX}"'] &
\end{tikzcd}$$

2) Set $[\alpha] = e_X^{-1}(Y \xrightarrow{\alpha} X)$. By (2.13.2), the isomorphism $Q_\alpha : (h'\downarrow[\alpha]) \to (h\downarrow Y)$ induces an isomorphism $\widetilde{Q}_\alpha^* : \widetilde{(h\downarrow Y)} \to \widetilde{(h'\downarrow[\alpha])}$. $\widetilde{\Theta} : \widetilde{\mathcal{C}}/aY \to (\widetilde{\mathcal{C}}/aX)/(a/X)(Y \xrightarrow{\alpha} X)$ denotes the functor given by $(Z \xrightarrow{p} aY) \mapsto ((Z \xrightarrow{a(\alpha)p} aX) \xrightarrow{p} (a/X)(Y \xrightarrow{\alpha} X))$. Note that $\widetilde{\Theta}$ is an isomorphism of categories. Then, the following diagram commutes.

$$\begin{tikzcd}
\widetilde{(h\downarrow Y)} \arrow[r, "\tilde{e}_Y"] \arrow[d, "\widetilde{Q}_\alpha^*"] & \widetilde{\mathcal{C}}/aY \arrow[r, "\widetilde{\Theta}"] & (\widetilde{\mathcal{C}}/aX)/(a/X)(Y \xrightarrow{\alpha} X) \\
\widetilde{(h'\downarrow[\alpha])} \arrow[d, "\tilde{e}_{[\alpha]}"] & & \\
\widetilde{(h\downarrow X)}/a_X[\alpha] \arrow[r, "\tilde{e}_X/a_X[\alpha]"] & (\widetilde{\mathcal{C}}/aX)/\tilde{e}_X a_X[\alpha] \arrow[r, "(a/X)e_X(\eta_{[\alpha]})^*"] & (\widetilde{\mathcal{C}}/aX)/(a/X)e_X[\alpha] \arrow[uu, "\Sigma_{(a/X)(\lambda_\alpha)}"']
\end{tikzcd}$$

Proof. 1) Since $i_X\tilde{e}_X^{-1} = e_X^{-1}\eta_X^*(i/aX)$ and there are natural equivalences $e_Xe_X^{-1} \to id_{\widehat{\mathcal{C}}/X}$, $\tilde{e}_X^{-1}\tilde{e}_X \to id_{\widetilde{(h\downarrow X)}}$, $\eta_X^*(i/aX)\tilde{e}_X$ is naturally equivalent to e_Xi_X.

By (A.3.12), a/X is a left adjoint of $\eta_X^*(i/aX)$. Hence the commutativity of the right rectangle of the upper diagram implies that $\tilde{e}_X^{-1}(a/X)e_X$ is a left adjoint of i_X. Since a_X is also a left adjoint of i_X, it is naturally equivalent to $\tilde{e}_X^{-1}(a/X)e_X$. Therefore $\tilde{e}_Xa_X$ is naturally equivalent to $(a/X)e_X$.

By (A.6.14), $\tilde{e}_X\widetilde{P}_X^* = (a/X)X^*i$. Since a is left exact, $(a/X)X^*$ is naturally equivalent to $(aX)^*a$. Hence $\tilde{e}_X\widetilde{P}_X^*$ is naturally equivalent to $(aX)^*ai \cong (aX)^*$.

Obviously, we have $\Sigma_{aX}(a/X) = a\Sigma_X$. If we define $\widetilde{P}_{X!}$ to be $aP_{X!}i_X$ ((2.11.2)), we have $\Sigma_{aX}\tilde{e}_X = \Sigma_{aX}(a/X)e_Xi_X = a\Sigma_Xe_Xi_X = aP_{X!}i_X = \widetilde{P}_{X!}$.

2) Since $e_X : \widehat{(h\downarrow X)} \to \widehat{\mathcal{C}}/X$ and $a/X : \widehat{\mathcal{C}}/X \to \widetilde{\mathcal{C}}/aX$ preserves pull-backs, the following diagram commutes up to natural equivalences.

$$\begin{tikzcd}
\widehat{(h\downarrow X)}/i_Xa_X[\alpha] \arrow[r, "e_X/i_Xa_X[\alpha]"] \arrow[d, "\eta_{[\alpha]}^*"] & (\widehat{\mathcal{C}}/X)/e_Xi_Xa_X[\alpha] \arrow[r, "(a/X)/e_Xi_Xa_X[\alpha]"] \arrow[d, "e_X(\eta_{[\alpha]})^*"] & (\widetilde{\mathcal{C}}/aX)/\tilde{e}_Xa_X[\alpha] \arrow[d, "(a/X)e_X(\eta_{[\alpha]})^*"] \\
\widehat{(h\downarrow X)}/[\alpha] \arrow[r, "e_X/[\alpha]"] & (\widehat{\mathcal{C}}/X)/e_X[\alpha] \arrow[r, "(a/X)/e_X[\alpha]"] & (\widetilde{\mathcal{C}}/aX)/(a/X)e_X[\alpha]
\end{tikzcd}$$

We also have the following commutative diagrams.

$$\begin{tikzcd}
\widehat{\mathcal{C}}/Y \arrow[r, "\Theta"] \arrow[d, "a/Y"] & (\widehat{\mathcal{C}}/X)/(Y \xrightarrow{\alpha} X) \arrow[d, "(a/X)/(Y \xrightarrow{\alpha} X)"] & (\widehat{\mathcal{C}}/X)/e_X[\alpha] \arrow[l, "\Sigma_{\lambda_\alpha}"'] \arrow[d, "(a/X)/e_X[\alpha]"] \\
\widetilde{\mathcal{C}}/aY \arrow[r, "\widetilde{\Theta}"] & (\widetilde{\mathcal{C}}/aX)/(a/X)(Y \xrightarrow{\alpha} X) & (\widetilde{\mathcal{C}}/a)/(a/X)(Y \xrightarrow{\alpha} X) \arrow[l, "\Sigma_{(a/X)\lambda_\alpha}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{(h{\downarrow}Y)} \arrow[r, "i_Y"] \arrow[d, "\widetilde{Q}^*_\alpha"] & \widehat{(h{\downarrow}Y)} \arrow[d, "Q^*_\alpha"] \\
\widetilde{(h'{\downarrow}[\alpha])} \arrow[r, "i_{[\alpha]}"] & \widehat{(h'{\downarrow}[\alpha])}
\end{tikzcd}$$

Then, the result follows from 1) and (A.6.18). □

2.14 Examples of Grothendieck topos

Definition 2.14.1 *A lattice is a set A with operations $\vee, \wedge : A \times A \to A$ and two distinguished elements $0, 1 \in A$ satisfying the following conditions.*

(1) $(x \vee y) \vee z = x \vee (y \vee z)$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$

(2) $x \vee y = y \vee x$, $x \wedge y = y \wedge x$

(3) $x \vee x = x$, $x \wedge x = x$

(4) $x \vee 0 = x$, $x \wedge 1 = x$

(5) $(x \wedge y) \vee y = y$, $x \wedge (x \vee y) = x$

If a lattice A satisfies the following condition, A is said to be distributive.

(6) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Let A and B be lattices. If a map $f : A \to B$ satisfies $f(0) = 0$, $f(1) = 1$, $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$ for any $x, y \in A$, f is called a morphism of lattices.

If a category $\mathcal{C}$ satisfies "For any $x, y \in \operatorname{Ob}\mathcal{C}$, $\mathcal{C}(x, y)$ has at most one element.", $\mathcal{C}$ is called a partially ordered set. We have a binary relation $\leq$ in $\operatorname{Ob}\mathcal{C}$ given by "$x \leq y \Leftrightarrow \mathcal{C}(x, y) \neq \emptyset$". $x \leq y$ also denotes the unique morphism from x to y. Moreover, a partially ordered set $\mathcal{C}$ satisfying "If neither $\mathcal{C}(x, y)$ nor $\mathcal{C}(y, x)$ is empty, then $x = y$." is called an ordered set.

Lemma 2.14.2 *Let $(A, \leq)$ be an partially ordered set.*

1) Every morphism in A is a monomorphism and epimorphism.

2) $(x_i \leq x)_{i \in I}$ is a strict epimorphic family if and only if $x = \sup\{x_i | \, i \in I\}$.

3) Suppose that A has finite products (resp. finite coproducts). Then A has finite limits (resp. finite colimits).

4) Suppose that A has finite products. We denote by $x \wedge y$ the product of x and y. A family $(x_i \leq x)_{i \in I}$ of morphisms in A is a universal strict epimorphic family if and only if $\sup\{x_i \wedge y | \, i \in I\} = y$ for any $y \in A$ such that $y \leq x$.

Proof. 1) Since $A(x, y)$ has at most one element for any $x, y \in A$, the assertion is obvious.

2) Suppose that $(x_i \leq x)_{i \in I}$ is a strict epimorphic family. If $x_i \leq y$ for any $i \in I$, we have a unique morphism from x to y. Hence $x \leq y$ and it follows that $x = \sup\{x_i | \, i \in I\}$. Conversely, assume $x = \sup\{x_i | \, i \in I\}$. If $x_i \leq y$ for any $i \in I$, then $x \leq y$ by the definition of supremum. Therefore $(x_i \leq x)_{i \in I}$ is a strict epimorphic family.

3) Since $A(x, y)$ has at most one element for any $x, y \in A$, the notions of equalizers and coeqalizers in A reduce to the identitiy morphisms.

4) For a morphism $y \leq x$, $x_i \wedge y \leq y$ is the pull-back of $x_i \leq x$ along $y \leq x$. By 3), $(x_i \wedge y \leq y)_{i \in I}$ is a strict epimorphic family if and only if $y = \sup\{x_i \wedge y | \, i \in I\}$. Then, the result follows from (2.2.7). □

Proposition 2.14.3 *Let A be a lattice.*

1) $x \vee y = y$ if and only if $x \wedge y = x$.

2) Define a relation $\leq$ in A by "$x \leq y \Leftrightarrow x \wedge y = x$". Then, $(A, \leq)$ is an ordered set.

3) If we regard the ordered set $(A, \leq)$ as a category, it has finite limits and colimits. In fact, $x \wedge y$ is a product of x and y, $x \vee y$ is a coproduct of x and y, 0 is the initial object, 1 is the terminal object.

4) A is distributive if and only if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ holds for any $x, y, z \in A$.

Proof. 1) If $x \vee y = y$, then $x \wedge y = x \wedge (x \vee y) = x$ by (5). If $x \wedge y = x$, then $x \vee y = (x \wedge y) \vee y = y$ by (5).

2) $x \leq x$ follows from (3). If $x \leq y$ and $y \leq x$, then $x = x \wedge y = y \wedge x = y$ by (2). If $x \leq y$ and $y \leq z$, then $x = x \wedge y = x \wedge (y \wedge z) = (x \wedge y) \wedge z = x \wedge z$, namely, $x \leq z$. Hence $(A, \leq)$ is an ordered set.

3) It is obvious from (4) of (2.14.1) that 0 is the initial object and 1 is the terminal object. For $x, y \in A$, suppose that $z \leq x$ and $z \leq y$ (resp. $x \leq z$ and $y \leq z$). Then we have $x \wedge z = z$ and $y \wedge z = z$ (resp. $x \vee z = z$

and $y \vee z = z$). Hence $(x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge z = z$ (resp. $(x \vee y) \vee z = x \vee (y \vee z) = x \vee z = z$) and we have $z \leq x \wedge y$ (resp. $x \vee y \leq z$).

4) Suppose that A is distributive. $(x \vee y) \wedge (x \vee z) = ((x \vee y) \wedge x) \vee ((x \vee y) \wedge z) = x \vee ((x \wedge z) \vee (y \wedge z)) = (x \vee (x \wedge z)) \vee (y \wedge z) = x \vee (y \wedge z)$. Conversely, assume that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ holds for any $x, y, z \in A$. $(x \wedge y) \vee (x \wedge z) = ((x \wedge y) \vee x) \wedge ((x \wedge y) \vee z) = x \wedge ((x \vee z) \wedge (y \vee z)) = (x \wedge (x \vee z)) \wedge (y \vee z) = x \wedge (y \vee z)$. □

Definition 2.14.4 *1) A lattice A satisfying the following properties (7) and (8) is called a frame.*

(7) For any family $(x_i)_{i \in I}$ of elements of A, the ordered set $(A, \leq)$ has a coproduct $\bigvee_{i \in I} x_i$.
(8) For any family $(y_i)_{i \in I}$ of elements of A and $x \in A$, $x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I}(x \wedge y_i)$.

A morphism $f : A \to B$ of frames is a morphism of lattices satisfying $f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)$ for any family $(x_i)_{i \in I}$ of elements of A. We denote by $\mathcal{F}r$ the category of frames.

2) The opposite category of $\mathcal{F}r$ is called the category of locales. We denote by $\mathcal{L}oc$ the category of locales and its object is called a locale. If X is a locale, the corresponding frame is denoted by $\mathcal{O}(X)$. For a morphism $f : X \to Y$ of locales, $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ denotes the corresponding morphism of frames.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ functors. Suppose that $\mathcal{D}$ is a partially ordered set. There is at most one natural transformation from F to G if and only if $\varphi(X) \leq \psi(X)$ for every $X \in \mathrm{Ob}\,\mathcal{C}$. Hence $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ is a partially ordered set (resp. an ordered set) if $\mathcal{D}$ is a partially ordered set (resp. an ordered set). It follows that $\mathcal{F}r$ is a 2-category such that $\mathcal{F}r(A, B)$ is an ordered set for every pair (A, B) of frames. Thus $\mathcal{L}oc$ is also a 2-category.

Proposition 2.14.5 *Let $\varphi : A \to B$ be a morphism of frames. Regarding ordered sets $(A, \leq)$, $(B, \leq)$ as categories, φ has a right adjoint.*

Proof. Define $\psi : B \to A$ by $\psi(y) = \bigvee_{\varphi(x) \leq y} x$. Clearly, $\psi(y) \leq \psi(z)$ if $y \leq z$ in B. Hence ψ is a functor. For $x \in A$ and $y \in B$, $x \leq \psi(y)$ if and only if $\varphi(x) \leq y$. In fact, if $x \leq \psi(y)$, then $\varphi(x) \leq \varphi\psi(y) = \varphi(\bigvee_{\varphi(w) \leq y} w) = \bigvee_{\varphi(w) \leq y} \varphi(w) \leq y$. It is obviuos that $\varphi(x) \leq y$ implies $x \leq \psi(y)$. Therefore ψ is a right adjoint of φ. □

The above ψ need not be a morphism of frames and it preserves limits. If $f : X \to Y$ is a morphism of locales, we denote by $f_* : \mathcal{O}(X) \to \mathcal{O}(Y)$ the right adjoint of $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$.

Example 2.14.6 *1) Let S be a topological space. We denote by $\mathcal{O}(S)$ the lattice of open sets of S. Then $\mathcal{O}(S)$ is a frame and we have a locale $\mathrm{Loc}(S)$ associated with S. If $f : S \to T$ is a continuous map of topological spaces, $O \mapsto f^{-1}(O)$ defines a morphism $f^{-1} : \mathcal{O}(T) \to \mathcal{O}(S)$ of frames. Thus we have a morphism $\mathrm{Loc}(f) : \mathrm{Loc}(S) \to \mathrm{Loc}(T)$ of frames. If we denote by $\mathcal{T}op$ the category of topological spaces and continuous maps, we have a functor $\mathrm{Loc} : \mathcal{T}op \to \mathcal{L}oc$.*

2) Let $\mathcal{E}$ be a $\mathcal{U}$-topos and F an object of $\mathcal{E}$. Then, the lattice $(\mathrm{Sub}(F), \cap, \cup)$ is a frame by (2.4.18).

For a locale X, we regard an ordered set $(\mathcal{O}(X), \leq)$ as a category and give the canonical topology J. We denote by $\mathrm{Sh}(X)$ the category of sheaves on $\mathcal{O}(X)$ for the canonical topology. If X is a topological space, we also denote $\mathrm{Sh}(\mathrm{Loc}(X))$ by $\mathrm{Sh}(X)$.

Proposition 2.14.7 *Let X be a locale.*

1) A family $(x_i \leq x)_{i \in I}$ of morphisms in $\mathcal{O}(X)$ is a strict epimorphic family if and only if $\bigvee_{i \in I} x_i = x$. A strict epimorphic family in $\mathcal{O}(X)$ is universal.

2) A family of morphisms $(x_i \leq x)_{i \in I}$ in $\mathcal{O}(X)$ is a covering for the canonical topology J if and only if $\bigvee_{i \in I} x_i = x$.

Proof. 1) Since $\bigvee_{i \in I} x_i = \sup\{x_i | \, i \in I\}$, the first assertion follows from (2.14.2). Let $(x_i \leq x)_{i \in I}$ be a strict epimorphic family in $\mathcal{O}(X)$ and $y \leq x$ a morphism. Then, $\sup\{x_i \wedge y | \, i \in I\} = \bigvee_{i \in I}(x_i \wedge y) = (\bigvee_{i \in I} x_i) \wedge y = x \wedge y = y$ by (8) of (2.14.4). Hence $(x_i \leq x)_{i \in I}$ is a universal strict epimorphic family by (2.14.2).

2) The assertion follows from 1) and (2.2.4). □

Definition 2.14.8 *A category which is equivalent to $\mathrm{Sh}(X)$ for some locale X is called a localic topos.*

Proposition 2.14.9 *For a Grothendieck topos $\mathcal{E}$, the following conditions are equivalent.*

i) $\mathcal{E}$ is localic.
ii) There exist a $\mathcal{U}$-site $(\mathcal{P}, J)$ for $\mathcal{E}$ such that $\mathcal{P}$ is a partially ordered set.

iii) $\mathcal{E}$ *is generated by the subobjects of its terminal object 1.*

Proof. Since a frame is an ordered set, $i)$ implies $ii)$.

$ii) \Rightarrow iii)$: For $x \in \mathcal{P}$, every morphism $h_x \to 1$ in $\widehat{\mathcal{P}}$ is a monomorphism. Since the associated sheaf functor $a : \widehat{\mathcal{P}} \to \widetilde{\mathcal{P}}$ is left exact, ah_x is regarded as a subobject of 1. Hence by (2.4.3), that $\widetilde{\mathcal{P}}$ is generated by the subobjects of 1. Since $\mathcal{E}$ is equivalent to $\widetilde{\mathcal{P}}$, $iii)$ holds.

$iii) \Rightarrow i)$: Let X be the locale corresponding to the frame $(\mathrm{Sub}(1), \cap, \cup)$ (2.14.6). Since $\mathrm{Sub}(1)$ generates $\mathcal{E}$ and is closed under taking subobjects in $\mathcal{E}$, $i)$ follows from (2.10.7). $\square$

Let X be a locale. For $x \in \mathcal{O}(X)$, the sheaf h_x represented by x is regarded as a subobject of the terminal object 1 of $\mathrm{Sh}(X)$, which is the sheaf represented by the terminal object 1 of $\mathcal{O}(X)$. In fact, $h_x(y) = \begin{cases} \{\le\} & \text{if } y \le x \\ \emptyset & \text{if } y \not\le x \end{cases}$ and $1(y) = \{\le\}$ for any $y \in \mathcal{O}(X)$. Thus the Yoneda embedding induces a map $\sigma : \mathcal{O}(X) \to \mathrm{Sub}_{\mathrm{Sh}(X)}(1)$, $x \mapsto h_x$.

Proposition 2.14.10 *σ is an isomorphism of frames and the inverse $\sigma^{-1} : \mathrm{Sub}_{\mathrm{Sh}(X)}(1) \to \mathcal{O}(X)$ is given by $\sigma^{-1}(F) = \bigvee_{F(x) \neq \emptyset} x$.*

Proof. Since the initial object 0 of $\mathcal{O}(X)$ is strict, $F(0)$ consists of a single element for any $F \in \mathrm{Ob}\,\mathrm{Sh}(X)$ by (2.2.9). Hence $\sigma(0) = h_0$ is an initial object 0 of $\mathrm{Sh}(X)$. It is clear that $\sigma(1) = h_1 = 1$. For $x, y, z \in \mathcal{O}(X)$, since $z \le x \wedge y$ if and only if $z \le x$ and $z \le y$, $\sigma(x \wedge y)(z) = h_{x\wedge y}(z) = h_x(z) \cap h_y(z) = \sigma(x)(z) \cap \sigma(y)(z) = (\sigma(x) \cap \sigma(y))(z)$ in $1(z)$. Hence $\sigma(x \wedge y) = \sigma(x) \cap \sigma(y)$ in $\mathrm{Sub}_{\mathrm{Sh}(X)}(1)$. Let $(x_i)_{i\in I}$ be a family of elements of $\mathcal{O}(X)$ and $z \in \mathcal{O}(X)$. Since $z \wedge (\bigvee_{i\in I} x_i) = \bigvee_{i\in I}(z \wedge x_i)$, $z \le \bigvee_{i\in I} x_i$ holds if and only if $(z \wedge x_i \le z)_{i\in I}$ is a covering of z by (2.14.7). On the other hand, it follows from (2.4.7) that $(z \wedge x_i \le z)_{i\in I}$ is a covering of z if and only if $(\sigma(z \wedge x_i) \subset \sigma(z))_{i\in I}$ is an epimorphic family, that is, $\bigcup_{i\in I} \sigma(z \wedge x_i) = \sigma(z)$ holds in $\mathrm{Sub}_{\mathrm{Sh}(X)}(1)$. In particular, if $z = \bigvee_{i\in I} x_i$, then $z \wedge x_i = x_i$ and we have $\sigma(\bigvee_{i\in I} x_i) = \bigcup_{i\in I} \sigma(z \wedge x_i) = \bigcup_{i\in I} \sigma(x_i)$. Thus we have shown that σ is a morphism of frames.

For $x \in \mathcal{O}(X)$, $\sigma^{-1}\sigma(x) = \sigma^{-1}(h_x) = \bigvee_{h_x(y)\neq\emptyset} y = \bigvee_{y \le x} y = x$. For any $x \in \mathcal{O}(X)$, since h_x is a subobject of the terminal object 1, every morphism whose domain is h_x is a monomorphism. Hence, for $F \in \mathrm{Ob}\,\mathrm{Sub}_{\mathrm{Sh}(X)}(1)$, $(hP\langle x, f\rangle \xrightarrow{f} F)_{\langle x,f\rangle \in \mathrm{Ob}\,(h\downarrow F)}$ is a colimiting cone in $\mathrm{Sub}_{\mathrm{Sh}(X)}(1)$. Note that there is a bijection $\chi : \mathrm{Ob}\,(h\downarrow F) \to \{x \in \mathcal{O}(X) | \, F(x) \neq \emptyset\}$ given by $\chi\langle x, f\rangle = x$. In fact, the inverse is defined by $\chi^{-1}(x) = \langle x, f_x\rangle$, where $f_x : h_x \to F$ maps the unique element of $h_x(y)$ to the image of the unique element of $F(x)$ by the map $F(x) \to F(y)$ induced by $y \le x$. Therefore, in $\mathrm{Sub}_{\mathrm{Sh}(X)}(1)$ we have $F = \bigcup_{\langle x,f\rangle \in \mathrm{Ob}\,(h\downarrow F)} hP\langle x, f\rangle = \bigcup_{F(x)\neq\emptyset} h_x = \bigcup_{F(x)\neq\emptyset} \sigma(x) = \sigma(\bigvee_{F(x)\neq\emptyset} x) = \sigma\sigma^{-1}(F)$. Therefore σ^{-1} is the inverse of σ. $\square$

Example 2.14.11 *Let G be a (discrete) group and X a topological space acting G on the left. A G-sheaf (of sets) on X is a sheaf whose etale space $p : E \to X$ is an G-equivariant map, that is, G acts on E on the left and p commutes with G-actions. We denote by $\mathrm{Sh}(X; G)$ the category of G-sheaves on X. Then, we can easily verify the conditions of (2.10.3). In fact, the conditions (0)∼(3) are straightforward and, for each open set U of X, define a map $p_U : G \times U \to X$ by $p_U(g, x) = gx$ and regard $G \times U$ as a left G-space by $(g', (g, x)) \mapsto (g'g, x)$. Then, $\{(p_U : G \times U \to X) \in \mathrm{Ob}\,\mathrm{Sh}(X; G) | \, U \text{ is an open set in } X\}$ is a small set of generators of $\mathrm{Sh}(X; G)$. Hence $\mathrm{Sh}(X; G)$ is a topos by Giraud's theorem.*

Example 2.14.12 *Let $\mathcal{E}$ be a topos and G an internal group in $\mathcal{E}$ with product $\mu : G \times G \to G$ and unit $\eta : 1_{\mathcal{E}} \to G$. A left G-object in $\mathcal{E}$ is a pair (X, α) of $X \in \mathrm{Ob}\,\mathcal{E}$ and a morphism $\alpha : G \times X \to X$ in $\mathcal{E}$ such that the following diagrams commute.*

$$\begin{tikzcd} G \times G \times X \arrow[r, "id_G \times \alpha"] \arrow[d, "\mu"] & G \times X \arrow[d, "\alpha"] \\ G \times X \arrow[r, "\alpha"] & X \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathcal{E}} \times X \arrow[r, "\eta \times id_X"] \arrow[dr, "\mathrm{pr}_2"] & G \times X \arrow[d, "\alpha"] \\ & X \end{tikzcd}$$

A morphism $\varphi : (X, \alpha) \to (Y, \beta)$ of left G-objects is a morphism $\varphi : X \to Y$ in $\mathcal{E}$ satisfying $\varphi\alpha = \beta(id_G \times \varphi)$. We denote by $\mathcal{E}^{G^{op}}$ the category of left G-objects in $\mathcal{E}$.

Note that there is a functor $U : \mathcal{E}^{G^{op}} \to \mathcal{E}$ forgetting the G-actions and a functor $F : \mathcal{E} \to \mathcal{E}^{G^{op}}$ given by $F(X) = (G \times X, \mu \times id_X)$ and $F(f) = id_G \times f$. A map $\mathcal{E}^{G^{op}}(F(X), (Y, \beta)) \to \mathcal{E}(X, U(Y, \beta))$ defined by $\varphi \mapsto \varphi(\eta, id_X)$ has an inverse $\psi \mapsto \beta(id_G \times \psi)$, hence F is a left adjoint of U. Consider the monad

$\boldsymbol{T} = (UF, \iota, U(\varepsilon))$ *on* $\mathcal{E}$ *given by the above adjunction, where* $\iota : id_{\mathcal{E}} \to UF$ *and* $\varepsilon : FU \to id_{\mathcal{E}^{G^{op}}}$ *are the unit and the counit. Then the category of* $\boldsymbol{T}$*-algebras* $\mathcal{E}^{\boldsymbol{T}}$ *is isomorphic to* $\mathcal{E}^{G^{op}}$ *and the forgetful functor* $\mathcal{E}^{\boldsymbol{T}} \to \mathcal{E}$ *is identified with* U*. Thus* U *creates limits by (A.3.7).*

We claim that $\mathcal{E}^{G^{op}}$ *is a topos by (2.10.3). Obviously,* $\mathcal{E}^{G^{op}}$ *is a* $\mathcal{U}$*-category. Since* $\mathcal{E}$ *has finite limits and* U *creates them,* $\mathcal{E}^{G^{op}}$ *has finite limits and they are preserved by* U*. Let* $((X_i, \alpha_i))_{i\in I}$ *be a family of* G*-objects indexed by a* $\mathcal{U}$*-small set* I*. Form a coproduct* $\coprod_{i\in I} X_i$ *in* $\mathcal{E}$ *and* $\iota_i : X_i \to \coprod_{i\in I} X_i$ *denotes the canonical morphism into the* i*-th summand. Then, by the universality of coproducts in* $\mathcal{E}$ *and (A.4.5), the morphism* $j : \coprod_{i\in I}(G \times X_i) \to G \times \left(\coprod_{i\in I} X_i\right)$ *induced by* $id_G \times \iota_i : G \times X_i \to G \times \left(\coprod_{i\in I} X_i\right)$ *is an isomorphism. Define* $\alpha : G \times \left(\coprod_{i\in I} X_i\right) \to \coprod_{i\in I} X_i$ *to be* $\left(\coprod_{i\in I} \alpha_i\right) j^{-1}$*. Then,* $\left(\coprod_{i\in I} X_i, \alpha\right)$ *is a coproduct of* $((X_i, \alpha_i))_{i\in I}$ *and it is clearly universal and disjoint. Let* $(R, \rho) \overset{f}{\underset{g}{\rightrightarrows}} (X, \alpha)$ *be an equivalence relation in* $\mathcal{E}^{G^{op}}$*. By (A.3.20),* $R \overset{f}{\underset{g}{\rightrightarrows}} X$ *is an equivalence relation in* $\mathcal{E}$*. Let* $p : X \to Y$ *be an coequalizer of* $R \overset{f}{\underset{g}{\rightrightarrows}} X$ *in* $\mathcal{E}$*, then it is universal and its kernel pair is* $R \overset{f}{\underset{g}{\rightrightarrows}} X$*. Hence by (A.4.5),* $id_G \times p : G \times X \to G \times Y$ *is a coequalizer of* $id_G \times f$*, and* $id_G \times g$*. Since* $p\alpha(id_G \times f) = pf\rho = pg\rho = p\alpha(id_G \times g)$*, there exists a unique morphism* $\beta : G \times Y \to Y$ *satisfying* $\beta(id_G \times p) = p\alpha$*. Then, it is easy to verify that* $p : (X, \alpha) \to (Y, \beta)$ *is a universal coequalizer of* $(R, \rho) \overset{f}{\underset{g}{\rightrightarrows}} (X, \alpha)$ *and* $(R, \rho) \overset{f}{\underset{g}{\rightrightarrows}} (X, \alpha)$ *is a kernel pair of* p*. Let* $\{Z_i\}_{i\in I}$ *be a* $\mathcal{U}$*-small set of generators of* $\mathcal{E}$*. Since* U *reflects isomorphisms, it follows from (A.4.15) that* $\{F(Z_i)\}_{i\in I}$ *is a set of generators of* $\mathcal{E}^{G^{op}}$*.*

Example 2.14.13 *Define a pretopology* P *on the category* $\mathcal{T}op$ *of* $\mathcal{U}$*-small topological spaces by* $P(X) = \{(s_i : U_i \to X)_{i\in I} |\, s_i$ *is an open immersion and* $\bigcup_{i\in I} s_i(U_i) = X\}$ *and regard* $\mathcal{T}op$ *as a site with the topology generated by* P*. If we choose a universe* $\mathcal{V}$ *such that* $\mathcal{U} \in \mathcal{V}$*,* $\mathcal{T}op$ *is a* $\mathcal{V}$*-site. For a topological space* X*, we give* $\mathcal{T}op/X$ *the topology induced by* $\varsigma_X : \mathcal{T}op/X \to \mathcal{T}op$*. We denote by* $\mathrm{TOP}(X)$ *the topos associated with the* $\mathcal{V}$*-site* $\mathcal{T}op/X$*. We call* $\mathrm{TOP}(X)$ *the big topos of* X*. Since the topology on* $\mathcal{T}op/X$ *is coarser than the canonical topology, the Yoneda embedding gives a fully faithful embedding* $\mathcal{T}op/X \to \mathrm{TOP}(X)$*. Hence the notion of the big topos of* X *is a generalization of the category of spaces over* X*.*

Example 2.14.14 *Let* $\mathcal{C}$ *be a* $\mathcal{U}$*-small category and give* $\mathcal{C}$ *the coarsest topology. Then, the category of presheaves* $\widehat{\mathcal{C}}$ *is a* $\mathcal{U}$*-topos. We note that* $\widehat{\mathcal{C}}$ *has a* $\mathcal{U}$*-small set of generators* $\{h_X |\, X \in \mathrm{Ob}\,\mathcal{C}\}$ *such that each member* h_X *is connected and projective. That is, an object* X *of a category* $\mathcal{D}$ *is said to be connected (resp. projective) if the functor* $Y \mapsto \mathcal{D}(X, Y)$ *preserves coproducts (resp. epimorphisms).*

Let $f : Y \to X$ *be a morphism in a category such that* Y *is connected and projective, and* $(g_i : X_i \to X)_{i\in I}$ *an epimorphic family. We denote by* $\iota_i : X_i \to \coprod_{i\in I} X_i$ *the canonical morphism and by* $g : \coprod_{i\in I} X_i \to X$ *the morphism induced by* g_i*'s. Then, for each* $i \in I$*,* $g_* : \mathcal{C}\left(Y, \coprod_{i\in I} X_i\right) \to \mathcal{C}(Y, X)$ *is an epimorphism and* $\iota_{i*} : \mathcal{C}(Y, X_i) \to \mathcal{C}\left(Y, \coprod_{i\in I} X_i\right)$ *induce a bijection* $\coprod_{i\in I} \mathcal{C}(Y, X_i) \to \mathcal{C}\left(Y, \coprod_{i\in I} X_i\right)$*. Hence there exist* $i \in I$ *and a morphism* $\varphi_i : Y \to X_i$ *such that* $f = g_i\varphi_i$*. Therefore in a topos* $\mathcal{E}$ *of the form* $\widehat{\mathcal{C}}$*, every covering of the form* $(h_{Z_k} \to X)_{k\in K}$ *is refined by an arbitrary covering of* X*.*

Example 2.14.15 *Let* I *be a directed set such that* $I \in \mathcal{U}$ *and* $\boldsymbol{G} = (G_i)_{i\in I}$ *a projective system of groups* G_i *with* $G_i \in \mathcal{U}$*. We suppose that this projective system is strict, that is, each transition map* $\rho_i^j : G_j \to G_i$ *is surjective. Let* E *be a* $\mathcal{U}$*-set with a filtration* $(E_i)_{i\in I}$ *such that* $E = \bigcup_{i\in I} E_i$ *and* $E_i \subset E_j$ *if* $i \leqq j$*. A left action of* $\boldsymbol{G}$ *on* E *is a family of operations* $(\alpha_i : G_i \times E_i \to E_i)_{i\in I}$ *satisfying* $\alpha_j(g, x) = \alpha_i(\rho_i^j(g), x)$ *for* $g \in G_j$*,* $x \in E_i$*. We call such* E *with a left* $\boldsymbol{G}$*-action a left* $\boldsymbol{G}$*-set. If a map* $\varphi : E \to F$ *between left* $\boldsymbol{G}$*-sets satisfies* $\varphi(E_i) \subset F_i$ *and* $\varphi|_{E_i} : E_i \to F_i$ *commutes with left* G_i *actions, we say that* φ *is a morphism of left* $\boldsymbol{G}$*-sets. We denote by* $B\boldsymbol{G}$ *the category of left* $\boldsymbol{G}$*-sets. We set* $\Gamma_i = \coprod_{j\leqq i} G_j$ *and* $\Gamma = \coprod_{i\in I} G_i$*. Then,* Γ_i *is regarded as a subset of* Γ_j *if* $i \leqq j$ *and* $\Gamma = \bigcup_{i\in I} \Gamma_i$*. For a fixed* $i \in I$*, compositions* $G_i \times G_j \xrightarrow{\rho_j^i \times 1} G_j \times G_j \xrightarrow{prod} G_j$ $(j \leqq i)$ *define a left* G_i *action on* Γ_i*. Hence* Γ *is a left* $\boldsymbol{G}$*-set. It is easy to verify that* Γ *is a generator of* $B\boldsymbol{G}$*. It follows from (2.10.3) that* $B\boldsymbol{G}$ *is a Grothendieck topos. We call* $B\boldsymbol{G}$ *the classifying topos of* $\boldsymbol{G}$*. Set* $G = \varprojlim G_i$*. Then we have a left* G*-action* $\alpha : G \times E \to E$ *defined by* $\alpha(g, x) = \alpha_j(g_j, x)$ *for* $g = (g_i)_{i\in I}$ *if* $x \in E_j$*.*

2.15 Geometric morphisms

Definition 2.15.1 *Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{U}$-topoi.*

1) A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ consists of functors $f_* : \mathcal{E} \to \mathcal{F}$, $f^* : \mathcal{F} \to \mathcal{E}$ and a bijection $\alpha_f = (\alpha_f)_{Y,X} : \mathcal{E}(f^*(Y), X) \to \mathcal{F}(Y, f_*(X))$ for each $X \in \mathrm{Ob}\,\mathcal{E}$ and $Y \in \mathrm{Ob}\,\mathcal{F}$ such that f^* is left exact and α_f is natural in X and Y. We call f_* the direct image and f^* the inverse image of f.

2) If $f = (f_*, f^*, \alpha_f), g = (g_*, g^*, \alpha_g) : \mathcal{E} \to \mathcal{F}$ are geometric morphisms, a morphism of geometric morphisms $\varphi : f \to g$ means a pair of natural transformations of functors $\varphi_* : g_* \to f_*$ and $\varphi^* : f^* \to g^*$ such that the following square commutes for any $X \in \mathrm{Ob}\,\mathcal{E}$ and $Y \in \mathrm{Ob}\,\mathcal{F}$.

$$\begin{tikzcd}
\mathcal{E}(g^*(Y), X) \arrow[r, "(\alpha_g)_{Y,X}"] \arrow[d, "(\varphi^*_Y)^*"] & \mathcal{F}(Y, g_*(X)) \arrow[d, "(\varphi_{*X})_*"] \\
\mathcal{E}(f^*(Y), X) \arrow[r, "(\alpha_f)_{Y,X}"] & \mathcal{F}(Y, f_*(X))
\end{tikzcd}$$

Remark 2.15.2 *1) If $\varphi = (\varphi_*, \varphi^*) : f \to g$ is a morphism of geometric morphisms, φ^* (resp. φ_*) uniquely determines φ_* (resp. φ^*) by (A.14.1).*

2) Topoi, geometric morphisms and natural transformations form a 2-category, which we denote by $\mathfrak{Top}$.

We often drop the adjonction and denote a geometric (f_*, f^*, α_f) by (f_*, f^*).

Example 2.15.3 *Let $(\mathcal{C}, J)$ be a site such that $\mathcal{C}$ is $\mathcal{U}$-small. Regard the category $\widehat{\mathcal{C}}$ of presheaves as a topos obtained from the coasest topology on $\mathcal{C}$. The pair of the inclusion functor $i : \widetilde{\mathcal{C}} \to \widehat{\mathcal{C}}$ and the accosiated sheaf functor $a : \widehat{\mathcal{C}} \to \widetilde{\mathcal{C}}$ defines a geometric morphism $(i, a) : \widetilde{\mathcal{C}} \to \widehat{\mathcal{C}}$.*

For categories $\mathcal{C}$ and $\mathcal{E}$, we denote by $\mathrm{Filt}(\mathcal{C}, \mathcal{E})$ a full subcategory of the functor category $\mathrm{Funct}(\mathcal{C}, \mathcal{E})$ consisting of filtering functors from $\mathcal{C}$ to $\mathcal{E}$. Let $\mathcal{C}$ be a $\mathcal{U}$-small category and $\mathcal{E}$ a topos which is $\mathcal{U}$-cocomplete. We give $\mathcal{C}$ the coarsest topology and regard $\widehat{\mathcal{C}}$ a $\mathcal{U}$-topos. Define a functor $\Psi : \mathrm{Filt}(\mathcal{C}, \mathcal{E}) \to \mathfrak{Top}(\mathcal{E}, \widehat{\mathcal{C}})$ as follows. For a filtering functor $K : \mathcal{C} \to \mathcal{E}$, $\Psi(K)_* : \mathcal{E} \to \widehat{\mathcal{C}}$ is a composition $\mathcal{E} \xrightarrow{h^{\mathcal{E}}} \widehat{\mathcal{E}} \xrightarrow{K^*} \widehat{\mathcal{C}}$ and $\Psi(K)^* : \widehat{\mathcal{C}} \to \mathcal{E}$ is the left Kan extension of K along the Yoneda embedding $h^{\mathcal{C}} : \mathcal{C} \to \widehat{\mathcal{C}}$. That is, for a presheaf F on $\widehat{\mathcal{C}}$, there is a colimiting cone

$$(KP\langle X, p\rangle \xrightarrow{\lambda(K)^F_{\langle X,p\rangle}} \Psi(K)^*(F))_{\langle X,p\rangle \in \mathrm{Ob}(h^{\mathcal{C}} \downarrow F)}.$$

We choose $\Psi(K)^*$ so that $\Psi(K)^* h^{\mathcal{C}} = K$ holds (2.9.6). Hence if $F = h^{\mathcal{C}}_Y$ for some $Y \in \mathrm{Ob}\,\mathcal{C}$, $\lambda(K)^{h^{\mathcal{C}}_X}_{\langle X,p\rangle} = K(p_X(id_X))$. It follows from (2.9.6) and (2.9.14) that $\Psi(K)^*$ is a left exact left adjoint of $\Psi(K)_*$. We note that the adjunction $\alpha(K)$ obtained from (2.9.6) is the unique natural map such that the following diagram commutes for any $F \in \mathrm{Ob}\,\widehat{\mathcal{C}}$, $\langle X, p\rangle \in \mathrm{Ob}(h^{\mathcal{C}} \downarrow F)$ and $Z \in \mathrm{Ob}\,\mathcal{E}$. Here, $\Theta : h^{\mathcal{E}}_Z(K(X)) \to \widehat{\mathcal{C}}(h^{\mathcal{C}}_X, h^{\mathcal{E}}_Z K)$ denotes the natural bijection (A.1.6).

$$\begin{tikzcd}
\mathcal{E}(\Psi(K)^*(F), Z) \arrow[rrr, "\alpha(K)_{F,Z}"] \arrow[d, "\lambda(K)^{F*}_{\langle X,p\rangle}"] & & & \widehat{\mathcal{C}}(F, \Psi(K)_*(Z)) \arrow[r, equal] & \widehat{\mathcal{C}}(F, h^{\mathcal{E}}_Z K) \arrow[d, "p^*"] \\
\mathcal{E}(KP\langle X, p\rangle, Z) \arrow[r, equal] & h^{\mathcal{E}}_Z(K(X)) \arrow[r, "\Theta"] & \widehat{\mathcal{C}}(h^{\mathcal{C}}_X, h^{\mathcal{E}}_Z K) \arrow[rr, equal] & & \widehat{\mathcal{C}}(h^{\mathcal{C}} P\langle X, p\rangle, h^{\mathcal{E}}_Z K)
\end{tikzcd}$$

Thus we have a geometric morphism $\Psi(K) = (\Psi(K)_*, \Psi(K)^*, \alpha(K))$. Suppose that $\theta : K \to K'$ is a natural transfomation of filtering functors. For $Z \in \mathrm{Ob}\,\mathcal{E}$, $\Psi(\theta)_{*Z} : \Psi(K')_* \to \Psi(K)_*$ is a morphism $h^{\mathcal{E}}_Z(\theta) : h^{\mathcal{E}}_Z K' \to h^{\mathcal{E}}_Z K$ in $\widehat{\mathcal{C}}$. For a presheaf F on $\mathcal{C}$, define $\Psi(\theta)^*_F : \Psi(K)^*(F) \to \Psi(K')^*(F)$ to be the unique morphism such that the following square on the left commutes for any $\langle X, p\rangle \in \mathrm{Ob}(h^{\mathcal{C}} \downarrow F)$.

$$\begin{tikzcd}
KP\langle X, p\rangle \arrow[r, "\lambda(K)^F_{\langle X,p\rangle}"] \arrow[d, "\theta_{P\langle X,p\rangle}"] & \Psi(K)^*(F) \arrow[d, "\Psi(\theta)^*_F"] \\
K'P\langle X, p\rangle \arrow[r, "\lambda(K')^F_{\langle X,p\rangle}"] & \Psi(K')^*(F)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}(\Psi(K')^*(F), Z) \arrow[r, "\alpha(K')_{F,Z}"] \arrow[d, "(\Psi(\theta)^*_F)^*"] & \widehat{\mathcal{C}}(F, \Psi(K')_*(Z)) \arrow[d, "(\Psi(\theta)_{*Z})_*"] \\
\mathcal{E}(\Psi(K)^*(F), Z) \arrow[r, "\alpha(K)_{F,Z}"] & \widehat{\mathcal{C}}(F, \Psi(K)_*(Z))
\end{tikzcd}$$

We claim that the above right diagram also commutes. In fact, for any $\langle X,p\rangle \in \mathrm{Ob}(h^{\mathcal{C}}\downarrow F)$, $p^*\alpha(K)_{F,Z}(\Psi(\theta)^*_F)^* = \Theta\lambda(K)^{F*}_{\langle X,p\rangle}(\Psi(\theta)^*_F)^* = \Theta\theta^*_{P\langle X,p\rangle}\lambda(K')^{F*}_{\langle X,p\rangle} = h^{\mathcal{E}}_Z(\theta)_*\Theta\lambda(K')^{F*}_{\langle X,p\rangle} = h^{\mathcal{E}}_Z(\theta)_*p^*\alpha(K')_{F,Z} = p^*(\Psi(\theta)_*)_*\alpha(K')_{F,Z}$. Since

$$(\widehat{\mathcal{C}}(F,\Psi(K)_*(Z)) \xrightarrow{p^*} \widehat{\mathcal{C}}(h^{\mathcal{C}}P\langle X,p\rangle,\Psi(K)_*(Z)))_{\langle X,p\rangle\in\mathrm{Ob}(h^{\mathcal{C}}\downarrow F)}$$

is a limiting cone, we have $\alpha(K)_{F,Z}(\Psi(\theta)^*_F)^* = (\Psi(\theta)_*)_*\alpha(K')_{F,Z}$. Thus $(\Psi(\theta)_*,\Psi(\theta)^*)$ is a morphism of geometric morphisms. It is easy to verify that Ψ is a functor.

Proposition 2.15.4 *$\Psi : \mathrm{Filt}(\mathcal{C},\mathcal{E}) \to \mathfrak{Top}(\mathcal{E},\widehat{\mathcal{C}})$ is an equivalence of the categories.*

Proof. For a geometric morphism $f : \mathcal{E} \to \widehat{\mathcal{C}}$, $f^* : \widehat{\mathcal{C}} \to \mathcal{E}$ is a left Kan extension of $f^*h^{\mathcal{C}} : \mathcal{C} \to \mathcal{E}$ along $h^{\mathcal{C}}$. In fact, since f^* has a right adjoint, it preserves colimits. For a presheaf F on $\widehat{\mathcal{C}}$, applying f^* to the colimiting cone $(h^{\mathcal{C}}P\langle X,p\rangle = h^{\mathcal{C}}_X \xrightarrow{p} F)_{\langle X,p\rangle\in\mathrm{Ob}(h^{\mathcal{C}}\downarrow F)}$, (A.4.2) we have a colimiting cone $(f^*h^{\mathcal{C}}P\langle X,p\rangle = f^*(h^{\mathcal{C}}_X) \xrightarrow{f^*(p)} f^*(F))_{\langle X,p\rangle\in\mathrm{Ob}(h^{\mathcal{C}}\downarrow F)}$. Since f^* is left exact, it follows from (2.9.14) that $f^*h^{\mathcal{C}} : \mathcal{C} \to \mathcal{E}$ is a filtering functor. The quasi-inverse $\Phi : \mathfrak{Top}(\mathcal{E},\widehat{\mathcal{C}}) \to \mathrm{Filt}(\mathcal{C},\mathcal{E})$ of Ψ is given by as follows. For a geometric morphism $f = (f_*, f^*, \alpha_f) : \mathcal{E} \to \widehat{\mathcal{C}}$, set $\Phi(f) = f^*h^{\mathcal{C}}$. Then, for a filtering functor $K : \mathcal{C} \to \mathcal{E}$, we have $\Phi\Psi(K) = \Psi(K)^*h^{\mathcal{C}} = K$. For a morphism $\varphi : f \to g$ of geometric morphisms, we put $\Phi(\varphi) = \varphi_{h^{\mathcal{C}}} : f^*h^{\mathcal{C}} \to g^*h^{\mathcal{C}}$. Let $\theta : K \to K'$ be a morphism of filtering functors and Y an object of $\mathcal{C}$. Since the following square commutes for any $\langle X,p\rangle \in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow h^{\mathcal{C}}_Y)$, we have $\Phi\Psi(\theta)_Y = \Psi(\theta)^*_{h^{\mathcal{C}}_Y} = \theta_Y$.

$$\begin{tikzcd}
KP\langle X,p\rangle \arrow[r, "\lambda(K)^{h^{\mathcal{C}}_Y}_{\langle X,p\rangle} = K(p_X(id_X))"] \arrow[d, "\theta_{P\langle X,p\rangle}"] & \Psi(K)^*(h^{\mathcal{C}}_Y) \arrow[r, equal] & K(Y) \arrow[d, "\theta_Y"] \\
K'P\langle X,p\rangle \arrow[r, "\lambda(K')^{h^{\mathcal{C}}_Y}_{\langle X,p\rangle} = K'(p_X(id_X))"] & \Psi(K')^*(h^{\mathcal{C}}_Y) \arrow[r, equal] & K'(Y)
\end{tikzcd}$$

Hence $\Psi\Phi = id_{\mathrm{Filt}(\mathcal{C},\mathcal{E})}$ holds. On the other hand, for a geometric morphism $f : \mathcal{E} \to \widehat{\mathcal{C}}$, since both f^* and $\Psi(f^*h^{\mathcal{C}})^*$ are left Kan extensions of $f^*h^{\mathcal{C}} : \mathcal{C} \to \mathcal{E}$ along $h^{\mathcal{C}}$ such that $\Psi(f^*h^{\mathcal{C}})^*h^{\mathcal{C}} = f^*h^{\mathcal{C}}$, there is a natural equivalence $\kappa^*_f : f^* \to \Phi(f^*h^{\mathcal{C}})^*$ such that, for any $F \in \mathrm{Ob}\,\widehat{\mathcal{C}}$ and $\langle X,p\rangle \in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow F)$, the following diagram commutes.

$$\begin{tikzcd}
f^*(h^{\mathcal{C}}_X) \arrow[r, "f^*(p)"] \arrow[d, equal] & f^*(F) \arrow[d, "(\kappa^*_f)_F"] \\
f^*h^{\mathcal{C}}P\langle X,p\rangle \arrow[r, "\lambda(f^*h^{\mathcal{C}})^F_{\langle X,p\rangle}"] & \Phi(f^*h^{\mathcal{C}})^*(F)
\end{tikzcd}$$

We define an equivalence $\kappa_{f*} : \Phi(f^*h^{\mathcal{C}})_* \to f_*$ as follows. For $Z \in \mathrm{Ob}\,\mathcal{E}$ and $X \in \mathrm{Ob}\,\mathcal{C}$, $(\kappa_{f*Z})_X : \Phi(f^*h^{\mathcal{C}})_*(Z)(X) \to f_*(Z)(X)$ is the composition

$$\Phi(f^*h^{\mathcal{C}})_*(Z)(X) = h^{\mathcal{E}}_Z(f^*(h^{\mathcal{C}}_X)) = \mathcal{E}(f^*(h^{\mathcal{C}}_X),Z) \xrightarrow[\cong]{(\alpha_f)_{h^{\mathcal{C}}_X,Z}} \widehat{\mathcal{C}}(h^{\mathcal{C}}_X, f_*(Z)) \xrightarrow[\cong]{\Theta^{-1}} f_*(Z)(X)$$

of the natural bijections. Then, the following diagram commutes.

$$\begin{tikzcd}
\widehat{\mathcal{C}}(F,\Phi(f^*h^{\mathcal{C}})_*(Z)) \arrow[r, "p^*"] \arrow[d, "(\kappa_{f*})_*"] & \widehat{\mathcal{C}}(h^{\mathcal{C}}P\langle X,p\rangle,\Phi(f^*h^{\mathcal{C}})_*(Z)) \arrow[d, "(\kappa_{f*})_*"] & \Psi(f^*h^{\mathcal{C}})_*(Z)(X) \arrow[l, "\Theta"'] \arrow[d, equal] \\
\widehat{\mathcal{C}}(F,f_*(Z)) \arrow[r, "p^*"] & \widehat{\mathcal{C}}(h^{\mathcal{C}}P\langle X,p\rangle,f_*(Z)) & \mathcal{E}(f^*(h^{\mathcal{C}}_X),Z) \arrow[l, "(\alpha_f)_{h^{\mathcal{C}}_X,Z}"']
\end{tikzcd}$$

We also have the following commutative diagram for each $\langle X,p\rangle \in \mathrm{Ob}\,(h^{\mathcal{C}}\downarrow F)$.

$$\begin{tikzcd}
\mathcal{E}(f^*(F),Z) \arrow[r, "(\alpha_f)_{F,Z}"] \arrow[d, "f^*(p)^*"] & \widehat{\mathcal{C}}(F,f_*(Z)) \arrow[d, "p^*"] \\
\mathcal{E}(f^*(h^{\mathcal{C}}_X),Z) \arrow[r, "(\alpha_f)_{h^{\mathcal{C}}_X,Z}"] & \widehat{\mathcal{C}}(h^{\mathcal{C}}_X,Z)
\end{tikzcd}$$

By the definition of the adjunction $\alpha(f^*h^{\mathcal{C}})$ and the comutativity of the diagrams above, $p^*(\kappa_{f*})_*\alpha(f^*h^{\mathcal{C}})_{F,Z} = (\kappa_{f*})_*p^*\alpha(f^*h^{\mathcal{C}})_{F,Z} = (\kappa_{f*})_*\Theta\lambda(f^*h^{\mathcal{C}})^{F*}_{\langle X,p\rangle} = (\alpha_f)_{h^{\mathcal{C}}_X,Z}f^*(p)^*(\kappa^*_f)^*$
$= p^*(\alpha_f)_{F,Z}(\kappa^*_f)^*$. Since

$$(\widehat{\mathcal{C}}(F,\Psi(f^*h^{\mathcal{C}})_*(Z)) \xrightarrow{p^*} \widehat{\mathcal{C}}(h^{\mathcal{C}}P\langle X,p\rangle,\Psi(f^*h^{\mathcal{C}})_*(Z)))_{\langle X,p\rangle\in\mathrm{Ob}(h^{\mathcal{C}}\downarrow F)}$$

is a limiting cone, it follows that $(\kappa_{f*})_*\alpha(f^*h^{\mathcal{C}})_{F,Z} = (\alpha_f)_{F,Z}(\kappa^*_f)^*$. Thus we have an isomorphism $\kappa_f = (\kappa_{f*},\kappa^*_f) : f \to \Psi\Phi(f)$ of geometric morphisms. Finally, we verify the naturality of κ_f in f. Let $\varphi : f \to g$ be a morphism in $\mathfrak{Top}(\mathcal{E},\widehat{\mathcal{C}})$. For a presheaf F on $\mathcal{C}$ and $\langle X,p\rangle \in \mathrm{Ob}(h^{\mathcal{C}}\downarrow F)$, consider the following diagram.

$$\begin{tikzcd}
f^*h^{\mathcal{C}}P\langle X,p\rangle \arrow[r, "f^*(p)"] \arrow[d, "\varphi^*_{h^{\mathcal{C}}_X}"] & f^*(F) \arrow[r, "(\kappa^*_f)_F"] \arrow[d, "\varphi^*_F"] & \Psi(f^*h^{\mathcal{C}})^*(F) \arrow[d, "\Psi(\varphi^*_{h^{\mathcal{C}}})^*_F"] \\
g^*h^{\mathcal{C}}P\langle X,p\rangle \arrow[r, "g^*(p)"] & g^*(F) \arrow[r, "(\kappa^*_g)_F"] & \Psi(g^*h^{\mathcal{C}})^*(F)
\end{tikzcd}$$

Clearly, the left square is commutative. Since $(\kappa^*_f)_Ff^*(p) = \lambda(f^*h^{\mathcal{C}})^F_{\langle X,p\rangle}$ and $(\kappa^*_g)_Fg^*(p) = \lambda(g^*h^{\mathcal{C}})^F_{\langle X,p\rangle}$, the outer rectangle also commutes. Thus we have $\Psi(\varphi^*_{h^{\mathcal{C}}})^*(\kappa^*_f)_Ff^*(p) = (\kappa^*_g)_F\varphi^*_Ff^*(p)$. Recall that $(f^*h^{\mathcal{C}}P\langle X,p\rangle \xrightarrow{f^*(p)} f^*(F))_{\langle X,p\rangle\in\mathrm{Ob}(h^{\mathcal{C}}\downarrow F)}$ is a colimiting cone. It follows that $\Psi(\varphi^*_{h^{\mathcal{C}}})^*(\kappa^*_f)_F = (\kappa^*_g)_F\varphi^*_Ff^*(p)$, namely the right square of the above diagram commutes. Hence the following diagram on the left commutes for any $F \in \mathrm{Ob}\,\widehat{\mathcal{C}}$ and $Z \in \mathrm{Ob}\,\mathcal{E}$.

$$\begin{tikzcd}
\mathcal{E}(\Psi(g^*h^{\mathcal{C}})^*(F),Z) \arrow[r, "(\kappa^*_g)^*_F"] \arrow[d, "(\Psi(\varphi^*_{h^{\mathcal{C}}})^*_F)^*"] & \mathcal{E}(g^*(F),Z) \arrow[d, "(\varphi^*_F)^*"] \\
\mathcal{E}(\Psi(f^*h^{\mathcal{C}})^*(F),Z) \arrow[r, "(\kappa^*_f)^*_F"] & \mathcal{E}(f^*(F),Z)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}(F,\Psi(g^*h^{\mathcal{C}})_*(Z)) \arrow[r, "(\kappa_{g*})_{F*}"] \arrow[d, "(\Psi(\varphi^*_{h^{\mathcal{C}}})_{*F})_*"] & \mathcal{E}(F,g_*(Z)) \arrow[d, "(\varphi_{*F})_*"] \\
\mathcal{E}(F,\Psi(f^*h^{\mathcal{C}})_*(Z)) \arrow[r, "(\kappa_{f*})_{F*}"] & \mathcal{E}(F,f_*(Z))
\end{tikzcd}$$

Taking the adjoints, we see that the right diagram also commutes. This shows the naturality of κ_f in f and we have a natural equivalence of functors $\kappa : id_{\mathfrak{Top}(\mathcal{E},\widehat{\mathcal{C}})} \to \Psi\Phi$. □

Lemma 2.15.5 *Let $f : \mathcal{E} \to \widehat{\mathcal{C}}$ be a geometric morphism and J a $\mathcal{U}$-topology on $\mathcal{C}$.*

1) Suppose that $f^*h^{\mathcal{C}} : \mathcal{C} \to \mathcal{E}$ maps coverings for J to epimorphic families. If a morphism $p : H \to K$ of presheaves on $\mathcal{C}$ is a covering (resp. bicovering) in the sense of (2.5.3), $f^*(p) : f^*(H) \to f^*(K)$ is an epimorphism (resp. isomorphism).

2) Let $i = (i,a) : \widetilde{\mathcal{C}} \to \widehat{\mathcal{C}}$ be the geometric morphism in (2.15.3). The following conditions are equivalent.

(i) There exist a geometric morphism $\tilde{f} : \mathcal{E} \to \widetilde{\mathcal{C}}$ and an isomorphism $f \to i\tilde{f}$.
(ii) $f^*h^{\mathcal{C}} : \mathcal{C} \to \mathcal{E}$ maps coverings for J to epimorphic families.

Proof. 1) We first show that, for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $R \in J(X)$, the morphism $f^*(\iota) : f^*(R) \to f^*(h_X)$ induced by the inclusion morphism $\iota : R \to h_X$ is an isomorphism. Since $f^* : \widehat{\mathcal{C}} \to \mathcal{E}$ is left exact, $f^*(\iota) : f^*(R) \to f^*(h_X)$ is a monomorhism. Suppose that $s,t : f^*(h_X) \to Z$ are morphisms in $\mathcal{E}$ such that $sf^*(\iota) = tf^*(\iota)$. We choose a covering $(p_i : X_i \to X)_{i\in I}$ which generates R. Then, $h_{p_i} : h_{X_i} \to h_X$ factors through ι and there is a unique morphism $p^\sharp_i : h_{X_i} \to R$ such that $h_{p_i} = \iota p^\sharp_i$. Hence $sf^*(h_{p_i}) = sf^*(\iota)f^*(p^\sharp_i) = tf^*(\iota)f^*(p^\sharp_i) = tf^*(h_{p_i})$. Since $(f^*(h_{p_i}) : f^*(h_{X_i}) \to f^*(h_X))_{i\in I}$ is an epimorphic family by the assumption, we have $s = t$. Thus $f^*(\iota)$ is also an epimorphism and it follows from (2.4.5) that $f^*(\iota)$ is an isomorphism.

Suppose that $p : H \to K$ is a covering and $s,t : f^*(K) \to Z$ are morphisms in $\mathcal{E}$ such that $sf^*(p) = tf^*(p)$. Let us denote by $s',t' : K \to f_*(Z)$ the adjoint of s, t respectively. For arbitrary $X \in \mathrm{Ob}\,\mathcal{C}$ and $\varphi : h_X \to K$, form a pull-back of p along φ as follows.

$$\begin{tikzcd}
H \times_K h_X \arrow[r, "\bar{p}"] \arrow[d, "\bar{\varphi}"] & h_X \arrow[d, "\varphi"] \\
H \arrow[r, "p"] & K
\end{tikzcd}$$

We denote by R the image of $\bar{p}$ and $H \times_K h_X \overset{\pi}{\twoheadrightarrow} R \overset{\iota}{\rightarrowtail} h_X$ the mono-epi factorization of $\bar{p}$. Then, R is a covering sieve by the assumption. Since f^* preserves epimorphisms by (A.3.13) and $f^*(\iota)$ is an isomorphism, $f^*(\bar{p})$ is an epimorphism. It follows from $sf^*(\varphi)f^*(\bar{p}) = sf^*(p)f^*(\bar{\varphi}) = tf^*(p)f^*(\bar{\varphi}) = tf^*(\varphi)f^*(\bar{p})$ that $sf^*(\varphi) = tf^*(\varphi)$. Taking the adjoint, we have $s'\varphi = t'\varphi$. Since φ is arbitrary, $s' = t'$ thus $s = t$. Therefore $f^*(p)$ is an epimorphism.

If $p : H \to K$ is a bicovering, then $f^*(p)$ is an epimorphism and the diagonal morphism $\Delta : H \to H \times_K H$ is a covering. Since f^* preserves pull-backs, it follows from the above result that the diagonal morphism $f^*(H) \to f^*(H) \times_{f^*(K)} f^*(H)$ is an epimorphism. This shows that $f^*(p)$ is also a monomorphism. Hence $f^*(p)$ is an isomorphism.

2) $(i) \Rightarrow (ii)$; Let $\tilde{f} : \mathcal{E} \to \widetilde{\mathcal{C}}$ be a geometric morphism, $\chi_f : f \to i\tilde{f}$ an isomorphism and $(p_i : X_i \to X)_{i\in I}$ a covering on $X \in \mathrm{Ob}\,\mathcal{C}$ for J. Then, $(\epsilon_J(p_i) : \epsilon_J(X_i) \to \epsilon_J(X))_{i\in I}$ is an epimorphic family in $\widetilde{\mathcal{C}}$ by (2.4.7). Since $\tilde{f}^* : \widetilde{\mathcal{C}} \to \mathcal{E}$ has a right adjoint $\tilde{f}_*$, it preserves epimorphic families. Hence $(\tilde{f}^*\epsilon_J(p_i) : \tilde{f}^*\epsilon_J(X_i) \to \tilde{f}^*\epsilon_J(X))_{i\in I}$ is an epimorphic family in $\mathcal{E}$. The assertion follows from the fact that $(\chi_f^*)_{h^{\mathcal{C}}} : f^*h^{\mathcal{C}} \to (i\tilde{f})^*h^{\mathcal{C}} = \tilde{f}^*ah^{\mathcal{C}} = \tilde{f}^*\epsilon_J$ is a natural equivalence.

$(ii) \Rightarrow (i)$; Let R be a covering sieve on $X \in \mathrm{Ob}\,\mathcal{C}$ and $\iota : R \to h_X$ denotes the inclusion morphism. Then, $f^*(\iota) : f^*(R) \to f^*(h_X)$ is an isomorphism by 1). By the commutativity of following diagram, the map $\iota^* : \widehat{\mathcal{C}}(h_X, f_*(Z)) \to \widehat{\mathcal{C}}(R, f_*(Z))$ induced by ι is bijective for any $Z \in \mathrm{Ob}\,\mathcal{E}$.

$$\begin{tikzcd}
\widehat{\mathcal{C}}(h_X, f_*(Z)) \arrow[r, "\iota"] \arrow[d, "\alpha_f"] & \widehat{\mathcal{C}}(R, f_*(Z)) \arrow[d, "\alpha_f"] \\
\mathcal{E}(f^*(h_X), Z) \arrow[r, "f^*(\iota)"] & \mathcal{E}(f^*(R), Z)
\end{tikzcd}$$

Therefore $f_*(Z)$ is a sheaf on $\mathcal{C}$ and there is a functor $\tilde{f}_* : \mathcal{E} \to \widetilde{\mathcal{C}}$ such that $f_* = i\tilde{f}_*$. Set $\tilde{f}^* = f^*i : \widetilde{\mathcal{C}} \to \mathcal{E}$, then $\tilde{f}^*$ is left exact and $\mathcal{E}(\tilde{f}^*(F), Z) = \mathcal{E}(f^*i(F), Z) \underset{\cong}{\xrightarrow{\alpha_f}} \widehat{\mathcal{C}}(i(F), f_*(Z)) = \widehat{\mathcal{C}}(i(F), i\tilde{f}_*(Z)) \overset{[}{\cong}] i^{-1}\widetilde{\mathcal{C}}(F, \tilde{f}_*(Z))$. Hence $\tilde{f} = (\tilde{f}_*, \tilde{f}^*) : \mathcal{E} \to \widetilde{\mathcal{C}}$ is a geometric morphism. Moreover, since $i\tilde{f} = (i\tilde{f}_*, \tilde{f}^*a) = (f_*, f^*ia)$, the unit $\eta : id_{\widehat{\mathcal{C}}} \to ia$ of the adjunction induces a morphism $(id_{f_*}, f^*(\eta)) : f \to i\tilde{f}$ of geometric morphisms. In fact, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{E}(f^*ia(F), Z) \arrow[r, "(\alpha_f)_{ia(F),Z}"] \arrow[d, "f^*(\eta)_F^*"] & \widehat{\mathcal{C}}(ia(F), f_*(Z)) \arrow[r, equal] \arrow[d, "\eta_F^*"] & \widetilde{\mathcal{C}}(a(F), \tilde{f}_*(Z)) \\
\mathcal{E}(f^*(F), Z) \arrow[r, "(\alpha_f)_{F,Z}"] & \widehat{\mathcal{C}}(F, f_*(Z)) & \widehat{\mathcal{C}}(ia(F), i\tilde{f}_*(Z)) \arrow[l, "ad"'] \arrow[u, "\cong"', "i"]
\end{tikzcd}$$

Since, for any presheaf F, $\eta_F : F \to ia(F)$ is a bicovering, $f^*(\eta_F)$ is an isomorphism by 1) and $(id_{f_*}, f^*(\eta))$ is an isomorphism of geometric morphisms. $\square$

We remark that the isomorphism $\chi_f = (id_{f_*}, f^*(\eta)) : f \to i\tilde{f}$ is natural in f. In fact, let $\varphi : f \to g$ be a morphism of geometric morphisms satisfying the condition (ii) of (2.15.5). Since $i : \widetilde{\mathcal{C}} \to \widehat{\mathcal{C}}$ is fully faithful, there is a unique natural transformation $\tilde{\varphi}_* : \tilde{g}_* \to \tilde{f}_*$ such that $i(\tilde{\varphi}_*) = \varphi_* : g_* \to f_*$. Set $\tilde{\varphi}^* = \varphi_i^* : \tilde{f}^* \to \tilde{g}^*$. Then, the following diagram commutes for any $F \in \mathrm{Ob}\,\widetilde{\mathcal{C}}$ and $Z \in \mathrm{Ob}\,\mathcal{E}$.

$$\begin{tikzcd}
\mathcal{E}(g^*i(F), Z) \arrow[r, "\alpha_g"] \arrow[d, "(\varphi_{i(F)}^*)^*"] & \widehat{\mathcal{C}}(i(F), g_*(Z)) \arrow[r, equal] \arrow[d, "(\varphi_{*Z})_*"] & \widehat{\mathcal{C}}(i(F), i\tilde{g}_*(Z)) \arrow[d, "i(\tilde{\varphi}_{*Z})_*"] & \widetilde{\mathcal{C}}(F, \tilde{g}_*(Z)) \arrow[l, "i", "\cong"'] \arrow[d, "(\tilde{\varphi}_{*Z})_*"] \\
\mathcal{E}(f^*i(F), Z) \arrow[r, "\alpha_f"] & \widehat{\mathcal{C}}(i(F), f_*(Z)) \arrow[r, equal] & \widehat{\mathcal{C}}(i(F), i\tilde{f}_*(Z)) & \widetilde{\mathcal{C}}(F, \tilde{f}_*(Z)) \arrow[l, "i", "\cong"']
\end{tikzcd}$$

Hence $\tilde{\varphi} = (\tilde{\varphi}_*, \tilde{\varphi}^*) : \tilde{f} \to \tilde{g}$ is a morphism of geometric morphisms and we have the following commutative diagrams, which show the naturality of χ_f.

$$\begin{tikzcd}
f^* \arrow[r, "f^*(\eta)"] \arrow[d, "\varphi^*"] & f^*ia \arrow[r, equal] \arrow[d, "\varphi_{ia}^*"] & (i\tilde{f})^* \arrow[d, "\tilde{\varphi}_a^*"] \\
g^* \arrow[r, "g^*(\eta)"] & g^*ia \arrow[r, equal] & b(i\tilde{g})^*
\end{tikzcd}
\qquad
\begin{tikzcd}
f_* & f_* \arrow[l, "id_{f_*}"'] \arrow[r, equal] & (i\tilde{f})_* \\
g_* \arrow[u, "\varphi_*"] & g_* \arrow[l, "id_{g_*}"'] \arrow[r, equal] \arrow[u, "\varphi_*"] & (i\tilde{g})_* \arrow[u, "i(\tilde{\varphi}_*)"']
\end{tikzcd}$$

For a $\mathcal{U}$-topology J on $\mathcal{C}$, let us denote by $\mathrm{Filt}_J(\mathcal{C}, \mathcal{E})$ the full subcategory of $\mathrm{Filt}(\mathcal{C}, \mathcal{E})$ consisting of filtering functors which maps coverings for J to epimorphic families. We denote by $j : \mathrm{Filt}_J(\mathcal{C}, \mathcal{E}) \to \mathrm{Filt}(\mathcal{C}, \mathcal{E})$ the inclusion functor.

Theorem 2.15.6 *$\Psi : \mathrm{Filt}(\mathcal{C},\mathcal{E}) \to \mathfrak{Top}(\mathcal{E},\widehat{\mathcal{C}})$ induces an equivalence $\widetilde{\Psi} : \mathrm{Filt}_J(\mathcal{C},\mathcal{E}) \to \mathfrak{Top}(\mathcal{E},\widetilde{\mathcal{C}})$ of categories with a quasi-inverse $\widetilde{\Phi} : \mathfrak{Top}(\mathcal{E},\widetilde{\mathcal{C}}) \to \mathrm{Filt}_J(\mathcal{C},\mathcal{E})$ such that the following diagram on the left commutes up to natual equivalence and the right one is commutative.*

$$\begin{tikzcd}
\mathrm{Filt}_J(\mathcal{C},\mathcal{E}) \arrow[r, "\widetilde{\Psi}"] \arrow[d, "j"] & \mathfrak{Top}(\mathcal{E},\widetilde{\mathcal{C}}) \arrow[d, "i_*"] \\
\mathrm{Filt}(\mathcal{C},\mathcal{E}) \arrow[r, "\Psi"] & \mathfrak{Top}(\mathcal{E},\widehat{\mathcal{C}})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{Top}(\mathcal{E},\widetilde{\mathcal{C}}) \arrow[r, "\widetilde{\Phi}"] \arrow[d, "i_*"] & \mathrm{Filt}_J(\mathcal{C},\mathcal{E}) \arrow[d, "j"] \\
\mathfrak{Top}(\mathcal{E},\widehat{\mathcal{C}}) \arrow[r, "\Phi"] & \mathrm{Filt}(\mathcal{C},\mathcal{E})
\end{tikzcd}$$

Proof. Since $\Psi(j(K))^*h^{\mathcal{C}} = j(K)$ for each $K \in \mathrm{Ob}\,\mathrm{Filt}_J(\mathcal{C},\mathcal{E})$, there exist a geometric morphism $\widetilde{\Psi}(K) : \mathcal{E} \to \widetilde{\mathcal{C}}$ such that $i\widetilde{\Psi}(K)_* = \Psi(j(K))_*$, $\widetilde{\Psi}(K)^* = \Psi(j(K))^*i$ and an isomorphism $\chi_{\Psi(j(K))} = (id_{\Psi(j(K))_*}, \Psi(j(K))^*(\eta)) : \Psi j(K) \to i\widetilde{\Psi}(K)$ by (2.15.5). If $\theta : K \to K'$ is a morphism in $\mathrm{Filt}_J(\mathcal{C},\mathcal{E})$, $\widetilde{\Psi}(\theta)_* : \widetilde{\Psi}(K')_* \to \widetilde{\Psi}(K)_*$ is the unique morphism satisfying $i(\widetilde{\Psi}(\theta)_*) = \Psi(j(\theta))_*$ and $\widetilde{\Psi}(\theta)^* : \widetilde{\Psi}(K)^* \to \widetilde{\Psi}(K')^*$ is defined by $\widetilde{\Psi}(\theta)^* = \Psi(j(\theta))^*_i$. Then, the following diagrams commute for any $F \in \mathrm{Ob}\,\widetilde{\mathcal{C}}$, $Z \in \mathrm{Ob}\,\mathcal{E}$ and $\widetilde{\Psi}(\theta) = (\widetilde{\Psi}(\theta)_*, \widetilde{\Psi}(\theta)^*) : \widetilde{\Psi}(K) \to \widetilde{\Psi}(K')$ is a morphism of geometric morphisms.

$$\begin{tikzcd}
\mathcal{E}(\widetilde{\Psi}(K')^*(F),Z) \arrow[r, equal] \arrow[d, "(\widetilde{\Psi}(\theta)^*_F)^*"] & \mathcal{E}(\Psi(j(K'))^*i(F),Z) \arrow[r, "\alpha(j(K'))"] \arrow[d, "(\Psi(j(\theta))^*_{i(F)})^*"] & \widehat{\mathcal{C}}(i(F),\Psi(j(K'))_*(Z)) \arrow[d, "(\Psi(j(\theta))_{*Z})_*"] \\
\mathcal{E}(\widetilde{\Psi}(K)^*(F),Z) \arrow[r, equal] & \mathcal{E}(\Psi(j(K))^*i(F),Z) \arrow[r, "\alpha(j(K))"] & \widehat{\mathcal{C}}(i(F),\Psi(j(K))_*(Z))
\end{tikzcd}$$

$$\begin{tikzcd}
\widehat{\mathcal{C}}(i(F),\Psi(j(K'))_*(Z)) \arrow[r, equal] \arrow[d, "(\Psi(j(\theta))_{*Z})_*"] & \widehat{\mathcal{C}}(i(F),i\widetilde{\Psi}(K')_*(Z)) \arrow[d, "(i\widetilde{\Psi}(\theta)_{*Z})_*"] & \widetilde{\mathcal{C}}(F,\widetilde{\Psi}(K')_*(Z)) \arrow[l, "i", "\cong"'] \arrow[d, "(\widetilde{\Psi}(\theta)_{*Z})_*"] \\
\widehat{\mathcal{C}}(i(F),\Psi(j(K))_*(Z)) \arrow[r, equal] & \widehat{\mathcal{C}}(i(F),i\widetilde{\Psi}(K)_*(Z)) & \widetilde{\mathcal{C}}(F,\widetilde{\Psi}(K)_*(Z)) \arrow[l, "i", "\cong"']
\end{tikzcd}$$

Thus we have a functor $\widetilde{\Psi} : \mathrm{Filt}_J(\mathcal{C},\mathcal{E}) \to \mathfrak{Top}(\mathcal{E},\widetilde{\mathcal{C}})$. As we remarked before, $\chi_{\Psi(j(K))} : \Psi j(K) \to i\widetilde{\Psi}(K)$ is natural in K. Hence we also have a natural equivalence $\chi : \Psi j \to i_*\widetilde{\Psi}$.

Let $f : \mathcal{E} \to \widetilde{\mathcal{C}}$ be a geometric morphism. Then, for any J-covering $(p_i : X_i \to X)_{i\in I}$, $(\epsilon_J(p_i) : \epsilon_J(X_i) \to \epsilon_J(X))_{i\in I}$ is an epimorphic family in $\widetilde{\mathcal{C}}$ by (2.4.7). Since $f^* : \widetilde{\mathcal{C}} \to \mathcal{E}$ has a right adjoint, it preserves epimorphic families. Hence $\Phi i_*(f) = (if)^*h^{\mathcal{C}} = f^*\epsilon_J$ satisfies the condition (ii) of (2.15.5) and there is a functor $\widetilde{\Phi} : \mathfrak{Top}(\mathcal{E},\widetilde{\mathcal{C}}) \to \mathrm{Filt}_J(\mathcal{C},\mathcal{E})$ such that $j\widetilde{\Phi} = \Phi i_*$. Then $\Phi(\chi_{\Psi(j(K))}) : j(K) = \Phi\Psi(j(K)) \to \Phi(i\widetilde{\Psi}(K)) = j\widetilde{\Phi}\widetilde{\Psi}(K)$ is an equivalence which is natural in K. Since j is fully faithful, we have a natural equivalence $\tilde{\chi} : id_{\mathrm{Filt}_J(\mathcal{C},\mathcal{E})} \to \widetilde{\Phi}\widetilde{\Psi}$ such that $j(\tilde{\chi}_K) = \Phi(\chi_{\Psi(j(K))})$.

For a geometric morphism $f : \mathcal{E} \to \widetilde{\mathcal{C}}$, we note that $(\widetilde{\Psi}\widetilde{\Phi}(f))^* = \Psi(j\widetilde{\Phi}(f))^*i = \Psi(\Phi i_*(f))^*i = (\Psi\Phi(if))^*i$ and $i(\widetilde{\Psi}\widetilde{\Phi}(f))_* = \Psi(j\widetilde{\Phi}(f))_* = \Psi(\Phi i_*(f))_* = (\Psi\Phi(if))_*$. Hence the isomorphism $\kappa_{if} : if \to \Psi\Phi(if)$ of geometric morphisms induces natural equivalences $(\kappa^*_{if})_i : f^*ai = (if)^*i \to (\widetilde{\Psi}\widetilde{\Phi}(f))^*$ and $\kappa_{if*} : i(\widetilde{\Psi}\widetilde{\Phi}(f))_* \to if_*$. Since the counit $\varepsilon : ai \to id_{\widetilde{\mathcal{C}}}$ is an equivalence and $i : \widetilde{\mathcal{C}} \to \widehat{\mathcal{C}}$ is fully faithful, we also have equivalences $f^*(\varepsilon) : f^*ai \to f^*$ and $\tilde{\kappa}_{f*} : (\widetilde{\Psi}\widetilde{\Phi}(f))_* \to f_*$ such that $i(\tilde{\kappa}_{f*}) = \kappa_{if*}$. Define an equivalence $\tilde{\kappa}^*_f : f^* \to (\widetilde{\Psi}\widetilde{\Phi}(f))^*$ by $\tilde{\kappa}^*_f = (\kappa^*_{if})_i f^*(\varepsilon)^{-1}$. Then, the following diagrams commute.

$$\begin{tikzcd}
\mathcal{E}(\Psi\Phi(if)^*i(F),Z) \arrow[r, "\alpha(\Phi(if))_{i(F),Z}"] \arrow[d, "(\kappa^*_{if})^*_{i(F)}"] & \widehat{\mathcal{C}}(i(F),\Psi\Phi(if)_*(Z)) \arrow[d, "(\kappa_{if*})_{Z*}"] & \widetilde{\mathcal{C}}(F,\widetilde{\Psi}\widetilde{\Phi}_*(Z)) \arrow[l, "i"'] \arrow[d, "(\tilde{\kappa}_{f*})_{Z*}"] \\
\mathcal{E}((if)^*i(F),Z) \arrow[r, "(\alpha_{if})_{i(F),Z}"] & \widetilde{\mathcal{C}}(F,f_*(Z)) & \widehat{\mathcal{C}}(i(F),(if)_*(Z)) \arrow[l, "i"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{E}((if)^*i(F),Z) \arrow[r, equal] \arrow[d, "(\alpha_{if})_{i(F),Z}"] & \mathcal{E}(f^*ai(F),Z) \arrow[d, "(\alpha_f)_{ai(F),Z}"] & \mathcal{E}(f^*(F),Z) \arrow[l, "f^*(\varepsilon)^*"'] \arrow[d, "(\alpha_f)_{F,Z}"] \\
\widehat{\mathcal{C}}(i(F),(if)_*(Z)) & \widetilde{\mathcal{C}}(ai(F),f_*(Z)) \arrow[l, "adj"'] & \widetilde{\mathcal{C}}(F,f_*(Z)) \arrow[l, "\varepsilon^*"']
\end{tikzcd}$$

Since a composition $\widetilde{\mathcal{C}}(F,f_*(Z)) \xrightarrow{\varepsilon^*} \widetilde{\mathcal{C}}(ai(F),f_*(Z)) \xrightarrow{adj} \widehat{\mathcal{C}}(i(F),if_*(Z)) = \widehat{\mathcal{C}}(i(F),(if)_*(Z))$ coincides with $i : \widetilde{\mathcal{C}}(F,f_*(Z)) \to \widehat{\mathcal{C}}(i(F),(if)_*(Z))$ and the adjunction $\tilde{\alpha}_f : \mathcal{E}(\widetilde{\Psi}\widetilde{\Phi}(f)^*(F),Z) \to \mathcal{E}(F,\widetilde{\Psi}\widetilde{\Phi}(f)_*(Z))$ is given by a composite

$$\mathcal{E}(\Psi\Phi(if)^*i(F),Z) \xrightarrow{\alpha(\Phi(if))} \widehat{\mathcal{C}}(i(F),\Psi\Phi(if)_*(Z)) = \widehat{\mathcal{C}}(i(F),i\widetilde{\Psi}\widetilde{\Phi}(f)_*(Z)) \xrightarrow{i^{-1}} \widetilde{\mathcal{C}}(F,f_*(Z)),$$

it follows from the commtativity of the above diagrams that the following square commutes.

$$\begin{tikzcd}
\mathcal{E}(\widetilde{\Psi}\widetilde{\Phi}(f)^*(F),Z) \arrow[r, "\tilde{\alpha}_f"] \arrow[d, "(\tilde{\kappa}_f^*)^*"] & \widetilde{\mathcal{C}}(F,\widetilde{\Psi}\widetilde{\Phi}(f)_*(Z)) \arrow[d, "(\tilde{\kappa}_{f*})_*"] \\
\mathcal{E}(f^*(F),Z) \arrow[r, "\alpha_f"] & \widetilde{\mathcal{C}}(F,f_*(Z))
\end{tikzcd}$$

Therefore $\tilde{\kappa}_f = (\tilde{\kappa}_{f*},\tilde{\kappa}_f^*) : f \to \widetilde{\Psi}\widetilde{\Phi}(f)$ is a isomorphism of geometric morphisms. By the naturality of κ_f in f, $\tilde{\kappa}_f$ is natural in f. Thus we have an equivalence $\tilde{\kappa} : id_{\mathfrak{Top}(\mathcal{E},\widetilde{\mathcal{C}})} \to \widetilde{\Psi}\widetilde{\Phi}$. □

Proposition 2.15.7 *Let $(\mathcal{C},J)$ and $(\mathcal{C}',J')$ be $\mathcal{U}$-sites. A functor $K : \mathcal{C} \to \widetilde{\mathcal{C}'}$ belongs to $\mathrm{Filt}_J(\mathcal{C},\widetilde{\mathcal{C}'})$ if and only if K satisfies the following conditions.*

(1) For $Y \in \mathrm{Ob}\,\mathcal{C}'$, $(p : Z \to Y \,|\, K(X)(Z) \neq \emptyset$ for some $X \in \mathrm{Ob}\,\mathcal{C})$ is a covering of Y.

(2) Let Y, Z be objects of $\mathcal{C}$ and W an object of $\mathcal{C}'$. For $y \in K(Y)(W)$ and $z \in K(Z)(W)$, $(p : V \to W \,|\, K(f)_V(v) = K(Y)(p)(y),\ K(g)_V(v) = K(Z)(p)(z)$ for some $f : X \to Y$, $g : X \to Z$ and $v \in K(X)(V))$ is a covering of W.

(3) Let $s,t : Y \to Z$ be morphisms in $\mathcal{C}$ and W an object of $\mathcal{C}'$. If $K(s)_W(w) = K(t)_W(w)$ for $w \in K(Y)(W)$, $(p : V \to W \,|\, sv = tv,\ K(v)_V(y) = K(Y)(p)(w)$ for some $v : X \to Y$ and $y \in K(X)(V))$ is a covering of W.

(4) For $R \in J(X)$, $Y \in \mathrm{Ob}\,\mathcal{C}'$ and $y \in K(X)(Y)$, $(p : Z \to Y \,|\, K(f)_Z(z) = K(X)(p)(y)$ for some $f \in R(W)$ and $z \in K(W)(Z))$ is a covering of Y.

Proof. Recall that a family of morphisms $(\varphi_i : F_i \to F)_{i\in I}$ in $\widetilde{\mathcal{C}'}$ is a covering for the canonical topology if and only if it is an epimorphic family (by (2.4.6)). The above condition (i) $(i = 1,2,3)$ is equivalent to the condition (i) of (2.9.5) for the canonical topology of $\widetilde{\mathcal{C}'}$. In fact, for $i = 1,2$, the equivalence is a direct consequece of (2.6.10). For morphisms $Y \underset{t}{\overset{s}{\rightrightarrows}} Z$ in $\mathcal{C}$, let $E \xrightarrow{e} K(Y)$ be an equalizer of $K(Y) \underset{K(t)}{\overset{K(s)}{\rightrightarrows}} K(Z)$ in $\widetilde{\mathcal{C}'}$. Then, $(f : K(X) \to E \,|\, X \in \mathrm{Ob}\,\mathcal{C}, {}^\exists v \in \mathcal{C}(X,Y)$ *such that* $sv = tv,\ ef = K(v))$ is an epimorphic family if and only if, for any $W \in \mathrm{Ob}\,\mathcal{C}'$ and $w \in E(W)$, a family of morphisms $(p : V \to W \,|\, V \in \mathrm{Ob}\,\mathcal{C}',\ f_V(y) = E(p)(w)$ *for some* $X \in \mathrm{Ob}\,\mathcal{C}$, $y \in K(X)(V)$, $f : K(X) \to E$, $v \in \mathcal{C}(X,Y)$ *such that* $sv = tv$, $ef = K(v))$ in $\mathcal{C}'$ is a covering. Regarding E as a subsheaf of $K(Y)$, $w \in E(W)$ if and only if $K(s)_W(w) = K(t)_W(w)$. For a morphism $p : V \to W$, $f_V(y) = E(p)(w)$ for some $X \in \mathrm{Ob}\,\mathcal{C}$, $y \in K(X)(V)$, $f : K(X) \to E$, $v \in \mathcal{C}(X,Y)$ such that $sv = tv$, $ef = K(v)$ if and only if $K(v)_V(y) = K(Y)(p)(w)$ for some $X \in \mathrm{Ob}\,\mathcal{C}$, $y \in K(X)(V)$, $v \in \mathcal{C}(X,Y)$ such that $sv = tv$. Hence the condition (3) above is equivalent to (3) of (2.9.5).

Note that a filtering functor $K : \mathcal{C} \to \widetilde{\mathcal{C}'}$ belongs to $\mathrm{Filt}_J(\mathcal{C},\widetilde{\mathcal{C}'})$ if and only if $(K(f) : K(\mathrm{dom}(f)) \to K(X))_{f\in R}$ is an epimorphic family for any $X \in \mathrm{Ob}\,\mathcal{C}$ and $R \in J(X)$. Fixing $X \in \mathrm{Ob}\,\mathcal{C}$ and $R \in J(X)$, it follows from (2.6.10) that, $(K(f) : K(\mathrm{dom}(f)) \to K(X))_{f\in R}$ is an epimorphic family if and only if, for any $Y \in \mathrm{Ob}\,\mathcal{C}'$ and $y \in K(X)(Y)$, a family of morphisms $(p : Z \to Y \,|\, Z \in \mathrm{Ob}\,\mathcal{C}',\ K(f)_Z(z) = K(p)(y)$ *for some* $f \in R(W)$, $z \in K(W)(Z))$ in $\mathcal{C}'$ is a covering, that is, the condition (4) above holds. □

Theorem 2.15.8 *Let $(\mathcal{C},J)$ and $(\mathcal{C}',J')$ be $\mathcal{U}$-sites and $u : \mathcal{C} \to \mathcal{C}'$ a functor. Suppose that $\mathcal{C}$ has finite limits, G is a $\mathcal{U}$-small topologically generating set $(\mathcal{C},J)$ and u is a left exact functor satisfying the following condition.*

$(*)$ For every covering $(f_i : X_i \to X)_{i\in I}$ of $X \in \mathrm{Ob}\,\mathcal{C}$ for J such that I is $\mathcal{U}$-small and $X_i \in G$, $(u(f_i) : u(X_i) \to u(X))_{i\in I}$ is a covering of $u(X)$ for J'.

Then, $\tilde{u}^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ has a left exact left adjoint $\tilde{u}_! : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$. Hence $(\tilde{u}^*,\tilde{u}_!) : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ is a geometric morphism.

Proof. By (2.11.6), u is $\mathcal{U}$-continuous. The assertion follows from (2.12.11). □

If $\mathcal{C}$ is $\mathcal{U}$-small, the above theorem is proved using (2.15.6) as follows. Since u is left exact, $\epsilon_{J'}u = a'h'u : \mathcal{C} \to \widetilde{\mathcal{C}'}$ is also left exact, hence filtering. Let $(f_i : X_i \to X)_{i\in I}$ is a covering of X. For each $i \in I$, there is a covering $(f_{ij} : X_{ij} \to X_i)_{j\in I_i}$ such that I_i is $\mathcal{U}$-small and $X_{ij} \in G$. Then, $(f_i f_{ij} : X_{ij} \to X)_{j\in I_i, i\in I}$ is a covering.

Since G is $\mathcal{U}$-small, there is a $\mathcal{U}$-small subset M of $\{(i,j)|\, i\in I,\ j\in I_i\}$ such that $(f_if_{ij}: X_{ij}\to X)_{(i,j)\in M}$ generates the same sieve as $(f_if_{ij}: X_{ij}\to X)_{j\in I_i, i\in I}$ does. Hence $(u(f_if_{ij}): u(X_{ij})\to u(X))_{(i,j)\in M}$ is a covering of $u(X)$ by the assumption and so is $(u(f_if_{ij}): u(X_{ij})\to u(X))_{j\in I_i, i\in I}$. Since the sieve generated by $(u(f_i): u(X_i)\to u(X))_{i\in I}$ contains the sieve generated by $(u(f_if_{ij}): u(X_{ij})\to u(X))_{j\in I_i, i\in I}$, $(u(f_i): u(X_i)\to u(X))_{i\in I}$ is a covering. Thus $(\epsilon_{J'}u(f_i): \epsilon_{J'}u(X_i)\to \epsilon_{J'}u(X))_{i\in I}$ is an epimorphic family. It follows that $\epsilon_{J'}u$ is an object of $\mathrm{Filt}_J(\mathcal{C},\widetilde{\mathcal{C}'})$. Since $h^{\mathcal{C}'*}a'^*h^{\widetilde{\mathcal{C}'}}(F) = h^{\widetilde{\mathcal{C}'}}_F a'h^{\mathcal{C}'} \cong h^{\widehat{\mathcal{C}'}}_{i'(F)}h^{\mathcal{C}'} \cong i'(F)$ for $F\in \mathrm{Ob}\,\widetilde{\mathcal{C}'}$, $i\widetilde{\Psi}(\epsilon_{J'}u)_* = (\epsilon_{J'}u)^*h^{\widetilde{\mathcal{C}'}} = u^*h^{\mathcal{C}'*}a'^*h^{\widetilde{\mathcal{C}'}} \cong u^*i' = i\tilde{u}^*$. Since $i:\widetilde{\mathcal{C}}\to\widehat{\mathcal{C}}$ is fully faithful, it follows that $\widetilde{\Psi}(\epsilon_{J'}u)_*$ is isomorphic to $\tilde{u}^*$. On the other hand, since $\tilde{u}_! = a'u_!i$ is a left adjoint of $\tilde{u}^*$, $\widetilde{\Psi}(\epsilon_{J'}u)^*$ is naturally equivalent to $\tilde{u}_!$.

Theorem 2.15.9 *Let $(\mathcal{C},J)$, $(\mathcal{C}',J')$ be $\mathcal{U}$-sites and $u:\mathcal{C}\to\mathcal{C}'$ a cocontinuous functor. Define $\bar{u}^*:\widetilde{\mathcal{C}'}\to\widetilde{\mathcal{C}}$ to be the composition $\widetilde{\mathcal{C}'}\xrightarrow{i'}\widehat{\mathcal{C}'}\xrightarrow{u^*}\widehat{\mathcal{C}}\xrightarrow{a}\widetilde{\mathcal{C}}$. Then, $\bar{u}^*$ is left exact and has a right adjoint $\tilde{u}_*:\widetilde{\mathcal{C}}\to\widetilde{\mathcal{C}'}$. Thus we have a geometric morphism $(\tilde{u}_*,\bar{u}^*):\widetilde{\mathcal{C}}\to\widetilde{\mathcal{C}'}$.*

Proof. This is a direct consequence of (2.12.12) and (2.11.11). $\square$

If $\mathcal{C}$ is $\mathcal{U}$-small, the above theorem is proved using (2.15.6) as follows. By (A.4.2), there is a colimitimg cone $(h'P\langle X,f\rangle \xrightarrow{f} F)_{\langle X,f\rangle\in\mathrm{Ob}(h'\downarrow F)}$ for $F\in\mathrm{Ob}\,\widehat{\mathcal{C}'}$. Since $u^*:\widehat{\mathcal{C}'}\to\widehat{\mathcal{C}}$ and $a:\widehat{\mathcal{C}}\to\widetilde{\mathcal{C}}$ preserves colimits, $(au^*h'P\langle X,f\rangle \xrightarrow{au^*(f)} au^*(F))_{\langle X,f\rangle\in\mathrm{Ob}(h'\downarrow F)}$ is a colimiting cone. It follows that $au^*:\widehat{\mathcal{C}'}\to\widetilde{\mathcal{C}}$ is a left Kan extension of $au^*h':\mathcal{C}'\to\widetilde{\mathcal{C}}$ along $h':\mathcal{C}'\to\widehat{\mathcal{C}'}$. Since au^* is left exact, au^*h' is filtering by (2.9.9). Let $(g_i: Y_i\to Y)_{i\in I}$ be a covering of $Y\in\mathcal{C}'$ and $R\in J'(Y)$ the sieve generated by $(g_i:Y_i\to Y)_{i\in I}$. Note that $h'^{-1}_f(R)\in J'(u(Y))$ for any $X\in\mathrm{Ob}\,\mathcal{C}$ and morphism $f:u(X)\to Y$. Since u is cocontinuous, $h'^{-1}_f(R)^u\in J(X)$. On the other hand, for $f\in u^*(h'_Y)(X)=\mathcal{C}'(u(X),Y)$, a morphism $p:Z\to X$ in $\mathcal{C}$ satisfies $g_iq = u^*(h'_{g_i})_Z(q) = u^*(h'_Y)(p)(f) = fu(p)$ for some $i\in I$ and $q\in u^*(h'_{Y_i})(Y)=\mathcal{C}'(u(Y_i),Y)$ if and only if $p\in h'^{-1}_f(R)^u(X)$. Thus a family of morphisms $(p:Z\to X|\, Y\in\mathrm{Ob}\,\mathcal{C},\ u^*(h'_{g_i})_Z(q) = u^*(h'_Y)(p)(f)$ *for some* $i\in I,\ q\in u^*(h'_{Y_i})(Y))$ is a covering. Applying (2.6.9) to a family $(u^*(h'_{g_i}): u^*(h'_{Y_i})\to u^*(h'_Y))_{i\in I}$ of morphisms in $\widehat{\mathcal{C}}$, $(au^*(h'_{g_i}): au^*(h'_{Y_i})\to au^*(h'_Y))_{i\in I}$ is an epimorphic family. Hence au^*h' is an object of $\mathrm{Filt}_J(\mathcal{C}',\widetilde{\mathcal{C}})$ and, since $\widetilde{\Psi}(au^*h')^*$ is a left Kan extension of au^*h' along h', it is naturally equivalent to au^*.

Let $(\mathcal{C},J)$ be a $\mathcal{U}$-site. For each object X of $\mathcal{C}$, we give $\mathcal{C}/X$ the topology induced by $\Sigma_X:\mathcal{C}/X\to\mathcal{C}$. If $f:Y\to X$ is a morphism in $\mathcal{E}$, then $\Sigma_f:\mathcal{C}/Y\to\mathcal{C}/X$ is continuous and cocontinuous by (2.13.3). Then, $\Sigma_f^*:\widehat{\mathcal{C}/X}\to\widehat{\mathcal{C}/Y}$ induces $\widetilde{\Sigma}_f^*:\widetilde{\mathcal{C}/X}\to\widetilde{\mathcal{C}/Y}$ which is naturally equivalent to a composition $\widetilde{\mathcal{C}/X}\xrightarrow{i}\widehat{\mathcal{C}/X}\xrightarrow{\Sigma_f^*}\widehat{\mathcal{C}/Y}\xrightarrow{a}\widetilde{\mathcal{C}/Y}$. It follows from (2.15.9) that $\widetilde{\Sigma}_f^*$ is left exact and it has a right adjoint $\tilde{f}_*:\widetilde{\mathcal{C}/Y}\to\widetilde{\mathcal{C}/X}$.

Corollary 2.15.10 $(\tilde{f}_*,\widetilde{\Sigma}_f^*):\widetilde{\mathcal{C}/Y}\to\widetilde{\mathcal{C}/X}$ *is a geometric morphism.*

Lemma 2.15.11 *Let $f:\mathcal{E}\to\mathcal{E}'$ be a functor between $\mathcal{U}$-topoi which has a right adjoint $g:\mathcal{E}'\to\mathcal{E}$. Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are $\mathcal{U}$-small generating subcategories of $\mathcal{E}$ and $\mathcal{E}'$ such that $f(\mathrm{Ob}\,\mathcal{C})\subset\mathrm{Ob}\,\mathcal{C}'$. Consider the topologies J, J' on $\mathcal{C}$, $\mathcal{C}'$ induced by the canonical topologies on $\mathcal{E}$, $\mathcal{E}'$. Then, the functor $u:\mathcal{C}\to\mathcal{C}'$ induced by f is $\mathcal{U}$-continuous. Moreover, the functor $\tilde{u}^*:\widetilde{\mathcal{C}'}\to\widetilde{\mathcal{C}}$ induced by $u^*:\widehat{\mathcal{C}'}\to\widehat{\mathcal{C}}$ has a left adjoint $\tilde{u}_!$ and there are natural equivalences $\varphi:\mathcal{E}\to\widetilde{\mathcal{C}}$ and $\varphi':\mathcal{E}'\to\widetilde{\mathcal{C}'}$ such that the the following squares commutes up to natural equivalences.*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "f"] \arrow[d, "\varphi"] & \mathcal{E}' \arrow[d, "\varphi'"] \\
\widetilde{\mathcal{C}} \arrow[r, "\tilde{u}_!"] & \widetilde{\mathcal{C}'}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}' \arrow[r, "g"] \arrow[d, "\varphi'"] & \mathcal{E} \arrow[d, "\varphi"] \\
\widetilde{\mathcal{C}'} \arrow[r, "\tilde{u}^*"] & \widetilde{\mathcal{C}}
\end{tikzcd}$$

Proof. Let us denote by $K:\mathcal{C}\to\mathcal{E}$, $K':\mathcal{C}'\to\mathcal{E}'$ the inclusion functors and by $L:\widehat{\mathcal{C}}\to\mathcal{E}$, $L':\widehat{\mathcal{C}'}\to\mathcal{E}'$ the left Kan extensions of K, K' along the Yoneda embeddings $h^{\mathcal{C}}:\mathcal{C}\to\widehat{\mathcal{C}}$, $h^{\mathcal{C}'}:\mathcal{C}'\to\widehat{\mathcal{C}'}$ such that $K=Lh^{\mathcal{C}}$, $K'=Lh^{\mathcal{C}'}$. Set $R=K^*h^{\mathcal{E}}:\mathcal{E}\to\widehat{\mathcal{C}}$, $R'=K'^*h^{\mathcal{E}'}:\mathcal{E}'\to\widehat{\mathcal{C}'}$. Then, R and R' induce equivalences $\varphi:\mathcal{E}\to\widetilde{\mathcal{C}}$ and $\varphi':\mathcal{E}'\to\widetilde{\mathcal{C}'}$ such that $i\varphi=R$, $i'\varphi'=R'$, where $i:\widetilde{\mathcal{C}}\to\widehat{\mathcal{C}}$, $i':\widetilde{\mathcal{C}'}\to\widehat{\mathcal{C}'}$ are the inclusion functors. Recall that the quasi-inverses of φ, φ' are given by $Li:\widetilde{\mathcal{C}}\to\mathcal{E}$, $L'i':\widetilde{\mathcal{C}'}\to\mathcal{E}'$ and that the assosiated sheaf functors $a:\widehat{\mathcal{C}}\to\widetilde{\mathcal{C}}$, $a':\widehat{\mathcal{C}'}\to\widetilde{\mathcal{C}'}$ are given by $a=\varphi L$, $a'=\varphi'L'$. Define $\tilde{u}_!:\widetilde{\mathcal{C}}\to\widetilde{\mathcal{C}'}$ by $\tilde{u}_!=\varphi'fLi$. Then, the diagram of *iv*) of (2.11.2) commutes up to natural equivalence. In fact, since the counit $\varepsilon : LR \to id_{\mathcal{E}}$ of the adjunction of L and R is an equivalence, $\tilde{u}_!\epsilon_J = \varphi' f Li\varphi Lh^{\mathcal{C}} = \varphi' f LRK \cong \varphi' f K = \varphi' K' u = \varphi' L' h^{\mathcal{C}'} u = \epsilon_{J'} u$. Moreover, since f has a right adjoint g, $\tilde{u}_!$ has a right adjoint $\varphi g L' i'$. In particular, $\tilde{u}_!$ preserves colimits. It follows from (2.11.2) that u is ($\mathcal{U}$-)continuous. Then, $u^* : \widehat{\mathcal{C}'} \to \widehat{\mathcal{C}}$ induces $\tilde{u}^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$, which is a right adjoint of $\tilde{u}_!$ by (2.11.3). By the definition of $\tilde{u}_!$, $\tilde{u}_!\varphi$ is naturally equivalent to $\varphi' f$. Since g is a right adjoint of f and $L' i' \tilde{u}_!$ is naturally equivalent to $f Li$, $\tilde{u}^*\varphi'$ is naturally equivalent to φg. □

Proposition 2.15.12 *Let $f : \mathcal{E} \to \mathcal{E}'$ be a geometric morphism. The inverse image $f^* : \mathcal{E}' \to \mathcal{E}$ has a left adjoint $f_! : \mathcal{E} \to \mathcal{E}'$ if and only if there exist $\mathcal{U}$-sites $(\mathcal{C}, J)$, $(\mathcal{C}', J')$ and continuous and cocontinuous functor $u : \mathcal{C} \to \mathcal{C}'$ such that there are equivalences of categories $\varphi : \widetilde{\mathcal{C}} \to \mathcal{E}$ and $\varphi' : \widetilde{\mathcal{C}'} \to \mathcal{E}'$ making the following squares commutes up to natural equivalences.*

$$
\begin{tikzcd}
\mathcal{E} \arrow[r, "f_*"] \arrow[d, "\varphi"] & \mathcal{E}' \arrow[d, "\varphi'"] \\
\widetilde{\mathcal{C}} \arrow[r, "\tilde{u}_*"] & \widetilde{\mathcal{C}'}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{E}' \arrow[r, "f^*"] \arrow[d, "\varphi'"] & \mathcal{E} \arrow[d, "\varphi"] \\
\widetilde{\mathcal{C}'} \arrow[r, "\tilde{u}^*"] & \widetilde{\mathcal{C}}
\end{tikzcd}
$$

Proof. Suppose that f^* has a left adjoint $f_!$. We choose $\mathcal{U}$-small generating subcategories $\mathcal{C}_1$ and $\mathcal{C}'_1$ of $\mathcal{E}$ and $\mathcal{E}'$ such that $f_!(\mathrm{Ob}\,\mathcal{C}_1) \subset \mathrm{Ob}\,\mathcal{C}'_1$. Inductively, assume that we have full subcategories $\mathcal{C}_i$ and $\mathcal{C}'_i$ of $\mathcal{E}$ and $\mathcal{E}'$ for $i \leqq n$ such that $f_!(\mathrm{Ob}\,\mathcal{C}_i) \subset \mathrm{Ob}\,\mathcal{C}'_i$ and $f^*(\mathrm{Ob}\,\mathcal{C}'_{i-1}) \subset \mathrm{Ob}\,\mathcal{C}_i$. Let $\mathcal{C}_{i+1}$ be a full subcategory of $\mathcal{E}$ with objects $\mathrm{Ob}\,\mathcal{C}_i \cap f^*(\mathrm{Ob}\,\mathcal{C}'_i)$ and $\mathcal{C}'_{i+1}$ be a full subcategory of $\mathcal{E}'$ with objects $\mathrm{Ob}\,\mathcal{C}'_i \cap f_!(\mathrm{Ob}\,\mathcal{C}_{i+1})$. Thus we have increasing sequences $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots$, $\mathcal{C}'_1 \subset \mathcal{C}'_2 \subset \cdots$ of full subcategories of $\mathcal{E}$, $\mathcal{E}'$. Define full subcategories $\mathcal{C}$, $\mathcal{C}'$ of $\mathcal{E}$, $\mathcal{E}'$ by $\mathrm{Ob}\,\mathcal{C} = \bigcup_{i\geqq 1} \mathrm{Ob}\,\mathcal{C}_i$, $\mathrm{Ob}\,\mathcal{C}' = \bigcup_{i\geqq 1} \mathrm{Ob}\,\mathcal{C}'_i$. Then, we have $f_!(\mathrm{Ob}\,\mathcal{C}) \subset \mathrm{Ob}\,\mathcal{C}'$ and $f^*(\mathrm{Ob}\,\mathcal{C}') \subset \mathrm{Ob}\,\mathcal{C}$. Let $u : \mathcal{C} \to \mathcal{C}'$ and $v : \mathcal{C}' \to \mathcal{C}$ be the restrictions of $f_!$ and f^*. Then, u is a left adjoint of v. We give $\mathcal{C}$, $\mathcal{C}'$ topologies induced by the canonical topologies on $\mathcal{E}$, $\mathcal{E}'$. Since $f_!$ and f^* have right adjoints, both u and v are $\mathcal{U}$-continuous by (2.15.11). It follows from (2.11.12) that u is cocontinuous. We consider the natural equivalences $\varphi : \mathcal{E} \to \widetilde{\mathcal{C}}$ and $\varphi' : \mathcal{E}' \to \widetilde{\mathcal{C}'}$ given in (2.15.11) and use the same notations as in (2.15.11). Define $\tilde{u}_* : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}'}$ by $\tilde{u}_* = \varphi' f_* Li$. Then, $\tilde{u}_*\varphi$ is naturally equivalent to $\varphi' f_*$. We show the converse. By (2.11.13), $\tilde{u}^* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ has a left adjoint $\tilde{u}_!$. Let us denote by $\psi' : \widetilde{\mathcal{C}'} \to \mathcal{E}'$ the quasi-inverse of φ and define $f_! : \mathcal{E} \to \mathcal{E}'$ by $f_! = \psi' \tilde{u}_! \varphi$. Then, $f_!$ is a left adjoint of f^*. □

2.16 Localic topoi

Let X be a locale and J denotes the canonical topology on $\mathcal{O}(X)$. For a $\mathcal{U}$-cocomplete topos $\mathcal{E}$, there is an equivalence $\tilde{\Psi} : \mathrm{Filt}_J(\mathcal{O}(X), \mathcal{E}) \to \mathfrak{Top}(\mathcal{E}, \mathrm{Sh}(X))$ by (2.15.6). Since $\mathcal{O}(X)$ has finite limits, a functor $F : \mathcal{O}(X) \to \mathcal{E}$ belongs to $\mathrm{Filt}_J(\mathcal{O}(X), \mathcal{E})$ if and only if F is left exact ((2.9.14)) and $(F(U_i) \to F(\bigvee_{i\in I} U_i))_{i\in I}$ is an epimorphic family for any family $(U_i)_{i\in I}$ of elements of $\mathcal{O}(X)$ ((2.14.7)).

Let us denote by $\mathrm{Loc}(\mathcal{E})$ the locale such that $\mathcal{O}(\mathrm{Loc}(\mathcal{E})) = \mathrm{Sub}_{\mathcal{E}}(1_{\mathcal{E}})$. We construct an equivalence $\Xi : \mathcal{L}oc(\mathrm{Loc}(\mathcal{E}), X) = \mathcal{F}r(\mathcal{O}(X), \mathrm{Sub}_{\mathcal{E}}(1_{\mathcal{E}})) \to \mathrm{Filt}_J(\mathcal{O}(X), \mathcal{E})$ as follows. For a morphism $f^{-1} : \mathcal{O}(X) \to \mathrm{Sub}_{\mathcal{E}}(1_{\mathcal{E}})$ of frames and $U \in \mathcal{O}(X)$, we choose a monomorphism $U_f \rightarrowtail 1_{\mathcal{E}}$ representing the class of $f^{-1}(U) \in \mathrm{Ob}\,\mathrm{Sub}_{\mathcal{E}}(1_{\mathcal{E}})$. In particular, since $f^{-1}(1)$ is the class of the identity morphism of $1_{\mathcal{E}}$, we can choose $1_f = 1_{\mathcal{E}}$. If $U \leq V$ in $\mathcal{O}(X)$, then $f^{-1}(U) \leq f^{-1}(V)$ and there is a unique monomorphism $i^f_{UV} : U_f \rightarrowtail V_f$. Define $\Xi(f^{-1}) : \mathcal{O}(X) \to \mathcal{E}$ by $\Xi(f^{-1})(U) = U_f$ and $\Xi(f^{-1})(U \leq V) = (i^f_{UV} : U_f \rightarrowtail V_f)$. By the uniqeness of i^f_{UV}, $\Xi(f^{-1})$ is a functor. Since $1_f = 1_{\mathcal{E}}$, $\Xi(f^{-1})$ preserves terminal objects. For $U, V \in \mathcal{O}(X)$, since $f^{-1}(U \wedge V) = f^{-1}(U) \cap f^{-1}(V)$, $U_f \times V_f$, $(U \wedge V)_f$ is isomorphic to $U_f \times V_f$, $\Xi(f^{-1})$ preserves products, hence it is left exact (See 3) of (2.14.2)). For a family $(U_j)_{j\in I}$ of elements of $\mathcal{O}(X)$, since $f^{-1}(\bigvee_{j\in I} U_j) = \bigcup_{j\in I} f^{-1}(U_j)$, $((U_k)_f \xrightarrow{i^f_{U_k \bigvee_{j\in I} U_j}} (\bigvee_{j\in I} U_j)_f)_{k\in I}$ is an epimorphic family. Thus we have seen that $\Xi(f^{-1})$ is an object of $\mathrm{Filt}_J(\mathcal{O}(X), \mathcal{E})$.

For morphisms of frames $f^{-1}, g^{-1} : \mathcal{O}(X) \to \mathrm{Sub}_{\mathcal{E}}(1_{\mathcal{E}})$, suppose that $f^{-1} \leq g^{-1}$, that is, $f^{-1}(U) \leq g^{-1}(U)$ for every $U \in \mathcal{O}(X)$. There exists a unique monomorphism $i^{fg}_U : U_f \to U_g$ and this makes the following square commute if $U \leq V$.

$$
\begin{tikzcd}
U_f \arrow[r, "i^f_{UV}"] \arrow[d, "i^{fg}_U"] & V_f \arrow[d, "i^{fg}_V"] \\
U_g \arrow[r, "i^g_{UV}"] & V_g
\end{tikzcd}
$$

Thus we have a natural transformation $i^{fg} : \Xi(f^{-1}) \to \Xi(g^{-1})$. Putting $\Xi(f^{-1} \le g^{-1}) = i^{fg}$, we have a functor $\Xi : \mathcal{F}r(\mathcal{O}(X), \mathrm{Sub}_{\mathcal{E}}(1_{\mathcal{E}})) \to \mathrm{Filt}_J(\mathcal{O}(X), \mathcal{E})$.

Since there is at most one morphism between two morphisms of frames, Ξ is faithful. Suppose that there is a natural transfomation $\tau : \Xi(f^{-1}) \to \Xi(g^{-1})$. For $U \in \mathcal{O}(X)$, by the naturality of τ and $\Xi(f^{-1})(U) = \Xi(f^{-1})(V) = 1_{\mathcal{E}}$,

$$\begin{tikzcd} U_f \arrow[r, "i^f_{U1}"] \arrow[d, "\tau_U"'] & 1_{\mathcal{E}} \arrow[d, "\tau_1 = id_{1_{\mathcal{E}}}"] \\ U_g \arrow[r, "i^g_{U1}"] & 1_{\mathcal{E}} \end{tikzcd}$$

commutes and, since i^f_{U1} is a monomorphism, it follows that τ_U is a monomorphism. Note that τ_U is the unique morphism such that $i^g_{U1}\tau = i^f_{U1}$, for i^g_{U1} is a monomorphism. Therefore $f^{-1}(U) \le g^{-1}(U)$ for every U, hence by the uniqueness of τ, $\tau = i^{fg}_U = \Xi(f^{-1} \le g^{-1})$. We deduce that Ξ is fully faithful.

Let $F : \mathcal{O}(X) \to \mathcal{E}$ be an object of $\mathrm{Filt}_J(\mathcal{O}(X), \mathcal{E})$. Since F is left exact, F maps $U \le 1$ to a monomorphism $F(U) \rightarrowtail 1_{\mathcal{E}}$. We denote by $f^{-1}(U)$ the class of $\mathrm{Sub}_{\mathcal{E}}(1_{\mathcal{E}})$ represented by $F(U) \rightarrowtail 1_{\mathcal{E}}$. For $U, V \in \mathcal{O}(X)$, since $F(U) \xleftarrow{F((U\wedge V)\le U)} F(U \wedge V) \xrightarrow{F((U\wedge V)\le V)} F(V)$ is a product diagram, we have $f^{-1}(U \wedge V) = f^{-1}(U) \cap f^{-1}(V)$. For a family $(U_j)_{j\in I}$ of elements of $\mathcal{O}(X)$, since $(F(U_k) \xrightarrow{F(U_k \le \bigvee_{j\in I} U_j)} F(\bigvee_{j\in I} U_j))_{k\in I}$ is an epimorphic family, we see that $f^{-1}(\bigvee_{j\in I} U_j) = \bigcup_{j\in I} f^{-1}(U_j)$ holds. Hence $f^{-1} : \mathcal{O}(X) \to \mathrm{Sub}_{\mathcal{E}}(1_{\mathcal{E}})$ is a morphism of frames such that $\Xi(f^{-1})(U) = F(U)$ for every $U \in \mathcal{O}(X)$. If $U \le V$, since both $F(U)$ and $F(V)$ are subobjects of terminal objects, $F(U \le V)$ is the unique morphism from $F(U)$ to $F(V)$. On the other hand, $\Xi(f^{-1})(U \le V)$ is also a morphism from $F(U)$ to $F(V)$. Therefore $\Xi(f^{-1})(U \le V) = F(U \le V)$ and we have $\Xi(f^{-1}) = F$.

We have shown tha following result.

Proposition 2.16.1 *Let X be a locale and $\mathcal{E}$ a $\mathcal{U}$-cocomplete topos. There is a natural equivalence of categories $\Xi : \mathcal{L}oc(\mathrm{Loc}(\mathcal{E}), X) \to \mathrm{Filt}_J(\mathcal{O}(X), \mathcal{E})$. Hence we have an equivalence*

$$\tilde{\Psi}\Xi : \mathcal{L}oc(\mathrm{Loc}(\mathcal{E}), X) \to \mathfrak{Top}(\mathcal{E}, \mathrm{Sh}(X)).$$

Let X and Y be locales and consider the case $\mathcal{E} = \mathrm{Sh}(Y)$. Since $\mathrm{Sub}_{\mathrm{Sh}(Y)}(1)$ is isomorphic to $\mathcal{O}(Y)$ by (2.14.10), the above result implies the following.

Corollary 2.16.2 *Let X and Y be locales. There is a natural equivalence $\mathcal{L}oc(Y, X) \to \mathfrak{Top}(\mathrm{Sh}(Y), \mathrm{Sh}(X))$ of categories.*

Explicitly, if $f : \mathrm{Loc}(\mathcal{E}) \to X$ is a morphism of locales, the direct image $f_* : \mathcal{E} \to \mathrm{Sh}(X)$ is given by $f_*(X)(x) = \mathcal{E}(f^{-1}(x), X)$ for $X \in \mathrm{Ob}\,\mathcal{E}$, $x \in \mathcal{O}(X)$. If $x \le y$ in $\mathcal{O}(X)$, then $f^{-1}(x) \subset f^{-1}(y)$ in $\mathrm{Sub}_{\mathcal{E}}(1_{\mathcal{E}})$ and this inclusion induces $f_*(X)(y) \to f_*(X)(x)$. For a morphism $u : X \to Y$ in $\mathcal{E}$, $f_*(u) : f_*(X) \to f_*(Y)$ is given by $f_*(u)_x = u_* : \mathcal{E}(f^{-1}(x), X) \to \mathcal{E}(f^{-1}(x), Y)$. If $f : Y \to X$ is a morphism of locales, the direct image $f_* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ is given by $f_*(F) = Ff^{-1}$ for $F \in \mathrm{Ob}\,\mathrm{Sh}(Y)$.

The category of frames has an initial object $\{0, 1\}$. In fact, for any frame A, a morphism $f : \{0, 1\} \to A$ given by $f(0) = 0$, $f(1) = 1$ is the unique morphism of frames. We denote by 1 the locale such that $\mathcal{O}(1) = \{0, 1\}$. Thus 1 is a terminal object of $\mathcal{L}oc$.

Definition 2.16.3 *Let X be a locale. We call a morphism $p : 1 \to X$ of locales a point of X.*

Let $p : 1 \to X$ be a point of X and consider the morphism $p^{-1} : \mathcal{O}(X) \to \{0, 1\}$. We put $K = \{U \in \mathcal{O}(X) |\, p^{-1}(U) = 0\}$ and call this the kernel of p^{-1}. It is easy to verify the following fact.

Proposition 2.16.4 *Above K has the following properties.*

i) $1 \notin K$
ii) $U \wedge V \in K$ *if and only if* $U \in K$ *or* $V \in K$.
iii) $\bigvee U_i \in K$ *if and only if* $U_i \in K$ *for all* i.

Proposition 2.16.5 *If a subset K of $\mathcal{O}(X)$ has the properties of the above proposition, there exists a unique point p of X such that K is the kernel of p^{-1}.*

Proof. For $K \subset \mathcal{O}(X)$ having the above properties, define p^{-1} by $p^{-1}(U) = \begin{cases} 0 & U \in K \\ 1 & U \notin K \end{cases}$. Then, p^{-1} is a morphism of frames whose kernel is K. The uniqueness of p is obvious. □

For a partially ordered set A and $x \in A$, put $\downarrow x = \{y \in A \mid y \leq x\}$ and call this the downward closure of x.

Proposition 2.16.6 *Let X be a locale.*

1) For a subset K of $\mathcal{O}(X)$ satisfying the conditions in (2.16.4), put $P = \bigvee K = \bigvee\limits_{U \in K} U$. Then, P has the following properties.

i) $P \neq 1$
ii) If $U \wedge V \leq P$, $U \leq P$ or $V \leq P$.

2) For an element P satisfying the conditions above, put $K = \downarrow P$. Then, K has the properties in (2.16.4).

3) For $P \in \mathcal{O}(X)$, $\bigvee(\downarrow P) = P$ holds. If $K \subset \mathcal{O}(X)$ satisfies the condition iii) of (2.16.4), $\downarrow(\bigvee K) = K$ holds.

Proof. 1) Since $P = \bigvee\limits_{U \in K} U \in K$ by *iii*) of (2.16.4), it follows from *i*) of (2.16.4) that $P \neq 1$. If $U \wedge V \leq P$, then $P \vee (U \wedge V) = P \in K$. Hence, by *iii*) of (2.16.4), $U \wedge V \in K$. By *ii*) of (2.16.4), $U \in K$ or $V \in K$, namely, $U \leq P$ or $V \leq P$.

2) Since $1 \in K$ implies $P = 1$, the condition *i*) implies $1 \notin K$. If $U \wedge V \in K$, then $U \wedge V \leq P$ and it follows from *ii*) that $U \leq P$ or $V \leq P$. Therefore $U \in K$ or $V \in K$. Conversely, assume $U \in K$ or $V \in K$. Then, $U \leq P$ or $V \leq P$. Since $U \wedge V \leq U$ and $U \wedge V \leq V$, we have $U \wedge V \leq P$. Hence $U \wedge V \in K$. If $\bigvee U_i \in K$, then $(\bigvee U_i) \wedge 1 = \bigvee U_i \leq P$. By *i*) and *ii*), $\bigvee U_i \leq P$ hence $U_i \leq P$ for any i. It follows that $U_i \in K$ for all i. Conversely, assume that $U_i \in K$ for all i. Then, $U_i \leq P$ for any i and this implies $\bigvee U_i \leq P$. Thus $\bigvee U_i \in K$.

3) Since $P \in \downarrow P$ and $Q \leq P$ for any $Q \in \downarrow P$, we have $\bigvee(\downarrow P) = P$. $U \in \downarrow(\bigvee K)$ if and only if $U \leq \bigvee K$, which is equivalent to $U \vee (\bigvee K) = \bigvee K$. By the assumption, $U \vee (\bigvee K) = \bigvee K$ implies $U \in K$. It is obvious that $U \in K$ implies $U \in \downarrow(\bigvee K)$. □

If $P \in \mathcal{O}(X)$ satisfies the condition *ii*) of (2.16.6), we call it a prime elememt of $\mathcal{O}(X)$. If it also satisfies *ii*) of (2.16.6), it is called a proper prime element.

Proposition 2.16.7 *Let S be a topological space, $\mathcal{O}(S)$ the frame of open sets of S and s a point of S.*

1) $S - \overline{\{s\}}$ is a proper prime element of $\mathcal{O}(S)$.

2) $K_s = \{U \in \mathcal{O}(S) \mid U \not\ni s\}$ satisfies the conditions in (2.16.4).

3) Define $p_s^{-1} : \mathcal{O}(S) \to \{0, 1\}$ by $p_s^{-1}(U) = \begin{cases} 1 & U \ni s \\ 0 & U \not\ni s \end{cases}$. Then, p_s^{-1} is a morphism of frames.

4) $K_s = \downarrow(S - \overline{\{s\}})$ and $S - \overline{\{s\}} = \bigcup K_s$ hold. Moreover, K_s is the kernel of p_s.

Proof. 1) If $U \cap V \subset S - \overline{\{s\}}$, $(S - U) \cup (S - V) \supset \overline{\{s\}}$. Hence $s \in S - U$ or $s \in S - V$. Since both $S - U$ and $S - V$ are closed sets, $\overline{\{s\}} \subset S - U$ or $\overline{\{s\}} \subset S - V$ holds.

2) This assertion is clear.

3) This assertion is also clear.

4) $U \in K_s \Leftrightarrow U \not\ni s \Leftrightarrow s \in S - U \Leftrightarrow \overline{\{s\}} \subset S - U \Leftrightarrow U \subset S - \overline{\{s\}} \Leftrightarrow U \in \downarrow(S - \overline{\{s\}})$
$x \in S - \overline{\{s\}} \Leftrightarrow x \notin \overline{\{s\}} \Leftrightarrow \exists U \in K_s,\ x \in U \Leftrightarrow x \in \bigcup K_s$
It is obvious that K_s is the kernel of p_s. □

For a locale X, let us denote by $\mathrm{pt}(X)$ the set of points of X. If $U \in \mathcal{O}(X)$, we put $\mathrm{pt}(U) = \{p \in \mathrm{pt}(X) \mid p^{-1}(U) = 1\}$. Then $\mathrm{pt}(1) = \mathrm{pt}(X)$, $\mathrm{pt}(0) = \emptyset$.

Proposition 2.16.8 *Let X be a locale. Then, the following identities holds.*

i) $\mathrm{pt}(U \wedge V) = \mathrm{pt}(U) \cap \mathrm{pt}(V)$ for $U, V \in \mathcal{O}(X)$
ii) $\mathrm{pt}(\bigvee\limits_i U_i) = \bigcup\limits_i \mathrm{pt}(U_i)$ for $U_i \in \mathcal{O}(X)$

Proof. *i*) $p \in \mathrm{pt}(U \wedge V) \Leftrightarrow p^{-1}(U \wedge V) = 1 \Leftrightarrow p^{-1}(U) \wedge p^{-1}(V) = 1$ in $\{0, 1\} \Leftrightarrow p^{-1}(U) = 1$ and $p^{-1}(V) = 1$ in $\{0, 1\} \Leftrightarrow p \in \mathrm{pt}(U)$ and $p \in \mathrm{pt}(V) \Leftrightarrow p \in \mathrm{pt}(U) \cap \mathrm{pt}(V)$

ii) $p \in \mathrm{pt}(\bigvee_i U_i) \Leftrightarrow p^{-1}(\bigvee_i U_i) = 1 \Leftrightarrow \bigvee_i p^{-1}(U_i) = 1$ in $\{0,1\} \Leftrightarrow p^{-1}(U_i) = 1$ for some i in $\{0,1\} \Leftrightarrow p \in \mathrm{pt}(U_i)$ for some $i \Leftrightarrow p \in \bigcup_i \mathrm{pt}(U_i)$ □

By virtue of the above result, we can give a topology on $\mathrm{pt}(X)$ so that $\{\mathrm{pt}(U) | U \in \mathcal{O}(X)\}$ is the set of open sets. Let $f : X \to Y$ be a morphism of locales. Define $\mathrm{pt}(f) : \mathrm{pt}(X) \to \mathrm{pt}(Y)$ by $\mathrm{pt}(f)(p) = fp$ (the composition $1 \xrightarrow{p} X \xrightarrow{f} Y$ of morphisms of locales). Then, for $V \in \mathcal{O}(Y)$, $\mathrm{pt}(f)^{-1}(\mathrm{pt}(V)) = \{p \in \mathrm{pt}(X) | fp \in \mathrm{pt}(V)\} = \{p \in \mathrm{pt}(X) | p^{-1}f^{-1}(V) = 1\} = \mathrm{pt}(f^{-1}(V))$. Hence $\mathrm{pt}(f)$ is continuous and we have a functor $\mathrm{pt} : \mathcal{L}oc \to \mathcal{T}op$.

Let S be a topological space. Define a map $\eta_S : S \to \mathrm{pt}\circ\mathrm{Loc}(S)$ by $\eta_S(s) = p_s$, where p_s is the point of $\mathrm{Loc}(S)$ corresponding to the morphism of frames $p_s^{-1} : \mathcal{O}(S) \to \{0,1\}$ given in (2.16.7). For $U \in \mathcal{O}(S)$, since $\eta_S^{-1}(\mathrm{pt}(U)) = \{s \in S | p_s^{-1}(U) = 1\} = U$, η_S is continuous. If $f : S \to T$ is a continuous map and $s \in S$, since $p_s^{-1}f^{-1}(U) = \begin{cases} 1 & f^{-1}(U) \ni s \\ 0 & f^{-1}(U) \not\ni s \end{cases} = \begin{cases} 1 & U \ni f(s) \\ 0 & U \not\ni f(s) \end{cases} = p_{f(s)}^{-1}(U)$ for any $U \in \mathcal{O}(T)$, we have $p_s^{-1}f^{-1} = p_{f(s)}^{-1}$ as morphisms of frames. Hence $\mathrm{Loc}(f)p_s = p_{f(s)}$ as morphisms of locales. It follows that $\mathrm{pt}(\mathrm{Loc}(f))\eta_S(s) = \mathrm{pt}(\mathrm{Loc}(f))(p_s) = \mathrm{Loc}(f)p_s = p_{f(s)} = \eta_T f(s)$ and $\eta : id_{\mathcal{T}op} \to \mathrm{pt}\circ\mathrm{Loc}$ is a natural transformation.

Let X be a locale. Define a map $\varepsilon_X^{-1} : \mathcal{O}(X) \to \mathcal{O}(\mathrm{pt}(X))$ by $\varepsilon_X^{-1}(U) = \mathrm{pt}(U)$. Then ε_X^{-1} is a morphism of frames by (2.16.8) and, by the definition of the topology on $\mathrm{pt}(X)$, it is surjective. We denote by $\varepsilon_X : \mathrm{Loc}\circ\mathrm{pt}(X) \to X$ the corresponding morphism of locales. For a morphism $f : X \to Y$ of frames, $\varepsilon_X^{-1}f^{-1}(U) = \mathrm{pt}(f^{-1}(U)) = \mathrm{pt}(f)^{-1}(\mathrm{pt}(U)) = \mathrm{pt}(f)^{-1}\varepsilon_Y^{-1}(U)$ for any $U \in \mathcal{O}(Y)$. Hence $f\varepsilon_X = \varepsilon_Y \mathrm{Loc}\circ\mathrm{pt}(f)$ and $\varepsilon : \mathrm{Loc}\circ\mathrm{pt} \to id_{\mathcal{L}oc}$ is a natural transformation.

Proposition 2.16.9 $\mathrm{pt} : \mathcal{L}oc \to \mathcal{T}op$ *is a right adjoint of* $\mathrm{Loc} : \mathcal{T}op \to \mathcal{L}oc$.

Proof. Let S be a topological space. For $U \in \mathcal{O}(S)$, $\eta_S^{-1}\varepsilon_{\mathrm{Loc}(S)}^{-1}(U) = \eta_S^{-1}(\mathrm{pt}(U)) = \eta_S^{-1}(\mathrm{pt}(U)) = \{s \in S | p_s \in \mathrm{pt}(U)\} = \{s \in S | p_s^{-1}(U) = 1\} = U$. Hence $\eta_S^{-1}\varepsilon_{\mathrm{Loc}(S)}^{-1} = id_{\mathcal{O}(S)}$ and this means $\varepsilon_{\mathrm{Loc}(S)}\mathrm{Loc}(\eta_S) = id_{\mathrm{Loc}(S)}$.

Let X be a locale. For $q \in \mathrm{pt}(X)$ and $U \in \mathcal{O}(X)$, $p_q^{-1}\varepsilon_X^{-1}(U) = p_q^{-1}(\mathrm{pt}(U)) = \begin{cases} 1 & \mathrm{pt}(U) \ni q \\ 0 & \mathrm{pt}(U) \not\ni q \end{cases} = q^{-1}(U)$. We have $p_q^{-1}\varepsilon_X^{-1} = q^{-1}$ as morphisms of frames. This implies $\varepsilon_X p_q = q$ as morphisms of locales. Hence $\mathrm{pt}(\varepsilon_X)\eta_{\mathrm{pt}(X)}(q) = \mathrm{pt}(\varepsilon_X)(p_q) = \varepsilon_X p_q = q$, that is, $\mathrm{pt}(\varepsilon_X)\eta_{\mathrm{pt}(X)} = id_{\mathrm{pt}(X)}$. □

Definition 2.16.10 *Let S be a topological space.*

1) A subset F of S is said to be irreducible if it satisfies the following condition (Irr).

(Irr) If $F = A \cup B$ for subsets A, B of S closed in F, then $F = A$ or $F = B$ holds.

2) For a closed subset F of S, a point x of S is called a generic point of F if $F = \overline{\{y\}}$.

3) S is said to be sober if every irreducible closed subset of S has a unique generic point.

Lemma 2.16.11 *Let S be a topological space.*

1) A a closed subset F of S is irreducible if and only if $P = S - F$ satisfies the following condition; "If open sets U, V satisfies $U \cap V \subset P$, then $U \subset P$ or $V \subset P$"

2) S is a T_0-space if and only if $\overline{\{x\}} = \overline{\{y\}}$ for $x, y \in S$ implies $x = y$. In particular, a sober topological space is a T_0-space.

3) If $f : S \to T$ is a continuous map and F is an irreducible closed set of S, $\overline{f(F)}$ is an irreducible closed set of T.

Proof. 1) Note that $U \cap V \subset P$ holds if and only if $F = (F \cap (S - U)) \cup (F \cap (S - V))$. If F is irreducible and $U \cap V \subset P$, then $F \subset S - U$ or $F \subset S - V$, that is, $U \subset P$ or $V \subset P$.

Suppose that P satisfies the condition and $F = A \cup B$ for some closen sets A and B. Then, $P = (S - A) \cap (S - B)$ and it follows that $S - A \subset P$ or $S - B \subset P$. Hence $F \subset A$ or $F \subset B$.

2) $\overline{\{x\}} = \overline{\{y\}}$ for $x, y \in S$ holds if and only if the set of open sets containing x coincides with the set of open sets containing y. Thus the assertion follows.

3) Suppose $\overline{f(F)} = A \cup B$ for closed subsets A, B of T. Then, $F = (f^{-1}(A) \cap F) \cup (f^{-1}(B) \cap F)$ and it follows that $F \subset f^{-1}(A)$ or $F \subset f^{-1}(B)$ since F is irreducible. Hence $f(F) \subset A$ or $f(F) \subset B$. Since both A and B are closed, we have $\overline{f(F)} \subset A$ or $\overline{f(F)} \subset B$. □

Proposition 2.16.12 *For a locale X, the topological space $\mathrm{pt}(X)$ is sober.*

Proof. Let F be an irreducible closed subset of $\mathrm{pt}(X)$, then $\mathrm{pt}(X) - F = \mathrm{pt}(P)$ for some $P \in \mathcal{O}(X)$. Suppose that there exists a point p of X such that $F = \overline{\{p\}}$. Then, $\mathrm{pt}(P) = \mathrm{pt}(X) - \overline{\{p\}}$ and this means that every open set of $\mathrm{pt}(X)$ which does not contain p is contained in $\mathrm{pt}(P)$. Hence, for $U \in \mathcal{O}(X)$, $\mathrm{pt}(U) \subset \mathrm{pt}(P)$ if and only if $p \notin \mathrm{pt}(U)$, equivalently, $p^{-1}(U) = 0$. Thus p is uniquely determined by

$$p^{-1}(U) = \begin{cases} 0 & \mathrm{pt}(U) \subset \mathrm{pt}(P) \\ 1 & \mathrm{pt}(U) \not\subset \mathrm{pt}(P) \end{cases} \quad \cdots (*)$$

We show that $p^{-1} : \mathcal{O}(X) \to \{0,1\}$ given by $(*)$ is a morphism of frames. Since $\mathrm{pt}(P) \neq \mathrm{pt}(X) = \mathrm{pt}(1)$ and $\mathrm{pt}(0) = \emptyset$, we have $p^{-1}(1) = 1$ and $p^{-1}(0) = 0$. For $U, V \in \mathcal{O}(X)$, since F is irreducible, $\mathrm{pt}(U \wedge V) = \mathrm{pt}(U) \cap \mathrm{pt}(V) \subset \mathrm{pt}(P)$ if and only if $\mathrm{pt}(U) \subset \mathrm{pt}(P)$ or $\mathrm{pt}(V) \subset \mathrm{pt}(P)$ by (2.16.10). Hence $p^{-1}(U \wedge V) = 0$ if and only if $p^{-1}(U) = 0$ or $p^{-1}(V) = 0$. Thus we have $p^{-1}(U \wedge V) = p^{-1}(U) \wedge p^{-1}(V)$. For $U_i \in \mathcal{O}(X)$ $(i \in I)$, $p^{-1}(\bigvee\limits_{i \in I} U_i) = 0$ if and only if $\bigcup\limits_{i \in I} \mathrm{pt}(U_i) = \mathrm{pt}(\bigvee\limits_{i \in I} U_i) \subset \mathrm{pt}(P)$, namely $\mathrm{pt}(U_i) \subset \mathrm{pt}(P)$ for all $i \in I$. Hence $p^{-1}(\bigvee\limits_{i \in I} U_i) = 0$ if and only if $p^{-1}(U_i) = 0$ for all $i \in I$. Therefore $p^{-1}(\bigvee\limits_{i \in I} U_i) = \bigvee\limits_{i \in I} p^{-1}(U_i)$. □

Proposition 2.16.13 *Let S be a topological space.*

1) For an open set U of S, $\eta_S(U) = \mathrm{pt}(U) \cap \eta_S(S)$. Hence $\eta_S : S \to \mathrm{pt}(\mathrm{Loc}(S))$ is an open map onto its image.

2) For $s, t \in S$, $\eta_S(s) = \eta_S(t)$ if and only if $\overline{\{s\}} = \overline{\{t\}}$. Hence $\eta_S : S \to \mathrm{pt}(\mathrm{Loc}(S))$ is injective if and only if S is a T_0-space.

3) $p \in \mathrm{pt}(\mathrm{Loc}(S))$ belongs to the image of $\eta_S : S \to \mathrm{pt}(\mathrm{Loc}(S))$ if and only if $S - \bigcup\limits_{U \in K} U$ has a generic point, where K is the kernel of p. Hence $\eta_S : S \to \mathrm{pt}(\mathrm{Loc}(S))$ is surjective if and only if every irreducible closed subset of S has a generic point.

4) S is sober if and only if $\eta_S : S \to \mathrm{pt}(\mathrm{Loc}(S))$ is a homeomorphism.

Proof. 1) $\mathrm{pt}(U) \cap \eta_S(S) = \{p_s \in \mathrm{pt}(\mathrm{Loc}(S)) \,|\, p_s^{-1}(U) = 1\} = \{p_s \in \mathrm{pt}(\mathrm{Loc}(S)) \,|\, s \in U\} = \eta_S(U)$.

2) For $s, t \in S$, $\eta_S(s) = \eta_S(t) \Leftrightarrow p_s = p_t \Leftrightarrow$ "For $U \in \mathcal{O}(S)$, $s \in U \Leftrightarrow t \in U$" $\Leftrightarrow \overline{\{s\}} = \overline{\{t\}}$. Thus the second assertion follows from 2) of (2.16.11).

3) For $s \in S$, $S - \bigcup\limits_{U \in K_s} U = \overline{\{s\}}$ by 4) of (2.16.7). Assume $S - \bigcup\limits_{U \in K} U = \overline{\{s\}}$, then $\bigcup\limits_{U \in K} U = S - \overline{\{s\}} = \bigcup\limits_{U \in K_s} U$. By 3) of (2.16.6), $K = \downarrow(\bigcup\limits_{U \in K} U) = \downarrow(S - \overline{\{s\}}) = \downarrow(\bigcup\limits_{U \in K_s} U) = K_s$. Thus we have $p = p_s$.

4) If S is sober, η_S is bijective by 2) and 3) above and it is an open map by 1). Hence η_s is a homeomorphism. The converse follows from (2.16.12). □

Let S be a topological space. Define a binary relation $\leq$ on S by "$x \leq y \Leftrightarrow x \in \overline{\{y\}}$". Then, $(S, \leq)$ is a partially ordered set and it is an ordered set if and only if S is a T_0-space. For topological spaces S, T, define a relation $\leq$ on $\mathcal{T}op(S,T)$ by "$f \leq g \Leftrightarrow f(x) \leq g(x)$ for all $x \in S$". Then, $\mathrm{Loc} : \mathcal{T}op(S,T) \to \mathcal{L}oc(\mathrm{Loc}(S), \mathrm{Loc}(T))$ preserves order. In fact, suppose $f \leq g$ for continuous maps $f, g : S \to T$. For $U \in \mathcal{O}(T)$ and $s \in f^{-1}(U)$, since $f(s) \in \overline{\{g(s)\}}$, U containes $g(s)$, in other words, $s \in g^{-1}(U)$. Thus $f^{-1}(U) \subset g^{-1}(U)$ for every $U \in \mathcal{O}(T)$.

Corollary 2.16.14 *Let T be a topological space.*

1) $\mathrm{Loc} : \mathcal{T}op(S,T) \to \mathcal{L}oc(\mathrm{Loc}(S), \mathrm{Loc}(T))$ is injective for every topological space S if and only if T is a T_0-space.

2) $\mathrm{Loc} : \mathcal{T}op(S,T) \to \mathcal{L}oc(\mathrm{Loc}(S), \mathrm{Loc}(T))$ is bijective for every topological space S if and only if T is sober.

Proof. Since $\mathrm{Loc} : \mathcal{T}op(S,T) \to \mathcal{L}oc(\mathrm{Loc}(S), \mathrm{Loc}(T))$ is composition

$$\mathcal{T}op(S,T) \xrightarrow{(\eta_T)_*} \mathcal{T}op(S, \mathrm{pt}(\mathrm{Loc}(T))) \xrightarrow[\cong]{\text{adjoint}} \mathcal{L}oc(\mathrm{Loc}(S), \mathrm{Loc}(T)),$$

Loc is injective (resp. bijective) for every topological space S if and only if η_T is a monomorphism (resp. isomorphism). Generally, a continuous map $f : X \to Y$ is a monomorphism (resp. isomorphism) if and only if f is injective (resp. homeomorphism). □

Combining the above result with (2.16.2), we have the following result.

Proposition 2.16.15 *Let S and T be topological spaces.*

1) If T is a T_0-space, $\tilde{\Psi}\Xi\mathrm{Loc} : \mathcal{T}op(S,T) \to \mathfrak{Top}(\mathrm{Sh}(S), \mathrm{Sh}(T))$ is fully faithful.

2) If T is sober, $\tilde{\Psi}\Xi\mathrm{Loc} : \mathcal{T}op(S,T) \to \mathfrak{Top}(\mathrm{Sh}(S), \mathrm{Sh}(T))$ is an equivalence.

Definition 2.16.16 *We say that a locale X has enough points if $U \neq V$ $(U, V \in \mathcal{O}(X))$, there exists a point p of X such that $p^{-1}(U) \neq p^{-1}(V)$.*

Proposition 2.16.17 *Let S be a topological space. Then,* $\mathrm{Loc}(S)$ *has enough points.*

Proof. Suppose $U \neq V$ $(U, V \in \mathcal{O}(S))$. Then, $U - V \neq \emptyset$ or $V - U \neq \emptyset$. In the former case, take $s \in U - V$, then $p_s^{-1}(U) = 1$ and $p_s^{-1}(V) = 0$. In the latter case, take $s \in V - U$, then $p_s^{-1}(U) = 0$ and $p_s^{-1}(V) = 1$. Hence $p_s^{-1}(U) \neq p_s^{-1}(V)$ in both cases. □

Proposition 2.16.18 *Let X be a locale. The following conditions are equivalent.*

i) X has enough points.

ii) $\mathrm{pt}(U) = \mathrm{pt}(V)$ *for $U, V \in \mathcal{O}(X)$ implies $U = V$.*

iii) $\varepsilon_X : \mathrm{Loc}(\mathrm{pt}(X)) \to X$ is an isomorphism of locales.

Proof. *i)* ⇒ *ii)*; $\mathrm{pt}(U) = \mathrm{pt}(V)$ holds if and only if $p^{-1}(U) = p^{-1}(V)$ for every point p of X.

ii) ⇒ *iii)*; Recall that $\varepsilon_X^{-1} : \mathcal{O}(X) \to \mathcal{O}(\mathrm{pt}(X))$ is always surjective. Since $\varepsilon_X^{-1}(U) = \mathrm{pt}(U)$, ε_X^{-1} is injective by the assumption. Hence ε_X^{-1} is an isomorphism of frames.

iii) ⇒ *i)*; This follows from (2.16.17). □

Let us denote by $\mathcal{S}ob$ the full subcategory of $\mathcal{T}op$ consisting of sober topological spaces. For a topological space S, we denote by $\sigma(S)$ the set of all irreducible closed subsets of S. For a closed subset A of S, put $\tilde{A} = \{F \in \sigma(S) \,|\, F \subset A\}$. Then, $\tilde{\emptyset} = \emptyset$, $\tilde{S} = \sigma(S)$ and $\widetilde{A \cup B} = \tilde{A} \cup \tilde{B}$, $\widetilde{\bigcap_i A_i} = \bigcap_i \tilde{A}_i$ hold. In fact, first and second equalities are clear. It is also clear that $\widetilde{A \cup B} \supset \tilde{A} \cup \tilde{B}$, $\widetilde{\bigcap_i A_i} \subset \bigcap_i \tilde{A}_i$ hold. If $F \in \widetilde{A \cup B}$, $F \subset A \cup B$, hence $F = (A \cap F) \cup (B \cap F)$. Since F is irreducible and both A and B is closed, $F = A \cap F$ or $F = B \cap F$, namely $F \subset A$ or $F \subset B$ holds. Therefore $F \in \tilde{A} \cup \tilde{B}$. If $F \in \bigcap_i \tilde{A}_i$, $F \subset A_i$ for all i. Then, $F \subset \bigcap_i A_i$ and this implies $F \in \widetilde{\bigcap_i A_i}$. We can give $\sigma(S)$ a topology such that $\{\tilde{A} \,|\, A \text{ is closed set of } S\}$ is the set of all closed sets of $\sigma(S)$.

Define a map $u_S : S \to \sigma(S)$ by $u_S(s) = \overline{\{s\}}$. Let us denote by $\mathcal{A}(X)$ the lattice of all closed subsets of a topological space X.

Lemma 2.16.19 *1) For a closed subset A of S, $u_S^{-1}(\tilde{A}) = A$ and $u_S(A) = \tilde{A} \cap u_S(S)$ hold. Moreover, $u_S(A)$ is dense in $\tilde{A}$.*

2) $u_S^{-1} : \mathcal{A}(\sigma(S)) \to \mathcal{A}(S)$ is an isomorphism of lattices. Hence u_S^{-1} maps the subset of irreducible closed subsets of $\sigma(S)$ bijectively onto the subset of irreducible closed subsets of S.

3) $u_S^{-1} : \mathcal{O}(\sigma(S)) \to \mathcal{O}(S)$ is an isomorphism of frames. In particular, u_S is continuous.

4) If A is an irreducible subset of S, $\overline{\{A\}} = \tilde{A}$ in $\sigma(S)$.

Proof. 1) If $s \in A$, since $\overline{\{s\}}$ is irreducible and contained in A, $u_S(s) \in \tilde{A}$. Suppose $u_S(s) \in \tilde{A}$, then $s \in \overline{\{s\}} \subset A$. Hence we have $u_S^{-1}(\tilde{A}) = A$. It is clear that $u_S(A) \subset \tilde{A} \cap u_S(S)$. If $u_S(s) \in \tilde{A}$, then $s \in \overline{\{s\}} \subset A$. Thus $u_S(s) \in u_S(A)$ and we have $u_S(A) = \tilde{A} \cap u_S(S)$.

Let U be an open set of $\sigma(S)$ then $\sigma(S) - U = \tilde{B}$ for some closed set B in S. Suppose $u_S(A) \cap U = \emptyset$, that is, $u_S(A) \subset \tilde{B}$. For $s \in A$, since $u_S(s) = \overline{\{s\}} \in u_S(A) \subset \tilde{B}$, $s \in \overline{\{s\}} \subset B$. Hence $A \subset B$ and we have $\tilde{A} \subset \tilde{B}$, in other words, $U \cap \tilde{A} = \emptyset$. Therefore $u_S(A)$ is dense in $\tilde{A}$.

2) By the first equality of 1) above, the inverse of $u_S^{-1} : \mathcal{A}(\sigma(S)) \to \mathcal{A}(S)$ is given by $A \mapsto \tilde{A}$.

3) This is a direct consequence of 2).

4) Since A is irreducible, $A \in \tilde{A}$, thus $\overline{\{A\}} \subset \tilde{A}$. For $F \in \tilde{A}$ and a neighborhood U of F, then $F \subset A$ and there exists a closed subset B of S such that $\sigma(S) - U = \tilde{B}$. Suppose $A \in \tilde{B}$. We have $F \subset A \subset B$ and this implies $F \in \tilde{B} = \sigma(S) - U$, which contradicts $F \in U$. Therefore $A \in U$ and we see $F \in \overline{\{A\}}$. Hence $\overline{\{A\}} = \tilde{A}$. □

Lemma 2.16.20 *1) $\sigma(S)$ is sober.*

2) If S is sober, u_S is a homeomorphism.

Proof. 1) If $\tilde{A}$ ($A \in \mathcal{A}(S)$) is an irreducible closed subset of $\sigma(S)$, by 2) of (2.16.19), $u_S^{-1}(\tilde{A}) = A$ is an irreducible closed subset of S, namely, A is regarded as an element of $\sigma(S)$. Then, by 4) of (2.16.19), $\overline{\{A\}} = \tilde{A}$. It follows that every closed subset of $\sigma(S)$ has a generic point. If $\overline{\{F\}} = \tilde{A}$ for $F \in \sigma(S)$, then $\tilde{F} = . = \overline{\{F\}} = \tilde{A}$ and we have $F = u_S^{-1}(\tilde{F}) = u_S^{-1}(\tilde{A}) = A$. Thus we see that $\sigma(S)$ is sober.

2) If S is sober, it is clear that u_S is bijective. Then, by the second equality of 1), u_S is a closed map. $\square$

Lemma 2.16.21 *Let $f : S \to T$ be a continuous map and A a subset A of S. Then, $\overline{f(\bar{A})} = \overline{f(A)}$.*

Proof. Since $f(\bar{A}) \supset f(A)$, $\overline{f(\bar{A})} \supset \overline{f(A)}$. By the continuity of f, we have $f(\bar{A}) \subset \overline{f(A)}$. It follows that $\overline{f(\bar{A})} \subset \overline{f(A)}$. $\square$

Proposition 2.16.22 *$\mathcal{S}ob$ is a reflexive subcategory of $\mathcal{T}op$, that is, the inclusion functor $\iota : \mathcal{S}ob \to \mathcal{T}op$ has a left adjoint $\sigma : \mathcal{T}op \to \mathcal{S}ob$ such that there is a natural equivalence $u : \mathrm{Loc} \to \mathrm{Loc}\sigma$.*

Proof. Let $f : S \to T$ be a continuous map. For $F \in \sigma(S)$, since $\overline{f(F)}$ is an irreducible closed subset of T by 3) of (2.16.11), we can define a map $\sigma(f) : \sigma(S) \to \sigma(T)$ by $\sigma(f)(F) = \overline{f(F)}$. Let B be a closed subset of T. We show $\sigma(f)^{-1}(\tilde{B}) = \widetilde{f^{-1}(B)}$. In fact, $F \in \sigma(f)^{-1}(\tilde{B}) \Leftrightarrow \overline{f(F)} \in \tilde{B} \Leftrightarrow \overline{f(F)} \subset B \Leftrightarrow f(F) \subset B \Leftrightarrow F \subset f^{-1}(B) \Leftrightarrow F \in \widetilde{f^{-1}(B)}$. Hence $\sigma(f)$ is continuous. For a continuous map $g : T \to W$ and an irreducible closed subset F of S, since $\overline{g(\overline{f(F)})} = \overline{gf(F)}$ by (2.16.21), we have $\sigma(gf) = \sigma(g)\sigma(f)$. Therefore correspondences $S \mapsto \sigma(S)$, $f \mapsto \sigma(f)$ give a functor $\sigma : \mathcal{T}op \to \mathcal{S}ob$ by 1) of (2.16.20).

For $s \in S$, $u_T(f(s)) = \overline{f(s)} = \overline{f(\overline{\{s\}})} = \sigma(f)(\overline{\{s\}}) = \sigma(f)u_S(s)$ by (2.16.21). Thus we have a natural transformation $u : id_{\mathcal{T}op} \to \iota\sigma$. For a sober space T, since $u_T : T \to \sigma(T)$ is a homeomorphism by 2) of (2.16.20), define $e_T : \sigma\iota(T) \to T$ by $e_T = u_T^{-1}$. Then, $\iota(e_T)u_{\iota(T)} = u_T^{-1}u_T = id_T$. For a topological space S and an irreducible subset F of S, since $u_{\sigma(S)}(F) = \overline{\{F\}} = \tilde{F} = \overline{u_S(F)}$ by 1), 4) of (2.16.19), $e_{\sigma(S)}\sigma(u_S)(F) = u^{-1}_{\sigma(S)}(\overline{u_S(F)}) = u^{-1}_{\sigma(S)}(u_{\sigma(S)}(F)) = F$. Therefore, σ is a left adjoint of ι.

By 3) of (2.16.19), $u_S^{-1} : \mathcal{O}(\sigma(S)) \to \mathcal{O}(S)$ gives a natural isomorphism $u_S : \mathrm{Loc}(S) \to \mathrm{Loc}(\sigma(S))$ of locales. Hence we have a natural equivalence $u : \mathrm{Loc} \to \mathrm{Loc}\sigma$. $\square$

Chapter 3

Elementary topos

Retold version of P. T. Johnstone's book "Topos Theory" Part I

3.1 Definitions

Definition 3.1.1 *A subobject classifier of a category $\mathcal{E}$ with a terminal object 1 is an object Ω with a morphism $t : 1 \to \Omega$ which has the following property.*

($*$) *For each monomorphism $\sigma : Y \rightarrowtail X$ in $\mathcal{E}$, there is a unique morphism $\phi_\sigma : X \to \Omega$ (the classifying map of σ) making the following square a pull-back.*

$$\begin{array}{ccc} Y & \longrightarrow & 1 \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ X & \xrightarrow{\phi_\sigma} & \Omega \end{array}$$

Definition 3.1.2 *A category $\mathcal{E}$ is called an elementary topos if it satisfies the following axioms.*

(*i*) *$\mathcal{E}$ has all finite limits, that is, $\mathcal{E}$ has pull-back and a terminal object.*
(*ii*) *$\mathcal{E}$ is cartesian closed.*
(*iii*) *$\mathcal{E}$ has a subobject classifier $t : 1 \to \Omega$.*

Definition 3.1.3 *Let $\mathcal{E}$ and $\mathcal{F}$ be topoi.*

(1) A functor $F : \mathcal{E} \to \mathcal{F}$ is called logical if it is left exact and preserves exponentials and the subobject classifier.

(2) A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ consists of a pair of functors $f_* : \mathcal{E} \to \mathcal{F}$, $f^* : \mathcal{F} \to \mathcal{E}$ and a natural bijection $\alpha_f : \mathcal{E}(f^*(Y), X) \to \mathcal{F}(Y, f_*(X))$ such that f^* is left exact. We call f_* the direct image and f^* the inverse image of f.

(3) A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is said to be essential if f^* has a left adjoint $f_!$.

(4) If $f = (f_*, f^*, \alpha_f), g = (g_*, g^*, \alpha_g) : \mathcal{E} \to \mathcal{F}$ are geometric morphisms, a morphism of geometric morphisms $\varphi : f \to g$ means a pair of natural transformations of functors $\varphi_* : g_* \to f_*$ and $\varphi^* : f^* \to g^*$ such that the following square commutes for any $X \in \mathrm{Ob}\,\mathcal{E}$ and $Y \in \mathrm{Ob}\,\mathcal{F}$.

$$\begin{array}{ccc} \mathcal{E}(g^*(Y), X) & \xrightarrow{(\varphi^*_Y)^*} & \mathcal{E}(f^*(Y), X) \\ \big\downarrow{\scriptstyle\alpha_g} & & \big\downarrow{\scriptstyle\alpha_f} \\ \mathcal{F}(Y, g_*(X)) & \xrightarrow{(\varphi_{*X})_*} & \mathcal{F}(Y, f_*(X)) \end{array}$$

If $\varphi = (\varphi_*, \varphi^*) : f \to g$ is a morphism of geometric morphisms, φ^* (resp. φ_*) uniquely determines φ_* (resp. φ^*) by (A.14.1). We often drop the adjunction and denote a geometric morphism (f_*, f^*, α_f) by (f_*, f^*).

Proposition 3.1.4 *Topoi, geometric morphisms and morphisms of geometric morphisms form a 2-category, which we denote by $\mathfrak{Top}$.*

Proof. Let $\mathcal{E}$ and $\mathcal{F}$ be topoi and $f = (f_*, f^*, \alpha_f), g = (g_*, g^*, \alpha_g), h = (h_*, h^*, \alpha_h) : \mathcal{E} \to \mathcal{F}$ geomrtric morphisms. For morphisms $\varphi = (\varphi_*, \varphi^*) : f \to g$, $\psi = (\psi_*, \psi^*) : g \to h$ of geomrtric morphisms, the composition $\psi\varphi : f \to h$ of is given by $(\varphi_*\psi_*, \psi^*\varphi^*)$. In fact, since the left and right squares of the following diagrams commutes, so does the outer rectangle.

$$\begin{tikzcd}
\mathcal{E}(h^*(Y), X) \arrow[r, "(\psi_Y^*)^*"] \arrow[d, "\alpha_h"] & \mathcal{E}(g^*(Y), X) \arrow[r, "(\varphi_Y^*)^*"] \arrow[d, "\alpha_g"] & \mathcal{E}(f^*(Y), X) \arrow[d, "\alpha_f"] \\
\mathcal{F}(Y, h_*(X)) \arrow[r, "(\psi_{*X})_*"] & \mathcal{F}(Y, g_*(X)) \arrow[r, "(\varphi_{*X})_*"] & \mathcal{F}(Y, f_*(X))
\end{tikzcd}$$

Hence geometric morphisms from $\mathcal{E}$ to $\mathcal{F}$ and morphisms between them form a category, which we denote by $\mathfrak{Top}(\mathcal{E}, \mathcal{F})$.

For topoi $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, define a functor $\mu : \mathfrak{Top}(\mathcal{E}, \mathcal{F}) \times \mathfrak{Top}(\mathcal{F}, \mathcal{G}) \to \mathfrak{Top}(\mathcal{E}, \mathcal{G})$ as follows. Let $f = (f_*, f^*, \alpha_f) : \mathcal{E} \to \mathcal{F}$ and $g = (g_*, g^*, \alpha_g) : \mathcal{F} \to \mathcal{G}$ be geometric morphisms. For $X \in \operatorname{Ob}\mathcal{E}$ and $Z \in \operatorname{Ob}\mathcal{G}$, define a bijection $(\alpha_{gf})_{X,Z} : \mathcal{E}(f^*g^*(Z), X) \to \mathcal{G}(Z, g_*f_*(X))$ to be the composition of $(\alpha_f)_{X,g^*(Z)} : \mathcal{E}(f^*g^*(Z), X) \to \mathcal{F}(g^*(Z), f_*(X))$ and $(\alpha_g)_{f_*(X),Z} : \mathcal{F}(g^*(Z), f_*(X)) \to \mathcal{G}(Z, g_*f_*(X))$. Obviously, $(\alpha_{gf})_{X,Z}$ is natural in X and Z. We define the composition $gf : \mathcal{E} \to \mathcal{G}$ of geometric morphisms by $gf = (g_*f_*, f^*g^*, \alpha_{gf})$ and set $\mu(f, g) = gf$. Let $\varphi : f_1 \to f_2$ (resp. $\psi : g_1 \to g_2$) be a morphism of geometric morphisms from $\mathcal{E}$ to $\mathcal{F}$ (resp. from $\mathcal{F}$ to $\mathcal{G}$). We note that, for any $X \in \operatorname{Ob}\mathcal{E}$ and $Z \in \operatorname{Ob}\mathcal{G}$, the following squares commute.

$$\begin{tikzcd}
g_{2*}f_{2*}(X) \arrow[r, "g_{2*}(\varphi_{*X})"] \arrow[d, "\psi_{*f_{2*}(X)}"] & g_{2*}f_{1*}(X) \arrow[d, "\psi_{*f_{1*}(X)}"] \\
g_{1*}f_{2*}(X) \arrow[r, "g_{1*}(\varphi_{*X})"] & g_{1*}f_{1*}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
f_1^*g_1^*(Z) \arrow[r, "f_1^*(\psi_Z^*)"] \arrow[d, "\varphi^*_{g_1^*(Z)}"] & f_1^*g_2^*(Z) \arrow[d, "\varphi^*_{g_2^*(Z)}"] \\
f_2^*g_1^*(Z) \arrow[r, "f_2^*(\psi_Z^*)"] & f_2^*g_2^*(Z)
\end{tikzcd}$$

Since the following diagram commutes by the assumption, a pair of natural transformations $\psi_{*f_{1*}}g_{2*}(\varphi_*) : g_{2*}f_{2*} \to g_{1*}f_{1*}$ and $\varphi^*_{g_2^*}f_1^*(\psi^*) : f_1^*g_1^* \to f_2^*g_2^*$ gives a morphism $\psi{\cdot}\varphi = (\psi_{*f_{1*}}g_{2*}(\varphi_*), \varphi^*_{g_2^*}f_1^*(\psi^*)) : g_1f_1 \to g_2f_2$.

$$\begin{tikzcd}
\mathcal{E}(f_2^*g_2^*(Z), X) \arrow[r, "(\varphi^*_{g_2^*(Z)})^*"] \arrow[d, "\alpha_{f_2}"] & \mathcal{E}(f_1^*g_2^*(Z), X) \arrow[r, "(f_1^*(\psi_Z^*))^*"] \arrow[d, "\alpha_{f_1}"] & \mathcal{E}(f_1^*g_1^*(Z), X) \arrow[d, "\alpha_{f_1}"] \\
\mathcal{F}(g_2^*(Z), f_{2*}(X)) \arrow[r, "(\varphi_{*X})_*"] \arrow[d, "\beta_{f_2}"] & \mathcal{F}(g_2^*(Z), f_{1*}(X)) \arrow[r, "(\psi_Z^*)^*"] \arrow[d, "\beta_{f_2}"] & \mathcal{F}(g_1^*(Z), f_{1*}(X)) \arrow[d, "\beta_{f_1}"] \\
\mathcal{G}(Z, g_{*2}f_{2*}(X)) \arrow[r, "(g_{2*}(\varphi_{*X}))_*"] & \mathcal{G}(Z, g_{*2}f_{1*}(X)) \arrow[r, "(\psi_{*f_{1*}(X)})_*"] & \mathcal{G}(Z, g_{*1}f_{1*}(X))
\end{tikzcd}$$

We set $\mu(\varphi, \psi) = \psi{\cdot}\varphi$. Let $\varphi : f_1 \to f_2$, $\zeta : f_2 \to f_3$ be morphisms in $\mathfrak{Top}(\mathcal{E}, \mathcal{F})$ and $\psi : g_1 \to g_2$, $\xi : g_2 \to g_3$ morphisms in $\mathfrak{Top}(\mathcal{F}, \mathcal{G})$. Then, by the above definition, we have

$$\begin{aligned}
(\xi{\cdot}\zeta)(\psi{\cdot}\varphi) &= (\psi_{*f_{1*}}g_{2*}(\varphi_*)\xi_{*f_{2*}}g_{3*}(\zeta_*), \zeta^*_{g_3^*}f_2^*(\xi^*)\varphi^*_{g_2^*}f_1^*(\psi^*)) = (g_{1*}(\varphi_*)\psi_{*f_{2*}}g_{2*}(\zeta_*)\xi_{*f_{3*}}, f_3^*(\xi^*)\zeta^*_{g_2^*}f_2^*(\psi^*)\varphi^*_{g_1^*}) \\
&= (g_{1*}(\varphi_*)g_{1*}(\zeta_*)\psi_{*f_{3*}}\xi_{*f_{3*}}, f_3^*(\xi^*)f_3^*(\psi^*)\zeta^*_{g_1^*}\varphi^*_{g_1^*}) = (g_{1*}(\varphi_*\zeta_*)(\psi_*\xi_*)_{f_{3*}}, f_3^*(\xi^*\psi^*)(\zeta^*\varphi^*)_{g_1^*}) \\
&= (g_{1*}((\zeta\varphi)_*)((\xi\psi)_*)_{f_{3*}}, f_3^*((\xi\psi)^*)((\zeta\varphi)^*)_{g_1^*}) = (\xi\psi){\cdot}(\zeta\varphi).
\end{aligned}$$

Let $id_f : f \to f$ and $id_g : g \to g$ the identity morphisms of geometric morphisms $f : \mathcal{E} \to \mathcal{F}$ and $g : \mathcal{F} \to \mathcal{G}$. Then, it is clear that $\mu(id_f, id_g) = id_g{\cdot}id_f$ is the identity morphism of $gf : \mathcal{E} \to \mathcal{G}$. Hence μ is a functor.

For each topos $\mathcal{E}$, there is an identity geometric morphism $Id_{\mathcal{E}} = (id_{\mathcal{E}}, id_{\mathcal{E}}, id) : \mathcal{E} \to \mathcal{E}$. We denote by $\mathbf{1}$ the category with a single object 1 and a single morphism id_1. Define a functor $u_{\mathcal{E}} : \mathbf{1} \to \mathfrak{Top}(\mathcal{E}, \mathcal{E})$ by $u_{\mathcal{E}}(1) = Id_{\mathcal{E}}$. We claim that the following diagrams commute.

$$\begin{tikzcd}
\mathfrak{Top}(\mathcal{E}, \mathcal{F}) \times \mathfrak{Top}(\mathcal{F}, \mathcal{G}) \times \mathfrak{Top}(\mathcal{G}, \mathcal{H}) \arrow[r, "\mu \times 1"] \arrow[d, "1 \times \mu"] & \mathfrak{Top}(\mathcal{E}, \mathcal{G}) \times \mathfrak{Top}(\mathcal{G}, \mathcal{H}) \arrow[d, "\mu"] \\
\mathfrak{Top}(\mathcal{E}, \mathcal{F}) \times \mathfrak{Top}(\mathcal{F}, \mathcal{H}) \arrow[r, "\mu"] & \mathfrak{Top}(\mathcal{E}, \mathcal{H})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{Top}(\mathcal{E}, \mathcal{G}) \times \mathbf{1} \arrow[d, "1 \times u_{\mathcal{G}}"] \arrow[dr, "\mathrm{pr}_1"] & \\
\mathfrak{Top}(\mathcal{E}, \mathcal{G}) \times \mathfrak{Top}(\mathcal{G}, \mathcal{G}) \arrow[r, "\mu"] & \mathfrak{Top}(\mathcal{E}, \mathcal{G})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \times \mathfrak{Top}(\mathcal{E}, \mathcal{G}) \arrow[d, "u_{\mathcal{E}} \times 1"] \arrow[dr, "\mathrm{pr}_2"] & \\
\mathfrak{Top}(\mathcal{E}, \mathcal{E}) \times \mathfrak{Top}(\mathcal{E}, \mathcal{G}) \arrow[r, "\mu"] & \mathfrak{Top}(\mathcal{E}, \mathcal{G})
\end{tikzcd}$$

In fact, the commutativity on objects is clear. Let $f_1, f_2 : \mathcal{E} \to \mathcal{F}$, $g_1, g_2 : \mathcal{F} \to \mathcal{G}$, $h_1, h_2 : \mathcal{G} \to \mathcal{H}$ be geometric morphisms and $\varphi : f_1 \to f_2$, $\psi : g_1 \to g_2$, $\chi : h_1 \to h_2$ morphisms between them. Then,

$$\begin{aligned}
\chi\cdot(\psi\cdot\varphi) &= (\chi_{*(g_1 f_1)_*} h_{2*}((\psi\cdot\varphi)_*), (\psi\cdot\varphi)^*_{h_2^*}(g_1 f_1)^*(\chi^*)) = (\chi_{*g_{1*}f_{1*}} h_{2*}(\psi_{*f_{1*}} g_{2*}(\varphi_*)), (\varphi^*_{g_2^*} f_1^*(\psi^*))_{h_2^*} f_1^* g_1^*(\chi^*)) \\
&= (\chi_{*g_{1*}f_{1*}} h_{2*}(\psi_{*f_{1*}}) h_{2*} g_{2*}(\varphi_*), \varphi^*_{g_2^* h_2^*} f_1^*(\psi^*_{h_2^*}) f_1^* g_1^*(\chi^*)) \\
&= ((\chi_{*g_{1*}} h_{2*}(\psi_*))_{f_{1*}} h_{2*} g_{2*}(\varphi_*), \varphi^*_{g_2^* h_2^*} f_1^*(\psi^*_{h_2^*} g_1^*(\chi^*))) \\
&= ((\chi\cdot\psi)_{*f_{1*}} (h_2 g_2)_*(\varphi_*), \varphi^*_{(h_2 g_2)^*} f_1^*((\chi\cdot\psi)^*)) = (\chi\cdot\psi)\cdot\varphi
\end{aligned}$$

Hence the upper diagram also commutes on morphisms. It is easy to verify that the lower diagrams commute on morphisms. □

Proposition 3.1.5 *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be topoi.*

1) The product category $\mathcal{E}_1 \times \mathcal{E}_2$ is a topos and the projection functor $P_i : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_i$ $(i = 1, 2)$ is the inverse image of a geometric morphism $s_i : \mathcal{E}_i \to \mathcal{E}_1 \times \mathcal{E}_2$, whose direct image is the functor given by $X \mapsto (X, 1)$ if $i = 1$, $Y \mapsto (1, Y)$ if $i = 2$.

2) The geometric morphisms $s_i : \mathcal{E}_i \to \mathcal{E}_1 \times \mathcal{E}_2$ induce an equivalence $(s_1^*, s_2^*) : \mathfrak{Top}(\mathcal{E}_1 \times \mathcal{E}_2, \mathcal{F}) \to \mathfrak{Top}(\mathcal{E}_1, \mathcal{F}) \times \mathfrak{Top}(\mathcal{E}_2, \mathcal{F})$ for any topos $\mathcal{F}$. Hence $\mathcal{E}_1 \times \mathcal{E}_2$ is a coporoduct of $\mathcal{E}_1$ and $\mathcal{E}_2$ in $\mathfrak{Top}$.

Proof. 1) Finite limits in $\mathcal{E}_1 \times \mathcal{E}_2$ are given componentwise. It is easy to check that $(Y_1, Y_2)^{(X_1, X_2)} = (Y_1^{X_1}, Y_2^{X_2})$ and that the subobject classifier for $\mathcal{E}_1 \times \mathcal{E}_2$ is (Ω_1, Ω_2). P_i is left exact by (A.4.7). Since $(\mathcal{E}_1 \times \mathcal{E}_2)((X, Y), (Z, W)) = \mathcal{E}_1(X, Z) \times \mathcal{E}_2(Y, W)$ for $X, Z \in \operatorname{Ob}\mathcal{E}_1$ and $Y, W \in \operatorname{Ob}\mathcal{E}_2$, $(\mathcal{E}_1 \times \mathcal{E}_2)((X, Y), (Z, 1))$ and $(\mathcal{E}_1 \times \mathcal{E}_2)((X, Y), (1, W))$ are naturally equivalent to $\mathcal{E}_1(P_1(X, Y), Z)$ and $\mathcal{E}_2(P_2(X, Y), W)$, respectively.

2) We define a functor $F : \mathfrak{Top}(\mathcal{E}_1, \mathcal{F}) \times \mathfrak{Top}(\mathcal{E}_2, \mathcal{F}) \to \mathfrak{Top}(\mathcal{E}_1 \times \mathcal{E}_2, \mathcal{F})$ as follows. Let $f : \mathcal{E}_1 \to \mathcal{F}$ and $g : \mathcal{E}_2 \to \mathcal{F}$ be geometric morphisms. Set $h^* = (f^*, g^*) : \mathcal{F} \to \mathcal{E}_1 \times \mathcal{E}_2$ and define a functor $h_* : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{F}$ by $h_*(X, Y) = f_*(X) \times g_*(Y)$ for $(X, Y) \in \operatorname{Ob}\mathcal{E}_1 \times \mathcal{E}_2$ and $h_*(s, t) = f_*(s) \times g_*(t)$ for $(s, t) \in \operatorname{Mor}\mathcal{E}_1 \times \mathcal{E}_2$. Clearly, h^* is left exact since f^* and g^* are so. For $(X, Y) \in \operatorname{Ob}\mathcal{E}_1 \times \mathcal{E}_2$ and $Z \in \operatorname{Ob}\mathcal{F}$, there are natural bijections $(\mathcal{E}_1 \times \mathcal{E}_2)(h^*(Z), (X, Y)) = (\mathcal{E}_1 \times \mathcal{E}_2)((f^*(Z), g^*(Z)), (X, Y)) = \mathcal{E}_1(f^*(Z), X) \times \mathcal{E}_2(g^*(Z), Y) \xrightarrow[\cong]{\alpha_f \times \alpha_g} \mathcal{F}(Z, f_*(X)) \times \mathcal{F}(Z, g_*(Y)) \cong \mathcal{F}(Z, h_*(X, Y))$. Hence $h = (h_*, h^*) : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{F}$ is a geometric morphism and set $F(f, g) = h$. Let $\varphi : f \to k$ and $\psi : g \to l$ be morphisms in $\mathfrak{Top}(\mathcal{E}_1, \mathcal{F})$ and $\mathfrak{Top}(\mathcal{E}_2, \mathcal{F})$, respectively. Define $\theta_* : F(k, l)_* \to F(f, g)_*$ by $\theta_{*(X,Y)} = \varphi_{*X} \times \psi_{*Y} : k_*(X) \times l_*(Y) \to f_*(X) \times g_*(Y)$. $\theta^* : F(f, g)^* \to F(k, l)^*$ is defined by $\theta^*_Z = (\varphi^*_Z, \psi^*_Y) : (f^*(Z), g^*(Z)) \to (k^*(Z), l^*(Z))$ for $Z \in \mathcal{F}$. Since the following diagram commutes, $\theta = (\theta_*, \theta^*) : F(f, g) \to F(k, l)$ is a morphism of geometric morphisms.

$$\begin{tikzcd}
(\mathcal{E}_1 \times \mathcal{E}_2)(F(k,l)^*(Z), (X,Y)) \arrow[r, "(\theta_Z^*)^*"] \arrow[d, equal] & (\mathcal{E}_1 \times \mathcal{E}_2)(F(f,g)^*(Z), (X,Y)) \arrow[d, equal] \\
\mathcal{E}_1(k^*(Z), X) \times \mathcal{E}_2(l^*(Z), Y) \arrow[r, "(\varphi_Z^*)^* \times (\psi_Z^*)^*"] \arrow[d, "\alpha_k \times \alpha_l"] & \mathcal{E}_1(f^*(Z), X) \times \mathcal{E}_2(g^*(Z), Y) \arrow[d, "\alpha_f \times \alpha_g"] \\
\mathcal{F}(Z, k_*(X)) \times \mathcal{F}(Z, l_*(Y)) \arrow[r, "(\varphi_{*X})_* \times (\psi_{*Y})_*"] & \mathcal{F}(Z, f_*(X)) \times \mathcal{F}(Z, g_*(Y)) \\
\mathcal{F}(Z, k_*(X) \times l_*(Y)) \arrow[r, "(\varphi_{*X} \times \psi_{*Y})_*"] \arrow[u, "(\mathrm{pr}_{1*}, \mathrm{pr}_{2*})", "\cong"'] \arrow[d, equal] & \mathcal{F}(Z, f_*(X) \times g_*(Y)) \arrow[u, "(\mathrm{pr}_{1*}, \mathrm{pr}_{2*})", "\cong"'] \arrow[d, equal] \\
\mathcal{F}(Z, F(k,l)_*(X,Y)) \arrow[r, "(\theta_{*(X,Y)})_*"] & \mathcal{F}(Z, F(f,g)_*(X,Y))
\end{tikzcd}$$

Thus we set $F(\varphi, \psi) = \theta$. It is easy to verify that F is a functor.

Let $h = (h_*, h^*) : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{F}$ be a geometric morphism. For $(X, Y) \in \operatorname{Ob}\mathcal{E}_1 \times \mathcal{E}_2$, since h_* is left exact, a morphism $\beta = (h_*(id_X, o_Y), h_*(o_X, id_Y)) : h_*(X, Y) \to h_*(X, 1) \times h_*(1, Y)$ is an isomorphism. Here $o_X : X \to 1$ and $o_Y : Y \to 1$ denote the unique morphisms. Since $(F(s_1^*, s_2^*)(h))_*(X, Y) = F(hs_1, hs_2)_*(X, Y) = (hs_1)_*(X) \times (hs_2)_*(Y) = h_*(X, 1) \times h_*(1, Y)$, we have an equivalence $\kappa_{h*} : h_* \to (F(s_1^*, s_2^*)(h))_*$. For $Z \in \operatorname{Ob}\mathcal{F}$, we have $(F(s_1^*, s_2^*)(h))^*(Z) = F(hs_1, hs_2)^*(Z) = ((hs_1)^*(Z), (hs_2)^*(Z)) = (P_1 h^*(Z), P_2 h^*(Z)) = h^*(Z)$. We claim that $\kappa_h = (\kappa_{h*}, id_{h^*}) : F(s_1^*, s_2^*)(h) \to h$ is a morphism of geometric morphisms, that is, the following diagram commutes.

$$\begin{tikzcd}
(\mathcal{E}_1 \times \mathcal{E}_2)((F(s_1^*, s_2^*)(h))^*(Z), (X, Y)) \arrow[d, equal] & \\
\mathcal{E}_1(P_1 h^*(Z), X) \times \mathcal{E}_2(P_2 h^*(Z), Y) \arrow[r, equal] \arrow[d, "\alpha_{s_1} \times \alpha_{s_2}"] & (\mathcal{E}_1 \times \mathcal{E}_2)(h^*(Z), (X, Y)) \arrow[d, "\alpha_h"] \\
(\mathcal{E}_1 \times \mathcal{E}_2)(h^*(Z), s_{1*}(X)) \times (\mathcal{E}_1 \times \mathcal{E}_2)(h^*(Z), s_{2*}(Y)) \arrow[d, "\alpha_h \times \alpha_h"] & \mathcal{F}(Z, h_*(X, Y)) \arrow[d, "(\kappa_{h*})_*"] \\
\mathcal{F}(Z, h_* s_{1*}(X)) \times \mathcal{F}(Z, h_* s_{2*}(Y)) & \mathcal{F}(Z, h_* s_{1*}(X) \times h_* s_{2*}(Y)) \arrow[l, "(\mathrm{pr}_{1*}, \mathrm{pr}_{2*})"', "\cong"] \arrow[d, equal] \\
 & \mathcal{F}(Z, (F(s_1^*, s_2^*)(h))_*(X, Y))
\end{tikzcd}$$

For a morphism $(u, v) : h^*(Z) \to (X, Y)$ in $\mathcal{E}_1 \times \mathcal{E}_2$ ($u : P_1 h^*(Z) \to X$, $v : P_2 h^*(Z) \to Y$), we have

$$\mathrm{pr}_{1*}(\kappa_{h*})_* \alpha_h(u, v) = h_*(id_X, o_Y)\alpha_h(u, v) = \alpha_h(u, o_Y v) = \alpha_h \alpha_{s_1}(u)$$
$$\mathrm{pr}_{2*}(\kappa_{h*})_* \alpha_h(u, v) = h_*(o_X, id_Y)\alpha_h(u, v) = \alpha_h(o_X u, v) = \alpha_h \alpha_{s_2}(v)$$

by the naturality of α_h. Clearly, κ_h is natural in h. Thus we have an equivalence $\kappa : F(s_1^*, s_2^*) \to id_{\mathfrak{Top}(\mathcal{E}_1 \times \mathcal{E}_2, \mathcal{F})}$.

We note that $f_*(1)$ and $g_*(1)$ are terminal object in $\mathcal{F}$ and that $(F(f,g)s_1)_*(X) = F(f,g)_* s_{1*}(X) = F(f,g)_*(X, 1) = f_*(X) \times g_*(1)$ and $(F(f,g)s_2)_*(Y) = F(f,g)_* s_{2*}(Y) = F(f,g)_*(1, Y) = f_*(1) \times g_*(Y)$ hold. Hence the projections $\mathrm{pr}_1 : f_*(X) \times g_*(1) \to f_*(X)$, $\mathrm{pr}_2 : f_*(1) \times g_*(Y) \to g_*(Y)$ define equivalences $\pi^1_{(f,g)} : (F(f,g)s_1)_* \to f_*$, $\pi^2_{(f,g)} : (F(f,g)s_2)_* \to g_*$. Moreover, since $(F(f,g)s_1)^*(Z) = P_1 F(f,g)^*(Z) = P_1(f^*(Z), g^*(Z)) = f^*(Z)$ and $(F(f,g)s_2)^*(Z) = P_2 F(f,g)^*(Z) = P_2(f^*(Z), g^*(Z)) = g^*(Z)$ hold, we have isomorphisms of geometric morphisms $\rho^1_{(f,g)} = (\pi^1_{(f,g)}, id_{f^*}) : f \to F(f,g)s_1$ and $\rho^2_{(f,g)} = (\pi^2_{(f,g)}, id_{f^*}) : g \to F(f,g)s_2$. Therefore an isomorphism $(\rho^1_{(f,g)}, \rho^2_{(f,g)}) : (f, g) \to (F(f,g)s_1, F(f,g)s_2) = (s_1^*, s_2^*)F(f,g)$ in $\mathfrak{Top}(\mathcal{E}_1, \mathcal{F}) \times \mathfrak{Top}(\mathcal{E}_2, \mathcal{F})$ defines an equivalence $\rho : id_{\mathfrak{Top}(\mathcal{E}_1, \mathcal{F}) \times \mathfrak{Top}(\mathcal{E}_2, \mathcal{F})} \to (s_1^*, s_2^*)F$. $\square$

Proposition 3.1.6 *Let $\mathcal{E}$ be a category with finite limits. If there is a monomorphism $\tau : T \rightarrowtail \Omega$ in $\mathcal{E}$ having the following property, T is a terminal object of $\mathcal{E}$.*

For each monomorphism $\sigma : Y \rightarrowtail X$ in $\mathcal{E}$, there is a unique $\phi_\sigma : X \to \Omega$ such that the following square is a pull-back.

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "\sigma"] & T \arrow[d, "\tau"] \\
X \arrow[r, "\phi_\sigma"] & \Omega
\end{tikzcd}$$

Proof. Let 1 be a terminal object of $\mathcal{E}$ and $\phi_1 : 1 \to \Omega$ the morphism such that the right square of the following diagram is a pull-back.

$$\begin{tikzcd}
T \arrow[r, "o"] \arrow[d, "id_T"] & 1 \arrow[r, "p"] \arrow[d, "id_1"] & T \arrow[d, "\tau"] \\
T \arrow[r, "o"] & 1 \arrow[r, "\phi_1"] & \Omega
\end{tikzcd}$$

Obviously, the left square is also a pull-back. Hence the outer rectangle is a pull-back by (A.3.1). On the other hand, since τ is a monomorphism, the kernel pair of τ is a pair of the identity morphisms of T. By the uniqueness, we have $\tau = \phi_1 o$ and it follows that $po = id_T$. Clearly, $op = id_1$ and the unique morphism $o : T \to 1$ is an isomorphism. $\square$

Proposition 3.1.7 *The axiom (2) of (3.1.2) can be replaced by the following axiom (4) and the axioms (2) and (3) of (3.1.2) can be replaced by the following single axiom (5).*

(4) For any $X \in \mathrm{Ob}\,\mathcal{E}$, there exist a power object PX and a morphism $e_X : PX \times X \to \Omega$ such that, for any morphism $f : Y \times X \to \Omega$, there exists a unique morphism $\hat{f} : Y \to PX$ satisfying $f = e_X(\hat{f} \times id_X)$.

(5) For any $X \in \mathrm{Ob}\,\mathcal{E}$, there exist a power object PX and a subobject $\in_X \overset{i_X}{\rightarrowtail} PX \times X$ such that, for each $Y \in \mathrm{Ob}\,\mathcal{E}$ and each subobject $R \overset{j}{\rightarrowtail} Y \times X$, there exists a unique morphism $r : Y \to PX$ such that the following square is a pull-back.

$$\begin{tikzcd} R \arrow[r] \arrow[d, "j"] & \in_X \arrow[d, "i_X"] \\ Y \times X \arrow[r, "r \times id_X"] & PX \times X \end{tikzcd}$$

Proof. First, we show that the axioms (1), (2), (3) of (3.1.2) imply (5). Set $PX = \Omega^X$ and let $i_X : \in_X \rightarrowtail PX \times X$ be the monomorphism classified by the evaluation map $ev : PX \times X = \Omega^X \times X \to \Omega$. For $Y \in \mathrm{Ob}\,\mathcal{E}$ and a subobject $R \overset{j}{\rightarrowtail} Y \times X$, let $\phi : Y \times X \to \Omega$ be the morphism that classifies j and $r : Y \to PX$ the exponential transpose of ϕ. Since $ev(r \times id_X) = \phi$, there is a unique morphism $j' : R \to \in_X$ such that left square of the following diagram commutes.

$$\begin{tikzcd} R \arrow[r, "j'"] \arrow[d, "j"] & \in_X \arrow[r] \arrow[d, "i_X"] & 1 \arrow[d, "t"] \\ Y \times X \arrow[r, "r \times id_X"] & PX \times X \arrow[r, "ev"] & \Omega \end{tikzcd}$$

Note that the right square and the outer rectangle are pull-backs. Hence the left square is also a pull-back by (A.3.1).

Next, we show that the axioms (1) and (5) imply (3) and (4). Set $\Omega = P(1)$. Since the projection $Y \times 1 \to Y$ is an isomorphism natural in Y, (5) implies that the monomorphism $t : \in_1 \rightarrowtail \Omega \times 1 \cong \Omega$ have the property of (3.1.6). Thus $\in_1$ is a terminal object 1 of $\mathcal{E}$ and $\mathcal{E}$ satisfies (3) of (3.1.2). Let $e_X : PX \times X \to \Omega$ be the unique morphism such that the following square is a pull-back.

$$\begin{tikzcd} \in_X \arrow[r, "o_1"] \arrow[d, "i_X"] & 1 \arrow[d, "t"] \\ PX \times X \arrow[r, "e_X"] & \Omega \end{tikzcd}$$

Let $j : R \rightarrowtail Y \times X$ be the pull-back of $t : 1 \to \Omega$ along f. By (5), there exists a unique morphism $\hat{f} : Y \to PX$ such that the left square of the following diagram is a pull-back.

$$\begin{tikzcd} R \arrow[r] \arrow[d, "j"] & \in_X \arrow[r] \arrow[d, "i_X"] & 1 \arrow[d, "t"] \\ Y \times X \arrow[r, "\hat{f} \times id_X"] & PX \times X \arrow[r, "e_X"] & \Omega \end{tikzcd}$$

Since the right square is also a pull-back, so is the outer rectangle. Hence j is a pull-back of t along $e_X(\hat{f} \times id_X)$ and we obtain $f = e_X(\hat{f} \times id_X)$ (by (3)). Suppose that $f = e_X(f' \times id_X)$. Then, there is a morphism $g : R \to \in_X$ such that $gi_X = (f' \times id_X)j$. Since the outer rectangle and right square of the following diagram are pull-backs, so is the left square.

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, "j"] & \in_X \arrow[r] \arrow[d, "i_X"] & 1 \arrow[d, "t"] \\ Y \times X \arrow[r, "f' \times id_X"] & PX \times X \arrow[r, "e_X"] & \Omega \end{tikzcd}$$

It follows from (5) that such f' is unique.

Finally, we show that the axioms (1), (3) and (4) imply (5) and (2). We denote by $i_X : \in_X \rightarrowtail PX \times X$ the pull-back of $t : 1 \rightarrowtail \Omega$ along $e_X : PX \times X \to \Omega$. Let $j : R \rightarrowtail Y \times X$ be a monomorphism. By (3), there exists a unique morphism $f : Y \times X \to \Omega$ such that j is a pull-back of t along f. Then, there exists a unique morphism $r : Y \to PX$ satisfying $f = e_X(r \times id_X)$ by (4). Thus, $(r \times id_X)j : R \to PX \times X$ and the unique morphism $R \to 1$ induces $\bar{r} : R \to \in_X$ satisfying $(r \times id_X)j = i_X\bar{r}$. Since the right square and the outer rectangle of the following diagram are pull-backs, so is the left square.

$$\begin{tikzcd} R \arrow[r, "\bar{r}"] \arrow[d, "j"] & \in_X \arrow[r] \arrow[d, "i_X"] & 1 \arrow[d, "t"] \\ Y \times X \arrow[r, "r \times id_X"] & PX \times X \arrow[r, "e_X"] & \Omega \end{tikzcd}$$

If a pair of morphisms $r' : Y \to PX$ and $\bar{r}' : R \to \in_X$ also makes the left square of the above diagram pull-back, j is a pull-back of t along $e_X(r' \times id_X)$. Hence, by (3), we have $e_X(r' \times id_X) = e_X(r \times id_X)$ and this implies $r = r'$ by (4). Therefore (5) holds.

Consider a monomorphism $\mathrm{pr}_2^{-1} : X \rightarrowtail 1 \times X$ induced by $X \to 1$ and id_X, the diagonal morphism $\Delta : X \rightarrowtail X \times X$ and a monomorphism $i_{X\times Y} : \in_{X\times Y} \rightarrowtail P(X \times Y) \times X \times Y$. Let $\ulcorner X \urcorner : 1 \to PX$, $\{\} : X \to PX$ and $r : P(X \times Y) \times X \to PY$ be the unique morphisms such that the following squares are pull-backs.

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[d, "\mathrm{pr}_2^{-1}"] & \in_X \arrow[d, "i_X"] \\
1 \times X \arrow[r, "\ulcorner X \urcorner \times id_X"] & PX \times X
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d, "\Delta"] & \in_X \arrow[d, "i_X"] \\
X \times X \arrow[r, "\{\} \times id_X"] & PX \times X
\end{tikzcd}
\quad
\begin{tikzcd}
\in_{X\times Y} \arrow[r, "\rho"] \arrow[d, "i_{X\times Y}"] & \in_Y \arrow[d, "i_Y"] \\
P(X \times Y) \times X \arrow[r, "r \times id_Y"] & PY \times Y
\end{tikzcd}$$

Note that $i_{X\times Y} : \in_{X\times Y} \to P(X \times Y) \times X \times Y$ is a pull-back of $t : 1 \to \Omega$ along both $e_Y(r \times id_Y)$ and $e_{X\times Y}$. Therefore $e_Y(r \times id_Y) = e_{X\times Y}$.

We claim that $\{\}$ is a monomorphism. Let $f, g : Z \to X$ be morphisms such that $\{\}f = \{\}g$ and $\gamma : G \rightarrowtail Z \times X$ the pull-back of i_X along $\{\}f \times id_X$. Then, there exist morphisms $\bar{f}, \bar{g} : G \to X$ such that $\Delta\bar{f} = (f \times id_X)\gamma$, $\Delta\bar{g} = (g \times id_X)\gamma$. Hence we have $\bar{f} = \mathrm{pr}_2\Delta\bar{f} = \mathrm{pr}_2(f \times id_X)\gamma = \mathrm{pr}_2\gamma = \mathrm{pr}_2(g \times id_X)\gamma = \mathrm{pr}_2\Delta\bar{g} = \bar{g}$. By (A.3.1), the left square of the following diagram is a pull-back for $h = f, g$. It is clear that the right square is also a pull-back.

$$\begin{tikzcd}
G \arrow[r, "\gamma"] \arrow[d, "\bar{f} = \bar{g}"] & Z \times X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "h \times id_X"] & Z \arrow[d, "h"] \\
X \arrow[r, "\Delta"] & X \times X \arrow[r, "\mathrm{pr}_1"] & X
\end{tikzcd}$$

Since $\mathrm{pr}_1\Delta = id_X$ and the outer rectangle is a pull-back, $\mathrm{pr}_1\gamma$ is an isomorphism. Then, we have $f\mathrm{pr}_1\gamma = \bar{f} = \bar{g} = g\mathrm{pr}_1\gamma$ and this implies $f = g$.

Let $k : Q \to P(X \times Y) \times X$ be the pull-back of $\{\}$ along r. Then, k is a monomorphism and there exists a unique $q : P(X \times Y) \to PX$ such that the following square in the middle is a pull-back. We define an object Y^X of $\mathcal{E}$ by the following pull-back square on the right.

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, "k"] & Y \arrow[d, "\{\}"] \\
P(X \times Y) \times X \arrow[r, "r"] & PY
\end{tikzcd}
\quad
\begin{tikzcd}
Q \arrow[r] \arrow[d, "k"] & \in_X \arrow[d, "i_X"] \\
P(X \times Y) \times X \arrow[r, "q \times id_X"] & PX \times X
\end{tikzcd}
\quad
\begin{tikzcd}
Y^X \arrow[r, "o_2"] \arrow[d, "\zeta"] & 1 \arrow[d, "\ulcorner X \urcorner"] \\
P(X \times Y) \arrow[r, "q"] & PX
\end{tikzcd}$$

Hence k is a pull-back of t along $e_X(q \times id_X)$ by (A.3.1). Since $\{\} : Y \to PY$ is a monomorphism, there exists a unique morphism $\sigma : PY \to \Omega$ such that the following square is a pull-back.

$$\begin{tikzcd}
Y \arrow[r, "o_3"] \arrow[d, "\{\}"] & 1 \arrow[d, "t"] \\
PY \arrow[r, "\sigma"] & \Omega
\end{tikzcd}$$

Then, k is also a pull-back of t along σr and this implies $\sigma r = e_X(q \times id_X)$. By the definition of $\ulcorner X \urcorner$, the composition of $Y^X \times X \to 1$ and t equals to $to_2\iota\mathrm{pr}_2(o_2 \times id_X) = e_X i_X \iota \mathrm{pr}_2(o_2 \times id_X) = e_X(\ulcorner X \urcorner \times id_X)(o_2 \times id_X) = e_X(q \times id_X)(\zeta \times id_X) = \sigma r(\zeta \times id_X)$. Hence there exists a unique morphism $\varepsilon : Y^X \times X \to Y$ satisfying $r(\zeta \times id_X) = \{\}\varepsilon$.

Suppose that, for morphisms $g_1, g_2 : Z \to Y^X$, $\varepsilon(g_1 \times id_X) = \varepsilon(g_2 \times id_X)$ holds. Composing $\{\} : Y \to PY$, we have $r(\zeta g_1 \times id_X) = r(\zeta g_2 \times id_X)$. Thus $e_{X\times Y}(\zeta g_1 \times id_{X\times Y}) = e_Y(r \times id_Y)(\zeta g_1 \times id_X \times id_Y) = e_Y(r(\zeta g_1 \times id_X) \times id_Y) = e_Y(r(\zeta g_2 \times id_X) \times id_Y) = e_Y(r \times id_Y)(\zeta g_2 \times id_X \times id_Y) = e_{X\times Y}(\zeta g_2 \times id_{X\times Y})$. It follows from the fact we have shown above that $\zeta g_1 = \zeta g_2$. Since ζ is a monomorphism, we have $g_1 = g_2$.

Let $f : Z \times X \to Y$ be a morphism in $\mathcal{E}$. There exists a unique morphism $h : Z \to P(X \times Y)$ such that $e_Y(\{\} \times id_Y)(f \times id_Y) = e_{X\times Y}(h \times id_{X\times Y})$. The right hand side equals to $e_Y(r \times id_Y)(h \times id_{X\times Y}) = e_Y(r(h \times id_X) \times id_Y)$. Thus we have $\{\}f = r(h \times id_X)$. Composing $\sigma : PY \to \Omega$, $to_3 f = \sigma r(h \times id_X) = e_X(qh \times id_X)$. On the other hand, $to_3 f = to_1\iota\mathrm{pr}_2 = e_X i_X \iota \mathrm{pr}_2 = e_X(\ulcorner X \urcorner o_4 \times id_X)$ ($o_4 : Z \to 1$). Hence $qh = \ulcorner X \urcorner o_4$ and there is a unique morphism $g : Z \to Y^X$ satisfying $\zeta g = h$. We show that $f = \varepsilon(g \times id_X)$. In fact, $\{\}\varepsilon(g \times id_X) = r(\zeta g \times id_X) = r(h \times id_X) = \{\}f$. Since $\{\}$ is a monomorphism, the assertion follows. □

Proposition 3.1.8 *Let $\mathcal{E}$ be a category with finite products and a subobject classifier $t : 1 \to \Omega$. If a pull-back of t along an arbitrary morphism $f : X \to \Omega$ exists, $\mathcal{E}$ has finite limits.*

Proof. It suffices to show that an equalizer of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$. Let $\delta : Y \times Y \to \Omega$ be the classifying map of the diagonal morphism $\Delta : Y \rightarrowtail Y \times Y$ and $e : E \to X$ a pull-back of t along $\delta(f, g)$. Since the right square of the following diagram is a pull-back, there is a unique morphism $h : E \to Y$ such that the left square commutes.

$$\begin{tikzcd}
E \arrow[r, "h"] \arrow[d, "e"] & Y \arrow[r] \arrow[d, "\Delta"] & 1 \arrow[d, "t"] \\
X \arrow[r, "{(f,g)}"] & Y \times Y \arrow[r, "\delta"] & \Omega
\end{tikzcd}$$

Since the outer rectangle is a pull-back, so is the left square. Thus $e : E \to X$ is an equalizer of f and g. □

Proposition 3.1.9 *Let $\mathcal{E}$ be a topos, $\alpha : \Omega \rightarrowtail \Omega$ a monomorphism and $m : U \rightarrowtail \Omega$ the subobject classified by α.*

1) The following left diagram commutes and the right one is a pull-back.

$$\begin{tikzcd}
U \arrow[r, "m"] \arrow[d, "o"] & \Omega \\
1 \arrow[r, "t"] & \Omega \arrow[u, "\alpha"]
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "id_U"] \arrow[d, "m"] & U \arrow[d, "m"] \\
\Omega \arrow[r, "\alpha^2"] & \Omega
\end{tikzcd}$$

2) $\alpha^2 = id_\Omega$.

Proof. 1) Let $\beta : V \rightarrowtail U$ be a pull-back of t along m. Each morphism of the following diagram is a monomorphism and it follows that the left square is a pull-back. The middle and the right squares are also pull-backs.

$$\begin{tikzcd}
V \arrow[r, "id_V"] \arrow[d, "\beta"] & V \arrow[r, "\beta"] \arrow[d] & U \arrow[r, "o"] \arrow[d, "m"] & 1 \arrow[d, "t"] \\
U \arrow[r, "o"] & 1 \arrow[r, "t"] & \Omega \arrow[r, "\alpha"] & \Omega
\end{tikzcd}$$

Hence β is a pull-back of t along αto and this implies $\alpha to = m$. Since α is a monomorphism, the lower right square of the following diagram is a pull-back. The other five squares are pull-backs.

$$\begin{tikzcd}
U \arrow[r, "id_U"] \arrow[d, "id_U"] & U \arrow[r, "m"] \arrow[d, "o"] & \Omega \arrow[r, "id_\Omega"] \arrow[d, "\alpha"] & \Omega \arrow[d, "\alpha"] \\
U \arrow[r, "o"] \arrow[d, "id_U"] & 1 \arrow[r, "t"] \arrow[d, "id_U"] & \Omega \arrow[r, "id_\Omega"] \arrow[d, "id_\Omega"] & \Omega \arrow[d, "\alpha"] \\
U \arrow[r, "o"] & 1 \arrow[r, "t"] & \Omega \arrow[r, "\alpha"] & \Omega
\end{tikzcd}$$

Hence the outer big rectangle is a pull-back and the assertion follows.

2) By the above result and the definition of m, the outer rectangle of the following diagram is a pull-back.

$$\begin{tikzcd}
U \arrow[r, "id_U"] \arrow[d, "m"] & U \arrow[r, "o"] \arrow[d, "m"] & 1 \arrow[d, "t"] \\
\Omega \arrow[r, "\alpha^2"] & \Omega \arrow[r, "\alpha"] & \Omega
\end{tikzcd}$$

Then, m is a pull-back of t along α^3. Therefore we have $\alpha^3 = \alpha$. Since α is a monomorphism, $\alpha^2 = id_\Omega$. □

3.2 Equivalence relations and partial maps

Proposition 3.2.1 *In a topos, every monomorphism is an equalizer.*

Proof. Let $\nu : \Omega \to \Omega$ be the composition $\Omega \to 1 \xrightarrow{t} \Omega$. Then, $1 \xrightarrow{t} \Omega \overset{id_\Omega}{\underset{\nu}{\rightrightarrows}} \Omega$ is an equalizer. If $\sigma : Y \rightarrowtail X$ is a monomorphism, it is a pull-back of an equalizer t along the classifying map $\phi_\sigma : X \to \Omega$. Hence σ is an equalizer of ϕ_σ and $\nu\phi_\sigma$ by (A.3.2). $\square$

The next result follows from (A.8.11).

Corollary 3.2.2 *A topos is balanced.*

Corollary 3.2.3 *In a topos, equivalence relations are effective.*

Proof. Let $R \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} X$ be an equivalence relation. We denote by $\phi : X \times X \to \Omega$ the classifying map of $(\alpha_1, \alpha_2) : R \rightarrowtail X \times X$ and $\bar{\phi} : X \to \Omega^X$ its exponential transpose. We show that $\bar{\phi}\alpha_1 = \bar{\phi}\alpha_2$ holds. Taking the exponential transpose, this equality is equivalent to $\phi(\alpha_1 \times id_X) = \phi(\alpha_2 \times id_X)$. Let $\sigma_i : S_i \to R \times X$ be a pull-back of (α_1, α_2) along $\alpha_i \times id_X$ respectively. Then, the upper square of the following commutative diagram is a cartesian square of sets for each object Z of $\mathcal{E}$.

$$\begin{tikzcd}
\mathcal{E}(Z, S_i) \arrow[rr] \arrow[d, "\sigma_{i*}"] & & \mathcal{E}(Z, R) \arrow[d, "(\alpha_1,\alpha_2)_*"] \\
\mathcal{E}(Z, R \times X) \arrow[rr, "(\alpha_i \times id_X)_*"] \arrow[d, "\cong"] & & \mathcal{E}(Z, X \times X) \arrow[d, "\cong"] \\
\mathcal{E}(Z, R) \times \mathcal{E}(Z, X) \arrow[rr, "\alpha_{i*} \times id_{\mathcal{E}(Z,X)}"] \arrow[d, "(\alpha_{1*},\alpha_{2*}) \times id_{\mathcal{E}(Z,X)}"] & & \mathcal{E}(Z, X) \times \mathcal{E}(Z, X) \\
\mathcal{E}(Z, X) \times \mathcal{E}(Z, X) \times \mathcal{E}(Z, X) \arrow[urr, "\mathrm{pr}_i \times id_{\mathcal{E}(Z,X)}"'] & &
\end{tikzcd}$$

Since the image of $(\alpha_{1*}, \alpha_{2*}) : \mathcal{E}(Z, R) \to \mathcal{E}(Z, X) \times \mathcal{E}(Z, X)$ is an equivalence relation on $\mathcal{E}(Z, X)$, the image of the composition of the left vertical arrows of the above diagram consists of elements (x, y, z) such that x, y and z are equivalent each other. Hence the images of $h_{\sigma_1} : h_{S_1} \to h_{R\times X}$ and $h_{\sigma_2} : h_{S_2} \to h_{R\times X}$ are the same. Therefore σ_1 and σ_2 are equivalent monomorphisms by (A.3.4). Since $S_i \overset{\sigma_i}{\rightarrowtail} R \times X$ is classified by $\phi(\alpha_i \times id_X)$, we have $\phi(\alpha_1 \times id_X) = \phi(\alpha_2 \times id_X)$.

Suppose that $\bar{\phi}f = \bar{\phi}g$ holds for $f, g : Y \to X$. Taking the exponential transpose, we have $\phi(f \times id_X) = \phi(g \times id_X) : Y \times X \to \Omega$, hence $\phi(f, g) = \phi(g, g) : Y \to \Omega$. Since $(g, g) = \Delta g : Y \to X \times X$ factors through a monomorphism $(\alpha_1, \alpha_2) : R \rightarrowtail X \times X$, the both squares of the following diagram are pull-backs.

$$\begin{tikzcd}
Y \arrow[r] \arrow[dr, "(g,g)"'] & R \arrow[r] \arrow[d, "t"] & 1 \arrow[d, "(\alpha_1,\alpha_2)"] \\
& X \times X \arrow[r, "\phi"] & \Omega
\end{tikzcd}$$

Thus $\phi(g, g) : Y \to \Omega$ is the classifying map of $id_Y : Y \to Y$. If $h : Y' \to Y$ is a pull-back of (α_1, α_2) along (f, g), the both squares of the following diagram are pull-backs.

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d, "h"] & R \arrow[r] \arrow[d, "(\alpha_1,\alpha_2)"] & 1 \arrow[d, "t"] \\
Y \arrow[r, "(f,g)"] & X \times X \arrow[r, "\phi"] & \Omega
\end{tikzcd}$$

Since Y' is classified by $\phi(f, g) = \phi(g, g)$, which classifies the maximal subobject Y, h is an isomorphism. Hence (f, g) factors through (α_1, α_2). $\square$

Definition 3.2.4 *We define the singleton map* $\{\} : X \to \Omega^X$ *to be the exponential transpose of the classifying map* $\delta : X \times X \to \Omega$ *of the diagonal map* $\Delta : X \rightarrowtail X \times X$*. The above proof shows that the kernel pair of* $\{\}$ *is* (id_X, id_X)*, hence* $\{\}$ *is a monomorphism.*

Definition 3.2.5 *We say that diagrams* $X \overset{d}{\leftarrowtail} X' \xrightarrow{f} Y$ *and* $Z \overset{c}{\leftarrowtail} Z' \xrightarrow{g} W$ *in a category* $\mathcal{E}$ *are equivalent if* $X = Z$, $Y = W$ *and there exists an isomorphism* $\varphi : X' \to Z'$ *satisfying* $c\varphi = d$ *and* $g\varphi = f$*. An equivalence class of diagrams in* $\mathcal{E}$ *of the form* $X \overset{d}{\leftarrowtail} X' \xrightarrow{f} Y$ *is called a partial map from* X *to* Y *and denoted by* $X \overset{f}{\rightharpoonup} Y$*.*

We say that partial maps with codomain Y are representable if there exists a monomorphism $Y \overset{\eta}{\rightarrowtail} \widetilde{Y}$ such that, for any partial map $X \overset{f}{\rightharpoonup} Y$, there exists a unique $\tilde{f} : X \to \widetilde{Y}$ making

$$\begin{tikzcd} X' \arrow[r, tail, "d"] \arrow[d, "f"] & X \arrow[d, "\tilde{f}"] \\ Y \arrow[r, tail, "\eta"] & \widetilde{Y} \end{tikzcd}$$

a pull-back diagram.

Theorem 3.2.6 *In a topos, all partial maps are representable.*

Proof. Let $\phi : \Omega^Y \times Y \to \Omega$ the classifying map of the graph of the singleton map $(\{\}, id_Y) : Y \rightarrowtail \Omega^Y \times Y$. We denote by $\bar{\phi} : \Omega^Y \to \Omega^Y$ the exponential transpose of ϕ. Define $\widetilde{Y} \overset{e}{\rightarrowtail} \Omega^Y$ to be the equalizer of $\bar{\phi}$ and id_Y.

Since $\{\}$ is a monomorphism, the following square is a pull-back by (A.3.5).

$$\begin{tikzcd} Y \arrow[r, "id_Y"] \arrow[d, "\Delta"] & Y \arrow[d, "(\{\}, id_Y)"] \\ Y \times Y \arrow[r, "\{\} \times id_Y"] & \Omega^Y \times Y \end{tikzcd}$$

This implies that $\phi(\{\} \times id_Y)$ classifies the diagonal subobject of $Y \times Y$, hence its exponential transpose $\bar{\phi}\{\}$ coincides with the singleton map. Thus the singleton map factors through $e : \widetilde{Y} \to \Omega^Y$ and gives a monomorphism $\eta : Y \rightarrowtail \widetilde{Y}$.

For a partial map $X \overset{f}{\rightharpoonup} Y$, define $\bar{\psi} : X \to \Omega^Y$ to be the exponential transpose of the classifying map $\psi : X \times Y \to \Omega$ of $(d, f) : X' \rightarrowtail X \times Y$. We show that $\bar{\psi}$ factors through $e : \widetilde{Y} \to \Omega^Y$, that is, $\bar{\phi}\bar{\psi} = \bar{\phi}$ or its exponential transpose $\phi(\bar{\psi} \times id_Y) = \psi$. Hence it suffices to show that the following diagram on the left is a cartesian square.

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "(d,f)"] & Y \arrow[d, "(\{\}, id_Y)"] \\ X \times Y \arrow[r, "\bar{\psi} \times id_Y"] & \Omega^Y \times Y \end{tikzcd} \qquad \begin{tikzcd} X \times Y \arrow[r, "\bar{\psi} \times id_Y"] \arrow[d, "\mathrm{pr}_1"] & \Omega^Y \times Y \arrow[d, "\mathrm{pr}_1"] \\ X \arrow[r, "\bar{\psi}"] & \Omega^Y \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "d"] & Y \arrow[d, "\{\}"] \\ X \arrow[r, "\bar{\psi}"] & \Omega^Y \end{tikzcd}$$

Since the above diagram in the middle is a pull-back, it suffices to show that the above diagram on the right is a cartesian square. Suppose that morphisms $a : U \to X$ and $b : U \to Y$ are given so that $\bar{\psi}a = \{\}b$, or equivalently $\psi(a \times id_Y) = \delta(b \times id_Y)$ hold. Then, $\psi(a, b) = \delta(b, b) = \delta\Delta b = (U \to 1 \xrightarrow{t} \Omega)$ and this implies that $(a, b) : U \to X \times Y$ factors through $(d, f) : X' \to X \times Y$ since

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "(d,f)"] & 1 \arrow[d, "t"] \\ X \times Y \arrow[r, "\psi"] & \Omega \end{tikzcd}$$

is a pull-back. Thus we have shown that $\bar{\psi}$ factors through e and gives a morphism $\tilde{f} : X \to \widetilde{Y}$. By (A.3.6),

$$\begin{tikzcd} X' \arrow[r, tail, "d"] \arrow[d, "f"] & X \arrow[d, "\tilde{f}"] \\ Y \arrow[r, tail, "\eta"] & \widetilde{Y} \end{tikzcd}$$

is a pull-back diagram.

Finally, we show the uniqueness of $\tilde{f}$. Suppose that $\tilde{f}_1$ and $\tilde{f}_2$ satisfy the condition. Since e is a monomorphism, it suffices to show that $e\tilde{f}_1 = e\tilde{f}_2$. Put $e\tilde{f}_i = g_i$. Since the following diagram on the left is a pull-back by assumption and (A.3.6), so is the right diagram by (A.3.1).

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "d"] & Y \arrow[d, "\{\}"] \\ X \arrow[r, "g_i"] & \Omega^Y \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "(d,f)"] & Y \arrow[d, "(\{\}, id_Y)"] \\ X \times Y \arrow[r, "g_i \times id_Y"] & \Omega^Y \times Y \end{tikzcd}$$

Hence $\phi(g_i \times id_Y)$ classifies (d, f) and we have $\phi(g_1 \times id_Y) = \phi(g_2 \times id_Y)$. Taking the exponential transpose, $\bar{\phi}g_1 = \bar{\phi}g_2$. On the other hand, since $\bar{\phi}e = e$, $\bar{\phi}g_i = \bar{\phi}e\tilde{f}_i = e\tilde{f}_i = g_i$. ∎

By the uniqueness of $\tilde{f}$, we have a functor $T : \mathcal{E} \to \mathcal{E}$ defined by $T(Y) = \widetilde{Y}$, $T(f : X \to Y) =$ (*the classifying map of* $\widetilde{X} \overset{\eta}{\leftarrowtail} X \xrightarrow{f} Y$) and a natural transformation $\eta : 1_{\mathcal{E}} \to T$. In particular, the exponential transpose of the classifying map of $(\eta, id_Y) : Y \to \widetilde{Y} \times Y$ coincides with e.

Corollary 3.2.7 *An object of the form $\widetilde{Y}$ is injective.*

Proof. Let $k : X' \to X$ be a monomorphism and $f : X' \to \widetilde{Y}$ a morphism. Form a pull-back

$$\begin{tikzcd} X'' \arrow[r, tail, "d"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, tail, "\eta"] & \widetilde{Y} \end{tikzcd},$$

then f is the classifying map of a partial map $X' \overset{d}{\leftarrowtail} X'' \xrightarrow{g} Y$. Let $\tilde{g} : X \to \widetilde{Y}$ be the classifying map of a partial map $X \overset{kd}{\leftarrowtail} X'' \xrightarrow{g} Y$. Since k is a monomorphism and the following diagram on the left is a pull-back, so is the diagram on the right.

$$\begin{tikzcd} X'' \arrow[r, "kd"] \arrow[d, "g"] & X \arrow[d, "\tilde{g}"] \\ Y \arrow[r, "\eta"] & \widetilde{Y} \end{tikzcd} \qquad \begin{tikzcd} X'' \arrow[r, "d"] \arrow[d, "g"] & X \arrow[d, "\tilde{g}k"] \\ Y \arrow[r, "\eta"] & \widetilde{Y} \end{tikzcd}$$

Hence $k\tilde{g}$ is also the classifying map of a partial map $X' \overset{d}{\leftarrowtail} X'' \xrightarrow{g} Y$ and we have $k\tilde{g} = f$. ∎

Corollary 3.2.8 *If the following square is a push-out and f is a monomorphism, then g is also a monomorphism and the square is also a pull-back.*

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"] & Y \arrow[d, "g"] \\ Z \arrow[r, "q"] & T \end{tikzcd}$$

Proof. Let $h : Z \to \widetilde{Y}$ be the classifying map of the partial map $Z \overset{f}{\leftarrowtail} X \xrightarrow{p} Y$. Then, $\eta : Y \to \widetilde{Y}$ and h induces $k : T \to \widetilde{Y}$ satisfying $kg = \eta$ and $kq = h$. Since η is a monomorphism, so is g, and since

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"] & Y \arrow[d, "\eta"] \\ Z \arrow[r, "h"] & \widetilde{Y} \end{tikzcd}$$

is a pull-back, so is the given square by (A.3.6). ∎

Let $\mathcal{E}$ be a category with pull-backs. For partial maps $X \overset{f}{\rightharpoonup} Y$ and $Y \overset{g}{\rightharpoonup} Z$ represented by diagrams $X \overset{d}{\leftarrowtail} X' \xrightarrow{f} Y$ and $Y \overset{c}{\leftarrowtail} Y' \xrightarrow{f} Z$, we define a composition $X \overset{gf}{\rightharpoonup} Z$ as follows. Form a pull-back

$$\begin{tikzcd} E \arrow[r, "k"] \arrow[d, "e"] & Y' \arrow[d, "c"] \\ X' \arrow[r, "f"] & Y \end{tikzcd}$$

and the composition is the class of the diagram $X \overset{de}{\leftarrowtail} E \xrightarrow{gk} Y$. It is easy to verify that this definition does not depend on the choice of the representatives and that the composition law of this operation holds. $\mathcal{E}^p$ denotes a category with $\operatorname{Ob}\mathcal{E}^p = \operatorname{Ob}\mathcal{E}$ whose morphisms are partial maps in $\mathcal{E}$. We call $\mathcal{E}^p$ the category of partial maps in $\mathcal{E}$. Note that there is a faithful functor $\Phi : \mathcal{E} \to \mathcal{E}^p$ given by $X \mapsto X$ and $(f : X \to Y) \mapsto (X \overset{id_X}{\leftarrowtail} X \xrightarrow{f} Y)$.

Lemma 3.2.9 *Let $\mathcal{E}$ be a topos and $(Y \xrightarrow{f} X)$, $(Z \xrightarrow{g} X)$ partial maps with classifying maps $\theta : Y \to \widetilde{X}$, $\rho : Z \to \widetilde{X}$, respectively. Then, for a morphism $\psi : Y \to Z$ in $\mathcal{E}$, $\rho\psi = \theta$ holds if and only if $g\Phi(\psi) = f$ holds in $\mathcal{E}^p$.*

Proof. Let $Y \xleftarrow{d} Y' \xrightarrow{f} X$ and $Z \xleftarrow{c} Z' \xrightarrow{g} X$ be the diagrams which represents given partial maps.

Suppose that $\rho\psi = \theta$ holds. Since $\eta f = \theta d = \rho\psi d$ and

$$\begin{array}{ccc} Z' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \eta} \\ Z & \xrightarrow{\rho} & \widetilde{X} \end{array}$$

is a pull-back, there is a unique morphism $\psi' : Y' \to Z'$ such that $g\psi' = f$, $c\psi' = \psi d$. By applying (A.3.1) to

$$\begin{array}{ccccc} Y' & \xrightarrow{\psi'} & Z' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \eta} \\ Y & \xrightarrow{\psi} & Z & \xrightarrow{\rho} & \widetilde{X} \end{array},$$

we see that

$$\begin{array}{ccc} Y' & \xrightarrow{\psi'} & Z' \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle c} \\ Y & \xrightarrow{\psi} & Z \end{array}$$

is a cartesian square. Hence $Y \xleftarrow{d} Y' \xrightarrow{g\psi'=f} X$ represents the composition $g\Phi(\psi)$.

Conversely, suppose that $g\Phi(\psi) = f$ holds in $\mathcal{E}^p$. Form a pull-back

$$\begin{array}{ccc} W & \xrightarrow{\phi} & Z' \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle c} \\ Y & \xrightarrow{\psi} & Z \end{array}.$$

By the assumption, there is an isomorphism $h : Y' \to W$ such that $kh = d$, $g\phi h = f$. Put $\psi' = \phi h$. The diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\psi'} & Z' \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle c} \\ Y & \xrightarrow{\psi} & Z \end{array}$$

obtained form the above diagram by replacing W by Y' is a cartesian square. Therefore $\rho\psi$ classifies the partial map $Y \xleftarrow{d} Y' \xrightarrow{g\psi'=f} X$ and we have $\rho\psi = \theta$. □

For an object X of $\mathcal{E}$, let $(\mathcal{E}^p/X)_*$ be a subcategory of $\mathcal{E}^p/X$ with $\mathrm{Ob}(\mathcal{E}^p/X)_* = \mathrm{Ob}\,\mathcal{E}^p/X$ and

$$(\mathcal{E}^p/X)_*((Y \xrightarrow{f} X), (Z \xrightarrow{g} X)) = \{\psi \in \mathcal{E}(Y, Z) |\, g\Phi(\psi) = f \textit{ in } \mathcal{E}^p\}.$$

Proposition 3.2.10 *Let $\mathcal{E}$ be a topos and X an object of $\mathcal{E}$. Then $\mathcal{E}/\widetilde{X}$ is isomorphic to $(\mathcal{E}^p/X)_*$.*

Proof. Define a functor $F : \mathcal{E}/\widetilde{X} \to (\mathcal{E}^p/X)_*$ as follows. For $Y \xrightarrow{\theta} \widetilde{X}$, form a pull-back

$$\begin{array}{ccc} Y' & \xrightarrow{f} & X \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \eta} \\ Y & \xrightarrow{\theta} & \tilde{X} \end{array}$$

and $F(Y \xrightarrow{\theta} \widetilde{X})$ is the partial map $Y \xrightarrow{f} X$ represented by the diagram $Y \xleftarrow{d} Y' \xrightarrow{f} X$. If $\psi : (Y \xrightarrow{\theta} \widetilde{X}) \to (Z \xrightarrow{\rho} \widetilde{X})$ is a morphism in $\mathcal{E}/\widetilde{X}$, $\Phi(\psi)$ gives a morphism from $Y \xrightarrow{f} X$ to $Z \xrightarrow{g} X$ by (3.2.9). We set $F(\psi) = \psi \in (\mathcal{E}^p/X)_*((Y \xrightarrow{f} X), (Z \xrightarrow{g} X))$. It follows from (3.2.6) and (3.2.9) that F is an isomorphism. □

Proposition 3.2.11 *Let $Y \overset{f}{\rightharpoonup} X$ and $\widetilde{Z} \overset{g}{\rightharpoonup} X$ be partial maps in $\mathcal{E}$ represented by diagrams $Y \overset{d}{\leftarrowtail} Y' \xrightarrow{f} X$ and $\widetilde{Z} \overset{\eta}{\leftarrowtail} Z \xrightarrow{g} X$. Then, there is a natural bijection*

$$\Psi : (\mathcal{E}^p/X)_*((Y \overset{f}{\rightharpoonup} X), (\widetilde{Z} \overset{g}{\rightharpoonup} X)) \to \mathcal{E}/X((Y' \xrightarrow{f} X), (Z \xrightarrow{g} X)).$$

Proof. For $\psi \in (\mathcal{E}^p/X)_*((Y \overset{f}{\rightharpoonup} X), (\widetilde{Z} \overset{g}{\rightharpoonup} X))$, by the same argument as in the proof of (3.2.9), there exists a unique morphism $\psi' : Y' \to Z$ such that

$$\begin{tikzcd} Y' \arrow[r, "\psi'"] \arrow[d, "d"] & Z \arrow[d, "\eta"] \\ Y \arrow[r, "\psi"] & \widetilde{Z} \end{tikzcd}$$

is a pull-back and that $g\psi' = f$. We set $\Psi(\psi) = \psi'$.

For $\phi \in \mathcal{E}/X((Y' \xrightarrow{f} X), (Z \xrightarrow{g} X))$, let $\tilde{\phi} : Y \to \widetilde{Z}$ be the classifying map of a partial map $Y \overset{d}{\leftarrowtail} Y' \xrightarrow{\phi} Z$. Then, the inverse of Ψ is given by $\phi \mapsto \tilde{\phi}$.

Let $W \overset{h}{\rightharpoonup} X$ and $\widetilde{V} \overset{k}{\rightharpoonup} X$ be partial maps represented by diagrams $W \overset{c}{\leftarrowtail} W' \xrightarrow{h} X$ and $\widetilde{V} \overset{\eta}{\leftarrowtail} V \xrightarrow{k} X$. It is easy to verify the commutativity of the following diagrams for morphisms $\alpha : (W \overset{h}{\rightharpoonup} X) \to (Y \overset{f}{\rightharpoonup} X)$ in $(\mathcal{E}^p)_*$ and $\beta : (Z \xrightarrow{g} X) \to (V \xrightarrow{k} X)$ in $\mathcal{E}/X$.

$$\begin{tikzcd} (\mathcal{E}^p/X)_*((Y \overset{f}{\rightharpoonup} X), (\widetilde{Z} \overset{g}{\rightharpoonup} X)) \arrow[r, "\Psi"] \arrow[d, "\alpha^*"] & \mathcal{E}/X((Y' \xrightarrow{f} X), (Z \xrightarrow{g} X)) \arrow[d, "(\alpha')^*"] \\ (\mathcal{E}^p/X)_*((W \overset{h}{\rightharpoonup} X), (\widetilde{Z} \overset{g}{\rightharpoonup} X)) \arrow[r, "\Psi"] & \mathcal{E}/X((W' \xrightarrow{h} X), (Z \xrightarrow{g} X)) \end{tikzcd}$$

$$\begin{tikzcd} (\mathcal{E}^p/X)_*((Y \overset{f}{\rightharpoonup} X), (\widetilde{Z} \overset{g}{\rightharpoonup} X)) \arrow[r, "\Psi"] \arrow[d, "\tilde{\beta}_*"] & \mathcal{E}/X((Y' \xrightarrow{f} X), (Z \xrightarrow{g} X)) \arrow[d, "\beta_*"] \\ (\mathcal{E}^p/X)_*((Y \overset{f}{\rightharpoonup} X), (\widetilde{V} \overset{k}{\rightharpoonup} X)) \arrow[r, "\Psi"] & \mathcal{E}/X((Y' \xrightarrow{f} X), (V \xrightarrow{k} X)) \end{tikzcd}$$

□

Proposition 3.2.12 *1) Let $\mathcal{E}$ be an arbitrary category. If $X \in \operatorname{Ob}\mathcal{E}$ is injective and Y is a retract of X, Y is also injective.*

2) Let $\mathcal{E}$ be a cartesian closed category. If $X \in \operatorname{Ob}\mathcal{E}$ is injective, X^Y is injective for any $Y \in \operatorname{Ob}\mathcal{E}$.

3) Let $\mathcal{E}$ be a topos. An object Y of $\mathcal{E}$ is injective if and only if Y is a retract of Ω^X for some X.

Proof. 1) Let $i : Y \to X$ be a morphism with a retraction $r : X \to Y$. For a monomorphism $\sigma : Z \rightarrowtail W$ and a morphism $f : Z \to Y$, there is a morphism $g : W \to X$ such that $g\sigma = if$. Since $ri = id_Y$, we have $rg\sigma = f$.

2) For a monomorphism $\sigma : Z \rightarrowtail W$ and a morphism $f : Z \to X^Y$, let $\bar{f} : Z \times Y \to X$ be the transpose of f. Since $\sigma \times id_Y : Z \times Y \to W \times Y$ is a monomorphism and X is injective, there is a morphism $\bar{g} : W \times Y \to X$ such that $\bar{g}(\sigma \times id_Y) = \bar{f}$. Then, the transpose $g : W \to X^Y$ of $\bar{g}$ satisfies $g\sigma = f$.

3) Suppose that Y is injective. Since the singleton map $\{\} : Y \to \Omega^Y$ is a monomorphism (3.2.4), there is a retraction $r : \Omega^Y \to Y$. The converse follows from 1), 2) and the fact that Ω is injective (3.2.7). □

3.3 The opposite category of a topos

Assume that $\mathcal{E}$ is a topos. We set $P = P_\Omega : \mathcal{E}^{op} \to \mathcal{E}$ and call this the contravariant power set functor. Let us denote by $\in_X \rightarrowtail \Omega^X \times X$ the subobject classified by the evaluation map $ev : \Omega^X \times X \to \Omega$.

Lemma 3.3.1 *1) If $f : X \to Y$ is a morphism in $\mathcal{E}$, then the exponential transpose of the composite $Y \xrightarrow{\{\}} \Omega^Y \xrightarrow{Pf} \Omega^X$ is the classifying map of the graph $(f, id_X) : X \to Y \times X$ of f.*

2) $P : \mathcal{E}^{op} \to \mathcal{E}$ is faithful, hence it reflects monomorphisms and epimorphisms. Moreover, it reflects isomorphisms.

Proof. 1) The transpose of the $Pf\{\} : Y \to \Omega^X$ is $ev(id_{\Omega^X} \times f)(\{\} \times id_X) = ev(\{\} \times id_Y)(id_Y \times f)$. Since $ev(\{\} \times id_Y) = \delta : Y \times Y \to \Omega$ classifies the diagonal subobject $\Delta : Y \rightarrowtail Y \times Y$, there exists a unique morphism $Y \to \in_Y$ such that the square in the middle of the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "{(f,id_X)}"] & Y \arrow[r] \arrow[d, "\Delta"] & \in_Y \arrow[r] \arrow[d] & 1 \arrow[d, "t"] \\
Y \times X \arrow[r, "1\times f"] & Y \times Y \arrow[r, "\{\}\times 1"] & \Omega^Y \times Y \arrow[r, "ev"] & \Omega
\end{tikzcd}$$

By (A.3.1), the middle square is a pull-back, and by (A.3.5), the left square is also a pull-back. Hence the outer rectangle is a pull-back.

2) Let $f, g : X \to Y$ be morphisms such that $Pf = Pg$. Then, $\{\}Pf = \{\}Pg$ and it follows from the above result that there is an isomorphism $h : X \to X$ satisfying $(g, id_X)h = (f, id_Y)$. Hence we have $h = id_X$ and $g = f$. It follows from (3.2.2) that P reflects isomorphisms. □

For a monomorphism $f : X \to Y$, define a morphism $\exists f : \Omega^X \to \Omega^Y$ to be the transpose of the classifying map of the monomorphism $\in_X \rightarrowtail \Omega^X \times X \xrightarrow{1\times f} \Omega^X \times Y$.

If $f : X \to Y$ is a monomorphism in $\mathcal{E}$, define a map $f_\sharp : \mathcal{E}(X, \Omega) \to \mathcal{E}(Y, \Omega)$ as follows. For a morphism $\phi : X \to \Omega$, let $\sigma : V \rightarrowtail X$ be the subobject classified by ϕ. We set $f_\sharp(\phi) = ($*the classifying map of* $V \overset{f\sigma}{\rightarrowtail} Y)$.

Lemma 3.3.2 *For a monomorphism $f : X \to Y$ and an object U, the following square is commutative, where the vertical arrows are adjoint isomorphisms.*

$$\begin{tikzcd}
\mathcal{E}(U \times X, \Omega) \arrow[r, "(1\times f)_\sharp"] \arrow[d, "\cong"] & \mathcal{E}(U \times Y, \Omega) \arrow[d, "\cong"] \\
\mathcal{E}(U, \Omega^X) \arrow[r, "(\exists f)_*"] & \mathcal{E}(U, \Omega^Y)
\end{tikzcd}$$

Proof. Let $\sigma : V \rightarrowtail U \times X$ be the subobject classified by $\phi : U \times X \to \Omega$. Then, since the following diagram commutes

$$\begin{tikzcd}
U \times X \arrow[r, "\iota\times id_X"] & (U \times X)^X \times X \arrow[r, "\phi^X\times 1"] \arrow[d, "ev"] & \Omega^X \times X \arrow[d, "ev"] \\
V \arrow[u, "\sigma"] \arrow[r, "\sigma"] & U \times X \arrow[r, "\phi"] & \Omega
\end{tikzcd}$$

and $\phi\sigma$ factors through $t : 1 \to \Omega$, $(\phi^X \times 1)(\iota \times 1)\sigma$ factors through $\in_X \rightarrowtail \Omega^X \times X$. Hence we have a commutative diagram

$$\begin{tikzcd}
V \arrow[rr] \arrow[d, "\sigma"] & & \in_X \arrow[r] \arrow[d] & 1 \arrow[d, "t"] \\
U \times X \arrow[r, "\iota\times 1"] & (U \times X)^X \times X \arrow[r, "\phi^X\times 1"] & \Omega^X \times X \arrow[r, "ev"] & \Omega
\end{tikzcd}$$

and it follows from (A.3.1) that the left rectangle of the above diagram is a pull-back. Again by (A.3.1), the outer rectangle of the following diagram is a pull-back.

$$\begin{tikzcd}
V \arrow[rr] \arrow[d, "\sigma"] & & \in_X \arrow[r] \arrow[d] & 1 \arrow[dd, "t"] \\
U \times X \arrow[r, "\iota\times 1"] \arrow[d, "id_U\times f"] & (U \times X)^X \times X \arrow[r, "\phi^X\times 1"] \arrow[d, "id_{(U\times X)^X}\times f"] & \Omega^X \times X \arrow[d, "id_{\Omega^X}\times f"] & \\
U \times Y \arrow[r, "\iota\times id_Y"] & (U \times X)^X \times Y \arrow[r, "\phi^X\times id_Y"] & \Omega^X \times Y \arrow[r, "\overline{\exists f}"] & \Omega
\end{tikzcd},$$

where $\overline{\exists f} : \Omega^X \times Y \to \Omega$ is the transpose of $\exists f$. Therefore the exponential transpose of $\exists f \phi^X \iota$ classifies $(1 \times f)\sigma : V \rightarrowtail U \times Y$ and this shows the assertion. □

Lemma 3.3.3 *If the following square on the left is a pull-back and g, h are monomorphisms, then the right one commutes.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "h"] \\ Z \arrow[r, "k"] & W \end{tikzcd} \qquad \begin{tikzcd} \Omega^Y \arrow[r, "Pf"] \arrow[d, "\exists h"] & \Omega^X \arrow[d, "\exists g"] \\ \Omega^W \arrow[r, "Pk"] & \Omega^Z \end{tikzcd}$$

Proof. By (A.16.5) and (3.3.2), it suffices to show that the commutativity of the following square for any object U.

$$\begin{tikzcd} \mathcal{E}(U \times Y, \Omega) \arrow[r, "(1\times f)^*"] \arrow[d, "(1\times h)_\sharp"] & \mathcal{E}(U \times X, \Omega) \arrow[d, "(1\times g)_\sharp"] \\ \mathcal{E}(U \times W, \Omega) \arrow[r, "(1\times k)^*"] & \mathcal{E}(U \times Z, \Omega) \end{tikzcd}$$

For a morphism $\phi : U \times Y \to \Omega$. Let $\sigma : V \rightarrowtail U \times Y$ be the subobject classified by ϕ. Then $(1\times h)_\sharp(\phi) : U \times W \to \Omega$ classifies $(1 \times h)\sigma : V \rightarrowtail U \times W$. Let $\tau : W \rightarrowtail U \times W$ be the pull-back of σ along $1 \times f : U \times X \to U \times Y$, then the outer square of the following commutative diagram is a pull-back.

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\tau"] & V \arrow[r] \arrow[d, "\sigma"] & 1 \arrow[dd, "t"] \\ U \times X \arrow[r, "id_U \times f"] \arrow[d, "id_U \times g"] & U \times Y \arrow[d, "1\times h"] & \\ U \times Z \arrow[r, "id_U \times k"] & U \times W \arrow[r, "(1\times h)_\sharp(\phi)"] & \Omega \end{tikzcd}$$

This shows that $(1 \times h)_\sharp(\phi)(1 \times k) = (1 \times k)^*(1 \times h)_\sharp(\phi)$ classifies $(1 \times g)\tau : W \rightarrowtail U \times Z$ Since the classifying map of $\tau : W \rightarrowtail U \times W$ is $\phi(1 \times f) = (1 \times f)^*(\phi)$, $(1 \times g)_\sharp(1 \times f)^*(\phi)$, also classifies $(1 \times g)\tau : W \rightarrowtail U \times Z$. Thus we have shown $(1 \times k)^*(1 \times h)_\sharp(\phi) = (1 \times g)_\sharp(1 \times f)^*(\phi)$. □

Lemma 3.3.4 *If $f : X \to Y$ is a monomorphism, $Pf\exists f = id_{\Omega^X}$.*

Proof. Apply the above result to the pull-back

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "id"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}.$$

□

Theorem 3.3.5 *The functor $P_* : \mathcal{E} \to \mathcal{E}^{op}$ is monadic. That is, $\mathcal{E}^{op}$ is equivalent to the category of algebras for the monad in $\mathcal{E}$ defined by the adjunction* $\mathcal{E} \underset{P}{\overset{P_*}{\rightleftarrows}} \mathcal{E}^{op}$.

Proof. By (A.13.4), it is sufficient to prove that $\mathcal{E}^{op}$ has coequalizers of reflexive pairs, P preserves them and P reflects isomorphisms. Since $\mathcal{E}$ has finite limits, the first requirement is obviously satisfied. The third requirement is satisfied by (3.3.1).

Let $X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z$ be a coreflexive equalizer in $\mathcal{E}$, namely a reflexive coequalizer in $\mathcal{E}^{op}$. Since there is a morphism $d : Z \to Y$ such that $dg = dh = id_Y$, g and h are monomorphisms. Applying (3.3.3) to a pull-back diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, "g"] \\ Y \arrow[r, "h"] & Z \end{tikzcd},$$

we have $\exists f Pf = Ph\exists g$. Since f and g are monomorphisms, we have $Pf\exists f = id_{PX}$ and $Pg\exists g = id_{PY}$ by (3.3.4). Hence $PZ \underset{Ph}{\overset{Pg}{\rightrightarrows}} PY \xrightarrow{Pf} PX$ is a split fork. □

Corollary 3.3.6 *Let $D : \mathcal{D} \to \mathcal{E}$ be a functor and D^{op} the same data regarded as a functor $\mathcal{D}^{op} \to \mathcal{E}^{op}$. If a functor $PD^{op} : \mathcal{D}^{op} \to \mathcal{E}$ has a limit, D has a colimit. In particular, since $\mathcal{E}$ has finite limits, $\mathcal{E}$ has finite colimits.*

Proof. Let $\boldsymbol{G}$ be the monad in $\mathcal{E}$ defined by the adjunction $\mathcal{E} \underset{P}{\overset{P_*}{\rightleftarrows}} \mathcal{E}^{op}$. We denote by $U : \mathcal{E}^{\boldsymbol{G}} \to \mathcal{E}$ and $K : \mathcal{E}^{op} \to \mathcal{E}^{\boldsymbol{G}}$ the forgetful functor and the comparison functor, respectively. Since U creates limits and $P = UK$, $\varprojlim KD^{op}$ exists in $\mathcal{E}^{\boldsymbol{G}}$. Hence $\varprojlim D^{op}$ exist in $\mathcal{E}^{op}$ and it is nothing but the colimit of D in $\mathcal{E}$. □

We remark that the unique morphism $\nu : PP_*(1) \to 1$ defines a structure of $\boldsymbol{G}$-algebra on the terminal object 1 and that $\langle 1, \nu \rangle$ is a terminal object of $\mathcal{E}^{\boldsymbol{G}}$. It follows from (A.13.1) and (3.3.5) that the initial object of $\mathcal{E}$ is the equalizer of $P(1) \underset{\eta_{P(1)}}{\overset{P(\nu)}{\rightrightarrows}} PPP(1)$, where $\eta : id_{\mathcal{E}} \to PP_* = PP$ is the unit of the adjunction.

Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be monads in topoi $\mathcal{E}$ and $\mathcal{T}$ defined from the adjunctions $\mathcal{E} \underset{P_{\mathcal{E}}}{\overset{P_{\mathcal{E}*}}{\rightleftarrows}} \mathcal{E}^{op}$ and $\mathcal{T} \underset{P_{\mathcal{T}}}{\overset{P_{\mathcal{T}*}}{\rightleftarrows}} \mathcal{T}^{op}$. We denote by $U_{\boldsymbol{G}} : \mathcal{E}^{\boldsymbol{G}} \to \mathcal{E}$, $U_{\boldsymbol{H}} : \mathcal{T}^{\boldsymbol{H}} \to \mathcal{H}$ the forgetful functors and by $K_{\boldsymbol{G}} : \mathcal{E}^{op} \to \mathcal{E}^{\boldsymbol{G}}$, $K_{\boldsymbol{H}} : \mathcal{T}^{op} \to \mathcal{T}^{\boldsymbol{H}}$ the comparison functors.

For a functor $T : \mathcal{E} \to \mathcal{T}$, T^{op} denotes a functor T regarded as a functor $\mathcal{E}^{op} \to \mathcal{T}^{op}$. If T is logical, consider the natural isomorphism $\xi^X_Z : T(Z^X) \to T(Z)^{T(X)}$ defined in (A.16.7). Define natural transformations $\alpha : P_{\mathcal{T}*}T \to T^{op}P_{\mathcal{E}*}$ and $\beta : P_{\mathcal{T}}T^{op} \to TP_{\mathcal{E}}$ by $\alpha_X = \xi^X_\Omega$ and $\beta_X = (\xi^X_\Omega)^{-1}$. Then, α and β satisfies the conditions of (A.14.10) by (A.16.13).

By (A.14.4), (A.14.10) and (3.3.5), we have a functor $\overline{T} : \mathcal{E}^{\boldsymbol{G}} \to \mathcal{T}^{\boldsymbol{H}}$ such that $U_{\boldsymbol{H}}\overline{T} = TU_{\boldsymbol{G}}$ defined from $\lambda = \beta_{P_{\mathcal{E}*}}P_{\mathcal{T}}(\alpha) : P_{\mathcal{T}}P_{\mathcal{T}*}T \to TP_{\mathcal{E}}P_{\mathcal{E}*}$.

Corollary 3.3.7 *Let $T : \mathcal{E} \to \mathcal{T}$ be a logical functor.*

1) T preserves finite colimits.

2) T has a left adjoint if and only if it has a right adjoint.

Proof. 1) Let D and D^{op} be as in (3.3.6). Suppose that a functor $P_{\mathcal{E}}D^{op} : \mathcal{D}^{op} \to \mathcal{E}$ has a limit and that T preserves it. Then, since $U_{\boldsymbol{H}}\overline{T} = TU_{\boldsymbol{G}}$ and $U_{\boldsymbol{H}}$ creates limits, $K_{\boldsymbol{G}}D^{op}$ has a limit and $\overline{T}$ preserves it Since $K_{\boldsymbol{G}}$ and $K_{\boldsymbol{H}}$ are equivalences, D^{op} has a limit and T^{op} preserves it. In other words, D has a colimit and T preserves it.

2) Since $K_{\boldsymbol{G}}$ and $K_{\boldsymbol{H}}$ are equivalences of categories and α, β are natural equivalences, the conditions of (A.15.3) and (A.15.4) are satisfied. Hence if T has a left adjoint, T^{op} also has a left adjoint which is nothing but a right adjoint of T. Conversely, if T has a right adjoint, T^{op} also has a right adjoint which is nothing but a left adjoint of T. □

3.4 Pull-back functors

Theorem 3.4.1 *Let $\mathcal{E}$ be a topos and X an object of $\mathcal{E}$, then $\mathcal{E}/X$ is a topos and the pull-back functor $X^* : \mathcal{E} \to \mathcal{E}/X$ along $X \to 1$ is logical.*

Proof. (1) A product of $(Y \xrightarrow{f} X)$ and $(Z \xrightarrow{g} X)$ in $\mathcal{E}$ is given by the pull-back of g along f. Since $\Sigma_X : \mathcal{E}/X \to \mathcal{E}$ creates equalizers, $\mathcal{E}/X$ has equalizers. It follows from (A.3.9) that X^* preserves limits.

(2) A morphism φ in $\mathcal{E}/X$ is a monomorphism if and only if $\Sigma_X(\varphi)$ is a monomorphism. In fact, if $\varphi : (Y \xrightarrow{f} X) \to (Z \xrightarrow{g} X)$ is a monomorphism in $\mathcal{E}/X$ and $a, b : W \to Y$ are morphisms in $\mathcal{E}$ such that $\Sigma_X(\varphi)a = \Sigma_X(\varphi)b$, then a and b give morphisms $a : (W \xrightarrow{fa} X) \to (Y \xrightarrow{f} X)$ and $b : (W \xrightarrow{fb} X) \to (Y \xrightarrow{f} X)$ such that $\varphi a = \varphi b$ in $\mathcal{E}/X$. Hence $a = b$ in $\mathcal{E}/X$, thus $a = b$ in $\mathcal{E}$. Therefore $\Sigma_X(\varphi)$ is a monomorphism. Since Σ_X is faithful, converse is obvious by (A.3.3).

We show that $X^*(\Omega) = (\Omega \times X \xrightarrow{\mathrm{pr}_1} X)$ is the object classifier in $\mathcal{E}/X$. Let $\sigma : (Y \xrightarrow{f} X) \to (Z \xrightarrow{g} X)$ be a monomorphism in $\mathcal{E}/X$. Since $\sigma : Y \to Z$ is a monomorphism in $\mathcal{E}$, we have the classifying map $\phi : Z \to \Omega$ of σ. By applying (A.3.1) to the following diagram, the left square is a pull-back.

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\sigma"] & X \cong 1 \times X \arrow[r] \arrow[d, "t \times id_X"] & 1 \arrow[d, "t"] \\
Z \arrow[r, "{(\phi, g)}"] & \Omega \times X \arrow[r, "\mathrm{pr}_1"] & \Omega
\end{tikzcd}$$

Since $\mathrm{pr}_2(\phi, g) = g$, (ϕ, g) is a morphism in $\mathcal{E}/X$ and this is a classifying map of $\sigma : (Y \xrightarrow{f} X) \to (Z \xrightarrow{g} X)$. If $\varphi : Z \to \Omega \times X$ is a morphism in $\mathcal{E}/X$ (that is, $\mathrm{pr}_2\varphi = g$) such that the left square of

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\sigma"] & X \cong 1 \times X \arrow[r] \arrow[d, "t\times id_X"] & 1 \arrow[d, "t"] \\ Z \arrow[r, "\varphi"] & \Omega \times X \arrow[r, "\mathrm{pr}_1"] & \Omega \end{tikzcd}$$

is a pull-back, then the outer rectangle is a pull-back and it follows that $\mathrm{pr}_1\varphi = \phi$. This shows the uniqueness of the classifying map.

(3) Finally we show that $\mathcal{E}/X$ is cartesian closed. For objects $Y \xrightarrow{f} X$ and $Z \xrightarrow{g} X$, let $\theta : X \times Y \to \widetilde{X}$ be the classifying map of a partial map represented by $X \times Y \xleftarrow{(f, id_Y)} Y \xrightarrow{f} X$ and form a pull-back

$$\begin{tikzcd} E_g^f \arrow[r, "\xi_g^f"] \arrow[d, "p_g^f"] & \widetilde{Z}^Y \arrow[d, "\tilde{g}^Y"] \\ X \arrow[r, "\bar{\theta}"] & \widetilde{X}^Y \end{tikzcd} \qquad (*)$$

where $\bar{\theta} : X \to \widetilde{X}^Y$ is the transpose of θ. It suffices to show that E_g^f represents a functor $(T \xrightarrow{h} X) \mapsto \mathcal{E}/X((T \times_X Y \to X), (Z \xrightarrow{g} X))$.

Since $(*)$ is a pull-back, $\psi \mapsto \xi_g^f\psi$ gives a natural bijection

$$\mathcal{E}/X((T \xrightarrow{h} X), (E_g^f \xrightarrow{p_g^f} X)) \to \mathcal{E}/\widetilde{X}^Y((T \xrightarrow{\bar{\theta}h} \widetilde{X}^Y), (\widetilde{Z}^Y \xrightarrow{\tilde{g}^Y} \widetilde{X}^Y)).$$

A morphism $\varphi : T \to \widetilde{Z}^Y$ makes the the following left diagram commute if and only if its transpose $\bar{\varphi}$ makes the right one commute.

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[d, "h"] & \widetilde{Z}^Y \arrow[d, "\tilde{g}^Y"] \\ X \arrow[r, "\bar{\theta}"] & \widetilde{X}^Y \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\bar{\varphi}"] \arrow[d, "h\times id_Y"] & \widetilde{Z} \arrow[d, "\tilde{g}"] \\ X \times Y \arrow[r, "\theta"] & \widetilde{X} \end{tikzcd}$$

Thus we have a natural bijection

$$\mathcal{E}/\widetilde{X}^Y((T \xrightarrow{\bar{\theta}h} \widetilde{X}^Y), (\widetilde{Z}^Y \xrightarrow{\tilde{g}^Y} \widetilde{X}^Y)) \to \mathcal{E}/\widetilde{X}((T \times Y \xrightarrow{\theta(h\times id_Y)} \widetilde{X}), (\widetilde{Z} \xrightarrow{\tilde{g}} \widetilde{X})).$$

The outer rectangle and the right square of the following left diagram are pull-backs, the outer rectangle of the right diagram is a pull-back.

$$\begin{tikzcd} T \times_X Y \arrow[r] \arrow[d, "\mathrm{pr}_2"] & T \times Y \arrow[r, "\mathrm{pr}_1"] \arrow[d, "h\times id_Y"] & T \arrow[d, "h"] \\ Y \arrow[r, "(f, id_Y)"] & X \times Y \arrow[r, "\mathrm{pr}_1"] & X \end{tikzcd} \qquad \begin{tikzcd} T \times_X Y \arrow[r, "\mathrm{pr}_2"] \arrow[d] & Y \arrow[r, "f"] \arrow[d, "(f, id_Y)"] & X \arrow[d, "\eta"] \\ T \times Y \arrow[r, "h\times id_Y"] & X \times Y \arrow[r, "\theta"] & \widetilde{X} \end{tikzcd}$$

It follows from (3.2.10) that there is a natural bijection

$$\mathcal{E}/\widetilde{X}((T \times Y \xrightarrow{\theta(h\times id_Y)} \widetilde{X}), (\widetilde{Z} \xrightarrow{\tilde{g}} \widetilde{X})) \to (\mathcal{E}^p/X)_*((T \times Y \overset{f\mathrm{pr}_2}{\rightharpoondown} X), (\widetilde{Z} \overset{g}{\rightharpoondown} X)),$$

where $T \times Y \overset{f\mathrm{pr}_2}{\rightharpoondown} X$ and $\widetilde{Z} \overset{g}{\rightharpoondown} X$ are partial maps represented by $T \times Y \leftarrowtail T \times_X Y \xrightarrow{\mathrm{pr}_2} Y$ and $\widetilde{Z} \overset{\eta}{\leftarrowtail} Z \xrightarrow{g} X$, respectively.

It follows from (3.2.11) that we have a natural bijection

$$(\mathcal{E}^p/X)_*((T \times Y \overset{f\mathrm{pr}_2}{\rightharpoondown} X), (\widetilde{Z} \overset{g}{\rightharpoondown} X)) \to \mathcal{E}^p/X((T \times_X Y \xrightarrow{f\mathrm{pr}_2} X), (Z \overset{g}{\rightharpoondown} X)).$$

If $Y = A \times X$, $Z = B \times X$ and $f = \mathrm{pr}_2$, $g = \mathrm{pr}_2$, it follows from (A.3.9) that there are natural bijections $\mathcal{E}/X((T \xrightarrow{h} X), X^*(B^A)) \cong \mathcal{E}(T, B^A) \cong \mathcal{E}(T \times A, B) \cong \mathcal{E}((T \xrightarrow{h} X) \times_X X^*(A) \to X), X^*(B))$. Hence $X^*(B^A)$ is naturally isomorphic to $X^*(B)^{X^*(A)}$ by (A.3.8). $\square$

If $\alpha : (Y' \xrightarrow{f'} X) \to (Y \xrightarrow{f} X)$ and $\beta : (Z \xrightarrow{g} X) \to (Z' \xrightarrow{g'} X')$ are morphisms in $\mathcal{E}/X$, $\theta(id_X \times \alpha)$ classifies $X \times Y' \xleftarrow{(f', id_{Y'})} Y' \xrightarrow{f'} X$ by applying (A.3.5) and (A.3.1) to the following diagram on the left, and the right diagram commutes.

$$\begin{tikzcd}
Y' \arrow[r, "\alpha"] \arrow[d, "(f', id_{Y'})"] & Y \arrow[r, "f"] \arrow[d, "(f, id_Y)"] & X \arrow[d, "\eta"] \\
X \times Y \arrow[r, "id_X \times \alpha"] & X \times Y \arrow[r, "\theta"] & \widetilde{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{Z}^Y \arrow[r, "\tilde{\beta}^\alpha"] \arrow[d, "\tilde{g}^Y"] & \widetilde{Z'}^{Y'} \arrow[d, "\tilde{g'}^{Y'}"] \\
\widetilde{X}^Y \arrow[r, "\widetilde{X}^\alpha"] & \widetilde{X}^{Y'}
\end{tikzcd}$$

Since the transpose of $\theta(id_X \times \alpha)$ is $\widetilde{X}^\alpha \bar{\theta}$, we have a morphism $e^\beta_\alpha : E^f_g \to E^{f'}_{g'}$ satisfying $e^\beta_\alpha p^{f'}_{g'} = p^f_g$ and $\xi^{f'}_{g'} e^\beta_\alpha = \xi^f_g \tilde{\beta}^\alpha$.

Corollary 3.4.2 *Let $\mathcal{E}$ be a topos and $f : X \to Y$ a morphism in $\mathcal{E}$, then the pull-back functor $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ is logical and it has a right adjoint $\Pi_f : \mathcal{E}/X \to \mathcal{E}/Y$.*

Proof. Regarding f as an object of $\mathcal{E}/Y$, there is an isomorphism of categories $F : \mathcal{E}/X \to (\mathcal{E}/Y)/f$ defined by $F(Z \xrightarrow{g} X) = ((Z \xrightarrow{fg} Y) \xrightarrow{g} (X \xrightarrow{f} Y))$ and $F(\varphi : (W \xrightarrow{h} X) \to (Z \xrightarrow{g} X)) = (\varphi : F(W \xrightarrow{h} X) \to F(Z \xrightarrow{g} X))$. Then the composition $\mathcal{E}/Y \xrightarrow{f^*} \mathcal{E}/X \xrightarrow{F} (\mathcal{E}/Y)/f$ is f^* in the sense of the preceding theorem, hence logical. Therefore the pull-back functor $f^* : \mathcal{E}/Y \to \mathcal{E}/X$ is logical and it has a right adjoint by (A.3.9) and (3.3.7). □

Corollary 3.4.3 *Each morphism $f : X \to Y$ in a topos $\mathcal{E}$ induces an essential geometric morphism $\mathcal{E}/X \xrightarrow{f} \mathcal{E}/Y$ with $f_* = \Pi_f$, $f^* =$ (the pull-back functor). Thus we have a functor $\mathcal{E} \to \mathfrak{Top}/\mathcal{E}$ which maps an object X to $(\mathcal{E}/X \xrightarrow{\Sigma_X} \mathcal{E})$.*

Theorem 3.4.4 *Let $\mathcal{F} \xrightarrow{f} \mathcal{E}$ be an essential geometric morphism such that f^* is logical and that the left adjoint $f_!$ of f^* preserves equalizers. Then there exists an object X of $\mathcal{E}$ (unique up to isomorphism) such that there exists an equivalence $\Psi : \mathcal{F} \to \mathcal{E}/X$ satisfying $\Sigma_X \Psi = f_!$, $\Psi f^* \cong X^*$ and $\Pi_X \Psi \cong f_*$.*

Proof. Since f^* is logical, $f^*(\Omega_{\mathcal{E}})$ is the subobject classifier $\Omega_{\mathcal{F}}$ of $\mathcal{F}$. Hence the adjoint isomorphism gives a natural isomorphism $\mathcal{E}(f_!(Y), \Omega_{\mathcal{E}}) \xrightarrow{\cong} \mathcal{F}(Y, \Omega_{\mathcal{F}})$. Suppose that $\sigma : Y' \to Y$ is a morphism in $\mathcal{F}$ such that $f_!(\sigma)$ is an isomorphism. By the commutativity of

$$\begin{tikzcd}
\mathcal{E}(f_!(Y), \Omega_{\mathcal{E}}) \arrow[r, "\cong"] \arrow[d, "f_!(\sigma)^*"] & \mathcal{F}(Y, \Omega_{\mathcal{F}}) \arrow[d, "\sigma^*"] \\
\mathcal{E}(f_!(Y'), \Omega_{\mathcal{E}}) \arrow[r, "\cong"] & \mathcal{F}(Y', \Omega_{\mathcal{F}})
\end{tikzcd},$$

$\sigma^* : \mathcal{E}(Y, \Omega_{\mathcal{F}}) \to \mathcal{F}(Y', \Omega_{\mathcal{F}})$ is an isomorphism. Moreover, suppose that σ is a monomorphism classified by $\phi_\sigma : Y \to \Omega_{\mathcal{F}}$. Since the pull-backs of σ and id_Y along σ are both the identity morphism of Y', we have $\phi_{id_{Y'}} = \phi_\sigma \sigma = \phi_{id_Y} \sigma$. Hence $\phi_\sigma = \phi_{id_Y}$ and this implies that σ is an isomorphism.

Let $\alpha, \beta : Y \to Z$ be morphisms in $\mathcal{F}$ such that $f_!(\alpha) = f_!(\beta)$ and consider the equalizer $Y' \overset{\sigma}{\rightarrowtail} Y \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Z$.

Since $f_!$ preserves equalizers, $f_!(Y') \overset{f_!(\sigma)}{\rightarrowtail} f_!(Y) \underset{f_!(\beta)}{\overset{f_!(\alpha)}{\rightrightarrows}} f_!(Z)$ is an equalizer. It follows from $f_!(\alpha) = f_!(\beta)$ that $f_!(\sigma)$ is an isomorphism. Thus σ is an isomorphism and we have $\alpha = \beta$, namely $f_!$ is faithful. By (A.3.3), $f_!$ reflects monomorphisms and epimorphisms, hence by (3.2.2), $f_!$ reflects isomorphisms.

Let $\boldsymbol{G} = (f_! f^*, \varepsilon, f_!(\eta_{f^*}))$ be the comonad on $\mathcal{E}$ obtained from the adjunction $\mathcal{F} \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathcal{E}$. Applying the opposite of (A.13.4) to this adjunction, $f_!$ is comonadic. It follows from (A.16.15) that there exists an equivalence of categories $\Psi : \mathcal{F} \to \mathcal{E}/X$ such that $\Sigma_X \Psi = f_!$ and that Ψf^* is naturally equivalent to X^*, where $X = f_! f^*(1_{\mathcal{E}})$.

Let $\Phi : \mathcal{E}/X \to \mathcal{F}$ be a quasi-inverse of Ψ, then Φ is a left adjoint of Ψ and f^* is naturally equivalent to ΦX^*. For any object Z of $\mathcal{E}$ and Y of $\mathcal{F}$, we have the following chain of natural isomorphisms.

$$\mathcal{E}(Z, f_*(Y)) \cong \mathcal{F}(f^*(Z), Y) \cong \mathcal{F}(\Phi X^*(Z), Y) \cong \mathcal{E}/X(X^*(Z), \Psi(Y)) \cong \mathcal{E}(Z, \Pi_X \Psi(Y)).$$

Hence f_* is naturally equivalent to $\Pi_X \Psi$. □

3.5 Image factorization

Let $\mathcal{E}$ be a topos and X an object of $\mathcal{E}$. Suppose that $((X_i \xrightarrow{p_i} X) \xrightarrow{\alpha^i_j} (X_j \xrightarrow{p_j} X))$ is a diagram in $\mathcal{E}/X$ such that $(X_i \xrightarrow{\alpha^i_j} X_j)$ has a limiting cone $(X_i \xrightarrow{\pi_i} Z)$ in $\mathcal{E}$. Since $\Sigma_X : \mathcal{E}/X \to \mathcal{E}$ creates colimits, there exists a unique morphism $\rho : Z \to X$ satisfying $\rho\pi_i = p_i$ for any i and $((X_i \xrightarrow{p_i} X) \xrightarrow{\pi_i} (Z \xrightarrow{\rho} X))$ is a limiting cone in $\mathcal{E}/X$.

Let $f : Y \to X$ be a morphism in $\mathcal{E}$. Since $f^* : \mathcal{E}/X \to \mathcal{E}/Y$ and $\Sigma_Y : \mathcal{E}/Y \to \mathcal{E}$ have right adjoints, they preserve colimits. Hence, by applying $\Sigma_Y f^*$ to the above diagrams in $\mathcal{E}/X$, we see that $(X_i \times_X Y \xrightarrow{\pi_i \times 1_Y} Z \times_X Y)$ is a limiting cone of the "pulled-back" diagram $(X_i \times_X Y \xrightarrow{\alpha^i_j \times 1_Y} X_j \times_X Y)$ in $\mathcal{E}$.

In particular, if $Z = X$ and $p_i = \pi_i$, we have the following result.

Proposition 3.5.1 *In a topos, colimits are universal (See (A.3.5)). In particular, a topos satisfy R3 of (A.8.1).*

By (3.2.3), (3.3.6) and (3.5.1), we have the following result.

Corollary 3.5.2 *A topos is an exact category (See (A.8.1)).*

Proposition 3.5.3 *In a topos, the initial object is strict, that is, any morphism whose codomain is an initial object is an isomorphism.*

Proof. Let $\mathcal{E}$ be a topos with initial object 0. Suppose that there is a morphism $f : X \to 0$. In $\mathcal{E}/0$, $0 \xrightarrow{id_0} 0$ is both initial and terminal object. Since the pull-back functor $f^* : \mathcal{E}/0 \to \mathcal{E}/X$ has both right and left adjoints by (3.4.2) and (A.3.9), f preserves the initial and terminal object. Hence $f^*(0 \xrightarrow{id_0} 0)$ is both initial and terminal object in $\mathcal{E}/X$. By (3.4.1), $\mathcal{E}/X$ is a topos and it follows from (A.16.17) that every object of $\mathcal{E}/X$ is isomorphic to $(X \xrightarrow{id_X} X)$. In particular, $(0 \to X)$ is isomorphic to $(X \xrightarrow{id_X} X)$. Thus X is an isomorphic to 0, namely X is an initial object. $\square$

Corollary 3.5.4 *Finite coproducts in a topos are disjoint.*

Proof. Let X_i $(i = 1, 2, \ldots, n)$ be objects of a topos $\mathcal{E}$. By definition, the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X_i \arrow[d, "\iota_i"] \\ X_j \arrow[r, "\iota_j"] & X_i \coprod X_j \end{tikzcd}$$

is a push-out. By (3.5.3), $0 \to X_i$ and $0 \to X_j$ are monomorphisms. It follows from (3.2.8) that the above square is a pull-back and that ι_i and ι_j are monomorphisms. Hence the canonical inclusions $\nu_i : X_i \to \coprod_{j=1}^n X_j$ are monomorphisms and so are $\nu_{ij} : X_i \coprod X_j \to \coprod_{j=1}^n X_j$. Apply (A.3.6) to the above square and ν_{ij}, we see that finite coproducts are disjoint. $\square$

If $f : X \to Y$ is a morphism in $\mathcal{E}$, define a map $f_\sharp : \mathcal{E}(X, \Omega) \to \mathcal{E}(Y, \Omega)$ as follows. For a morphism $\phi : X \to \Omega$, let $\sigma : V \rightarrowtail X$ be the subobject classified by ϕ. We set $f_\sharp(\phi) = $ (*the classifying map of the image of* $V \xrightarrow{f\sigma} Y$).

Proposition 3.5.5 *Let $f : X \to Y$ be a morphism in a topos. Then, the following squares commute (See (A.9.3)).*

$$\begin{tikzcd} \mathcal{E}(Y,\Omega) \arrow[r, "f^*"] \arrow[d, "\cong"] & \mathcal{E}(X,\Omega) \arrow[d, "\cong"] \\ \mathrm{Sub}(Y) \arrow[r, "f^*"] & \mathrm{Sub}(X) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}(X,\Omega) \arrow[r, "f_\sharp"] \arrow[d, "\cong"] & \mathcal{E}(Y,\Omega) \arrow[d, "\cong"] \\ \mathrm{Sub}(X) \arrow[r, "f_!"] & \mathrm{Sub}(Y) \end{tikzcd}$$

Here the vertical maps are given by pull-backs of $t : 1 \rightarrowtail \Omega$.

Proof. For $\phi \in \mathcal{E}(Y, \Omega)$, let $\sigma : Z \rightarrowtail Y$ be a pull-back of $t : 1 \to \Omega$ along ϕ and set $f^*(Z \overset{\sigma}{\rightarrowtail} Y) = (W \overset{\bar{\sigma}}{\rightarrowtail} X)$. Then, the both squares of the following diagram are pull-backs.

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\bar{\sigma}"] & Z \arrow[r] \arrow[d, "\sigma"] & 1 \arrow[d, "t"] \\ X \arrow[r, "f"] & Y \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

Hence $\bar{\sigma}$ is a pull-back of t along $f^*(\phi) = \phi f$ and this implies the commutativity of the left square. The commutativity of the right square is obvious from the definitions of $f_\sharp$ and $f_!$. □

For a morphism $f : X \to Y$ in a topos $\mathcal{E}$, we define a morphism $\exists f : \Omega^X \to \Omega^Y$ to be the exponential transpose of the classifying map of the image of a composition $\in_X \rightarrowtail \Omega^X \times X \xrightarrow{1\times f} \Omega^X \times Y$.

Proposition 3.5.6 *Let $\mathcal{E}$ be a topos.*

1) For any object U of $\mathcal{E}$, the following diagram commutes, where τ denotes the exponential transpose.

$$\begin{tikzcd} \mathcal{E}(U, \Omega^X) \arrow[r, "(\exists f)_*"] \arrow[d, "\tau"] & \mathcal{E}(U, \Omega^Y) \arrow[d, "\tau"] \\ \mathcal{E}(U \times X, \Omega) \arrow[r, "(id_U \times f)_\sharp"] & \mathcal{E}(U \times Y, \Omega) \end{tikzcd}$$

2) The correspondence $X \mapsto \Omega^X$, $f \mapsto \exists f$ gives a functor $\mathcal{E} \to \mathcal{E}$.

3) If the following square on the left is a pull-back, the right one commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "h"] \\ Z \arrow[r, "k"] & W \end{tikzcd} \qquad \begin{tikzcd} \Omega^Y \arrow[r, "Pf"] \arrow[d, "\exists h"] & \Omega^X \arrow[d, "\exists g"] \\ \Omega^W \arrow[r, "Pk"] & \Omega^Z \end{tikzcd}$$

4) If f is an epimorphism, then $\exists f Pf = id_{\Omega^Y}$.

5) The functor in 2) preserves pull-backs of monomorphisms.

Proof. 1) For any morphism $g : V \to U$ in $\mathcal{E}$, the following left square is a pull-back.

$$\begin{tikzcd} V \times X \arrow[r, "id_V \times f"] \arrow[d, "g\times id_X"] & V \times Y \arrow[d, "g\times id_Y"] \\ U \times X \arrow[r, "id_U \times f"] & U \times Y \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sub}(U \times X) \arrow[r, "(id_U \times f)_!"] \arrow[d, "(g\times id_X)^*"] & \mathrm{Sub}(U \times Y) \arrow[d, "(g\times id_Y)^*"] \\ \mathrm{Sub}(V \times X) \arrow[r, "(id_V \times f)_!"] & \mathrm{Sub}(V \times Y) \end{tikzcd}$$

It follows from (A.9.4) that the right square above commutes. Hence, by (3.5.5), the following square commutes.

$$\begin{tikzcd} \mathcal{E}(U \times X, \Omega) \arrow[r, "(id_U \times f)_\sharp"] \arrow[d, "(g\times id_X)^*"] & \mathcal{E}(U \times Y, \Omega) \arrow[d, "(g\times id_Y)^*"] \\ \mathcal{E}(V \times X, \Omega) \arrow[r, "(id_V \times f)_\sharp"] & \mathcal{E}(V \times Y, \Omega) \end{tikzcd}$$

Consider the case $U = \Omega^X$. Since $\tau(id_{\Omega^X}) = ev$ and this classifies $\in_X \rightarrowtail \Omega^X \times X$, we have $(id_{\Omega^X} \times f)_\sharp\tau(id_{\Omega^X}) = \tau(\exists f)$. For arbitrary $U \in \mathrm{Ob}\,\mathcal{E}$ and $\varphi \in \mathcal{E}(U, \Omega^X)$, by the commutativity of the above square, we have $\tau(\exists f)_*(\varphi) = \tau(\exists f \varphi) = \tau(\exists f)(\varphi \times id_Y) = (\varphi \times id_Y)^*(id_{\Omega^X} \times f)_\sharp\tau(id_{\Omega^X}) = (id_U \times f)_\sharp(\varphi \times id_X)^*\tau(id_{\Omega^X}) = (id_U \times f)_\sharp\tau(\varphi)$.

2) It is obvious from the definition that $\exists id_X = id_{\Omega^X}$. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{E}$ and U an object of $\mathcal{E}$. By 1) and (3.5.5), we have the following commutative diagrams.

$$\begin{tikzcd} \mathcal{E}(U, \Omega^X) \arrow[r, "\exists f"] \arrow[d, "\cong"] & \mathcal{E}(U, \Omega^Y) \arrow[r, "\exists g"] \arrow[d, "\cong"] & \mathcal{E}(U, \Omega^Y) \arrow[d, "\cong"] \\ \mathrm{Sub}(U \times X) \arrow[r, "(id_U \times f)_!"] & \mathrm{Sub}(U \times Y) \arrow[r, "(id_U \times g)_!"] & \mathrm{Sub}(U \times Z) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}(U, \Omega^X) \arrow[r, "\exists(gf)"] \arrow[d, "\cong"] & \mathcal{E}(U, \Omega^Z) \arrow[d, "\cong"] \\ \mathrm{Sub}(U \times X) \arrow[r, "(id_U \times gf)_!"] & \mathrm{Sub}(U \times Z) \end{tikzcd}$$

Since $(id_U \times g)_!(id_U \times f)_! = (id_U \times gf)_!$, we have $(\exists g)(\exists f) = \exists(gf)$.

3) By the assumption, the left diagram below is a pull-back for any $U \in \mathrm{Ob}\,\mathcal{E}$.

$$\begin{array}{ccc} U \times X & \xrightarrow{id_U \times f} & U \times Y \\ \downarrow{\scriptstyle id_U \times g} & & \downarrow{\scriptstyle id_U \times h} \\ U \times Z & \xrightarrow{id_U \times k} & U \times W \end{array} \qquad \begin{array}{ccc} \mathrm{Sub}(U \times Y) & \xrightarrow{(id_U \times f)^*} & \mathrm{Sub}(U \times X) \\ \downarrow{\scriptstyle (id_U \times h)_!} & & \downarrow{\scriptstyle (id_U \times g)_!} \\ \mathrm{Sub}(U \times W) & \xrightarrow{(id_U \times k)^*} & \mathrm{Sub}(U \times Z) \end{array}$$

Hence, by (A.9.4), the right diagram commutes. It follows from (A.16.5) and (3.5.5) that the following left square commutes.

$$\begin{array}{ccc} \mathcal{E}(U, \Omega^Y) & \xrightarrow{Pf_*} & \mathcal{E}(U, \Omega^X) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \mathrm{Sub}(U \times Y) & \xrightarrow{(id_U \times f)^*} & \mathrm{Sub}(U \times X) \end{array} \qquad \begin{array}{ccc} \mathcal{E}(U, \Omega^Y) & \xrightarrow{Pf_*} & \mathcal{E}(U, \Omega X) \\ \downarrow{\scriptstyle (\exists h)_*} & & \downarrow{\scriptstyle (\exists g)_*} \\ \mathcal{E}(U, \Omega^W) & \xrightarrow{Pk_*} & \mathcal{E}(U, \Omega Z) \end{array}$$

Together with 1), we see the commutativity of the right one. Since U is arbitrary, the assertion follows.

4) Since an epimorphism in $\mathcal{E}$ is regular epimorphism and $\mathcal{E}$ is a regular category, $id_U \times f : U \times X \to U \times Y$ is a regular epimorphism. Hence by (A.9.3), $(id_U \times f)_!(id_U \times f)^* = id_{\mathrm{Sub}(U \times Y)}$. On the other hand, the following diagram commutes by (A.16.5) and 1) of (3.5.5), 1).

$$\begin{array}{ccccc} \mathcal{E}(U, \Omega^Y) & \xrightarrow{(Pf)_*} & \mathcal{E}(U, \Omega^X) & \xrightarrow{(\exists f)_*} & \mathcal{E}(U, \Omega^Y) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \mathrm{Sub}(U \times Y) & \xrightarrow{(id_U \times f)^*} & \mathrm{Sub}(U \times X) & \xrightarrow{(id_U \times f)_!} & \mathrm{Sub}(U \times Y) \end{array}$$

Thus we have $\exists f Pf = id_{\Omega^Y}$.

5) Suppose that the following diagram on the left is a pull-back and k is a monomorphism. Then, the right one is also a pull-back.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ Z & \xrightarrow{g} & W \end{array} \qquad \begin{array}{ccc} U \times X & \xrightarrow{id_U \times f} & U \times Y \\ \downarrow{\scriptstyle id_U \times h} & & \downarrow{\scriptstyle id_U \times k} \\ U \times Z & \xrightarrow{id_U \times g} & U \times W \end{array}$$

By (A.9.4), the left diagram below is a pull-back.

$$\begin{array}{ccc} \mathrm{Sub}(U \times X) & \xrightarrow{(id_U \times f)_!} & \mathrm{Sub}(U \times Y) \\ \downarrow{\scriptstyle (id_U \times h)_!} & & \downarrow{\scriptstyle (id_U \times k)_!} \\ \mathrm{Sub}(U \times W) & \xrightarrow{(id_U \times g)^*} & \mathrm{Sub}(U \times Z) \end{array} \qquad \begin{array}{ccc} \mathcal{E}(U, \Omega^X) & \xrightarrow{(\exists f)_*} & \mathcal{E}(U, \Omega^Y) \\ \downarrow{\scriptstyle (\exists h)_*} & & \downarrow{\scriptstyle (\exists k)_*} \\ \mathcal{E}(U, \Omega^Z) & \xrightarrow{(\exists g)_*} & \mathcal{E}(U, \Omega^W) \end{array}$$

Hence, by (3.5.5) and 1), the right diagram above is a pull-back and the result follows. □

Chapter 4

Topologies and sheaves

4.1 Topologies

Let $\mathcal{E}$ be a topos and $\wedge : \Omega \times \Omega \to \Omega$ the classifying map of subobject $(t, t) : 1 \to \Omega \times \Omega$.

Proposition 4.1.1 *1) Let $\alpha : Y \rightarrowtail X$ and $\beta : Z \rightarrowtail X$ be subobjects of X with classifying maps $\phi_\alpha, \phi_\beta : X \to \Omega$. Then, the classifying map of $Y \cap Z \rightarrowtail X$ is given by a composition $X \xrightarrow{(\phi_\alpha, \phi_\beta)} \Omega \times \Omega \xrightarrow{\wedge} \Omega$.*

2) Let $f : Y \rightarrowtail X$ and $g : Z \rightarrowtail W$ be subobjects of X and W classified by $\phi : X \to \Omega$ and $\psi : W \to \Omega$, respectively. Then, the classifying map of $f \times g : Y \times Z \rightarrowtail X \times W$ is given by a composition $X \times Z \xrightarrow{\phi \times \psi} \Omega \times \Omega \xrightarrow{\wedge} \Omega$.

Proof. 1) Recall that $Y \cap Z$ is defined by the pull-back square

$$\begin{tikzcd} Y \cap Z \arrow[r, "\bar{\alpha}"] \arrow[d, "\bar{\beta}"] & Z \arrow[d, "\beta"] \\ Y \arrow[r, "\alpha"] & X \end{tikzcd} .$$

It suffices to show that

$$\begin{tikzcd} Y \cap Z \arrow[r] \arrow[d, "\alpha\bar{\beta}"] & 1 \arrow[d, "{(t,t)}"] \\ X \arrow[r, "{(\phi_\alpha, \phi_\beta)}"] & \Omega \times \Omega \end{tikzcd}$$

is a pull-back. Suppose that a morphism $f : W \to X$ is given so that $(\phi_\alpha, \phi_\beta) f$ factors through (t, t). Then $\phi_\alpha f$ and $\phi_\beta f$ factor through t, there exist morphisms $g : W \to Y$ and $h : W \to Z$ such that $f = \alpha g = \beta h$. Hence there is a unique morphism $k : W \to Y \cap Z$ satisfying $\bar{\beta} k = g$ and $\bar{\alpha} k = h$. Thus we have $\alpha\bar{\beta} k = \alpha g = f$.

2) Since the following square is a pull-back, $Y \times Z = (Y \times W) \cap (X \times Z)$.

$$\begin{tikzcd} Y \times Z \arrow[r, "f \times id_Z"] \arrow[d, "id_Y \times g"] & X \times Z \arrow[d, "id_X \times g"] \\ Y \times W \arrow[r, "f \times id_W"] & X \times W \end{tikzcd}$$

Note that $\mathrm{pr}_1^*(Y) = Y \times W$ and $\mathrm{pr}_2^*(Z) = X \times Z$ hold in $\mathrm{Sub}(X \times W)$, where $\mathrm{pr}_1 : X \times W \to X$ and $\mathrm{pr}_2 : X \times W \to W$ are projections. Hence it follows from 1) that $Y \times Z$ is classified by $\wedge(\phi \mathrm{pr}_1, \psi \mathrm{pr}_2) = \wedge(\phi \times \psi)$. $\square$

Definition 4.1.2 *A topology on a topos $\mathcal{E}$ is a morphism $j : \Omega \to \Omega$ satisfying $jt = t$ and $jj = j$ and making the following diagram commute.*

$$\begin{tikzcd} \Omega \times \Omega \arrow[r, "\wedge"] \arrow[d, "j \times j"] & \Omega \arrow[d, "j"] \\ \Omega \times \Omega \arrow[r, "\wedge"] & \Omega \end{tikzcd}$$

We denote by $J \overset{\bar{t}}{\rightarrowtail} \Omega$ *the subobject classified by* j *and by* $\Omega_j \to \Omega$ *the equalizer of* $\Omega \overset{j}{\underset{id_\Omega}{\rightrightarrows}} \Omega$.

Definition 4.1.3 *Let* $\mathcal{C}$ *be a category with pull-backs of monomorphisms. A universal closure operation on* $\mathcal{C}$ *is a collection of maps* $(cl_X : \mathrm{Sub}(X) \to \mathrm{Sub}(X))_{X \in \mathrm{Ob}\,\mathcal{C}}$ *satisfying the following properties.*

C1) For any $X \in \mathrm{Ob}\,\mathcal{C}$ *and* $Y \in \mathrm{Sub}(X)$, $Y \subset cl_X(Y)$.
C2) For any $X \in \mathrm{Ob}\,\mathcal{C}$, cl_X *preserves the order.*
C3) For any $X \in \mathrm{Ob}\,\mathcal{C}$, $cl_X cl_X = cl_X$.
C4) For any morphism $f : Y \to X$ *and* $Z \in \mathrm{Sub}(X)$, $f^* cl_X(Z) = cl_Y f^*(Z)$.

For $Y \in \mathrm{Sub}(X)$, $cl_X(Y)$ *is called the closure of* Y *and denoted by* $\overline{Y}$. *We say that a subobject* Y *of* X *is dense if* $\overline{Y} = X$ *and closed if* $\overline{Y} = Y$.

Proposition 4.1.4 *Let* $\mathcal{C}$ *be a category with pull-backs of monomorphisms. Suppose that a universal closure operation is defined on* $\mathcal{C}$.

1) Let $\sigma : Y \rightarrowtail X$ *be a monomorphism. Then,* $cl_Y = \sigma^* cl_X \sigma_!$. *If* $Y \in \mathrm{Sub}(X)$ *is closed,* $cl_X \sigma_! = \sigma_! cl_Y$. *Hence if* $Z \in \mathrm{Sub}(Y)$ *is closed in* Y, $\sigma_!(Z)$ *is closed in* X.

2) Let $f : W \to X$ *be a morphism in* $\mathcal{C}$. *If* $Y \in \mathrm{Sub}(X)$ *is dense (resp. closed) in* X, *so is* $f^*(Y) \in \mathrm{Sub}(W)$ *in* W.

3) For $Z, Y \in \mathrm{Sub}(X)$, *if* Z *is dense in* Y *and* Y *is dense in* X, *then* Z *is dense in* X.

4) If $Z \subset Y$ *in* $\mathrm{Sub}(X)$, *then* $Y = cl_X(Z)$ *holds in* $\mathrm{Sub}(X)$ *if and only if* Y *is closed in* X *and* Z *is dense in* Y.

5) If $Y, Z \in \mathrm{Sub}(X)$, *then* $cl_X(Y \cap Z) = cl_X(Y) \cap cl_X(Z)$.

Proof. 1) For $Z \in \mathrm{Sub}(Y)$, since $\sigma^*\sigma_!(Z) = Z \cap Y = Z$, we have $\sigma^* cl_X \sigma_!(Z) = cl_Y \sigma^*\sigma_!(Z) = cl_Y(Z)$. Suppose that $Y \in \mathrm{Sub}(X)$ is closed. Since cl_X preserves the order, $cl_X(\sigma_!(Z)) \subset cl_X(Y) = Y$. Hence $\sigma_! cl_Y(Z) = \sigma_!\sigma^* cl_X \sigma_!(Z) = (cl_X \sigma_!(Z)) \cap Y = cl_X \sigma_!(Z)$.

2) $cl_W f^*(Y) = f^* cl_X(Y) = \begin{cases} f^*(X) = W \text{ if } Y \text{ is dense in } X \\ f^*(Y) = W \text{ if } Y \text{ is closed in } X \end{cases}$

3) Let $\sigma : Y \rightarrowtail X$ be the monomorphism representing $Y \in \mathrm{Sub}(X)$ and we identify $Z \in \mathrm{Sub}(Y)$ with $\sigma_!(Z) \in \mathrm{Sub}(X)$. Since $cl_Y(Z) = Y$ and $cl_X(Y) = X$, $X = cl_X(\sigma_! cl_Y(Z)) = cl_X(\sigma_!\sigma^* cl_X \sigma_!(Z)) = cl_X(cl_X(Z) \cap Y) \subset cl_X cl_X(X) = cl_X(Z)$. Thus we have $X = cl_X(Z)$.

4) Let $\sigma : Y \rightarrowtail X$ be as above. Suppose $Y = cl_X(Z)$. Then, we have $cl_X(Y) = cl_X cl_X(Z) = cl_X(Z) = Y$ and $cl_Y(Z) = \sigma^* cl_X \sigma_!(Z) = cl_X(Z) \cap Y = Y$. Thus Y is closed in X and Z is dense in Y. Conversely, suppose that Y is closed in X and Z is dense in Y. Then, $cl_X(Z) = cl_X \sigma_!(Z) = \sigma_! cl_Y(Z) = \sigma_!(Y) = Y$ by 1).

5) Squares
$$\begin{array}{ccc} Y \cap Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ cl_X(Y) \cap Z & \longrightarrow & cl_X(Y) \end{array} \quad \text{and} \quad \begin{array}{ccc} cl_X(Y) \cap Z & \longrightarrow & Z \\ \downarrow & & \downarrow \\ cl_X(Z) & \longrightarrow & cl_X(Y) \cap cl_X(Z) \end{array}$$
are pull-backs by (A.3.1).

Since Y and Z are dense in $cl_X(Y)$ and $cl_X(Z)$ respectively, it follows from 2) that $Y \cap Z$ and $cl_X(Y) \cap Z$ are dense in $cl_X(Y) \cap Z$ and $cl_X(Y) \cap cl_X(Z)$ respectively. Hence by 3), $Y \cap Z$ is dense in $cl_X(Y) \cap cl_X(Z)$. On the other hand $cl_X(Y) \cap cl_X(Z)$ is closed in $cl_X(Y)$ by 2), thus it is closed in X by 1). The result follows from 4). □

Let $\mathcal{E}$ be a topos and X an object of $\mathcal{E}$. There is a bijection $S_X : \mathcal{E}(X, \Omega) \to \mathrm{Sub}(X)$ defined by $S_X(\phi) =$ (*the subobject of* X *represented by the pull-back of* $t : 1 \to \Omega$ *along* ϕ).
For a morphism $f : Y \to X$ and a monomorphism $i : Z \to X$ the following diagrams commute.

$$\begin{array}{ccc} \mathcal{E}(X,\Omega) & \xrightarrow{f^*} & \mathcal{E}(Y,\Omega) \\ \downarrow{\scriptstyle S_X} & & \downarrow{\scriptstyle S_Y} \\ \mathrm{Sub}(X) & \xrightarrow{f^*} & \mathrm{Sub}(Y) \end{array} \qquad \begin{array}{ccc} \mathcal{E}(Z,\Omega) & \xrightarrow{i_\sharp} & \mathcal{E}(X,\Omega) \\ \downarrow{\scriptstyle S_Z} & & \downarrow{\scriptstyle S_X} \\ \mathrm{Sub}(Z) & \xrightarrow{i_!} & \mathrm{Sub}(X) \end{array}$$

Proposition 4.1.5 *Let* $j : \Omega \to \Omega$ *be a topology on a topos* $\mathcal{E}$. *Define* $cl^j_X : \mathrm{Sub}(X) \to \mathrm{Sub}(X)$ *by* $cl^j_X = S_X j_* S_X^{-1}$. *Then,* $(cl^j_X : \mathrm{Sub}(X) \to \mathrm{Sub}(X))_{X \in \mathrm{Ob}\,\mathcal{C}}$ *is a universal closure operation.*

Proof. Let Y be a subobject of X represented by a monomorphism $\sigma : Y \rightarrowtail X$ and $\phi_\sigma : X \to \Omega$ the classifying map of σ. We denote by $\bar{\sigma} : cl_X^j(Y) \rightarrowtail X$ the monomorphism representing $cl_X^j(Y)$. Then, $\bar{\sigma}$ is the pull-back of $\bar{t} : J \rightarrowtail \Omega$ along ϕ_σ by (A.3.1).

Since $jt = t = tid_1$, there is a unique morphism $t' : 1 \to J$ satisfying $\bar{t}t' = t$ and the subobject of Ω represented by $1 \xrightarrow{t} \Omega$ is contained in the one represented by $J \xrightarrow{\bar{t}} \Omega$. Hence $Y = \phi_\sigma^*(1 \xrightarrow{t} \Omega) \subset \phi_\sigma^*(J \xrightarrow{\bar{t}} \Omega) = cl_X^j(Y)$.

$cl_X^j cl_X^j = cl_X^j$ is obvious from $jj = j$.

For a morphism $f : Z \to X$, we have $f^* cl_X^j = f^* S_X j_* S_X^{-1} = S_Z f^* j_* S_X^{-1} = S_Z j_* f^* S_X^{-1} = S_Z j_* S_Z^{-1} f^* = cl_Z^j f^*$.

Let $\tau : Z \rightarrowtail X$ be a subobject of X with classifying map $\phi_\tau : X \to \Omega$. Since $Y \cap Z$ is classified by $\wedge(\phi_\sigma, \phi_\tau)$ by (A.15.1), $cl_X^j(Y \cap Z)$ is classified by $j \wedge (\phi_\sigma, \phi_\tau) = \wedge(j\phi_\sigma, j\phi_\tau)$, which classifies $cl_X^j(Y) \cap cl_X^j(Z)$. Thus we have $cl_X^j(Y \cap Z) = cl_X^j(Y) \cap cl_X^j(Z)$. In particular, if $Z \subset Y$, $cl_X^j(Z) = cl_X^j(Y) \cap cl_X^j(Z) \subset cl_X^j(Y)$. □

Proposition 4.1.6 *Let $\mathcal{E}$ be a topos with a universal closure operation $(cl_X : \mathrm{Sub}(X) \to \mathrm{Sub}(X))_{X \in \mathrm{Ob}\,\mathcal{E}}$. Let $j : \Omega \to \Omega$ be the classifying map of $cl_\Omega(1 \xrightarrow{t} \Omega) \in \mathrm{Sub}(\Omega)$. Then, j is a topology on $\mathcal{E}$.*

Proof. We set $J = cl_\Omega(1)$. Let $\bar{t} : J \rightarrowtail \Omega$ be the monomorphism representing the subobject J of Ω. Then, $\bar{t}$ is a pull-back of t along j and we have $j^*(1 \xrightarrow{t} \Omega) = (J \xrightarrow{\bar{t}} \Omega)$ in $\mathrm{Sub}(\Omega)$. It follows that $j^*(1) = J = cl_\Omega(1) = cl_\Omega cl_\Omega(1) = cl_\Omega j^*(1) = j^* cl_\Omega(1) = j^* j^*(1) = (jj)^*(1)$. This implies that both j and jj classify $J \xrightarrow{\bar{t}} \Omega$. Therefore we have $jj = j$.

Since $1 \subset cl_\Omega(1)$, the pull-back of $\bar{t}$ along t is the identity morphism of 1. By the commutativity of the following diagram, we have $jt = t$.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "id_1"] & J \arrow[r] \arrow[d, "\bar{t}"] & 1 \arrow[d, "t"] \\ 1 \arrow[r, "t"] & \Omega \arrow[r, "j"] & \Omega \end{tikzcd}$$

By (4.1.1), $J \times J \rightarrowtail \Omega \times \Omega$ is classified by $\wedge(j \times j)$. On the other hand, by (4.1.4) and the following pull-back diagrams,

$$\begin{tikzcd} 1 \times \Omega \arrow[r, "\mathrm{pr}_1"] \arrow[d, "t \times id_\Omega"] & 1 \arrow[d, "t"] \\ \Omega \times \Omega \arrow[r, "\mathrm{pr}_1"] & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Omega \times 1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "id_\Omega \times t"] & 1 \arrow[d, "t"] \\ \Omega \times \Omega \arrow[r, "\mathrm{pr}_2"] & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "(t, id_1)"] \arrow[d, "(id_1, t)"] & \Omega \times 1 \arrow[d, "id_\Omega \times t"] \\ 1 \times \Omega \arrow[r, "t \times id_\Omega"] & \Omega \times \Omega \end{tikzcd}$$

$J \times J = \mathrm{pr}_1^*(J) \cap \mathrm{pr}_2^*(J) = cl_{\Omega\times\Omega}\mathrm{pr}_1^*(1) \cap cl_{\Omega\times\Omega}\mathrm{pr}_2^*(1) = cl_{\Omega\times\Omega}(\mathrm{pr}_1^*(1) \cap \mathrm{pr}_2^*(1)) = cl_{\Omega\times\Omega}((1 \times \Omega) \cap (\Omega \times 1)) = cl_{\Omega\times\Omega}(1 \xrightarrow{(t,t)} \Omega\times\Omega) = cl_{\Omega\times\Omega}\wedge^*(1 \xrightarrow{t} \Omega) = \wedge^*(J) = \wedge^* j^*(1) = (j\wedge)^*(1)$. Hence $j\wedge$ also classifies $\mathrm{pr}_1^*(J) \cap \mathrm{pr}_2^*(J)$. Thus we have $\wedge(j \times j) = j\wedge$. □

Proposition 4.1.7 *The correspondence $j \mapsto (cl_X^j : \mathrm{Sub}(X) \to \mathrm{Sub}(X))_{X \in \mathrm{Ob}\,\mathcal{E}}$ gives a bijection from the set of topologies on a topos $\mathcal{E}$ to the set of universal closure operations on $\mathcal{E}$.*

Proof. We show that $(cl_X : \mathrm{Sub}(X) \to \mathrm{Sub}(X))_{X \in \mathrm{Ob}\,\mathcal{E}} \mapsto$ (*the classifying map of* $cl_\Omega(1)$) is the inverse correspondence. By the definition of cl_X^j, $cl_\Omega^j(1)$ is the subobject of Ω classified by j. Conversely, suppose that a universal closure operation $(cl_X : \mathrm{Sub}(X) \to \mathrm{Sub}(X))_{X \in \mathrm{Ob}\,\mathcal{E}}$ on $\mathcal{E}$ is given and let j be the classifying map of J. We note that, if Y is a subobject of X classified by $\phi : X \to \Omega$, we have $cl_X(Y) = cl_X \phi^*(1) = \phi^* J = \phi^* j^*(1) = (j\phi)^*(1)$. Thus $cl_X(Y)$ is classified by $j\phi$. □

Let j be a topology on a topos $\mathcal{E}$ and X an object of $\mathcal{E}$. We say that $Y \in \mathrm{Sub}(X)$ is (j-)closed (resp. dense) if $cl_X^j(Y) = X$ (resp. $cl_X^j(Y) = Y$) in $\mathrm{Sub}(X)$. The next result follows from the definition.

Proposition 4.1.8 *Let j be a topology on $\mathcal{E}$ and Y an subobject of X classified by $\phi : X \to \Omega$. Then Y is closed (resp. dense) if and only if $j\phi = \phi$ (resp. ϕ factors through $J \xrightarrow{\bar{t}} \Omega$).*

Let D be a subobject of Ω represented by a monomorphism $D \xrightarrow{d} \Omega$. We denote by Ξ_D the class of monomorphisms whose classifying map factors through $D \xrightarrow{d} \Omega$. It is obvious that Ξ_D is stable under pull-backs, that is, for a morphism $f : Y \to X$, the pull-back of $(\sigma : Z \rightarrowtail X) \in \Xi_D$ along f is contained in Ξ_D.

Proposition 4.1.9 *Let D be a subobject of Ω such that $1 \xrightarrow{t} \Omega \in \Xi_D$. Then, Ξ_D is stable under the formation of push-outs.*

Proof. Let
$$\begin{tikzcd} X \arrow[r, tail, "\sigma"] \arrow[d, "f"] & Z \arrow[d, "g"] \\ Y \arrow[r, "\tau"] & W \end{tikzcd}$$
be a push-out diagram with $\sigma \in \Xi_D$. This square is a pull-back and τ is a monomorphism by (3.2.8). Let $\phi : W \to \Omega$ be the classifying map of $Y \overset{\tau}{\rightarrowtail} W$, then $\phi\tau$ factors through t, hence factors through $D \overset{d}{\rightarrowtail} \Omega$, that is, $\phi\tau = d\psi$ for some $\psi : Y \to D$. On the other hand, ϕg classifies $X \overset{\sigma}{\rightarrowtail} Z$, hence it also factors through d that is, $\phi g = d\psi'$ for some $\psi' : Z \to D$. Since the above square is a push-out and $d\psi f = \phi\tau f = \phi g\sigma = d\psi'\sigma$ implies $\psi f = \psi'\sigma$, we have $\alpha : W \to D$ satisfying $\alpha\tau = \psi$ and $\alpha g = \psi'$. Therefore we have $\phi\tau = d\alpha\tau$ and $\phi g = d\alpha g$ which imply $\phi = d\alpha$ namely, $\tau \in \Xi_D$. $\square$

We note that $(1 \xrightarrow{t} \Omega) \in \Xi_D$ if and only if Ξ_D contains every isomorphism in $\mathcal{E}$. In fact, let $\phi : X \to \Omega$ be the classifying map of an isomorphism $X' \overset{f}{\rightarrowtail} X$, then ϕ factors through $1 \xrightarrow{t} \Omega$. If $(1 \xrightarrow{t} \Omega) \in \Xi_D$, $t = dt'$ for some $t' : 1 \to D$ and it follows that ϕ factors through t. Hence Ξ_D contains f. Conversely, $1 \overset{id_1}{\rightarrowtail} 1 \in \Xi_D$ implies that t factors through d.

Proposition 4.1.10 *Let J be a subobject of Ω classified by $j : \Omega \to \Omega$. Then, J is a topology if and only if Ξ_J contains all isomorphisms and satisfies "$\sigma\tau \in \Xi_J \Leftrightarrow \sigma \in \Xi_J$ and $\tau \in \Xi_J$".*

Proof. Suppose that j is a topology, then Ξ_J is the class of j-dense monomorphisms by (4.1.8). By 3) of (4.1.4), $(\sigma : Y \rightarrowtail X) \in \Xi_J$ and $(\tau : Z \rightarrowtail Y) \in \Xi_J$ imply $\sigma\tau \in \Xi_J$. If $\sigma\tau \in \Xi_J$, that is, $cl^j_X\sigma_!(Z) = X$, then we have $cl^j_Y(Z) = \sigma^* cl^j_X\sigma_!(Z) = \sigma^*(X) = Y$ and $cl^j_X(Y) \supset cl^j_X\sigma_!(Z) = X$. Hence Z is dense in Y and Y is dense in X, thus $\sigma, \tau \in \Xi_J$.

Conversely, suppose that Ξ_J satisfies the conditions. Since Ξ_J contains all isomorphisms, $t : 1 \to \Omega$ factors through $\bar{t} : J \rightarrowtail \Omega$. Let $\tilde{t} : 1 \to J$ the morphism satisfying $\bar{t}\tilde{t} = t$. Then, the left square of the following diagram is a pull-back.

$$\begin{tikzcd} 1 \arrow[r, "\tilde{t}"] \arrow[d, "id_1"] & J \arrow[r] \arrow[d, "\bar{t}"] & 1 \arrow[d, "t"] \\ 1 \arrow[r, "t"] & \Omega \arrow[r, "j"] & \Omega \end{tikzcd}$$

In fact, if $f : X \to J$ is a morphism such that $\bar{t}f = to$ $(o : X \to 1)$, then $\bar{t}\tilde{t}o = to = \bar{t}f$. Since $\bar{t}$ is a monomorphism, we have $\tilde{t}o = f$. By (A.3.1), the outer rectangle is a pull-back and this implies that jt is a classifying map of $id_1 : 1 \to 1$. Therefore $jt = t$

In order to show $jj = j$, it suffices to show that the left square of the following diagram is a pull-back, where $o : J \to 1$ is the unique morphism.

$$\begin{tikzcd} J \arrow[r, "\tilde{t}o"] \arrow[d, "\bar{t}"] & J \arrow[r] \arrow[d, "\bar{t}"] & 1 \arrow[d, "t"] \\ \Omega \arrow[r, "j"] & \Omega \arrow[r, "j"] & \Omega \end{tikzcd}$$

Then, since the right square is a pull-back, jj also classifies $J \overset{\bar{t}}{\rightarrowtail} \Omega$. Since $\bar{t}\tilde{t} = t$, $\bar{t}\tilde{t}o = to = j\bar{t}$ and the square is commutative. Suppose that $\alpha : X \to \Omega$ and $\beta : X \to J$ are morphisms such that $j\alpha = \bar{t}\beta$. Let $Y \overset{\sigma}{\rightarrowtail} X$ be the pull-back of $\bar{t}$ along α. Then, we have the following diagrams, where each square is a pull-back.

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\tau"] & 1 \arrow[d, "\tilde{t}"] & \\ Y \arrow[r, "\rho"] \arrow[d, "\sigma"] & J \arrow[r, "o"] \arrow[d, "\bar{t}"] & 1 \arrow[d, "t"] \\ X \arrow[r, "\alpha"] & \Omega \arrow[r, "j"] & \Omega \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r] \arrow[d, "\tau"] & 1 \arrow[d, "\tilde{t}"] \arrow[dr, "t"] & \\ Y \arrow[r, "\rho"] & J \arrow[r, "\bar{t}"] & \Omega \end{tikzcd}$$

Since the outer rectangle of the above right diagram is a pull-back, $Z \overset{\tau}{\rightarrowtail} Y$ is classified by $\bar{t}\rho$, hence $\tau \in \Xi_J$. Similarly, $Y \overset{\sigma}{\rightarrowtail} X$ is classified by $j\alpha = \bar{t}\beta$, hence $\sigma \in \Xi_J$. It follows from the assumption that $\sigma\tau \in \Xi_J$. Since $Z \overset{\sigma\tau}{\rightarrowtail} X$ is classified by α, α factors through $\bar{t}$, namely, $\alpha = \bar{t}\gamma$ for some $\gamma : X \to J$. Then, $\bar{t}\beta = j\alpha = j\bar{t}\gamma = \bar{t}\tilde{t}o\gamma$ and since $\bar{t}$ is a monomorphism, we have $\beta = \tilde{t}o\gamma$.

Since each square of the following diagrams is cartesian by (A.3.1) and the preceding argument, $\bar{t}\mathrm{pr}_1$ classifies $\tilde{t} \times id_\Omega$ and $\bar{t}\mathrm{pr}_2$ classifies $id_\Omega \times \tilde{t}$, hence $id_\Omega \times \tilde{t}, \tilde{t} \times id_\Omega \in \Xi_J$. Therefore $id_1 \times \tilde{t}, \tilde{t} \times id_J \in \Xi_J$ and it follows from the assumption that $\tilde{t} \times \tilde{t} \in \Xi_J$.

$$\begin{tikzcd}
1 \times \Omega \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\tilde{t}\times id_\Omega"] & 1 \arrow[r, "id_1"] \arrow[d, "\tilde{t}"] & 1 \arrow[d, "t"] \\
J \times \Omega \arrow[r, "\mathrm{pr}_1"] & J \arrow[r, "\bar{t}"] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega \times 1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "id_\Omega\times\tilde{t}"] & 1 \arrow[r, "id_1"] \arrow[d, "\tilde{t}"] & 1 \arrow[d, "t"] \\
\Omega \times J \arrow[r, "\mathrm{pr}_2"] & J \arrow[r, "\bar{t}"] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \times 1 \arrow[r, "id_1\times\tilde{t}"] \arrow[d, "\tilde{t}\times id_1"] & 1 \times J \arrow[r, "id_1\times\bar{t}"] \arrow[d, "\tilde{t}\times id_J"] & 1 \times \Omega \arrow[d, "\tilde{t}\times id_\Omega"] \\
J \times 1 \arrow[r, "id_J\times\tilde{t}"] \arrow[d, "\bar{t}\times id_1"] & J \times J \arrow[r, "id_J\times\bar{t}"] \arrow[d, "\bar{t}\times id_J"] & J \times \Omega \arrow[d, "\bar{t}\times id_\Omega"] \\
\Omega \times 1 \arrow[r, "id_\Omega\times\tilde{t}"] & \Omega \times J \arrow[r, "id_\Omega\times\bar{t}"] & \Omega \times \Omega
\end{tikzcd}$$

We note that $1 \xrightarrow{\Delta} 1 \times 1$ is an isomorphism, hence belongs to Ξ_J and we set $\sigma = (\tilde{t} \times \tilde{t})\Delta : 1 \rightarrowtail J \times J$. Since Ξ_J contains isomorphisms and it is closed under composition, we have $\sigma \in \Xi_J$. Each square of the following diagram is cartesian.

$$\begin{tikzcd}
1 \arrow[r, "\Delta"] \arrow[d, "(id_1\times\tilde{t})\Delta"] & 1 \times 1 \arrow[r] \arrow[d, "id_1\times t"] & 1 \arrow[r] \arrow[dd, "(t,t)"] & 1 \arrow[dd, "t"] \\
1 \times J \arrow[r, "id_1\times\bar{t}"] \arrow[d, "\tilde{t}\times id_J"] & 1 \times \Omega \arrow[d, "\tilde{t}\times id_\Omega"] & & \\
J \times J \arrow[r, "id_J\times\bar{t}"] & J \times \Omega \arrow[r, "\bar{t}\times id_\Omega"] & \Omega \times \Omega \arrow[r, "\wedge"] & \Omega
\end{tikzcd}$$

Then, it follows from the above diagram that σ is classified by $\wedge(\bar{t} \times \bar{t})$, hence there exists a morphism $\mu : J \times J \to J$ such that $\wedge(\bar{t} \times \bar{t}) = \bar{t}\mu$.

We show that the left square of the following diagram is a pull-back. Then, since the right square is also a pull-back, it follows that $j\wedge$ is the classifying map of $\bar{t} \times \bar{t} : J \times J \rightarrowtail \Omega \times \Omega$. On the other hand, $\wedge(j \times j)$ also classifies $\bar{t} \times \bar{t} : J \times J \rightarrowtail \Omega \times \Omega$. Therefore we have $\wedge(j \times j) = j\wedge$.

$$\begin{tikzcd}
J \times J \arrow[r, "\mu"] \arrow[d, "\bar{t}\times\bar{t}"] & J \arrow[r] \arrow[d, "\bar{t}"] & 1 \arrow[d, "t"] \\
\Omega \times \Omega \arrow[r, "\wedge"] & \Omega \arrow[r, "j"] & \Omega
\end{tikzcd}$$

Let $(\phi_1, \phi_2) : X \to \Omega \times \Omega$ and $f : X \to J$ be morphisms such that $\wedge(\phi_1, \phi_2) = \bar{t}b$. We denote by $Y_i \overset{\tau_i}{\rightarrowtail} X$ the monomorphism classified by ϕ_i and $Y_1 \cap Y_2 \overset{\tau_i'}{\rightarrowtail} Y_i$ denotes the inclusion morphism. Since $\tau_i\tau_i' : Y_1 \cap Y_2 \rightarrowtail X$ is classified by $\wedge(\phi_1, \phi_2) = \bar{t}f$ by (A.15.1), we have $\tau_i\tau_i' \in \Xi_J$. It follows from the assumption that $\tau_i \in \Xi_J$, thus $\phi_i = \bar{t}\psi_i$ for some $\psi_i : X \to J$. Therefore we have $(\bar{t} \times \bar{t})(\psi_1, \psi_2) = (\phi_1, \phi_2)$. Moreover, $\bar{t}f = \wedge(\phi_1, \phi_2) = \wedge(\bar{t} \times \bar{t})(\psi_1, \psi_2) = \bar{t}\mu(\psi_1, \psi_2)$ and since $\bar{t}$ is a monomorphism we have $f = \mu(\psi_1, \psi_2)$. □

Lemma 4.1.11 *Let $\sigma : X \rightarrowtail Z$ and $\tau : Y \rightarrowtail W$ be monomorphisms in a topos such that σ is dense and τ is closed. Suppose that we have morphisms $f : X \to Y$ and $g : Z \to W$ satisfying $g\sigma = \tau f$. Then, there exists a unique morphism $h : Z \to Y$ such that $g = \tau h$.*

Proof. Consider the pull-back square

$$\begin{tikzcd}
g^*(Y) \arrow[r, "\bar{g}"] \arrow[d, "g^*(\tau)"] & Y \arrow[d, "\tau"] \\
Z \arrow[r, "g"] & W
\end{tikzcd}.$$

Then we have $(X \overset{\sigma}{\rightarrowtail} Z) \subset g^*(Y \overset{\tau}{\rightarrowtail} W)$ in $\mathrm{Sub}(Z)$. Since $g^*(Y)$ is closed in $\mathrm{Sub}(Z)$ by (4.1.4) and X is dense in $\mathrm{Sub}(Z)$, it follows that $g^*(Y) = Z$, that is, the pull-back $g^*(\tau) : g^*(Y) \rightarrowtail Z$ of τ is an isomorphism. We set $h = \bar{g}g^*(\tau)^{-1} : Z \to Y$. Then, $\tau h = \tau\bar{g}g^*(\tau)^{-1} = g$. Since τ is a monomorphism, h is unique. □

4.2 Sheaves

Definition 4.2.1 *Let j be a topology in a topos $\mathcal{E}$ and F an object of $\mathcal{E}$.*

1) We say that F is j-separated if, for any j-dense monomorphism $\sigma : Y \rightarrowtail X$, $\sigma^* : \mathcal{E}(X, F) \to \mathcal{E}(Y, F)$ is injective.

2) We say that F is a j-sheaf if, for any j-dense monomorphism $\sigma : Y \rightarrowtail X$, $\sigma^* : \mathcal{E}(X, F) \to \mathcal{E}(Y, F)$ is bijective.

We write $\mathrm{Sh}_j(\mathcal{E})$ for the full subcategory of $\mathcal{E}$ whose objects are sheaves.

Lemma 4.2.2 *$\mathrm{Sh}_j(\mathcal{E})$ has finite limits and the inclusion functor $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$ preserves them.*

Proof. Let $(F \xrightarrow{p_i} F_i)_{i\in I}$ be a limiting cone in $\mathcal{E}$ of a finite diagram $(F_j \xrightarrow{f_i^j} F_i)_{i,j\in I}$in $\mathrm{Sh}_j(\mathcal{E})$. We claim that F is a j-sheaf. For any j-dense monomorphism $\sigma : Y \rightarrowtail X$ in $\mathcal{E}$, $\sigma^* : \mathcal{E}(X, F_i) \to \mathcal{E}(Y, F_i)$ $(i \in I)$ are bijective. Since $(\mathcal{E}(X, F) \xrightarrow{p_{i*}} \mathcal{E}(X, F_i))_{i\in I}$ and $(\mathcal{E}(Y, F) \xrightarrow{p_{i*}} \mathcal{E}(Y, F_i))_{i\in I}$ are limiting cones of $(\mathcal{E}(X, F_j) \xrightarrow{(f_i^j)_*} \mathcal{E}(X, F_i))_{i,j\in I}$ and $(\mathcal{E}(Y, F_j) \xrightarrow{(f_i^j)_*} \mathcal{E}(Y, F_i))_{i,j\in I}$ respectively, and $(\sigma^* : \mathcal{E}(X, F_i) \to \mathcal{E}(Y, F_i))_{i\in I}$ is an isomorphism of the these diagrams, $\sigma^* : \mathcal{E}(X, F) \to \mathcal{E}(Y, F)$ is bijective. □

Proposition 4.2.3 *If F is a sheaf (resp. separated object), so is F^X for any object X of $\mathcal{E}$.*

Proof. Let $\sigma : Z \rightarrowtail Y$ be a dense monomorphism. Since $\sigma \times id_X : Z \times X \rightarrowtail Y \times X$ is a pull-back of σ along $\mathrm{pr}_1 : Y \times X \to Y$, it is also dense by (4.1.4). Then, by (A.16.2) and the assumption, $\sigma^* : \mathcal{E}(Y, F^X) \to \mathcal{E}(Z, F^X)$ is bijective (resp. injective). □

By (4.2.2) and (4.2.3), we have the following result.

Corollary 4.2.4 *$\mathrm{Sh}_j(\mathcal{E})$ is cartesian closed and the inclusion functor $\mathrm{Sh}_j(\mathcal{E}) \to \mathcal{E}$ preserves exponentials.*

Lemma 4.2.5 *1) A subobject of a separated object is separated.*

2) A closed subobject of a sheaf is a sheaf.

3) If G is a separated object and F is a subobject of G which is a sheaf, then F is closed in G.

Proof. 1) Let $\sigma : Y \rightarrowtail X$ and $\tau : F \rightarrowtail G$ be monomorphisms such that σ is dense and G is separated. Since the vertical maps and lower horizontal map in the following commutative diagram are injective, so is the upper map. Hence F is separated.

$$\begin{array}{ccc} \mathcal{E}(X,F) & \xrightarrow{\sigma^*} & \mathcal{E}(Y,F) \\ \downarrow{\scriptstyle \tau_*} & & \downarrow{\scriptstyle \tau_*} \\ \mathcal{E}(X,G) & \xrightarrow{\sigma^*} & \mathcal{E}(Y,G) \end{array}$$

2) Let $\tau : F \to G$ is a closed monomorphism such that G is a sheaf. For any dense monomorphism $\sigma : Y \rightarrowtail X$ and $f \in \mathcal{E}(Y, F)$, there exists $g \in \mathcal{E}(X, G)$ such that $g\sigma = \tau f$. It follows from (4.1.11) that there exists $h \in \mathcal{E}(X, F)$ satisfying $\tau h = g$. Since τ is a monomorphism and $\tau h\sigma = g\sigma = \tau f$, we have $h\sigma = f$, hence $\sigma^* : \mathcal{E}(X, F) \to \mathcal{E}(Y, F)$ is surjective. By 1), F is separated, thus $\sigma^* : \mathcal{E}(X, F) \to \mathcal{E}(Y, F)$ is injective.

3) Let $\overline{F} = cl_G^j(F)$ be the closure of F in G. Since $i : F \rightarrowtail \overline{F}$ is dense and F is a sheaf, there exists a unique $r : \overline{F} \to F$ such that $ri = id_F$. Then, $iri = i$ and $\overline{F}$ is separated by 1), hence $ir = id_{\overline{F}}$. Thus i is an isomorphism and $F = \overline{F}$ in $\mathrm{Sub}(G)$, that is, F is closed in G. □

Lemma 4.2.6 *Ω_j is a sheaf.*

Proof. Let us denote by $\mathrm{Cl}(F)$ the set of closed subobjects of F. By (4.1.8), $\Psi_F : \widehat{\mathcal{C}}(F, \Omega_j) \to \mathrm{Cl}(F)$ defined by $\Psi_F(\phi) = (e\phi)^*(1 \overset{t}{\rightarrowtail} \Omega)$ is a bijection, where $e : \Omega_j \rightarrowtail \Omega$ is the equalizer of $\Omega \underset{id_\Omega}{\overset{j}{\rightrightarrows}} \Omega$. For a morphism $f : F \to G$, define $\mathrm{Cl}(f) : \mathrm{Cl}(G) \to \mathrm{Cl}(F)$ by $\mathrm{Cl}(f)(H) = f^*(H)$. Then,

$$\begin{array}{ccc} \widehat{\mathcal{C}}(G,\Omega_j) & \xrightarrow{f^*} & \widehat{\mathcal{C}}(F,\Omega_j) \\ \downarrow{\scriptstyle \Psi_G} & & \downarrow{\scriptstyle \Psi_F} \\ \mathrm{Cl}(G) & \xrightarrow{\mathrm{Cl}(f)} & \mathrm{Cl}(F) \end{array}$$

commutes. Hence it suffices to show if $F \overset{\sigma}{\rightarrowtail} G$ is a dense monomorphism, $\mathrm{Cl}(\sigma) : \mathrm{Cl}(G) \to \mathrm{Cl}(F)$ is bijective. For $K \in \mathrm{Cl}(F)$, we have $\mathrm{Cl}(\sigma)(cl^j_G(K)) = \sigma^*(cl^j_G(K)) = cl^j_F\sigma^*(K) = K$. For $H \in \mathrm{Cl}(G)$, since F is dense, we have $H = cl^j_G(H) = cl^j_G(H) \cap G = cl^j_G(H) \cap cl^j_G(F) = cl^j_G(H \cap F) = cl^j_G(\mathrm{Cl}(\sigma)(H))$. Therefore $\mathrm{Cl}(\sigma)$ has a two sided inverse $K \mapsto cl^j_G(K)$. □

Proposition 4.2.7 *Ω_j is a subobject classifier of $\mathrm{Sh}_j(\mathcal{E})$.*

Proof. Let F be a sheaf and G a subobject of F which is a sheaf. By (4.2.5), G is closed in F and the classifying map $\phi_G : F \to \Omega$ uniquely factors through $e : \Omega_j \to \Omega$. Conversely, a morphism $\phi : F \to \Omega_j$ defines a closed subfunctor of F, which is a sheaf by (4.2.5). □

By (4.2.2), (4.2.4) and (4.2.7), we have the following result.

Theorem 4.2.8 *$\mathrm{Sh}_j(\mathcal{E})$ is a topos.*

Proposition 4.2.9 *Let F be an object of $\mathcal{E}$. The following conditions are equivalent.*

(1) F is separated.
(2) The diagonal morphism $\Delta : F \rightarrowtail F \times F$ is closed.
(3) There exists a monomorphism $F \rightarrowtail G$, where G is a sheaf.

Proof. (1)⇒(2): Let $\overline{F} \xrightarrow{(a,b)} F \times F$ be the closure of $F \overset{\Delta}{\rightarrowtail} F \times F$. Since the inclusion morphism $i : F \rightarrowtail \overline{F}$ is dense and $(a, b)i = \Delta$, that is, $ai = bi = id_F$, it follows from the assumption that $a = b$. This means that (a, b) factors through Δ. Hence $\overline{F} = F$ in $\mathrm{Sub}(F \times F)$.

(2)⇒(3): Since Δ is closed, its classifying map $\delta : F \times F \to \Omega$ factors through $e : \Omega_j \to \Omega$, namely, $\delta = e\tilde{\delta}$ for some $\tilde{\delta} : F \times F \to \Omega_j$. Let us denote by $\tilde{\{\}} : F \to \Omega^F_j$ the exponential transpose of $\tilde{\delta}$, then we have $\{\} = e^F\tilde{\{\}}$. Since the singleton map $\{\} : F \to \Omega^F$ is a monomorphism, so is $\tilde{\{\}}$. Moreover, Ω^F_j is a sheaf by (4.2.3) and (4.2.6).

(3)⇒(1): Straightforward from (4.2.5). □

4.3 Grothendieck topos

Let $\mathcal{C}$ be a $\mathcal{U}$-small category. For presheaves F, G of $\mathcal{U}$-small set on $\mathcal{C}$, we define a presheaf F^G on $\mathcal{C}$ by $F^G(U) = \widehat{\mathcal{C}}(h_U \times G, F)$ for $U \in \mathrm{Ob}\,\mathcal{C}$ and $F^G(f : U \to V) = ((h_f \times id_G)^* : \widehat{\mathcal{C}}(h_V \times G, F) \to \widehat{\mathcal{C}}(h_U \times G, F))$ for $(f : U \to V) \in \mathrm{Mor}\,\mathcal{C}$. For morphisms $\alpha : F \to F'$ and $\beta : G' \to G$ of $\mathcal{C}$, we define morphisms of presheaves $\alpha^G : F^G \to F'^G$ and $F^\beta : F^G \to F^{G'}$ by $\alpha^G_U = \alpha_* : \widehat{\mathcal{C}}(h_U \times G, F) \to \widehat{\mathcal{C}}(h_U \times G, F')$ and $F^\beta_U = (id_U \times \beta)^*$, respectively. Thus we have a functor $\widehat{\mathcal{C}}^{op} \times \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ which maps $(G, F) \in \mathrm{Ob}\,(\widehat{\mathcal{C}}^{op} \times \widehat{\mathcal{C}})$ to F^G.

Define morphims of presheaves $\iota_F : F \to (F \times G)^G$ and $ev_F : F^G \times G \to F$ as follows. Let U be an object of $\mathcal{C}$. For $V \in \mathrm{Ob}\,\mathcal{C}$, $x \in F(U)$ and $f \in h_U(V)$, $y \in G(V)$, we put $((\iota_F)_U(x))_V(f, y) = (F(f)(x), y)$. For $\varphi \in \widehat{\mathcal{C}}(h_U \times G, F)$ and $x \in G(U)$, we put $(ev_F)_U(\varphi, x) = \varphi_U(id_U, x)$.

Proposition 4.3.1 *$(ev_F)^G\iota_{F^G} = id_{F^G}$ and $ev_{F\times G}(\iota_F \times id_G) = id_{F\times G}$ hold. Hence the functor $\widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ defined by $F \mapsto F^G$ and $\alpha \mapsto \alpha^G$ is a right adjoint of the functor $\widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ defined by $F \mapsto F \times G$ and $\alpha \mapsto \alpha \times id_G$, that is, $\widehat{\mathcal{C}}$ is cartesian closed.*

Proof. For $U, V \in \mathrm{Ob}\,\mathcal{C}$, $x \in F^G(U) = \widehat{\mathcal{C}}(h_U \times G, F)$ and $(f, y) \in h_U(V) \times G(V)$, we have

$$((\iota_{F^G})_U(x))_V(f, y) = (F^G(f)(x), y) = (x(h_f \times id_G), y) \in \widehat{\mathcal{C}}(h_V \times G, F) \times G(V) = (F^G \times G)(V).$$

Hence the following equality holds, which shows $(ev_F)^G\iota_{F^G} = id_{F^G}$.

$$((ev^G_F)_U(\iota_{F^G})_U(x))_V(f, y) = (ev_F(\iota_{F^G})_U(x))_V(f, y) = (ev_F)_V(x(h_f \times id_G), y) = (x(h_f \times id_G))_V(id_V, y) = x_V(f, y)$$

For $U \in \mathrm{Ob}\,\mathcal{C}$, $(x, y) \in F(U) \times G(U)$, the following equality holds.

$$(ev_{F\times G})_U(\iota_F \times id_G)_U(x, y) = (ev_{F\times G})_U((\iota_F)_U(x), y) = ((\iota_F)_U(x))_U(id_U, y) = (F(id_U)(x), y) = (x, y)$$

Thus $ev_{F\times G}(\iota_F \times id_G) = id_{F\times G}$ follows. □

Let Ω be a presheaf on $\mathcal{C}$ defined by $\Omega(X) = (\textit{the set of all sieves on } X)$, $\Omega(f)(R) = h_f^{-1}(R)$ for $f : Y \to X$, $R \subset h_X$. Since $\mathcal{C}$ is $\mathcal{U}$-small, so is $\Omega(X)$. We denote 1 the terminal object of $\widehat{\mathcal{C}}$ given by $1(X) = \{h_X\}$ and define a morphism $t : 1 \to \Omega$ by $t_X(h_X) = h_X$.

Lemma 4.3.2 *For $R \in \Omega(X)$, let $i : R \hookrightarrow h_X$ be the inclusion morphism. Then,*

$$\begin{tikzcd} R \arrow[r] \arrow[d, "i"] & 1 \arrow[d, "t"] \\ h_X \arrow[r, "\theta_\Omega(R)"] & \Omega \end{tikzcd}$$

is a pull-back square.

Proof. It suffices to show that

$$\begin{tikzcd} R(Y) \arrow[r] \arrow[d, "i_Y"] & 1 \arrow[d, "t_Y"] \\ h_X(Y) \arrow[r, "(\theta_\Omega(R))_Y"] & \Omega(Y) \end{tikzcd}$$

is a pull-back square. $(\theta_\Omega(R))_Y^{-1}(h_Y) = \{f \in h_X(Y) |\, h_f^{-1}(R) = h_X\}$ and $h_f^{-1}(R) = h_X$ holds if and only if $f \in R$. □

Proposition 4.3.3 *Ω is a subobject classifier of $\widehat{\mathcal{C}}$.*

Proof. Let G be a subfunctor of F and $i : G \hookrightarrow F$ the inclusion morphism. For $x \in F(X)$, the pull-back $h_X \times_F G \to h_X$ of $i : G \hookrightarrow F$ along $\theta_F(x) : h_X \to F$ is regarded as an inclusion morphism. Define ϕ_G by $(\phi_G)_X(x) = h_X \times_F G = \{f \in h_X |\, F(f)(x) \in G(\mathrm{dom}(f))\}$ for an object X of $\mathcal{C}$, then, we have a pull-back square

$$\begin{tikzcd} G(X) \arrow[r] \arrow[d, "i_X"] & 1(X) \arrow[d, "t_X"] \\ F(X) \arrow[r, "(\phi_G)_X"] & \Omega(X) \end{tikzcd} .$$

In fact, $(\phi_G)_X^{-1}(h_X)$ is the subset of $F(X)$ consisting of element x such that the pull-back of i along $\theta_F(x) : h_X \to F$ is an isomorphism and this coincides with $\{x \in F(X) |\, \theta_F(x)(h_X) \subset G\} = G(X)$. Suppose that $\psi : F \to \Omega$ also satisfies the same condition as ϕ_G and that $\psi \neq \phi_G$. Then, $\psi_X(x) \neq (\phi_G)_X(x)$ for some $X \in \mathrm{Ob}\,\mathcal{C}$, $x \in F(X)$ and the pull-back of t along ψ is $i : G \hookrightarrow F$. Note that $\theta_\Omega(\psi_X(x)) = \psi\theta_F(x) \neq \phi_G\theta_F(x) = \theta_\Omega((\phi_G)_X(x))$ holds and that by (4.3.2), there are pull-back squares

$$\begin{tikzcd} \psi_X(x) \arrow[r] \arrow[d] & 1 \arrow[d, "t"] \\ h_X \arrow[r, "\psi\theta_F(x)"] & \Omega \end{tikzcd} \qquad \begin{tikzcd} (\phi_G)_X(x) \arrow[r] \arrow[d] & 1 \arrow[d, "t"] \\ h_X \arrow[r, "\phi_G\theta_F(x)"] & \Omega \end{tikzcd} .$$

However, the inclusion morphisms $\psi_X(x) \hookrightarrow h_X$ and $(\phi_G)_X(x) \hookrightarrow h_X$ are both pull-backs of i along $\theta_F(x)$ and this contradicts $\psi_X(x) \neq (\phi_G)_X(x)$. □

The classifying map of a sieve R on X is $\theta_\Omega(R) : h_X \to \Omega$ by (4.3.2).

Proposition 4.3.4 *Let F be a presheaf, G a subfunctor of F and $f : H \to F$ a morphism of presheaves. The classifying map of a subfunctor $f^{-1}(G)$ of H is given by $\phi_G f$.*

Proof. The result follows from the pull-back squares below.

$$\begin{tikzcd} f^{-1}(G) \arrow[r] \arrow[d] & G \arrow[d, "i"] \\ H \arrow[r, "f"] & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r] \arrow[d, "i"] & 1 \arrow[d, "t"] \\ F \arrow[r, "\phi_G"] & \Omega \end{tikzcd} .$$

□

Proposition 4.3.5 *Let $\wedge : \Omega \times \Omega \to \Omega$ be the classifying map of $(t, t) : 1 \to \Omega \times \Omega$ (See (4.1.1)). Then, $\wedge$ is given by $\wedge_X(R, S) = R \cap S$ for $R, S \in \Omega(X)$.*

Proof. We have $\wedge_X(R, S) = \{f \in h_X \mid (h_f^{-1}(R), h_f^{-1}(S)) = (h_{\mathrm{dom}(f)}, h_{\mathrm{dom}(f)})\}$ by the proof of (4.3.3). Since $h_f^{-1}(R) = h_{\mathrm{dom}(f)}$ holds if and only if $f \in R$, the assertion follows. □

Suppose that for each $X \in \mathrm{Ob}\,\mathcal{C}$, a set $J(X)$ of sieves on X is given. Then, a correspondence $X \mapsto J(X)$ defines a subfunctor of Ω if and only if the condition T2 of (G.1.2) is satisfied.

Proposition 4.3.6 *Let J be a subfunctor of Ω and $j : \Omega \to \Omega$ the classifying map of J.*
1) T1 holds if and only if $jt = t$.
2) T3 holds if and only if $j_X(S) \supset R$ for some $R \in J(X)$ implies $S \in J(X)$.
3) The following statements are equivalent.
i) For any object X and $R, S \in \Omega(X)$, $R \cap S \in J(X)$ holds if and only if $R, S \in J(X)$.
ii) A diagram $\begin{array}{ccc} \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \\ \downarrow{\scriptstyle j \times j} & & \downarrow{\scriptstyle j} \\ \Omega \times \Omega & \xrightarrow{\wedge} & \Omega \end{array}$ *commutes.*
4) J is a Grothendieck topology on $\mathcal{C}$ if and only if j is a topology on a topos $\widehat{\mathcal{C}}$.

Proof. 1) $jt = t$ holds if and only if $j_X(h_X) = h_X$ and this is equivalent to $h_X \in j_X^{-1}(h_X) = J(X)$.

2) We note that j is given by $j_X(R) = \{f \in h_X \mid h_f^{-1}(R) \in J(\mathrm{dom}(f))\}$. Hence T3 holds if and only if, for $S \in \Omega(X)$ and $R \in J(X)$, $R \subset j_X(S)$ implies $S \in J(X)$.

3) Since $j_X(R \cap S) = \{f \in h_X \mid h_f^{-1}(R \cap S) \in J(\mathrm{dom}(f))\}$ and $j_X(R) \cap j_X(S) = \{f \in h_X \mid h_f^{-1}(R), h_f^{-1}(S) \in J(\mathrm{dom}(f))\}$, the diagram commutes if the condition i) holds. Suppose that the diagram commutes. $R \cap S \in J(X)$ if and only if $id_X \in j_X(R \cap S) = j_X(R) \cap j_X(S)$ and this is equivalent to $R, S \in J(X)$.

4) Suppose that J is a Grothendieck topology on $\mathcal{C}$. If $j_X(R) \in J(X)$ for $R \in \Omega(X)$, it follows from T3 and 2) that $R \in J(X)$. Thus $j_X^{-1}(J(X)) \subset J(X)$. Since $j_X^{-1}(J(X)) \supset J(X)$ by T1, we have $j_X^{-1}(J(X)) = J(X)$, namely, $(jj)_X^{-1}(h_X) = j_X^{-1}(h_X)$. This means that jj is also the classifying map of J. Therefore $jj = j$ by the uniqueness. The commutativity of the diagram follows from T4, (4.1.2) and 3). Suppose that $jt = t$ and $jj = j$ hold and that the diagram commutes. We note that the commutativity of the diagram implies that $j_X(R) \subset j_X(S)$ if $R \subset S$. Assume $R \subset j_X(S)$ for $S \in \Omega(X)$ and $R \in J(X)$, then $h_X = j_X(R) \subset (jj)_X(S) = j_X(S)$. Hence $j_X(S) = h_X$ and this means $S \in J(X)$. By 2), T3 follows. □

Remark 4.3.7 *By the above proof, J is a Grothendieck topology if $jt = t$ and $jj = j$ hold and $R \subset S$ implies $j_X(R) \subset j_X(S)$.*

We note that a sieve R on X is dense if and only if $R \in J(X)$ by (4.3.2).

Proposition 4.3.8 *F is a separated presheaf (resp. sheaf) if and only if $\iota^* : \widehat{\mathcal{C}}(G, F) \to \widehat{\mathcal{C}}(H, F)$ is injective (resp. bijective) for any presheaf G and dense subfunctor $H \overset{\iota}{\hookrightarrow} G$.*

Proof. For a morphism $\alpha : h_X \to G$, $\alpha^{-1}(H)$ is dense, hence $\alpha^{-1}(H) \in J(X)$. Consider the category $(h{\downarrow}G)$ and a functor $\Phi : (h{\downarrow}G) \to \widehat{\mathcal{C}}$ as in (A.4.2). Then, $(\Phi\langle Y, \alpha\rangle \overset{\alpha}{\to} G)_{\langle Y,\alpha\rangle \in \mathrm{Ob}(h\downarrow G)}$ is a colimiting cone of a diagram $((h{\downarrow}G), \Phi)$. Hence there is an isomorphism $\varphi : \widehat{\mathcal{C}}(G, F) \to \varprojlim_{(h\downarrow G)} \widehat{\mathcal{C}}(h_Y, F)$ induced by $\alpha^* : \widehat{\mathcal{C}}(G, F) \to \widehat{\mathcal{C}}(h_Y, F) = \widehat{\mathcal{C}}(\Phi\langle Y, \alpha\rangle, F)$. Define a functor $\Psi : (h{\downarrow}G) \to \widehat{\mathcal{C}}$ by $\Psi\langle Y, \alpha\rangle = \alpha^{-1}(H)$ $\Psi(f) =$ (*the restriction of h_f to $\alpha^{-1}(H)$*) for $f : \langle Y, \alpha\rangle \to \langle Z, \beta\rangle$). The inclusion morphism $\iota_\alpha \alpha^{-1}(H) \hookrightarrow h_X$ defines a natural transformation $\eta : \Psi \to \Phi$. Since colimits commute with pull-backs in $\widehat{\mathcal{C}}$, we have a colimiting cone $(\Psi\langle Y, \alpha\rangle \overset{\alpha}{\to} H)_{\langle Y,\alpha\rangle \in \mathrm{Ob}(h\downarrow G)}$ and an isomorphism $\psi : \widehat{\mathcal{C}}(H, F) \to \varprojlim_{(h\downarrow G)} \widehat{\mathcal{C}}(\alpha^{-1}(H), F)$ induced by $\alpha^* : \widehat{\mathcal{C}}(H, F) \to \widehat{\mathcal{C}}(\alpha^{-1}(H), F) = \widehat{\mathcal{C}}(\Psi\langle Y, \alpha\rangle, F)$. It follows from a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{C}}(G, F) & \xrightarrow{\varphi} & \varprojlim_{(h\downarrow G)} \widehat{\mathcal{C}}(h_X, F) \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \varprojlim \iota_\alpha^*} \\ \widehat{\mathcal{C}}(H, F) & \xrightarrow{\psi} & \varprojlim_{(h\downarrow G)} \widehat{\mathcal{C}}(\alpha^{-1}(H), F) \end{array}$$

that $\iota^* : \widehat{\mathcal{C}}(G, F) \to \widehat{\mathcal{C}}(H, F)$ is injective (resp. bijective) if $\iota_\alpha^* : \widehat{\mathcal{C}}(h_X, F) \to \widehat{\mathcal{C}}(h_f^{-1}(H), F)$ is injective (resp. bijective) for any $\langle X, \alpha \rangle \in (h{\downarrow}G)$. $\square$

By (4.2.8) and the above results, we have the following result.

Theorem 4.3.9 *A Grothendieck topos is an elementary topos.*

Chapter 5

Internal category theory

Retold version of P. T. Johnstone's book "Topos Theory" Part II

5.1 Internal categories and diagrams

Definition 5.1.1 *Let $\mathcal{E}$ be a category with finite limits. An internal category $\boldsymbol{C}$ in $\mathcal{E}$ consists of the following objects and morphisms.*

(1) A pair of objects C_0 (the object-of-objects) and C_1 (the object-of-morphisms).

(2) Four morphisms $\sigma : C_1 \to C_0$ (domain), $\tau : C_1 \to C_0$ (codomain), $\varepsilon : C_0 \to C_1$ (identity), $\mu : C_1 \times_{C_0} C_1 \to C_1$ (composition), where $C_1 \xleftarrow{\mathrm{pr}_1} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1$ is a limit of diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma} C_1$, such that $\sigma\varepsilon = \tau\varepsilon = id_{C_0}$ and the following diagrams commute.

$$\begin{tikzcd}
C_1 \arrow[d, "\sigma"] & C_1 \times_{C_0} C_1 \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mu"] & C_1 \arrow[d, "\tau"] \\
C_0 & C_1 \arrow[l, "\sigma"'] \arrow[r, "\tau"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "\mu \times id_{C_1}"] \arrow[d, "id_{C_1} \times \mu"] & C_1 \times_{C_0} C_1 \arrow[d, "\mu"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\mu"] & C_1
\end{tikzcd}$$

$$\begin{tikzcd}
C_1 \times_{C_0} C_0 \arrow[r, "id_{C_1} \times \varepsilon"] \arrow[dr, "\mathrm{pr}_1"'] & C_1 \times_{C_0} C_1 \arrow[d, "\mu"] & C_0 \times_{C_0} C_1 \arrow[l, "\varepsilon \times id_{C_1}"'] \arrow[dl, "\mathrm{pr}_2"] \\
& C_1 &
\end{tikzcd}$$

Here, $C_1 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_i} C_1$ $(i = 1, 2, 3)$ is a limit of diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma} C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma} C_1$ and $C_1 \xleftarrow{\mathrm{pr}_1} C_1 \times_{C_0} C_0 \xrightarrow{\mathrm{pr}_2} C_0$ is a limit of diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{id_{C_0}} C_0$, $C_0 \xleftarrow{\mathrm{pr}_1} C_0 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1$ is a limit of diagram $C_0 \xrightarrow{id_{C_0}} C_0 \xleftarrow{\sigma} C_1$. We denote by $(C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ an internal category $\boldsymbol{C}$ whose object-of-objects and object-of-morphisms are C_0 and C_1, respectively, with structure morphisms σ, τ, ε, μ.

A morphism $f : \boldsymbol{C} \to \boldsymbol{D}$ of internal categories (internal functor) consists of two morphisms $f_0 : C_0 \to D_0$ and $f_1 : C_1 \to D_1$ in $\mathcal{E}$ such that the following diagrams commute.

$$\begin{tikzcd}
C_0 \arrow[d, "f_0"] & C_1 \arrow[l, "\sigma"'] \arrow[r, "\tau"] \arrow[d, "f_1"] & C_0 \arrow[d, "f_0"] \\
D_0 & D_1 \arrow[l, "\sigma"'] \arrow[r, "\tau"] & D_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times_{C_0} C_1 \arrow[r, "\mu"] \arrow[d, "f_1 \times f_1"] & C_1 \arrow[d, "f_1"] & C_0 \arrow[l, "\varepsilon"'] \arrow[d, "f_0"] \\
D_1 \times_{D_0} D_1 \arrow[r, "\mu"] & D_1 & D_0 \arrow[l, "\varepsilon"']
\end{tikzcd}$$

The above internal functor f is denoted by (f_0, f_1). If both f_0 and f_1 are monomorphisms, $\boldsymbol{D}$ is regarded as an internal subcategory of $\boldsymbol{C}$.

An internal natural transformation $\varphi : f \to g$ of internal functors $f, g : \boldsymbol{C} \to \boldsymbol{D}$ is a morphism $\varphi : C_0 \to D_1$ in $\mathcal{E}$ making the following diagrams commute.

$$\begin{tikzcd} & C_0 \arrow[dl, "f_0"'] \arrow[d, "\varphi"] \arrow[dr, "g_0"] & \\ D_0 & D_1 \arrow[l, "\sigma"'] \arrow[r, "\tau"] & D_0 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "{(f_1,\varphi\tau)}"] \arrow[d, "{(\varphi\sigma,g_1)}"] & D_1 \times_{D_0} D_1 \arrow[d, "\mu"] \\ D_1 \times_{D_0} D_1 \arrow[r, "\mu"] & D_1 \end{tikzcd}$$

We denote by $\boldsymbol{cat}(\mathcal{E})$ *the category of internal categories in* $\mathcal{E}$*.*

Proposition 5.1.2 $\boldsymbol{cat}(\mathcal{E})$ *is a 2-category whose 1-arrows are internal functors and whose 2-arrows are internal natural transformations.*

Proof. If $f = (f_0, f_1) : \boldsymbol{C} \to \boldsymbol{D}$ and $g = (g_0, g_1) : \boldsymbol{D} \to \boldsymbol{E}$ are internal functors, the compostion is defined to be $gf = (g_0f_0, g_1f_1) : \boldsymbol{C} \to \boldsymbol{E}$. Then, it is clear that $\boldsymbol{cat}(\mathcal{E})$ is a category.

Let $f, g, h : \boldsymbol{C} \to \boldsymbol{D}$ be internal functors and $\varphi : f \to g$, $\psi : g \to h$ internal natural transformations. The composition $\psi{\cdot}\varphi : f \to h$ is the morphism $C_0 \to D_1$ in $\mathcal{E}$ given as follows. Since $\tau\varphi = g_0 = \sigma\psi$, there is a morphism $(\varphi, \psi) : C_0 \to D_1 \times_{D_0} D_1$. We set $\psi{\cdot}\varphi = \mu(\varphi, \psi)$. Then, $\sigma(\psi{\cdot}\varphi) = \sigma\mu(\varphi, \psi) = \sigma\mathrm{pr}_1(\varphi, \psi) = \sigma\varphi = f_0$, $\tau(\psi{\cdot}\varphi) = \tau\mu(\varphi, \psi) = \tau\mathrm{pr}_2(\varphi, \psi) = \tau\psi = h_0$ and $\mu((\psi{\cdot}\varphi)\sigma, h_1) = \mu(\mu(\varphi, \psi)\sigma, h_1) = \mu(\mu \times id_{D_1})(\varphi\sigma, \psi\sigma, h_1) = \mu(id_{D_1} \times \mu)(\varphi\sigma, \psi\sigma, h_1) = \mu(\varphi\sigma, \mu(\psi\sigma, h_1)) = \mu(\varphi\sigma, \mu(g_1, \psi\tau)) = \mu(id_{D_1} \times \mu)(\varphi\sigma, g_1, \psi\tau) = \mu(\mu \times id_{D_1})(\varphi\sigma, g_1, \psi\tau) = \mu(\mu(\varphi\sigma, g_1), \psi\tau) = \mu(\mu(f_1, \varphi\tau), \psi\tau) = \mu(\mu \times id_{D_1})(f_1, \varphi\tau, \psi\tau) = \mu(id_{D_1} \times \mu)(f_1, \varphi\tau, \psi\tau) = \mu(f_1, \mu(\varphi, \psi)\tau) = \mu(f_1, (\psi{\cdot}\varphi)\tau)$. Thus $\psi{\cdot}\varphi$ is an internal natural transformation. By the associativity of the composition μ of $\mathcal{D}$, the compositions of internal natural transformations are associative. For an internal functor $f : \boldsymbol{C} \to \boldsymbol{D}$, we set $id_f = \varepsilon f_0 : C_0 \to D_1$. Then, $\sigma id_f = \sigma\varepsilon f_0 = f_0$, $\tau id_f = \tau\varepsilon f_0 = f_0$, and $\mu(id_f\sigma, f_1) = \mu(\varepsilon f_0\sigma, f_1) = \mu(\varepsilon\sigma f_1, f_1) = \mu(\varepsilon \times id_{D_1})(\sigma f_1, f_1) = \mathrm{pr}_2(\sigma f_1, f_1) = f_1 = \mathrm{pr}_1(f_1, \tau f_1) = \mu(id_{D_1} \times \varepsilon)(f_1, \tau f_1) = \mu(f_1, \varepsilon\tau f_1) = \mu(f_1, \varepsilon f_0\tau) = \mu(f_1, id_f\tau)$. Hence id_f is an internal natural transformation. If $\varphi : f \to g$ is an internal natural transformation, then we have $id_g{\cdot}\varphi = \mu(\varphi, \varepsilon g_0) = \mu(id_{D_1} \times \varepsilon)(\varphi, g_0) = \mathrm{pr}_1(\varphi, g_0) = \varphi$ and $\varphi{\cdot}id_f = \mu(\varepsilon f_0, \varphi) = \mu(\varepsilon \times id_{D_1})(f_0, \varphi) = \mathrm{pr}_2(f_0, \varphi) = \varphi$. Therefore, id_f is the identity internal natural transformation of f and we have shown that $\boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{D})$ is a category.

For $\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E} \in \mathrm{Ob}\,\boldsymbol{cat}(\mathcal{E})$, we define a functor $c : \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{D}) \times \boldsymbol{cat}(\mathcal{E})(\boldsymbol{D}, \boldsymbol{E}) \to \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{E})$ as follows. If $(f, g) \in \mathrm{Ob}\,\boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{D}) \times \boldsymbol{cat}(\mathcal{E})(\boldsymbol{D}, \boldsymbol{E})$, $c(f, g)$ is the composition of internal functors gf. Let $f, f' : \boldsymbol{C} \to \boldsymbol{D}$, $g, g' : \boldsymbol{D} \to \boldsymbol{E}$ be internal functors and $\varphi : f \to f'$, $\psi : g \to g'$ internal natural transformations. Then, $\tau g_1\varphi = g_0\tau\varphi = g_0f'_0 = \sigma\psi f'_0$, $\tau\psi f_0 = g'_0f_0 = g'_0\sigma\varphi = \sigma g'_1\varphi$. Hence there are morphisms $(g_1\varphi, \psi f'_0), (\psi f_0, g'_1\varphi) : C_0 \to E_1 \times_{E_0} E_1$. We put $\varphi{*}\psi = \mu(g_1\varphi, \psi f'_0)$. Note that $\mu(g_1\varphi, \psi f'_0) = \mu(g_1\varphi, \psi\tau\varphi) = \mu(g_1, \psi\tau)\varphi = \mu(\psi\sigma, g'_1)\varphi = \mu(\psi\sigma\varphi, g'_1\varphi) = \mu(\psi f_0, g'_1\varphi)$ and that $\sigma(\varphi{*}\psi) = \sigma\mu(g_1\varphi, \psi f'_0) = \sigma\mathrm{pr}_1(g_1\varphi, \psi f'_0) = \sigma g_1\varphi = g_0\sigma\varphi = g_0f_0$, $\tau(\varphi{*}\psi) = \tau\mu(g_1\varphi, \psi f'_0) = \tau\mathrm{pr}_2(g_1\varphi, \psi f'_0) = \tau\psi f'_0 = g'_0f'_0$, $\mu((\varphi{*}\psi)\sigma, g'_1f'_1) = \mu(\mu(g_1\varphi, \psi f'_0)\sigma, g'_1f'_1) = \mu(\mu \times id_{E_1})(g_1\varphi\sigma, \psi f'_0\sigma, g'_1f'_1) = \mu(id_{E_1} \times \mu)(g_1\varphi\sigma, \psi\sigma f'_1, g'_1f'_1) = \mu(g_1\varphi\sigma, \mu(\psi\sigma, g'_1)f'_1) = \mu(g_1\varphi\sigma, \mu(g_1, \psi\tau)f'_1) = \mu(id_{E_1} \times \mu)(g_1\varphi\sigma, g_1f'_1, \psi\tau f'_1) = \mu(\mu \times id_{E_1})(g_1\varphi\sigma, g_1f'_1, \psi f'_0\tau) = \mu(\mu(g_1 \times g_1)(\varphi\sigma, f'_1), \psi f'_0\tau) =$
$\mu(g_1\mu(\varphi\sigma, f'_1), \psi f'_0\tau) = \mu(g_1\mu(f_1, \varphi\tau), \psi f'_0\tau) = \mu(\mu(g_1 \times g_1)(f_1, \varphi\tau), \psi f'_0\tau) = \mu(\mu \times id_{E_1})(g_1f_1, g_1\varphi\tau, \psi f'_0\tau) = \mu(id_{E_1} \times \mu)(g_1f_1, g_1\varphi\tau, \psi f'_0\tau) = \mu(g_1f_1, \mu(g_1\varphi\tau, \psi f'_0\tau)) = \mu(g_1f_1, \mu(g_1\varphi, \psi f'_0)\tau) = \mu(g_1f_1, (\varphi{*}\psi)\tau)$. Thus $\varphi{*}\psi$ is an internal natural transformation from gf to $g'f'$ and we set $c(\varphi, \psi) = \varphi{*}\psi$. For internal functors $f : \boldsymbol{C} \to \boldsymbol{D}$ and $g : \boldsymbol{D} \to \boldsymbol{E}$, $c(id_f, id_g) = id_f{*}id_g = \mu(g_1id_f, id_gf_0) = \mu(g_1\varepsilon f_0, \varepsilon g_0f_0) = \mu(\varepsilon g_0f_0, \varepsilon g_0f_0) = \mu(\varepsilon \times id_{E_1})(g_0f_0, \varepsilon g_0f_0) = \mathrm{pr}_2(g_0f_0, \varepsilon g_0f_0) = \varepsilon g_0f_0 = id_{gf}$. Let $f, f', f'' : \boldsymbol{C} \to \boldsymbol{D}$, $g, g', g'' : \boldsymbol{D} \to \boldsymbol{E}$ be internal functors and $\varphi : f \to f'$, $\zeta : f' \to f''$, $\psi : g \to g'$, $\xi : g' \to g''$ internal natural transformations. Then, $c(\zeta{\cdot}\varphi, \xi{\cdot}\psi) = (\zeta{\cdot}\varphi){*}(\xi{\cdot}\psi) = \mu(g_1\mu(\varphi, \zeta), \mu(\psi, \xi)f''_0) = \mu(id_{E_1} \times \mu)(\mu(g_1\varphi, g_1\zeta), \psi f''_0, \xi f''_0) = \mu(\mu \times \mu)(g_1\varphi, g_1\zeta, \psi\tau\zeta, \xi f''_0) = \mu(\mu \times id_{E_1})(g_1\varphi, \mu(g_1, \psi\tau)\zeta, \xi f''_0) = \mu(\mu \times id_{E_1})(g_1\varphi, \mu(\psi\sigma, g'_1)\zeta, \xi f''_0)$
$= \mu(\mu \times \mu)(g_1\varphi, \psi f'_0, g'_1\zeta, \xi f''_0) = \mu(\mu(g_1\varphi, \psi f'_0), \mu(g'_1\zeta, \xi f''_0)) = (\zeta{*}\xi){\cdot}(\varphi{*}\psi) = c(\zeta, \xi){\cdot}c(\varphi, \psi)$. It follows that c is a functor.

For an internal category $\boldsymbol{C}$, there is the identity internal functor $id_{\boldsymbol{C}} = (id_{C_0}, id_{C_1}) : \boldsymbol{C} \to \boldsymbol{C}$. We denote by $\mathbf{1}$ the category with a single object 1 and a single morphism id_1. Define a functor $u_{\boldsymbol{C}} : \mathbf{1} \to \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{C})$ by $u_{\boldsymbol{C}}(1) = id_{\boldsymbol{C}}$. We claim that the following diagrams commute.

$$\begin{tikzcd} \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{D}) \times \boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{E}) \times \boldsymbol{cat}(\mathcal{E})(\boldsymbol{E},\boldsymbol{F}) \arrow[r, "c \times id_{\boldsymbol{cat}(\mathcal{E})(\boldsymbol{E},\boldsymbol{F})}"] \arrow[d, "id_{\boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{D})} \times c"] & \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{E}) \times \boldsymbol{cat}(\mathcal{E})(\boldsymbol{E},\boldsymbol{F}) \arrow[d, "c"] \\ \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{D}) \times \boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{F}) \arrow[r, "c"] & \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{F}) \end{tikzcd}$$

$$\begin{tikzcd} \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{D}) \times \mathbf{1} \arrow[d, "id_{\boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{D})} \times u_{\boldsymbol{D}}"] \arrow[dr, "\mathrm{pr}_1"] & \\ \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{D}) \times \boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{D}) \arrow[r, "c"] & \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{D}) \end{tikzcd} \quad \begin{tikzcd} \mathbf{1} \times \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{D}) \arrow[d, "u_{\boldsymbol{C}} \times id_{\boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{D})}"] \arrow[dr, "\mathrm{pr}_2"] & \\ \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{C}) \times \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{D}) \arrow[r, "c"] & \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{D}) \end{tikzcd}$$

In fact, the commutativity on objects is clear. Let $f, f' : \boldsymbol{C} \to \boldsymbol{D}$, $g, g' : \boldsymbol{D} \to \boldsymbol{E}$, $h, h' : \boldsymbol{E} \to \boldsymbol{F}$ be internal functors and $\varphi : f \to f'$, $\psi : g \to g'$, $\chi : h \to h'$ internal natural transformations between them. Then, we have $\chi{*}(\psi{*}\varphi) = \mu(h_1(\psi{*}\varphi), \chi g_0' f_0') = \mu(h_1\mu(g_1\varphi, \psi f_0'), \chi g_0' f_0') = \mu(\mu(h_1 \times h_1)(g_1\varphi, \psi f_0'), \chi g_0' f_0') =$ $\mu(\mu \times id_{F_1})(h_1 g_1 \varphi, h_1 \psi f_0', \chi g_0' f_0') = \mu(id_{F_1} \times \mu)(h_1 g_1 \varphi, h_1 \psi f_0', \chi g_0' f_0') = \mu(h_1 g_1 \varphi, \mu(h_1\psi, \chi g_0') f_0') =$ $\mu(h_1 g_1 \varphi, (\chi{*}\psi) f_0') = (\chi{*}\psi){*}\varphi$. Moreover, $id_{id_{\boldsymbol{D}}}{*}\varphi = \mu(id_{D_1}\varphi, id_{id_{\boldsymbol{D}}} f_0') = \mu(id_{D_1}\varphi, \varepsilon id_{D_0} f_0') = \mu(id_{D_1}{\times}\varepsilon)(\varphi, f_0')$ $= \mathrm{pr}_1(\varphi, f_0') = \varphi$, $\psi{*}id_{id_{\boldsymbol{C}}} = \mu(g_1 id_{id_{\boldsymbol{C}}}, \psi id_{C_0}) = \mu(g_1 \varepsilon id_{C_0}, \psi) = \mu(\varepsilon g_0, \psi) = \mu(\varepsilon{\times}id_{D_1})(g_0, \psi) = \mathrm{pr}_2(g_0, \psi) =$ ψ. Therefore the above diagrams commute also on morphisms. □

For an object C of $\mathcal{E}$, we denote by $\boldsymbol{cat}(\mathcal{E}; C)$ a subcategory of $\boldsymbol{cat}(\mathcal{E})$ whose objects are internal categories with object-of-objects C and morphisms of the form (id_C, f_1). We note that $\boldsymbol{cat}(\mathcal{E}; C)$ has an initial object $\mathbf{0}_C = (C, C; id_C, id_C, id_C, \mu)$ where $\mu : C \times_C C \to C$ is the isomorphism induced by the projection $\mathrm{pr}_1 : C \times C \to C$ and a terminal object $\mathbf{1}_C = (C \times C, C; \mathrm{pr}_1, \mathrm{pr}_2, \delta, \mathrm{pr}_1 \times_C \mathrm{pr}_1)$ where $\delta : C \to C \times C$ is the diagonal morphism and $\mathrm{pr}_1 \times_C \mathrm{pr}_1 : (C \times C) \times_C (C \times C) \to C \times C$ is the morphism induced by $\mathrm{pr}_1 \times \mathrm{pr}_1 : (C \times C) \times_C (C \times C) \to C \times C$. In fact, for an internal category $\boldsymbol{C} = (C, C_1; \sigma, \tau, \varepsilon, \mu)$, $(id_C, \varepsilon) : \mathbf{0}_C \to \boldsymbol{C}$ and $(id_C, (\sigma, \tau)) : \boldsymbol{C} \to \mathbf{1}_C$ are the unique morphisms.

For an internal category $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$, an opposite category $\boldsymbol{C}^{op}$ of $\boldsymbol{C}$ is defined to be an internal category $(C_0, C_1; \tau, \sigma, \varepsilon, \mu T)$, where $T = (\mathrm{pr}_2, \mathrm{pr}_1) : C_1 \times_{C_0} C_1 \to C_1 \times_{C_0} C_1$.

Remark 5.1.3 *1) Let $\mathcal{U}$ be a fixed universe. For a category $\mathcal{C}$ such that $\mathrm{Ob}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{C} \in \mathcal{U}$, the structure maps domain, codomain $: \mathrm{Mor}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}$, identity $: \mathrm{Ob}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{C}$ and composition $: \mathrm{Mor}\,\mathcal{C} \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{C}$ are morphisms in the category $\mathcal{U}$-**Ens** of $\mathcal{U}$-sets. Hence we can associate an internal category $\boldsymbol{C}(\mathcal{C}) = (\mathrm{Ob}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{C};$ domain,codomain,identity,composition$)$ in $\mathcal{U}$-**Ens**. If $F : \mathcal{C} \to \mathcal{D}$ is a functor between categories such that $\mathrm{Ob}\,\mathcal{C}, \mathrm{Ob}\,\mathcal{D}, \mathrm{Mor}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{D} \in \mathcal{U}$, maps $F_{ob} : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ and $F_{mor} : \mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{D}$ define an internal functor $\boldsymbol{C}(F) : \boldsymbol{C}(\mathcal{C}) \to \boldsymbol{C}(\mathcal{D})$. Moreover, if $\varphi : F \to G$ is a natural transformation between functors $F, G : \mathcal{C} \to \mathcal{D}$, $\varphi : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{D}$ defines an internal natural transformation $\boldsymbol{C}(\varphi) : \boldsymbol{C}(F) \to \boldsymbol{C}(G)$. Thus we have a 2-functor from the 2-category of categories whose objects and morphisms belong to $\mathcal{U}$ to* $\mathbf{cat}(\mathcal{U}\text{-}\mathbf{Ens})$.

2) If $F : \mathcal{E} \to \mathcal{F}$ is a left exact functor and $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ is an internal category in $\mathcal{E}$, then $F(\boldsymbol{C}) = (F(C_0), F(C_1); F(\sigma), F(\tau), F(\varepsilon), F(\mu)\kappa^{-1})$ is an internal category in $\mathcal{F}$, where $\kappa : F(C_1 \times_{C_0} C_1) \to F(C_1) \times_{F(C_0)} F(C_1)$ is the isomorphism induced by $F(\mathrm{pr}_1), F(\mathrm{pr}_2) : F(C_1 \times_{C_0} C_1) \to F(C_1)$. In particular, if we choose a universe $\mathcal{U}$ such that $\mathcal{E}$ is a $\mathcal{U}$-category and $\widehat{\mathcal{E}}$ denotes the category of $\mathcal{U}$-presheaves $\mathrm{Funct}(\mathcal{E}^{op}, \mathcal{U}\text{-}\mathbf{Ens})$, *since the Yoneda embedding $h : \mathcal{E} \to \widehat{\mathcal{E}}$ preserves limits, $(h_{C_0}, h_{C_1}; h_\sigma, h_\tau, h_\varepsilon h_\mu \kappa^{-1})$ is an internal category in $\widehat{\mathcal{E}}$. Moreover, for each object U of $\mathcal{E}$, $(h_{C_0}(U), h_{C_1}(U); h_{\sigma U}, h_{\tau U}, h_{\varepsilon U}, h_{\mu U}\kappa_U^{-1})$ is an internal category in $\mathcal{U}$-**Ens**.*

We assume that $\mathcal{E}$ is a category with finite limits below.

Definition 5.1.4 *1) An internal groupoid $\boldsymbol{G}$ in $\mathcal{E}$ is an internal category in $\mathcal{E}$ with a morphism $\iota : G_1 \to G_1$ (inverse) such that $\sigma\iota = \tau$, $\tau\iota = \sigma$ and the following diagram commutes.*

$$\begin{tikzcd}
G_1 \arrow[r, "(id_{G_1}{,}\iota)"] \arrow[d, "\sigma"] & G_1 \times_{G_0} G_1 \arrow[d, "\mu"] & G_1 \arrow[l, "(\iota{,}id_{G_1})"'] \arrow[d, "\tau"] \\
G_0 \arrow[r, "\varepsilon"] & G_1 & G_0 \arrow[l, "\varepsilon"']
\end{tikzcd}$$

2) We say that an internal category $(C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ is an internal poset if $(\sigma, \tau) : C_1 \to C_0 \times C_0$ is a monomorphism.

3) We say that an internal category $(C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ is discrete if $\varepsilon : C_0 \to C_1$ is an isomorphism.

Lemma 5.1.5 *Let $\boldsymbol{C}$ be an internal category in $\mathcal{E}$. Suppose that morphisms $\alpha, \beta, \gamma : D \to C_1$ satisfy $\tau\alpha = \sigma\beta$, $\sigma\gamma = \tau\beta$ and make the following diagram commute, then $\alpha = \gamma$.*

$$\begin{tikzcd}
D \arrow[r, "(\beta{,}\gamma)"] \arrow[d, "\sigma\beta"] & C_1 \times_{C_0} C_1 \arrow[d, "\mu"] & D \arrow[l, "(\alpha{,}\beta)"'] \arrow[d, "\tau\beta"] \\
C_0 \arrow[r, "\varepsilon"] & C_1 & C_0 \arrow[l, "\varepsilon"']
\end{tikzcd}$$

Proof. $\alpha = \mu(id_{C_1}, \varepsilon\tau)\alpha = \mu(\alpha, \varepsilon\tau\alpha) = \mu(\alpha, \varepsilon\sigma\beta) = \mu(\alpha, \mu(\beta, \gamma)) = \mu(\mu(\alpha, \beta), \gamma) = \mu(\varepsilon\tau\beta, \gamma) = \mu(\varepsilon\sigma\gamma, \gamma) =$ $\mu(\varepsilon\sigma, id_{C_1})\gamma = \gamma$. □

Proposition 5.1.6 *Let $\boldsymbol{G} = (G_0, G_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$ and $\boldsymbol{H} = (H_0, H_1; \sigma', \tau', \varepsilon', \mu')$ an internal groupoid in $\mathcal{E}$ with inverse ι'. Suppose that morphisms $f_0 : G_0 \to H_0$ and $f_1 : G_1 \to H_1$ make the following diagrams commute.*

$$\begin{tikzcd} G_0 \arrow[d, "f_0"] & G_1 \arrow[l, "\sigma"'] \arrow[r, "\tau"] \arrow[d, "f_1"] & G_0 \arrow[d, "f_0"] \\ H_0 & H_1 \arrow[l, "\sigma'"] \arrow[r, "\tau'"'] & H_0 \end{tikzcd} \qquad \begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, "\mu"] \arrow[d, "f_1 \times f_1"] & G_1 \arrow[d, "f_1"] \\ H_1 \times_{H_0} H_1 \arrow[r, "\mu'"'] & H_1 \end{tikzcd}$$

1) The following diagram commute.

$$\begin{tikzcd} G_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & G_1 \arrow[d, "f_1"] \\ H_0 \arrow[r, "\varepsilon'"'] & H_1 \end{tikzcd}$$

2) If G is an internal groupoid in $\mathcal{E}$ with inverse ι, the following diagram commute.

$$\begin{tikzcd} G_1 \arrow[r, "\iota"] \arrow[d, "f_1"] & G_1 \arrow[d, "f_1"] \\ H_1 \arrow[r, "\iota'"'] & H_1 \end{tikzcd}$$

In particular, if $G_i = H_i$, $f_i = id_{G_i}$ $(i = 0, 1)$, $\sigma = \sigma'$, $\tau = \tau'$ and $\mu = \mu'$, then we have $\varepsilon = \varepsilon'$ and $\iota = \iota'$. It follows that we can regard the category of internal groupoids in $\mathcal{E}$ as a full subcategory of $\boldsymbol{cat}(\mathcal{E})$.

Proof. 1) By the commutativity of the third diagram of (5.1.1), we have $\mu(\varepsilon, \varepsilon) = \mu(id_{G_1} \times \varepsilon)(\varepsilon, id_{G_0}) = \mathrm{pr}_1(\varepsilon, id_{G_0}) = \varepsilon$. Hence $\mu'(f_1\varepsilon, f_1\varepsilon) = \mu'(f_1 \times f_1)(\varepsilon, \varepsilon) = f_1\mu(\varepsilon, \varepsilon) = f_1\varepsilon$. By the commutativity of the diagram of (5.1.4), we have $\mu'(\iota' f_1\varepsilon, f_1\varepsilon) = \mu'(\iota', id_{H_1})f_1\varepsilon = \varepsilon'\tau' f_1\varepsilon = \varepsilon' f_0\tau\varepsilon = \varepsilon' f_0$. Therefore the commutativity of the second diagram of (5.1.1) and the diagram of (5.1.4) imply $\mu'(\iota' f_1\varepsilon, f_1\varepsilon) = \mu'(\iota' f_1\varepsilon, \mu'(f_1\varepsilon, f_1\varepsilon)) = \mu'(\mu'(\iota' f_1\varepsilon, f_1\varepsilon), f_1\varepsilon) = \mu'(\varepsilon' f_0, f_1\varepsilon) = \mu'(\varepsilon' \times id_{H_1})(f_0, f_1\varepsilon) = \mathrm{pr}_2(f_0, f_1\varepsilon) = f_1\varepsilon$. Thus $\varepsilon' f_0 = f_1\varepsilon$.

2) We note that $\tau' f_1\iota = f_0\tau\iota = f_0\sigma = \sigma' f_1$ and $\sigma'\iota' f_1 = \tau' f_1$. By the commutativity of the diagram of (5.1.4) and the above, we have $\mu'(f_1\iota, f_1) = \mu'(f_1 \times f_1)(\iota, id_{G_1}) = f_1\mu(\iota, id_{G_1}) = f_1\varepsilon\tau = \varepsilon' f_0\tau = \varepsilon'\tau' f_1$ and $\mu'(f_1, \iota' f_1) = \mu'(id_{H_1}, \iota')f_1 = \varepsilon'\sigma' f_1$. Hence the assumptions of (5.1.5) are satisfied for $\boldsymbol{C} = \boldsymbol{H}$ and $\alpha = f_1\iota$, $\beta = f_1$, $\gamma = \iota' f_1$. Thus we have $f_1\iota = \iota' f_1$. $\square$

We denote by $\boldsymbol{gr}(\mathcal{E})$ the category of internal groupoids in $\mathcal{E}$. For an object C of $\mathcal{E}$, we denote by $\boldsymbol{gr}(\mathcal{E}; C)$ a full subcategory of $\boldsymbol{cat}(\mathcal{E}; C)$ whose objects are internal groupoids. We note that the initial object $\mathbf{0}_C$ and the terminal object $\mathbf{1}_C$ of $\boldsymbol{cat}(\mathcal{E}; C)$ belong to $\boldsymbol{gr}(\mathcal{E}; C)$. Hence $\boldsymbol{gr}(\mathcal{E}; C)$ has initial and terminal objects.

Proposition 5.1.7 *Let $\boldsymbol{C}$ be an internal category in $\mathcal{E}$ and $\boldsymbol{G}$ an internal groupoid in $\mathcal{E}$. Then, $\boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{G})$ is a groupoid.*

Proof. Let $f, g : \boldsymbol{C} \to \boldsymbol{G}$ be internal functors and $\varphi : f \to g$ an internal natural transformation. If $\iota : G_1 \to G_1$ is the inverse of $\boldsymbol{G}$, we show that $\iota\varphi : C_0 \to G_1$ is an an internal natural transformation from g to f and that it is the inverse of φ.

We put $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{G} = (G_0, G_1; \sigma', \tau', \varepsilon', \mu')$ By the definition of internal natural transformations, we have $\sigma'\iota\varphi = \tau'\varphi = g_0$, $\tau'\iota\varphi = \sigma'\varphi = f_0$ and $\mu'(f_1, \varphi\tau) = \mu'(\varphi\sigma, g_1)$. Since $\sigma'\mu'(f_1, \varphi\tau) = f_0\sigma$ and $\tau'\iota\varphi\sigma = \sigma'\varphi\sigma = f_0\sigma$, we can compose $\iota\varphi\sigma$ and $\mu'(f_1, \varphi\tau)$ and we have

$$\begin{aligned} \mu'(\iota\varphi\sigma, \mu'(f_1, \varphi\tau)) &= \mu'(\iota\varphi\sigma, \mu'(\varphi\sigma, g_1)) = \mu'(\mu'(\iota\varphi\sigma, \varphi\sigma), g_1) = \mu'(\mu'(\iota, id_{G_1})\varphi\sigma, g_1) = \mu'(\varepsilon'\tau'\varphi\sigma, g_1) \\ &= \mu'(\varepsilon' g_0\sigma, g_1) = \mu'(\varepsilon'\sigma' g_1, g_1) = \mu'(\varepsilon' \times id_{G_1})(\sigma' g_1, g_1) = \mathrm{pr}_2(\sigma' g_1, g_1) = g_1 \end{aligned}$$

Since $\tau' g_1 = g_0\tau$ and $\sigma'\iota\varphi\tau = \tau'\varphi\tau = g_0\tau$, we can compose g_1 and $\iota\varphi\tau$. By the above result, we have

$$\begin{aligned} \mu'(g_1, \iota\varphi\tau) &= \mu'(\mu'(\iota\varphi\sigma, \mu'(f_1, \varphi\tau)), \iota\varphi\tau) = \mu'(\iota\varphi\sigma, \mu'(\mu'(f_1, \varphi\tau), \iota\varphi\tau)) = \mu'(\iota\varphi\sigma, \mu'(f_1, \mu'(\varphi\tau, \iota\varphi\tau))) \\ &= \mu'(\iota\varphi\sigma, \mu'(f_1, \mu'(id_{G_1}, \iota)\varphi\tau)) = \mu'(\iota\varphi\sigma, \mu'(f_1, \varepsilon'\sigma'\varphi\tau)) = \mu'(\iota\varphi\sigma, \mu'(f_1, \varepsilon' f_0\tau)) \\ &= \mu'(\iota\varphi\sigma, \mu'(f_1, f_1\varepsilon\tau)) = \mu'(\iota\varphi\sigma, f_1\mu(id_{C_1}, \varepsilon\tau)) = \mu'(\iota\varphi\sigma, f_1\mu(id_{C_1} \times \varepsilon)(id_{C_1}, \tau)) \\ &= \mu'(\iota\varphi\sigma, f_1\mathrm{pr}_1(id_{C_1}, \tau)) = \mu'(\iota\varphi\sigma, f_1). \end{aligned}$$

Hence $\iota\varphi$ is an an internal natural transformation from g to f.

By the definition of the composition of internal natural transformations, we have $\iota\varphi{\cdot}\varphi = \mu'(\iota\varphi, \varphi) = \mu'(\iota, id_{G_1})\varphi = \varepsilon'\tau'\varphi = \varepsilon' g_0 = id_g$ and $\varphi{\cdot}\iota\varphi = \mu'(\varphi, \iota\varphi) = \mu'(id_{G_1}, \iota)\varphi = \varepsilon'\sigma'\varphi = \varepsilon' f_0 = id_f$. Therefore, $\iota\varphi$ is the inverse of φ. $\square$

Proposition 5.1.8 *$\boldsymbol{gr}(\mathcal{E})$ is a coreflexive subcategory of $\boldsymbol{cat}(\mathcal{E})$, that is, the inclusion functor $\boldsymbol{gr}(\mathcal{E}) \hookrightarrow \boldsymbol{cat}(\mathcal{E})$ has a right adjoint. Similarly, $\boldsymbol{gr}(\mathcal{E}; C)$ is a coreflexive subcategory of $\boldsymbol{cat}(\mathcal{E}; C)$ for any object C of $\mathcal{E}$.*

Proof. Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category and $\mathrm{pr}_{12}, \mathrm{pr}_{23} : C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to C_1 \times_{C_0} C_1$ the projections onto the first and the second (resp. the second and the third) components. Define a morphism $e : C_1^i \to C_1 \times_{C_0} C_1 \times_{C_0} C_1$ to be the pull-back of $\varepsilon \times \varepsilon : C_0 \times C_0 \to C_1 \times C_1$ along $(\mu\mathrm{pr}_{12}, \mu\mathrm{pr}_{23}) : C_1 \times_{C_0} C_1 \times_{C_0} C_1 \to C_1 \times C_1$. Then, it follows from (5.1.5) that $\mathrm{pr}_1 e = \mathrm{pr}_3 e : C_1^i \to C_1$. We put $\lambda = \mathrm{pr}_2 e$, $\nu = \mathrm{pr}_1 e$. By considering the functors represented by C_1 and C_0, we see that λ and ν are monomorphisms. We also see that $\varepsilon : C_0 \to C_1$ and $\nu : C_1^i \to C_1$ and $C_1^i \times_{C_0} C_1^i \xrightarrow{\lambda\times\lambda} C_1 \times_{C_0} C_1 \xrightarrow{\mu} C_1$ lift to maps $\varepsilon^i : C_0 \to C_1^i$, $\iota : C_1^i \to C_1^i$ and $\mu^i : C_1^i \times_{C_0} C_1^i \to C_1^i$ so that $(C_0, C_1^i; \sigma\lambda, \tau\lambda, \varepsilon^i, \mu^i)$ is an internal groupoid and (λ, id_{C_0}) is an internal functor. We set $G(\boldsymbol{C}) = (C_0, C_1^i; \sigma\lambda, \tau\lambda, \varepsilon^i, \mu^i)$ and $\eta = (id_{C_0}, \lambda) : G(\boldsymbol{C}) \to \boldsymbol{C}$, then $G(\boldsymbol{C})$ is regarded as an internal subcategory of $\boldsymbol{C}$ via η.

If $\boldsymbol{G}$ is an internal groupoid and $f : \boldsymbol{G} \to \boldsymbol{C}$ is an internal functor, there is an unique internal functor $\bar{f} : \boldsymbol{G} \to G(\boldsymbol{C})$ such that $f = \eta\bar{f}$. Note that since η is a morphism in $\boldsymbol{cat}(\mathcal{E}; C_0)$, if f is a morphism in $\boldsymbol{cat}(\mathcal{E}; C_0)$, $\bar{f}$ is a morphism of $\boldsymbol{gr}(\mathcal{E}; C_0)$. This proves the assertion. $\square$

Definition 5.1.9 *Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$. We call a pair of morphisms $(\pi : X \to C_0, \alpha : X \times_{C_0} C_1 \to X)$ of $\mathcal{E}$ an internal diagram on $\boldsymbol{C}$ if it makes the following diagrams commute, where $X \times_{C_0} C_1$ is the fibered product of $\pi : X \to C_0$ and $\sigma : C_1 \to C_0$.*

$$\begin{tikzcd} X \times_{C_0} C_1 \arrow[r, "\alpha"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d, "\pi"] \\ C_1 \arrow[r, "\tau"] & C_0 \end{tikzcd} \qquad \begin{tikzcd} X \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "\alpha \times id_{C_1}"] \arrow[d, "id_X \times \mu"] & X \times_{C_0} C_1 \arrow[d, "\alpha"] \\ X \times_{C_0} C_1 \arrow[r, "\alpha"] & X \end{tikzcd} \qquad \begin{tikzcd} X \times_{C_0} C_1 \arrow[r, "\alpha"] & X \\ X \times_{C_0} C_0 \arrow[u, "id_X \times \varepsilon"] \arrow[ur, "\mathrm{pr}_1"'] & \end{tikzcd}$$

Let $(\pi : X \to C_0, \alpha)$ and $(\rho : Y \to C_0, \beta)$ be internal diagrams on $\boldsymbol{C}$. A morphism $f : X \to Y$ in $\mathcal{E}$ is called a morphism of internal diagrams if it satisfies $\rho f = \pi$ and $\beta(f \times id_{C_0}) = f\alpha$. We denote by $\mathcal{E}^{\boldsymbol{C}}$ the category of internal diagrams on $\boldsymbol{C}$.

Similarly, we also consider an "opposite diagram" defined as follows. A pair of morphisms $(\pi : X \to C_0, \alpha : C_1 \times_{C_0} X \to X)$ in $\mathcal{E}$ is called an internal presheaf on $\boldsymbol{C}$ if it makes the following diagrams commute, where $C_1 \times_{C_0} X$ is the fibered product of $\pi : X \to C_0$ and $\tau : C_1 \to C_0$.

$$\begin{tikzcd} C_1 \times_{C_0} X \arrow[r, "\alpha"] \arrow[d, "\mathrm{pr}_1"] & X \arrow[d, "\pi"] \\ C_1 \arrow[r, "\sigma"] & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_1 \times_{C_0} C_1 \times_{C_0} X \arrow[r, "id_{C_1} \times \alpha"] \arrow[d, "\mu \times id_X"] & C_1 \times_{C_0} X \arrow[d, "\alpha"] \\ C_1 \times_{C_0} X \arrow[r, "\alpha"] & X \end{tikzcd} \qquad \begin{tikzcd} C_1 \times_{C_0} X \arrow[r, "\alpha"] & X \\ C_0 \times_{C_0} X \arrow[u, "\varepsilon \times id_X"] \arrow[ur, "\mathrm{pr}_2"'] & \end{tikzcd}$$

The notion of morphisms of internal presheaves is defined similarly. The category of internal presheaves on $\boldsymbol{C}$ is isomorphic to the category of internal diagrams on $\boldsymbol{C}^{op}$.

Remark 5.1.10 *1) There is a terminal object in $\mathcal{E}^{\boldsymbol{C}}$ given by $(id_{C_0} : C_0 \to C_0, \tau\mathrm{pr}_2 : C_0 \times_{C_0} C_1 \to C_0)$. For an internal diagram $(\pi : X \to C_0, \alpha)$, $\pi : X \to C_0$ is the unique morphism to the terminal object.*

2) If $(\pi : X \to C_0, \alpha)$ is an internal diagram on $\boldsymbol{C}$, we put $C_\alpha = X \times_{C_0} C_1$, $\sigma_\alpha = \mathrm{pr}_1, \tau_\alpha = \alpha : C_\alpha \to X$ and define $\varepsilon_\alpha : X \to C_\alpha$ to be $X \xrightarrow[\cong]{\mathrm{pr}_1^{-1}} X \times_{C_0} C_0 \xrightarrow{id_X \times \varepsilon} C_\alpha$. We note that the following diagram commutes and that $\mathrm{pr}_1 : (X \times_{C_0} C_1) \times_X (X \times_{C_0} C_1) \to X \times_{C_0} C_1$ is a pull-back of σ along $\tau\mathrm{pr}_2$, where $X \times_{C_0} C_1$ on the right (resp. left) factor of the pull-back over X is regarded as having structure map $\sigma_\alpha = \mathrm{pr}_1$ (resp. $\tau_\alpha = \alpha$).

$$\begin{tikzcd} (X \times_{C_0} C_1) \times_X (X \times_{C_0} C_1) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & X \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_2"] & C_1 \arrow[d, "\sigma"] \\ X \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_2"] & C_1 \arrow[r, "\tau"] & C_0 \end{tikzcd}$$

Hence $(\mathrm{pr}_1, \mathrm{pr}_2\mathrm{pr}_2) : (X \times_{C_0} C_1) \times_X (X \times_{C_0} C_1) \to (X \times_{C_0} C_1) \times_{C_0} C_1$ *is an isomorphism. Define* $\mu_\alpha : C_\alpha \times_X C_\alpha \to C_\alpha$ *to be the composition* $C_\alpha \times_X C_\alpha \cong X \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{1\times\mu} C_\alpha$. *Then we have an internal category* $(X, C_\alpha; \sigma_\alpha, \tau_\alpha, \varepsilon_\alpha, \mu_\alpha)$. *Note that* $(\pi, \mathrm{pr}_2) : (X, C_\alpha; \sigma_\alpha, \tau_\alpha, \varepsilon_\alpha, \mu_\alpha) \to (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ *is an internal functor. If* $\boldsymbol{C}$ *is an internal groupoid,* $\iota_\alpha : C_\alpha \to C_\alpha$ *is defined to be* $(\alpha, \iota\mathrm{pr}_2)$ *and* $(X, C_\alpha; \sigma_\alpha, \tau_\alpha, \varepsilon_\alpha, \mu_\alpha)$ *is an internal groupoid. Let* $f : (\pi : X \to C_0, \alpha) \to (\rho : Y \to C_0, \beta)$ *be a morphism of internal diagrams on* $\boldsymbol{C}$. *Then,* $(f : X \to Y, f \times id_{C_1} : C_\alpha \to C_\beta)$ *is a morphism in* $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{C}$. *Hence we have a functor* $\mathcal{E}^{\boldsymbol{C}} \to \boldsymbol{cat}(\mathcal{E})/\boldsymbol{C}$ $(\mathcal{E}^{\boldsymbol{C}} \to \boldsymbol{gr}(\mathcal{E})/\boldsymbol{C}$ *if* $\boldsymbol{C}$ *is an internal groupoid) given by* $(\pi : X \to C_0, \alpha) \mapsto ((X, C_\alpha; \sigma_\alpha, \tau_\alpha, \varepsilon_\alpha, \mu_\alpha) \xrightarrow{(\pi,\mathrm{pr}_2)} (C_0, C_1; \sigma, \tau, \varepsilon, \mu))$. *Obviously, this functor is faithful. Suppose that* $(f_0, f_1) : (X, C_\alpha; \sigma_\alpha, \tau_\alpha, \varepsilon_\alpha, \mu_\alpha) \to (Y, C_\beta; \sigma_\beta, \tau_\beta, \varepsilon_\beta, \mu_\beta)$ *is a morphism in* $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{C}$. *Then, it follows from* $\mathrm{pr}_1 f_1 = f_0\mathrm{pr}_1$ *and* $\mathrm{pr}_2 f_1 = \mathrm{pr}_2$ *that* $f_1 = f_0 \times id_{C_1}$. *Moreover, since* $\beta f_1 = f_0\alpha$ *and* $\rho f = \pi$, f_0 *is a morphism of internal diagrams. Therefore the functor* $\mathcal{E}^{\boldsymbol{C}} \to \boldsymbol{cat}(\mathcal{E})/\boldsymbol{C}$ *is fully faithful and* $\mathcal{E}^{\boldsymbol{C}}$ *is regarded as a full subcategory of* $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{C}$. *An object* $(f_0, f_1) : \boldsymbol{D} \to \boldsymbol{C}$ *is isomorphic to an object belonging to the image of the functor* $\mathcal{E}^{\boldsymbol{C}} \to \boldsymbol{cat}(\mathcal{E})/\boldsymbol{C}$ *if and only if the square*

$$\begin{array}{ccc} D_1 & \xrightarrow{\sigma} & D_0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ C_1 & \xrightarrow{\sigma} & C_0 \end{array}$$

is a pull-back. We call such an internal functor (f_0, f_1) *a discrete opfibration.*

Dually, for an internal presheaf $(\pi : X \to C_0, \alpha : C_1 \times_{C_0} X \to X)$, *put* $C^\alpha = C_1 \times_{C_0} X$, $\sigma^\alpha = \mathrm{pr}_2, \tau^\alpha = \alpha : C_\alpha \to X$ *and define* $\varepsilon^\alpha : X \to C^\alpha$ *to be* $X \xrightarrow{\mathrm{pr}_2^{-1}} C_0 \times_{C_0} X \xrightarrow{\varepsilon\times id_X} C^\alpha$. *We define* $\mu^\alpha : C^\alpha \times C^\alpha \to C^\alpha$ *as above and we have an internal category* $(C^\alpha, X; \sigma^\alpha, \tau^\alpha, \varepsilon^\alpha, \mu^\alpha)$. *An internal functor which corresponds to an internal presheaf is called a discrete fibration.*

3) Let $\mathcal{C}$ *be a category such that* $\mathrm{Ob}\,\mathcal{C}, \mathrm{Mor}\,\mathcal{C} \in \mathcal{U}$ *and* $(\pi : X \to \mathrm{Ob}\,\mathcal{C}, \alpha : X \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Mor}\,\mathcal{C} \to X)$ *an internal diagram on* $\boldsymbol{C}(\mathcal{C})$ *(5.1.3). Define a functor* $F_\alpha : \mathcal{C} \to \mathcal{U}$-$\mathbf{Ens}$ *by* $F_\alpha(Z) = \pi^{-1}(Z)$, $F_\alpha(f)(x) = \alpha(x, f)$ $(f \in \mathcal{C}(Z, W),\ x \in \pi^{-1}(Z))$. *If* $\psi : (\pi : X \to \mathrm{Ob}\,\mathcal{C}, \alpha) \to (\rho : Y \to \mathrm{Ob}\,\mathcal{C}, \beta)$ *is a morphism of internal diagrams, we define a natural transformation* $T_\psi : F_\alpha \to F_\beta$ *by* $(T_\psi)_Z(x) = \psi(x)$ *for* $x \in \pi^{-1}(Z)$. *It is easy to verify that a correspondence* $(\pi : X \to \mathrm{Ob}\,\mathcal{C}, \alpha) \mapsto F_\alpha$, $\psi \mapsto T_\psi$ *gives an isomorphism of categories* $(\mathcal{U}\text{-}\mathbf{Ens})^{\boldsymbol{C}(\mathcal{C})} \to \mathrm{Funct}(\mathcal{C}, \mathcal{U}\text{-}\mathbf{Ens})$. *In fact, for a functor* $F : \mathcal{C} \to \mathcal{U}$-$\mathbf{Ens}$, *let* $\pi_F : \coprod_{Z\in\mathrm{Ob}\,\mathcal{C}} F(Z) \to \mathrm{Ob}\,\mathcal{C}$ *be the map given by* $\pi_F(F(Z)) = \{Z\}$ *and* $\alpha_F : (\coprod_{Z\in\mathrm{Ob}\,\mathcal{C}} F(Z)) \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Mor}\,\mathcal{C} \to \coprod_{Z\in\mathrm{Ob}\,\mathcal{C}} F(Z)$ *the map given by* $\alpha(x, f) = F(f)(x)$ $(x \in F(Z),\ f \in \mathcal{C}(Z, W))$, *then the inverse is given by* $F \mapsto (\pi_F : \coprod_{Z\in\mathrm{Ob}\,\mathcal{C}} F(Z) \to \mathrm{Ob}\,\mathcal{C}, \alpha_F)$ *and* $(T : F \to G) \mapsto \coprod_{Z\in\mathrm{Ob}\,\mathcal{C}} T_Z$.

Definition 5.1.11 *Let* $(\pi : X \to C_0, \alpha)$ *be an internal diagram on* $\boldsymbol{C}$ *and* $i : Y \rightarrowtail X$ *a subobject of* X. *If* $\alpha(i \times id_{C_1}) : Y \times_{C_0} C_1 \to X$ *lifts to* Y, Y *is said to be an invariant subobject of* X.

There are faithful functors $\Phi : \boldsymbol{cat}(\mathcal{E}) \to \mathcal{E} \times \mathcal{E}$ and $\Phi_C : \boldsymbol{cat}(\mathcal{E}; C) \to \mathcal{E}$ given by $(C_0, C_1; \sigma, \tau, \varepsilon, \mu) \mapsto (C_0, C_1)$, $(f_0, f_1) \mapsto (f_0, f_1)$ and $(C, C_1; \sigma, \tau, \varepsilon, \mu) \mapsto C_1$, $(id_C, f) \mapsto f$. We denote by $\Phi' : \boldsymbol{gr}(\mathcal{E}) \to \mathcal{E} \times \mathcal{E}$, $\Phi'_C : \boldsymbol{gr}(\mathcal{E}; C) \to \mathcal{E}$ the compositions $\boldsymbol{gr}(\mathcal{E}) \hookrightarrow \boldsymbol{cat}(\mathcal{E}) \xrightarrow{\Phi} \mathcal{E} \times \mathcal{E}$, $\boldsymbol{gr}(\mathcal{E}; C) \hookrightarrow \boldsymbol{cat}(\mathcal{E}; C) \xrightarrow{\Phi_C} \mathcal{E}$.

Proposition 5.1.12 *1)* $\Phi : \boldsymbol{cat}(\mathcal{E}) \to \mathcal{E} \times \mathcal{E}$ *and* $\Phi' : \boldsymbol{gr}(\mathcal{E}) \to \mathcal{E} \times \mathcal{E}$ *creates limits.*

2) $\Phi_C : \boldsymbol{cat}(\mathcal{E}; C) \to \mathcal{E}$ *and* $\Phi'_C : \boldsymbol{gr}(\mathcal{E}; C) \to \mathcal{E}$ *creates limits of functors from connected categories.*

Proof. 1) Let $D : \mathcal{D} \to \boldsymbol{cat}(\mathcal{E})$ be a functor and $((L_0, L_1) \xrightarrow{(p_i^0, p_i^1)} \Phi D(i))_{i\in\mathrm{Ob}\,\mathcal{D}}$ a limiting cone of $\Phi D : \mathcal{D} \to \mathcal{E}\times\mathcal{E}$. Suppose that $D(i) = (C_{i0}, C_{i1}; \sigma_i, \tau_i, \varepsilon_i, \mu_i)$ and consider the projection functors $P_1, P_2 : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$. By (A.4.7), $(L_\nu \xrightarrow{p_i^\nu} C_{i\nu})_{i\in\mathrm{Ob}\,\mathcal{D}}$ $(\nu = 0, 1)$ are limiting cones of $P_\nu \Phi D$. It is easy to verify that $(L_1 \xrightarrow{\sigma_i p_i^1} C_{i0})_{i\in\mathrm{Ob}\,\mathcal{D}}$, $(L_1 \xrightarrow{\tau_i p_i^1} C_{i0})_{i\in\mathrm{Ob}\,\mathcal{D}}$ and $(L_0 \xrightarrow{\varepsilon_i p_i^0} C_{i1})_{i\in\mathrm{Ob}\,\mathcal{D}}$ are cones and there are unique morphisms $\sigma, \tau : L_1 \to L_0$, $\varepsilon : L_0 \to L_1$ satisfying $p_i^0\sigma = \sigma_i p_i^1$, $p_i^0\tau = \tau_i p_i^1$, $p_i^1\varepsilon = \varepsilon_i p_i^0$. Then, $(L_1 \times_{L_0} L_1 \xrightarrow{\mu_i(p_i^1\times p_i^1)} C_{i1})_{i\in\mathrm{Ob}\,\mathcal{D}}$ is also a cone and there is a unique morphism $\mu : L_1 \times_{L_0} L_1 \to L_1$ satisfying $p_i^1\mu = \mu_i(p_i^1 \times p_i^1)$. Hence the pair (L_0, L_1) has a unique structure of internal category with structure maps σ, τ, ε, μ such that $((L_0, L_1; \sigma, \tau, \varepsilon, \mu) \xrightarrow{(p_i^0, p_i^1)} D(i))_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of D. Thus Φ creates limits.

If each $D(i)$ is a groupoid with inverse ι_i, $(L_1 \xrightarrow{\iota_i p_i^1} C_{i1})_{i\in\mathrm{Ob}\,\mathcal{D}}$, is a cone and there is a unique morphism $\iota : L_1 \to L_1$ satisfying $\iota_i p_i^1 = p_i^1\iota$. Therefore $(L_0, L_1; \sigma, \tau, \varepsilon, \mu)$ is a groupoid and it follows that Φ' creates limits.

2) Let $D : \mathcal{D} \to \boldsymbol{cat}(\mathcal{E}; C)$ be a functor such that $\mathcal{D}$ is a connected category and $(L \xrightarrow{p_i} \Phi_C D(i))_{i \in \operatorname{Ob} \mathcal{D}}$ a limiting cone of $\Phi_C D : \mathcal{D} \to \mathcal{E}$. We denote by σ_i, τ_i, ε_i, μ_i the structure maps of $D(i)$ and put $D_i = \Phi_C D(i)$. If $\theta : i \to j$ is a morphism in $\mathcal{D}$, we have $\sigma_j p_j = \sigma_j D(\theta) p_i = \sigma_i p_i$. Similarly, $\tau_j p_j = \tau_i p_i$. Since $\mathcal{D}$ is connected, $\sigma_j p_j = \sigma_i p_i$ and $\tau_j p_j = \tau_i p_i$ for any pair of objects (i, j) of $\mathcal{D}$. We set $\sigma = \sigma_i p_i$ and $\tau = \tau_i p_i$. Then, p_i induces $p_i \times p_i : L \times_C L \to D_i \times_C D_i$ and we have a cone $(L \times_C L \xrightarrow{\mu_i(p_i \times p_i)} D_i)_{i \in \operatorname{Ob} \mathcal{D}}$ of $\Phi_C D$. There is a unique morphism $\mu : L \times_C L \to L$ satisfying $\mu_i(p_i \times p_i) = p_i \mu$ for any $i \in \operatorname{Ob} \mathcal{D}$. Since $(C \xrightarrow{\varepsilon_i} D_i)_{i \in \operatorname{Ob} \mathcal{D}}$ is a cone of $\Phi_C D$, there is a unique morphism $\varepsilon : C \to L$ satisfying $\varepsilon_i = p_i \varepsilon$ for any $i \in \operatorname{Ob} \mathcal{D}$. Hence the pair (C, L) has a unique structure of internal category with structure maps σ, τ, ε, μ such that $((C, L; \sigma, \tau, \varepsilon, \mu) \xrightarrow{(id_C, p_i)} D(i))_{i \in \operatorname{Ob} \mathcal{D}}$ is a limiting cone of D. Thus Φ_C creates limits. By the same argument as in 1), we can show that Φ'_C creates limits. $\square$

Proposition 5.1.13 *1)* $\boldsymbol{cat}(\mathcal{E})$ *and* $\boldsymbol{gr}(\mathcal{E})$ *have finite limits and the inclusion functor* $\boldsymbol{gr}(\mathcal{E}) \hookrightarrow \boldsymbol{cat}(\mathcal{E})$ *is left exact.*

2) $\boldsymbol{cat}(\mathcal{E}; C)$ *and* $\boldsymbol{gr}(\mathcal{E}; C)$ *have finite limits and the inclusion functor* $\boldsymbol{gr}(\mathcal{E}; C) \hookrightarrow \boldsymbol{cat}(\mathcal{E}; C)$ *is left exact.*

3) The inclusion functors $\boldsymbol{cat}(\mathcal{E}; C) \hookrightarrow \boldsymbol{cat}(\mathcal{E})$ *and* $\boldsymbol{gr}(\mathcal{E}; C) \hookrightarrow \boldsymbol{gr}(\mathcal{E})$ *preserves limits of finite connected diagrams.*

4) The inclusion functor $\mathcal{E}^{\boldsymbol{C}} \to \boldsymbol{cat}(\mathcal{E})/\boldsymbol{C}$ *creates finite limits. Hence the category* $\mathcal{E}^{\boldsymbol{C}}$ *has finite limits.*

Proof. 1) Since $\mathcal{E} \times \mathcal{E}$ has finite limits, so do $\boldsymbol{cat}(\mathcal{E})$ and $\boldsymbol{gr}(\mathcal{E})$ by (5.1.12). Hence $\Phi : \boldsymbol{cat}(\mathcal{E}) \to \mathcal{E} \times \mathcal{E}$ and $\Phi' : \boldsymbol{gr}(\mathcal{E}) \to \mathcal{E} \times \mathcal{E}$ are left exact. Again by (5.1.12), $\boldsymbol{gr}(\mathcal{E}) \hookrightarrow \boldsymbol{cat}(\mathcal{E})$ is left exact.

2) Since $\mathcal{E}$ has pull-backs, so do $\boldsymbol{cat}(\mathcal{E}; C)$ and $\boldsymbol{gr}(\mathcal{E}; C)$ by (5.1.12). By the existence of the terminal object of $\boldsymbol{cat}(\mathcal{E}; C)$ which belongs to $\boldsymbol{gr}(\mathcal{E}; C)$, $\boldsymbol{cat}(\mathcal{E}; C)$, $\boldsymbol{gr}(\mathcal{E}; C)$ have finite limits. Note that $\Phi : \boldsymbol{cat}(\mathcal{E}) \to \mathcal{E} \times \mathcal{E}$ and $\Phi' : \boldsymbol{gr}(\mathcal{E}) \to \mathcal{E} \times \mathcal{E}$ preserve pull-backs. Thus $\boldsymbol{gr}(\mathcal{E}) \hookrightarrow \boldsymbol{cat}(\mathcal{E})$ preserves pull-backs by (5.1.12) and it also preserves the terminal object. Hence the inclusion functor is left exact.

3) The assertion is a direct consequence of (5.1.12).

4) We denote by $\Psi : \mathcal{E}^{\boldsymbol{C}} \to \boldsymbol{cat}(\mathcal{E})/\boldsymbol{C}$ the inclusion functor. Let $D : \mathcal{D} \to \mathcal{E}^{\boldsymbol{C}}$ be a functor such that $\mathcal{D}$ is a connected finite category and set $D(i) = (\pi_i : X_i \to C_0, \alpha_i)$ for $i \in \operatorname{Ob} \mathcal{D}$. Suppose that $(\boldsymbol{L} \xrightarrow{(p_i^0, p_i^1)} \Psi D(i))_{i \in \operatorname{Ob} \mathcal{D}}$ is a limiting cone of ΨD. We denote by $(l_0, l_1) : \boldsymbol{L} \to \boldsymbol{C}$ the structure map of $\boldsymbol{L}$ as an object of $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{C}$. Then, it is a composition $\boldsymbol{L} \xrightarrow{(p_i^0, p_i^1)} (X_i, C_{\alpha_i}; \sigma_{\alpha_i}, \tau_{\alpha_i}, \varepsilon_{\alpha_i}, \mu_{\alpha_i}) \xrightarrow{(\pi_i, \mathrm{pr}_2)} \boldsymbol{C}$. We set $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{L} = (L_0, L_1; \bar{\sigma}, \bar{\tau}, \bar{\varepsilon}, \bar{\mu})$. We claim that (l_0, l_1) is a discrete opfibration. Suppose that $a : Y \to C_1$ and $b : Y \to L_0$ are morphisms satisfying $\sigma a = l_0 b$, then we have $\sigma a = \pi_i p_i^0 b$ for each $i \in \operatorname{Ob} \mathcal{D}$. Since $\mathrm{pr}_2 : X_i \times_{C_0} C_1 \to C_1$ is a pull-back of π_i along σ , there is a unique morphism $a_i : Y \to X_i \times_{C_0} C_1$ satisfying $a = \mathrm{pr}_2 a_i$ and $p_i^0 b = \mathrm{pr}_1 a_i$. It follows from the uniqueness of a_i that $(Y \xrightarrow{a_i} X_i \times_{C_0} C_1)_{i \in \operatorname{Ob} \mathcal{D}}$ is a cone of the composition of the functors $\mathcal{D} \xrightarrow{D} \mathcal{E}^{\boldsymbol{C}} \xrightarrow{\Psi} \boldsymbol{cat}(\mathcal{E})/\boldsymbol{C} \xrightarrow{\Sigma_{\boldsymbol{C}}} \boldsymbol{cat}(\mathcal{E}) \xrightarrow{\Phi} \mathcal{E} \times \mathcal{E} \xrightarrow{P_1} \mathcal{E}$. By (A.3.11), (A.4.7), (5.1.12) and above 1), $(L_1 \xrightarrow{p_i^1} X_i \times_{C_0} C_1)_{i \in \operatorname{Ob} \mathcal{D}}$ is a limiting cone of $P_1 \Phi \Sigma_{\boldsymbol{C}} \Psi D$. There is a unique morphism $c : Y \to L_1$ such that $a_i = p_i^1 c$, hence $a = \mathrm{pr}_2 a_i = \mathrm{pr}_2 p_i^1 c = l_1 c$. Since $p_i^0 \bar{\sigma} c = \mathrm{pr}_1 p_i^1 c = \mathrm{pr}_1 a_i = p_i^0 b$ for any $i \in \operatorname{Ob} \mathcal{D}$ and $(L_0 \xrightarrow{p_i^0} X_i)_{i \in \operatorname{Ob} \mathcal{D}}$ is a limiting cone of $P_2 \Phi \Sigma_{\boldsymbol{C}} \Psi D$, we have $\bar{\sigma} c = b$. If $c' : Y \to L_1$ is a morphism satisfying $l_1 c' = a$ and $\bar{\sigma} c' = b$, then $\mathrm{pr}_2 p_i^1 c' = l_1 c' = a = l_1 c = \mathrm{pr}_2 p_i^1 c$ and $\mathrm{pr}_1 p_i^1 c' = p_i^0 \bar{\sigma} c' = p_i^0 b = p_i^0 \bar{\sigma} c = \mathrm{pr}_1 p_i^1 c$ for any $i \in \operatorname{Ob} \mathcal{D}$. Hence $p_i^1 c' = p_i^1 c$ and this implies that $c' = c$. Thus (l_0, l_1) is a discrete opfibration and Ψ creates finite limits of connected diagrams.
Since $(id_{C_0} : C_0 \to C_0, \tau \mathrm{pr}_2 : C_0 \times_{C_0} C_1 \to C_0)$ is a terminal object of $\mathcal{E}^{\boldsymbol{C}}$ and Ψ maps this to

$$(C_0, C_0 \times_{C_0} C_1; \sigma \mathrm{pr}_2, \tau \mathrm{pr}_2, (id_{C_0}, \varepsilon), (id_{C_0} \times \mu)(id_{C_0} \times \mathrm{pr}_1)) \xrightarrow{(id_{C_0}, \mathrm{pr}_2)} \boldsymbol{C},$$

which is isomorphic to $id_{\boldsymbol{C}} : \boldsymbol{C} \to \boldsymbol{C}$, Ψ preserves terminal objects. Therefore Ψ creates arbitrary finite limits. $\square$

The next result follows from the definition of discrete opfibration and (A.3.1).

Proposition 5.1.14 *Let* $f : \boldsymbol{C} \to \boldsymbol{D}$ *and* $g : \boldsymbol{D} \to \boldsymbol{E}$ *be morphisms in* $\boldsymbol{cat}(\mathcal{E})$.

1) If f *and* g *are discrete opfibrations, so is the composite* gf.

2) If gf *and* g *are discrete opfibrations, so is* f.

Proposition 5.1.15 *For* $(\pi : X \to C_0, \alpha) \in \operatorname{Ob} \mathcal{E}^{\boldsymbol{C}}$, *let* $f : \boldsymbol{X} \to \boldsymbol{C}$ *be the corresponding discrete opfibration. Then,* $\mathcal{E}^{\boldsymbol{C}}/(\pi : X \to C_0, \alpha)$ *is isomorphic to* $\mathcal{E}^{\boldsymbol{X}}$.

Proof. Define a functor $F : \mathcal{E}^{\boldsymbol{X}} \to \mathcal{E}^{\boldsymbol{C}}/(\pi : X \to C_0, \alpha)$ as follows. For $(\rho : Y \to X, \beta : Y \times_X (X \times_{C_0} C_1) \to Y) \in \operatorname{Ob}\mathcal{E}^{\boldsymbol{X}}$, we denote by $\beta' : Y \times_{C_0} C_1 \to Y$ the composition $Y \times_{C_0} C_1 \xrightarrow{\cong} Y \times_X (X \times_{C_0} C_1) \xrightarrow{\beta} Y$. We set $F(\rho : Y \to X, \beta) = ((\pi\rho : Y \to C_0, \beta') \xrightarrow{\rho} (\pi : X \to C_0, \alpha))$. If $\varphi : (\rho : Y \to X, \beta) \to (\lambda : Z \to X, \gamma)$ is a morphism in $\mathcal{E}^{\boldsymbol{X}}$, φ is also regarded as a morphism in $\mathcal{E}^{\boldsymbol{C}}/(\pi : X \to C_0, \alpha)$ and we set $F(\varphi) = \varphi$. Then, F is an isomorphism of the categories. □

Proposition 5.1.16 *Let $f : \boldsymbol{C} \to \boldsymbol{D}$ be a morphism in $\boldsymbol{cat}(\mathcal{E})$. Then, the pull-back functor $f^* : \boldsymbol{cat}(\mathcal{E})/\boldsymbol{D} \to \boldsymbol{cat}(\mathcal{E})/\boldsymbol{C}$ preserves discrete opfibration, hence it induces a functor $f^* : \mathcal{E}^{\boldsymbol{D}} \to \mathcal{E}^{\boldsymbol{C}}$ which is left exact.*

Proof. Let $(g_0, g_1) : (E_0, E_1; \sigma, \tau, \varepsilon, \mu) \to (D_0, D_1; \sigma, \tau, \varepsilon, \mu)$ be a discrete opfibration. Form pull-backs of g_i along f_i $(i = 0, 1)$.

$$\begin{tikzcd}
F_1 \arrow[r, "\bar{f}_1"] \arrow[d, "\bar{g}_1"] & E_1 \arrow[r, "\sigma''"] \arrow[d, "g_1"] & E_0 \arrow[d, "g_0"] & F_0 \arrow[l, "\bar{f}_0"'] \arrow[d, "\bar{g}_0"] \\
C_1 \arrow[r, "f_1"] & D_1 \arrow[r, "\sigma'"] & D_0 & C_0 \arrow[l, "f_0"']
\end{tikzcd}$$

Then, $f_0\sigma\bar{g}_1 = \sigma' f_1\bar{g}_1 = \sigma' f_1\bar{g}_1 = \sigma' g_1\bar{f}_1 = g_0\sigma''\bar{f}_1 = g_0\sigma''\bar{f}_1$ and there is a unique morphism $\bar{\sigma} : F_1 \to F_0$ such that $\sigma\bar{g}_1 = g_0\bar{\sigma}$ and $\sigma''\bar{f}_1 = \bar{f}_0\bar{\sigma}$. Since each square of the above diagram is a pull-back, it follows from (A.3.1) that $f^*(\boldsymbol{E} \xrightarrow{g} \boldsymbol{D}) = (\boldsymbol{F} \xrightarrow{\bar{g}} \boldsymbol{C})$ is a discrete opfibration. Since the pull-back functor $f^* : \boldsymbol{cat}(\mathcal{E})/\boldsymbol{D} \to \boldsymbol{cat}(\mathcal{E})/\boldsymbol{C}$ has a left adjoint Σ_f, it is left exact. Hence $f^* : \mathcal{E}^{\boldsymbol{D}} \to \mathcal{E}^{\boldsymbol{C}}$ is also left exact by (5.1.13). □

Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$. For morphisms $f : D \to C_0$, $g : E \to C_0$ of $\mathcal{E}$, we denote by $C_{f,g}$ a limit of a diagram $D \xrightarrow{f} C_0 \xleftarrow{\sigma} C_1 \xrightarrow{\tau} C_0 \xleftarrow{g} E$. $C_{f,g}$ is also denoted by $D \times_{C_0} C_1 \times_{C_0} E$. Define $\sigma_{f,g} : C_{f,g} \to D$, $f{*}g : C_{f,g} \to C_1$ and $\tau_{f,g} : C_{f,g} \to E$ to be the projections onto each component. If $\boldsymbol{C}$ is an internal groupoid, let $\iota_{f,g} : C_{f,g} \to C_{g,f}$ be the morphism induced by $\tau_{f,g}$, $\iota(f{*}g) : C_{f,g} \to C_1$ and $\sigma_{f,g}$.

Let $h : F \to C_0$ a morphism of $\mathcal{E}$. We define $\mu_{f,g,h} : C_{f,g} \times_E C_{g,h} \to C_{f,h}$ to be the composition $C_{f,g} \times_E C_{g,h} \cong D \times_{C_0} C_1 \times_{C_0} C_1 \times_{C_0} F \xrightarrow{id_D \times \mu \times id_F} C_{f,h}$.

In the case $D = E = F$ and $f = g = h$, we set $C_{f,f} = D_{f1}$, $\sigma_{f,f} = \sigma_f$, $\tau_{f,f} = \tau_f$, $\mu_{f,f,f} = \mu_f$, $f{*}f = \tilde{f}$, ($\iota_{f,f} = \iota_f$ if $\boldsymbol{C}$ is an internal groupoid) and denote by $\varepsilon_f : D \to D_{f1}$ the morphism induced by $id_D : D \to D$ and $\varepsilon f : D \to C_1$. Then we have an internal category $\boldsymbol{C}_f = (D, D_{f1}; \sigma_f, \tau_f, \varepsilon_f, \mu_f)$ with structure maps σ_f, τ_f, ε_f, μ_f and also have an internal functor $(f, \tilde{f}) : \boldsymbol{C}_f \to \boldsymbol{C}$.

Definition 5.1.17 *We call $\boldsymbol{C}_f$ a pull-back of $\boldsymbol{C}$ along f. Note that $\boldsymbol{C}_f$ is an internal groupoid with inverse ι_f if $\boldsymbol{C}$ is so.*

The following fact is easily verified.

Proposition 5.1.18 *1) Let $\boldsymbol{D} = (D_0, D_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$ and $(f_0, f_1) : \boldsymbol{D} \to \boldsymbol{C}$ an internal functor, then there is a unique morphism $h : D_1 \to D_{f_01}$ such that $(id_{D_0}, h) : \boldsymbol{D} \to \boldsymbol{C}_{f_0}$ is an internal functor and $f_1 = \tilde{f}_0 h$.*

2) Let $f' : D' \to D$ and $g' : E' \to E$ be morphisms of $\mathcal{E}$. Then, there is a natural isomorphism $C_{ff',gg'} \cong (C_{f,g})_{f',g'}$ which commutes with various structure maps.

Definition 5.1.19 *An internal functor $f = (f_0, f_1) : \boldsymbol{D} \to \boldsymbol{C}$ is said to be faithful (resp. full, fully faithful) if the induced morphism $h : D_1 \to D_{f_01}$ is a monomorphism (resp. epimorphism, isomorphism). An internal subcategory $\boldsymbol{D}$ of $\boldsymbol{C}$ consists of subobjects $D_0 \rightarrowtail C_0$, $D_1 \rightarrowtail C_1$ such that these monomorphism gives an internal functor. If this internal functor is fully faithful, $\boldsymbol{D}$ is called a full subcategory.*

Remark 5.1.20 *If f and g are monomorphisms, then $f{*}g : C_{f,g} \to C_1$ is also a monomorphism. If $\mathcal{E}$ is the category of sets, the image of $f{*}g$ consists of morphisms of C_0 whose sources and targets belong to the images of f and g, respectively.*

Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$. We define an internal category $\boldsymbol{C}^2$ as follows. Set $C_0^2 = C_1$ and let $C_1^2 \overset{p}{\underset{q}{\rightrightarrows}} C_1 \times_{C_0} C_1$ be the kernel pair of $\mu : C_1 \times_{C_0} C_1 \to C_1$. Define $\sigma^2, \tau^2 : C_1^2 \to C_0^2$ by $\sigma^2 = \mathrm{pr}_1 p$, $\tau^2 = \mathrm{pr}_2 q$. Since $\mu(id_{C_1}, \varepsilon\tau) = \mu(\varepsilon\sigma, id_{C_1}) = id_{C_1}$, there is a unique morphism $\varepsilon^2 : C_0^2 =$

$C_1 \to C_1^2$ satisfying $p\varepsilon^2 = (id_{C_1}, \varepsilon\tau)$ and $q\varepsilon^2 = (\varepsilon\sigma, id_{C_1})$. We note that a composition $C_1^2 \times_{C_0^2} C_1^2 \xrightarrow{\mathrm{pr}_1} C_1^2 \xrightarrow{p} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1 \xrightarrow{\tau} C_0$ coincides with $C_1^2 \times_{C_0^2} C_1^2 \xrightarrow{\mathrm{pr}_2} C_1^2 \xrightarrow{p} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1 \xrightarrow{\sigma} C_0$. In fact, this is easily verified by applying the Yoneda embedding $h : \mathcal{E} \to \widehat{\mathcal{E}}$ to these morphisms (5.1.3). Hence there is a unique morphism $m_1 : C_1^2 \times_{C_0^2} C_1^2 \to C_1 \times_{C_0} C_1$ satisfying $\mathrm{pr}_1 m_1 = \mathrm{pr}_2 p\mathrm{pr}_1$ and $\mathrm{pr}_2 m_1 = \mathrm{pr}_2 p\mathrm{pr}_2$. Since a composition $C_1^2 \times_{C_0^2} C_1^2 \xrightarrow{\mathrm{pr}_1} C_1^2 \xrightarrow{p} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_1} C_1 \xrightarrow{\tau} C_0$ coincides with $\sigma\mu m_1$, we have a morphism $m_1' : C_1^2 \times_{C_0^2} C_1^2 \to C_1 \times_{C_0} C_1$ satisfying $\mathrm{pr}_1 m_1' = \mathrm{pr}_1 p\mathrm{pr}_1$ and $\mathrm{pr}_2 m_1' = \mu m_1$. Similarly, a composition $C_1^2 \times_{C_0^2} C_1^2 \xrightarrow{\mathrm{pr}_1} C_1^2 \xrightarrow{q} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_1} C_1 \xrightarrow{\tau} C_0$ coincides with $C_1^2 \times_{C_0^2} C_1^2 \xrightarrow{\mathrm{pr}_2} C_1^2 \xrightarrow{q} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_1} C_1 \xrightarrow{\sigma} C_0$. We also have a unique morphism $m_2 : C_1^2 \times_{C_0^2} C_1^2 \to C_1 \times_{C_0} C_1$ satisfying $\mathrm{pr}_1 m_2 = \mathrm{pr}_1 q\mathrm{pr}_1$ and $\mathrm{pr}_2 m_2 = \mathrm{pr}_1 q\mathrm{pr}_2$. Since a composition $C_1^2 \times_{C_0^2} C_1^2 \xrightarrow{\mathrm{pr}_2} C_1^2 \xrightarrow{q} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1 \xrightarrow{\sigma} C_0$ coincides with $\tau\mu m_2$, we have a morphism $m_2' : C_1^2 \times_{C_0^2} C_1^2 \to C_1 \times_{C_0} C_1$ satisfying $\mathrm{pr}_1 m_2' = \mu m_2$ and $\mathrm{pr}_2 m_1' = \mathrm{pr}_2 q\mathrm{pr}_2$. Moreover, we can verify that $\mu m_1' = \mu m_2'$ and we define $\mu^2 : C_1^2 \times_{C_0^2} C_1^2 \to C_1^2$ to be the morphism satisfying $p\mu^2 = m_1'$ and $q\mu^2 = m_2'$. We put $\boldsymbol{C}^2 = (C_0^2, C_1^2; \sigma^2, \tau^2, \varepsilon^2, \mu^2)$.

Set $\bar{\sigma} = \mathrm{pr}_1 q, \bar{\tau} = \mathrm{pr}_2 p : C_1^2 \to C_1$ and define internal functors $\sigma_{\boldsymbol{C}}, \tau_{\boldsymbol{C}} : \boldsymbol{C}^2 \to \boldsymbol{C}$ by $\sigma_{\boldsymbol{C}} = (\sigma, \bar{\sigma})$, $\tau_{\boldsymbol{C}} = (\tau, \bar{\tau})$. Since $\mu(\varepsilon\sigma, id_{C_1}) = \mu(id_{C_1}, \varepsilon\tau) = id_{C_1}$, there is a unique morphism $\bar{\varepsilon} : C_0^2 = C_1 \to C_1^2$ satisfying $p\bar{\varepsilon} = (\varepsilon\sigma, id_{C_1})$ and $q\bar{\varepsilon} = (id_{C_1}, \varepsilon\tau)$. Define $\varepsilon_{\boldsymbol{C}} : \boldsymbol{C} \to \boldsymbol{C}^2$ by $\varepsilon_{\boldsymbol{C}} = (\varepsilon, \bar{\varepsilon})$. In $\boldsymbol{cat}(\mathcal{E})$, form a pull-back

$$\begin{tikzcd}
\boldsymbol{C}^2 \times_{\boldsymbol{C}} \boldsymbol{C}^2 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & \boldsymbol{C}^2 \arrow[d, "\sigma_{\boldsymbol{C}}"] \\
\boldsymbol{C}^2 \arrow[r, "\tau_{\boldsymbol{C}}"] & \boldsymbol{C}
\end{tikzcd}$$

and define an internal functor $\mu_{\boldsymbol{C}} : \boldsymbol{C}^2 \times_{\boldsymbol{C}} \boldsymbol{C}^2 \to \boldsymbol{C}^2$ as follows. We note that a composition $C_1^2 \times_{C_1} C_1^2 \xrightarrow{\mathrm{pr}_1} C_1^2 \xrightarrow{p} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_1} C_1 \xrightarrow{\tau} C_0$ coincides with $C_1^2 \times_{C_1} C_1^2 \xrightarrow{\mathrm{pr}_2} C_1^2 \xrightarrow{p} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_1} C_1 \xrightarrow{\sigma} C_0$. Hence there is a unique morphism $\bar{m}_1 : C_1^2 \times_{C_1} C_1^2 \to C_1 \times_{C_0} C_1$ satisfying $\mathrm{pr}_1 \bar{m}_1 = \mathrm{pr}_1 p\mathrm{pr}_1$ and $\mathrm{pr}_2 \bar{m}_1 = \mathrm{pr}_1 p\mathrm{pr}_2$. Since a composition $C_1^2 \times_{C_1} C_1^2 \xrightarrow{\mathrm{pr}_2} C_1^2 \xrightarrow{p} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1 \xrightarrow{\sigma} C_0$ coincides with $\tau\mu\bar{m}_1$, we have a morphism $\bar{m}_1' : C_1^2 \times_{C_1} C_1^2 \to C_1 \times_{C_0} C_1$ satisfying $\mathrm{pr}_1 \bar{m}_1' = \mu\bar{m}_1$ and $\mathrm{pr}_2 \bar{m}_1' = \mathrm{pr}_2 p\mathrm{pr}_2$. Similarly, a composition $C_1^2 \times_{C_1} C_1^2 \xrightarrow{\mathrm{pr}_1} C_1^2 \xrightarrow{q} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1 \xrightarrow{\tau} C_0$ coincides with $C_1^2 \times_{C_1} C_1^2 \xrightarrow{\mathrm{pr}_2} C_1^2 \xrightarrow{q} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1 \xrightarrow{\sigma} C_0$. We also have a unique morphism $\bar{m}_2 : C_1^2 \times_{C_1} C_1^2 \to C_1 \times_{C_0} C_1$ satisfying $\mathrm{pr}_1 \bar{m}_2 = \mathrm{pr}_2 q\mathrm{pr}_1$ and $\mathrm{pr}_2 \bar{m}_2 = \mathrm{pr}_2 q\mathrm{pr}_2$. Since a composition $C_1^2 \times_{C_1} C_1^2 \xrightarrow{\mathrm{pr}_1} C_1^2 \xrightarrow{q} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_1} C_1 \xrightarrow{\tau} C_0$ coincides with $\sigma\mu\bar{m}_2$, we have a morphism $\bar{m}_2' : C_1^2 \times_{C_1} C_1^2 \to C_1 \times_{C_0} C_1$ satisfying $\mathrm{pr}_1 \bar{m}_2' = \mathrm{pr}_1 q\mathrm{pr}_1$ and $\mathrm{pr}_2 \bar{m}_1' = \mu\bar{m}_2$. Moreover, we can verify that $\mu\bar{m}_1' = \mu\bar{m}_2'$ and we define $\bar{\mu} : C_1^2 \times_{C_1} C_1^2 \to C_1^2$ to be the morphism satisfying $p\bar{\mu} = \bar{m}_1'$ and $q\bar{\mu} = \bar{m}_2'$. Now, $\mu_{\boldsymbol{C}}$ is defined by $\mu_{\boldsymbol{C}} = (\mu, \bar{\mu})$.

Proposition 5.1.21 *1) $(\boldsymbol{C}, \boldsymbol{C}^2; \sigma_{\boldsymbol{C}}, \tau_{\boldsymbol{C}}, \varepsilon_{\boldsymbol{C}}, \mu_{\boldsymbol{C}})$ is an internal category in $\boldsymbol{cat}(\mathcal{E})$. If $\boldsymbol{C}$ is an internal groupoid, so is $\boldsymbol{C}^2$.*

2) Let $f, g : \boldsymbol{D} \to \boldsymbol{C}$ be a pair of internal functors and $\mathrm{Nat}(f, g)$ denotes the set of internal natural transformations from f to g. For each $\varphi \in \mathrm{Nat}(f, g)$, define an internal functor $d_\varphi : \boldsymbol{D} \to \boldsymbol{C}^2$ by $d_\varphi = (\varphi, \bar{\varphi})$, where $\bar{\varphi} : D_1 \to C_1^2$ is the unique morphism satisfying $p\bar{\varphi} = (\varphi\sigma, g_1)$ and $q\bar{\varphi} = (f_1, \varphi\tau)$. Then, the correspondence $\varphi \mapsto d_\varphi$ gives a bijection from $\mathrm{Nat}(f, g)$ onto the set of internal functors $h : \boldsymbol{D} \to \boldsymbol{C}^2$ such that $\sigma_{\boldsymbol{C}} h = f$, $\tau_{\boldsymbol{C}} h = g$.

Proof. 1) The first assertion follows from a routine verification. Suppose that $\boldsymbol{C}$ is an internal groupoid with inverse $\iota : C_1 \to C_1$. Since $\tau\mathrm{pr}_2 q = \tau\mu q = \tau\mu p = \tau\mathrm{pr}_2 p = \sigma\iota\mathrm{pr}_2 p$ and $\tau\iota\mathrm{pr}_1 q = \sigma\mathrm{pr}_1 q = \sigma\mu q = \sigma\mu p = \sigma\mathrm{pr}_1 p$, there are morphisms $\bar{p}, \bar{q} : C_1^2 \to C_1 \times_{C_0} C_1$ such that $\mathrm{pr}_1 \bar{p} = \mathrm{pr}_2 q$, $\mathrm{pr}_2 \bar{p} = \iota\mathrm{pr}_2 p$, $\mathrm{pr}_1 \bar{q} = \iota\mathrm{pr}_1 q$, $\mathrm{pr}_2 \bar{q} = \mathrm{pr}_1 p$. Since $\mu(\mathrm{pr}_1 p, \mathrm{pr}_2 p) = \mu p = \mu q = \mu(\mathrm{pr}_1 q, \mathrm{pr}_2 q)$, we have $\mu(\iota\mathrm{pr}_1 q, \mathrm{pr}_1 p) = \mu(\iota\mathrm{pr}_1 q, \mu(\mathrm{pr}_1 p, \varepsilon\sigma\mathrm{pr}_2 p))$
$= \mu(\iota\mathrm{pr}_1 q, \mu(\mathrm{pr}_1 p, \mu(\mathrm{pr}_2 p, \iota\mathrm{pr}_2 p))) = \mu(\iota\mathrm{pr}_1 q, \mu(\mu(\mathrm{pr}_1 p, \mathrm{pr}_2 p), \iota\mathrm{pr}_2 p))$
$= \mu(\mu(\iota\mathrm{pr}_1 q, \mu(\mathrm{pr}_1 p, \mathrm{pr}_2 p)), \iota\mathrm{pr}_2 p) = \mu(\mu(\iota\mathrm{pr}_1 q, \mu(\mathrm{pr}_1 q, \mathrm{pr}_2 q)), \iota\mathrm{pr}_2 p)$
$= \mu(\mu(\mu(\iota\mathrm{pr}_1 q, \mathrm{pr}_1 q), \mathrm{pr}_2 q), \iota\mathrm{pr}_2 p) = \mu(\mu(\varepsilon\tau\mathrm{pr}_1 q, \mathrm{pr}_2 q), \iota\mathrm{pr}_2 p) = \mu(\mathrm{pr}_2 q, \iota\mathrm{pr}_2 p)$. Hence $\mu\bar{p} = \mu(\mathrm{pr}_1 \bar{p}, \mathrm{pr}_2 \bar{p}) = \mu(\mathrm{pr}_2 q, \iota\mathrm{pr}_2 p) = \mu(\iota\mathrm{pr}_1 q, \mathrm{pr}_1 p) = \mu(\mathrm{pr}_1 \bar{q}, \mathrm{pr}_2 \bar{q}) = \mu\bar{q}$ and it follows that there is a morphism $\iota^2 : C_1^2 \to C_1^2$ satisfying $p\iota^2 = \bar{p}$, $q\iota^2 = \bar{q}$. It is easy to verify that ι is an inverse of C_1^2.

2) It is easy to verify that $d_\varphi : \boldsymbol{D} \to \boldsymbol{C}^2$ is an internal functor satisfying $\sigma_{\boldsymbol{C}} d_\varphi = f$, $\tau_{\boldsymbol{C}} d_\varphi = g$. Suppose that $h : \boldsymbol{D} \to \boldsymbol{C}^2$ is an internal functor such that $\sigma_{\boldsymbol{C}} h = f$, $\tau_{\boldsymbol{C}} h = g$. Then, $h_0 : D_0 \to C_0^2 = C_1$ gives an internal natural transformation $f \to g$ and $h \mapsto h_0$ is the inverse correspondence of $\varphi \mapsto d_\varphi$. $\square$

Let $\mathcal{E}$ be a cartesian closed category with finite limits and $\boldsymbol{C}$ (resp. $\boldsymbol{D}$) an internal category in $\mathcal{E}$ with structure maps σ, τ, ε, μ (resp. σ', τ', ε', μ'). We form pull-backs

$$\begin{tikzcd} S \arrow[r, "p_1"] \arrow[d, "p_0"'] & D_1^{C_1} \arrow[d, "(\sigma')^{C_1}"] \\ D_0^{C_0} \arrow[r, "D_0^{\sigma}"] & D_0^{C_1} \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "q_1"] \arrow[d, "q_0"'] & D_1^{C_1} \arrow[d, "(\tau')^{C_1}"] \\ D_0^{C_0} \arrow[r, "D_0^{\tau}"] & D_0^{C_1} \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "r_1"] \arrow[d, "r_0"'] & D_1^{C_1} \arrow[d, "D_1^{\varepsilon}"] \\ D_0^{C_0} \arrow[r, "(\varepsilon')^{C_0}"] & D_1^{C_0} \end{tikzcd}$$

and regard S, T, E as subobjects of $D_0^{C_0} \times D_1^{C_1}$. Let $i : S \cap T \to S$ and $j : S \cap T \to T$ be the inclusion morphisms. Then, $p_l i = q_l j$ $(l = 0, 1)$ and there is a unique morphism $\theta : (S \cap T) \times (C_1 \times_{C_0} C_1) \to D_1 \times_{D_0} D_1$ such that $\mathrm{pr}_1\theta = ev(q_1 j \times \mathrm{pr}_1)$ and $\mathrm{pr}_2\theta = ev(p_1 i \times \mathrm{pr}_2)$. In fact, $\tau' ev(q_1 j \times \mathrm{pr}_1) = ev((\tau')^{C_1} q_1 j \times \mathrm{pr}_1) = ev(D_0^{\tau} q_0 j \times \mathrm{pr}_1) = ev(D_0^{\tau} \times id_{C_1})(q_0 j \times \mathrm{pr}_1) = ev(id_{D_0^{C_0}} \times \tau)(q_0 j \times \mathrm{pr}_1) = ev(q_0 j \times \tau\mathrm{pr}_1) = ev(p_0 i \times \sigma\mathrm{pr}_2) = ev(id_{D_0^{C_0}} \times \sigma)(p_0 i \times \mathrm{pr}_2) = ev(D_0^{\sigma} \times id_{C_1})(p_0 i \times \mathrm{pr}_2) = ev(D_0^{\sigma} p_0 i \times \mathrm{pr}_2) = ev((\sigma')^{C_1} p_1 i \times \mathrm{pr}_2) = \sigma' ev(p_1 i \times \mathrm{pr}_2)$. We denote by $\bar{\theta} : S \cap T \to (D_1 \times_{D_0} D_1)^{C_1 \times_{C_0} C_1}$ the transpose of θ. We denote by $k : S \cap T \cap E \to S \cap T$ the inclusion morphism and let $e_0 : M_0 \to S \cap T \cap E$ be an equalizer of $(\mu')^{C_1 \times_{C_0} C_1}\bar{\theta}k$ and $D_1^{\mu} p_1 ik$. Consider the following diagram, where $d : M_0 \to D_0^{C_0}$ denotes the composition $M_0 \xrightarrow{e_0} S \cap T \cap E \xrightarrow{k} S \cap T \xrightarrow{i} S \xrightarrow{p_0} D_0^{C_0}$ and each square is a pull-back.

$$\begin{tikzcd} N \arrow[r, "t"] \arrow[d, "s"'] & T' \arrow[r, "q_1'"] \arrow[d, "q_2'"] & M_0 \arrow[d, "d"] \\ S' \arrow[r, "p_2'"] \arrow[d, "p_1'"'] & D_1^{C_0} \arrow[r, "(\tau')^{C_0}"] \arrow[d, "(\sigma')^{C_0}"] & D_0^{C_0} \\ M_0 \arrow[r, "d"] & D_0^{C_0} & \end{tikzcd}$$

There are morphisms $\zeta, \xi : N \times C_1 \to D_1 \times_{D_0} D_1$ such that $\mathrm{pr}_1\zeta = ev(p_1 ike_0 p_1' s \times id_{C_1})$, $\mathrm{pr}_2\zeta = ev(p_2' s \times \tau)$, $\mathrm{pr}_1\xi = ev(p_2' s \times \sigma)$, $\mathrm{pr}_2\xi = ev(p_1 ike_0 q_1' t \times id_{C_1})$. In fact, we have $\tau' ev(p_1 ike_0 p_1' s \times id_{C_1}) = ev((\tau')^{C_1} q_1 jke_0 p_1' s \times id_{C_1}) = ev(D_0^{\tau} q_0 jke_0 p_1' s \times id_{C_1}) = ev(D_0^{\tau} dp_1' s \times id_{C_1}) = ev(D_0^{\tau}(\sigma')^{C_0} p_2' s \times id_{C_1}) = ev((\sigma')^{C_1} \times id_{C_1})(D_1^{\tau} p_2' s \times id_{C_1}) = \sigma' ev(D_1^{\tau} \times id_{C_1})(p_2' s \times id_{C_1}) = \sigma' ev(p_2' s \times \tau)$, $\tau' ev(p_2' s \times \sigma) = \tau' ev(D_1^{\sigma} \times id_{C_1})(q_2' t \times id_{C_1}) = ev((\tau')^{C_1} \times id_{C_1})(D_1^{\sigma} q_2' t \times id_{C_1}) = ev(D_0^{\sigma}(\tau')^{C_0} q_2' t \times id_{C_1}) = ev(D_0^{\sigma} dq_1' t \times id_{C_1}) = ev(D_0^{\sigma} p_0 ike_0 q_1' t \times id_{C_1}) = ev((\sigma')^{C_1} p_1 ike_0 p_1' s \times id_{C_1}) = \sigma' ev(p_1 ike_0 p_1' s \times id_{C_1})$. Let us denote by $\bar{\zeta}, \bar{\xi} : N \to (D_1 \times_{D_0} D_1)^{C_1}$ the transposes of ζ, ξ and let $e_1 : M_1 \to N$ be the equalizer of $(\mu')^{C_1}\bar{\zeta}, (\mu')^{C_1}\bar{\xi} : N \to D_1^{C_1}$.

We define an internal category $\boldsymbol{D}^{\boldsymbol{C}} = (M_1, M_0; \tilde{\sigma}, \tilde{\tau}, \tilde{\varepsilon}, \tilde{\mu})$ as follows. Set $\tilde{\sigma} = p_1' se_1$, $\tilde{\tau} = q_1' te_1$. Since $d = (\sigma')^{C_0}(\varepsilon')^{C_0} d = (\tau')^{C_0}(\varepsilon')^{C_0} d$, there are morphisms $\varepsilon_s : M_0 \to S$, $\varepsilon_t : M_0 \to T$ satisfying $p_1'\varepsilon_s = q_1'\varepsilon_t = id_{M_0}$, $p_2'\varepsilon_s = q_2'\varepsilon_t = (\varepsilon')^{C_0} d$. Then, we have a morphism $\varepsilon_N : M_0 \to N$ satisfying $s\varepsilon_N = \varepsilon_s$ and $t\varepsilon_N = \varepsilon_t$. Since $\mathrm{pr}_1\zeta(\varepsilon_N \times id_{C_1}) = ev(p_1 ike_0 \times id_{C_1})$, $\mathrm{pr}_2\zeta(\varepsilon_N \times id_{C_1}) = ev((\varepsilon')^{C_0} p_0 ike_0 \times \tau) = ev(D_1^{\tau}(\varepsilon')^{C_0} q_0 jke_0 \times id_{C_1}) = ev((\varepsilon')^{C_1} D_0^{\tau} q_0 jke_0 \times id_{C_1}) = \varepsilon' ev((\tau')^{C_1} q_1 jke_0 \times id_{C_1})$ and $\mathrm{pr}_1\xi(\varepsilon_N \times id_{C_1}) = ev((\varepsilon')^{C_0} p_0 ike_0 \times \sigma) = ev(D_1^{\sigma}(\varepsilon')^{C_0} p_0 ike_0 \times id_{C_1}) = ev((\varepsilon')^{C_1} D_0^{\sigma} p_0 ike_0 \times id_{C_1}) = \varepsilon' ev((\sigma')^{C_1} p_1 ike_0 \times id_{C_1})$, $\mathrm{pr}_2\xi(\varepsilon_N \times id_{C_1}) = ev(p_1 ike_0 \times id_{C_1})$, we have $\mu'\zeta(\varepsilon_N \times id_{C_1}) = \mu'(id_{D_1} \times \varepsilon')(ev(p_1 ike_0 \times id_{C_1}), ev((\tau')^{C_1} q_1 jke_0 \times id_{C_1})) = ev(p_1 ike_0 \times id_{C_1})$ and $\mu'\xi(\varepsilon_N \times id_{C_1}) = \mu'(\varepsilon' \times id_{D_1})(ev((\sigma')^{C_1} p_1 ike_0 \times id_{C_1}), ev(p_1 ike_0 \times id_{C_1})) = ev(p_1 ike_0 \times id_{C_1})$. Taking the exponential transposes, $(\mu')^{C_1}\bar{\zeta}\varepsilon_N = (\mu')^{C_1}\bar{\xi}\varepsilon_N = p_1 ike_0$. Hence there is a unique morphism $\tilde{\varepsilon} : M_0 \to M_1$ satisfying $e_1\tilde{\varepsilon} = \varepsilon_N$. Then, we have $\tilde{\sigma}\tilde{\varepsilon} = p_1' se_1\tilde{\varepsilon} = p_1' s\varepsilon_N = p_1'\varepsilon_s = id_{M_0}$ and $\tilde{\tau}\tilde{\varepsilon} = q_1' te_1\tilde{\varepsilon} = q_1' t\varepsilon_N = q_1'\varepsilon_t = id_{M_0}$. There is a unique morphism $\chi : (M_1 \times_{M_0} M_1) \times C_0 \to D_1 \times_{D_0} D_1$ such that $\mathrm{pr}_1\chi = ev(p_2' se_1\mathrm{pr}_1 \times id_{C_0})$ and $\mathrm{pr}_2\chi = ev(p_2' se_1\mathrm{pr}_2 \times id_{C_0})$. In fact, $\tau' ev(p_2' se_1\mathrm{pr}_1 \times id_{C_0}) = ev((\tau')^{C_0} q_2' te_1\mathrm{pr}_1 \times id_{C_0}) = ev(d\tilde{\tau}\mathrm{pr}_1 \times id_{C_0}) = ev(d\tilde{\sigma}\mathrm{pr}_2 \times id_{C_0}) = ev((\sigma')^{C_0} p_2' se_1\mathrm{pr}_2 \times id_{C_0}) = \sigma' ev(p_2' se_1\mathrm{pr}_2 \times id_{C_0})$. We denote by $\bar{\chi} : M_1 \times_{M_0} M_1 \to (D_1 \times_{D_0} D_1)^{C_0}$ the transpose of χ. Since the transpose of $(\sigma')^{C_0}(\mu')^{C_0}\bar{\chi} : M_1 \times_{M_0} M_1 \to D_0^{C_0}$ is $\sigma'\mu'\chi = \sigma'\mathrm{pr}_1\chi = \sigma' ev(p_2' se_1\mathrm{pr}_1 \times id_{C_0}) = ev(d\tilde{\sigma}\mathrm{pr}_1 \times id_{C_0})$, we have $(\sigma')^{C_0}(\mu')^{C_0}\bar{\chi} = d\tilde{\sigma}\mathrm{pr}_1$. Similarly, we have $(\tau')^{C_0}(\mu')^{C_0}\bar{\chi} = d\tilde{\tau}\mathrm{pr}_2$. Hence there is a unique morphism $m : M_1 \times_{M_0} M_1 \to N$ such that $\tilde{\sigma}\mathrm{pr}_1 = p_1' sm$, $\tilde{\tau}\mathrm{pr}_2 = q_1' tm$ and $(\mu')^{C_0}\bar{\chi} = p_2' sm$. Moreover, we show that $(\mu')^{C_1}\bar{\zeta}m = (\mu')^{C_1}\bar{\xi}m$ holds. Then, we have a unique morphism $\tilde{\mu} : M_1 \times_{M_0} M_1 \to M_1$ satisfying $m = e_1\tilde{\mu}$. Taking the transposes of the both hand sides, it suffices to show that $\mu'\zeta(m \times id_{C_1}) = \mu'\xi(m \times id_{C_1})$. We note that there are chains of equalities $\mathrm{pr}_1\zeta(m \times id_{C_1}) = ev(p_1 ike_0 p_1' sm \times id_{C_1}) = ev(p_1 ike_0\tilde{\sigma}\mathrm{pr}_1 \times id_{C_1}) = ev(p_1 ike_0 p_1' se_1\mathrm{pr}_1 \times id_{C_1}) = ev(p_1 ike_0 p_1' s \times id_{C_1})(e_1\mathrm{pr}_1 \times id_{C_1}) = \mathrm{pr}_1\zeta(e_1\mathrm{pr}_1 \times id_{C_1})$, $\mathrm{pr}_2\zeta(m \times id_{C_1}) = ev(p_2' sm \times \tau) = ev((\mu')^{C_0}\bar{\chi} \times \tau) = ev(D_1^{\tau}(\mu')^{C_0}\bar{\chi} \times id_{C_1}) = \mu'\chi(id_{M_1 \times_{M_0} M_1} \times \tau) = \mu'(ev(p_2' s \times \tau)(e_1\mathrm{pr}_1 \times id_{C_1}), ev(p_2' s \times \tau)(e_1\mathrm{pr}_2 \times id_{C_1})) = \mu'(\mathrm{pr}_2\zeta(e_1\mathrm{pr}_1 \times id_{C_1}), \mathrm{pr}_2\zeta(e_1\mathrm{pr}_2 \times id_{C_1}))$, $\mathrm{pr}_1\xi(m \times id_{C_1}) = ev(p_2' sm \times \sigma) = ev((\mu')^{C_0}\bar{\chi} \times \sigma) = ev(D_1^{\sigma}(\mu')^{C_0}\bar{\chi} \times$

$id_{C_1}) = \mu'\chi(id_{M_1\times_{M_0}M_1}\times\sigma) = \mu'(ev(p_2's\times\sigma)(e_1\mathrm{pr}_1\times id_{C_1}), ev(p_2's\times\sigma)(e_1\mathrm{pr}_2\times id_{C_1})) = \mu'(\mathrm{pr}_1\xi(e_1\mathrm{pr}_1\times id_{C_1}), \mathrm{pr}_1\xi(e_1\mathrm{pr}_2\times id_{C_1}))$, $\mathrm{pr}_2\xi(m\times id_{C_1}) = ev(p_1ike_0q_1'tm\times id_{C_1}) = ev(p_1ike_0\tilde\tau\mathrm{pr}_2\times id_{C_1}) = ev(p_1ike_0q_1'te_1\mathrm{pr}_3\times id_{C_1}) = ev(p_1ike_0q_1't\times id_{C_1})(e_1\mathrm{pr}_1\times id_{C_1}) = \mathrm{pr}_2\xi(e_1\mathrm{pr}_2\times id_{C_1})$ and $\mathrm{pr}_2\xi(e_1\mathrm{pr}_1\times id_{C_1}) = ev(p_1ike_0\tilde\tau\mathrm{pr}_1\times id_{C_1}) = ev(p_1ike_0\tilde\sigma\mathrm{pr}_2\times id_{C_1}) = \mathrm{pr}_1\zeta(e_1\mathrm{pr}_2\times id_{C_1})$. Therefore we have $\mu'\zeta(m\times id_{C_1}) = \mu'(id_{D_1}\times\mu')(\mathrm{pr}_1\zeta(e_1\mathrm{pr}_1\times id_{C_1}), \mathrm{pr}_2\zeta(e_1\mathrm{pr}_1\times id_{C_1}), \mathrm{pr}_2\zeta(e_1\mathrm{pr}_2\times id_{C_1}))$
$= \mu'(\mu'\times id_{D_1})(\mathrm{pr}_1\zeta(e_1\mathrm{pr}_1\times id_{C_1}), \mathrm{pr}_2\zeta(e_1\mathrm{pr}_1\times id_{C_1}), \mathrm{pr}_2\zeta(e_1\mathrm{pr}_2\times id_{C_1}))$
$= \mu'(\mu'\zeta(e_1\mathrm{pr}_1\times id_{C_1}), \mathrm{pr}_2\zeta(e_1\mathrm{pr}_2\times id_{C_1})) = \mu'(\mu'\xi(e_1\mathrm{pr}_1\times id_{C_1}), \mathrm{pr}_2\zeta(e_1\mathrm{pr}_2\times id_{C_1}))$
$= \mu'(\mu'\times id_{D_1})(\mathrm{pr}_1\xi(e_1\mathrm{pr}_1\times id_{C_1}), \mathrm{pr}_2\xi(e_1\mathrm{pr}_1\times id_{C_1}), \mathrm{pr}_2\zeta(e_1\mathrm{pr}_2\times id_{C_1}))$
$= \mu'(id_{D_1}\times\mu')(\mathrm{pr}_1\xi(e_1\mathrm{pr}_1\times id_{C_1}), \mathrm{pr}_1\zeta(e_1\mathrm{pr}_2\times id_{C_1}), \mathrm{pr}_2\zeta(e_1\mathrm{pr}_2\times id_{C_1}))$
$= \mu'(\mathrm{pr}_1\xi(e_1\mathrm{pr}_1\times id_{C_1}), \mu'\zeta(e_1\mathrm{pr}_2\times id_{C_1})) = \mu'(\mathrm{pr}_1\xi(e_1\mathrm{pr}_1\times id_{C_1}), \mu'\xi(e_1\mathrm{pr}_2\times id_{C_1}))$
$= \mu'(id_{D_1}\times\mu')(\mathrm{pr}_1\xi(e_1\mathrm{pr}_1\times id_{C_1}), \mathrm{pr}_1\xi(e_1\mathrm{pr}_2\times id_{C_1}), \mathrm{pr}_2\xi(e_1\mathrm{pr}_2\times id_{C_1}))$
$= \mu'(\mu'\times id_{D_1})(\mathrm{pr}_1\xi(e_1\mathrm{pr}_1\times id_{C_1}), \mathrm{pr}_1\xi(e_1\mathrm{pr}_2\times id_{C_1}), \mathrm{pr}_2\xi(e_1\mathrm{pr}_2\times id_{C_1})) = \mu'\xi(m\times id_{C_1})$. We note that $\tilde\sigma\tilde\mu = p_1'se_1\tilde\mu = p_1'se_1m = \tilde\sigma\mathrm{pr}_1$ and $\tilde\tau\tilde\mu = q_1'te_1\tilde\mu = q_1'te_1m = \tilde\tau\mathrm{pr}_2$.

Moreover, $\tilde\mu(id_{M_1}\times\tilde\varepsilon) = \mathrm{pr}_1$, $\tilde\mu(\tilde\varepsilon\times id_{M_1}) = \mathrm{pr}_2$ and $\tilde\mu(\tilde\mu\times id_{M_1}) = \tilde\mu(id_{M_1}\times\tilde\mu)$ hold. Since e_1 is a monomorphism and $e_1\tilde\mu = m$, it suffices to show that $m(id_{M_1}\times\tilde\varepsilon) = e_1\mathrm{pr}_1$, $m(\tilde\varepsilon\times id_{M_1}) = e_1\mathrm{pr}_2$ and $m(\tilde\mu\times id_{M_1}) = m(id_{M_1}\times\tilde\mu)$. By the definition of m, $p_1'sm(id_{M_1}\times\tilde\varepsilon) = \tilde\sigma\mathrm{pr}_1(id_{M_1}\times\tilde\varepsilon) = p_1'se_1\mathrm{pr}_1$, $p_1'sm(\tilde\varepsilon\times id_{M_1}) = \tilde\sigma\mathrm{pr}_1(\tilde\varepsilon\times id_{M_1}) = \tilde\sigma\tilde\varepsilon\mathrm{pr}_1 = \mathrm{pr}_1 = \tilde\sigma\mathrm{pr}_2 = p_1'se_1\mathrm{pr}_2$, $q_1'tm(id_{M_1}\times\tilde\varepsilon) = \tilde\tau\mathrm{pr}_2(id_{M_1}\times\tilde\varepsilon) = \tilde\tau\tilde\varepsilon\mathrm{pr}_2 = \mathrm{pr}_2 = \tilde\tau\mathrm{pr}_1 = q_1'te_1\mathrm{pr}_1$, $q_1'tm(\tilde\varepsilon\times id_{M_1}) = \tilde\tau\mathrm{pr}_2(\tilde\varepsilon\times id_{M_1}) = \tilde\tau\mathrm{pr}_2 = q_1'te_1\mathrm{pr}_2$, $p_2'sm(id_{M_1}\times\tilde\varepsilon) = (\mu')^{C_0}\bar\chi(id_{M_1}\times\tilde\varepsilon)$, $p_2'sm(\tilde\varepsilon\times id_{M_1}) = (\mu')^{C_0}\bar\chi(\tilde\varepsilon\times id_{M_1})$. The transpose of $(\mu')^{C_0}\bar\chi(id_{M_1}\times\tilde\varepsilon)$ is $\mu\chi(id_{M_1}\times\tilde\varepsilon\times id_{C_0}) = \mu(ev(p_2'se_1\mathrm{pr}_1\times id_{C_0}), ev(p_2'se_1\tilde\varepsilon\mathrm{pr}_2\times id_{C_0})) = \mu(ev(p_2'se_1\mathrm{pr}_1\times id_{C_0}), ev(p_2'\varepsilon_s\mathrm{pr}_2\times id_{C_0})) = \mu(ev(p_2'se_1\mathrm{pr}_1\times id_{C_0}), ev((\varepsilon')^{C_0}d\mathrm{pr}_2\times id_{C_0})) = \mu(ev(p_2'se_1\mathrm{pr}_1\times id_{C_0}), \varepsilon'ev(d\mathrm{pr}_2\times id_{C_0})) = ev(p_2'se_1\mathrm{pr}_1\times id_{C_0})$. Similarly, the transpose of $(\mu')^{C_0}\bar\chi(\tilde\varepsilon\times id_{M_1})$ is $\mu\chi(\tilde\varepsilon\times id_{M_1}\times id_{C_0}) = \mu(ev(p_2'se_1\tilde\varepsilon\mathrm{pr}_1\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_2\times id_{C_0})) = \mu(ev(p_2'\varepsilon_s\mathrm{pr}_1\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_2\times id_{C_0})) = \mu(ev((\varepsilon')^{C_0}d\mathrm{pr}_1\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_2\times id_{C_0})) = \mu(\varepsilon'ev(d\mathrm{pr}_1\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_2\times id_{C_0})) = ev(p_2'se_1\mathrm{pr}_2\times id_{C_0}))$. Hence we have $p_2'sm(id_{M_1}\times\tilde\varepsilon) = p_2'se_1\mathrm{pr}_1$ and $p_2'sm(\tilde\varepsilon\times id_{M_1}) = p_2'se_1\mathrm{pr}_2$ and equalities $m(id_{M_1}\times\tilde\varepsilon) = e_1\mathrm{pr}_1$, $m(\tilde\varepsilon\times id_{M_1}) = e_1\mathrm{pr}_2$ follow. For the third one, $p_1'sm(\tilde\mu\times id_{M_1}) = \tilde\sigma\mathrm{pr}_1(\tilde\mu\times id_{M_1}) = \tilde\sigma\tilde\mu(\mathrm{pr}_1,\mathrm{pr}_2) = \tilde\sigma\mathrm{pr}_1(\mathrm{pr}_1,\mathrm{pr}_2) = \tilde\sigma\mathrm{pr}_1 = \tilde\sigma\mathrm{pr}_1(id_{M_1}\times\tilde\mu) = p_1'sm(id_{M_1}\times\tilde\mu)$, $q_1'tm(\tilde\mu\times id_{M_1}) = \tilde\tau\mathrm{pr}_2(\tilde\mu\times id_{M_1}) = \tilde\tau\mathrm{pr}_3 = \tilde\tau\mathrm{pr}_2(\mathrm{pr}_2,\mathrm{pr}_3) = \tilde\tau\tilde\mu(\mathrm{pr}_2,\mathrm{pr}_3) = \tilde\tau\mathrm{pr}_2(id_{M_1}\times\tilde\mu) = q_1'tm(id_{M_1}\times\tilde\mu)$. Since the transposes of $p_2'sm(\tilde\mu\times id_{M_1}) = (\mu')^{C_0}\bar\chi(\tilde\mu\times id_{M_1})$ and $p_2'sm(id_{M_1}\times\tilde\mu) = (\mu')^{C_0}\bar\chi(id_{M_1}\times\tilde\mu)$ are given by $\mu'\chi(\tilde\mu\times id_{M_1}\times id_{C_0})$ and $\mu'\chi(id_{M_1}\times\tilde\mu\times id_{C_0})$ respectively, it remains to check that both of them coincide. Recalling the definition of χ, we have $\mu'\chi(\tilde\mu\times id_{M_1}\times id_{C_0})$
$= \mu'(ev(p_2'se_1\mathrm{pr}_1(\tilde\mu\times id_{M_1})\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_2(\tilde\mu\times id_{M_1})\times id_{C_0}))$
$= \mu'(ev(p_2'se_1\tilde\mu(\mathrm{pr}_1,\mathrm{pr}_2)\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_3\times id_{C_0}))$
$= \mu'(ev(p_2'sm(\mathrm{pr}_1,\mathrm{pr}_2)\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_3\times id_{C_0}))$
$= \mu'(ev((\mu')^{C_0}\bar\chi(\mathrm{pr}_1,\mathrm{pr}_2)\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_3\times id_{C_0}))$
$= \mu'(\mu'\chi((\mathrm{pr}_1,\mathrm{pr}_2)\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_3\times id_{C_0}))$
$= \mu'(\mu'(ev(p_2'se_1\mathrm{pr}_1(\mathrm{pr}_1,\mathrm{pr}_2)\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_2(\mathrm{pr}_1,\mathrm{pr}_2)\times id_{C_0})), ev(p_2'se_1\mathrm{pr}_3\times id_{C_0}))$
$= \mu'(\mu'\times id_{M_1})(ev(p_2'se_1\mathrm{pr}_1\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_2\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_3\times id_{C_0}))$
$= \mu'(id_{M_1}\times\mu')(ev(p_2'se_1\mathrm{pr}_1\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_2\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_3\times id_{C_0}))$
$= \mu'(ev(p_2'se_1\mathrm{pr}_1\times id_{C_0}), \mu'(ev(p_2'se_1\mathrm{pr}_1(\mathrm{pr}_2,\mathrm{pr}_3)\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_2(\mathrm{pr}_2,\mathrm{pr}_3)\times id_{C_0})))$
$= \mu'(ev(p_2'se_1\mathrm{pr}_1\times id_{C_0}), \mu'\chi((\mathrm{pr}_2,\mathrm{pr}_3)\times id_{C_0}))$
$= \mu'(ev(p_2'se_1\mathrm{pr}_1\times id_{C_0}), ev((\mu')^{C_0}\bar\chi(\mathrm{pr}_2,\mathrm{pr}_3)\times id_{C_0}))$
$= \mu'(ev(p_2'se_1\mathrm{pr}_1\times id_{C_0}), ev(p_2'sm(\mathrm{pr}_2,\mathrm{pr}_3)\times id_{C_0}))$
$= \mu'(ev(p_2'se_1\mathrm{pr}_1\times id_{C_0}), ev(p_2'se_1\tilde\mu(\mathrm{pr}_2,\mathrm{pr}_3)\times id_{C_0}))$
$= \mu'(ev(p_2'se_1\mathrm{pr}_1(id_{M_1}\times\tilde\mu)\times id_{C_0}), ev(p_2'se_1\mathrm{pr}_2(id_{M_1}\times\tilde\mu)\times id_{C_0})) = \mu'\chi(id_{M_1}\times\tilde\mu)$. Thus $\boldsymbol{D}^{\boldsymbol{C}}$ is an internal category in $\mathcal{E}$.

Proposition 5.1.22 *Suppose that $\mathcal{E}$ is a cartesian closed category.*

1) There is a natural bijection $\boldsymbol{cat}(\mathcal{E})(\boldsymbol{E}\times\boldsymbol{C},\boldsymbol{D}) \to \boldsymbol{cat}(\mathcal{E})(\boldsymbol{E},\boldsymbol{D}^{\boldsymbol{C}})$ for any internal category $\boldsymbol{E}$ in $\mathcal{E}$. Hence $\boldsymbol{cat}(\mathcal{E})$ is cartesian closed.

2) If $\boldsymbol{D}$ is an internal groupoid, so is $\boldsymbol{D}^{\boldsymbol{C}}$.

Proof. 1) We denote by σ'', τ'', ε'', μ'' the structure maps of $\boldsymbol{E}$. Let $f = (f_0, f_1) : \boldsymbol{E}\times\boldsymbol{C}\to\boldsymbol{D}$ be an internal functor and $\bar f_l : E_l \to D_l^{C_l}$ $(l = 0,1)$ denotes the transpose of $f_l : E_l\times C_l \to D_l$. Taking the transposes of the equalities $f_0(\sigma''\times\sigma) = \sigma'f_1$, $f_0(\tau''\times\tau) = \tau'f_1$ and $\varepsilon'f_0 = f_1(\varepsilon''\times\varepsilon)$, we have $D_0^\sigma\bar f_0\sigma'' = (\sigma')^{C_1}\bar f_1$, $D_0^\tau\bar f_0\tau'' = (\tau')^{C_1}\bar f_1$ and $(\varepsilon')^{C_0}\bar f_0 = D_1^\varepsilon\bar f_1\varepsilon''$. Hence there are morphisms $p : E_1\to S$, $q : E_1\to T$, $r : E_0\to E$ such that $\bar f_1 = p_1p = q_1q$, $\bar f_1\varepsilon'' = r_1r$, $\bar f_0\sigma'' = p_0p$, $\bar f_0\tau'' = q_0q$, $\bar f_0 = r_0r$. Thus we have $\bar f_0 = p_0p\varepsilon'' = q_0q\varepsilon'' = r_0r$ and morphisms $p\varepsilon'' : E_0\to S$, $q\varepsilon'' : E_0\to T$, $r : E_0\to E$ induce $u : E_0\to S\cap T\cap E$ such that $iku = p\varepsilon''$,

$jku = q\varepsilon''$, $hu = r$ ($h : S \cap T \cap E \to E$ denotes the inclusion morphism). Since the following square is a pull-back,

$$\begin{tikzcd}
(E_1 \times_{E_0} E_1) \times (C_1 \times_{C_0} C_1) \arrow[r, "\mathrm{pr}_2 \times \mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1 \times \mathrm{pr}_1"] & E_1 \times C_1 \arrow[d, "\sigma'' \times \sigma"] \\
E_1 \times C_1 \arrow[r, "\tau'' \times \tau"] & E_0 \times C_0
\end{tikzcd}$$

there is a unique isomorphism $\rho : (E_1 \times C_1) \times_{(E_0 \times C_0)} (E_1 \times C_1) \to (E_1 \times_{E_0} E_1) \times (C_1 \times_{C_0} C_1)$ such that $(\mathrm{pr}_l \times \mathrm{pr}_l)\rho = \mathrm{pr}_l : (E_1 \times C_1) \times_{(E_0 \times C_0)} (E_1 \times C_1) \to E_1 \times C_1$ $(l = 1, 2)$. We note that the composition $(E_1 \times C_1) \times_{(E_0 \times C_0)} (E_1 \times C_1) \to E_1 \times C_1$ of $\boldsymbol{E} \times \boldsymbol{C}$ is given by $(\mu'' \times \mu)\rho$. Hence we have $f_1(\mu'' \times \mu)\rho = \mu'(f_1 \times f_1)$. We claim that $(\mu')^{C_1 \times_{C_0} C_1}\bar{\theta}ku = D_1^{\mu} p_1 iku$. The right hand side is $D_1^{\mu}\bar{f}_1\varepsilon''$ by the definition of u. Taking the exponential transposes, we show that $\mu'\theta(ku \times id_{C_1 \times_{C_0} C_1}) = f_1(\varepsilon'' \times \mu)$. For $l = 1, 2$, $(\mathrm{pr}_l \times \mathrm{pr}_l)\rho(\varepsilon'' \times \mathrm{pr}_1, \varepsilon'' \times \mathrm{pr}_2) = \mathrm{pr}_l(\varepsilon'' \times \mathrm{pr}_1, \varepsilon'' \times \mathrm{pr}_2) = \varepsilon'' \times \mathrm{pr}_l = (\mathrm{pr}_l \times \mathrm{pr}_l)((\varepsilon'', \varepsilon'') \times id_{C_1 \times_{C_0} C_1})$. Then, we have $\rho(\varepsilon'' \times \mathrm{pr}_1, \varepsilon'' \times \mathrm{pr}_2) = (\varepsilon'', \varepsilon'') \times id_{C_1 \times_{C_0} C_1}$. On the other hand, $\mathrm{pr}_1\theta(ku \times id_{C_1 \times_{C_0} C_1}) = ev(q_1 jku \times \mathrm{pr}_1) = ev(\bar{f}_1\varepsilon'' \times \mathrm{pr}_1) = ev(\bar{f}_1 \times id_{C_1})(\varepsilon'' \times \mathrm{pr}_1) = f_1(\varepsilon'' \times \mathrm{pr}_1)$, $\mathrm{pr}_2\theta(ku \times id_{C_1 \times_{C_0} C_1}) = ev(p_1 iku \times \mathrm{pr}_2) = ev(\bar{f}_1\varepsilon'' \times \mathrm{pr}_2) = ev(\bar{f}_1 \times id_{C_1})(\varepsilon'' \times \mathrm{pr}_2) = f_1(\varepsilon'' \times \mathrm{pr}_2)$. Hence, $\mu'\theta(ku \times id_{C_1 \times_{C_0} C_1}) = \mu'(f_1 \times f_1)(\varepsilon'' \times \mathrm{pr}_1, \varepsilon'' \times \mathrm{pr}_2) = f_1(\mu'' \times \mu)\rho(\varepsilon'' \times \mathrm{pr}_1, \varepsilon'' \times \mathrm{pr}_2) = f_1(\mu'' \times \mu)((\varepsilon'', \varepsilon'') \times id_{C_1 \times_{C_0} C_1}) = f_1(\mu''(\varepsilon'', \varepsilon'') \times \mu) = f_1(\varepsilon'' \times \mu)$. Thus we have a unique morphism $\tilde{f}_0 : E_0 \to M_0$ such that $e_0\tilde{f}_0 = u$.

Since $d\tilde{f}_0\sigma'' = p_0 ike_0\tilde{f}_0\sigma'' = p_0 iku\sigma'' = p_0 p\varepsilon''\sigma'' = \bar{f}_0\sigma''$, $(\sigma')^{C_0}D_1^{\varepsilon}\bar{f}_1 = D_0^{\varepsilon}(\sigma')^{C_1}\bar{f}_1 = D_0^{\varepsilon}D_0^{\sigma}\bar{f}_1\sigma'' = D_0^{\sigma\varepsilon}\bar{f}_1\sigma'' = \bar{f}_1\sigma''$, $(\tau')^{C_0}D_1^{\varepsilon}\bar{f}_1 = D_0^{\varepsilon}(\tau')^{C_1}\bar{f}_1 = D_0^{\varepsilon}D_0^{\tau}\bar{f}_1\tau'' = D_0^{\tau\varepsilon}\bar{f}_1\tau'' = \bar{f}_1\tau''$, $d\tilde{f}_0\tau'' = p_0 ike_0\tilde{f}_0\tau'' = q_0 jku\tau'' = q_0 q\varepsilon''\tau'' = \bar{f}_0\tau''$, there is a morphism $v : E_1 \to N$ such that $D_1^{\varepsilon}\bar{f}_1 = p_2' sv$, $\tilde{f}_0\sigma'' = p_1' sv$, $\tilde{f}_0\tau'' = q_1' tv$. We show that $(\mu')^{C_1}\bar{\zeta}v = (\mu')^{C_1}\bar{\xi}v$. Considering the transposes of the both sides, it suffices to show that $\mu'\zeta(v \times id_{C_1}) = \mu'\xi(v \times id_{C_1})$. Since $\mathrm{pr}_1\zeta(v \times id_{C_1}) = ev(p_1 ike_0 p_1' sv \times id_{C_1}) = ev(p_1 ike_0\tilde{f}\sigma'' \times id_{C_1}) = ev(p_1 iku\sigma'' \times id_{C_1}) = ev(\bar{f}_1\varepsilon''\sigma'' \times id_{C_1}) = f_1(\varepsilon''\sigma'' \times id_{C_1})$, $\mathrm{pr}_2\zeta(v \times id_{C_1}) = ev(p_2' sv \times \tau) = ev(D_1^{\varepsilon}\bar{f}_1 \times \tau) = f_1(id_{E_1} \times \varepsilon\tau)$, $\mathrm{pr}_1\xi(v \times id_{C_1}) = ev(p_2' sv \times \sigma) = ev(D_1^{\varepsilon}\bar{f}_1 \times \sigma) = f_1(id_{E_1} \times \varepsilon\sigma)$, $\mathrm{pr}_2\xi(v \times id_{C_1}) = ev(p_1 ike_0 q_1' tv \times id_{C_1}) = ev(p_1 ike_0\tilde{f}\tau'' \times id_{C_1}) = ev(p_1 iku\tau'' \times id_{C_1}) = ev(\bar{f}_1\varepsilon''\tau'' \times id_{C_1}) = f_1(\varepsilon''\tau'' \times id_{C_1})$, it follows that $\mu'\zeta(v \times id_{C_1}) = \mu'(f_1 \times f_1)(\varepsilon''\sigma'' \times id_{C_1}, id_{E_1} \times \varepsilon\tau) = f_1(\mu'' \times \mu)\rho(\varepsilon''\sigma'' \times id_{C_1}, id_{E_1} \times \varepsilon\tau) = f_1(\mu'' \times \mu)((\varepsilon''\sigma'', id_{E_1}) \times (id_{C_1}, \varepsilon\tau)) = f_1(\mu''(\varepsilon''\sigma'', id_{E_1}) \times \mu(id_{C_1}, \varepsilon\tau)) = f_1$ and $\mu'\xi(v \times id_{C_1}) = \mu'(f_1 \times f_1)(id_{E_1} \times \varepsilon\sigma, \varepsilon''\tau'' \times id_{C_1}) = f_1(\mu'' \times \mu)\rho(id_{E_1} \times \varepsilon\sigma, \varepsilon''\tau'' \times id_{C_1}) = f_1(\mu'' \times \mu)((id_{E_1}, \varepsilon''\tau'') \times (\varepsilon\sigma, id_{C_1})) = f_1(\mu''(id_{E_1}, \varepsilon''\tau'') \times \mu(\varepsilon\sigma, id_{C_1})) = f_1$. Therefore we have a unique morphism $\tilde{f}_1 : E_1 \to M_1$ such that $e_1\tilde{f}_1 = v$. Next we show that $\tilde{f} = (\tilde{f}_1, \tilde{f}_0) : \boldsymbol{E} \to \boldsymbol{D}^{\boldsymbol{C}}$ is an internal functor, that is, we verify $\tilde{\sigma}\tilde{f}_1 = \tilde{f}_0\sigma''$, $\tilde{\tau}\tilde{f}_1 = \tilde{f}_0\tau''$, $\tilde{\varepsilon}\tilde{f}_0 = \tilde{f}_1\varepsilon''$ and $\tilde{\mu}(\tilde{f}_1 \times \tilde{f}_1) = \tilde{f}_1\mu''$. Since $e_0\tilde{\sigma}\tilde{f}_1 = e_0 p_1' se_1\tilde{f}_1 = e_0 p_1' sv = e_0\tilde{f}_0\sigma''$, $e_0\tilde{\tau}\tilde{f}_1 = e_0 p_1' se_1\tilde{f}_1 = e_0 p_1' sv = e_0\tilde{f}_0\tau''$ and $e_0 : M_0 \to S \cap T \cap E$ is a monomorphism, the first and the second equalities follow. For the third and the fourth ones, since $e_1\tilde{\varepsilon}\tilde{f}_0 = \varepsilon_N\tilde{f}_0$, $e_1\tilde{f}_1\varepsilon'' = v\varepsilon''$, $e_1\tilde{\mu}(\tilde{f}_1 \times \tilde{f}_1) = m(\tilde{f}_1 \times \tilde{f}_1)$, $e_1\tilde{f}_1\mu'' = v\mu''$ and $e_1 : M_1 \to N$ is a monomorphism, it suffices to show $\varepsilon_N\tilde{f}_0 = v\varepsilon''$ and $m(\tilde{f}_1 \times \tilde{f}_1) = v\mu''$. By the definition of ε_N, $p_1' s\varepsilon_N\tilde{f}_0 = p_1'\varepsilon_s\tilde{f}_0 = \tilde{f}_0$, $q_1' t\varepsilon_N\tilde{f}_0 = q_1'\varepsilon_t\tilde{f}_0 = \tilde{f}_0$, and $p_2' s\varepsilon_N\tilde{f}_0 = p_2'\varepsilon_s\tilde{f}_0 = (\varepsilon')^{C_0}d\tilde{f}_0 = (\varepsilon')^{C_0}p_0 ike_0\tilde{f}_0 = (\varepsilon')^{C_0}p_0 iku = (\varepsilon')^{C_0}p_0 p\varepsilon'' = (\varepsilon')^{C_0}\bar{f}_0$. On the other hand, $p_1' sv\varepsilon'' = \tilde{f}_0\sigma''\varepsilon'' = \tilde{f}_0$, $q_1' tv\varepsilon'' = \tilde{f}_0\tau''\varepsilon'' = \tilde{f}_0$, $p_2' sv\varepsilon'' = D_1^{\varepsilon}\bar{f}_1\varepsilon'' = (\varepsilon')^{C_0}\bar{f}_0$. Hence we have $p_1' s\varepsilon_N\tilde{f}_0 = p_1' sv\varepsilon''$, $q_1' t\varepsilon_N\tilde{f}_0 = q_1' tv\varepsilon''$, $p_2' s\varepsilon_N\tilde{f}_0 = p_2' sv\varepsilon''$ which imply $\varepsilon_N\tilde{f}_0 = v\varepsilon''$. Since $p_1' sm(\tilde{f}_1 \times \tilde{f}_1) = \tilde{\sigma}\mathrm{pr}_1(\tilde{f}_1 \times \tilde{f}_1) = \tilde{\sigma}\tilde{f}_1\mathrm{pr}_1 = \tilde{f}_0\sigma''\mathrm{pr}_1 = \tilde{f}_0\sigma''\mu'' = p_1' sv\mu''$ and $q_1' tm(\tilde{f}_1 \times \tilde{f}_1) = \tilde{\tau}\mathrm{pr}_2(\tilde{f}_1 \times \tilde{f}_1) = \tilde{\tau}\tilde{f}_1\mathrm{pr}_2 = \tilde{f}_0\tau''\mathrm{pr}_2 = \tilde{f}_0\tau''\mu'' = q_1' tv\mu''$, it suffices to show $p_2' sm(\tilde{f}_1 \times \tilde{f}_1) = p_2' sv\mu''$. The left hand side is $(\mu')^{C_0}\bar{\chi}(\tilde{f}_1 \times \tilde{f}_1)$ and its transpose is $\mu'\chi((\tilde{f}_1 \times \tilde{f}_1) \times id_{C_0})$. The right hand side is $D_1^{\varepsilon}\bar{f}_1\mu''$ whose transpose is $f_1(\mu'' \times \varepsilon)$. For $l = 1, 2$, $\mathrm{pr}_l\chi((\tilde{f}_1 \times \tilde{f}_1) \times id_{C_0}) = ev(p_2' se_1\mathrm{pr}_l(\tilde{f}_1 \times \tilde{f}_1) \times id_{C_0}) = ev(p_2' se_1\tilde{f}_1\mathrm{pr}_l \times id_{C_0}) = ev(p_2' sv\mathrm{pr}_l \times id_{C_0}) = ev(D_1^{\varepsilon}\bar{f}_1 \times id_{C_0})(\mathrm{pr}_l \times id_{C_0}) = f_1(\mathrm{pr}_l \times \varepsilon)$. Then, $\mu'\chi((\tilde{f}_1 \times \tilde{f}_1) \times id_{C_0}) = \mu'(f_1 \times f_1)(\mathrm{pr}_1 \times \varepsilon, \mathrm{pr}_2 \times \varepsilon) = f_1(\mu'' \times \mu)\rho(\mathrm{pr}_1 \times \varepsilon, \mathrm{pr}_2 \times \varepsilon) = f_1(\mu'' \times \mu)(id_{E_1 \times_{E_0} E_1} \times (\varepsilon, \varepsilon)) = f_1(\mu'' \times \mu(\varepsilon, \varepsilon)) = f_1(\mu'' \times \varepsilon)$.

Thus we have a map $\alpha : \boldsymbol{cat}(\mathcal{E})(\boldsymbol{E} \times \boldsymbol{C}, \boldsymbol{D}) \to \boldsymbol{cat}(\mathcal{E})(\boldsymbol{E}, \boldsymbol{D}^{\boldsymbol{C}})$ given by $\alpha(f_0, f_1) = (\tilde{f}_0, \tilde{f}_1)$. In order to define the inverse of α, we construct the evaluation map $\mathbf{ev} = (\widetilde{ev}_1, \widetilde{ev}_0) : \boldsymbol{D}^{\boldsymbol{C}} \times \boldsymbol{C} \to \boldsymbol{D}$ as follows. $\widetilde{ev}_0 : M_0 \times C_0 \to D_0$ is defined to be the transpose of $d = p_0 ike_0 : M_0 \to D_0^{C_0}$ and $\widetilde{ev}_1 : M_1 \times C_1 \to D_1$ is defined to be $\mu'\zeta(e_1 \times id_{C_1})$. Then, $\sigma'\widetilde{ev}_1 = \sigma'\mu'\zeta(e_1 \times id_{C_1}) = \sigma'\mathrm{pr}_1\zeta(e_1 \times id_{C_1}) = \sigma' ev(p_1 ike_0 p_1' se_1 \times id_{C_1}) = ev((\sigma')^{C_1}p_1 ike_0\tilde{\sigma} \times id_{C_1}) = ev(D_0^{\sigma}p_0 ike_0\tilde{\sigma} \times id_{C_1}) = ev(d\tilde{\sigma} \times \sigma) = ev(d \times id_{C_0})(\tilde{\sigma} \times \sigma) = \widetilde{ev}_0(\tilde{\sigma} \times \sigma)$, $\tau'\widetilde{ev}_1 = \tau'\mu'\zeta(e_1 \times id_{C_1}) = \tau'\mathrm{pr}_2\zeta(e_1 \times id_{C_1}) = \tau' ev(p_2' se_1 \times \tau) = ev((\tau')^{C_0}q_2' te_1 \times \tau) = ev(d\tilde{\tau} \times \tau) = ev(d \times id_{C_0})(\tilde{\tau} \times \tau) = \widetilde{ev}_0(\tilde{\tau} \times \tau)$ and $\widetilde{ev}_1(\tilde{\varepsilon} \times \varepsilon) = \mu'\zeta(e_1\tilde{\varepsilon} \times \varepsilon) = \mu'(ev(p_1 ike_0 p_1' se_1\tilde{\varepsilon} \times \varepsilon), ev(p_2' se_1\tilde{\varepsilon} \times \tau\varepsilon)) = \mu'(ev(p_1 ike_0 p_1'\varepsilon_s \times \varepsilon), ev(p_2'\varepsilon_s \times id_{C_0})) = \mu'(ev(p_1 ike_0 \times \varepsilon), ev((\varepsilon')^{C_0}d \times id_{C_0})) = \mu'(ev(D_1^{\varepsilon}p_1 ike_0 \times id_{C_0}), \varepsilon' ev(d \times id_{C_0})) = ev(D_1^{\varepsilon}r_1 he_0 \times id_{C_0}) = ev((\varepsilon')^{C_0}r_0 he_0 \times id_{C_0}) = \varepsilon' ev(d \times id_{C_0}) = \varepsilon'\widetilde{ev}_0$. We note that $\mathrm{pr}_1\zeta(e_1\tilde{\mu} \times \mu) = ev(p_1 ike_0 p_1' se_1\tilde{\mu} \times \mu), = ev(D_1^{\mu}p_1 ike_0\tilde{\sigma}\tilde{\mu} \times id_{C_1 \times_{C_0} C_1})$
$= ev((\mu')^{C_1 \times_{C_0} C_1}\bar{\theta}ke_0\tilde{\sigma}\mathrm{pr}_1 \times id_{C_1 \times_{C_0} C_1}) = \mu'\theta(ke_0\tilde{\sigma}\mathrm{pr}_1 \times id_{C_1 \times_{C_0} C_1})$
$= \mu'(ev(p_1 ike_0\tilde{\sigma}\mathrm{pr}_1 \times \mathrm{pr}_1), ev(p_1 ike_0\tilde{\sigma}\mathrm{pr}_1 \times \mathrm{pr}_2))$, $\mathrm{pr}_2\zeta(e_1\tilde{\mu} \times \mu) = ev(p_2' se_1\tilde{\mu} \times \tau\mu) = ev(p_2' sm \times \tau\mathrm{pr}_2) =$

$ev((\mu')^{C_0}\bar{\chi}\times\tau\mathrm{pr}_2) = \mu'\chi(id_{M_1\times_{M_0}M_1}\times\tau\mathrm{pr}_2) = \mu'(ev(p_2'se_1\mathrm{pr}_1\times\tau\mathrm{pr}_2), ev(p_2'se_1\mathrm{pr}_2\times\tau\mathrm{pr}_2))$. By the associativity of μ', $\mu'(\mu'\times\mu') = \mu'(id_{D_1}\times\mu')(\mu'\times id_{D_1\times_{D_0}D_1}) = \mu'(\mu'\times id_{D_1})(\mu'\times id_{D_1\times_{D_0}D_1}) = \mu'(\mu'\times id_{D_1})(id_{D_1}\mu'\times id_{D_1})$. Hence we have $\widetilde{ev}_1(\tilde{\mu}\times\mu)\rho = \mu'\zeta(e_1\tilde{\mu}\times\mu)\rho = \mu'(\mu'\times\mu')$
$(ev(p_1ike_0\tilde{\sigma}\mathrm{pr}_1\times\mathrm{pr}_1), ev(p_1ike_0\tilde{\sigma}\mathrm{pr}_1\times\mathrm{pr}_2), ev(p_2'se_1\mathrm{pr}_1\times\tau\mathrm{pr}_2), ev(p_2'se_1\mathrm{pr}_2\times\tau\mathrm{pr}_2))\rho$
$= \mu'(\mu'\times id_{D_1})(ev(p_1ike_0\tilde{\sigma}\mathrm{pr}_1\times\mathrm{pr}_1), \mu'(ev(p_1ike_0p_1'se_1\mathrm{pr}_1\times\mathrm{pr}_2), ev(p_2'se_1\mathrm{pr}_1\times\tau\mathrm{pr}_2)), ev(p_2'se_1\mathrm{pr}_2\times\tau\mathrm{pr}_2))\rho$
$= \mu'(\mu'\times id_{D_1})(ev(p_1ike_0\tilde{\sigma}\mathrm{pr}_1\times\mathrm{pr}_1), \mu'\zeta(e_1\times id_{C_1})(\mathrm{pr}_1\times\mathrm{pr}_2), ev(p_2'se_1\mathrm{pr}_2\times\tau\mathrm{pr}_2))\rho$
$= \mu'(\mu'\times id_{D_1})(ev(p_1ike_0\tilde{\sigma}\mathrm{pr}_1\times\mathrm{pr}_1), \mu'\xi(e_1\times id_{C_1})(\mathrm{pr}_1\times\mathrm{pr}_2), ev(p_2'se_1\mathrm{pr}_2\times\tau\mathrm{pr}_2))\rho$
$= \mu'(\mu'\times id_{D_1})(ev(p_1ike_0\tilde{\sigma}\mathrm{pr}_1\times\mathrm{pr}_1), \mu'(ev(p_2'se_1\mathrm{pr}_1\times\sigma\mathrm{pr}_2), ev(p_1ike_0q_1'te_1\mathrm{pr}_1\times\mathrm{pr}_2), ev(p_2'se_1\mathrm{pr}_2\times\tau\mathrm{pr}_2))\rho$
$= \mu'(\mu'\times\mu')(ev(p_1ike_0\tilde{\sigma}\mathrm{pr}_1\times\mathrm{pr}_1), ev(p_2'se_1\mathrm{pr}_1\times\sigma\mathrm{pr}_2), ev(p_1ike_0\tilde{\tau}\mathrm{pr}_1\times\mathrm{pr}_2), ev(p_2'se_1\mathrm{pr}_2\times\tau\mathrm{pr}_2))\rho$
$= \mu'(\mu'\times\mu')(ev(p_1ike_0\tilde{\sigma}\mathrm{pr}_1\times\mathrm{pr}_1), ev(p_2'se_1\mathrm{pr}_1\times\tau\mathrm{pr}_1), ev(p_1ike_0\tilde{\sigma}\mathrm{pr}_2\times\mathrm{pr}_2), ev(p_2'se_1\mathrm{pr}_2\times\tau\mathrm{pr}_2))\rho$
$= \mu'(\mu'\times\mu')(ev(p_1ike_0p_1's\times id_{C_1})(e_1\mathrm{pr}_1\times\mathrm{pr}_1), ev(p_2's\times\tau)(e_1\mathrm{pr}_1\times\mathrm{pr}_1), ev(p_1ike_0p_1's\times id_{C_1})(e_1\mathrm{pr}_2\times\mathrm{pr}_2),$
$ev(p_2's\times\tau)(e_1\mathrm{pr}_2\times\mathrm{pr}_2))\rho$
$= \mu'(\mu'\times\mu')(\zeta(e_1\mathrm{pr}_1\times\mathrm{pr}_1)\rho, \zeta(e_1\mathrm{pr}_2\times\mathrm{pr}_2)\rho) = \mu'(\mu'\zeta(e_1\times id_{C_1})\mathrm{pr}_1, \mu'\zeta(e_1\times id_{C_1})\mathrm{pr}_2) = \mu'(\widetilde{ev}_1\times\widetilde{ev}_1)$. Therefore $\mathbf{ev} = (\widetilde{ev}_1, \widetilde{ev}_0)$ is an internal functor.

Define a map $\beta : \boldsymbol{cat}(\mathcal{E})(\boldsymbol{E}, \boldsymbol{D}^{\boldsymbol{C}}) \to \boldsymbol{cat}(\mathcal{E})(\boldsymbol{E}\times\boldsymbol{C}, \boldsymbol{D})$ by $\beta(g) = \mathbf{ev}(g\times id_{\boldsymbol{C}})$. Let $f = (f_0, f_1) : \boldsymbol{E}\times\boldsymbol{C}\to\boldsymbol{D}$ be an internal functor. Since $d\tilde{f}_0 = p_0ike_0\tilde{f}_0 = p_0iku = p_0p\varepsilon'' = \bar{f}_0$, $\widetilde{ev}_0(\tilde{f}_0\times id_{C_0}) = ev(d\tilde{f}_0\times id_{C_0}) = ev(\bar{f}_0\times id_{C_0}) = f_0$. Moreover, $\widetilde{ev}_1(\tilde{f}_1\times id_{C_1}) = \mu'\zeta(e_1\tilde{f}_1\times id_{C_1}) = \mu'(ev(p_1ike_0p_1'sv\times id_{C_1}), ev(p_2'sv\times\tau)) = \mu'(ev(p_1ike_0\tilde{f}_0\sigma''\times id_{C_1}), ev(D_1^{\varepsilon}\bar{f}_1\times\tau)) = \mu'(ev(p_1iku\sigma''\times id_{C_1}), ev(\bar{f}_1\times\varepsilon\tau)) = \mu'(ev(p_1p\varepsilon''\sigma''\times id_{C_1}), ev(\bar{f}_1\times\varepsilon\tau)) = \mu'(ev(\bar{f}_1\varepsilon''\sigma''\times id_{C_1}), ev(\bar{f}_1\times\varepsilon\tau)) = \mu'(f_1(\varepsilon''\sigma''\times id_{C_1}), f_1(id_{E_1}\times\varepsilon\tau)) = \mu'(f_1\times f_1)(\varepsilon''\sigma''\times id_{C_1}, id_{E_1}\times\varepsilon\tau) = f_1(\mu''\times\mu)\rho(\varepsilon''\sigma''\times id_{C_1}, id_{E_1}\times\varepsilon\tau) = f_1(\mu''\times\mu)((\varepsilon''\sigma'', id_{E_1})\times(id_{C_1}, \varepsilon\tau)) = f_1(\mu''(\varepsilon''\sigma'', id_{E_1})\times\mu(id_{C_1}, \varepsilon\tau)) = f_1$. Hence $\beta\alpha$ is the identity map of $\boldsymbol{cat}(\mathcal{E})(\boldsymbol{E}\times\boldsymbol{C}, \boldsymbol{D})$. Let $g = (g_0, g_1) : \boldsymbol{E}\to\boldsymbol{D}^{\boldsymbol{C}}$ be an internal functor. We put $f_l = \widetilde{ev}_l(g_l\times id_{C_l})$ $(l = 1, 2)$ and denote by $\bar{f}_l$ the transpose of f_l. Since $f_0 = ev(dg_0\times id_{C_0})$, we have $\bar{f}_0 = dg_0$. There are morphisms $p : E_1\to S$, $q : E_1\to T$, $r : E_0\to E$ such that $\bar{f}_1 = p_1p = q_1q$, $\bar{f}_1\varepsilon'' = r_1r$, $\bar{f}_0\sigma'' = p_0p$, $\bar{f}_0\tau'' = q_0q$, $\bar{f}_0 = r_0r$. It follows that $p_0ike_0g_0 = p_0p\varepsilon''$, $q_0jke_0g_0 = q_0q\varepsilon''$ and $r_0he_0g_0 = r_0r$. On the other hand, $f_1(\varepsilon''\times id_{C_1}) = \mu'\zeta(e_1g_1\varepsilon''\times id_{C_1}) = \mu'\zeta(e_1\tilde{\varepsilon}g_0\times id_{C_1}) = \mu'\zeta(\varepsilon_Ng_0\times id_{C_1}) = \mu'(ev(p_1ike_0p_1's\varepsilon_Ng_0\times id_{C_1}), ev(p_2's\varepsilon_Ng_0\times\tau)) = \mu'(ev(p_1ike_0p_1'\varepsilon_sg_0\times id_{C_1}), ev(p_2'\varepsilon_sg_0\times\tau)) = \mu'(ev(p_1ike_0g_0\times id_{C_1}), ev((\varepsilon')^{C_0}dg_0\times\tau)) = \mu'(ev(p_1ike_0g_0\times id_{C_1}), \varepsilon'ev(dg_0\times\tau)) = ev(p_1ike_0g_0\times id_{C_1})$. Considering the transpose, we have $p_1ike_0g_0 = \bar{f}_1\varepsilon'' = p_1p\varepsilon''$. Thus we also have $q_1jke_0g_0 = q_1q\varepsilon''$ and $r_1he_0g_0 = r_1r$. It follows that $ike_0g_0 = p\varepsilon''$, $jke_0g_0 = q\varepsilon''$, $he_0g_0 = r$, which show that $e_0g_0 = u = e_0\tilde{f}_0$, hence $\tilde{f}_0 = g_0$. To show $\tilde{f}_1 = g_1$, it suffices to show $e_1g_1 = v$, namely, g_1 satisfies $\tilde{f}_0\sigma'' = p_1'se_1g_1$, $\tilde{f}_0\tau'' = q_1'te_1g_1$ and $D_1^{\varepsilon}\bar{f}_1 = p_2'se_1g_1$. The first and the second ones are obtained easily. In fact, $\tilde{f}_0\sigma'' = \tilde{\sigma}g_1 = p_1'se_1g_1$, $\tilde{f}_0\tau'' = \tilde{\tau}g_1 = q_1'te_1g_1$. The transpose of $D_1^{\varepsilon}\bar{f}_1$ is $f_1(id_{E_1}\times\varepsilon) = \mu'\zeta(e_1g_1\times\varepsilon) = \mu'(ev(p_1ike_0p_1'se_1g_1\times\varepsilon), ev(p_2'se_1g_1\times\tau\varepsilon)) = \mu'(ev(p_1ike_0\tilde{\sigma}g_1\times\varepsilon), ev(p_2'se_1g_1\times id_{C_0})) = \mu'(ev(D_1^{\varepsilon}p_1ike_0g_0\sigma''\times id_{C_0}), ev(p_2'se_1g_1\times id_{C_0})) = \mu'(ev(D_1^{\varepsilon}\bar{f}_1\varepsilon''\sigma''\times id_{C_0}), ev(p_2'se_1g_1\times id_{C_0})) = \mu'(ev((\varepsilon')^{C_0}\bar{f}_0\sigma''\times id_{C_0}), ev(p_2'se_1g_1\times id_{C_0})) = \mu'(\varepsilon'f_0(\sigma''\times id_{C_0}), ev(p_2'se_1g_1\times id_{C_0})) = ev(p_2'se_1g_1\times id_{C_0})$, the third one follows. Therefore β is the inverse of α.

2) We denote by $\iota' : D_1\to D_1$ the inverse of $\boldsymbol{D}$. Since $dq_1't = (\tau')^{C_0}q_2't = (\sigma')^{C_0}(\iota')^{C_0}p_2's$ and $dp_1's = (\sigma')^{C_0}p_2's = (\tau')^{C_0}(\iota')^{C_0}p_2's$, there is a unique morphism $\iota_N : N\to N$ such that $p_2's\iota_N = (\iota')^{C_0}p_2's$, $p_1's\iota_N = q_1't$, $q_1't\iota_N = p_1's$. Then, $\mu'\zeta(\iota_Ne_1\times id_{C_1}) = \mu'(ev(p_1ike_0p_1's\iota_Ne_1\times id_{C_1}), ev(p_2's\iota_Ne_1\times\tau)) = \mu'(ev(p_1ike_0q_1'te_1\times id_{C_1}), ev((\iota')^{C_0}p_2'se_1\times\tau)) = \mu'(\mathrm{pr}_2\xi(e_1\times id_{C_1}), \iota'ev(p_2'se_1\times\tau)) = \mu'(\mathrm{pr}_2\zeta(e_1\times id_{C_1}), \iota'\mathrm{pr}_2\zeta(e_1\times id_{C_0})) = \varepsilon'\sigma'\mathrm{pr}_2\zeta(e_1\times id_{C_1}) = \varepsilon'\tau'\mathrm{pr}_1\xi(e_1\times id_{C_1}) = \mu'(\iota'\mathrm{pr}_1\xi(e_1\times id_{C_0}), \mathrm{pr}_1\xi(e_1\times id_{C_1})) = \mu'(\iota'ev(p_2'se_1\times\sigma), \mathrm{pr}_1\zeta(e_1\times id_{C_1})) = \mu'(ev((\iota')^{C_0}p_2'se_1\times\sigma), ev(p_1ike_0p_1'se_1\times id_{C_1})) = \mu'(ev(p_2's\iota_Ne_1\times\sigma), ev(p_1ike_0q_1't\iota_Ne_1\times id_{C_1})) = \mu'\xi(\iota_Ne_1\times id_{C_1})$. Hence $(\mu')^{C_1}\bar{\zeta}\iota_Ne_1 = (\mu')^{C_1}\bar{\xi}\iota_Ne_1$ and there is a morphism $\tilde{\iota} : M_1\to M_1$ such that $e_1\tilde{\iota} = \iota_Ne_1$. We claim that $\tilde{\iota}$ is the inverse of $\boldsymbol{D}^{\boldsymbol{C}}$. First, $\tilde{\sigma}\tilde{\iota} = p_1'se_1\tilde{\iota} = p_1's\iota_Ne_1 = q_1'te_1 = \tilde{\tau}$, $\tilde{\tau}\tilde{\iota} = q_1'te_1\tilde{\iota} = q_1't\iota_Ne_1 = p_1'se_1 = \tilde{\sigma}$. We also have $p_1'se_1\tilde{\mu}(id_{M_1}, \tilde{\iota}) = \tilde{\sigma}\tilde{\mu}(id_{M_1}, \tilde{\iota}) = \tilde{\sigma}\mathrm{pr}_1(id_{M_1}, \tilde{\iota}) = \tilde{\sigma} = \tilde{\sigma}\tilde{\varepsilon}\tilde{\sigma} = p_1'se_1\tilde{\varepsilon}\tilde{\sigma}$, $q_1'te_1\tilde{\mu}(id_{M_1}, \tilde{\iota}) = \tilde{\tau}\tilde{\mu}(id_{M_1}, \tilde{\iota}) = \tilde{\tau}\mathrm{pr}_2(id_{M_1}, \tilde{\iota}) = \tilde{\tau}\tilde{\iota} = \tilde{\sigma} = \tilde{\tau}\tilde{\varepsilon}\tilde{\sigma} = q_1'te_1\tilde{\varepsilon}\tilde{\sigma}$. Since the exponential transpose of $p_2'se_1\tilde{\mu}(id_{M_1}, \tilde{\iota}) = p_2'sm(id_{M_1}, \tilde{\iota}) = (\mu')^{C_0}\bar{\chi}(id_{M_1}, \tilde{\iota})$ is $\mu'\chi((id_{M_1}, \tilde{\iota})\times id_{C_0}) = \mu'(ev(p_2'se_1\times id_{C_0}), ev(p_2'se_1\tilde{\iota}\times id_{C_0}))$
$= \mu'(ev(p_2'se_1\times id_{C_0}), ev((\iota')^{C_0}p_2'se_1\times id_{C_0})) = \mu'(ev(p_2'se_1\times id_{C_0}), \iota'ev(p_2'se_1\times id_{C_0}))$
$= \varepsilon'\sigma'ev(p_2'se_1\times id_{C_0}) = ev((\varepsilon')^{C_0}(\sigma')^{C_0}p_2'se_1\times id_{C_0})$, it follows $p_2'se_1\tilde{\mu}(id_{M_1}, \tilde{\iota})$
$= (\varepsilon')^{C_0}(\sigma')^{C_0}p_2'se_1 = (\varepsilon')^{C_0}dp_1'se_1 = p_2'\varepsilon_s\tilde{\sigma} = p_2'se_1\tilde{\varepsilon}\tilde{\sigma}$. Thus we have $\tilde{\mu}(id_{M_1}, \tilde{\iota}) = \tilde{\varepsilon}\tilde{\sigma}$. Similarly, we can verify $\tilde{\mu}(\tilde{\iota}, id_{M_1}) = \tilde{\varepsilon}\tilde{\tau}$. □

5.2 Internal limits and colimits

Proposition 5.2.1 *Let $f, g : \boldsymbol{C}\to\boldsymbol{D}$ be morphisms in $\boldsymbol{cat}(\mathcal{E})$ and $\varphi : f\to g$ an internal natural transformation (5.1.1). Then, φ defines a natural transformation $f^*\to g^*$ of functors from $\mathcal{E}^{\boldsymbol{D}}\to\mathcal{E}^{\boldsymbol{C}}$. Thus we have a functor $\boldsymbol{cat}(\mathcal{E})\to\mathfrak{Top}/\mathcal{E}$ between 2-categories.*

Proof. Let $(\pi : X \to D_0, \alpha)$ be an object of $\mathcal{E}^{\boldsymbol{D}}$ and $\boldsymbol{X} = (X \times_{D_0} D_1, X; \sigma_X, \tau_X, \varepsilon_X, \mu_X) \xrightarrow{(\mathrm{pr}_2,\pi)} \boldsymbol{D}$ the corresponding discrete opfibration. $\square$

5.3 Internal fibered category

Let $\mathcal{E}$ be a category with finite limits.

For an internal functor $p = (p_1, p_0) : \boldsymbol{C} \to \boldsymbol{B}$, let $\left(M(p) \xrightarrow{m_p} B_1, M(p) \xrightarrow{\sigma_p} C_0, M(p) \xrightarrow{\tau_p} C_0\right)$ be the limiting cone of a diagram $C_0 \xrightarrow{p_0} B_0 \xleftarrow{\sigma} B_1 \xrightarrow{\tau} B_0 \xleftarrow{p_0} C_0$. Then, morphims $p_1 : C_1 \to B_1$ and $\sigma, \tau : C_1 \to C_0$ induce a unique morphism $\pi : C_1 \to M(p)$ satisfying $m_p\pi = p_1$, $\sigma_p\pi = \sigma$, $\tau_p\pi = \tau$. For morphims $M, N : U \to C_0$ and $f : U \to B_1$ satisfying $p_0M = \sigma f$ and $p_0N = \tau f$, let $\xi : U \to M(p)$ be the unique morphism satisfying $\sigma_p\xi = M$, $\tau_p\xi = N$ and $m_p\xi = f$. Consider the map $\pi_* : \mathcal{E}(U, C_1) \to \mathcal{E}(U, M(p))$ induced by π and we define a subset $\boldsymbol{C}_f(M, N)$ of $\mathcal{E}(U, C_1)$ by $\boldsymbol{C}_f(M, N) = \pi_*^{-1}(\xi)$. We note that a morphism $\varphi : U \to C_1$ belongs to $\boldsymbol{C}_f(M, N)$ if and only if φ satisfies $\sigma\varphi = M$, $\tau\varphi = N$ and $p_1\varphi = f$.

For a morphism $\varphi : V \to U$, since the following diagram commutes, $\varphi^* : \mathcal{E}(U, C_1) \to \mathcal{E}(V, C_1)$ induces $\varphi^* : \boldsymbol{C}_f(M, N) \to \boldsymbol{C}_{f\varphi}(M\varphi, N\varphi)$.

$$\begin{tikzcd}
\mathcal{E}(U, C_1) \arrow[r, "\pi_*"] \arrow[d, "\varphi^*"] & \mathcal{E}(U, M(p)) \arrow[d, "\varphi^*"] \\
\mathcal{E}(V, C_1) \arrow[r, "\pi_*"] & \mathcal{E}(V, M(p))
\end{tikzcd}$$

Let $L, M, N : U \to C_0$ and $f, g : U \to B_1$ be morphims satisfying $p_0L = \sigma g$, $p_0M = \tau g = \sigma f$ and $p_0N = \tau f$. Let $fg : U \to B_1$ be the composition of g and f, namely, fg is defined to be the composition $U \xrightarrow{(g,f)} B_1 \times_{B_0} B_1 \xrightarrow{\mu} B_1$. Define a map $\mu^{g,f}_{L,M,N} : \boldsymbol{C}_g(L, M) \times \boldsymbol{C}_f(M, N) \to \boldsymbol{C}_{fg}(L, N)$ as follows. Let $\xi, \zeta : U \to M(p)$ be morphisms satisfying $\sigma_p\xi = \tau_p\zeta = M$, $\sigma_p\zeta = L$, $\tau_p\xi = N$, $m_p\xi = f$ and $m_p\zeta = g$. Suppose $(\beta, \alpha) \in \boldsymbol{C}_g(L, M) \times \boldsymbol{C}_f(M, N)$. Then, $\tau\beta = \tau_p\pi\beta = \tau_p\zeta = M = \sigma_p\xi = \sigma_p\pi\alpha = \sigma\alpha$ and there is a unique morphism $(\beta, \alpha) : U \to C_1 \times_{C_0} C_1$ satisfying $\mathrm{pr}_1(\beta, \alpha) = \beta$ and $\mathrm{pr}_2(\beta, \alpha) = \alpha$. Thus we have $\sigma_p\pi\mu(\beta, \alpha) = \sigma\pi\mu(\beta, \alpha) = \sigma\beta = L$, $\tau_p\pi\mu(\beta, \alpha) = \tau\pi\mu(\beta, \alpha) = \tau\alpha = N$, $m_p\pi\mu(\beta, \alpha) = p_1\mu(\beta, \alpha) = \mu(p_1\beta, p_1\alpha) = \mu(m_p\pi\beta, m_p\pi\alpha) = \mu(m_p\zeta, m_p\xi) = \mu(g, f) = fg$. Therefore $\mu(\beta, \alpha)$ belongs to $\boldsymbol{C}_{fg}(L, N)$ and we put $\mu^{g,f}_{L,M,N}(\beta, \alpha) = \mu(\beta, \alpha)$.

Consider the case $p_0L = p_0M$ and $g = \varepsilon p_0M$. Put $X = p_0L = p_0M$ and $\boldsymbol{C}_g(L, M) = \boldsymbol{C}_X(L, M)$. We say that $\alpha : U \to C_1$ is cartesian if the map $\alpha_* : \boldsymbol{C}_X(L, M) \to \boldsymbol{C}_f(L, N)$ given by $\alpha_*(\beta) = \mu^{g,f}_{L,M,N}(\beta, \alpha)$ is bijective.

Proposition 5.3.1 *Let $\alpha_1, \alpha_2, \varphi : U \to C_1$ be morphisms in $\mathcal{E}$ satisfying $p_1\alpha_1 = p_1\alpha_2$, $\tau\alpha_1 = \sigma\varphi$, $\tau\alpha_2 = \tau\varphi$ and $p_1\varphi = \varepsilon p_0\tau\alpha_1$. If α_2 is cartesian, there exists a unique morphism $\psi : U \to C_1$ satisfying $\sigma\psi = \sigma\alpha_1$, $\tau\psi = \sigma\alpha_2$, $p_1\psi = \varepsilon p_0\sigma\alpha_1$ and $\mu(\psi, \alpha_2) = \mu(\alpha_1, \varphi)$.*

Proof. Put $L = \sigma\alpha_1$, $M = \sigma\alpha_2$, $N = \tau\alpha_2$, $X = p_0M$ and $f = p_1\alpha_2$. Then, $p_0L = p_0\sigma\alpha_1 = \sigma p_1\alpha_1 = \sigma p_1\alpha_2 = p_0\sigma\alpha_2 = p_0M$, $p_0L = \sigma f$, $p_0N = p_0\tau\alpha_2 = \tau p_1\alpha_2 = \tau f$, hence we can consider the map $\alpha_{2*} : \boldsymbol{C}_X(L, M) \to \boldsymbol{C}_f(L, N)$. Since $p_1\mu(\alpha_1, \varphi) = \mu(p_1\alpha_1, p_1\varphi) = \mu(f, \varepsilon p_0\tau\alpha_1) = f$, $\sigma\mu(\alpha_1, \varphi) = \sigma\alpha_1 = L$ and $\tau\mu(\alpha_1, \varphi) = \tau\varphi = \tau\alpha_2 = N$, $\mu(\alpha_1, \varphi) \in \boldsymbol{C}_f(L, N)$. Hence there exists a unique morphism $\psi \in \boldsymbol{C}_X(L, M)$ satisfying $\mu(\psi, \alpha_2) = \mu(\alpha_1, \varphi)$. $\square$

Corollary 5.3.2 *Let $\alpha_1, \alpha_2 : U \to C_1$ be morphisms in $\mathcal{E}$ satisfying $p_1\alpha_1 = p_1\alpha_2$ and $\tau\alpha_1 = \tau\alpha_2$. If both α_1 and α_2 are cartesian, the unique morphism $\psi : U \to C_1$ satisfying $\sigma\psi = \sigma\alpha_1$, $\tau\psi = \sigma\alpha_2$, $p_1\psi = \varepsilon p_0\sigma\alpha_1$ and $\mu(\psi, \alpha_2) = \alpha_1$ is invertible, namely, there is a morphism $\psi' : U \to C_1$ satisfying $\sigma\psi' = \tau\psi$, $\tau\psi' = \sigma\psi$, $\mu(\psi, \psi') = \varepsilon\sigma\psi$ and $\mu(\psi', \psi) = \varepsilon\tau\psi$.*

Consider a cartesian square

$$\begin{tikzcd}
B_1 \times_{B_0} C_0 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & C_0 \arrow[d, "p_0"] \\
B_1 \arrow[r, "\tau"] & B_0
\end{tikzcd}$$

Let $K(p_0) \overset{s}{\underset{t}{\rightrightarrows}} C_0$ be the kernel pair of $p_0 : C_0 \to B_0$. There is a unique morphism $u : K(p_0) \to M(p)$ satisfying $\sigma_p u = s$, $\tau_p u = t$ and $m_p u = \varepsilon p_0 s = \varepsilon p_0 t$.

For $\alpha : U \to C_1$, put $\sigma\alpha = M, \tau\alpha = N : U \to C_0$, $p_0\sigma\alpha = X : U \to B_0$ and $p_0\alpha = f : U \to B_1$. $L : U \to C_0$ is a morphism satisfying $p_0L = p_0M$.

Definition 5.3.3 *Let $\boldsymbol{B} = (B_1, B_0; \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be internal categories in $\mathcal{E}$ and $p = (p_1, p_0) : \boldsymbol{C} \to \boldsymbol{B}$ an internal functor. If the internal functor in $\widehat{\mathcal{E}}$ represented by p takes values in the category of fibered categories, we call p an internal fibered category.*

For an internal category $\boldsymbol{B}$ in $\mathcal{E}$, $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}$ is regarded as a 2-category as follows. If $\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}$ and $\boldsymbol{D} \xrightarrow{q} \boldsymbol{B}$ are objects of $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}$, let $P : \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{D}) \to \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{B})$ be the constant functor $P(f) = p$, $P(\varphi) = id_p = \varepsilon p_0$ and $q_* : \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{D}) \to \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{B})$ the functor given by $q_*(f) = c(f, q) = qf$, $q_*(\varphi) = c(\varphi, id_q) = \mu(q_1\varphi, \varepsilon q_0 f_0') = q_1\varphi$ for a morphism $\varphi : f \to f'$ in $\boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{D})$. We define $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}), (\boldsymbol{D} \xrightarrow{q} \boldsymbol{B}))$ to be the subcategory of $\boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{D})$ equalizing functors P and q_*, that is, it consists of objects $\{f \in \operatorname{Ob} \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{D}) | qf = p\}$ and morphisms $\varphi : f \to g$ satisfying $q_1\varphi = \varepsilon p_0$.

For $f, g, h \in \operatorname{Ob} \boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}), (\boldsymbol{D} \xrightarrow{q} \boldsymbol{B}))$ and morphisms $\varphi : f \to g$, $\psi : g \to h$ in $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}), (\boldsymbol{D} \xrightarrow{q} \boldsymbol{B}))$, it is clear that $\psi{\cdot}\varphi : f \to g$ is a morphism in $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}), (\boldsymbol{D} \xrightarrow{q} \boldsymbol{B}))$. The composition $c : \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{D}) \times \boldsymbol{cat}(\mathcal{E})(\boldsymbol{D}, \boldsymbol{E}) \to \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C}, \boldsymbol{E})$ induces a composition $c : \boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}), (\boldsymbol{D} \xrightarrow{q} \boldsymbol{B})) \times \boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{D} \xrightarrow{p} \boldsymbol{B}), (\boldsymbol{E} \xrightarrow{r} \boldsymbol{B})) \to \boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}), (\boldsymbol{E} \xrightarrow{r} \boldsymbol{B}))$. In fact, for $f \in \operatorname{Ob} \boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}), (\boldsymbol{D} \xrightarrow{q} \boldsymbol{B}))$ and $g \in \operatorname{Ob} \boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{D} \xrightarrow{q} \boldsymbol{B}), (\boldsymbol{E} \xrightarrow{r} \boldsymbol{B}))$, since $rc(f, g) = rgf = qf = p$, $c(f, g)$ is an object of $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}), (\boldsymbol{E} \xrightarrow{r} \boldsymbol{B}))$. For a morphism $\varphi : f \to f'$ in $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}), (\boldsymbol{D} \xrightarrow{q} \boldsymbol{B}))$ and a morphism $\psi : g \to g'$ in $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{D} \xrightarrow{q} \boldsymbol{B}), (\boldsymbol{E} \xrightarrow{r} \boldsymbol{B}))$, $r_1c(\varphi, \psi) = r_1\mu(g_1\varphi, \psi f_0') = \mu(r_1g_1\varphi, r_1\psi f_0') = \mu(q_1\varphi, \varepsilon q_0 f_0') = \mu(id_{B_1} \times \varepsilon)(\varepsilon p_0, q_0 f_0') = \mathrm{pr}_1(\varepsilon p_0, q_0 f_0') = \varepsilon p_0$. Hence $c(\varphi, \psi) : gf \to g'f'$ is a morphism in $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}((\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}), (\boldsymbol{E} \xrightarrow{r} \boldsymbol{B}))$. Therefore, $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}$ is a 2-category.

For an object X of $\mathcal{E}$, we denote by $\boldsymbol{T}_X = (X, X; id_X, id_X, id_X, \mathrm{pr}_1)$ the trivial internal category. Let $\boldsymbol{T} : \mathcal{E} \to \boldsymbol{cat}(\mathcal{E})$ be the functor defined by $X \mapsto \boldsymbol{T}_X$ on objects and $f \mapsto (f, f)$ on morphisms. We denote by $ob : \boldsymbol{cat}(\mathcal{E}) \to \mathcal{E}$ the composition of functors $\boldsymbol{cat}(\mathcal{E}) \xrightarrow{\Phi} \mathcal{E} \times \mathcal{E} \xrightarrow{\mathrm{pr}_2} \mathcal{E}$.

Proposition 5.3.4 *$\boldsymbol{T} : \mathcal{E} \to \boldsymbol{cat}(\mathcal{E})$ is a left adjoint of $ob : \boldsymbol{cat}(\mathcal{E}) \to \mathcal{E}$.*

Proof. If $\boldsymbol{B}$ is an internal category, define $\kappa : \mathcal{E}(X, ob(\boldsymbol{B})) \to \boldsymbol{cat}(\mathcal{E})(\boldsymbol{T}_X, \boldsymbol{B})$ by $\kappa(\varphi) = (\varepsilon\varphi, \varphi)$. It is easy to see that κ is a natural bijection. □

Definition 5.3.5 *For an object $\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}$ of $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}$ and a morphism $\varphi : X \to B_0$, let $\boldsymbol{C} \times_{\boldsymbol{B}} \boldsymbol{T}_X \xrightarrow{p_\varphi} \boldsymbol{T}_X$ be the pull-back of p along $\kappa(\varphi)$.*

$$\begin{tikzcd}
\boldsymbol{C} \times_{\boldsymbol{B}} \boldsymbol{T}_X \arrow[r, "i_\varphi"] \arrow[d, "p_\varphi"] & \boldsymbol{C} \arrow[d, "p"] \\
\boldsymbol{T}_X \arrow[r, "\kappa(\varphi)"] & \boldsymbol{B}
\end{tikzcd}$$

We call $\boldsymbol{C} \times_{\boldsymbol{B}} \boldsymbol{T}_X$ the fiber category of $\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}$ over φ and denote this by $\boldsymbol{C}_\varphi$.

Let $f : \boldsymbol{B}' \to \boldsymbol{B}$ and $p : \boldsymbol{C} \to \boldsymbol{B}$ be morphisms in $\boldsymbol{cat}(\mathcal{E})$. Consider the pull-back $p' : \boldsymbol{C} \times_{\boldsymbol{B}} \boldsymbol{B}' \to \boldsymbol{B}'$ of p along f.

$$\begin{tikzcd}
\boldsymbol{C} \times_{\boldsymbol{B}} \boldsymbol{B}' \arrow[r, "\bar{f}"] \arrow[d, "p'"] & \boldsymbol{C} \arrow[d, "p"] \\
\boldsymbol{B}' \arrow[r, "f"] & \boldsymbol{B}
\end{tikzcd}$$

We set $\boldsymbol{C}' = \boldsymbol{C} \times_{\boldsymbol{B}} \boldsymbol{B}'$. For a morphism $\varphi : X \to B_0'$, since $\kappa(f_0\varphi) = f\kappa(\varphi)$ and the both squares of the following diagram are pull-backs,

$$\begin{tikzcd}
\boldsymbol{C}'_\varphi \arrow[r, "i_\varphi"] \arrow[d, "p'_\varphi"] & \boldsymbol{C}' \arrow[r, "\bar{f}"] \arrow[d, "p'"] & \boldsymbol{C} \arrow[d, "p"] \\
\boldsymbol{T}_X \arrow[r, "\kappa(\varphi)"] & \boldsymbol{B}' \arrow[r, "f"] & \boldsymbol{B}
\end{tikzcd}$$

there is a unique isomorphism $f_\varphi : \boldsymbol{C}'_\varphi \to \boldsymbol{C}_{f_0\varphi}$ satisfying $p_{f_0\varphi} f_\varphi = p'_\varphi$ and $i_{f_0\varphi} f_\varphi = \bar{f} i_\varphi$.

Proposition 5.3.6 *Let $g : (\boldsymbol{C} \xrightarrow{p} \boldsymbol{B}) \to (\boldsymbol{D} \xrightarrow{q} \boldsymbol{B})$ be a morphism in $\boldsymbol{cat}(\mathcal{E})/\boldsymbol{B}$ and $f : \boldsymbol{B}' \to \boldsymbol{B}$ an internal functor. Suppose that g is fully faithful regarded as an internal functor $\boldsymbol{C} \to \boldsymbol{D}$. Then, for any morphism $\varphi : X \to B'_0$,*

5.4 Filtered category

Definition 5.4.1 *Let $\mathcal{C}$ be a category with finite limits and $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ an internal category in $\mathcal{C}$. We say that $\boldsymbol{C}$ a filtered category if the following conditions hold.*

(1) The unique morphism $C_0 \to 1$ is an epimorphism.

(2) Consider a pull-back

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & C_1 \arrow[d, "\tau"] \\ C_1 \arrow[r, "\tau"] & C_0 \end{tikzcd} .$$

Then, $(\sigma p_1, \sigma p_2) : P \to C_0 \times C_0$ is an epimorphism.

(3) Consider the pull-backs

$$\begin{tikzcd} C_1 \times_{C_0 \times C_0} C_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & C_1 \arrow[d, "(\sigma,\tau)"] \\ C_1 \arrow[r, "(\sigma,\tau)"] & C_0 \times C_0 \end{tikzcd} \qquad \begin{tikzcd} (C_1 \times_{C_0 \times C_0} C_1) \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & C_1 \arrow[d, "\sigma"] \\ C_1 \times_{C_0 \times C_0} C_1 \arrow[r, "\tau \mathrm{pr}_1 = \tau \mathrm{pr}_2"] & C_0 \end{tikzcd} .$$

Let $e : E \to (C_1 \times_{C_0 \times C_0} C_1) \times_{C_0} C_1$ be the equalizer of $\mu(\mathrm{pr}_1 \times 1), \mu(\mathrm{pr}_2 \times 1) : (C_1 \times_{C_0 \times C_0} C_1) \times_{C_0} C_1 \to C_1$. Then the composition $E \xrightarrow{e} (C_1 \times_{C_0 \times C_0} C_1) \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_1} C_1 \times_{C_0 \times C_0} C_1$ is an epimorphism.

Let $\boldsymbol{G} = (G_1, G_0; \sigma, \tau, \varepsilon, \mu)$ be an internal groupoid in $\mathcal{C}$.

Lemma 5.4.2 *We denote by $\Delta : G_1 \to G_1 \times_{G_0 \times G_0} G_1$ the diagonal morphism. Then, $\Delta \times id_{G_1} : G_1 \times_{G_0} G_1 \to (G_1 \times_{G_0 \times G_0} G_1) \times_{G_0} G_1$ is an equalizer of $\mu(\mathrm{pr}_1 \times 1), \mu(\mathrm{pr}_2 \times 1) : (G_1 \times_{G_0 \times G_0} G_1) \times_{G_0} G_1 \to G_1$.*

Proof. It is clear that $\mu(\mathrm{pr}_1 \times 1)(\Delta \times id_{G_1}) = \mu(\mathrm{pr}_2 \times 1)(\Delta \times id_{G_1})$ holds. Since $(\mathrm{pr}_1 \times id_{G_1})(\Delta \times id_{G_1})$ is the identity morphism of $G_1 \times_{G_0} G_1$, $\Delta \times id_{G_1}$ is a monomorphism. Suppose that there is a morphism $f : X \to (G_1 \times_{G_0 \times G_0} G_1) \times_{G_0} G_1$ such that $\mu(\mathrm{pr}_1 \times 1) f = \mu(\mathrm{pr}_2 \times 1) f$. Put $f_i = \mathrm{pr}_i \mathrm{pr}_1 f : X \to G_1$ $(i = 1, 2)$ and $f_3 = \mathrm{pr}_2 f : X \to G_1$. Then, we have $\sigma f_1 = \sigma f_2$, $\tau f_1 = \tau f_2 = \sigma f_3$ and $\mu(f_1, f_3) = \mu(f_2, f_3)$. Since $\boldsymbol{G}$ is an internal groupoid, it follows that $f_1 = \mu(\mu(f_1, f_3), \iota f_3) = \mu(\mu(f_2, f_3), \iota f_3) = f_2$. Hence there is a morphism $g : X \to G_1$ such that $\mathrm{pr}_1 f = \Delta g$. Thus we have $f = (\Delta \times id_{G_1})(g, \mathrm{pr}_2 f)$. □

Proposition 5.4.3 *(1) The condition (2) in (5.4.1) holds for $\boldsymbol{G}$ if and only if $(\sigma, \tau) : G_1 \to G_0 \times G_0$ is an epimorphism.*

(2) The condition (3) in (5.4.1) holds for $\boldsymbol{G}$ if and only if $\boldsymbol{G}$ is a poset, hence an equivalence relation.

Proof. (1) Since $\tau = \sigma\iota$, there exists a unique morphism $\varphi : P \to G_1 \times_{G_0} G_1$ satisfying $\mathrm{pr}_1 \varphi = p_1$ and $\mathrm{pr}_2 \varphi = \iota p_2$. Then, $(\sigma, \tau)\mu\varphi = (\sigma\mu\varphi, \tau\mu\varphi) = (\sigma \mathrm{pr}_1 \varphi, \tau \mathrm{pr}_2 \varphi) = (\sigma p_1, \tau\iota p_2) = (\sigma p_1, \sigma p_2)$ and φ is an isomorphism. Since μ is a split epimorphism, it follows that $(\sigma p_1, \sigma p_2)$ is an epimorphism if and only if (σ, τ) is so.

(2) Suppose that the condition (3) holds. It follows from (5.4.2) that $\mathrm{pr}_1(\Delta \times id_{G_1}) = \Delta \mathrm{pr}_1$ is an epimorphism, hence so is Δ. Then, (A.3.2) implies that $(\sigma, \tau) : G_1 \to G_0 \times G_0$ is a monomorphism.

Conversely, if $\boldsymbol{G}$ is a poset, Δ is an epimorphism by (A.3.2). Since the composition $G_1 \times_{G_0 \times G_0} G_1 \cong (G_1 \times_{G_0 \times G_0} G_1) \times_{G_0} G_0 \xrightarrow{id_{G_1 \times_{G_0 \times G_0} G_1} \times \varepsilon} (G_1 \times_{G_0 \times G_0} G_1) \times_{G_0} G_1 \xrightarrow{\mathrm{pr}_1} G_1 \times_{G_0 \times G_0} G_1$ is the identity morphism, $\mathrm{pr}_1 : (G_1 \times_{G_0 \times G_0} G_1) \times_{G_0} G_1 \to G_1 \times_{G_0 \times G_0} G_1$ is an epimorphism. Therefore $\mathrm{pr}_1(\Delta \times id_{G_1}) = \Delta \mathrm{pr}_1$ is an epimorphism and the condition holds by (5.4.2). □

Corollary 5.4.4 *An internal groupoid $\boldsymbol{G} = (G_1, G_0; \sigma, \tau, \varepsilon, \mu)$ in $\mathcal{C}$ is filtered if and only if $G_0 \to 1$ is an epimorphism and $(\sigma, \tau) : G_1 \to G_0 \times G_0$ is both an epimorphism and monomorphism.*

Definition 5.4.5 *Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{C}$ and $(\pi : X \to C_0, \alpha : C_1 \times_{C_0} X \to X)$ an internal presheaf on $\boldsymbol{C}$. If the internal category $(C^X, X; \sigma^X, \tau^X, \varepsilon^X, \mu^X)$ associated with internal presheaf $(\pi : X \to C_0, \alpha : C_1 \times_{C_0} X \to X)$ (5.1.10) is a filtered category, we call $(\pi : X \to C_0, \alpha : C_1 \times_{C_0} X \to X)$ a flat presheaf.*

Proposition 5.4.6 *If $\boldsymbol{G} = (G, 1; \sigma, \sigma, \varepsilon, \mu)$ is an internal group and $\alpha : G \times X \to X$ is a left G-object, then α is a flat presheaf on $\boldsymbol{G}$ if and only if the following conditions hold.*

(1) The unique morphism $X \to 1$ is an epimorphism.
(2) $(\mathrm{pr}_2, \alpha) : G \times X \to X \times X$ is an epimorphism and a monomorphism.

Chapter 6

An introduction to Galois category and its fundamental group

6.1 Pro-objects

Let $\mathcal{C}$ be a $\mathcal{U}$-category. For $X \in \mathrm{Ob}\,\mathcal{C}$, let $h^X : \mathcal{C} \to \mathcal{U}\text{-}\mathbf{Ens}$ be the functor given by $h^X(Y) = \mathcal{C}(X, Y)$. Set $\check{\mathcal{C}} = \mathrm{Funct}(\mathcal{C}, \mathcal{U}\text{-}\mathbf{Ens})$. We define the contravariant Yoneda embedding $h^{op} : \mathcal{C}^{op} \to \check{\mathcal{C}}$ by $h^{op}(X) = h^X$. By the dual of Yoneda's lemma, for $F \in \mathrm{Ob}\,\check{\mathcal{C}}$ and $X \in \mathrm{Ob}\,\mathcal{C}$, the map $\check{\mathcal{C}}(h^X, F) \to F(X)$ defined by $f \mapsto f_X(id_X)$ is bijective.

For $F, G \in \mathrm{Ob}\,\check{\mathcal{C}}$, we define a topology on $\check{\mathcal{C}}(F, G)$ as follows. Choose a functor $D : \mathcal{D} \to \mathcal{C}$ such that $(h^{D(i)} \xrightarrow{\lambda_i} F)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone (for example, $D = P : (h^{op}{\downarrow}F) \to \mathcal{C}$ (A.4.2)) of $h^{op}D : \mathcal{D}^{op} \to \check{\mathcal{C}}$. Give each $\check{\mathcal{C}}(h^{D(i)}, G)$ the discrete topology and we give a topology on $\check{\mathcal{C}}(F, G)$ such that $(\check{\mathcal{C}}(F, G) \xrightarrow{\lambda_i^*} \check{\mathcal{C}}(h^{D(i)}, G))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone in the category of topological spaces. We denote this topological space by $\check{\mathcal{C}}(F, G)_D$ until we show that this topology does not depend on the choice of D. Take a functor $E : \mathcal{E} \to \mathcal{C}$ such that $(h^{E(j)} \xrightarrow{\mu_j} G)_{j \in \mathrm{Ob}\,\mathcal{E}}$ is a colimiting cone.

Lemma 6.1.1 *The composition map $c : \check{\mathcal{C}}(F, G)_D \times \check{\mathcal{C}}(G, H)_E \to \check{\mathcal{C}}(F, H)_D$ is continuous.*

Proof. The following diagram commutes for any $i \in \mathrm{Ob}\,\mathcal{D}$.

$$\begin{array}{ccc}
\check{\mathcal{C}}(F, G)_D \times \check{\mathcal{C}}(G, H)_E & \xrightarrow{\quad c \quad} & \check{\mathcal{C}}(F, H)_D \\
\big\downarrow{\scriptstyle \lambda_i^* \times id} & & \big\downarrow{\scriptstyle \lambda_i^*} \\
\check{\mathcal{C}}(h^{D(i)}, G) \times \check{\mathcal{C}}(G, H)_E & \xrightarrow{\quad c \quad} & \check{\mathcal{C}}(h^{D(i)}, H)
\end{array}$$

For $\alpha \in \check{\mathcal{C}}(h^{D(i)}, G)$ and $\beta \in \check{\mathcal{C}}(G, H)$, choose $j \in \mathrm{Ob}\,\mathcal{E}$ and $\gamma : E(j) \to D(i)$ such that $\alpha_{D(i)}(id_{D(i)}) = \mu_{j\,D(i)}(\gamma)$, that is, $\alpha = \mu_j h^\gamma$. Suppose $\beta' \in \mu_j^{*-1}(\beta\mu_j)$. Then, $\beta'\alpha = \beta'\mu_j h^\gamma = \mu_j^*(\beta')h^\gamma = \beta\mu_j h^\gamma = \beta\alpha$. Set $U = \{\alpha\} \times \mu_j^{*-1}(\beta\mu_j)$. Then, U is a neighborhood of (α, β) in $\check{\mathcal{C}}(h^{D(i)}, G) \times \check{\mathcal{C}}(G, H)_E$ such that $c(U) = \{\beta\alpha\}$. It follows that the lower composition map of the above diagram is continuous. Since

$$(\check{\mathcal{C}}(F, G)_D \times \check{\mathcal{C}}(G, H)_E \xrightarrow{c(\lambda_i^* \times id)} \check{\mathcal{C}}(h^{D(i)}, H))_{i \in \mathrm{Ob}\,\mathcal{D}}$$

is a cone, the upper composition map is also continuous. $\square$

Let $D : \mathcal{D} \to \mathcal{C}$ and $D' : \mathcal{D}' \to \mathcal{C}$ be functors such that $(h^{D(i)} \xrightarrow{\lambda_i} F)_{i \in \mathrm{Ob}\,\mathcal{D}}$ and $(h^{D'(i')} \xrightarrow{\lambda'_{i'}} F)_{i' \in \mathrm{Ob}\,\mathcal{D}'}$ are colimiting cones. Since $c : \check{\mathcal{C}}(F, F)_D \times \check{\mathcal{C}}(F, G)_{D'} \to \check{\mathcal{C}}(F, G)_D$ is continuous, the identity map $id_F^* : \check{\mathcal{C}}(F, G)_{D'} \to \check{\mathcal{C}}(F, G)_D$ is continuous. Similarly, $id_F^* : \check{\mathcal{C}}(F, G)_D \to \check{\mathcal{C}}(F, G)_{D'}$ is also continuous. Hence the topologies on $\check{\mathcal{C}}(F, G)$ defined from D and D' are the same. We call this topology on $\check{\mathcal{C}}(F, G)$ the *natural* topology.

Proposition 6.1.2 *1) The composition map $c : \check{\mathcal{C}}(F, G) \times \check{\mathcal{C}}(G, H) \to \check{\mathcal{C}}(F, H)$ is continuous for the natural topologies on $\check{\mathcal{C}}(F, G)$, $\check{\mathcal{C}}(G, H)$ and $\check{\mathcal{C}}(F, H)$.*

2) Let $D : \mathcal{D} \to \check{\mathcal{C}}$ *be a functor and* $(D(i) \xrightarrow{\iota_i} F)_{i\in \mathrm{Ob}\,\mathcal{D}}$ *a colimiting cone of* D*. Then, for* $G \in \mathrm{Ob}\,\check{\mathcal{C}}$*,* $(\check{\mathcal{C}}(F,G) \xrightarrow{\iota_i^*} \check{\mathcal{C}}(D(i),G))_{i\in \mathrm{Ob}\,\mathcal{D}}$ *is a limiting cone in the category of topological spaces.*

3) If $T : \check{\mathcal{C}} \to \check{\mathcal{C}}'$ *is a functor preserving colimits, then, for* $F, G \in \mathrm{Ob}\,\check{\mathcal{C}}$*,* $T : \check{\mathcal{C}}(F,G) \to \check{\mathcal{C}}'(T(F),T(G))$ *is continuous.*

Proof. 1) This is a a direct consequence of (6.1.1).

2) For each $i \in \mathrm{Ob}\,\mathcal{D}$, choose a functor $D_i : \mathcal{D}_i \to \mathcal{C}$ such that $(h^{D_i(k)} \xrightarrow{\lambda^i_k} D(i))_{k\in \mathrm{Ob}\,\mathcal{D}_i}$ is a colimiting cone. Let $\alpha : i \to j$ be a morphism in $\mathcal{D}$. For $k \in \mathrm{Ob}\,\mathcal{D}_i$ and $l \in \mathrm{Ob}\,\mathcal{D}_j$, set $R(\alpha; k, l) = \{\beta \in \mathcal{C}(D_j(l), D_i(k))|\ \lambda^j_l h^\beta = D(\alpha)\lambda^i_k\}$. Since $(h^{D_j(l)}(D_i(k)) \xrightarrow{(\lambda^j_l)_{D_i(k)}} D(j)(D_i(k)))_{l\in \mathrm{Ob}\,\mathcal{D}_i}$ is a colimiting cone, $R(\alpha; k, l)$ is not empty for some $l \in \mathrm{Ob}\,\mathcal{D}_i$. Define a category $\mathcal{E}$ and a functor $E : \mathcal{E}^{op} \to \mathcal{C}$ as follows. Set $\mathrm{Ob}\,\mathcal{E} = \{(i, k)|\ i \in \mathrm{Ob}\,\mathcal{D},\ k \in \mathrm{Ob}\,\mathcal{D}_i\}$. For $i, j \in \mathrm{Ob}\,\mathcal{D}$ and $k \in \mathrm{Ob}\,\mathcal{D}_i$, $l \in \mathrm{Ob}\,\mathcal{D}_j$, set $\mathcal{E}((i, k), (j, l)) = \{(\alpha, \beta)|\ \alpha \in \mathcal{D}(i, j),\ \beta \in R(\alpha; k, l)\}$. Suppose that $(\alpha, \beta) \in \mathcal{E}((i, k), (j, l))$, $(\gamma, \delta) \in \mathcal{E}((j, l), (m, n))$. We note that $\beta\delta \in R(\gamma\alpha; k, n)$. We define the composition of (γ, δ) and (α, β) to be $(\gamma\alpha, \beta\delta)$. It is clear that $(id_i, id_{D_i(k)})$ is the identity morphism of (i, k). Set $E(i, k) = D_i(k)$ and $E(\alpha, \beta) = \beta$ for $(\alpha, \beta) \in \mathcal{E}((i, k), (j, l))$. Then, for a morphism $(\alpha, \beta) : (i, k) \to (j, l)$, $\iota_j\lambda^j_l h^{E(\alpha,\beta)} = \iota_j\lambda^j_l h^\beta = \iota_j D(\alpha)\lambda^i_k = \iota_i\lambda^i_k$. Hence $(h^{E(i,k)} \xrightarrow{\iota_i\lambda^i_k} F)_{(i,k)\in \mathrm{Ob}\,\mathcal{E}}$ is a cone.

We claim that $(h^{E(i,k)} \xrightarrow{\iota_i\lambda^i_k} F)_{(i,k)\in \mathrm{Ob}\,\mathcal{E}}$ is a colimiting cone of $h^{op}E : \mathcal{E} \to \check{\mathcal{C}}$. Since $(D(i) \xrightarrow{\iota_i} F)_{i\in \mathrm{Ob}\,\mathcal{D}}$ and $(h^{D_i(k)} \xrightarrow{\lambda^i_k} D(i))_{k\in \mathrm{Ob}\,\mathcal{D}_i}$ are colimiting cones, $(\iota_i\lambda^i_k)_{(i,k)\in \mathrm{Ob}\,\mathcal{E}}$ is an epimorphic family. Let $(h^{E(i,k)} \xrightarrow{\mu_{i,k}} H)_{(i,k)\in \mathrm{Ob}\,\mathcal{E}}$ be a cone. Since $R(id_i; k, l) = \{\beta \in \mathcal{C}(D_i(l), D_i(k))|\ \lambda^i_l h^\beta = \lambda^i_k\}$ contains $\{D_i(f)|\ f \in \mathcal{D}_i(l, k)\}$, fixing i, $(h^{E(i,k)} \xrightarrow{\mu_{i,k}} H)_{k\in \mathrm{Ob}\,\mathcal{D}_i}$ is a cone of $h^{op}D_i$. Hence there is a unique morphism $\nu_i : D(i) \to H$ such that $\mu_{i,k} = \nu_i\lambda^i_k$ for any $k \in \mathrm{Ob}\,\mathcal{D}_i$. Let $\alpha : i \to j$ be a morphism in $\mathcal{D}$. For any $k \in \mathrm{Ob}\,\mathcal{D}_i$, there exist $\beta \in R(\alpha; k, l)$ for some $l \in \mathrm{Ob}\,\mathcal{D}_j$. Then, $\nu_j\alpha\lambda^i_k = \nu_j\lambda^j_l h^\beta = \mu_{j,l}h^{E(\alpha,\beta)} = \mu_{i,k} = \nu_i\lambda^i_k$. It follows that $\nu_j\alpha = \nu_i$ and there is a unique morphism $\psi : F \to H$ such that $\nu_i = \psi\iota_i$. Thus ψ is the morphism satisfying $\psi\iota_i\lambda^i_k = \mu_{i,k}$ for any $(i, k) \in \mathrm{Ob}\,\mathcal{E}$.

We deduce that $(\check{\mathcal{C}}(F, G) \xrightarrow{(\lambda^i_k)^*\iota^*_i} \check{\mathcal{C}}(h^{E(i,k)}, G))_{(i,k)\in \mathrm{Ob}\,\mathcal{E}}$ is a limiting cone in the category of topological spaces. Suppose that $(X \xrightarrow{p_i} \check{\mathcal{C}}(D(i), G))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone in the category of topological spaces. Then, $(X \xrightarrow{(\lambda^i_k)^* p_i} \check{\mathcal{C}}(h^{E(i,k)}, G))_{(i,k)\in \mathrm{Ob}\,\mathcal{E}}$ is a cone. In fact, for a morphism $(\alpha, \beta) : (i, k) \to (j, l)$, since $\lambda^j_l h^\beta = D(\alpha)\lambda^i_k$, we have $h^{E(\alpha,\beta)*}(\lambda^j_l)^* p_j = h^{\beta*}(\lambda^j_l)^* p_j = (\lambda^j_l h^\beta)^* p_j = (D(\alpha)\lambda^i_k)^* p_j = (\lambda^i_k)^* D(\alpha)^* p_j = (\lambda^i_k)^* p_i$. There is a unique continuous map $\varphi : X \to \check{\mathcal{C}}(F, G)$ satisfying $(\lambda^i_k)^* p_i = (\lambda^i_k)^*\iota^*_i\varphi$ for any $(i, k) \in \mathrm{Ob}\,\mathcal{E}$. Since $(\check{\mathcal{C}}(D(i), G) \xrightarrow{(\lambda^i_k)^*} \check{\mathcal{C}}(h^{D_i(k)}, G))_{k\in \mathrm{Ob}\,\mathcal{D}_i}$ is a colimiting cone, $p_i = \iota^*_i\varphi$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. Hence the assertion follows.

3) Let $D : \mathcal{D} \to \mathcal{C}$ be a functor such that $(h^{D(i)} \xrightarrow{\lambda_i} F)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $h^{op}D$. By the assumption, $(T(h^{D(i)}) \xrightarrow{T(\lambda_i)} T(F))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $Th^{op}D$. It follows from 2) that $(\check{\mathcal{C}}'(T(F), T(G)) \xrightarrow{T(\lambda_i)^*} \check{\mathcal{C}}'(T(h^{D(i)}), T(G)))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone in the category of topological spaces. We note that the following diagram commutes.

$$\begin{tikzcd}
\check{\mathcal{C}}(F,G) \arrow[r, "T"] \arrow[d, "\lambda_i^*"] & \check{\mathcal{C}}'(T(F),T(G)) \arrow[d, "T(\lambda_i)^*"] \\
\check{\mathcal{C}}(h^{D(i)},G) \arrow[r, "T"] & \check{\mathcal{C}}'(T(h^{D(i)}),T(G))
\end{tikzcd}$$

Since $\check{\mathcal{C}}(h^{D(i)}, G)$ has the discrete topology, the lower map T of the above diagram is continuous. Hence $(\check{\mathcal{C}}(F, G) \xrightarrow{T\lambda^*_i} \check{\mathcal{C}}'(T(h^{D(i)}), T(G)))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone in the category of topological spaces and $T : \check{\mathcal{C}}(F, G) \to \check{\mathcal{C}}'(T(F), T(G))$ is the unique map making the above diagram commute for any $i \in \mathrm{Ob}\,\mathcal{D}$. Therefore T is continuous. □

A filtered category $\mathcal{D}$ is said to be essentially $\mathcal{U}$-small if it contains a cofinal $\mathcal{U}$-small subcategory. Note that, since $\mathcal{U}$-**Ens** is $\mathcal{U}$-cocomplete, so is $\check{\mathcal{C}}$. If $\mathcal{D}$ is an essentially $\mathcal{U}$-small filtered category and $D : \mathcal{D}^{op} \to \mathcal{C}$ is a functor, define a functor $L(D) : \mathcal{C} \to \mathcal{U}$-**Ens** to be the colimit of $h^{op}D : \mathcal{D} \to \check{\mathcal{C}}$.

Definition 6.1.3 *For a* $\mathcal{U}$*-category* $\mathcal{C}$*, a functor* $D : \mathcal{D}^{op} \to \mathcal{C}$ *from an opposite category of a filtered category* $\mathcal{D}$ *is called a pro-object of* $\mathcal{C}$*. We call* $\mathcal{D}$ *the domain of* D*. If* $\mathcal{D}$ *is essentially* $\mathcal{U}$*-small, we call* D *a* $\mathcal{U}$*-pro-object (or simply, pro-object). We define the category of pro-objects* $\mathrm{Pro}(\mathcal{C})$ *of* $\mathcal{C}$ *as follows.* $\mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$ *consists of*

$\mathcal{U}$-pro-objects. For $D, E \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$, we set $\mathrm{Pro}(\mathcal{C})(D,E) = \check{\mathcal{C}}(L(E), L(D))$. Hence there is a fully faithful functor $L : \mathrm{Pro}(\mathcal{C}) \to \check{\mathcal{C}}^{op}$. We say that an object F of $\check{\mathcal{C}}$ is pro-representable if F is isomorphic to $L(D)$ for some pro-object D. In other words, F is pro-representable if and only if there exist a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ and an element $(\xi_i)_{i\in \mathrm{Ob}\,\mathcal{D}} \in \varprojlim FD$ such that $(h^{D(i)} \xrightarrow{\xi_i^\sharp} F)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $h^{op}D$, where $\xi_i^\sharp : h^{D(i)} \to F$ is a morphism in $\check{\mathcal{C}}$ given by $(\xi_i^\sharp)_X(f) = F(f)(\xi_i)$ for $X \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathcal{C}(D(i), X)$.

For a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ of $\mathcal{C}$, we put $D_i = D(i)$ for $i \in \mathrm{Ob}\,\mathcal{D}$. There is a natural bijection $\theta : \mathrm{Pro}(\mathcal{C})(D,E) \to \varprojlim_j L(D)(E_j) = \varprojlim_j \varinjlim_i \mathcal{C}(D_i, E_j)$ given as follows. Let $f : D \to E$ be a morphism in $\mathrm{Pro}(\mathcal{C})$ and $\mathcal{E}$ the domain of E. For $Y \in \mathrm{Ob}\,\mathcal{C}$ and $k \in \mathrm{Ob}\,\mathcal{E}$, $\lambda_k^Y : \mathcal{C}(E_k, Y) \to \varinjlim_j \mathcal{C}(E_j, Y) = L(E)(Y)$ denotes the canonical morphism. Set $c_k = f_{E_k}\lambda_k^{E_k}(id_{E_k}) \in L(D)(E_k)$. If $\tau : E_k \to E_l$ is a transition map, $L(D)(\tau)(c_k) = L(D)(\tau)(f_{E_k}\lambda_k^{E_k}(id_{E_k})) = f_{E_l}L(E)(\tau)(\lambda_k^{E_k}(id_{E_k})) = f_{E_l}\lambda_k^{E_l}(\tau) = f_{E_l}\lambda_l^{E_l}(id_{E_l}) = c_l$. Hence $(c_k)_{k\in \mathrm{Ob}\,\mathcal{E}}$ is an element of $\varprojlim_j L(D)(E_j)$. We set $\theta(f) = (c_k)_{k\in\mathrm{Ob}\,\mathcal{E}}$. For $(c_k)_{k\in\mathrm{Ob}\,\mathcal{E}} \in \varprojlim_j L(D)(E_j)$ and $Y \in \mathrm{Ob}\,\mathcal{C}$, define $f_{Y,k} : \mathcal{C}(E_k, Y) \to L(D)(Y)$ by $f_{Y,k}(\varphi) = L(D)(\varphi)(c_k)$. If $\tau : E_k \to E_l$ is a transition map, then, for $\psi \in \mathcal{C}(E_l, Y)$, $f_{Y,k}\tau^*(\psi) = f_{Y,k}(\psi\tau) = L(D)(\psi\tau)(c_k) = L(D)(\psi)(c_j) = f_{Y,l}(\psi)$. Thus we have a unique morphism $f_Y : L(E)(Y) \to L(D)(Y)$ such that $f_Y\lambda_k^Y = f_{Y,k}$ for any $k \in \mathrm{Ob}\,\mathcal{E}$. It is easy to verify that f_Y is natural in Y and that $\theta^{-1} : \varprojlim_j L(D)(E_j) \to \mathrm{Pro}(\mathcal{C})(D,E)$ is given by $\theta^{-1}((c_k)_{k\in\mathrm{Ob}\,\mathcal{E}})_Y = f_Y$. We note that, if $\bar{c}_k \in \mathcal{C}(D_i, E_k)$ is a representative of $c_k \in L(D)(E_k)$, the following square commutes, where $\mu_k^Y : \mathcal{C}(E_k, Y) \to L(E)(Y)$ is the canonical map.

$$\begin{tikzcd}
\mathcal{C}(E_k, Y) \arrow[r, "\bar{c}_k^*"] \arrow[d, "\mu_k^Y"] & \mathcal{C}(D_i, Y) \arrow[d, "\lambda_i^Y"] \\
L(E)(Y) \arrow[r, "f"] & L(D)(Y)
\end{tikzcd}$$

We define a morphism $\epsilon_X : \varprojlim_j L(D)(E_j) \times L(E)(X) \to L(D)(X)$ for $X \in \mathrm{Ob}\,\mathcal{C}$ as follows. For $(c_k)_{k\in\mathrm{Ob}\,\mathcal{E}} \in \varprojlim_j L(D)(E_j)$ and $d \in L(E)(X)$, choose representatives $\bar{c}_k \in \mathcal{C}(D_{i(k)}, E_k)$ and $\bar{d} \in \mathcal{C}(E_j, X)$ of c_k and d. $\epsilon_X((c_k)_{k\in\mathrm{Ob}\,\mathcal{E}}, d)$ is the class represented by $\bar{d}\bar{c}_j$. It is easy to verify that the class of $\bar{d}\bar{c}_j$ does not depend on the choice of representatives of c_k and d. Let $F : \mathcal{F}^{op} \to \mathcal{C}$ be a pro-object and $p_l : \varprojlim_l L(D)(F_l) \to L(D)(F_l)$, $q_l : \varprojlim_l L(E)(F_l) \to L(E)(F_l)$ denote the canonical projections. Then,

$$(\varprojlim_j L(D)(E_j) \times \varprojlim_l L(E)(F_l) \xrightarrow{\epsilon_{F_l}(id\times q_l)} L(D)(F_l))_{l\in\mathrm{Ob}\,\mathcal{F}}$$

is a cone. Hence there exists a unique map

$$\hat{\epsilon} : \varprojlim_j L(D)(E_j) \times \varprojlim_l L(E)(F_l) \to \varprojlim_l L(D)(F_l)$$

such that the following diagram commutes.

$$\begin{tikzcd}
\varprojlim_j L(D)(E_j) \times \varprojlim_l L(E)(F_l) \arrow[r, "\hat{\epsilon}"] \arrow[d, "id\times q_l"] & \varprojlim_l L(D)(F_l) \arrow[d, "p_l"] \\
\varprojlim_j L(D)(E_j) \times L(E)(F_l) \arrow[r, "\epsilon_{F_l}"] & L(D)(F_l)
\end{tikzcd}$$

Let $f : D \to E$ and $g : E \to F$ be morphisms in $\mathrm{Pro}(\mathcal{C})$. We set $\theta(f) = (c_k)_{k\in\mathrm{Ob}\,\mathcal{E}}$, $\theta(g) = (d_l)_{l\in\mathrm{Ob}\,\mathcal{F}}$ and $\theta(gf) = (e_l)_{l\in\mathrm{Ob}\,\mathcal{F}}$. We choose representatives $\bar{c}_k \in \mathcal{C}(D_{i(k)}, E_k)$ and $\bar{d}_l \in \mathcal{C}(E_{j(l)}, F_l)$ of c_k and d_l. Set $\bar{e}_l = \bar{d}_l\bar{c}_{j(l)}$ and regard f, g as morphisms $L(E) \to L(D)$, $L(F) \to L(E)$ in $\check{\mathcal{C}}$. By the definition of θ, we have $e_l = (fg)_{F_l}\lambda_l^{F_l}(id_{F_l}) = f_{F_l}(d_l) = f_{F_l}\lambda_{j(l)}^{F_l}(\bar{d}_l) = f_{F_l,j(l)}(\bar{d}_l) = L(D)(\bar{d}_l)(c_{j(l)}) = L(D)(\bar{d}_l)\lambda_{i(j(l))}^{E_{j(l)}}(\bar{c}_{j(l)}) = \lambda_{i(j(l))}^{F_l}(\bar{e}_l)$. Hence $\bar{e}_l$ represents $e_l \in L(D)(F_l)$. In other words, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Pro}(\mathcal{C})(D,E) \times \mathrm{Pro}(\mathcal{C})(E,F) \arrow[r] \arrow[d, "\theta\times\theta"] & \mathrm{Pro}(\mathcal{C})(D,F) \arrow[d, "\theta"] \\
\varprojlim_j L(D)(E_j) \times \varprojlim_l L(E)(F_l) \arrow[r, "\hat{\epsilon}"] & \varprojlim_l L(D)(F_l)
\end{tikzcd}$$

For pro-objects $D : \mathcal{D}^{op} \to \mathcal{C}$ and $E : \mathcal{E}^{op} \to \mathcal{C}$, we give a topology on $\mathrm{Pro}(\mathcal{C})(D, E)$ so that $L : \mathrm{Pro}(\mathcal{C})(D, E) \to \check{\mathcal{C}}(L(E), L(D))$ is a homeomorphism. Note that there is a colimiting cone $(h^{E_j} \xrightarrow{\lambda_j} L(E))_{j\in\mathrm{Ob}\,\mathcal{D}}$ and that there is a natural bijection $\psi : \check{\mathcal{C}}(h^{E_j}, L(D)) \to L(D)(E_j)$. It follows that there is a limiting cone $(\check{\mathcal{C}}(L(E), L(D)) \xrightarrow{\psi\lambda_j^*} L(D)(E_j))_{j\in\mathrm{Ob}\,\mathcal{D}}$. Thus, the topology on $\mathrm{Pro}(\mathcal{C})(D, E)$ is also described as follows. Give the discrete topology on $L(D)(E_j)$ for each $j \in \mathrm{Ob}\,\mathcal{E}$ and consider the product topology on $\prod_{j\in\mathrm{Ob}\,\mathcal{E}} L(D)(E_j)$. The topology on $\varprojlim_j L(D)(E_j)$ is the one as a closed subspace of $\prod_{j\in\mathrm{Ob}\,\mathcal{E}} L(D)(E_j)$. The topology on $\mathrm{Pro}(\mathcal{C})(D, E)$ is the one that makes the bijection $\theta : \mathrm{Pro}(\mathcal{C})(D, E) \to \varprojlim_{j\in\mathrm{Ob}\,\mathcal{E}} L(D)(E_j)$ a homeomorphism. By (6.1.2), the composition map $\mathrm{Pro}(\mathcal{C})(D, E) \times \mathrm{Pro}(\mathcal{C})(E, F) \to \mathrm{Pro}(\mathcal{C})(D, F)$ is continuous.

Proposition 6.1.4 *Filtered colimits of pro-representable functors are pro-representable. Hence* $\mathrm{Pro}(\mathcal{C})$ *is closed under filtered limits and* $L : \mathrm{Pro}(\mathcal{C}) \to \check{\mathcal{C}}^{op}$ *preserves and reflects them.*

Proof. Let $\mathcal{F}$ be a filtered category and $D : \mathcal{F}^{op} \to \mathrm{Pro}(\mathcal{C})$ a functor. $\mathcal{D}_i$ denotes the domain of $D(i)$ for $i \in \mathrm{Ob}\,\mathcal{F}$. We show that the colimit of $LD : \mathcal{F} \to \check{\mathcal{C}}$ is pro-representable. For each $\alpha \in \mathrm{Mor}\,\mathcal{F}$, put $\theta(D(\alpha)) = (\alpha_k)_{k\in\mathrm{Ob}\,\mathcal{D}_i} \in \varprojlim_k L(D(j))(D(i)_k)$ ($i = \mathrm{dom}(\alpha)$, $j = \mathrm{codom}(\alpha)$). For $k \in \mathrm{Ob}\,\mathcal{D}_i$ and $l \in \mathrm{Ob}\,\mathcal{D}_j$, let $R(\alpha; k, l)$ be the set of all representatives of $\alpha_k \in L(D(j))(D(i)_k) = \varinjlim_l \mathcal{C}(D(j)_l, D(i)_k)$ which belong to $\mathcal{C}(D(j)_l, D(i)_k)$. Define a category $\mathcal{E}$ and a functor $E : \mathcal{E}^{op} \to \mathcal{C}$ as follows. Set $\mathrm{Ob}\,\mathcal{E} = \{(i, k) | \, i \in \mathrm{Ob}\,\mathcal{F},\ k \in \mathrm{Ob}\,\mathcal{D}_i\}$. For $i, j \in \mathrm{Ob}\,\mathcal{F}$ and $k \in \mathrm{Ob}\,\mathcal{D}_i$, $l \in \mathrm{Ob}\,\mathcal{D}_j$, set $\mathcal{E}((i, k), (j, l)) = \{(\alpha, \beta) | \, \alpha \in \mathcal{F}(i, j),\ \beta \in R(\alpha; k, l)\}$. Suppose that $(\alpha, \beta) \in \mathcal{E}((i, k), (j, l))$, $(\gamma, \delta) \in \mathcal{E}((j, l), (m, n))$. We note that $\beta\delta \in R(\gamma\alpha; k)$. We define the composition of (γ, δ) and (α, β) to be $(\gamma\alpha, \beta\delta)$. It is clear that $(id_i, id_{D(i)_k})$ is the identity morphism of (i, k). Set $E(i, k) = D(i)_k$ and $E(\alpha, \beta) = \beta$ for $(\alpha, \beta) \in \mathcal{E}((i, k), (j, l))$.

We claim that $\mathcal{E}$ is a filtered category. For $(i, k), (j, l) \in \mathrm{Ob}\,\mathcal{E}$, there exist morphisms $\alpha : i \to m$, $\gamma : j \to m$ and we choose $\beta \in R(\alpha; k, p)$, $\delta \in R(\gamma; l, q)$. Since $\mathcal{D}_m$ is a filtered category, there are transition morphisms $\tau : D(m)_n \to D(m)_p$ and $\tau' : D(m)_n \to D(m)_q$. Then, $(\alpha, \beta\tau) \in \mathcal{E}((i, k), (m, n))$ and $(\gamma, \delta\tau') \in \mathcal{E}((j, l), (m, n))$. For $(\alpha, \beta), (\gamma, \delta) \in \mathcal{E}((i, k), (j, l))$, there is a morphism $\varepsilon : j \to m$ such that $\varepsilon\alpha = \varepsilon\gamma$ and we choose $\eta \in R(\varepsilon; l, p)$. Then, $\beta\eta, \gamma\eta \in R(\varepsilon\alpha; l, p) = R(\varepsilon\gamma; l, p)$. It follows that there is a transition morphism $\tau : D(m)_n \to D(m)_p$ such that $\beta\eta\tau = \delta\eta\tau$. Hence $(\varepsilon, \eta\tau) \in \mathcal{E}((j, l), (m, n))$ and $(\varepsilon, \eta\tau)(\alpha, \beta) = (\varepsilon, \eta\tau)(\gamma, \delta)$.

We define a functor $\iota_i : \mathcal{D}_i \to \mathcal{E}$ by $\iota_i(k) = (i, k)$ and $\iota_i(\nu) = (id_i, D(i)(\nu))$ for $\nu \in \mathrm{Mor}\,\mathcal{D}_i$. For $X \in \mathrm{Ob}\,\mathcal{C}$ and $(i, k) \in \mathrm{Ob}\,\mathcal{E}$, $\lambda^X_{(i,k)} : \mathcal{C}(E(i, k), X) \to L(E)(X))_{(i,k)\in\mathrm{Ob}\,\mathcal{E}}$ denotes the canonical morphism. Since $(\mathcal{C}(D(i)_k, X) = \mathcal{C}(E\iota_i(k), X) \xrightarrow{\lambda^X_{\iota_i(k)}} L(E)(X))_{k\in\mathrm{Ob}\,\mathcal{D}_i}$ is a cone, there is a unique morphism $\tilde{\iota}_{iX} : L(D(i))(X) \to L(E)(X)$ such that $\tilde{\iota}_{iX}\lambda(i)^X_k = \lambda^X_{\iota_i(k)}$ for any $k \in \mathrm{Ob}\,\mathcal{D}_i$, where $\lambda(i)^X_k : \mathcal{C}(D(i)_k, X) \to L(D(i))(X)$ is the canonical morphism. It is easy to verify that $\tilde{\iota}_{iX}$ is natural in X and we have a morphism $\tilde{\iota}_i : L(D(i)) \to L(E)$. It remains to show that $(L(D(i)) \xrightarrow{\tilde{\iota}_i} L(E))_{i\in\mathrm{Ob}\,\mathcal{F}}$ is a colimiting cone in $\check{\mathcal{C}}$. Suppose that $(L(D(i))(X) \xrightarrow{f_i} S)_{i\in\mathrm{Ob}\,\mathcal{F}}$ is a cone in the category of $\mathcal{U}$-sets. For $x \in L(E)(X)$, there exist $(i, k) \in \mathrm{Ob}\,\mathcal{E}$ and $\varphi \in \mathcal{C}(E(i, k), X)$ such that $\lambda^X_{(i,k)}(\varphi) = x$. Then, $\tilde{\iota}_{iX}\lambda(i)^X_k(\varphi) = \lambda^X_{(i,k)}(\varphi) = x$ and this implies that $(\tilde{\iota}_{iX} : L(D(i))(X) \to L(E)(X))_{i\in\mathrm{Ob}\,\mathcal{F}}$ is an epimorphic family. Define a map $f : L(E)(X) \to S$ by $f(x) = f_i\lambda(i)^X_k(\varphi)$. Let $(\alpha, \beta) : (i, k) \to (j, l)$ be a morphism in $\mathcal{E}$. Then, $L(D(\alpha))_X\lambda(i)^X_k(\varphi) = L(D(\alpha))_X\lambda(i)^X_k\varphi_*(id_{D(i)_k}) = L(D(\alpha))_X L(D(i))(\varphi)\lambda(i)^{D(i)_k}_k(id_{D(i)_k}) = L(D(j))(\varphi)L(D(\alpha))_{D(i)_k}\lambda(i)^{D(i)_k}_k(id_{D(i)_k}) = L(D(j))(\varphi)(\alpha_k) = L(D(j))(\varphi)\lambda(j)^{D(i)_k}_l(\beta) = \lambda(j)^X_l\varphi_*(\beta) = \lambda(j)^X_l(\varphi\beta)$. Therefore we have $f_j\lambda(j)^X_l(\varphi E(\alpha, \beta)) = f_j\lambda(j)^X_l(\varphi\beta) = f_j L(D(\alpha))_X\lambda(i)^X_k(\varphi) = f_i\lambda(i)^X_k(\varphi)$ and this implies that the definition of f does not depend on the choice of φ. By the definition of f, $f\tilde{\iota}_{iX}\lambda(i)^X_k = f\lambda^X_{\iota_i(k)} = f_i\lambda(i)^X_k$ for any $i \in \mathrm{Ob}\,\mathcal{F}$ and $k \in \mathrm{Ob}\,\mathcal{D}_i$. It follows that $f\tilde{\iota}_{iX} = f_i$. $\square$

There is a fully faithful functor $\kappa : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ such that $L\kappa = h^{op}$. In fact, for $X \in \mathrm{Ob}\,\mathcal{C}$, $\{X\}$ denotes the category with a single object X and a single morphism id_X. $\kappa(X) : \{X\} \to \mathcal{C}$ is the inclusion functor. Then, $L(\kappa(X)) = h^X$. If $f : X \to Y$ is a morphism in $\mathcal{C}$, $\kappa(f) : \kappa(X) \to \kappa(Y)$ is defined to be $h^f : L(\kappa(Y)) = h^Y \to h^X = L(\kappa(X))$. It follows from the (dual of) Yoneda's lemma that κ is fully faithful.

Proposition 6.1.5 *Let* $D : \mathcal{D}^{op} \to \mathcal{C}$ *be a pro-object. For* $i \in \mathrm{Ob}\,\mathcal{D}$, $\lambda_i : L(\kappa(D_i)) = h^{D_i} \to L(D)$ *denotes the canonical morphism and regarding this as a morphism* $D \to \kappa(D_i)$ *in* $\mathrm{Pro}(\mathcal{C})$. *Then,* $(D \xrightarrow{\lambda_i} \kappa(D_i))_{i\in\mathrm{Ob}\,\mathcal{D}}$ *is a limiting cone of* $\kappa D : \mathcal{D}^{op} \to \mathrm{Pro}(\mathcal{C})$. *Hence the image of* κ *is a generating subcategory of* $\mathrm{Pro}(\mathcal{C})^{op}$ *by strict epimorphisms.*

Proof. Let $(E \xrightarrow{\pi_i} \kappa(D_i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ be a cone of κD. Then $(h^{D_i} = L(\kappa(D_i)) \xrightarrow{\pi_i} L(E))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of $L\kappa D = h^{op}D : \mathcal{D}^{op} \to \check{\mathcal{C}}$. Since $(h^{D_i} \xrightarrow{\lambda_i} L(D))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $h^{op}D$, there is a unique morphism $f : L(D) \to L(E)$ in $\check{\mathcal{C}}$ such that $f\lambda_i = \pi_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. Hence there is a unique morphism $f : E \to D$ in $\mathrm{Pro}(\mathcal{C})$ such that $\lambda_i f = \pi_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. $\square$

We note that, for a pro-object E, $\mathrm{Pro}(\mathcal{C})(E, \kappa(D_i)) \underset{\cong}{\xrightarrow{\theta}} L(E)(D_i)$ has a discrete topology and that $(\mathrm{Pro}(\mathcal{C})(E, D) \xrightarrow{\lambda_{i*}} \mathrm{Pro}(\mathcal{C})(E, \kappa(D_i)))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone in the category of topological spaces.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor. We define a functor $\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{C}')$ as follows. For $D \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$, put $\mathrm{Pro}(F)(D) = FD$. Suppose that $f : D \to E$ is a morphism of pro-objects $\mathcal{C}$ and set $\theta(f) = (c_k)_{k\in \mathrm{Ob}\,\mathcal{E}} \in \varprojlim_j L(D)(E_j)$. Let us denote by $(\varinjlim F)_Y : L(D)(Y) \to L(FD)(F(Y))$ the map induced by $F : \mathcal{C}(D_i, Y) \to \mathcal{C}'(F(D_i), F(Y))$. Then, $\mathrm{Pro}(F)(f) : \mathrm{Pro}(F)(D) \to \mathrm{Pro}(F)(E)$ is given by $\theta(\mathrm{Pro}(F)(f)) = ((\varinjlim F)_{E_k}(c_k))_{k\in \mathrm{Ob}\,\mathcal{E}}$. That is, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Pro}(\mathcal{C})(D,E) \arrow[r, "\mathrm{Pro}(F)"] \arrow[d, "\theta"] & \mathrm{Pro}(\mathcal{C}')(FD,FE) \arrow[d, "\theta"] \\
\varprojlim_j L(D)(E_j) \arrow[r, "\varprojlim_j (\varinjlim F)_{E_j}"] & \varprojlim_j L(FD)(F(E_j))
\end{tikzcd}$$

It follows that $\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C})(D,E) \to \mathrm{Pro}(\mathcal{C}')(FD,FE)$ is continuous. If $\bar{c}_k \in \mathcal{C}(D_i, E_k)$ is a representative of c_k, the following square commutes for $Y \in \mathrm{Ob}\,\mathcal{C}'$, the vertical maps are the canonical maps.

$$\begin{tikzcd}
\mathcal{C}'(F(E_k), Y) \arrow[r, "F(\bar{c}_k)^*"] \arrow[d] & \mathcal{C}'(F(D_i), Y) \arrow[d] \\
L(FE)(Y) \arrow[r, "L(\mathrm{Pro}(F)(f))_Y"] & L(FD)(Y)
\end{tikzcd}$$

Proposition 6.1.6 *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between $\mathcal{U}$-categories.*

1) If $\mathcal{V}$ is a universe containing $\mathcal{U}$ such that $\mathcal{C}$ is $\mathcal{V}$-small. Then, the following diagram commutes up to natural equivalence. Here $\check{\mathcal{C}}_{\mathcal{V}}$ and $\check{\mathcal{C}}'_{\mathcal{V}}$ denote the functor categories $\mathrm{Funct}(\mathcal{C}, \mathcal{V}\text{-}\mathbf{Ens})$ *and* $\mathrm{Funct}(\mathcal{C}', \mathcal{V}\text{-}\mathbf{Ens})$*, respectively.*

$$\begin{tikzcd}
\mathrm{Pro}(\mathcal{C}) \arrow[r, "\mathrm{Pro}(F)"] \arrow[d, "L"] & \mathrm{Pro}(\mathcal{C}') \arrow[d, "L"] \\
\check{\mathcal{C}}_{\mathcal{V}}^{op} \arrow[r, "F_!"] & (\check{\mathcal{C}}'_{\mathcal{V}})^{op}
\end{tikzcd}$$

2) $\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{C}')$ *preserves filtered limits and the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\kappa"] & \mathcal{C}' \arrow[d, "\kappa"] \\
\mathrm{Pro}(\mathcal{C}) \arrow[r, "\mathrm{Pro}(F)"] & \mathrm{Pro}(\mathcal{C}')
\end{tikzcd}$$

3) $\mathrm{Pro}(F)$ *is faithful (resp. fully faithful) if and only if F is so. If F is fully faithful,* $\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C})(D,E) \to \mathrm{Pro}(\mathcal{C}')(FD,FE)$ *is a homeomorphism.*

4) $\mathrm{Pro}(F)$ *is an equivalence of categories if and only if F is fully faithful and, for any $Y \in \mathrm{Ob}\,\mathcal{C}'$, there exist $X \in \mathrm{Ob}\,\mathcal{C}$ and a split monomorphism $s : Y \to F(X)$ in $\mathcal{C}'$.*

Proof. 1) Since $F_!$ has a right adjoint F^* (A.6.7), $F_!$ preserves colimits. By (A.6.12), $F_!$ can be chosen so that $F_!(h^X) = h^{F(X)}$ for each $X \in \mathrm{Ob}\,\mathcal{C}$. Moreover, by the definition of $L(D)$, there is a colimiting cone $(h^{D_i} \xrightarrow{\lambda_i} L(D))_i$. Hence, for $D \in \mathrm{Pro}(\mathcal{C})$, $(h^{F(D_i)} = F_!(h^{D_i}) \xrightarrow{F_!(\lambda)} F_!(L(D)))_i$ is a colimiting cone. Thus we have a natural isomorphism $L(FD) \to F_!(L(D))$.

2) Let $D : \mathcal{F}^{op} \to \mathrm{Pro}(\mathcal{C})$ be a functor such that $\mathcal{F}$ is a $\mathcal{U}$-small filtered category. By the preceding result, there is a limiting cone $(C \xrightarrow{\pi_i} D(i))_{i\in \mathrm{Ob}\,\mathcal{F}}$ in $\mathrm{Pro}(\mathcal{C})$ and $(L(D(i)) \xrightarrow{L(\pi_i)} L(C))_{i\in \mathrm{Ob}\,\mathcal{F}}$ is a colimiting cone in

$\check{\mathcal{C}}$. Since $F_! : \check{\mathcal{C}}_{\mathcal{V}} \to \check{\mathcal{C}}'_{\mathcal{V}}$ has a right adjoint, it preserves colimits. Hence $(F_! L(D(i)) \xrightarrow{F_! L(\pi_i)} F_! L(C))_{i \in \mathrm{Ob}\,\mathcal{F}}$ is a colimiting cone in $\check{\mathcal{C}}'_{\mathcal{V}}$. It follows from 1) that $(L\mathrm{Pro}(F)(D(i)) \xrightarrow{L\mathrm{Pro}(F)(\pi_i)} L\mathrm{Pro}(F)(C))_{i \in \mathrm{Ob}\,\mathcal{F}}$ is a colimiting cone in $\check{\mathcal{C}}'_{\mathcal{V}}$. Again by (6.1.4), $(\mathrm{Pro}(F)(C) \xrightarrow{\mathrm{Pro}(F)(\pi_i)} \mathrm{Pro}(F)(D(i)))_{i \in \mathrm{Ob}\,\mathcal{F}}$ is a limiting cone in $\mathrm{Pro}(\mathcal{C}')$. The second assertion is straightforward from the definitions of κ and $\mathrm{Pro}(F)$.

3) Since κ is fully faithful, F is faithful (resp. fully faithful) if $\mathrm{Pro}(F)$ is so by the commutativity of the diagram of 2). Suppose that F is faithful (resp. fully faithful). Since $F : \mathcal{C}(D_i, Y) \to \mathcal{C}'(F(D_i), F(Y))$ is injective (resp. bijective) for $Y \in \mathrm{Ob}\,\mathcal{C}$ and filtered colimits in $\mathcal{U}$-**Ens** preserves injections, $(\varinjlim F)_Y : L(D)(Y) \to L(FD)(F(Y))$ is injective (resp. bijective). Hence the map $\varprojlim_j L(D)(E_j) \to \varprojlim_j L(FD)(F(E_j))$ induced by $(\varinjlim F)_{E_j}$'s is injective (resp. a homeomorphism). Hence by the definition of $\mathrm{Pro}(F)$, the assertion follows.

4) Suppose that F is fully faithful and, for any $Y \in \mathrm{Ob}\,\mathcal{C}'$, there exist $X \in \mathrm{Ob}\,\mathcal{C}$ and a split monomorphism $s : Y \to F(X)$ in $\mathcal{C}'$. Let $p : F(X) \to Y$ be a morphism such that $ps = id_Y$. There is a unique morphism $e : X \to X$ such that $F(e) = sp$. Then, since F is faithful and $spsp = sp$, $e^2 = e$. We regard the set of natural numbers $\boldsymbol{N}$ as a directed set and consider a pro-object $D : \boldsymbol{N}^{op} \to \mathcal{C}$ defined by $D(n) = X$ and $D(n \to n+1) = e$. It is clear that $(Y \xrightarrow{s} FD(n))_{n \in \boldsymbol{N}}$ is a cone. We show that $(\mathcal{C}(FD(n), Z) \xrightarrow{s^*} \mathcal{C}(Y, Z))_{n \in \boldsymbol{N}}$ is a colimiting cone for $Z \in \mathrm{Ob}\,\mathcal{C}'$. In fact, for a cone $(\mathcal{C}(FD(n), Z) \xrightarrow{f_n} S)_{n \in \boldsymbol{N}}$, define $f : \mathcal{C}(Y, Z) \to S$ by $f(t) = f_0(tp)$. Since $tp = tpsp = F(e)^*(tp)$, $f_{n+1}(tp) = f_{n+1}F(e)^*(tp) = f_n(tp)$. Thus $f(t) = f_n(tp)$ for any $n \in \boldsymbol{N}$ and $fs^*(u) = f(us) = f_n(usp) = f_n(u)$ for any $u \in \mathcal{C}(FD(n), Z)$. Since s is a split monomorphism, $s^* : \mathcal{C}(FD(n), Z) \to \mathcal{C}(Y, Z)$ is surjective and it follows that F is unique. Hence we have shown that $L(\mathrm{Pro}(F)(D))$ is isomorphic to $h^Y = L(\kappa(Y))$, namely, $\mathrm{Pro}(F)(D)$ is isomorphic to $\kappa(Y)$.

Let $E : \mathcal{E}^{op} \to \mathcal{C}'$ be a pro-object. For $j \in \mathrm{Ob}\,\mathcal{E}$, $\lambda_j : L(\kappa(E_j)) = h^{E_j} \to L(E)$ denotes the canonical morphism. Regarding λ_j as a morphism in $\mathrm{Pro}(\mathcal{C}')$, $(E \xrightarrow{\lambda_j} \kappa(E_j))_{j \in \mathrm{Ob}\,\mathcal{E}}$ is a limiting cone of $\kappa E : \mathcal{E}^{op} \to \mathrm{Pro}(\mathcal{C}')$ by (6.1.5). By the above result, there is an isomorphism $\xi_j : \kappa(E_j) \to \mathrm{Pro}(F)(D_j)$ for some $D_j \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$. Since $\mathrm{Pro}(F)$ is fully faithful by 3), for each morphism $\tau : j \to k$ in $\mathcal{E}$, there exist a unique morphism $D(\tau) : D_k \to D_j$ such that $\mathrm{Pro}(F)(D(\tau))\xi_k = \xi_j \kappa(E(\tau))$. Hence we have a functor $D : \mathcal{E}^{op} \to \mathrm{Pro}(\mathcal{C})$ and a limiting cone $(E \xrightarrow{\xi_j \lambda_j} \mathrm{Pro}(F)(D_j))_{j \in \mathrm{Ob}\,\mathcal{E}}$ of $\mathrm{Pro}(F)D : \mathcal{E}^{op} \to \mathrm{Pro}(\mathcal{C}')$. By (6.1.4), there is a limiting cone $(\bar{D} \xrightarrow{\pi_j} D_j)_{j \in \mathrm{Ob}\,\mathcal{E}}$ of D in $\mathrm{Pro}(\mathcal{C})$. It follows from 2) that $(\mathrm{Pro}(F)(\bar{D}) \xrightarrow{\mathrm{Pro}(F)(\pi_j)} \mathrm{Pro}(F)(D_j))_{j \in \mathrm{Ob}\,\mathcal{E}}$ is a limiting cone of $\mathrm{Pro}(F)D : \mathcal{E}^{op} \to \mathrm{Pro}(\mathcal{C}')$. Therefore E is isomorphic to $\mathrm{Pro}(F)(\bar{D})$ and $\mathrm{Pro}(F)$ is an equivalence of categories.

Conversely, assume that $\mathrm{Pro}(F)$ is an equivalence of categories. Then, F is fully faithful by 3). For $Y \in \mathrm{Ob}\,\mathcal{C}'$, there is an isomorphism $\zeta : \mathrm{Pro}(F)(D) \to \kappa(Y)$ for some $D \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$. Regard ζ as a morphism $h^Y = L(\kappa(Y)) \to L(\mathrm{Pro}(F)(D)) = L(FD)$ in $\check{\mathcal{C}}$. Let $\lambda_j^Z : \mathcal{C}'(F(D_j), Z) \to L(FD)(Z)$ be the canonical morphism. We choose a morphism $p : F(D_i) \to Y$ such that $\lambda_i^Y(p) = \zeta_Y(id_Y)$. There is a unique morphism $s : Y \to F(D_i)$ such that $\zeta_{F(D_i)}(s) = \lambda_i^{F(D_i)}(id_{F(D_i)})$. Then, $\zeta_Y(ps) = \zeta_Y p_*(s) = L(FD)(p)\zeta_{F(D_i)}(s) = L(FD)(p)\lambda_i^{F(D_i)}(id_{F(D_i)}) = \lambda_i^Y p_*(id_{F(D_i)}) = \lambda_i^Y(p) = \zeta_Y(id_Y)$. Thus we have $ps = id_Y$. □

Suppose that $\mathcal{C}$ is a $\mathcal{U}$-complete category. We define a functor $\varprojlim_{\mathcal{C}} : \mathrm{Pro}(\mathcal{C}) \to \mathcal{C}$ as follows. For $D \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$, $\varprojlim_{\mathcal{C}}(D) = \varprojlim_i D_i$. For a morphism $f : D \to E$ in $\mathrm{Pro}(\mathcal{C})$, let $f_j : D_{i(j)} \to E_j$ be a representative of the image of id_{E_j} by the composition $\mathcal{C}(E_j, E_j) \xrightarrow{\lambda_j^{E_j}} L(E)(E_j) \xrightarrow{f_{E_j}} L(D)(E_j)$. We denote by $\pi_l : \varprojlim_i D_i \to D_l$ the projection onto the l-th component. For a transition morphism $\tau : E_j \to E_k$, since $L(D)(\tau)f_{E_j}\lambda_j^{E_j}(id_{E_j}) = f_{E_k}L(E)(\tau)\lambda_j^{E_j}(id_{E_j}) = f_{E_k}\lambda_j^{E_k}(\tau) = f_{E_k}\lambda_k^{E_k}(id_{E_k})$, $\tau f_j \in \mathcal{C}(D_{i(j)}, E_k)$ and $f_k \in \mathcal{C}(D_{i(k)}, E_k)$ represent the same element in $L(D)(E_k) = \varinjlim_i \mathcal{C}(D_i, E_k)$. Hence there exist transition morphisms $\alpha : D_m \to D_{i(j)}$ and $\beta : D_m \to D_{i(k)}$ such that $\tau f_j \alpha = f_k \beta$. It follows that $\tau f_j \pi_{i(j)} = \tau f_j \alpha \pi_m = f_k \beta \pi_m = f_k \pi_{i(k)}$. In particular, in the case $\tau = id_{E_i}$, we see that $f_j \pi_{i(j)}$ does not depend on the choice of $i(j)$. We put $\xi_j = f_j \pi_{i(j)}$. Thus $(\varprojlim_i D_i \xrightarrow{\xi_j} E_j)_{j \in \mathrm{Ob}\,\mathcal{E}}$ is a cone in $\mathcal{C}$ and there is a unique morphism $\varprojlim_{\mathcal{C}}(f) : \varprojlim_i D_i \to \varprojlim_j E_j$ such that $\xi_j = \pi'_j \varprojlim_{\mathcal{C}}(f)$, where $\pi'_j : \varprojlim_j E_j \to E_j$ denotes the projection onto the j-th component.

Proposition 6.1.7 *If $\mathcal{C}$ is $\mathcal{U}$-complete, $\varprojlim_{\mathcal{C}} : \mathrm{Pro}(\mathcal{C}) \to \mathcal{C}$ is a right adjoint of $\kappa : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$.*

Proof. For $X \in \mathrm{Ob}\,\mathcal{C}$ and $\mathcal{D} \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$, we define a map $\Phi : \mathrm{Pro}(\mathcal{C})(\kappa(X), D) \to \mathcal{C}(X, \varprojlim_{\mathcal{C}}(D))$ as follows. The domain of D is denoted by $\mathcal{D}$ and $\lambda_i^Y : \mathcal{C}(D_i, Y) \to L(D)(Y)$ ($Y \in \mathrm{Ob}\,\mathcal{C}$, $i \in \mathrm{Ob}\,\mathcal{D}$) denotes the canonical map. Let $f : \kappa(X) \to D$ be a morphism in $\mathrm{Pro}(\mathcal{C})$ and $f_i : X \to D_i$ ($i \in \mathrm{Ob}\,\mathcal{D}$) the image of id_{D_i} by a composition $\mathcal{C}(D_i, D_i) \xrightarrow{\lambda_i^{D_i}} L(D)(D_i) \xrightarrow{f_{D_i}} L(\kappa(X))(D_i) = \mathcal{C}(X, D_i)$. For a transition morphism $\tau : D_i \to D_k$, $\tau f_i = \tau_* f_{D_i}\lambda_i^{D_i}(id_{D_i}) = f_{D_k} L(D)(\tau)\lambda_i^{D_i}(id_{D_i}) = f_{D_k}\lambda_i^{D_k}(\tau) = f_{D_k}\lambda_k^{D_k}(id_{D_k}) = f_k$. Hence $(X \xrightarrow{f_i} D_i)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone and there is a unique morphism $\Phi(f) : X \to \varprojlim_{\mathcal{C}}(D)$ such that $f_i = \Phi(f)\pi_i$, where $\pi_i : \varprojlim_{\mathcal{C}}(D) \to D_i$ denotes the canonical projection onto the i-th component. For a morphism $g : X \to \varinjlim_{\mathcal{C}}(D)$ and $Y \in \mathrm{Ob}\,\mathcal{C}$, since $(\mathcal{C}(D_i, Y) \xrightarrow{(\pi_i g)^*} \mathcal{C}(X, Y))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone, there is a unique morphism $g_Y : L(D)(Y) \to L(\kappa(X))(Y) = \mathcal{C}(X, Y)$ such that $g_Y \lambda_i^Y = (\pi_i g)^*$. Clearly, g_Y is natural in Y and $\Phi^{-1}(g) : \kappa(X) \to D$ is given by $\Phi^{-1}(g)_Y = g_Y$. □

For each $X \in \mathrm{Ob}\,\mathcal{C}$, since $(X \xrightarrow{id_X} X)$ is a limiting cone of the constant pro-object $\kappa(X)$, we can choose $\varprojlim_{\mathcal{C}} : \mathrm{Pro}(\mathcal{C}) \to \mathcal{C}$ so that $\varprojlim_{\mathcal{C}} \kappa = id_{\mathcal{C}}$. Then, the unit $id_{\mathcal{C}} \to \varprojlim_{\mathcal{C}} \kappa$ of the above adjunction is the identity morphism.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor. If $\mathcal{C}'$ is $\mathcal{U}$-complete, we put $\bar{F} = \varprojlim_{\mathcal{C}'} \mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C}) \to \mathcal{C}'$. Then, $\bar{F}\kappa = \varprojlim_{\mathcal{C}'} \mathrm{Pro}(F)\kappa = \varprojlim_{\mathcal{C}'} \kappa F = F$. Since κ is fully faithful, F is fully faithful if the restriction of $\bar{F}$ to the image of κ is fully faithful. Note that $\mathrm{Pro}(F)$ preserves filtered limits by (6.1.6). Since $\varprojlim_{\mathcal{C}'}$ has a left adjoint by (6.1.7), it preserves limits. Hence $\bar{F}$ preserves filtered limits.

For a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$, $\pi_i : \bar{F}(D) = \varprojlim_i F(D_i) \to F(D_i)$ ($i \in \mathrm{Ob}\,\mathcal{D}$) denotes the canonical morphism. Let $E : \mathcal{E}^{op} \to \mathcal{C}$ a pro-object and $f : D \to E$ a morphism in $\mathrm{Pro}(\mathcal{C})$. Put $\theta(f) = (c_k)_{k\in \mathrm{Ob}\,\mathcal{D}}$ and choose a representative $\bar{c}_k \in \mathcal{C}(D_i, E_k)$ of $c_k \in L(D)(E_k)$. Then, the following square commutes.

$$\begin{tikzcd}
\bar{F}(D) \arrow[r, "\bar{F}(f)"] \arrow[d, "\pi_i"] & \bar{F}(E) \arrow[d, "\nu_k"] \\
D_i \arrow[r, "\bar{c}_k"] & E_k
\end{tikzcd}$$

Proposition 6.1.8 *Let $\mathcal{C}$ be a $\mathcal{U}$-category and $\mathcal{P}$ a full subcategory of $\mathrm{Pro}(\mathcal{C})$ containing the image of $\kappa : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$. We denote by $\iota : \mathcal{P} \to \mathrm{Pro}(\mathcal{C})$ the inclusion functor and consider the following conditions.*

*(i) For any pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ belonging to $\mathcal{P}$, $(\mathcal{C}(D_i, X) \xrightarrow{\pi_i^* F} \mathcal{C}'(\bar{F}(D), F(X)))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of the functor $h_X D : \mathcal{D} \to \mathcal{U}$-**Ens**.*

(ii) For any $Y \in \mathrm{Ob}\,\mathcal{C}'$, the opposite category of $(Y{\downarrow}F)$ is filtered and essentially $\mathcal{U}$-small.

(iii) Regarding F as a functor $\mathcal{C}^{op} \to (\mathcal{C}')^{op}$, the image of F is a generating subcategory of $(\mathcal{C}')^{op}$ by strict epimorphisms.

1) $\bar{F}\iota : \mathcal{P} \to \mathcal{C}'$ is fully faithful if and only if F is fully faithful and every object X of $\mathcal{C}$ satisfies the condition (i) above.

2) $\bar{F} : \mathrm{Pro}(\mathcal{C}) \to \mathcal{C}'$ is an equivalence if and only if F is fully faithful, every object X of $\mathcal{C}$ satisfies the condition (i) and the conditions (ii), (iii) are also satisfied.

Proof. 1) For a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$, we denote by $\lambda_i : h^{D_i} \to L(D)$ the canonical morphism. We claim that the composition $\mathcal{C}(D_i, X) \xrightarrow{h^{op}} \check{\mathcal{C}}(h^X, h^{D_i}) \xrightarrow{\lambda_{i*}} \check{\mathcal{C}}(h^X, L(D)) = \check{\mathcal{C}}(L(\kappa(X)), L(D)) = \mathrm{Pro}(\mathcal{C})(D, \kappa(X)) \xrightarrow{\bar{F}} \mathcal{C}'(\bar{F}(D), \bar{F}\kappa(X)) = \mathcal{C}'(\bar{F}(D), F(X))$ coincides with $\pi_i^* F : \mathcal{C}(D_i, X) \to \mathcal{C}'(\bar{F}(D), F(X))$. By the definitions of the functors $\mathrm{Pro}(F)$ and $\varprojlim_{\mathcal{C}}$, the following diagram commutes. Here, λ_i^X and $(\lambda_i')^{F(X)}$ denote the canonical maps.

$$\begin{tikzcd}
\mathrm{Pro}(\mathcal{C})(D, \kappa(X)) \arrow[r, "\mathrm{Pro}(F)"] \arrow[d, "\theta"] & \mathrm{Pro}(\mathcal{C}')(FD, F\kappa(X)) \arrow[r, "\varprojlim_{\mathcal{C}'}"] & \mathcal{C}'(\varprojlim_{\mathcal{C}'} FD, F(X)) \\
L(D)(X) \arrow[r, "(\varinjlim F)_X"] & L(FD)(F(X)) \arrow[u, "\theta^{-1}"] & \mathcal{C}'(F(D_i), F(X)) \arrow[l, "(\lambda_i')^{F(X)}"'] \arrow[u, "\pi_i^*"] \\
\mathcal{C}(D_i, X) \arrow[u, "\lambda_i^X"] \arrow[r, "F"] & \mathcal{C}'(F(D_i), F(X)) \arrow[u, "(\lambda_i')^{F(X)}"] &
\end{tikzcd}$$

For $f \in \mathcal{C}(D_i, X)$, $\theta\lambda_{i*}h^{op}(f) = \theta(\lambda_i h^f) = (\lambda_i h^f)_X(id_X) = \lambda_i^X(f)$. Hence $\bar{F}\lambda_{i*}h^{op}(f) = \bar{F}\theta^{-1}\theta\lambda_{i*}h^{op}(f) = \varinjlim_{\mathcal{C}'} \mathrm{Pro}(F)\theta^{-1}\lambda_i^X(f) = \varinjlim_{\mathcal{C}'} \theta^{-1}(\lambda_i')^{F(X)}F(f) = \pi_i^* F(f)$.

Let us denote by $y : L(D)(X) \to \check{\mathcal{C}}(h^X, L(D)) = \check{\mathcal{C}}(L\kappa(X), L(D))$ be the bijection defined by $y(u)_Y(\varphi) = (L(D)(\varphi))(u)$. Then, the following square commutes.

$$
\begin{tikzcd}
\mathcal{C}(D_i, X) \arrow[r, "\lambda_i^X"] \arrow[d, "h^{op}"] & L(D)(X) \arrow[d, "y"] \\
\check{\mathcal{C}}(h^X, h^{D_i}) \arrow[r, "\lambda_{i*}"] & \check{\mathcal{C}}(L\kappa(X), L(D))
\end{tikzcd}
$$

Since h^{op} is bijective, $(\mathcal{C}(D_i, X) \xrightarrow{\lambda_{i*}h^{op}} \check{\mathcal{C}}(L\kappa(X), L(D)) = \mathrm{Pro}(\mathcal{C})(D, \kappa(X)))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of the functor $h_X D : \mathcal{D} \to \mathcal{U}$-**Ens**. On the other hand, $(\mathcal{C}(D_i, X) \xrightarrow{\pi_i^* F} \mathcal{C}'(\bar{F}(D), F(X)))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of $h_X D$ and $\bar{F} : \mathrm{Pro}(\mathcal{C})(D, \kappa(X)) \to \mathcal{C}'(\bar{F}(D), F(X))$ is the unique morphism satisfying $\bar{F}\lambda_{i*}h^{op} = \pi_i^* F$ for each $i \in \mathrm{Ob}\,\mathcal{D}$.

Suppose that F is fully faithful and (i) is satisfied for any $X \in \mathrm{Ob}\,\mathcal{C}$. Then, for any pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ in $\mathcal{P}$, both $(\mathcal{C}(D_i, X) \xrightarrow{\lambda_{i*}h^{op}} \mathrm{Pro}(\mathcal{C})(D, \kappa(X)))_{i\in \mathrm{Ob}\,\mathcal{D}}$ and $(\mathcal{C}(D_i, X) \xrightarrow{\pi_i^* F} \mathcal{C}'(\bar{F}(D), F(X)))_{i\in \mathrm{Ob}\,\mathcal{D}}$ are colimiting cones of $h_X D$. Hence, for any $D \in \mathrm{Ob}\,\mathcal{P}$ and $X \in \mathrm{Ob}\,\mathcal{C}$, $\bar{F} : \mathrm{Pro}(\mathcal{C})(D, \kappa(X)) \to \mathcal{C}'(\bar{F}(D), F(X))$ is bijective. Let $E : \mathcal{E}^{op} \to \mathcal{C}$ be a pro-object. Regard the canonical morphism $\mu_j : L(\kappa(E_j)) = h^{E_j} \to L(E)$ $(j \in \mathrm{Ob}\,\mathcal{E})$ as a morphism in $\mathrm{Pro}(\mathcal{C})$. Then, $(E \xrightarrow{\mu_j} \kappa(E_j))_{j\in \mathrm{Ob}\,\mathcal{E}}$ is a limiting cone of $\kappa E : \mathcal{E}^{op} \to \mathrm{Pro}(\mathcal{C})$ by (6.1.5) and it follows that $(\mathrm{Pro}(\mathcal{C})(D, E) \xrightarrow{\mu_{j*}} \mathrm{Pro}(\mathcal{C})(D, \kappa(E_j)))_{j\in \mathrm{Ob}\,\mathcal{E}}$ is a limiting cone for any $D \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$. Since $\bar{F}$ preserves filtered limits, $(\bar{F}(E) \xrightarrow{\bar{F}(\mu_j)} \bar{F}\kappa(E_j))_{j\in \mathrm{Ob}\,\mathcal{E}}$ is a limiting cone of $\bar{F}\kappa E : \mathcal{E}^{op} \to \mathcal{C}'$. It follows that $(\mathcal{C}'(\bar{F}(D), \bar{F}(E)) \xrightarrow{\bar{F}(\mu_j)_*} \mathcal{C}'(\bar{F}(D), \bar{F}\kappa(E_j)))_{j\in \mathrm{Ob}\,\mathcal{E}}$ is a limiting cone. Since the right vertical map of the following commutative diagram is bijective for any $D \in \mathrm{Ob}\mathcal{P}$, $\bar{F} : \mathrm{Pro}(\mathcal{C})(D, E) \to \mathcal{C}'(\bar{F}(D), \bar{F}(E))$ is also bijective.

$$
\begin{tikzcd}
\mathrm{Pro}(\mathcal{C})(D, E) \arrow[r, "\mu_{j*}"] \arrow[d, "\bar{F}"] & \mathrm{Pro}(\mathcal{C})(D, \kappa(E_j)) \arrow[d, "\bar{F}"] \\
\mathcal{C}'(\bar{F}(D), \bar{F}(E)) \arrow[r, "\bar{F}(\mu_j)_*"] & \mathcal{C}'(\bar{F}(D), \bar{F}\kappa(E_j))
\end{tikzcd}
$$

Therefore $\bar{F}\iota$ is fully faithful.

Conversely, suppose that $\bar{F}\iota$ is fully faithful. Since $\mathcal{P}$ contains the image of κ, $F = \bar{F}\kappa$ is fully faithful. For any pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ and $X \in \mathrm{Ob}\,\mathcal{C}$, $(\mathcal{C}(D_i, X) \xrightarrow{\lambda_i^X} L(D)(X))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone. Since $h^{op} : \mathcal{C}(D_i, X) \to \check{\mathcal{C}}(h^X, h^{D_i})$, $y : L(D)(X) \to \check{\mathcal{C}}(L\kappa(X), L(D))$ and $\bar{F} : \check{\mathcal{C}}(L\kappa(X), L(D)) = \mathrm{Pro}(D, \kappa(X)) \to \mathcal{C}'(\bar{F}(D), F(X))$ are bijective if $D \in \mathrm{Ob}\mathcal{P}$, the condition (i) follows from the commutativity of the second diagram in this proof and the equality $\bar{F}\lambda_{i*}h^{op} = \pi_i^* F$.

2) Suppose that F is fully faithful and that (i) for any $X \in \mathrm{Ob}\,\mathcal{C}$ and (ii), (iii) are satisfied. Then, $\bar{F}$ is fully faithful by 1). By the dual of 1) of (A.4.10), (iii) holds if and only if, for any $Y \in \mathrm{Ob}\,\mathcal{C}'$, $(Y \xrightarrow{f} FP\langle f, X\rangle)_{\langle f,X\rangle\in \mathrm{Ob}(Y\downarrow F)}$ is a limiting cone of $FP : (Y\downarrow F) \to \mathcal{C}'$. For $Y \in \mathrm{Ob}\,\mathcal{C}$, the canonical functor $P : (Y\downarrow F) \to \mathcal{C}$ is a pro-object in $\mathcal{C}$ with domain $(Y\downarrow F)^{op}$ by (ii). Since $(Y \xrightarrow{f} FP\langle f, X\rangle)_{\langle f,X\rangle\in \mathrm{Ob}(Y\downarrow F)}$ is a limiting cone of $FP = \mathrm{Pro}(F)(P)$ by (iii), Y is isomorphic to $\bar{F}(P) = \varprojlim_{\mathcal{C}'} \mathrm{Pro}(F)(P)$.

Suppose that $\bar{F}$ is an equivalence. Then, F is fully faithful and (i) is satisfied by 1). For any $Y \in \mathrm{Ob}\,\mathcal{C}'$, there exists a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ such that Y is isomorphic to $\bar{F}(D)$. In other words, there is a limiting cone $(Y \xrightarrow{\nu_i} F(D_i))_{i\in \mathrm{Ob}\,\mathcal{D}}$. Hence there is a colimiting cone $(F(D_i) \xrightarrow{\nu_i} Y)_{i\in \mathrm{Ob}\,\mathcal{D}}$ in $(\mathcal{C}')^{op}$ and (iii) holds. We claim that $\{\langle\nu_i, D_i\rangle |\, i \in \mathrm{Ob}\,\mathcal{D}\}$ is cofinal in $(Y\downarrow F)^{op}$. Let $\chi : Y \to \bar{F}(D)$ be the isomorphism such that $\pi_i\chi = \nu_i$ $(i \in \mathrm{Ob}\,\mathcal{D})$ and $\langle f, X\rangle$ an object of $(Y\downarrow F)$. Then, we have a colimiting cone $(\mathcal{C}(D_i, X) \xrightarrow{\nu_i^* F} \mathcal{C}'(Y, F(X)))_{i\in \mathrm{Ob}\,\mathcal{D}}$ of $h_X D$ by (i). Hence there exist $i \in \mathrm{Ob}\,\mathcal{D}$ and $\alpha \in \mathcal{C}(D_i, X)$ such that $F(\alpha)\nu_i = f$. Thus we have a morphism $\alpha : \langle\nu_i, D_i\rangle \to \langle f, X\rangle$ in $(Y\downarrow F)$. For $\langle f, X\rangle, \langle g, Z\rangle \in \mathrm{Ob}\,(Y\downarrow F)$, there are morphisms $\alpha : \langle\nu_i, D_i\rangle \to \langle f, X\rangle$ and $\beta : \langle\nu_j, D_j\rangle \to \langle g, Z\rangle$ in $(Y\downarrow F)$. Since $(Y \xrightarrow{\nu_i} F(D_i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of FD and $\mathcal{D}$ is filtered, there are morphisms $\tau : \langle\nu_k, D_k\rangle \to \langle\nu_i, D_i\rangle$ $\sigma : \langle\nu_k, D_k\rangle \to \langle\nu_j, D_j\rangle$. Thus we have morphisms $\alpha\tau : \langle\nu_k, D_k\rangle \to \langle f, X\rangle$ and $\beta\sigma : \langle\nu_k, D_k\rangle \to \langle g, Z\rangle$. Let $\varphi, \psi : \langle f, X\rangle \to \langle g, Z\rangle$ be morphisms in $(Y\downarrow F)$. We choose a morphism $\alpha : \langle\nu_i, D_i\rangle \to \langle f, X\rangle$. Then, $\nu_i^* F(\varphi\alpha) = F(\varphi)F(\alpha)\nu_i = F(\varphi)f = g = F(\psi)f = F(\psi)F(\alpha)\nu_i = \nu_i^* F(\psi\alpha)$ and this implies that there is a transition morphism $\tau : D_j \to D_i$ such that $\varphi\alpha\tau = \psi\alpha\tau$ in $\mathcal{C}$. Therefore $\alpha\tau : \langle\nu_j, D_j\rangle \to \langle f, X\rangle$ is a morphism satisfying $\varphi\alpha\tau = \psi\alpha\tau$ in $(Y\downarrow F)$ and this completes the verification of the condition (ii). □

Remark 6.1.9 *Suppose that F is fully faithful and every object X of $\mathcal{C}$ satisfies the condition (i). Then, by the proof of 1) above, $\bar{F} : \mathrm{Pro}(\mathcal{C})(D, E) \to \mathcal{C}'(\bar{F}(D), \bar{F}(E))$ is bijective if D is an object of $\mathcal{P}$.*

Generally, for a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$, $(h^{D_i} \xrightarrow{\pi_i^D} h^{\bar{F}(D)}F)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of the functor $h^{op}D : \mathcal{D} \to \check{\mathcal{C}}$, where π_i^D is a morphism in $\check{\mathcal{C}}$ given by $(\pi_i^D)_X = \pi_i^* F : \mathcal{C}(D_i, X) \to \mathcal{C}'(\bar{F}(D), F(X))$. Hence there is a unique morphism $F_D : L(D) \to h^{\bar{F}(D)}F = F^*(h^{\bar{F}(D)})$ satisfying $F_D\lambda_i = \pi_i^D$ for every $i \in \mathrm{Ob}\,\mathcal{D}$.

Lemma 6.1.10 *Let $f : E \to D$ be a morphism in* $\mathrm{Pro}(\mathcal{C})$.
1) The following square commutes.

$$\begin{tikzcd} L(D) \arrow[r, "F_D"] \arrow[d, "L(f)"] & F^*(h^{\bar{F}(D)}) \arrow[d, "F^*(h^{\bar{F}(f)})"] \\ L(E) \arrow[r, "F_E"] & F^*(h^{\bar{F}(E)}) \end{tikzcd}$$

2) Let us denote by $\pi_i : \bar{F}(D) \to D_i$ and $\rho_j : \bar{F}(E) \to E_j$ the canonical projections. Suppose that f is an isomorphism. For $X \in \mathrm{Ob}\,\mathcal{C}$, $(\mathcal{C}(D_i, X) \xrightarrow{\pi_i^* F} \mathcal{C}'(\bar{F}(D), F(X)))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of the functor $h_X D : \mathcal{D} \to \mathcal{U}$-$\mathbf{Ens}$, if and only of $(\mathcal{C}(E_j, X) \xrightarrow{\rho_j^* F} \mathcal{C}'(\bar{F}(E), F(X)))_{j\in \mathrm{Ob}\,\mathcal{E}}$ is a colimiting cone of the functor $h_X E : \mathcal{E} \to \mathcal{U}$-$\mathbf{Ens}$.

Proof. 1) The assertion follows from the definitions of functors $\mathrm{Pro}(F)$ and $\varprojlim_{\mathcal{C}'}$.

2) $(\mathcal{C}(D_i,X) \xrightarrow{\pi_i^* F} \mathcal{C}'(\bar{F}(D),F(X)))_{i\in \mathrm{Ob}\,\mathcal{D}}$ (resp. $(\mathcal{C}(E_j,X) \xrightarrow{\rho_j^* F} \mathcal{C}'(\bar{F}(E),F(X)))_{j\in \mathrm{Ob}\,\mathcal{E}}$) is a colimiting cone of the functor $h_X D$ (resp. $h_X E$) if and only if $(F_D)_X : L(D)(X) \to \mathcal{C}'(\bar{F}(D), F(X))$ (resp. $(F_E)_X : L(E)(X) \to \mathcal{C}'(\bar{F}(E), F(X)))$ is bijective. Since f is an isomorphism, both $L(f)_X : L(D)(X) \to L(E)(X)$ and $F^*(h^{\bar{F}(f)})_X = \bar{F}(f)^* : \mathcal{C}'(\bar{F}(D),F(X)) \to \mathcal{C}'(\bar{F}(E), F(X))$ are bijective. Hence $(F_D)_X$ is bijective if and only if $(F_E)_X$ is so by 1). □

We denote by $\mathcal{C}_F$ the category with objects $\{(X,\xi)|\, X \in \mathrm{Ob}\,\mathcal{C},\ \xi \in F(X)\}$ and morphisms $\mathcal{C}_F((X,\xi),(Y,\zeta)) = \{f \in \mathcal{C}(X,Y)|\, F(f)(\xi) = \zeta\}$ (See (A.4.2)).

Proposition 6.1.11 *For an object F of $\check{\mathcal{C}}$, the following conditions are equivalent.*
(i) F is pro-representable.
(ii) $\mathcal{C}_F^{op}$ is filtered and essentially $\mathcal{U}$-small.
(iii) (Providing that $\mathcal{C}$ has finite limits,) F is left exact and $\mathrm{Ob}\,\mathcal{C}_F^{op}$ *has $\mathcal{U}$-small cofinal subset.*

Proof. $(i) \Rightarrow (ii)$; Suppose that F is pro-representable. There exist a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ and an element $(\xi_i)_{i\in \mathrm{Ob}\,\mathcal{D}} \in \varprojlim FD$ such that $(h^{D_i} \xrightarrow{\xi_i^\sharp} F)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $h^{op}D$, where $\xi_i^\sharp : h^{D_i} \to F$ is a morphism in $\check{\mathcal{C}}$ given by $(\xi_i^\sharp)_X(f) = F(f)(\xi_i)$ for $X \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathcal{C}(D(i), X)$. We may assume that $\mathcal{D}$ is $\mathcal{U}$-small. For $(X,\alpha),(Y,\beta) \in \mathrm{Ob}\,\mathcal{C}_F$, there are morphisms $f : D_i \to X$ and $g : D_j \to Y$ such that $F(f)(\xi_i) = \xi_i^\sharp(f) = \alpha$ and $F(g)(\xi_j) = \xi_j^\sharp(g) = \beta$. Then, $f : (D_i,\xi_i) \to (X,\alpha)$ and $g : (D_j,\xi_j) \to (Y,\beta)$ are morphisms in $\mathcal{C}_F$. Moreover, since $\mathcal{D}$ is filtered, there are transition morphisms $\sigma : D_k \to D_i$ and $\tau : D_k \to D_j$. Hence $F(\sigma)(\xi_k) = \xi_i$, $F(\tau)(\xi_k) = \xi_j$ and $\sigma : (D_k,\xi_k) \to (D_i,\xi_i)$, $\tau : (D_k,\xi_k) \to (D_j,\xi_j)$ are also regarded as morphisms in $\mathcal{C}_F$. Thus we have morphisms $\alpha\sigma : (D_k,\xi_k) \to (X,f)$ and $\beta\tau : (D_k,\xi_k) \to (Y,g)$ in $\mathcal{C}_F$. Let $f,g : (X,\alpha) \to (Y,\beta)$ be morphisms in $\mathcal{C}_F$. There is a morphism $p : (D_i,\xi) \to (X,\alpha)$ for some $i \in \mathrm{Ob}\,\mathcal{D}$. Then, $\xi_i^\sharp(fp) = F(fp)(\xi_i) = F(f)F(p)(\xi_i) = F(f)(\alpha) = \beta = F(g)(\alpha) = F(g)F(p)(\xi_i) = F(gp)(\xi_i) = \xi_i^\sharp(gp)$ and it follows that there are transition morphisms $\sigma : D_k \to D_i$ and $\tau : D_k \to D_j$ such that $fp\sigma = gp\tau$. Moreover, since $\mathcal{D}$ is filtered, there is a transition morphism $\rho : D_l \to D_k$ such that $\sigma\rho = \tau\rho$. We set $q = p\sigma\rho$, which can be regarded as a morphism $(D_l,\xi_l) \to (X,\alpha)$ in $\mathcal{C}_F$. Hence we have a morphism q equalizing f and g. Thus we have shown that $\mathcal{C}_F^{op}$ is filtered and it has a $\mathcal{U}$-small cofinal set $\{(D_i,\xi_i)|\, i \in \mathrm{Ob}\,\mathcal{D}\}$.

$(ii) \Rightarrow (i)$; Since $\mathcal{C}_F^{op}$ is filtered and essentially $\mathcal{U}$-small, we can define a pro-object $D : \mathcal{C}_F \to \mathcal{C}$ by $D(X,\alpha) = X$ and $D(f) = f$. Then, D represents F by the dual of (A.4.2).

$(i) \Rightarrow (iii)$; Let $D : \mathcal{D}^{op} \to \mathcal{C}$ be a pro-object. For each $i \in \mathrm{Ob}\,\mathcal{D}$, the functor $h^{D_i} : \mathcal{C} \to \mathcal{U}$-$\mathbf{Ens}$ is left exact. Since filtered colimits commutes with finite limits in $\mathcal{U}$-$\mathbf{Ens}$ (A.4.4), $L(D) = \varinjlim h^{op}D$ is left exact. Hence a pro-representable functor is left exact. It is a part of the conditions of (ii) that $\mathrm{Ob}\,\mathcal{C}_F^{op}$ has $\mathcal{U}$-small cofinal subset.

$(iii) \Rightarrow (ii)$; For $(X,\alpha),(Y,\beta) \in \mathrm{Ob}\,\mathcal{C}_F$, since projection morphisms $\mathrm{pr}_1 : X\times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$ induce a bijection $(F(\mathrm{pr}_1), F(\mathrm{pr}_2)) : F(X\times Y) \to F(X)\times F(Y)$, there exists $\gamma \in F(X\times Y)$ such that $F(\mathrm{pr}_1)(\gamma) =$

α and $F(\mathrm{pr}_2)(\gamma) = \beta$. Hence there are morphisms $\mathrm{pr}_1 : (X \times Y, \gamma) \to (X, \alpha)$ and $\mathrm{pr}_2 : (X \times Y, \gamma) \to (Y, \beta)$ in $\mathcal{C}_F$. Let $f, g : (X, \alpha) \to (Y, \beta)$ be morphisms in $\mathcal{C}_F$ and $e : Z \to X$ an equalizer of $f, g : X \to Y$ in $\mathcal{C}$. Then, $F(e) : F(Z) \to F(X)$ is an equalizer of $F(f), F(g) : F(X) \to F(Y)$. Since $F(f)(\alpha) = \beta = F(g)(\alpha)$, there exists a unique $\gamma \in F(Z)$ such that $F(e)(\gamma) = \alpha$. Thus $e : (Z, \gamma) \to (X, \alpha)$ is regarded as a morphism in $\mathcal{C}_F$ which equalizes f and g. $\square$

Definition 6.1.12 *Let $\mathcal{C}$ be a $\mathcal{U}$-category.*

1) A pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ is said to be strict if, for any $\varphi \in \mathrm{Mor}\,\mathcal{D}$, $D(\varphi)$ is an epimorphism. We denote by $\mathrm{Pro}^s(\mathcal{C})$ a full subcategory of $\mathrm{Pro}(\mathcal{C})$ consisting of strict pro-objects.

*2) A functor $F : \mathcal{C} \to \mathcal{U}$-**Ens** is said to be strictly pro-representable if there exist a strict pro-object D in $\mathcal{C}$ such that F is isomorphic to $L(D)$.*

*3) Let $F : \mathcal{C} \to \mathcal{U}$-**Ens** be a functor and (X, ξ) an object of $\mathcal{C}_F$. If there exists a morphism $u : (X, \xi) \to (Y, \zeta)$ in $\mathcal{C}_F$, we say that (X, ξ) dominates (Y, ζ).*

4) We say that (X, ξ) is minimal (resp. weakly minimal) if, for $(Z, \chi) \in \mathrm{Ob}\,\mathcal{C}_F$, a monomorphism (resp. a regular monomorphism (A.1.12)) $v : Z \to X$ in $\mathcal{C}$ satisfying $\xi = F(v)(\chi)$ is an isomorphism.

5) An object X of $\mathcal{C}$ is said to be artinian if each descending chain $X_1 \supset X_2 \supset \cdots \supset X_i \supset X_{i+1} \supset \cdots$ of subobjects of X is stationary, that is, there exists N such that $X_i = X_N$ if $i \geqq N$. If every object of $\mathcal{C}$ is artinian, $\mathcal{C}$ is said to be artinian.

6) An object X of $\mathcal{C}$ is said to be connected if X is not isomorphic to the coproduct of two objects which are not initial.

Lemma 6.1.13 *Let $\mathcal{C}$ be a $\mathcal{U}$-category with finite limits and $F : \mathcal{C} \to \mathcal{U}$-**Ens** a left exact functor. Suppose that $(X, \xi) \in \mathrm{Ob}\,\mathcal{C}_F$ is weakly minimal.*

1) A map $\mathcal{C}(X, Y) \to F(Y)$ given by $u \mapsto F(u)(\xi)$ is injective for any $Y \in \mathrm{Ob}\,\mathcal{C}$. In other words, there is at most one morphism $(X, \xi) \to (Y, \zeta)$ in $\mathcal{C}_F$ for each $(Y, \zeta) \in \mathrm{Ob}\,\mathcal{C}_F$.

2) If (Y, ζ) is weakly minimal and there are morphisms $f : (X, \xi) \to (Y, \zeta)$, $g : (Y, \zeta) \to (X, \xi)$ in $\mathcal{C}_F$, then f is an isomorphism with inverse g.

3) If (Z, χ) dominates (X, ξ), a map $v : Z \to X$ satisfying $\xi = F(v)(\chi)$ is an epimorphism in $\mathcal{C}$.

Proof. 1) Suppose that $F(u)(\xi) = F(w)(\xi)$ for $u, w : X \to Y$. Let $v : Z \to X$ be an equalizer of u and w. By the assumption, $F(v) : F(Z) \to F(X)$ is an equalizer of $F(u)$ and $F(w)$. Then, $\xi = F(v)(\chi)$ for some $\chi \in F(Z)$. Since (X, ξ) is weakly minimal, v is an isomorphism, namely, $u = w$.

2) Since $F(gf)(\xi) = \xi = F(id_X)(\xi)$ and $F(fg)(\zeta) = \zeta = F(id_Y)(\zeta)$, it follows from 1) that $gf = id_X$, $fg = id_Y$.

3) Let $u, w : X \to Y$ be morphisms satisfying $uv = wv$. Consider an equalizer $e : W \to X$ of u and w. Then, we have a morphism $s : Z \to W$ satisfying $v = es$. Hence $\xi = F(e)(F(s)(\chi))$ and it follows that e is an isomorphism. This implies that $u = w$. $\square$

Proposition 6.1.14 *Let $\mathcal{C}$ be a $\mathcal{U}$-category with finite limits. A functor $F : \mathcal{C} \to \mathcal{U}$-**Ens** is strictly pro-representable if and only if F has the following properties.*

i) F is left exact.
ii) Each $(X, \xi) \in \mathrm{Ob}\,\mathcal{C}_F$ is dominated by a weakly minimal object.
iii) The full subcategory of $\mathcal{C}_F$ consisting of weakly minimal objects is equivalent to a $\mathcal{U}$-small category.

Proof. Suppose that F is strictly pro-representable. Then F is left exact by (6.1.11). There exist a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ and an element $(\xi_i)_{i \in \mathrm{Ob}\,\mathcal{D}} \in \varprojlim FD$ such that $(h^{D(i)} \xrightarrow{\xi_i^\sharp} F)_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $h^{op}D$, where $\xi_i^\sharp : h^{D(i)} \to F$ is a morphism in $\check{\mathcal{C}}$ given by $(\xi_i^\sharp)_X(f) = F(f)(\xi_i)$ for $X \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathcal{C}(D(i), X)$. We claim that (D_i, ξ_i) is weakly minimal. For $(Z, \chi) \in \mathrm{Ob}\,\mathcal{C}_F$, let $v : Z \to D_i$ be a regular monomorphism which is an equalizer of morphisms $u, w : D_i \to W$ satisfying $\xi_i = F(v)(\chi)$. Since $\xi_i^\sharp(u) = F(u)(\xi_i) = F(uv)(\chi) = F(wv)(\chi) = F(w)(\xi_i) = \xi_i^\sharp(v)$, there exists a transition morphism $\tau : D_j \to D_i$ such that $u\tau = w\tau$. Note that τ is an epimorphism by (6.1.13) and it follows that $u = w$. Hence v is an isomorphism. For each $X \in \mathrm{Ob}\,\mathcal{C}$ and $\xi \in F(X)$, there exist $i \in I$ and a morphism $u : D_i \to X$ such that $\xi = F(u)(\xi_i)$ by the assumption. Thus (X, ξ) is dominated by a weakly minimal object (D_i, ξ_i). If (X, ξ) is an arbitrary minimal object in $\mathcal{C}_F$, there is a morphism $u : (D_i, \xi_i) \to (X, \xi)$ for some $i \in \mathrm{Ob}\,\mathcal{D}$. Then u is an isomorphism by (6.1.13) and the condition iii) holds.

Conversely, suppose that F satisfies i), ii) and iii). For $(X, \xi), (Y, \zeta) \in \mathrm{Ob}\,\mathcal{C}_F$, there exists a unique element $\pi \in F(X \times Y)$ satisfying $F(\mathrm{pr}_1)(\pi) = \xi$ and $F(\mathrm{pr}_2)(\pi) = \zeta$ by i). Choose a weakly minimal $(Z, \chi) \in \mathrm{Ob}\,\mathcal{C}_F$

dominating $(X \times Y, \pi)$. Thus we have a weakly minimal object (Z, χ) dominating both (X, ξ) and (Y, ζ). Moreover, if (X, ξ) is weakly minimal, there is at most one morphism $(X, \xi) \to (Y, \zeta)$ in $\mathcal{C}_F$ for each $(Y, \zeta) \in \operatorname{Ob}\mathcal{C}_F$ by (6.1.13). Let $\mathcal{D}$ be the opposite category of a skeleton of the full subcategory of $\mathcal{C}_F$ consisting objects of weakly minimal objects. Then $\mathcal{D}$ is a $\mathcal{U}$-small directed set. Define a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ by $D(X, \xi) = X$ and $D(f) = f$. If $f : (X, \xi) \to (Y, \zeta)$ is a morphism in $\mathcal{D}^{op}$, $f : X \to Y$ is an epimorphism in $\mathcal{C}$ by (6.1.13). Hence D is strict. For each $(X, \xi) \in \operatorname{Ob}\mathcal{D}$ and $Y \in \operatorname{Ob}\mathcal{C}$, the map $\lambda^Y_{(X,\xi)} : \mathcal{C}(D(X, \xi), Y) \to F(Y)$ defined by $\lambda^Y_{(X,\xi)}(u) = F(u)(\xi)$ is injective by $i)$ and (6.1.13). Suppose that $f \in \mathcal{C}(D(X, \xi), Y)$ and $g \in \mathcal{C}(D(Z, \zeta), Y)$ satisfy $\lambda^Y_{(X,\xi)}(f) = \lambda^Y_{(Z,\zeta)}(g)$. Take morphisms $\sigma : (W, \chi) \to (X, \xi)$ and $\tau : (W, \chi) \to (Z, \zeta)$ in $\mathcal{D}^{op}$. Then $\lambda^Y_{(W,\chi)}(f\sigma) = F(f\sigma)(\chi) = F(f)F(\sigma)(\chi) = F(f)(\xi) = \lambda^Y_{(X,\xi)}(f) = \lambda^Y_{(Z,\zeta)}(g) = F(g)(\zeta) = F(g)F(\tau)(\chi) = F(g\tau)(\chi) = \lambda^Y_{(W,\chi)}(g\tau)$ and we have $f\sigma = g\tau$. By $ii)$, $(\mathcal{C}(D(X, \xi), Y) \xrightarrow{\lambda^Y_{(X,\xi)}} F(Y))_{i \in \operatorname{Ob}\mathcal{D}}$ is an epimorphic family and it is a colimiting cone of $h_Y D : \mathcal{D} \to \mathcal{U}$-**Ens**. □

We summarize the second half of the above proof. Suppose that $F \in \operatorname{Ob}\check{\mathcal{C}}$ satisfies the conditions $i)$, $ii)$ and $iii)$. Then, F is represented by a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ defined as follows. $\mathcal{D}$ is the opposite category of a skeleton of the full subcategory of $\mathcal{C}_F$ consisting objects of weakly minimal objects and D is defined by $D(X, \xi) = X$ and $D(f) = f$. Then, $\mathcal{D}$ is a $\mathcal{U}$-small directed set and D is strict. For $(X, \xi) \in \operatorname{Ob}\mathcal{D}$ and $Y \in \operatorname{Ob}\mathcal{C}$, define $\lambda^Y_{(X,\xi)} : \mathcal{C}(D(X, \xi), Y) \to F(Y)$ by $\lambda^Y_{(X,\xi)}(f) = F(f)(\xi)$. Then

$$(\mathcal{C}(D(X, \xi), Y) \xrightarrow{\lambda^Y_{(X,\xi)}} F(Y))_{(X,\xi) \in \operatorname{Ob}\mathcal{D}}$$

is a colimiting cone of $h_Y D : \mathcal{D} \to \mathcal{U}$-**Ens**.

Proposition 6.1.15 *Let $F : \mathcal{C} \to \mathcal{U}$-**Ens** be a functor and X an artinian object of $\mathcal{C}$. For any $\xi \in F(X)$, $(X, \xi) \in \operatorname{Ob}\mathcal{C}_F$ is dominated by a minimal object.*

Proof. We set $S_\xi = \{Y \in \operatorname{Sub}(X) \,|\, (Y, \zeta) \text{ dominates}(X, \xi) \text{ for some} \zeta \in F(Y)\}$. Then, $X \in S_\xi$ and S_ξ is an ordered set whose descending chains are stationary. Hence S_ξ has a minimal element Z and $\chi \in F(Z)$ such that $F(\iota)(\chi) = \xi$, where $\iota : Z \to X$ denotes the inclusion morphism. It is obvious from the choice of Z that a pair (Z, χ) is minimal. □

Corollary 6.1.16 *Let $\mathcal{C}$ be an artinian $\mathcal{U}$-category with finite limits which is equivalent to a $\mathcal{U}$-small category. Then, $F \in \operatorname{Ob}\check{\mathcal{C}}$ is strictly pro-representable if and only if it is left exact.*

Proof. This is a direct consequence of (6.1.14) and (6.1.15). □

In the above case, the proof of (6.1.14) shows that a left exact functor $F : \mathcal{C} \to \mathcal{U}$-**Ens** is represented by a strict pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ defined as follows. $\mathcal{D}$ is the opposite category of a skeleton of the full subcategory of $\mathcal{C}_F$ consisting objects of minimal objects and D is defined by $D(X, \xi) = X$ and $D(f) = f$. In particular, we have the following result.

Corollary 6.1.17 *Let $\mathcal{C}$ be an artinian $\mathcal{U}$-category with finite limits which is equivalent to a $\mathcal{U}$-small category. For a pro-object D in $\mathcal{C}$, there is a strict pro-object $E : \mathcal{E}^{op} \to \mathcal{C}$ such that E is isomorphic to D and $\mathcal{E}$ is a $\mathcal{U}$-small directed set.*

Proposition 6.1.18 *If $\mathcal{C}$ is a category with finite coproducts, then $\operatorname{Pro}(\mathcal{C})$ has finite coproduct and $L : \operatorname{Pro}(\mathcal{C}) \to \check{\mathcal{C}}^{op}$ preserves them.*

Proof. Let $D : \mathcal{D}^{op} \to \mathcal{C}$ and $E : \mathcal{E}^{op} \to \mathcal{C}$ be pro-objects. Clearly, the product category $\mathcal{D} \times \mathcal{E}$ is filtered and define a pro-object $D \coprod E : (\mathcal{D} \times \mathcal{E})^{op} \to \mathcal{C}$ by $(D \coprod E)(i, j) = D_i \coprod E_i$ and $(D \coprod E)(f, g) = D(f) \coprod E(g)$ for $f \in \mathcal{D}(i, k)$, $g \in \mathcal{E}(j, l)$. Then, for $X \in \operatorname{Ob}\mathcal{C}$, there is a colimiting cone $(\mathcal{C}(D_i, X) \times \mathcal{C}(E_j, X) \xrightarrow{\mu^X_{ij}} L(D \coprod E)(X))_{(i,j) \in \operatorname{Ob}\mathcal{D} \times \mathcal{E}}$. Let us denote by $\lambda^X_i : \mathcal{C}(D_i, X) \to L(D)(X)$ and $\nu^X_j : \mathcal{C}(E_j, X) \to L(E)(X)$ the canonical morphisms. We claim that $(\mathcal{C}(D_i, X) \times \mathcal{C}(E_j, X) \xrightarrow{\lambda^X_i \times \nu^X_j} L(D)(X) \times L(E)(X))_{(i,j) \in \operatorname{Ob}\mathcal{D} \times \mathcal{E}}$ is also a colimiting cone. In fact, since products of epimorphic family is also an epimorphic family in the category of sets, the above cone is an epimorphic family. Suppose that $(\mathcal{C}(D_i, X) \times \mathcal{C}(E_j, X) \xrightarrow{\alpha_{ij}} S)_{(i,j) \in \operatorname{Ob}\mathcal{D} \times \mathcal{E}}$ is a cone, choose $(s, t) \in \mathcal{C}(D_i, X) \times \mathcal{C}(E_j, X)$ such that $\lambda^X_i(s) = x$, $\nu^X_j(t) = y$ and define a map $\psi : L(D)(X) \times L(E)(X) \to S$ by

$\psi(x,y) = \alpha_{ij}(s,t)$. If $\lambda_i^X(s) = \lambda_k(s') = x$ and $\nu_j^X(t) = \nu_l^X(t') = y$, there exist morphisms $\xi : i \to m$, $\xi' : k \to m$ in $\mathcal{D}$ and $\zeta : j \to n$, $\zeta' : l \to n$ in $\mathcal{E}$ such that $D(\xi)^*(s) = D(\xi')^*(s')$ and $E(\zeta)^*(t) = E(\zeta')^*(t')$. Hence $\alpha_{ij}(s,t) = \alpha_{mn}(D(\xi)^* \times E(\zeta)^*)(s,t) = \alpha_{mn}(D(\xi')^* \times E(\zeta')^*)(s,t) = \alpha_{kl}(s',t')$ and the definition of ψ does not depend on the choice of s and t. Therefore we have a bijection $\beta_X : L(D\coprod E)(X) \to L(D)(X) \times L(E)(X)$ satisfying $\beta_X \mu_{ij}^X = \lambda_i^X \times \nu_j^X$. It is obvious that β_X is natural in X and we have an isomorphism $\beta : L(D\coprod E) \to L(D) \times L(E)$ in $\check{\mathcal{C}}$. Hence $L(D\coprod E)$ is a coproduct of $L(D)$ and $L(E)$ in $\check{\mathcal{C}}^{op}$. Since L is fully faithful, $D\coprod E$ is a coproduct of D and E in $\mathrm{Pro}(\mathcal{C})$.

It remains to show that there is a pro-object ϕ such that $L(\phi)$ is a terminal object of $\check{\mathcal{C}}$. Let $\mathcal{O}$ be the category with a single object and a single morphism and $\phi : \mathcal{O}^{op} \to \mathcal{C}$ be the functor associating the unique object of $\mathcal{O}^{op}$ to the initial object 0 of $\mathcal{C}$. Then $L(\phi)(X) = \mathcal{C}(0, X)$ consists of a single element for any $X \in \mathrm{Ob}\,\mathcal{C}$. □

Proposition 6.1.19 *Let $\mathcal{C}$ be a category with finite coproducts which are disjoint and universal. If $D : \mathcal{D}^{op} \to \mathcal{C}$ is a pro-object such that D_i is connected for every $i \in \mathrm{Ob}\,\mathcal{D}$, then D is connected in* $\mathrm{Pro}(\mathcal{C})$.

Proof. Suppose that there is an isomorphism $f : D \to D'\coprod D''$ in for pro-objects D' and D'' with domains $\mathcal{D}'$, $\mathcal{D}''$. Set $\theta(f) = (c_{(i,j)})_{(i,j)\in \mathrm{Ob}\,\mathcal{D}'\times\mathcal{D}''} \in \varprojlim_{(i,j)} L(D)(D'_i\coprod D''_j)$, $\theta(f^{-1}) = (d_k)_{k\in\mathrm{Ob}\,\mathcal{D}} \in \varprojlim_k L(D'\coprod D'')(D_k)$ and choose representatives $\alpha_{i,j} : D_{k(i,j)} \to D'_i\coprod D''_j$, $\beta_k : D'_{i(k)}\coprod D''_{j(k)} \to D_k$ of $c_{(i,j)}$, d_k. Since $ff^{-1} = id_{D'\coprod D''}$, $\alpha_{i,j}\beta_{k(i,j)} : D'_{i(k(i,j))}\coprod D''_{j(k(i,j))} \to D'_i\coprod D''_j$ is equivalent to the identity morphism of $D'_i\coprod D''_j$ in $L(D'\coprod D'')(D'_i\coprod D''_j)$. Hence there are transition morphisms $\sigma' : D'_m \to D'_{i(k(i,j))}$, $\sigma'' : D''_n \to D''_{j(k(i,j))}$, $\tau' : D'_m \to D'_i$ and $\tau'' : D''_n \to D''_j$ such that $\alpha_{i,j}\beta_{k(i,j)}(\sigma'\coprod\sigma'') = \tau'\coprod\tau''$. Since $D_{k(i,j)}$ is connected, we may assume that $\alpha_{i,j}$ factors through the canonical morphism $\iota_1 : D'_i \to D'_i\coprod D''_j$ by (A.8.25). Thus, $\tau'\coprod\tau'' = \iota_1\gamma$ for some morphism $\gamma : D'_m\coprod D''_n \to D'_i$. Let $\iota_2 : D''_j \to D'_i\coprod D''_j$ and $\iota'_2 : D''_n \to D'_m\coprod D''_n$ denote the canonical morphisms. Then, $\iota_2\tau'' = (\tau'\coprod\tau'')\iota'_2 = \iota_1\gamma\iota'_2$. Since finite coproducts in $\mathcal{C}$ is disjoint, there is a unique morphism $\zeta : D''_n \to 0$ to the initial object such that both τ'' and $\gamma\iota'_2$ factor through it. By (A.3.16), initial objects are strict and it follows that D''_n is an initial object. For $l \in \mathrm{Ob}\,\mathcal{D}''$, there are morphisms $\lambda : l \to r$ and $\mu : n \to r$. Again, since initial objects are strict, $D''(\mu) : D''_r \to D''_n$ is an isomorphism. Then, D''_r is also an initial object and $D''(\lambda) : D''_r \to D''_n$ is an isomorphism. Hence D''_l is an initial object in $\mathcal{C}$ for any $l \in \mathrm{Ob}\,\mathcal{D}''$. Therefore $L(D'')$ is a terminal object in $\check{\mathcal{C}}$ and it follows that D'' is an initial object in $\mathrm{Pro}(\mathcal{C})$. □

Proposition 6.1.20 *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor.*

1) If $\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{C}')$ has a left adjoint $G : \mathrm{Pro}(\mathcal{C}') \to \mathrm{Pro}(\mathcal{C})$, the following diagram commutes up to a natural equivalence.

$$\begin{tikzcd}
\mathrm{Pro}(\mathcal{C}') \arrow[r, "G"] \arrow[d, "L"] & \mathrm{Pro}(\mathcal{C}) \arrow[d, "L"] \\
\check{\mathcal{C}}' \arrow[r, "F^*"] & \check{\mathcal{C}}
\end{tikzcd}$$

Hence $F^* : \check{\mathcal{C}}' \to \check{\mathcal{C}}$ maps each pro-representable functor to a pro-representable functor and $G : \mathrm{Pro}(\mathcal{C}')(D,E) \to \mathrm{Pro}(\mathcal{C})(G(D),G(E))$ is continuous.

2) If $\mathcal{C}$ is $\mathcal{U}$-small and $F^* : \check{\mathcal{C}}' \to \check{\mathcal{C}}$ maps each pro-representable functor to a pro-representable functor, $\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{C}')$ has a left adjoint.

Proof. 1) Let D be a pro-object of $\mathcal{C}'$ and X an object of $\mathcal{C}$. Since $h^{F(X)} = L\kappa F(X) = L\mathrm{Pro}(F)\kappa(X)$ by (6.1.6), there is the following chain of isomorphisms which are natural in X and D. $F^*(L(D))(X) = L(D)(F(X)) \cong \check{\mathcal{C}}'(h^{F(X)}, L(D)) = \check{\mathcal{C}}'(L\mathrm{Pro}(F)\kappa(X), L(D)) = \mathrm{Pro}(\mathcal{C}')(D, \mathrm{Pro}(F)\kappa(X)) \cong \mathrm{Pro}(\mathcal{C})(G(D), \kappa(X)) =$
$\check{\mathcal{C}}(L\kappa(X), L(G(D))) = \check{\mathcal{C}}(h^X, L(G(D))) \cong L(G(D))(X)$. Since F^* preserves colimits, $F^* : \check{\mathcal{C}}'(L(E), L(D)) \to \check{\mathcal{C}}(F^*L(E), F^*L(D))$ is continuous by (6.1.2). Hence it follows from the continuity of the composition maps in $\check{\mathcal{C}}$ and the natural equivalence $F^*L \cong LG$ that $G : \mathrm{Pro}(\mathcal{C}')(D,E) \to \mathrm{Pro}(\mathcal{C})(G(D),G(E))$ is continuous.

2) Assume that $\mathcal{C}$ is $\mathcal{U}$-small and $F^* : \check{\mathcal{C}}' \to \check{\mathcal{C}}$ maps each pro-representable functor to a pro-representable functor. Since $L : \mathrm{Pro}(\mathcal{C}) \to \check{\mathcal{C}}$ gives an equivalence from $\mathrm{Pro}(\mathcal{C})$ to the full subcategory of $\check{\mathcal{C}}$ consisting of pro-representable functors, there is a functor $G : \mathrm{Pro}(\mathcal{C}') \to \mathrm{Pro}(\mathcal{C})$ such that $LG : \mathrm{Pro}(\mathcal{C}') \to \check{\mathcal{C}}$ is naturally equivalent to F^*L. We claim that G is a left adjoint of $\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{C}')$. For $D \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C}')$ and $E \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$, since F^* has a left adjoint $F_!$ and $F_!L : \mathrm{Pro}(\mathcal{C}) \to \check{\mathcal{C}}'$ is equivalent to $L\mathrm{Pro}(F)$ by (6.1.6), we have $\mathrm{Pro}(\mathcal{C})(G(D),E) = \check{\mathcal{C}}(L(E), LG(D)) \cong \check{\mathcal{C}}(L(E), F^*L(D)) \cong \check{\mathcal{C}}'(F_!L(E), L(D)) \cong \check{\mathcal{C}}'(L\mathrm{Pro}(F)(E), L(D)) =$
$\mathrm{Pro}(\mathcal{C}')(D, \mathrm{Pro}(F)(E))$. □

6.2 Topological groups, rings and modules

We fix a universe $\mathcal{U}$ and we only deal with $\mathcal{U}$-small groups, rings and modules unless otherwise stated. We denote by $\mathcal{U}$-**Ens** the category of $\mathcal{U}$-small sets.

Definition 6.2.1 *For a topological group G, we denote by $\mathcal{N}_G$ the set of all open normal subgroups of G. We denote by $\mathcal{TopGr}$ the full subcategory of the category of topological groups and continuous homomorphisms consiting of toplogical groups G such that $\mathcal{N}_G$ is a fundamental system of the neighborhoods of the unit.*

Definition 6.2.2 *Let A be a commutative topological ring and M a topological A-module. We denote by $\mathcal{N}_A$ the set of all open ideals of A and by $\mathcal{N}_M$ the set of all open submodules of M.*

1) If $\mathcal{N}_A$ is a fundamental system of neighborhood of $0 \in A$, we say that A is linearly topologized. Let us denote by $\mathcal{TopAlg}$ the category of linearly topologized rings and continuous ring homomorphisms.

2) If $\mathcal{N}_M$ is a fundamental system of neighborhood of $0 \in M$, we say that M is linearly topologized. We denote by $\mathcal{TopMod}(A)$ the category of linearly topologized A-modules and continuous A-module homomorphisms.

We note that, if an A-module M has a topology such that there is a fundamental system of neighborhood of $0 \in M$ consisting of submodules of M, then M is automatically a topological A-module.

The category of discrete topological groups (resp. discrete topological rings, discrete A-modules) is denoted by $\mathcal{Gr}$ (resp. $\mathcal{An}$, $\mathcal{Mod}(A)$) and we regard this as a full subcategory of $\mathcal{TopGr}$ (resp. $\mathcal{TopAlg}$, $\mathcal{TopMod}(A)$). We denote by $\iota_{\mathcal{Gr}} : \mathcal{Gr} \to \mathcal{TopGr}$, $\iota_{\mathcal{An}} : \mathcal{An} \to \mathcal{TopAlg}$ and $\iota_{\mathcal{Mod}(A)} : \mathcal{Mod}(A) \to \mathcal{TopMod}(A)$ the inclusion functors.

$\mathcal{C}$ denotes one of the categories $\mathcal{Gr}$, $\mathcal{An}$ or $\mathcal{Mod}(A)$ and $\mathcal{Top}\,\mathcal{C}$ denotes one of the categories $\mathcal{TopGr}$, $\mathcal{TopAlg}$ or $\mathcal{TopMod}(A)$.

For an object G of $\mathcal{Top}\,\mathcal{C}$, we regard the orderd set $\mathcal{N}_G$ as a category. Then, its opposite category $\mathcal{N}_G^{op}$ is a filtered category.

Proposition 6.2.3 *Let G be an object of $\mathcal{Top}\,\mathcal{C}$ and H a subgroup (resp. subring, submodule) of G if $\mathcal{C} = \mathcal{Gr}$ (resp. $\mathcal{C} = \mathcal{An}$, $\mathcal{C} = \mathcal{Mod}(A)$), then $\{H \cap N | \, N \in \mathcal{N}_G\}$ is cofinal in $\mathcal{N}_G^{op}$. Hence H is also an object of $\mathcal{Top}\,\mathcal{C}$.*

Proof. For $K \in \mathcal{N}_H$, there exists an open set O of G such that $K = H \cap O$. Moreover, since the unit is contained in $K \subset O$, there exists $N \in \mathcal{N}_G$ such that $N \subset O$. Thus we have $H \cap N \subset H \cap O = K$. □

Proposition 6.2.4 *Let G be an object of $\mathcal{TopGr}$ or $\mathcal{TopMod}(A)$. Suppose that K is a normal subgroup of G if G is an object of $\mathcal{TopGr}$ and that K is a submodule of G if G is an object of $\mathcal{TopMod}(A)$. We denote by $p : G \to G/K$ the quotient map.*

1) $\mathcal{N}_{G/K}$ is a fundamental system of the neighborhood of the unit of G/K. Hence G/K is an object of $\mathcal{TopGr}$ (resp. $\mathcal{TopMod}(A)$) if G is an object of $\mathcal{TopGr}$ (resp. $\mathcal{TopMod}(A)$).

2) Put $\mathcal{N}_{G,K} = \{N \in \mathcal{N}_G | \, N \supset K\}$. A map $\varphi : \mathcal{N}_{G/K} \to \mathcal{N}_{G,K}$ defined by $\varphi(L) = p^{-1}(L)$ is an isomorphism of categories.

Proof. We first show that $p(N) \in \mathcal{N}_{G/K}$ for any $N \in \mathcal{N}_G$. Since NK (resp. $N + K$) is an open subgroup (resp. submodule) of G containing K, it follows from $NK = p^{-1}(p(NK)) = p^{-1}(p(N))$ (resp. $N + K = p^{-1}(p(N + K)) = p^{-1}(p(N))$) that $p(N)$ an open subgroup (resp. submodule) of G/K. Moreover, since $p(N)$ is a normal subgroup of G/K, we have $p(N) \in \mathcal{N}_{G/K}$.

1) If O is an open set of G/K containing the unit, there exists $N \in \mathcal{N}_G$ satisfying $N \subset p^{-1}(O)$. Then, $p(N)$ is contained in O and $p(N) \in \mathcal{N}_{G/K}$.

2) Define a map $\psi : \mathcal{N}_{G,K} \to \mathcal{N}_{G/K}$ by $\psi(N) = p(N)$. Since p is surjective, we have $p(p^{-1}(L)) = L$ for any $L \in \mathcal{N}_{G/K}$. Hence $\psi \circ \varphi = id_{\mathcal{N}_{G/K}}$. If $N \in \mathcal{N}_{G,K}$, it follows $N = p^{-1}(p(N))$. Therefore $\varphi \circ \psi = id_{\mathcal{N}_{G,K}}$. □

Proposition 6.2.5 *$\mathcal{Top}\,\mathcal{C}$ is $\mathcal{U}$-complete for $\mathcal{C} = \mathcal{Gr}, \mathcal{An}, \mathcal{Mod}(A)$.*

Proof. For an open neighborhood V of the unit of $\prod\limits_{i \in I} G_i$, there exist $i_1, i_2, \ldots, i_m \in I$ and $N_k \in \mathcal{N}_{G_{i_k}}$ ($k = 1, 2, \ldots, m$) such that $\prod\limits_{k=1}^{m} N_k \times \prod\limits_{i \neq i_1, i_2, \ldots, i_m} G_i$ is comtaimed in V. Since $\prod\limits_{k=1}^{m} N_k \times \prod\limits_{i \neq i_1, i_2, \ldots, i_m} G_i$ is an open normal subgroup (resp. ideal, submodule) of $\prod\limits_{i \in I} G_i$, $\prod\limits_{i \in I} G_i$ is an object of $\mathcal{Top}\,\mathcal{C}$. Let $f, g : G \to H$ be morphisms of $\mathcal{Top}\,\mathcal{C}$. Then, the equalizer of f and g exists in $\mathcal{Top}\,\mathcal{C}$ by (6.2.3). □

By the above result, there are functors $\varprojlim_{\mathcal{T}op\,\mathcal{C}} : \mathrm{Pro}(\mathcal{T}op\,\mathcal{C}) \to \mathcal{T}op\,\mathcal{C}$ for $\mathcal{C} = \mathcal{G}r, \mathcal{A}n$ and $\mathcal{T}op\mathcal{M}od(A)$ as defined in the previous section. Define functors $\bar{\iota}_{\mathcal{C}} : \mathrm{Pro}(\mathcal{C}) \to \mathcal{T}op\,\mathcal{C}$ by $\bar{\iota}_{\mathcal{C}} = \varprojlim_{\mathcal{T}op\,\mathcal{C}} \mathrm{Pro}(\iota_{\mathcal{C}})$ for $\mathcal{C} = \mathcal{G}r, \mathcal{A}n$ and $\mathcal{T}op\mathcal{M}od(A)$.

Next we recall the notion of filter.

Let X be a set. We regard the set $\mathfrak{P}(X)$ of all subsets of X as a category.

A full subcategory $\mathfrak{F}$ of $\mathfrak{P}(X)$ is called a filter of X if it satisfies the following conditions.

i) $\mathfrak{F}$ does not contain the empty set.
ii) $\mathfrak{F}$ has finite products.
iii) $\mathfrak{F}^{op}$ is a sieve.

A full subcategory $\mathfrak{B}$ of $\mathfrak{P}(X)$ is called a filter basis of X if it satisfies the following conditions.

i) $\mathfrak{B}$ is not empty and does not contain the empty set.
ii) $\mathfrak{B}^{op}$ is a filtered category.

Clearly, a filter is a filter basis and, if $\mathfrak{B}$ is a filter basis of X, $\mathfrak{F}(\mathfrak{B}) = \{U \subset X |\, {}^{\exists}V \in \mathfrak{B}(U \supset V)\}$ is a filter of X. We call $\mathfrak{F}(\mathfrak{B})$ the filter generated by $\mathfrak{B}$.

If X is a topological space, we say that a filter basis $\mathfrak{B}$ converges to a point $x \in X$ if, for any neighborhood U of x, there exists $V \in \mathfrak{B}$ such that $V \subset U$. Note that a filter basis $\mathfrak{B}$ converges to $x \in X$ if and only if the filter generated by $\mathfrak{B}$ converges to x.

Next, we recall the notion of uniform spaces. Let X be a set and R, S subsets of $X \times X$. We put $R \circ S = \{(x, y) \in X \times X |\, (x, z) \in R,\ (z, y) \in S \textit{ for some } z \in X\}$ and $R^{-1} = \{(x, y) \in X \times X |\, (y, x) \in R\}$.

Definition 6.2.6 *For a set X, a set $\mathfrak{U}$ of subsets of $X \times X$ is called a uniform structure of X if it satisfies the following conditions.*

i) If $U, V \in \mathfrak{U}$, $U \cap V \in \mathfrak{U}$.
ii) If $U \in \mathfrak{U}$ and $U \subset V \subset X \times X$, $V \in \mathfrak{U}$.
iii) Every $U \in \mathfrak{U}$ contains the diagonal subset $\Delta = \{(x, x) \in X \times X |\, x \in X\}$.
iv) If $U \in \mathfrak{U}$, $U^{-1} \in \mathfrak{U}$.
v) If $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that $V \circ V \subset U$.

We call a set with a uniform structure a uniform space. We denote by $(X, \mathfrak{U})$ a set X with a uniform structure $\mathfrak{U}$.

Definition 6.2.7 *For a set X, a set $\mathfrak{B}$ of subsets of $X \times X$ is called a basis of uniform structure of X if it satisfies the following conditions.*

i) If $U, V \in \mathfrak{B}$, there exists $W \in \mathfrak{B}$ such that $W \subset U \cap V$.
ii) Every $U \in \mathfrak{B}$ contains the diagonal subset $\Delta = \{(x, x) \in X \times X |\, x \in X\}$.
iii) If $U \in \mathfrak{B}$, there exists $V \in \mathfrak{B}$ such that $V \subset U^{-1}$.
iv) If $U \in \mathfrak{B}$, there exists $V \in \mathfrak{B}$ such that $V \circ V \subset U$.

For $M \subset X$, $x \in X$ and $U \subset X \times X$, we put $U[M] = \{z \in X |\, (y, z) \in U \textit{ for some } y \in M\}$ and $U[x] = U[\{x\}]$. If X is a uniform space with a uniform structure $\mathfrak{U}$, we give a topology on X such that, for each $x \in X$, $\{U[x] |\, U \in \mathfrak{U}\}$ is a fundamental system of the neighborhood of X.

Let X and Y be uniform spaces with uniform structures $\mathfrak{U}$ and $\mathfrak{V}$, respectively. If a map $f : X \to Y$ satifies "$(f \times f)^{-1}(V) \in \mathfrak{U}$ for any $V \in \mathfrak{V}$", f is called a uniformly continuous map.

Definition 6.2.8 *Let X be a uniform space with a uniform structure $\mathfrak{U}$.*

1) A filter $\mathfrak{F}$ in X is called a Cauchy filter in X if, for any $U \in \mathfrak{U}$, there exists $V \in \mathfrak{F}$ such that $V \times V \subset U$.

2) A sequence $(a_\lambda)_{\lambda \in \Lambda}$ in X indexed by an essentially $\mathcal{U}$-small directed set Λ is called a Cauchy sequence in X if, for any neighborhood $U \in \mathfrak{U}$, there exists $\nu \in \Lambda$ such that $(a_\lambda, a_\mu) \in U$ for any $\lambda, \mu \geq \nu$.

3) X is said to be complete if every Cauchy filter in X converges.

Example 6.2.9 *Let G (resp. A, M) be an object of $\mathcal{T}op\mathcal{G}r$ (resp. $\mathcal{T}op\mathcal{A}lg$, $\mathcal{T}op\mathcal{M}od(A)$). For a subgroup H of G (resp. an ideal $\mathfrak{a}$ of A, a submodule N of M), put $U_H = \{(a, b) \in G \times G |\, ab^{-1} \in H\}$ (resp. $U_{\mathfrak{a}} = \{(a, b) \in A \times A |\, a - b \in \mathfrak{a}\}$, $U_N = \{(a, b) \in M \times M |\, a - b \in N\}$). We define a basis of a uniform structure $\mathfrak{B}_G$ on G (resp. $\mathfrak{B}_A$ on A, $\mathfrak{B}_M$ on M) by $\mathfrak{B}_G = \{U_H |\, H \in \mathcal{N}_G\}$ (resp. $\mathfrak{B}_A = \{U_{\mathfrak{a}} |\, \mathfrak{a} \in \mathcal{N}_A\}$, $\mathfrak{B}_M = \{U_N |\, N \in \mathcal{N}_M\}$). We denote by $\mathcal{T}op\mathcal{G}r^c$ (resp. $\mathcal{T}op\mathcal{A}lg^c$, $\mathcal{T}op\mathcal{M}od^c(A)$) the full subcategory of $\mathcal{T}op\mathcal{G}r$ (resp. $\mathcal{T}op\mathcal{A}lg$, $\mathcal{T}op\mathcal{M}od(A)$) consisting of complete Hausdorff groups (resp. rings, A-modules).*

Proposition 6.2.10 *A uniform space $(X, \mathfrak{U})$ is complete if and only if every Cauchy sequence in X converges.*

Proof. Suppose that X is complete and that $(a_\lambda)_{\lambda\in\Lambda}$ is a Cauchy sequence in X. For $\mu \in \Lambda$, put $V_\mu = \{a_\lambda | \lambda \geq \mu\}$. Define a filter basis $\mathfrak{B}$ by $\mathfrak{B} = \{V_\mu | \mu \in \Lambda\}$ and let $\mathfrak{F}$ be the filter generated by $\mathfrak{B}$. Then, $\mathfrak{F}$ is a Cauchy filter. In fact, for any $U \in \mathfrak{U}$, there exists $\nu \in \Lambda$ such that $(a_\lambda, a_\mu) \in U$ for any $\lambda, \mu \geq \nu$. Thus, there exists $x \in X$ such that $\mathfrak{B}$ converges to x. Then, for any neighborhood W of x, there exists $\mu \in \Lambda$ such that $V_\mu \subset W$. This shows $(a_\lambda)_{\lambda\in\Lambda}$ converges to x.

Conversely, assume that every Cauchy sequence in X converges. Let $\mathfrak{F}$ be a Cauchy filter in X. Choose a_V from each $V \in \mathfrak{F}$. Then, $(a_V)_{V\in\mathfrak{F}}$ is a Cauchy sequence in X and converges to a point x. For any $U \in \mathfrak{U}$, choose $U' \in \mathfrak{U}$ such that $U' \circ U' \subset U$. Since $\mathfrak{F}$ is a Cauchy filter and $(a_V)_{V\in\mathfrak{F}}$ converges to x, there exists $Z \in \mathfrak{F}$ such that $(a, b) \in U'$ and $(x, a_Z) \in U'$ if $a, b \in Z$. For $a \in Z$, since $(x, a_Z) \in U'$ and $a_Z \in Z$, $(a_Z, a) \in U'$ and $(x, a) \in U' \circ U' \subset U$. Therefore $U[x] \subset U$ and it follows that $\mathfrak{F}$ converges to x. □

Let $(X, \mathfrak{U})$ be a uniform space and Y a subset of X. We put $\mathfrak{U}|_Y = \{U \cap (Y \times Y) | U \in \mathfrak{U}\}$. Then $\mathfrak{U}|_Y$ is a uniform structure of Y and we call $(Y, \mathfrak{U}|_Y)$ the iuduced uniform structure. We note that the topology on Y defined from $\mathfrak{U}|_Y$ coincides with the topology as a subspace of X.

Proposition 6.2.11 *Let $(X, \mathfrak{U})$ be a uniform space and Y a subset of X.*
1) If X is complete and Y is a closed subspace of X, then Y is complete.
2) If X is a Hausdorff space and Y is complete, then Y is closed in X.

Proof. 1) Let $\mathfrak{F}$ be a Cauchy filter in Y. Then, $\mathfrak{F}$ is also a Cauchy filter in X and $\mathfrak{F}$ converges to a point x of X. Suppose $x \in X - Y$. Since $X - Y$ is an open neighborhood of x, there exists $V \in \mathfrak{F}$ such that $V \subset X - Y$. But V is not empty and contained in Y. This contradicts $V \subset X - Y$.

2) Let x be a point of a closure of Y and $\mathfrak{F}$ the set of all neighborhood of x. Then, $\mathfrak{F}|_Y = \{Y \cap U | U \in \mathfrak{F}\}$ is a filter in Y and it converges to x as a filter in X. Hence $\mathfrak{F}|_Y$ is a Cauchy filter in Y. By the completeness of Y, $\mathfrak{F}|_Y$ converges to a point y of Y. Since X is a Hausdorff space, we have $x = y \in Y$. □

Let $((X_i, \mathfrak{U}_i))_{i\in I}$ be a family of uniform spaces. We define a basis of uniform structure $\mathfrak{B}$ on the product $\prod_{i\in I} X_i$ as follows. Let M be the set of maps $\theta : I \to \coprod_{i\in I} \mathfrak{U}_i$ such that $\theta(i) \in \mathfrak{U}_i$ and $\theta(i) = X_i \times X_i$ except for finitely many i's. For $\theta \in M$, put $U(\theta) = \{(x, y) \in \prod_{i\in I} X_i \times \prod_{i\in I} X_i | (x(i), y(i)) \in \theta(i)\}$. Define $\mathfrak{B}$ by $\mathfrak{B} = \{U(\theta) | \theta \in M\}$. Let $\mathfrak{U}$ be the uniform structure generated by $\mathfrak{B}$. We call $(\prod_{i\in I} X_i, \mathfrak{U})$ the product of $((X_i, \mathfrak{U}_i))_{i\in I}$. Since, for $x \in \prod_{i\in I} X_i$, $U(\theta)[x] = \{y \in \prod_{i\in I} X_i | y(i) \in \theta(i)[x(i)]\}$ and $\theta(i)[x(i)] = X_i$ except for finitely many i's, the topology on $\prod_{i\in I} X_i$ defined from $\mathfrak{U}$ is the product topology of X_i's. We note that the projection map $p_i : \prod_{i\in I} X_i \to X_i$ is uniformly continuous.

Lemma 6.2.12 *Let $f : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ be a uniformly continuous map between uniform spaces. If $\mathfrak{B}$ is a filter basis of a Cauchy filter of X, then $f(\mathfrak{B}) = \{f(V) | V \in \mathfrak{B}\}$ is a filter basis a Cauchy filter of Y.*

Proof. It is clear that $f(\mathfrak{B})$ is a filter basis. For $U \in \mathfrak{V}$, since $(f \times f)^{-1}(U) \in \mathfrak{U}$, there exists $V \in \mathfrak{B}$ such that $(a, b) \in (f \times f)^{-1}(U)$ for $a, b \in V$. Hence $(c, d) \in U$ for $c, d \in f(V)$ and the assertion follows. □

Proposition 6.2.13 *The product $(\prod_{i\in I} X_i, \mathfrak{U})$ of $((X_i, \mathfrak{U}_i))_{i\in I}$ is complete if and only if every $(X_i, \mathfrak{U}_i)$ is complete.*

Proof. Suppose that every $(X_i, \mathfrak{U}_i)$ is complete. Let $\mathfrak{F}$ be a Cauchy filter in $\prod_{i\in I} X_i$. By (6.2.12), $p_i(\mathfrak{F})$ is a filter basis of a Cauchy filter in X_i. Then, $p_i(\mathfrak{F})$ converges to a point x_i of X_i by the completeness. We show that $\mathfrak{F}$ converges to $x = (x_i)_{i\in I}$. For $U(\theta) \in \mathfrak{B}$, suppose $\theta(i) = X_i \times X_i$ except for $i = i_1, i_2, \ldots, i_n$. Choose $V_i \in \mathfrak{F}$ such that $p_i(V_i) \subset \theta(i)[x_i]$ for $i = i_1, i_2, \ldots, i_n$ and put $V = \bigcap_{s=1}^n V_{i_s} \in \mathfrak{F}$. For $y \in V$, $y(i) = p_i(y) \in \theta(i)[x_i]$ for every $i \in I$, that is, $y \in U(\theta)[x]$. Therefore $V \subset U(\theta)[x]$ and we deduce that $\mathfrak{F}$ converges to x.

Conversely, assume that $(\prod_{i\in I} X_i, \mathfrak{U})$ is complete. Let $(a_\lambda)_{\lambda\in\Lambda}$ be a Cauchy sequence in X_i. Choose $b_j \in X_j$ for each $j \in I - \{i\}$ and let x_λ be a point of $\prod_{i\in I} X_i$ such that $x_\lambda(i) = a_\lambda$ and $x_\lambda(j) = b_j$ if $j \neq i$. Then, $(x_\lambda)_{\lambda\in\Lambda}$ is a Cauchy sequence in $\prod_{i\in I} X_i$ and it converges to a point b. For any $U_i \in \mathfrak{U}_i$, define $\theta : I \to \coprod_{j\in I} \mathfrak{U}_j$ by $\theta(i) = U_i$ and $\theta(j) = X_j \times X_j$ if $j \neq i$. There exists $\lambda_0 \in \Lambda$ such that $x_\lambda \in U(\theta)[b]$ if $\lambda \geq \lambda_0$. Hence $(b(i), a_\lambda) \in U_i$ if $\lambda \geq \lambda_0$. It follows that $(a_\lambda)_{\lambda\in\Lambda}$ converges to $b(i)$ and assertion follows from (6.2.10) □

Proposition 6.2.14 *Let D be a functor with domain $\mathcal{D}$ such that $D(i)$ is a complete Hausdorff space for every $i \in \operatorname{Ob}\mathcal{D}$. If $\mathcal{D}$ is $\mathcal{U}$-small and $(L \xrightarrow{p_i} D(i))_{i\in\operatorname{Ob}\mathcal{D}}$ is a limiting cone in the category of uniform spaces, L is a complete Haudorff space.*

Proof. Since L is a closed subspace of a complete Hausdorff space $\prod_{i\in \mathrm{Ob}\,\mathcal{D}} D(i)$ ((6.2.13)), L is a complete Haudorff space by 1) of (6.2.11). $\square$

Corollary 6.2.15 *$\bar{\iota}_{\mathcal{G}r} : \mathrm{Pro}(\mathcal{G}r) \to \mathcal{T}op\mathcal{G}r$ (resp. $\bar{\iota}_{\mathcal{A}n} : \mathrm{Pro}(\mathcal{A}n) \to \mathcal{T}op\mathcal{A}lg$, $\bar{\iota}_{\mathcal{M}od(A)} : \mathrm{Pro}(\mathcal{M}od(A)) \to \mathcal{T}op\mathcal{M}od(A)$) takes values in $\mathcal{T}op\mathcal{G}r^c$ (resp. $\mathcal{T}op\mathcal{A}lg^c$, $\mathcal{T}op\mathcal{M}od^c(A)$).*

Let us denote by $\mathcal{C}$ one of the categories $\mathcal{G}r$, $\mathcal{A}n$ or $\mathcal{M}od(A)$ and by $\mathcal{T}op\,\mathcal{C}$ one of the categories $\mathcal{T}op\mathcal{G}r$, $\mathcal{T}op\mathcal{A}lg$ or $\mathcal{T}op\mathcal{M}od(A)$. Define functors $P : \mathcal{T}op\,\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ for $\mathcal{C} = \mathcal{G}r, \mathcal{A}n, \mathcal{M}od(A)$ as follows.

For an object G of $\mathcal{T}op\,\mathcal{C}$, since the set $\mathcal{N}_G$ is ordered and $\mathcal{N}_G^{op}$ is a filtered category, $P_{\mathcal{C}}(G) : \mathcal{N}_G \to \mathcal{C}$ is a functor given by $P_{\mathcal{C}}(G)(N) = G/N$. Let $f : G \to H$ be a morphism in $\mathcal{T}op\,\mathcal{C}$ and K an object of $\mathcal{C}$. Consider limiting cones

$$(\mathcal{C}(G/N, K) \xrightarrow{\lambda_N^K} L(P_{\mathcal{C}}(G))(K))_{N\in\mathcal{N}_G} \qquad (\mathcal{C}(H/M, K) \xrightarrow{\lambda_M^K} L(P_{\mathcal{C}}(H))(K))_{M\in\mathcal{N}_H}.$$

For $M \in \mathcal{N}_H$, we define a map $\rho_M : \mathcal{C}(H/M, K) \to L(P_{\mathcal{C}}(G))(K)$. Since f is continuous, there exists $N_0 \in \mathcal{N}_G$ such that $f(N_0) \subset M$. We denote by $f_{N_0,M} : G/N_0 \to H/M$ the map induced by f and put $\rho_M(\varphi) = \lambda_{N_0}^K(\varphi f_{N_0,M})$ for $\varphi \in \mathcal{C}(H/M, K)$. It is easy to verify that $\rho_M(\varphi)$ does not depend on the choice of $N_0 \in \mathcal{N}_G$ contained in $f^{-1}(M)$. Suppose $L \subset M$ in $\mathcal{N}_H$. Let $\pi_{L,M} : H/L \to H/M$ be the canonical map. Then $\rho_L(\varphi\pi_{L,M}) = \lambda_{N_0}^K(\varphi\pi_{L,M} f_{N_0,L})\lambda_{N_0}^K(\varphi f_{N_0,M})$. It follows that $(\mathcal{C}(H/M, K) \xrightarrow{\rho_M} L(P_{\mathcal{C}}(G))(K))_{M\in\mathcal{N}_H}$ is a cone and we have a unique map $\tilde{f}_K : L(P_{\mathcal{C}}(H))(K) \to L(P_{\mathcal{C}}(G))(K)$ satisfying $\rho_M = \tilde{f}_K\lambda_M^K$ for any $M \in \mathcal{N}_H$. We note that $\tilde{f}_K$ is natural in K. Thus we have a morphism $P_{\mathcal{C}}(f) : P_{\mathcal{C}}(G) \to P_{\mathcal{C}}(H)$ in $\mathrm{Pro}(\mathcal{C})$ corresponding to the natural transformation $\tilde{f} : L(P_{\mathcal{C}}(H)) \to L(P_{\mathcal{C}}(H))$. We note that $P_{\mathcal{C}} : \mathcal{T}op\,\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ takes values in $\mathrm{Pro}^s(\mathcal{C})$.

Proposition 6.2.16 *$P_{\mathcal{C}} : \mathcal{T}op\,\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ is a left adjoint of $\bar{\iota}_{\mathcal{C}} : \mathrm{Pro}(\mathcal{C}) \to \mathcal{T}op\,\mathcal{C}$.*

Proof. We denote by $\pi_N^G : \bar{\iota}_{\mathcal{C}}P_{\mathcal{C}}(G) \to G/N$ the the canonical projection for $N \in \mathcal{N}_G$. For an object G of $\mathcal{T}op\,\mathcal{C}$, let $\eta_G : G \to \bar{\iota}_{\mathcal{C}}P_{\mathcal{C}}(G)$ be the unique morphism in $\mathcal{T}op\,\mathcal{C}$ such that $\pi_N^G\eta_G : G \to G/N$ is the quotient map.

For a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$, define a morphism $\varepsilon_D : P_{\mathcal{C}}\bar{\iota}_{\mathcal{C}}(D) \to D$ in $\mathrm{Pro}(\mathcal{C})$ as follows. First of all, we denote by $\pi_i : \bar{\iota}_{\mathcal{C}}(D) \to D(i)$, the canonical projection for $i \in \mathrm{Ob}\,\mathcal{D}$. For $H \in \mathrm{Ob}\,\mathcal{C}$, $i \in \mathrm{Ob}\,\mathcal{D}$ and a morphism $f : D(i) \to H$, since $f\pi_i : \bar{\iota}_{\mathcal{C}}(D) \to H$ is continuous and H is descrete, $\mathrm{Ker}\,f\pi_i$ is open in $\bar{\iota}_{\mathcal{C}}(D)$. Hence, if $N \in \mathcal{N}_{\bar{\iota}_{\mathcal{C}}(D)}$ contained in $\mathrm{Ker}\,f\pi_i$, $f\pi_i$ induces a morphism $f' : \bar{\iota}_{\mathcal{C}}(D)/N \to H$ in $\mathcal{C}$. Define a map $e_i^H : \mathcal{C}(D(i), H) \to L(P_{\mathcal{C}}\bar{\iota}_{\mathcal{C}}(D))(H)$ by $e_i^H(f) = \lambda_N^H(f')$, where $\lambda_N^H : \mathcal{C}(\bar{\iota}_{\mathcal{C}}(D)/N, H) \to L(P_{\mathcal{C}}\bar{\iota}_{\mathcal{C}}(D))(H)$ is the canonical map. Note that $e_i^H(f)$ does not depend on the choice of the open ideal N contained in $\mathrm{Ker}\,f\pi_i$. If $\alpha : i \to j$ is a morphism in $\mathcal{D}$, since $fD(\alpha)\pi_j = f\pi_i$, we have $e_j^H D(\alpha)^*(f) = e_j^H(fD(\alpha)) = e_i^H(f)$. Hence $(\mathcal{C}(D(i), H) \xrightarrow{e_i^H} L(P_{\mathcal{C}}\bar{\iota}_{\mathcal{C}}(D))(H))_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a cone and this induces $L(\varepsilon_D)_H : L(D)(H) \to L(P_{\mathcal{C}}\bar{\iota}_{\mathcal{C}}(D))(H)$. For a morphism $\varphi : H \to K$ in $\mathcal{C}$, since N is contained in $\mathrm{Ker}\,\varphi f\pi_i$, $e_i^K(\varphi f) = \lambda_N^K(\varphi f') = \varphi\lambda_N^H(f') = \varphi e_i^H(f)$. It follows that $L(\varepsilon_D)_H$ is natural in H and we have a morphism $L(\varepsilon_D) : L(D) \to L(P_{\mathcal{C}}\bar{\iota}_{\mathcal{C}}(D))$, which defines $\varepsilon_D : P_{\mathcal{C}}\bar{\iota}_{\mathcal{C}}(D) \to D$.

Let G be an object of $\mathcal{T}op\,\mathcal{C}$ and H an object of $\mathcal{C}$. For $N \in \mathcal{N}_G$ and $f \in \mathcal{C}(G/N, H)$, choose $L \in \mathcal{N}\bar{\iota}_{\mathcal{C}}P_{\mathcal{C}}(G)$ contained in $\mathrm{Ker}\,f\pi_N^G$ and $N_0 \in \mathcal{N}_G$ contained in $\eta_G^{-1}(L)$. Consider maps $f' : \bar{\iota}_{\mathcal{C}}P_{\mathcal{C}}(G)/L \to H$ and $\eta' : G/N_0 \to \bar{\iota}_{\mathcal{C}}P_{\mathcal{C}}(G)/L$ such that the following diagram commutes.

$$\begin{tikzcd}
G \arrow[r, "\eta_G"] \arrow[d] & \bar{\iota}_{\mathcal{C}}P_{\mathcal{C}}(G) \arrow[r, "\pi_N^G"] \arrow[d] & G/N \arrow[d, "f"] \\
G/N_0 \arrow[r, "\eta'"] & \bar{\iota}_{\mathcal{C}}P_{\mathcal{C}}(G)/L \arrow[r, "f'"] & H
\end{tikzcd}$$

Then, $L(\varepsilon_{P_{\mathcal{C}}(G)}P_{\mathcal{C}}(\eta_G))_H\lambda_N^H(f) = L(P_{\mathcal{C}}(\eta_G))_H L(\varepsilon_{P_{\mathcal{C}}(G)})_H\lambda_N^H(f) = L(P_{\mathcal{C}}(\eta_G))_H e_N^H(f) = L(P_{\mathcal{C}}(\eta_G))_H\lambda_L^H(f') = \rho_L^H(f') = \lambda_{N_0}^H(f'\eta')$. Since $\pi_N^G\eta_G : G \to G/N$ is the quotient map, the commutativity of the outer rectangle of the above diagram implies that $\lambda_N^H(f) = \lambda_{N_0}^H(f'\eta')$. Therefore $\varepsilon_{P_{\mathcal{C}}(G)}P_{\mathcal{C}}(\eta_G) = id_{P_{\mathcal{C}}(G)}$. Let $D : \mathcal{D}^{op} \to \mathcal{C}$ be a pro-object and $\pi_i' : \bar{\iota}_{\mathcal{C}}(D)/\mathrm{Ker}\,\pi_i \to D(i)$ the map induced by π_i. Then, $\pi_i\bar{\iota}_{\mathcal{C}}(\varepsilon_D)\eta_{\bar{\iota}_{\mathcal{C}}(D)} = \pi_i'\pi_{\mathrm{Ker}\,\pi_i}^{\bar{\iota}_{\mathcal{C}}(D)}\eta_{\bar{\iota}_{\mathcal{C}}(D)} = \pi_i$. Hence $\bar{\iota}_{\mathcal{C}}(\varepsilon_D)\eta_{\bar{\iota}_{\mathcal{C}}(D)} = id_{\bar{\iota}_{\mathcal{C}}(D)}$. $\square$

Proposition 6.2.17 *Let G be an object of $\mathcal{T}op\,\mathcal{C}$ ($\mathcal{C} = \mathcal{G}r, \mathcal{A}n, \mathcal{M}od(A)$). Then $\eta_G : G \to \bar{\iota}_{\mathcal{C}}P_{\mathcal{C}}(A)$ is an isomorphism (resp. a monomorphism) if and only if A is complete Hausdorff (resp. Hausdorff).*

Proof. Since $\operatorname{Ker}\eta_G = \bigcap_{N\in\mathcal{N}_G} N$, η_G is a monomorphism if and only if G is Hausdorff. By (6.2.14), G is complete Hausdorff if η_G is an isomorphism. Assume that G is complete Hausdorff. For $N \in \mathcal{N}_G$, let $\pi_N : \bar{\iota}_{\mathcal{C}} P_{\mathcal{C}}(G) \to G/N$ be the canonical projection and $p_N : G \to G/N$ the quotient map. For $x \in \bar{\iota}_{\mathcal{C}} P_{\mathcal{C}}(A)$ and $N \in \mathcal{N}_G$, put $V_N = p_N^{-1}(\pi_N(x))$ and $\mathfrak{B} = \{V_N | \, N \in \mathcal{N}_G\}$. Then $\mathfrak{B}$ is not empty and does not contain empty set. If $N \subset L$ $(N, L \in \mathcal{N}_G)$, then $V_N \subset V_L$. In fact, since $p_L = qp_N$ and $\pi_L = q\pi_N$ for $q : A/N \to A/L$, if $a \in V_N$, then $p_L(a) = qp_N(a) = q\pi_N(x) = \pi_L(x)$. Hence $\mathfrak{B}$ is a filter basis of a Cauchy filter of G and $\mathfrak{B}$ converges. Suppose that $\mathfrak{B}$ converges to $\alpha \in G$. For any $N \in \mathcal{N}_G$, there exists $L \in \mathcal{N}_G$ such that $V_L \subset \alpha + N$. We may assume that $L \subset N$. Then p_N maps every element of V_L to $\pi_N(x)$. Hence $V_L \subset \alpha + N$ implies $p_N(x) = \pi_N(\alpha) = \pi_N \eta_G(\alpha)$. Therefore $\eta_G(\alpha) = x$ and η_G is surjective. For every $N \in \mathcal{N}_G$, since $\eta_G(N) = \eta_G(p_N^{-1}(0)) = \pi_N^{-1}(0)$, η_G is an open map. Thus η_G is a homeomorphism. □

Corollary 6.2.18 *The restriction of $P_{\mathcal{C}} : \mathcal{T}\!op\, \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ to $\mathcal{T}\!op\, \mathcal{C}^c$ is fully faithful.*

Definition 6.2.19 *For an object G of $\mathcal{T}\!op\, \mathcal{C}$ ($\mathcal{C} = \mathcal{G}r, \mathcal{A}n, \mathcal{M}od(A)$), we denote $\bar{\iota}_{\mathcal{C}} P_{\mathcal{C}}(A)$ by $\hat{G}$ and call $\hat{G}$ the completion of G.*

By (6.2.14),(6.2.16) and (6.2.17), we have the following result.

Proposition 6.2.20 *For a morphism $f : G \to H$ in $\mathcal{T}\!op\, \mathcal{C}$ such that H (resp. N) is complete, there exists a unique morphism $\hat{f} : \hat{G} \to H$ in $\mathcal{T}\!op\, \mathcal{C}$ such that $f = \hat{f}\eta_G$.*

Corollary 6.2.21 *The inclusion functor $\mathcal{T}\!op\, \mathcal{C}^c \to \mathcal{T}\!op\, \mathcal{C}$ has a left adjoint $G \mapsto \hat{G}$. Hence $\mathcal{T}\!op\, \mathcal{C}^c$ is a reflexive subcategory of $\mathcal{T}\!op\, \mathcal{C}$.*

Lemma 6.2.22 *If D is a pro-object in $\mathcal{C}$ with domain $\mathcal{D}^{op}$, then $\{\operatorname{Ker}\pi_i | \, i \in \operatorname{Ob}\mathcal{D}\}$ is cofinal in $\mathcal{N}_{\bar{\iota}_{\mathcal{C}}(D)}$.*

Proof. For $N \in \mathcal{N}_{\bar{\iota}_{\mathcal{C}}(D)}$, there exist $i_1, i_2, \ldots, i_n \in \operatorname{Ob}\mathcal{D}$ such that $\bigcap_{s=1}^{n} \pi_{i_s}^{-1}(0) \subset N$. There also exist morphisms $\varphi_s : i_s \to j$ in $\mathcal{D}$. Then $\pi_j^{-1}(0) \subset \pi_j^{-1}(D(\varphi_s)^{-1}(0)) = \pi_{i_s}^{-1}(0)$. Hence $\pi_j^{-1}(0) \subset \bigcap_{s=1}^{n} \pi_{i_s}^{-1}(0)$ and the assertion follows. □

Proposition 6.2.23 *For $G \in \operatorname{Ob}\mathcal{T}\!op\, \mathcal{C}$, the image of $\eta_G : G \to \hat{G}$ is dense.*

Proof. Let $\pi_N : \hat{G} \to G/N$ be the canonical projection and $p_N : G \to G/N$ the quotient map for $N \in \mathcal{N}_G$. For any $x \in \hat{G}$, choose $x_N \in G$ such that $\pi_N^G(x) = p_N(x_N)$ for each $N \in \mathcal{N}_G$. Hence $\pi_N^G(x) = \pi_N^G \eta_G(x_N)$ and we have $\eta_G(x_N) \in x + \operatorname{Ker}\pi_N^G$. It follows from (6.2.22) that x belongs to the closure of the image of η_G. □

Proposition 6.2.24 *For a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$, let us denote by $\pi_i : \bar{\iota}_{\mathcal{C}}(D) \to D(i)$ ($i \in \operatorname{Ob}\mathcal{D}$) the canonical projection and by $p_N : \bar{\iota}_{\mathcal{C}}(D) \to \bar{\iota}_{\mathcal{C}}(D)/N$ ($N \in \mathcal{N}_{\bar{\iota}_{\mathcal{C}}(D)}$) the quotient map. Consider the following conditions.*

i) For any $j \in \operatorname{Ob}\mathcal{D}$, there exist $i \in \operatorname{Ob}\mathcal{D}$ and a morphism $\xi : D(i) \to \bar{\iota}_{\mathcal{C}}(D)/\operatorname{Ker}\pi_j$ satisfying $p_{\operatorname{Ker}\pi_j} = \xi\pi_i$.
ii) For any $G \in \operatorname{Ob}\mathcal{C}$, $i \in \operatorname{Ob}\mathcal{D}$ and morphisms $f, g : D(i) \to G$ satisfying $f\pi_i = g\pi_i$, there exists a morphism $\varphi : i \to j$ in $\mathcal{D}$ such that $fD(\varphi) = gD(\varphi)$.

1) $L(\varepsilon_D)_G : L(D)(G) \to L(P\bar{\iota}_{\mathcal{C}}(D))(G)$ is surjective for any $G \in \operatorname{Ob}\mathcal{C}$ if and only if D satisfies i).
2) $L(\varepsilon_D)_G : L(D)(G) \to L(P\bar{\iota}_{\mathcal{C}}(D))(G)$ is injective for any $G \in \operatorname{Ob}\mathcal{C}$ if and only if D satisfies ii).

Proof. The followings are colimiting cones.

$$(\mathcal{C}(D(i), G) \xrightarrow{\lambda_i^G} L(D)(G))_{i\in\operatorname{Ob}\mathcal{D}} \qquad (\mathcal{C}(\bar{\iota}_{\mathcal{C}}(D)/N, G) \xrightarrow{\lambda_N^G} L(P\bar{\iota}_{\mathcal{C}}(D))(G))_{N\in\mathcal{N}_{\bar{\iota}_{\mathcal{C}}(D)}}$$

1) Suppose that *i*) is satisfied. For $N \in \mathcal{N}_{\bar{\iota}_{\mathcal{C}}(D)}$ and a morphism $f' : \bar{\iota}_{\mathcal{C}}(D)/N \to G$, there exist $j \in \operatorname{Ob}\mathcal{D}$ such that $\operatorname{Ker}\pi_j \subset N$ by (6.2.22) and $i \in \operatorname{Ob}\mathcal{D}$, $\xi : D(i) \to \bar{\iota}_{\mathcal{C}}(D)/\operatorname{Ker}\pi_j$ such that $p_{\operatorname{Ker}\pi_j} = \xi\pi_i$ by the assumption. Take $\rho : \bar{\iota}_{\mathcal{C}}(D)/\operatorname{Ker}\pi_j \to \bar{\iota}_{\mathcal{C}}(D)/N$ such that $\rho p_{\operatorname{Ker}\pi_j} = p_N$ and put $f = f'\rho\xi$. Then, $L(\varepsilon_D)_G(\lambda_i^G(f)) = \lambda_N^G(f')$. Hence $L(\varepsilon_D)_G : L(D)(G) \to L(P\bar{\iota}_{\mathcal{C}}(D))(G)$ is surjective.

Coversely, suppose that $L(\varepsilon_D)_G : L(D)(G) \to L(P\bar{\iota}_{\mathcal{C}}(D))(G)$ is surjective for any $G \in \operatorname{Ob}\mathcal{C}$. Take $G = \bar{\iota}_{\mathcal{C}}(D)/\operatorname{Ker}\pi_j$. There exists $i \in \operatorname{Ob}\mathcal{D}$ and a morphism $\xi : D(i) \to G$ such that $L(\varepsilon_D)_G(\lambda_i^G(\xi)) = \lambda_{\operatorname{Ker}\pi_j}^G(id_G)$. Since $\operatorname{Ker}\pi_i \subset \operatorname{Ker}(\xi\pi_i)$, there is a map $\xi' : \bar{\iota}_{\mathcal{C}}(D)/\operatorname{Ker}\pi_i \to G$ satisfying $\xi\pi_i = \xi' p_{\operatorname{Ker}\pi_i}$. Then, $\lambda_{\operatorname{Ker}\pi_j}^G(id_G) = L(\varepsilon_D)_G(\lambda_i^G(\xi)) = \lambda_{\operatorname{Ker}\pi_i}^G(\xi')$ implies that $p_{\operatorname{Ker}\pi_j} = id_G p_{\operatorname{Ker}\pi_j} = \xi' p_{\operatorname{Ker}\pi_i} = \xi\pi_i$.

2) Suppose that $ii)$ is satisfied. For $f, g \in \mathcal{C}(D(i), G)$, suppose that $L(\varepsilon_D)_G(\lambda_i^G(f)) = L(\varepsilon_D)_G(\lambda_i^G(g))$. Then, $\lambda^G_{\mathrm{Ker}\,\pi_i}(fq_i) = \lambda^G_{\mathrm{Ker}\,\pi_i}(gq_i)$, where $q_i : \bar{\iota}_{\mathcal{C}}(D)/\mathrm{Ker}\,\pi_i \to D(i)$ be the map induced by π_i. Since the transition maps of the direct system $h^G P\bar{\iota}_{\mathcal{C}}(D) : \mathcal{N}_{\bar{\iota}_{\mathcal{C}}(D)} \to \mathcal{U}$-**Ens** is injective, we have $fq_i = gq_i$. Hence $f\pi_i = fq_ip_{\mathrm{Ker}\,\pi_i} = gq_ip_{\mathrm{Ker}\,\pi_i} = g\pi_i$ and, by the assumption, there exists a morphism $\varphi : i \to j$ such that $fD(\varphi) = gD(\varphi)$. Therefore $\lambda_i^G(f) = \lambda_i^G(g)$.

Suppose that $L(\varepsilon_D)_G : L(D)(G) \to L(P\bar{\iota}_{\mathcal{C}}(D))(G)$ is injective for any $G \in \mathrm{Ob}\,\mathcal{C}$ and that morphisms $f, g : D(i) \to G$ satisfy $f\pi_i = g\pi_i$. Then, $fq_ip_{\mathrm{Ker}\,\pi_i} = gq_ip_{\mathrm{Ker}\,\pi_i}$. Since $p_{\mathrm{Ker}\,\pi_i}$ is an epimorphism, we have $fq_i = gq_i$. Hence $L(\varepsilon_D)_G(\lambda_i^G(f)) = \lambda^G_{\mathrm{Ker}\,\pi_i}(fq_i) = \lambda^G_{\mathrm{Ker}\,\pi_i}(gq_i) = L(\varepsilon_D)_G(\lambda_i^G(g))$. It follows from the assumption that $\lambda_i^G(f) = \lambda_i^G(g)$. Hence there exists a morphism $\varphi : i \to j$ in $\mathcal{D}$ such that $fD(\varphi) = gD(\varphi)$. □

We note that, if $D = P_{\mathcal{C}}(G)$ for some $G \in \mathrm{Ob}\,\mathcal{T}op\,\mathcal{C}$, D satisfies both $i)$ and $ii)$. By (6.2.17) and (6.2.24), we have the following result.

Corollary 6.2.25 *For a pro-object $D : \mathcal{D}^{op} \to \mathcal{C}$ ($\mathcal{C} = \mathcal{G}r, \mathcal{A}n, \mathcal{M}od(A)$), $\varepsilon_D : P\bar{\iota}(D) \to D$ is an isomorphism if and only if D satisfies the both conditions in (6.2.24). Hence $\bar{\iota} : \mathrm{Pro}(\mathcal{C}) \to \mathcal{T}op\,\mathcal{C}$ induces an equivalence from a full subcategory of $\mathrm{Pro}(\mathcal{C})$ consisting of objects satisfying conditions $i)$ and $ii)$ in (6.2.11) to $\mathcal{T}op\,\mathcal{C}^c$.*

Let G and H be objects of $\mathcal{T}op\,\mathcal{C}$. We give $\mathcal{T}op\,\mathcal{C}(G, H)$ the uniform convergent topology. That is, for $N \in \mathcal{N}_H$, put $U_N = \{(f, g) \in \mathcal{T}op\,\mathcal{C}(G, H) \times \mathcal{T}op\,\mathcal{C}(G, H) |\, f(x) \equiv g(x)$ *modulo* N *for all* $x \in G\}$ and let $\mathfrak{U}$ be the uniform system generated by $\{U_N |\, N \in \mathcal{N}_H\}$. The uniform convergent topology on $\mathcal{T}op\,\mathcal{C}(G, H)$ is the topology defined from $\mathfrak{U}$. Note that, if H is discrete, the uniform convergent topology on $\mathcal{T}op\,\mathcal{C}(G, H)$ is discrete. In fact, since $\{0\} \in \mathcal{N}_H$, $U_0 \in \mathfrak{U}$ is the diagonal subset of $\mathcal{T}op\,\mathcal{C}(G, H) \times \mathcal{T}op\,\mathcal{C}(G, H)$. For each $N \in \mathcal{N}_H$ and $f \in \mathcal{T}op\,\mathcal{C}(G, H)$, we have $U_N[f] = \{g \in \mathcal{T}op\,\mathcal{C}(G, H) |\, p_N g(x) = p_N f(x)$ *for all* $x \in G\} = p_{N*}^{-1}(p_N f)$. Thus we have the following result.

Proposition 6.2.26 *The uniform convergent topology on $\mathcal{T}op\,\mathcal{C}(G, H)$ is generated by*

$$\{p_{N*}^{-1}(\alpha) |\, \alpha \in \mathcal{T}op\,\mathcal{C}(G, H/N),\ N \in \mathcal{N}_H\}.$$

We remark that the toplogy on $\mathcal{T}op\,\mathcal{C}(G, H)$ is the weakest topology such that the map $p_{N*} : \mathcal{T}op\,\mathcal{C}(G, H) \to \mathcal{T}op\,\mathcal{C}(G, H/N)$ induced by the quotient map $p_N : H \to H/N$ is continuous for every $N \in \mathcal{N}_H$ and that $\mathcal{T}op\,\mathcal{C}(G, C)$ is discrete if C is discrete.

Definition 6.2.27 *Let A be a linearly topologized ring and M an A-module. The topology of M defined by giving a fundamental system of neighborhood of 0 $\{\mathfrak{a}M |\, \mathfrak{a}$ is an open ideal of $A\}$ is called the topology of M induced by A.*

Proposition 6.2.28 *Let A be a linearly topologized ring.*

1) Let M be a topological A-module such that the topology of M is coarser than the topology induced by A. If V is an open submodule of M, there exists an open ideal $\mathfrak{a}$ of A such that $\mathfrak{a}(M/V) = \{0\}$, hence M/V is an $A/\mathfrak{a}$-module.

2) Let M be as above. Then, the completion $\hat{M}$ has a unique structure of $\hat{A}$-module such that the following diagram commute.

$$\begin{array}{ccc} A \times M & \longrightarrow & M \\ \big\downarrow{\scriptstyle \eta_A \times \eta_M} & & \big\downarrow{\scriptstyle \eta_M} \\ \hat{A} \times \hat{M} & \longrightarrow & \hat{M} \end{array}$$

Here $\eta_A : A \to \hat{A}$ and $\eta_M : M \to \hat{M}$ denote the canonical homomorphisms.

3) Let M be as above. Then, the topology on $\hat{M}$ is coarser than the topology induced by A.

4) Suppose that the topology on M coincides with the topology induced by A. For any $\mathfrak{a} \in \mathcal{N}_A$, the map $\bar{\eta}_M : M/\mathfrak{a}M \to \hat{M}/\mathfrak{a}\hat{M}$ induced by $\eta_M : M \to \hat{M}$ is an isomorphism. Moreover, the topology on $\hat{M}$ coincides with the topology induced by A.

5) Let M and N be topological A-modules. If the topology of N is coarser than the topology induced by A, every A-homomorphism $u : M \to N$ is continuous.

6) If B is a linearly topologized A-algebra, the topology on B is coarser than the topology induced by A. Hence the completion $\hat{B}$ of B has a structure of $\hat{A}$-algebra.

Proof. 1) By the assumption, there is an ideal $\mathfrak{a}$ of A such that $\mathfrak{a}M \subset N$.

2) For any $V \in \mathcal{N}_M$, there exist $\mathfrak{a} \in \mathcal{N}_A$ such that M/V is an $A/\mathfrak{a}$-module by 1). The composition $\hat{A} \times \hat{M} \to A/\mathfrak{a} \times M/V \to M/V$ induces the multiplication map $\hat{A} \times \hat{M} \to \hat{M}$.

3) Let $(\hat{M} \xrightarrow{\pi_W} M/W)_{W \in \mathcal{N}_M}$ be the limiting cone of $P(M) : \mathcal{N}_M^{op} \to \mathcal{M}od(A)$. For $W \in \mathcal{N}_M$, there exists $\mathfrak{a} \in \mathcal{N}_A$ such that $\mathfrak{a}M \subset W$. Then, $\pi_W(\mathfrak{a}\hat{M}) = \mathfrak{a}\pi_W(M) \subset \mathfrak{a}M/W = \{0\}$ and we have $\mathfrak{a}\hat{M} \subset \operatorname{Ker}\pi_W$. By (6.2.22), the topology on $\hat{M}$ is coarser than the topology induced by A.

4) Since $\mathfrak{a}M \in \mathcal{N}_M$, the composition of $\eta_M : M \to \hat{M}$ and the canonical projection $\pi_{\mathfrak{a}M} : \hat{M} \to M/\mathfrak{a}M$ coincides with the quotient map $p_{\mathfrak{a}M} : M \to M/\mathfrak{a}M$. Obviously, $\mathfrak{a}\hat{M}$ is contained in the kernel of $\pi_{\mathfrak{a}M}$ and we have a map $\bar{\pi}_{\mathfrak{a}M} : \hat{M}/\mathfrak{a}\hat{M} \to M/\mathfrak{a}M$. Hence the following diagram commutes.

$$\begin{tikzcd}
M \arrow[r, "\eta_M"] \arrow[d, "p_{\mathfrak{a}M}"] & \hat{M} \arrow[r, "\pi_{\mathfrak{a}M}"] \arrow[d, "\hat{p}_{\mathfrak{a}\hat{M}}"] & M/\mathfrak{a}M \\
M/\mathfrak{a}M \arrow[r, "\bar{\eta}_M"] & \hat{M}/\mathfrak{a}\hat{M} \arrow[ur, "\bar{\pi}_{\mathfrak{a}M}"'] &
\end{tikzcd}$$

Then, we have $\bar{\pi}_{\mathfrak{a}M}\bar{\eta}_M p_{\mathfrak{a}M} = \pi_{\mathfrak{a}M}\eta_M = p_{\mathfrak{a}M}$ and $\bar{\eta}_M\bar{\pi}_{\mathfrak{a}M}\hat{p}_{\mathfrak{a}\hat{M}}\eta_M = \bar{\eta}_M\pi_{\mathfrak{a}M}\eta_M = \bar{\eta}_M p_{\mathfrak{a}M} = \hat{p}_{\mathfrak{a}\hat{M}}\eta_M$. Since $p_{\mathfrak{a}M}$ and $\hat{p}_{\mathfrak{a}\hat{M}}$ are surjective and the image of η_M is dense, we have $\bar{\pi}_{\mathfrak{a}M}\bar{\eta}_M = id_{M/\mathfrak{a}M}$, $\bar{\eta}_M\bar{\pi}_{\mathfrak{a}M}\hat{p}_{\mathfrak{a}\hat{M}} = id_{\hat{M}/\mathfrak{a}\hat{M}}$ and $\bar{\eta}_M\pi_{\mathfrak{a}M} = \hat{p}_{\mathfrak{a}\hat{M}}$. Hence $\bar{\eta}_M : M/\mathfrak{a}M \to \hat{M}/\mathfrak{a}\hat{M}$ is an isomorphism.

For any $\mathfrak{a} \in \mathcal{N}_A$, since $\hat{p}_{\mathfrak{a}\hat{M}}(x) = \bar{\eta}_M\pi_{\mathfrak{a}M}(x) = 0$ if $x \in \operatorname{Ker}\pi_{\mathfrak{a}M}$, we have $\operatorname{Ker}\pi_{\mathfrak{a}M} \subset \mathfrak{a}\hat{M}$. This shows that the topology on $\hat{M}$ is finer than the topology induced by A.

5) For any neighborhood V of 0 in N, there exists an ideal $\mathfrak{a}$ of A such that $\mathfrak{a}N \subset V$, hence $u(\mathfrak{a}M) \subset \mathfrak{a}N \subset V$.

6) Let $\mathfrak{b}$ be an open ideal of B. By the continuity of the structure map $u : A \to B$, there exists an open ideal $\mathfrak{a}$ of A such that $u(\mathfrak{a}) \subset \mathfrak{b}$. Then, we have $\mathfrak{a}B \subset \mathfrak{b}$. □

Proposition 6.2.29 $\operatorname{Hom}^c_A(M,N)$ *is a linearly topologized abelian group such that a fundamental system of the neighborhood of the zero map is given by* $\{\operatorname{Im}(i_{W*} : \operatorname{Hom}^c_A(M,W) \to \operatorname{Hom}^c_A(M,N)) \mid W \in \mathcal{N}_N\}$. *Here,* $i_W : W \to N$ *denotes the inclusion map. Moreover, if the topology on* N *is coarser than the topology induced by* A*, the scalar multiplication on* $\operatorname{Hom}^c_A(M,N)$ *is continuous, hence* $\operatorname{Hom}^c_A(M,N)$ *is a topological* A*-module. In this case, the uniform convergent topology on* $\operatorname{Hom}^c_A(M,N)$ *is coarser than the topology induced by* A.

Proof. For $W \in \mathcal{N}_N$ and $f \in \operatorname{Hom}^c_A(M,N)$, we have $U_W[f] = \{g \in \operatorname{Hom}^c_A(M,N) \mid g(x) - f(x) \in W \textit{ for all } x \in M\} = f + \operatorname{Im} i_{W*}$. Hence $\operatorname{Hom}^c_A(M,N)$ is linearly topologized. For $f, g \in \operatorname{Hom}^c_A(M,N)$ and $W \in \mathcal{N}_N$, there exists $W' \in \mathcal{N}_N$ such that $W' + W' \subset W$. Then, $(f(x) + W') + (g(x) + W') \subset (f+g)(x) + W$ for any $x \in M$ and it follows that the addition map $+ : \operatorname{Hom}^c_A(M,N) \times \operatorname{Hom}^c_A(M,N) \to \operatorname{Hom}^c_A(M,N)$ is continuous. Suppose that the topology on N is coarser than the topology induced by A. For $f \in \operatorname{Hom}^c_A(M,N)$, $a \in A$ and $W \in \mathcal{N}_N$, there exists $\mathfrak{a} \in \mathcal{N}_A$ such that $\mathfrak{a}N \subset W$. Then, $(a + \mathfrak{a})(f(x) + W) \subset af(x) + W$ for any $x \in M$ and it follows that the scalar product $\cdot : A \times \operatorname{Hom}^c_A(M,N) \to \operatorname{Hom}^c_A(M,N)$ is continuous. For $W \in \mathcal{N}_N$, there exists $\mathfrak{a} \in \mathcal{N}_A$ such that $\mathfrak{a}N \subset W$. Hence $\mathfrak{a}\operatorname{Hom}^c_A(M,N) \subset \operatorname{Im} i_{W*}$. □

Proposition 6.2.30 *The compostion map* $\mu : \mathcal{T}op\,\mathcal{C}(G,H) \times \mathcal{T}op\,\mathcal{C}(H,K) \to \mathcal{T}op\,\mathcal{C}(G,K)$ *is continuous for* $\mathcal{C} = \mathcal{G}r, \mathcal{A}n, \mathcal{T}op\mathcal{M}od(A)$.

Proof. For every $N \in \mathcal{N}_K$, the following diagram commutes and the vartical maps are continuous.

$$\begin{tikzcd}
\mathcal{T}op\,\mathcal{C}(G,H) \times \mathcal{T}op\,\mathcal{C}(H,K) \arrow[r, "\mu"] \arrow[d, "id \times p_{N*}"] & \mathcal{T}op\,\mathcal{C}(G,K) \arrow[d, "p_{N*}"] \\
\mathcal{T}op\,\mathcal{C}(G,H) \times \mathcal{T}op\,\mathcal{C}(H,K/N) \arrow[r, "\mu"] & \mathcal{T}op\,\mathcal{C}(G,K/N)
\end{tikzcd}$$

By the definition of the topology on $\mathcal{T}op\,\mathcal{C}(G,K)$, it suffices to show that the lower horizontal map is continuous. Since $\mathcal{T}op\,\mathcal{C}(H,K/N)$ is discrete, it suffices to show that, for each $g \in \mathcal{T}op\,\mathcal{C}(H,K/N)$, $g_* : \mathcal{T}op\,\mathcal{C}(G,H) \to \mathcal{T}op\,\mathcal{C}(G,K/N)$ is continuous. Put $L = \operatorname{Ker} g$ and let $\bar{g} : H/L \to K/N$ be the map induced by g. Then, $L \in \mathcal{N}_H$ and $g = \bar{g}p_L$, hence g_* is a composition $\mathcal{T}op\,\mathcal{C}(G,H) \xrightarrow{p_{L*}} \mathcal{T}op\,\mathcal{C}(G,H/L) \xrightarrow{\bar{g}_*} \mathcal{T}op\,\mathcal{C}(G,K/N)$. $p_{L*} : \mathcal{T}op\,\mathcal{C}(G,H) \to \mathcal{T}op\,\mathcal{C}(G,H/L)$ is continuous by the definition of the topology on $\mathcal{T}op\,\mathcal{C}(G,H)$ and $\bar{g}_* : \mathcal{T}op\,\mathcal{C}(G,H/L) \to \mathcal{T}op\,\mathcal{C}(G,K/N)$ is continuous, for $\mathcal{T}op\,\mathcal{C}(G,K/N)$ is discrete. □

Lemma 6.2.31 *For* $G \in \operatorname{Ob}\mathcal{T}op\,\mathcal{C}$ *and a toplogical space* X*, a map* $f : X \to G$ *is continuous if and only if* $p_N f : X \to G/N$ *is continuous for any* $N \in \mathcal{N}_G$.

Proof. Suppose that $p_N f : X \to G/N$ is continuous for any $N \in \mathcal{N}_G$. For $x \in X$, $f^{-1}(p_N^{-1}(p_N(f(x)))) = (p_N f)^{-1}(p_N(f(x)))$ is an open set of X by the assumption. On the other hand, we have $f(f^{-1}(p_N^{-1}(p_N(f(x))))) \subset p_N^{-1}(p_N(f(x))) = f(x) + N$. Thus f is continuous. Since each quotient map p_N is continuous, the converse is clear. □

Proposition 6.2.32 *The evaluation map $e : \mathcal{T}op\,\mathcal{C}(G, H) \times G \to H$ is continuous.*

Proof. For $N \in \mathcal{N}_H$, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}op\,\mathcal{C}(G,H) \times G \arrow[r, "e"] \arrow[d, "p_{N*} \times id_H"] & H \arrow[d, "p_N"] \\
\mathcal{T}op\,\mathcal{C}(G,H/N) \times G \arrow[r, "e"] & H/N
\end{tikzcd}$$

Since $\mathcal{T}op\,\mathcal{C}(G, H/N)$ is discrete, $e : \mathcal{T}op\,\mathcal{C}(G, H/N) \times G \to H/N$ is continuous. Then, the assertion follows from (6.2.31). □

Proposition 6.2.33 *Let $F : \mathcal{D} \to \mathcal{T}op\,\mathcal{C}$ be a functor and $(H \xrightarrow{\pi_s} F(s))_{s \in \mathrm{Ob}\,\mathcal{D}}$ a limiting cone of F. For $G \in \mathrm{Ob}\,\mathcal{T}op\,\mathcal{C}$, $(\mathcal{T}op\,\mathcal{C}(G, H) \xrightarrow{\pi_{s*}} \mathcal{T}op\,\mathcal{C}(G, F(s)))_{s \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone in the category of uniform spaces and uniformly continuous maps.*

Proof. Let $\mathfrak{U}$ be the uniform structure of $\mathcal{T}op\,\mathcal{C}(G, H)$ and $\mathfrak{U}'$ the uniform structure of $\mathcal{T}op\,\mathcal{C}(G, H)$ such that $(\mathcal{T}op\,\mathcal{C}(G, H) \xrightarrow{\pi_{s*}} \mathcal{T}op\,\mathcal{C}(G, F(s)))_{s \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone in the category of uniform spaces. Since $(H \xrightarrow{\pi_s} F(s))_{s \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of F, for any $N \in \mathcal{N}_H$, there exist $s_1, \dots, s_n \in \mathrm{Ob}\,\mathcal{D}$ and $N_i \in \mathcal{N}_{F(s_i)}$ $(i = 1, \dots, n)$ such that $\bigcap_{i=1}^n p_{s_i}^{-1}(N_i) \subset N$. Suppose that $f, g \in \mathcal{T}op\,\mathcal{C}(G, H)$ satisfy $p_{s_i} f(x) \equiv p_{s_i} g(x)$ modulo N_i for all $x \in G$ and $i = 1, \dots, n$. Then, $f(x) \equiv g(x)$ modulo $\bigcap_{i=1}^n p_{s_i}^{-1}(\mathfrak{c}_i)(\subset N)$ for all $x \in G$. Hence $\mathfrak{U}$ is coarser than $\mathfrak{U}'$. Take arbitrary $s \in \mathrm{Ob}\,\mathcal{D}$ and $N \in \mathcal{N}_{F(s)}$. Suppose that $f, g \in \mathcal{T}op\,\mathcal{C}(G, H)$ satisfy $f(x) \equiv g(x)$ modulo $p_s^{-1}(N)$ for all $x \in G$. Then, $p_s f(x) \equiv p_s g(x)$ modulo N for all $x \in G$ and this implies that $\mathfrak{U}'$ is coarser than $\mathfrak{U}$. □

Corollary 6.2.34 *For $G \in \mathrm{Ob}\,\mathcal{T}op\,\mathcal{C}$ and $H \in \mathrm{Ob}\,\mathcal{T}op\,\mathcal{C}^c$, $\mathcal{T}op\,\mathcal{C}(G, H)$ is complete Hausdorff.*

Proof. Since $(\hat{H} \xrightarrow{\pi_N^H} H/N)_{N \in \mathcal{N}_H}$ is a limiting cone in $\mathcal{T}op\,\mathcal{C}$, $(\mathcal{T}op\,\mathcal{C}(G, \hat{H}) \xrightarrow{\pi_{N*}^H} \mathcal{T}op\,\mathcal{C}(G, H/N))_{N \in \mathcal{N}_H}$ is a limiting cone in the category of uniform spaces. Since $\mathcal{T}op\,\mathcal{C}(G, H/N)$ is discrete, hence complete Hausdorff, $\mathcal{T}op\,\mathcal{C}(G, \hat{H})$ is complete Hausdorff by (6.2.14). Since H is complete Hausdorff, $\eta_H : H \to \hat{H}$ is an isomorphism. Hence the assertion follows. □

Proposition 6.2.35 *Let $F : \mathcal{D} \to \mathcal{T}op\mathcal{M}od(A)$ be a functor and $(N \xrightarrow{\pi_s} F(s))_{s \in \mathrm{Ob}\,\mathcal{D}}$ a limiting cone of F. For $M \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od(A)$, $(\mathrm{Hom}_A^c(M, N) \xrightarrow{\pi_{s*}} \mathrm{Hom}_A^c(M, F(s)))_{s \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone in the category of topological abelian groups. If the topology on $F(s)$ is coarser than the topology induced by A for all $s \in \mathrm{Ob}\,\mathcal{D}$, the topology on N is coarser than the topology induced by A and $(\mathrm{Hom}_A^c(M, N) \xrightarrow{\pi_{s*}} \mathrm{Hom}_A^c(M, F(s)))_{s \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone in the category of topological A-modules.*

Proof. The first assertion can be proved in similar way as in (6.2.33). Suppose that the topology on $F(s)$ is coarser than the topology induced by A for all $s \in \mathrm{Ob}\,\mathcal{D}$. Note that $\{\mathrm{Im}(i_{\pi_s^{-1}(W)*} : \mathrm{Hom}_A^c(M, \pi_s^{-1}(W)) \to \mathrm{Hom}_A^c(M, N)) \,|\, s \in \mathrm{Ob}\,\mathcal{D},\ W \in \mathcal{N}_{F(s)}\}$ is a sub-basis of the neighborhood of the zero map. Since, for any $W \in \mathcal{N}_{F(s)}$, there exists $\mathfrak{a} \in \mathcal{N}_A$ such that $\mathfrak{a}N \subset W$ and $\mathfrak{a}F(s) \subset \pi_s^{-1}(W)$, $\mathfrak{a}\mathrm{Hom}_A^c(M, N) \subset \mathrm{Im}\, i_{\pi_s^{-1}(W)*}$. Hence the topology on N is coarser than the topology induced by A. □

Using the above result, the following assertion is proved in similar way as in (6.2.34).

Corollary 6.2.36 *For $M \in \mathrm{Ob}\,\mathcal{T}op\mathcal{M}od(A)$ and $N \in \mathcal{T}op\mathcal{M}od^c(A)$, $\mathrm{Hom}_A^c(M, N)$ is a complete Hausdorff abelian group. If the topology on N is coarser than the topology induced by A, $\mathrm{Hom}_A^c(M, N)$ is an object of $\mathcal{T}op\mathcal{M}od^c(A)$.*

Proposition 6.2.37 *For $G, H \in \mathrm{Ob}\,\mathcal{T}op\,\mathcal{C}$, $P_{\mathcal{C}} : \mathcal{T}op\,\mathcal{C}(G, H) \to \mathrm{Pro}(\mathcal{C})(P_{\mathcal{C}}(G), P_{\mathcal{C}}(H))$ is continuous.*

Proof. Let $(h^{H/N} \xrightarrow{\lambda_N^H} L(P_{\mathcal{C}}(H)))_{N\in\mathcal{N}_H}$ be the colimiting cone. For $N \in \mathcal{N}_H$, since $\{0\} \in \mathcal{N}_{H/N}$, the canonical map $h^{H/N} \xrightarrow{\lambda_0^{H/N}} L(P_{\mathcal{C}}(H/N))$ is an isomorphism. Hence $\lambda_0^{H/N*} : \check{\mathcal{C}}(L(P_{\mathcal{C}}(H/N)), L(P_{\mathcal{C}}(G))) \to \check{\mathcal{C}}(h^{H/N}, L(P_{\mathcal{C}}(G)))$ is a homeomorphism. Since H/N is discrete, so is $\mathcal{T}op\,\mathcal{C}(G, H/N)$ and this implies that $P_{\mathcal{C}} : \mathcal{T}op\,\mathcal{C}(G, H/N) \to \mathrm{Pro}(\mathcal{C})(P_{\mathcal{C}}(G), P_{\mathcal{C}}(H/N))$ is continuous. By the definition of $L(P_{\mathcal{C}}(p_N)) : L(P_{\mathcal{C}}(H/N)) \to L(P_{\mathcal{C}}(H))$, we have $L(P_{\mathcal{C}}(p_N))\lambda_0^{H/N} = \lambda_N^H$. Thus the following diagram commutes.

$$\begin{tikzcd}
\mathcal{T}op\,\mathcal{C}(G,H) \arrow[r, "P_{\mathcal{C}}"] \arrow[d, "p_{N*}"] & \mathrm{Pro}(\mathcal{C})(P_{\mathcal{C}}(G),P_{\mathcal{C}}(H)) \arrow[r, equal] \arrow[d, "P_{\mathcal{C}}(p_N)_*"] & \check{\mathcal{C}}(L(P_{\mathcal{C}}(H)), L(P_{\mathcal{C}}(G))) \arrow[r, "\lambda_N^{H*}"] \arrow[d, "L(P_{\mathcal{C}}(p_N))^*"] & \check{\mathcal{C}}(h^{H/N}, L(P_{\mathcal{C}}(G))) \\
\mathcal{T}op\,\mathcal{C}(G,H/N) \arrow[r, "P_{\mathcal{C}}"] & \mathrm{Pro}(\mathcal{C})(P_{\mathcal{C}}(G),P_{\mathcal{C}}(H/N)) \arrow[r, equal] & \check{\mathcal{C}}(L(P_{\mathcal{C}}(H/N)),L(P_{\mathcal{C}}(G))) \arrow[ur, "\lambda_0^{H/N*}"'] &
\end{tikzcd}$$

Therefore the composition of the vertical maps are continuous for every $N \in \mathcal{N}_H$ and the continuity of $P_{\mathcal{C}} : \mathcal{T}op\,\mathcal{C}(G, H) \to \mathrm{Pro}(\mathcal{C})(P_{\mathcal{C}}(G), P_{\mathcal{C}}(H))$ follows. □

Proposition 6.2.38 *For $D, E \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$, $\bar{\iota}_{\mathcal{C}} : \mathrm{Pro}(\mathcal{C})(D, E) \to \mathcal{T}op\,\mathcal{C}(\bar{\iota}_{\mathcal{C}}(D), \bar{\iota}_{\mathcal{C}}(E))$ is continuous.*

Proof. Let $\mathcal{D}^{op}$ (resp. $\mathcal{E}^{op}$) be the domain of D (resp. E). We denote by $\pi_i^D : \bar{\iota}_{\mathcal{C}}(D) \to D(i)$, $\pi_j^E : \bar{\iota}_{\mathcal{C}}(E) \to E(j)$ the canonical projections for $i \in \mathrm{Ob}\,\mathcal{D}$, $j \in \mathrm{Ob}\,\mathcal{E}$. By (6.2.33), $(\mathcal{T}op\,\mathcal{C}(\bar{\iota}_{\mathcal{C}}(D), \bar{\iota}_{\mathcal{C}}(E)) \xrightarrow{\pi_{j*}^E} \mathcal{T}op\,\mathcal{C}(\bar{\iota}_{\mathcal{C}}(D), E(j)))_{j\in\mathrm{Ob}\,\mathcal{E}}$ is a limiting cone. Hence it suffices to show that compositions $\pi_{j*}^E\bar{\iota}_{\mathcal{C}} : \mathrm{Pro}(\mathcal{C})(D, E) \to \mathcal{T}op\,\mathcal{C}(\bar{\iota}_{\mathcal{C}}(D), E(j))$ are continuous for all $j \in \mathrm{Ob}\,\mathcal{E}$. Let $(h^{D(i)} \xrightarrow{\lambda_i^D} L(D))_{i\in\mathrm{Ob}\,\mathcal{D}}$, $(h^{E(j)} \xrightarrow{\lambda_j^E} L(E))_{j\in\mathrm{Ob}\,\mathcal{E}}$ be the colimiting cones. Define a map $r_j : \check{\mathcal{C}}(h^{E(j)}, L(D)) \to \mathcal{T}op\,\mathcal{C}(\bar{\iota}_{\mathcal{C}}(D), E(j))$ as follows. For $\theta \in \check{\mathcal{C}}(h^{E(j)}, L(D))$, choose $i \in \mathrm{Ob}\,\mathcal{D}$ and a morphism $\zeta : D(i) \to E(j)$ in $\mathcal{C}$ so that $\lambda_i^D(\zeta) = \theta_{E(j)}(id_{E(j)})$. Put $r_j(\theta) = \zeta\pi_i^D$. Suppose $\lambda_k^D(\xi) = \theta_{E(j)}(id_{E(j)})$ for $k \in \mathrm{Ob}\,\mathcal{D}$ and $\xi : D(k) \to E(j)$. There exist morphisms $\alpha : i \to l$, $\beta : k \to l$ in $\mathcal{D}$ such that $\zeta D(\alpha) = \xi D(\beta)$. Hence $\zeta\pi_i^D = \zeta D(\alpha)\pi_l^D = \xi D(\beta)\pi_l^D = \xi\pi_k^D$ and $r_j(\theta)$ does not depend on the choice of i and ζ. Since $\check{\mathcal{C}}(h^{E(j)}, L(D))$ is discrete, r_j is continuous. For $\varphi \in \check{\mathcal{C}}(L(E), L(D))$, choose $i \in \mathrm{Ob}\,\mathcal{D}$ and a morphism $\zeta : D(i) \to E(j)$ so that $\lambda_i^D(\zeta) = (\varphi\lambda_j^E)_{E(j)}(id_{E(j)})$. Then, $r_j(\varphi\lambda_j^E) = \zeta\pi_i^D$. On the other hand, we have $\pi_j^E\bar{\iota}_{\mathcal{C}}(\varphi) = \zeta\pi_i^D$ by the definition of $\bar{\iota}_{\mathcal{C}}(\varphi)$. Hence the following diagram commutes and the continuity of $\pi_{j*}^E\bar{\iota}_{\mathcal{C}}$ follows.

$$\begin{tikzcd}
\check{\mathcal{C}}(L(E), L(D)) \arrow[r, equal] \arrow[d, "\lambda_j^E"] & \mathrm{Pro}(\mathcal{C})(D, E) \arrow[r, "\bar{\iota}_{\mathcal{C}}"] & \mathcal{T}op\,\mathcal{C}(\bar{\iota}_{\mathcal{C}}(D), \bar{\iota}_{\mathcal{C}}(E)) \arrow[d, "\pi_{j*}^E"] \\
\check{\mathcal{C}}(h^{E(j)}, L(D)) \arrow[r, equal] & \check{\mathcal{C}}(h^{E(j)}, L(D)) \arrow[r, "r_j"] & \mathcal{T}op\,\mathcal{C}(\bar{\iota}_{\mathcal{C}}(D), E(j))
\end{tikzcd}$$

□

Corollary 6.2.39 *The adjoint $\mathrm{Pro}(\mathcal{C})(P_{\mathcal{C}}(G), D) \cong \mathcal{T}op\,\mathcal{C}(G, \bar{\iota}_{\mathcal{C}}(D))$ shown in (6.2.16) are homeomorphism for $\mathcal{C} = \mathcal{G}r, \mathcal{A}n, \mathcal{M}od(A)$.*

Corollary 6.2.40 *1) If $H \in \mathrm{Ob}\,\mathcal{T}op\,\mathcal{C}^c$, $P_{\mathcal{C}} : \mathcal{T}op\,\mathcal{C}(G, H) \to \mathrm{Pro}(\mathcal{C})(P_{\mathcal{C}}(G), P_{\mathcal{C}}(H))$ is a homeomorphism for any $G \in \mathrm{Ob}\,\mathcal{T}op\,\mathcal{C}$.*

2) If $D \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$ satisfies the both conditions in (6.2.24), $\bar{\iota}_{\mathcal{C}} : \mathrm{Pro}(\mathcal{C})(D, E) \to \mathcal{T}op\,\mathcal{C}(\bar{\iota}_{\mathcal{C}}(D), \bar{\iota}_{\mathcal{C}}(E))$ is a homeomorphism for any $E \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$.

Let A be a linearly topologized ring and M, N topological A-modules which are linearly topologized. For submodules V, W of M, N, we set $U(V,W) = \mathrm{Im}(V \otimes_A N) + \mathrm{Im}(M \otimes_A W) \subset M \otimes_A N$. We give $M \otimes_A N$ the topology such that $\{U(V,W) | V \in \mathcal{N}_M,\ W \in \mathcal{N}_N\}$ is a fundamental system of neighborhood of 0. The completion of $M \otimes_A N$ with respect to this topology is called the complete tensor product of M and N over A, which is denoted by $(M \otimes_A N)\hat{}$. If A, M and N are complete Hausdorff, we also denote $(M \otimes_A N)\hat{}$ by $M\hat{\otimes}_A N$.

Let $f : M \to M'$ and $g : N \to N'$ be morphisms in $\mathcal{T}op\mathcal{M}od(A)$. For $V' \in \mathcal{N}_{M'}$ and $W' \in \mathcal{N}_{N'}$, we have $U(f^{-1}(V'), g^{-1}(W')) = (f \otimes g)^{-1}(U(V', W'))$, thus $f \otimes g : M \otimes_A N \to M' \otimes_A N'$ is continuous.

Suppose that the topologies of M and N are coarser than the topologies induced by A. Then, for any $V \in \mathcal{N}_M$, $W \in \mathcal{N}_N$, there exists $\mathfrak{a} \in \mathcal{N}_A$ such that $\mathfrak{a}M \subset V$, $\mathfrak{a}N \subset W$. It follows that $\mathfrak{a}(M \otimes_A N) \subset U(V,W)$ and the topology on $M \otimes_A N$ is coarser than the topology induced by A. By 2) of (6.2.28), $(M \otimes_A N)\hat{}$ has a structure of $\hat{A}$-module.

We set $I = \{(\mathfrak{a}, V, W) \in \mathcal{N}_A \times \mathcal{N}_M \times \mathcal{N}_N \,|\, \mathfrak{a}M \subset V,\ \mathfrak{a}N \subset W\}$. Note that, if $(\mathfrak{a}, V, W) \in I$, M/V and N/W are $A/\mathfrak{a}$-modules. Define an order $\leq$ in I by "$(\mathfrak{a}, V, W) \leq (\mathfrak{a}', V', W') \Leftrightarrow \mathfrak{a} \subset \mathfrak{a}',\ V \subset V',\ W \subset W'$". Then, I is a directed set and we have an inverse system $(M/V \otimes_{A/\mathfrak{a}} N/W)_{(\mathfrak{a},V,W)\in I}$. Since $M/V \otimes_{A/\mathfrak{a}} N/W$ is identified with $M/V \otimes_A N/W$, it is canonically isomorphic to the quotient module of $M \otimes_A N$ by $U(V, W)$. By the asummption on topologies on M and N, $(M \otimes_A N)\hat{}$ is the limit of the inverse system $(M/V \otimes_{A/\mathfrak{a}} N/W)_{(\mathfrak{a},V,W)\in I}$.

Definition 6.2.41 *For $M, N \in \operatorname{Ob} \mathcal{TopMod}(A)$, a subset S of $\operatorname{Hom}^c_A(M, N)$ is called an equi-continuous set if, for any $W \in \mathcal{N}_N$, there exists $V \in \mathcal{N}_M$ such that $f(V) \subset W$ for all $f \in S$.*

If the topology on M is finer than the topology induced by A and the topology on N is coarser than the topology induced by A, then $\operatorname{Hom}^c_A(M, N)$ itself is an equi-continuous set.

Lemma 6.2.42 *Let A be a linearly topologized ring and L, M, N topological A-modules. We denote by $E(L; M, N)$ the set of elements $g \in \operatorname{Hom}^c_A(L, \operatorname{Hom}^c_A(M, N))$ such that $\operatorname{Im} g$ is an equi-continuous set. Then, $E(L; M, N)$ is a submodule of $\operatorname{Hom}^c_A(L, \operatorname{Hom}^c_A(M, N))$ if the topology on N is coarser than the topology induced by A.*

1) For $f \in \operatorname{Hom}^c_A(L \otimes_A M, N)$, define a map $f^a : L \to \operatorname{Hom}^c_A(M, N)$ by $f^a(x)(y) = f(x \otimes y)$. Then, $f^a \in E(L; M, N)$.

2) For $g \in E(L; M, N)$, define a map $g_a : L \otimes_A M \to M$ by $g_a(x \otimes y) = g(x)(y)$. Then, $g_a \in \operatorname{Hom}^c_A(L \otimes_A M, N)$.

3) Define a map $\Phi : \operatorname{Hom}^c_A(L \otimes_A M, N) \to E(L; M, N)$ by $\Phi(f) = f^a$. Then, Φ is an isomorphism in $\mathcal{TopMod}(A)$.

Proof. 1) For $Z \in \mathcal{N}_N$, there exist $V \in \mathcal{N}_L$ and $W \in \mathcal{N}_M$ such that $f(U(V, W)) \subset Z$ by the continuity of f. Since $f^a(x)(W) = f(x \otimes W) \subset Z$ for $x \in L$, $f^a(x) : M \to L$ is continuous and $\operatorname{Im} f^a$ is equi-continuous. If $x \in V$, then $f^a(x)(y) = f(x \otimes y) \in Z$ for all $y \in M$. Hence f^a is continuous.

2) For $Z \in \mathcal{N}_N$, there exists $V \in \mathcal{N}_L$ such that $g(V) \subset \operatorname{Im} i_{Z*}$ by the continuity of g. Hence $g_a(\operatorname{Im}(V \otimes M)) \subset Z$. Since $\operatorname{Im} g$ is a equi-continuous set, there exists $W \in \mathcal{N}_M$ such that $g_a(x \otimes y) = g(x)(y) \in Z$ for any $x \in L$ and $y \in W$. Thus we see $g_a \in \operatorname{Hom}^c_A(L \otimes_A M, N)$.

3) For $Z \in \mathcal{N}_N$ and $f \in \operatorname{Hom}^c_A(L \otimes_A M, Z)$, since $(i_Z f)^a(x)(y) = f(x \otimes y) \in Z$ for any $x \in L$, $y \in M$, $(i_Z f)^a(x) \in \operatorname{Im} i_{Z*} \subset \operatorname{Hom}^c_A(M, N)$ for any $x \in L$. Thus Φ is continuous. Define $\Psi : E(L; M, N) \to \operatorname{Hom}^c_A(L \otimes_A M, N)$ by $\Psi(g) = g_a$. Clearly, Ψ is the inverse of Φ. For $Z \in \mathcal{N}_N$ and $g \in E(L; M, N)$ such that $g(x)(y) \in Z$ for any $x \in L$, $y \in M$, since $g_a(x \otimes y) = g(x)(y) \in Z$ for any $x \in L$, $y \in M$, $g_a \in \operatorname{Im} i_{Z*}$. Hence Ψ is continuous. □

Proposition 6.2.43 *Let A be a linearly topologized ring and L, M, N topological A-modules. If the topology on N is coarser than the topology induced by A and N is complete, there is an isomorphism $\hat{\Phi} : \operatorname{Hom}^c_A((L \otimes_A M)\hat{}, N) \to E(L; M, N)$. given by $\hat{\Phi} = \Phi \eta^*_{L \otimes_A M}$.*

For a morphism $f : A \to B$ in $\mathcal{TopAlg}$, define functors $f_* : \mathcal{TopMod}(B) \to \mathcal{TopMod}(A)$ and $f^* : \mathcal{TopMod}(A) \to \mathcal{TopMod}(B)$ as follows.

For $N \in \operatorname{Ob} \mathcal{TopMod}(B)$, $f_*(N) = N$ as a topological abelian group and the A-module structure on $f_*(N)$ is given by $A \times N \xrightarrow{f \times 1} B \times N \to N$. If $\varphi : N \to N'$ is a morphism in $\mathcal{TopMod}(B)$, $f_*(\varphi) = \varphi$ regarded as an A-module homomorphism. It is clear that f_* maps $\mathcal{TopMod}^c(B) \to \mathcal{TopMod}^c(A)$.

For $M \in \operatorname{Ob} \mathcal{TopMod}(M)$, $f^*(M) = B \otimes_A M$ as a topological abelian group and the B-module structure $\mu_B : B \times f^*(M) \to f^*(M)$ is given by $\mu_B(b, c \otimes x) = bc \otimes x$. For $(b, x) \in B \times f^*(M)$, $\mathfrak{b} \in \mathcal{N}_B$ and $V \in \mathcal{N}_M$, we have $\mu_B((b + \mathfrak{b}) \times (x + U(\mathfrak{b}, V)) \subset bx + U(\mathfrak{b}, V)$, hence $f^*(M)$ is an object of $\mathcal{TopMod}(B)$. If $\varphi : M \to M'$ is a morphism in $\mathcal{TopMod}(A)$, $f_*(\varphi) = id_B \otimes \varphi$.

Proposition 6.2.44 *Let $f : A \to B$ be a morphism in $\mathcal{TopAlg}$.*

1) For $M, N \in \operatorname{Ob} \mathcal{TopMod}(B)$, $f_* : \operatorname{Hom}^c_B(M, N) \to \operatorname{Hom}^c_A(f_*(M), f_*(N))$ is continuous.

2) For $M, N \in \operatorname{Ob} \mathcal{TopMod}(A)$, $f^* : \operatorname{Hom}^c_A(M, N) \to \operatorname{Hom}^c_B(f^*(M), f^*(N))$ is continuous.

Proof. 1) Observe that $\mathcal{N}_N$ is a subset of $\mathcal{N}_{f_*(N)}$. Since the topology on $f_*(N)$ coincides with that of N, $\mathcal{N}_N$ is cofinal in $\mathcal{N}_{f_*(N)}$. For $W \in \mathcal{N}_{f_*(N)}$, choose $V \in \mathcal{N}_N$ such that $V \subset W$. Then, f_* maps the image of $i_{V*} : \operatorname{Hom}^c_B(M, V) \to \operatorname{Hom}^c_B(M, N)$ into the image of $i_{W*} : \operatorname{Hom}^c_A(f_*(M), W) \to \operatorname{Hom}^c_A(f_*(M), f_*(N))$.

2) If the image of $\varphi \in \operatorname{Hom}^c_A(M, N)$ is contained in $W \in \mathcal{N}_N$, the image of $f^*(\varphi)$ is contained in the image of $B \otimes_A W \to B \otimes_A N$. □

For $M \in \mathrm{Ob}\,\mathcal{TopMod}(A)$ and $N \in \mathrm{Ob}\,\mathcal{TopMod}(B)$, define $u_M : M \to f_*f^*(M)$ and $c_N : f^*f_*(N) \to N$ by $u_M(x) = 1 \otimes x$ and $c_N(b \otimes y) = by$. It is clear that u_M is continuous. If the topology on N is coarser than the topology induced by A, c_N is continuous. Thus we have the following fact.

Proposition 6.2.45 *For $M \in \mathrm{Ob}\,\mathcal{TopMod}(A)$ and $N \in \mathrm{Ob}\,\mathcal{TopMod}(B)$, if the topology on N is coarser than the topology induced by A, there is a natural isomorphism*

$$\mathrm{Hom}^c_B(f^*(M), N) \to \mathrm{Hom}^c_A(M, f_*(N)).$$

For a morphism $f : A \to B$ in $\mathcal{TopAlg}$, let $\hat{f}_* : \mathcal{TopMod}^c(B) \to \mathcal{TopMod}^c(A)$ be the functor induced by f_*. Define $\hat{f}^* : \mathcal{TopMod}^c(A) \to \mathcal{TopMod}^c(B)$ to be the composition $\mathcal{TopMod}^c(A) \hookrightarrow \mathcal{TopMod}(A) \xrightarrow{f^*} \mathcal{TopMod}(B) \xrightarrow{\iota P} \mathcal{TopMod}^c(B)$. By (6.2.37), (6.2.38) and (6.2.44), we have the following result.

Proposition 6.2.46 *Let $f : A \to B$ be a morphism in $\mathcal{TopAlg}$.*
1) For $M, N \in \mathrm{Ob}\,\mathcal{TopMod}^c(B)$, $\hat{f}_* : \mathrm{Hom}^c_B(M, N) \to \mathrm{Hom}^c_A(f_*(M), f_*(N))$ is continuous.
2) For $M, N \in \mathrm{Ob}\,\mathcal{TopMod}^c(A)$, $\hat{f}^* : \mathrm{Hom}^c_A(M, N) \to \mathrm{Hom}^c_B(f^*(M), f^*(N))$ is continuous.

For $M \in \mathrm{Ob}\,\mathcal{TopMod}^c(A)$ and $N \in \mathrm{Ob}\,\mathcal{TopMod}^c(B)$, define $\hat{u}_M : M \to \hat{f}_*\hat{f}^*(M)$ and $\hat{c}_N : \hat{f}^*\hat{f}_*(N) \to N$ as follows. $\hat{u}_M$ is a composition $M \xrightarrow{u_M} B \otimes_A M \xrightarrow{\eta_{B\otimes_A M}} (B \otimes_A M)\hat{}$. Since N is complete and $c_N : B \otimes_A N \to N$ is continuous if the topology on N is coarser than the topology induced by A, $\hat{c}_N : (B \otimes_A N)\hat{} \to N$ is the unique morphism satisfying $\hat{c}_N \eta_{B\otimes_A N} = c_N$.

Proposition 6.2.47 *For $M \in \mathrm{Ob}\,\mathcal{TopMod}^c(A)$ and $N \in \mathrm{Ob}\,\mathcal{TopMod}^c(B)$, if the topology on N is coarser than the topology induced by A, there is a natural isomorphism*

$$\mathrm{Hom}^c_B(\hat{f}^*(M), N) \to \mathrm{Hom}^c_A(M, \hat{f}_*(N)).$$

If B and C are commutative topological A-algebras which are linearly topologized, by (6.2.28), $(B \otimes_A C)\hat{}$ has a structure of $\hat{A}$-algebra. Let $\hat{i}_1 : B \to (B \otimes_A C)\hat{}$ (resp. $\hat{i}_2 : C \to (B \otimes_A C)\hat{}$) be the composition of maps $i_1 : B \to B \otimes_A C$, $x \mapsto x \otimes 1$ (resp. $i_2 : C \to B \otimes_A C$, $y \mapsto 1 \otimes y$) and the canonical map $B \otimes_A C \to (B \otimes_A C)\hat{}$.

Proposition 6.2.48 *Let D be a commutative topological A-algebra which is linearly topologized, complete and Hausdorff. For continuous A-algebra homomorphisms $f : B \to D$ and $g : C \to D$, there is a unique continuous $\hat{A}$-algebra homomorphism $h : (B \otimes_A C)\hat{} \to D$ such that $f = h\hat{i}_1$, $g = h\hat{i}_2$.*

Proof. There is a unique A-algebra homomorphism $h' : B \otimes_A C \to D$ such that $f = h'i_1$, $g = h'i_2$. For any $Z \in \mathcal{N}_D$, there exist $V \in \mathcal{N}_B$, $W \in \mathcal{N}_C$ such that $f(V) \subset Z$, $g(W) \subset Z$ by the continuity of f, g. Then, $h'(U(V, W)) = f(V) + g(W) \subset Z$ and h' is continuous. Hence h' uniquely induces $h : (B \otimes_A C)\hat{} \to D$ such that $h = h'\rho$, where $\rho : B \otimes_A C \to (B \otimes_A C)\hat{}$ is the canonical map. □

Corollary 6.2.49 *Let R be a commutative topological ring which is complete, Hausdorff and linearly topologized. Let $\mathcal{TopAlg}^c_R$ denote the category of topological R-algebras which are linearly topologized, complete and Hausdorff. Then, $\mathcal{TopAlg}^c_R$ has finite colimits.*

6.3 Totally disconnected compact groups

Lemma 6.3.1 *Let X be a compact topological space and Δ a set of closed subsets of X. If O is an open subset of X such that $\bigcap \Delta \subset O$, there exists a finite subset Δ' such that $\bigcap \Delta' \subset O$. Moreover, if Δ is closed under taking finite intersections, $C \subset O$ for some $C \in \Delta$.*

Proof. $\{F - O \mid F \in \Delta\}$ is a set of closed subsets of X and $\bigcap\{F - O \mid F \in \Delta\} = (\bigcap \Delta) - O = \phi$. Since X is compact, there exists a finite subset Δ' such that $\bigcap\{F - O \mid F \in \Delta'\} = \phi$, that is, $\bigcap \Delta' \subset O$. If Δ is closed under taking finite intersections, $\bigcap \Delta' \in \Delta$. □

Proposition 6.3.2 *1) Let X be a compact Hausdorff space and K a connected component of X. Then, K is the intersection of all closed and open subsets of X containing K.*
2) Let X be a locally compact Hausdorff space and K a compact connected component of X. If O is an open set containing K, there exists an open compact subset C such that $K \subset C \subset O$.

Proof. 1) Let Δ be the set of all closed and open subsets of X containing K. Then, Δ is closed under taking finite intersections and $\bigcap \Delta$ is a closed subset containing K. We show that $\bigcap \Delta$ is connected. Suppose $\bigcap \Delta = A \cup B$, $A \cap B = \phi$ for some closed subsets A, B. By the connectivity of K, we may assume $Z \subset A$. Since X is normal, there exist open subsets U, V of X such that $U \supset A$, $V \supset B$ and $U \cap V = \phi$. Hence $\bigcap \Delta = A \cup B \subset U \cup V$ and by (6.3.1), $\bigcap \Delta \subset C \subset U \cup V$ for some $C \in \Gamma$. Clearly, $C \cap U$ and $C \cap V$ are open in X. Since $C = (C \cap U) \cup (C \cap V)$ and $(C \cap U) \cap (C \cap V) = \phi$, $C \cap U$ and $C \cap V$ are closed in C. Hence $C \cap U$ and $C \cap V$ are closed in X. Thus $K \subset C \cap U \in \Delta$ and this implies $\bigcap \Delta \subset C \cap U$. Then $B = B \cap (\bigcap \Delta) \subset B \cap C \cap U \subset V \cap C \cap U = \phi$. Hence $\bigcap \Delta$ is connected and the assertion follows.

2) For each $x \in K$, we can choose an open neighborhood U_x of x whose closure $\overline{U}_x$ is compact and contained in O. By the compactness of K, $K \subset \bigcup_{i=1}^n U_{x_i}$ for some $x_1, x_2, \ldots, x_n \in K$. Put $U = \bigcup_{i=1}^n U_{x_i}$, then $\overline{U}$ is compact and K is a connected component of $\overline{U}$. By 1), K is the intersection of all closed and open subsets of $\overline{U}$ containing K. Hence by (6.3.1), there exists a closed and open subset C in $\overline{U}$ such that $K \subset C \subset U$. Since C is closed in a compact subset $\overline{U}$ of X, it is compact. There is an open set V of X such that $C = \overline{U} \cap V$. Then, $C = U \cap C = U \cap \overline{U} \cap V = U \cap V$ and C is open in X. □

Proposition 6.3.3 *Let X be a topological space satisfying the following condition.*

($*$) If x and y are points of X which belong to different connected components, there exists a closed and open subset U of X such that $x \in U$ and $y \in X - U$.

Suppose that R is an equivalence relation on X such that, for $x, y \in X$, $(x, y) \in R$ if x and y belong to the same connected component of X. Then, the quotient space X/R is totally disconnected and it also satisfies the above condition ($*$).

Proof. We denote by $p : X \to X/R$ the quotient map. Suppose that a and b are distinct points of X/R and choose representatives x, y of a, b, respectively. Then, by the assumption on R, x and y belong to different connected components. Hence there exists a closed and open subset U of X such that $x \in U$ and $y \in X - U$. Obviously, U and $X - U$ are unions of connected components of X and it follows that $p^{-1}(p(U)) = U$, $p^{-1}(p(X - U)) = X - U$. Thus $p(U)$ and $p(X - U) = Y - p(U)$ are closed and open subsets of X/R containing a and b, respectively. □

By (6.3.2), if X is a locally compact Hausdorff space whose connected components are compact, X satisfies the condition ($*$).

Corollary 6.3.4 *Let X be a totally disconnected locally compact Hausdorff space and G a topological group acting on X. Then, the quotient space X/G is totally disconnected satisfying the condition ($*$). In particular, if G is a totally disconnected locally compact topological group and H is a closed subgroup of G, then G/H is totally disconnected.*

Lemma 6.3.5 *Let $(X_i)_{i \in I}$ be a family of topological spaces.*

1) If K_i is a connected component of X_i, $\prod_{i \in I} K_i$ is a connected component of $\prod_{i \in I} X_i$.

2) If each X_i is totally disconnected, so is $\prod_{i \in I} X_i$.

Proof. 1) First of all, $\prod_{i \in I} K_i$ is a connected subset of $\prod_{i \in I} X_i$. Let K be a connected subset of $\prod_{i \in I} X_i$ containing $\prod_{i \in I} K_i$. Since $\mathrm{pr}_i(K)$ is a connected subset of X_i containing K_i, we have $\mathrm{pr}_i(K) = K_i$. Hence $\prod_{i \in I} K_i \subset K \subset \bigcap_{i \in I} \mathrm{pr}_i^{-1}(K_i) = \prod_{i \in I} K_i$. Therefore $K = \prod_{i \in I} K_i$.

2) This is a direct consequence of 1). □

Let X be a topological space and Δ the set of all non-empty closed and open subsets of X. We set $\mathcal{F} = \{\{C_1, C_2, \ldots, C_n\} \subset \Delta \mid n \geqq 1,\ \bigcup_{i=1}^n C_i = X,\ i \neq j \Rightarrow C_i \cap C_j = \phi\}$. For each $\Gamma \in \mathcal{F}$, we give Γ the discrete topology and define $q_\Gamma : X \to \Gamma$ by $q_\Gamma(x) = C$ if $x \in C$. Then, q_Γ is a continuous surjection. Define an order $\leq$ in $\mathcal{F}$ by "$\Gamma \leq \Gamma' \Leftrightarrow \Gamma'$ *refines*Γ". It is easy to verify that $\sup\{\Gamma, \Gamma'\}$ is given by $\{C \cap D \mid C \in \Gamma,\ D \in \Gamma'\} - \{\phi\}$. In particular, $\mathcal{F}$ is a directed set. If $\Gamma \leq \Gamma'$, there is a surjection $\rho_\Gamma^{\Gamma'} : \Gamma' \to \Gamma$ given by $\rho_\Gamma^{\Gamma'}(D) = C$ if $D \subset C$. Then $\rho_\Gamma^{\Gamma'} q_{\Gamma'} = q_\Gamma$ and $(X \xrightarrow{q_\Gamma} \Gamma)_{\Gamma \in \mathcal{F}}$ is a cone of a projective system $(\Gamma, \rho_\Gamma^{\Gamma'})_{\Gamma, \Gamma' \in \mathcal{F}}$. Hence there exists a continuous map $p : X \to \varprojlim_{\Gamma \in \mathcal{F}} \Gamma$ such that $p_\Gamma p = q_\Gamma$, where $p_\Gamma : \varprojlim_{\Gamma \in \mathcal{F}} \Gamma \to \Gamma$ denotes the canonical projection onto the Γ-component.

Let $\mathcal{F}_2$ be the subset of $\mathcal{F}$ consisting of elements of the form $\{C, X - C\}$ $(C \in \Delta)$ and $p' : X \to \prod_{\Gamma \in \mathcal{F}_2} \Gamma$ the map induced by q_Γ's for $\Gamma \in \mathcal{F}_2$. We note that $\prod_{\Gamma \in \mathcal{F}_2} \Gamma$ is a totally disconnected compact Hausdorff space.

Proposition 6.3.6 *1) If $p' : X \to \prod\limits_{\Gamma \in \mathcal{F}_2} \Gamma$ is injective, X is totally disconnected. If X is a totally disconnected compact Hausdorff space, p' is injective.*

2) The following conditions are equivalent.

i) X is a totally disconnected compact Hausdorff space.
ii) $p : X \to \varprojlim_{\Gamma \in \mathcal{F}} \Gamma$ is a homeomorphism.
iii) X is a limit of an inverse system of finite discrete spaces.

Proof. 1) Suppose that p' is injective. If $x \neq y$ for $x, y \in X$, $q_\Gamma(x) \neq q_\Gamma(y)$ for some $\Gamma \in \mathcal{F}_2$. This mean that there exists $C \in \Delta$ such that $x \in C$ and $y \in X - C$. Hence x and y belong different component. Assume that X is a totally disconnected compact Hausdorff space. If $x \neq y$, there exists $C \in \Delta$ such that $x \in C$ and $y \in X - C$ by (6.3.1). Thus $p'(x) \neq p'(y)$.

2) *i)* $\Rightarrow$ *ii)*; There is a continuous map $r : \varprojlim_{\Gamma \in \mathcal{F}} \Gamma \to \prod\limits_{\Gamma \in \mathcal{F}_2} \Gamma$ such that $\mathrm{pr}_\Gamma r = p_\Gamma$. Then, $rp = p'$. Since p' is injective by 1), so is p. Take $(C_\Gamma)_{\Gamma \in \mathcal{F}} \in \varprojlim_{\Gamma \in \mathcal{F}} \Gamma$. Since $\Gamma \leq \Gamma'$ implies $C_{\Gamma'} \subset C_\Gamma$, the intersection of finitely many C_Γ's is not empty. By the compactness of X, $\bigcap_{\Gamma \in \mathcal{F}} C_\Gamma$ is not empty. If $x \in \bigcap_{\Gamma \in \mathcal{F}} C_\Gamma$, $p(x) = (C_\Gamma)_{\Gamma \in \mathcal{F}}$. Therefore p is a continuous bijection from a compact space to a Hausdorff space.

ii) $\Rightarrow$ *iii)* is obvious.

iii) $\Rightarrow$ *i)*; Suppose that X is a limit of a functor D from a small category $\mathcal{D}$ to the category of topological spaces, where $D(i)$ is a finite discrete space for each $i \in \operatorname{Ob}\mathcal{D}$. Since the inclusion morphism $e : \varprojlim_{i \in \operatorname{Ob}\mathcal{D}} D(i) \to \prod\limits_{i \in \operatorname{Ob}\mathcal{D}} D(i)$ is an equalizer of continuous maps $\alpha, \beta : \prod\limits_{i \in \operatorname{Ob}\mathcal{D}} D(i) \to \prod\limits_{f \in \operatorname{Mor}\mathcal{D}} D(\operatorname{codom}(f))$ defined by $\mathrm{pr}_f \alpha = \mathrm{pr}_{\operatorname{codom}(f)}$, $\mathrm{pr}_f \beta = D(f)\mathrm{pr}_{\operatorname{dom}(f)}$ and $\prod\limits_{i \in \operatorname{Ob}\mathcal{D}} D(i)$ is a Hausdorff space , $\varprojlim_{i \in \operatorname{Ob}\mathcal{D}} D(i)$ is a closed subset of $\prod\limits_{i \in \operatorname{Ob}\mathcal{D}} D(i)$. Moreover, $\prod\limits_{i \in \operatorname{Ob}\mathcal{D}} D(i)$ is totally disconnected and compact. Thus $\varprojlim_{i \in \operatorname{Ob}\mathcal{D}} D(i)$ is a totally disconnected compact Hausdorff space. $\square$

A topological space which is a limit of an inverse system of finite discrete spaces is called pro-finite. The above result shows that a topological space is pro-finite if and only if it is a totally disconnected compact Hausdorff space

Proposition 6.3.7 *Let G be a totally disconnected locally compact topological group. For any open neighborhood U of the unit e of G, there is a compact open subgroup H contained in U.*

Proof. Since $\{e\}$ is a connected component of G, there is a compact open neighborhood P of e contained in U by (6.3.2). We put $Q = \{g \in G | \, Pg \subset P\}$. For $g \in Q$ and $x \in P$, since $xg \in P$ and P is open, there are neighborhoods U_x, V_x of x, g such that $U_x V_x \subset P$. Cover P by U_x's. By the compactness of P, there are $x_1, x_2, \ldots, x_n \in P$ such that $P \subset \bigcup_{i=1}^n U_{x_i}$. Put $V = \bigcap_{i=1}^n V_{x_i}$, then V is a neighborhood of g and $U_{x_i} V \subset P$ for $i = 1, 2, \ldots, n$. Hence $PV \subset P$, which implies $V \subset Q$. Therefore Q is open. Suppose $h \in G - Q$. There is $p \in P$ such that $ph \in G - P$. Since $G - P$ is open, there exists a neighborhood W of h such that $pW \subset G - P$. Then, $W \subset G - Q$ and it follows that $G - Q$ is open.

Since $e \in P$, $g = eg \in P$ for any $g \in Q$. Hence $Q \subset P$. Moreover, since $Pe = P \subset P$, $e \in Q$. Thus Q is a compact open neighborhood of e contained in P. Put $H = Q \cap Q^{-1}$. H is also a compact open neighborhood of e contained in P. For $u, v \in H$, then $u, v^{-1} \in Q$ and we have $P(uv^{-1}) = (Pu)v^{-1} \subset Pv^{-1} \subset P$. Hence $uv^{-1} \in Q$. Similarly, $(uv^{-1})^{-1} = vu^{-1} \in Q$. Therefore $uv^{-1} \in H$ and H is a subgroup of G. $\square$

Note that if H is an open subgroup of G, the quotient space G/H is a discrete space.

Proposition 6.3.8 *Let G be a totally disconnected compact topological group. For any open neighborhood U of the unit e of G, there is a compact open normal subgroup N contained in U.*

Proof. There is a compact open subgroup H contained in U by (6.3.7). We put $N = \bigcap_{x \in G} x^{-1} H x$. Clearly, N is a closed normal subgroup of G. For $x \in G$, since $x^{-1} e x = e \in H$ and H is open, there are neighborhoods U_x, V_x of e, x such that $V_x^{-1} U_x V_x \subset H$. Cover G by V_x's. By the compactness of G, there are $x_1, x_2, \ldots, x_n \in G$ such that $G = \bigcup_{i=1}^n V_{x_i}$. Put $U = \bigcap_{i=1}^n U_{x_i}$, then U is a neighborhood of e and $x^{-1} U x \subset H$ for any $x \in G$. Hence $U \subset N$ and it follows that $Uy \subset N$ for any $y \in N$. Therefore N is open. $\square$

If N is an open normal subgroup of a compact group G, the quotient group G/H is a finite discrete group.

Corollary 6.3.9 *Let $\mathcal{N}$ be the set of all compact open normal subgroups of a totally disconnected compact topological group G. If $N \subset L$ for $N, L \in \mathcal{N}$, we denote by $\pi^N_L : G/N \to G/L$ and $\lambda_N : G \to G/N$ the canonical projections. Then, $(G \xrightarrow{\lambda_N} G/N)_{N\in\mathcal{N}}$ is a limiting cone of the inverse system $(G/N, \pi^N_L)_{N,L\in\mathcal{N}}$. In other words, the map $\lambda : G \to \varprojlim_{N\in\mathcal{N}} G/N$ given by $\lambda(g) = (\lambda_N(g))_{N\in\mathcal{N}}$ is an isomorphism.*

Proof. Since $\bigcap_{N\in\mathcal{N}} N = \{e\}$ by (6.3.8), λ is injective. For $(\nu_N)_{N\in\mathcal{N}} \in \varprojlim_{N\in\mathcal{N}} G/N$, choose $g_N \in G$ such that $\lambda_N(g_N) = \nu_N$ for each $N \in \mathcal{N}$. Then, $g_N N$ is a closed subset and if $N \subset L$, $g_N N \subset g_L L$. Hence, for $N_1, N_2, \ldots, N_k \in \mathcal{F}$, $\bigcap_{i=1}^k g_{N_i} N_i \supset g_M M \neq \phi$, where $M = \bigcap_{i=1}^k N_i$. By the compactness of G, $\bigcap_{N\in\mathcal{N}} g_N N$ is not empty. For $g \in \bigcap_{N\in\mathcal{N}} g_N N$, we have $gN = g_N N$ for any $N \in \mathcal{N}$. Thus $\lambda(g) = (\nu_N)_{N\in\mathcal{N}}$ and λ is surjective. Since G is compact and $\varprojlim_{N\in\mathcal{N}} G/N$ is Hausdorff, λ is a homeomorphism. □

Corollary 6.3.10 *Let $f : G \to H$ be a homomorphism of topological groups which induces an isomorphism of topological groups $G/\mathrm{Ker}\, f \to \mathrm{Im}\, f$ (for example, G is compact) and U an open subgroup of G containing $\mathrm{Ker}\, f$. If H is a totally disconnected locally compact (resp. compact) topological group, there exists an open (resp. open normal) subgroup V of H such that $f^{-1}(V) \subset U$.*

Proof. We denote by $\bar{f} : G \to \mathrm{Im}\, f$ the surjection induced by f. It follows from $U \supset \mathrm{Ker}\, f$ that $\bar{f}^{-1}(f(U)) = f^{-1}(f(U)) = U$. Since $\bar{f}$ is a quotient map by the assumption, $f(U)$ is an open set in $\mathrm{Im}\, f$. There is an open set O of H such that $f(U) = O \cap \mathrm{Im}\, f$. If H is a totally disconnected locally compact (resp. compact) topological group, there is an open (resp. open normal) subgroup V of H such that $V \subset O$ by (6.3.7) (resp. (6.3.8)). Hence $V \cap \mathrm{Im}\, f \subset f(U)$ and this implies $f^{-1}(V) = f^{-1}(V \cap \mathrm{Im}\, f) \subset f^{-1}(f(U)) = U$. □

Lemma 6.3.11 *Let G be a topological group and X a left G-space. If O is an open set of X and C is a compact subset of X contained in O, $\{g \in G |\, gC \subset O\}$ is an open neighborhood of the unit e of G.*

Proof. Let us denote by $\alpha : G \times X \to X$ the left G-action on X. We put $Z = \{g \in G |\, gC \subset O\}$. Since $eC = C \subset O$, $e \in Z$. Suppose that $g_0 \in Z$, in other words, $\{g_0\} \times C \subset \alpha^{-1}(O)$. For each $x \in C$, we choose neighborhoods U_x, V_x of g, x such that $U_x \times V_x \subset \alpha^{-1}(O)$. Cover C by V_x's. There exist $x_1, x_2, \ldots, x_n \in C$ such that $C \subset \bigcup_{i=1}^n V_{x_i}$. Put $U = \bigcap_{i=1}^n U_{x_i}$, $V = \bigcup_{i=1}^n V_{x_i}$. Then, U is a neighborhood of g_0 and $U \times V \subset \alpha^{-1}(O)$. Hence $U \subset Z$ and it follows that Z is an open set of G. □

Proposition 6.3.12 *Let G be a totally disconnected compact topological group and X a left G-space. If $C_1, C_2, \ldots, C_n$ are compact subsets of X and $O_1, O_2, \ldots, O_n$ are open sets of X such that $C_i \subset O_i$ ($i = 1, 2, \ldots, n$), there exists a compact open normal subgroup N of G such that $NC_i \subset O_i$.*

Proof. By (6.3.11), there is an open neighborhoods U_i of e such that $U_i C_i \subset O_i$. It follows from (6.3.8) that there is a compact open normal subgroup N_i contained in U_i. Set $N = \bigcap_{i=1}^n N_i$. □

Corollary 6.3.13 *Let G be a totally disconnected compact topological group and X a left G-space which is a totally disconnected compact Hausdorff space. For each $\Gamma \in \mathcal{F}$, there exists a compact open normal subgroup N of G such that $NC = C$ for any $C \in \Gamma$.*

Let G be a topological group and X a left G-space. An element $\Gamma \in \mathcal{F}$ is said to be G-stable if Γ has a left G-action such that $q_\Gamma : X \to \Gamma$ is G-equivariant. That is, for $g \in G$ and $C \in \Gamma$, gC is contained some $D \in \Gamma$. We define a left G-action $G \times \mathcal{F} \to \mathcal{F}$ by $g\Gamma = \{gC |\, C \in \Gamma\}$. Then, Γ is G-stable if and only if Γ is a fixed point of this action.

Proposition 6.3.14 *Let G and X be as in (6.3.13). The set of G-stable elements in $\mathcal{F}$ is cofinal.*

Proof. For $\Gamma = \{C_1, C_2, \ldots, C_n\} \in \mathcal{F}$, we take a compact open normal subgroup N of G such that $NC_i = C_i$ for any $i = 1, 2, \ldots, n$ and choose representatives $g_1, g_2, \ldots, g_m$ from each class of G/N. Put $\Gamma_1 = \{g_j C_i |\, 1 \leqq i \leqq n,\ 1 \leqq j \leqq m\}$ and $\Gamma_2 = \{\bigcap_{k=1}^l D_k |\, D_k \in \Gamma_1\} - \{\phi\}$. Then, Γ_2 is a finite set of non-empty compact open sets of X. Let Γ_3 be the set of all minimal elements of Γ_2.

First, we show that $\Gamma_3 \in \mathcal{F}$. Suppose that $E_1 \cap E_2$ is not empty for $E_1, E_2 \in \Gamma_3$. Then, $E_1 \cap E_2 \in \Gamma_2$. Since $E_1 \cap E_2 \subset E_1, E_2$ and E_1, E_2 are minimal, it follows that $E_1 = E_2$. For $x \in X$ and j ($1 \leqq j \leqq m$), there is a unique $i(j)$ ($1 \leqq i(j) \leqq n$) such that $x \in g_j C_{i(j)}$. Then $x \in \bigcap_{j=1}^m g_j C_{i(j)}$ and $\bigcap_{j=1}^m g_j C_{i(j)} \in \Gamma_2$. If $t \neq i(s)$, then $g_s C_{i(s)} \cap g_s C_t = \phi$ and it follows that $g_s C_t \cap (\bigcap_{j=1}^m g_j C_{i(j)})$ is empty. Hence $\bigcap_{j=1}^m g_j C_{i(j)} \in \Gamma_3$.

Suppose that $\bigcap_{k=1}^{l} D_k \in \Gamma_3$. Since $\bigcup_{i=1}^{n} C_i = X$, $C_i \cap (\bigcap_{k=1}^{l} D_k)$ is not empty for some i. Note that $C_i \cap (\bigcap_{k=1}^{l} D_k)$ is contained in $\bigcap_{k=1}^{l} D_k$ and it belongs to Γ_2. Hence $C_i \cap (\bigcap_{k=1}^{l} D_k) = \bigcap_{k=1}^{l} D_k$, namely $\bigcap_{k=1}^{l} D_k \subset C_i$. Thus Γ_3 refines Γ.

Finally, we show that Γ_3 is G-stable. For $g \in G$ and j $(1 \leqq j \leqq m)$, $gg_j = g_k h$ for some k $(1 \leqq k \leqq m)$ and $h \in N$. Since $hC_i = C_i$ by the choice of N, we have $gg_jC_i = g_kC_i$ for any i $(1 \leqq i \leqq n)$. Hence Γ_1 is closed under the G-action and so is Γ_2. Since the action of $g \in G$ on Γ_2 is an isomorphism of ordered sets, the set of minimal elements Γ_3 is invariant under this action. □

Proposition 6.3.15 *Let G be a topological group and S a discrete space. A left G-action $\alpha : G \times S \to S$ is continuous if and only if there is an open normal subgroup N of G such that the restriction of the G-action to N is S trivial.*

Proof. Suppose that α is continuous. Note that the space $\mathrm{Map}(S, S)$ of all continuous maps from S to S is a discrete space with respect to the compact-open topology. Since the adjoint $\bar{\alpha} : G \to \mathrm{Map}(S, S)$ is a continuous homomorphism of monoids, $\bar{\alpha}^{-1}(id_S)$ is an open normal subgroup of G. Conversely, assume that there is an open normal subgroup N of G such that the restriction of the G-action to N is S trivial. We denote by $\pi : G \to G/N$ the projection. Then, α factors through $\pi \times id_S : G \times N \to G/N \times S$. Since G/N is discrete, the map $\alpha' : G/N \times S \to S$ induced by α is continuous. Hence $\alpha = \alpha'(\pi \times id_S)$ is continuous. □

Let G and X be as in (6.3.13). We denote by $\mathcal{F}_G$ the set of G-stable elements of $\mathcal{F}$. For $\Gamma \in \mathcal{F}_G$, since there is a compact open normal subgroup N of G such that $NC = C$ for any $C \in \Gamma$, the left G-action $G \times \Gamma \to \Gamma$ $(g, C) \mapsto gC$ is continuous by (6.3.15). Thus each element of $\mathcal{F}_G$ is a left G-space. Moreover, if $\Gamma \leq \Gamma'$, the transition map $\rho_\Gamma^{\Gamma'} : \Gamma' \to \Gamma$ is a continuous G-equivariant map. In other words, $(\Gamma, \rho_\Gamma^{\Gamma'})_{\Gamma, \Gamma' \in \mathcal{F}_G}$ is a projective system in the category of left G-spaces. By (6.3.6) and (6.3.14), we have the following result.

Proposition 6.3.16 *Let G and X be as in (6.3.13). Then, $(X \xrightarrow{q_\Gamma} \Gamma)_{\Gamma \in \mathcal{F}_G}$ is a limiting cone of the projective system $(\Gamma, \rho_\Gamma^{\Gamma'})_{\Gamma, \Gamma' \in \mathcal{F}_G}$ in the category of left G-spaces.*

Corollary 6.3.17 *Let G be as in (6.3.13) and H a closed subgroup of G different from G. For any $g \in G - H$, there is an open subgroup U such that $H \subset U \subset G - \{g\}$. Hence H is the intersection of open subgroups of G containing H.*

Proof. By (6.3.4), the quotient space G/H is a totally disconnected compact Hausdorff left G-space. Let $\rho : G \to G/H$ be the quotient map. Then, $(G/H \xrightarrow{q_\Gamma} \Gamma)_{\Gamma \in \mathcal{F}_G}$ is a limiting cone of the projective system $(\Gamma, \rho_\Gamma^{\Gamma'})_{\Gamma, \Gamma' \in \mathcal{F}_G}$ in the category of left G-spaces. Since $\rho(g) \neq \rho(e)$, there exists $\Gamma \in \mathcal{F}_G$ such that $q_\Gamma(g) \neq q_\Gamma(e)$. Set $U = \rho^{-1} q_\Gamma^{-1}(q_\Gamma \rho(e))$. Then U is an open subgroup of G such that $H \subset U \subset G - \{g\}$. □

Lemma 6.3.18 *Let D be a pro-object with domain $\mathcal{D}$ taking values in the category of non-empty compact Hausdorff spaces. Suppose that $\mathcal{D}$ is a directed set. Then, $\varprojlim_n D_n$ is not empty and, for each $i \in \mathrm{Ob}\,\mathcal{D}$, the image of the canonical projection $\pi_i : \varprojlim_n D_n \to D_i$ coincides with $\bigcap_{j \geq i} \tau_i^j(D_j)$, where $\tau_i^j : D_j \to D_i$ $(j \geq i)$ denotes the transition map. In particular, if D is strict, π_i is surjective.*

Proof. For $j \geq i$, put $X_j = \tau_i^j(D_j)$. Then, X_j is compact hence closed in D_i and $X_k \subset X_j$ if $k \geq j$. It follows that the intersection of finite number of elements of $\{X_j | \, j \geq i\}$ is not empty. By the compactness of D_i, $\bigcap_{j \geq i} X_j$ is not empty. We take $y \in \bigcap_{j \geq i} X_j$. For $j \geq i$, we set $L_j = \{(x_n)_{n \in \mathrm{Ob}\,\mathcal{D}} \in \prod_{n \in \mathrm{Ob}\,\mathcal{D}} D_n | \, x_i = y$ *and* $\tau_k^l(x_l) = x_k$ *for* $k \leq l \leq j\}$. Since $\prod_{n \in \mathrm{Ob}\,\mathcal{D}} D_n$ is a Hausdorff space, L_j is a closed subset of $\prod_{n \in \mathrm{Ob}\,\mathcal{D}} D_n$. Choose $x_n \in D_n$ for each $n \not\leq j$ and $x \in (\tau_i^j)^{-1}(y)$. Set $x_n = \tau_n^j(x)$ for $n \leq j$. Then, $(x_n)_{n \in \mathrm{Ob}\,\mathcal{D}} \in L_j$ hence L_j is not empty. It is obvious that $L_j \supset L_m$ if $j \leq m$. It follows that the intersection of finite number of elements of $\{L_j | \, j \geq i\}$ is not empty. By the compactness of $\prod_{n \in \mathrm{Ob}\,\mathcal{D}} D_n$, $\bigcap_{j \geq i} L_j$ is not empty. $(x_n)_{n \in \mathrm{Ob}\,\mathcal{D}} \in \prod_{n \in \mathrm{Ob}\,\mathcal{D}} D_n$ belongs to $\bigcap_{j \geq i} L_j$ if and only if $\tau_k^l(x_l) = x_k$ holds for any $k \leq l$ and $x_i = y$. Therefore $\bigcap_{j \geq i} L_j = \pi_i^{-1}(y) \subset \varprojlim_n D_n$ and this implies that $\pi_i(\varprojlim_n D_n) \supset \bigcap_{j \geq i} X_j$. On the other hand, $\pi_i(\varprojlim_n D_n) \subset \bigcap_{j \geq i} X_j$ is clear. Thus we have $\pi_i(\varprojlim_n D_n) = \bigcap_{j \geq i} X_j \neq \phi$. □

Let G be a pro-finite group. We denote by B_cG the category of left G-spaces whose underlying spaces are totally disconnected compact Hausdorff spaces which is $\mathcal{U}$-small and B_fG denotes a full subcategory of B_cG consisting of finite discrete left G-spaces. The inclusion functor is denoted by $\iota : B_fG \to B_cG$. Clearly B_cG is $\mathcal{U}$-complete. Hence we can consider the functor $\bar{\iota} : \mathrm{Pro}(B_fG) \to B_cG$ as in (6.1.8).

Theorem 6.3.19 *For a pro-finite group G, $\bar{\iota} : \mathrm{Pro}(B_fG) \to B_cG$ is an equivalence of categories.*

Proof. We verify the conditions (i), (ii), (iii) of (6.1.8) for the inclusion functor $\iota : B_fG \to B_cG$.

Let $D : \mathcal{D}^{op} \to B_fG$ be a pro-object and X an object of B_fG. Clearly, B_fG satisfies the conditions of (6.1.17). There exists a strict pro-object $E : \mathcal{E}^{op} \to B_fG$ such that E is isomorphic to D and $\mathcal{E}$ is a $\mathcal{U}$-small directed set. By (6.3.18), the canonical projection map $\rho_k : \varprojlim_j E_j \to E_k$ is surjective. Put $I = \mathrm{Ob}\,\mathcal{E}$ and denote by Σ the set of all finite subset of I. For $J \in \Sigma$ and $(x_j)_{j\in J} \in \prod\limits_{j\in J} E_j$, we put $O(J;(x_j)_{j\in J}) = (\varprojlim_j E_j)\cap(\bigcap_{j\in J} \mathrm{pr}_j^{-1}(x_j))$, where $\mathrm{pr}_j : \prod\limits_{i\in I} E_i \to E_j$ the projection onto the j-th component. Then, $\{O(J;(x_j)_{j\in J})|\, J \in \Sigma,\ (x_j)_{j\in J} \in \prod\limits_{j\in J} E_j\}$ is a basis of open sets of $\varprojlim_j E_j$. Let $\varphi : \bar{\iota}(E) \to X$ be a morphism in B_cG. For each $x \in \varphi(\bar{\iota}(E))$, $\varphi^{-1}(x)$ is a non-empty closed and open subset of $\bar{\iota}(E) = \varprojlim_j E_j$. Hence there exists a finite covering $\{O(J(x,s);(z_j)_{j\in J(x,s)})|\, 1 \leqq s \leqq n_x\}$ of $\varphi^{-1}(x)$. Since $\mathcal{E}$ is a directed set, there exists $k \geq j$ for any $j \in \bigcup_{x\in\varphi(\bar{\iota}(E))}\bigcup_{1\leqq s\leqq n_x} J(x,s)$. Suppose that $\rho_k((u_j)_{j\in I}) = \rho_k((v_j)_{j\in I})$ for $(u_j)_{j\in I}, (v_j)_{j\in I} \in \varprojlim_j E_j$. Then, $u_k = v_k$ and it follows that $u_j = v_j$ for any $j \leq k$. Put $x = \varphi((u_j)_{j\in I})$ and $y = \varphi((v_j)_{j\in I})$. Then, $(u_j)_{j\in I} \in O(J(x,s);(z_j)_{j\in J(x,s)})$ and $(v_j)_{j\in I} \in O(J(y,t);(w_j)_{j\in J(y,t)})$ for some $1 \leqq s \leqq n_x$, $1 \leqq t \leqq n_y$. Hence $z_j = u_j = v_j = w_j$ for $j \in J(x,s)\cap J(y,t)$ and $O(J(x,s);(z_j)_{j\in J(x,s)}) \cap O(J(y,t);(w_j)_{j\in J(y,t)})$ is not empty. Since $O(J(x,s);(z_j)_{j\in J(x,s)}) \subset \varphi^{-1}(x)$ and $O(J(y,t);(w_j)_{j\in J(y,t)}) \subset \varphi^{-1}(y)$, we have $x = y$. Therefore φ factors through a canonical map ρ_k. Clearly, the unique map $\bar{\varphi} : E_k \to X$ satisfying $\bar{\varphi}\rho_k = \varphi$ preserves the left G-action. We conclude that $(B_fG(E_j, X) \xrightarrow{\rho_j^*\iota} B_cG(\bar{\iota}(E), X))_{i\in I}$ is an epimorphic family. Since ρ_j is surjective, $\rho_j^*\iota : B_fG(E_j, X) \to B_cG(\bar{\iota}(E), X)$ is injective. Thus $(B_fG(E_j, X) \xrightarrow{\rho_j^*\iota} B_cG(\bar{\iota}(E), X))_{i\in I}$ is a limiting cone of h_XE. Since there is an isomorphism $D \to E$, it follows from (6.1.10) that $(B_fG(D_i, X) \xrightarrow{\pi_i^*\iota} B_cG(\bar{\iota}(D), X))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of h_XD.

Let Y be an object of B_cG. Since B_fG is a category with finite limits and $\iota : B_fG \to B_cG$ is left exact, $(Y{\downarrow}\iota)^{op}$ is filtered. Clearly, B_fG is equivalent to a small category. Hence $(Y{\downarrow}\iota)^{op}$ is essentially $\mathcal{U}$-small. The condition (iii) is a direct consequence of (6.3.16). □

For a closed subgroup H of G, let $\mathcal{N}_H$ be the set of open subgroups of G containing H. We define an order $\leq$ in $\mathcal{N}_H$ by "$K_1 \leq K_2 \Leftrightarrow K_1 \supset K_2$". Then, $\mathcal{N}_H$ is an directed set and we have a strict pro-object $P_H : \mathcal{N}_H^{op} \to B_fG$ defined by $P_H(K) = G/K$.

Proposition 6.3.20 *1) $(G/H \xrightarrow{p_K} G/K)_{K\in\mathrm{Ob}\,\mathcal{N}_H}$ is a limiting cone of $\iota P_H : \mathcal{N}_H \to B_cG$. Here, $p_K : G/H \to G/K$ denotes the quotient map. Hence $\bar{\iota}(P_H) = G/H$.*

2) For $X \in \mathrm{Ob}\,B_fG$, $L(P_H)(X) = X^H = \{x \in X|\, hx = x$ for any $h \in H\}$.

*3) $L(P_H) : B_fG \to \mathcal{U}$-**Ens** reflects initial objects if and only if H is the trivial subgroup $\{e\}$ of G.*

Proof. 1) By virtue of (6.3.16), it suffices to show that $\{G/K|\, K \in \mathrm{Ob}\,\mathcal{N}_H\} = \mathcal{F}_G$ for $X = G/H$. For $\Gamma \in \mathcal{F}_G$, since $q_\Gamma : G/H \to \Gamma$ is a surjective G-map, we have a surjective G-map $Q_\Gamma : G \to \Gamma$. Let K be the isotropy subgroup of $Q_\Gamma(e) \in \Gamma$. Then, $K \in \mathrm{Ob}\,\mathcal{N}_H$ and q_Γ induces an isomorphism $G/K \to \Gamma$ which can be regarded as the identity map. Hence $\Gamma \in \{G/K|\, K \in \mathrm{Ob}\,\mathcal{N}_H\}$.

2) For $K \in \mathrm{Ob}\,\mathcal{N}_H$, let $e_K : B_fG(G/K, X) \to X$ be the evaluation map at $K \in G/K$. Then, e_K is an injection whose image is X^K. In fact, since G acts on G/K transitively, e_K is injective. For $\varphi \in B_fG(G/K, X)$ and $g \in K$, $ge_K(\varphi) = g\varphi(K) = \varphi(gK) = \varphi(K) = e_K(\varphi)$. Hence $e_K(\varphi) \in X^K$. For $x \in X^K$, define $\varphi \in B_fG(G/K, X)$ by $\varphi(gK) = gx$. Then, φ is well-defined and $e_K(\varphi) = x$. Moreover, if $x \in X^H$, define a map $\psi : G \to X$ by $\psi(g) = gx$. Then, ψ is a continuous G-map. Hence the isotropy group K of x is an object of $\mathcal{N}_H$ and it follows that X^H is the union of X^K for $K \in \mathcal{N}_H$. Therefore $(B_fG(G/K, X) \xrightarrow{e_K} X^H)_{K\in\mathrm{Ob}\,\mathcal{N}_H}$ is a colimiting cone.

3) Suppose that $L(P_H)$ reflects initial objects and $H \neq \{s\}$. There is an open normal subgroup K of π such that $H \not\subset K$. Then, $L(P_H)(G/K) = (G/K)^H = \phi$ by 2). But G/K is not an initial object and this contradicts the assumption. Hence $H = \{e\}$. The converse is obvious. □

Lemma 6.3.21 *Let D be a pro-object with domain $\mathcal{D}$ taking values in the category of topological spaces. Suppose that $(L \xrightarrow{\lambda_i} D_i)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of D and $(X \xrightarrow{p_i} D_i)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a cone such that X is compact and each p_i is surjective. Then, the unique morphism $\rho : X \to L$ satisfying $\lambda_i\rho = p_i$ $(i \in \mathrm{Ob}\,\mathcal{D})$ is surjective.*

Proof. For $y \in L$ and $i \in \mathrm{Ob}\,\mathcal{D}$, we set $A_i = p_i^{-1}(\lambda_i(y))$. Then, A_i is a non-empty closed subset of X. If there is a morphism $\sigma : i \to j$ in $\mathcal{D}$, then $A_i \supset A_j$. In fact, for $z \in A_j$, $p_i(z) = D(\sigma)p_j(z) = D(\sigma)\lambda_j(y) = \lambda_i(y)$.

For $i_1, i_2, \ldots, i_n \in \mathrm{Ob}\,\mathcal{D}$, since $\mathcal{D}$ is filtered, there are morphisms $\sigma_\nu : i_\nu \to m$ for some $m \in \mathrm{Ob}\,\mathcal{D}$. Hence $A_1 \cap A_2 \cap \cdots \cap A_n \supset A_m \neq \phi$ and it follows from the compactness of X that $\bigcap_{i \in \mathrm{Ob}\,\mathcal{D}} A_i$ is not empty. If $x \in \bigcap_{i \in \mathrm{Ob}\,\mathcal{D}} A_i$, $\lambda_i \rho(x) = p_i(x) = \lambda_i(y)$ for any $i \in \mathrm{Ob}\,\mathcal{D}$ and this implies that $\rho(x) = y$. □

Let G be a pro-finite group. For closed subgroups H and K of G, put $M(H;K) = \{g \in G |\, g^{-1}Hg \subset K\}$ and define a map $\bar{\mu} : M(H;K) \to B_cG(G/H, G/K)$ by $\bar{\mu}(g)(hH) = hgK$. Then, $\bar{\mu}$ is surjective and $\bar{\mu}(g) = \bar{\mu}(g')$ if and only if $g^{-1}g' \in K$. Hence if we define an equivalence relation $\sim$ on $M(H;K)$ by "$g \sim g' \Leftrightarrow g^{-1}g' \in K$", $\bar{\mu}$ factors through the quotient map $M(H;K) \to M(H;K)/\!\sim$ and induces a bijection $\mu : M(H;K)/\!\sim \to B_cG(G/H, G/K)$.

Definition 6.3.22 *We define a category $C(G)$ as follows. The set objects of $C(G)$ consists of closed subgroups of G. The set of morphisms $C(G)(H,K)$ is defined to be $M(H;K)/\!\sim$. The composition $C(G)(H,K) \times C(G)(K,P) \to C(G)(H,P)$ is the map induced by $(g,h) \mapsto gh$.*

Note that every morphism in $C(G)$ is an epimorphism and that a morphism $\alpha : H \to K$ represented by $g \in G$ is an isomorphism if and only if $g^{-1}Hg = K$. In particular, $\mathrm{Aut}_{C(G)}(H) = N(H)/H$ and if H is a normal subgroup of G, $C(G)(H,H) = \mathrm{Aut}_{C(G)}(H) = G/H$. There is a fully faithful functor $\Psi : C(G) \to B_cG$ defined by $\Psi(H) = G/H$ and $\Psi(\alpha)(hH) = hgK$ for $\alpha \in C(G)(H,K)$ represented by $g \in G$. Moreover, we give $M(H;K) \subset G$ the induced topology and $C(G)(H,K)$ the quotient topology. Clearly, $M(H;K)$ is a closed subset of G on which K acts on the right. It follows from (6.3.3) that $C(G)(H,K)$ is a totally disconnected compact Hausdorff space.

Proposition 6.3.23 *For $X, Y \in \mathrm{Ob}\,B_cG$, we give $B_cG(X,Y)$ the compact-open topology.*

1) The composition $B_cG(X,Y) \times B_cG(Y,Z) \to B_cG(X,Z)$ is continuous.

2) Let $D : \mathcal{D} \to B_cG$ be a functor and $(L \xrightarrow{\pi_i} D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$ a limiting cone of D. Then, for $X \in \mathrm{Ob}\,B_cG$, $(B_cG(X,L) \xrightarrow{\pi_{i*}} B_cG(X, D(i)))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone in the category of topological spaces.

3) For $D, E \in \mathrm{Ob}\,\mathrm{Pro}(B_fG)$, $\bar{\iota} : \mathrm{Pro}(B_fG)(D,E) \to B_cG(\varprojlim_i \iota(D_i), \varprojlim_j \iota(E_j))$ is a homeomorphism.

4) For $H, K \in \mathrm{Ob}\,C(G)$, $\Psi : C(G)(H,K) \to B_cG(G/H, G/K)$ is a homeomorphism.

Proof. 1) Since X and Y are locally compact, this is a general property of compact-open topology.

2) Forgetting the G-actions, $(L \xrightarrow{\pi_i} D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$ a limiting cone in the category of topological spaces. Let $(Y \xrightarrow{\mu_i} B_cG(X, D(i)))_{i \in \mathrm{Ob}\,\mathcal{D}}$ be a cone in the category of topological spaces. Since X is locally compact, the adjoint $\mu_i' : Y \times X \to D(i)$ is continuous and $(Y \times X \xrightarrow{\mu_i'} D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a cone of D. Hence there exists a unique $f : Y \times X \to L$ such that $\pi_i f = \mu_i'$ for every $i \in \mathrm{Ob}\,\mathcal{D}$. For $(y,x) \in Y \times X$, $g \in G$ and $i \in \mathrm{Ob}\,\mathcal{D}$, $\pi_i f(y, gx) = \mu_i'(y, gx) = \mu_i(y)(gx) = g\mu_i(y)(x) = g\mu_i'(y,x) = g\pi_i f(y,x) = \pi_i(gf(y,x))$. Thus we have $f(y,gx) = gf(y,x)$ and the adjoint of f gives a map $f' : Y \to B_cG(X,L)$. Taking the adjoints of the both sides of $\pi_i f = \mu_i'$, we have $\pi_{i*} f' = \mu_i$.

3) Let $\mu_j : E \to \kappa(E_j)$ be the canonical morphism (6.1.5) and $\mu_j' : \varprojlim_j \iota(E_j) \to \iota(E_j)$ denote the canonical projection. Since $(\mathrm{Pro}(\mathcal{C})(D,E) \xrightarrow{\mu_{j*}} \mathrm{Pro}(\mathcal{C})(D, \kappa(E_j)))_{j \in \mathrm{Ob}\,\mathcal{E}}$ and $(B_cG(\varprojlim_i \iota(D_i), \varprojlim_j \iota(E_j)) \xrightarrow{\mu_{j*}'} B_cG(\varprojlim_i \iota D, \iota(E_j)))_{j \in \mathrm{Ob}\,\mathcal{E}}$ are limiting cones in the category of topological spaces and the following diagram commutes, it suffices to show that $\bar{\iota} : \mathrm{Pro}(B_fG)(D, \kappa(E_j)) \to B_cG(\varprojlim_i \iota(D_i), \iota(E_j))$ is a homeomorphism.

$$\begin{CD}
\mathrm{Pro}(B_fG)(D,E) @>{\bar{\iota}}>> B_cG(\varprojlim_i \iota(D_i), \varprojlim_j \iota(E_j)) \\
@V{\mu_{j*}}VV @VV{\mu_{j*}'}V \\
\mathrm{Pro}(B_fG)(D, \kappa(E_j)) @>{\bar{\iota}}>> B_cG(\varprojlim_i \iota(D_i), \iota(E_j))
\end{CD}$$

We have already seen in (6.3.20) that $\bar{\iota}$ is bijective. Since $\mathrm{Pro}(B_fG)(D, \kappa(E_j)) \cong L(D)(E_j)$ has the discrete topology, it suffices to show that $B_cG(X,Y)$ also has the discrete topology if $Y \in \mathrm{Ob}\,B_fG$. For any $\varphi \in B_cG(X,Y)$, since X is compact, $\varphi^{-1}(y)$ is closed, hence compact for any $y \in Y$. Then, since Y is a finite discrete space, $\{\varphi\}$ is the intersection of finitely many open subsets $\{\psi \in B_cG(X,Y) |\, \psi(\varphi^{-1}(y)) \subset \{y\}\}$ for $y \in Y$. Therefore $\{\varphi\}$ is open in $B_cG(X,Y)$.

4) Let us denote by $\rho : M(H;K) \to C(G)(H,K)$ and $p_H : G \to G/H$ the quotient maps. For a compact subset C of G/H and an open set O of G/K, set $W(C,O) = \{f \in B_cG(G/H, G/K) |\, f(C) \subset O\}$. Then, $\rho^{-1}\Psi^{-1}(W(C,O)) = \{g \in M(H;K) |\, p_H^{-1}(C)g \subset p_K^{-1}(O)\}$. Since G is compact, so is $p_H^{-1}(C)$. Hence

if $g \in \rho^{-1}\Psi^{-1}(W(C,O))$, there is an open neighborhood U of g such that $p_H^{-1}(C)U \subset p_K^{-1}(O)$. It follows that $\rho^{-1}\Psi^{-1}(W(C,O))$ is an open set of $M(H;K)$. Therefore $\Psi^{-1}(W(C,O))$ is an open set of $C(G)(H,K)$. Since $W(C,O)$'s generates the compact-open topology of $B_cG(G/H,G/K)$, Ψ is continuous. Obviously, $B_cG(G/H,G/K)$ is a Hausdorff space and the assertion follows from the compactness of $C(G)(H,K)$. □

Let $u : G' \to G$ be a continuous homomorphism between pro-finite groups. We define a functor $u^\sharp : B_fG \to B_fG'$ as follows. For a finite left G-space $(X, \alpha : G \times X \to X)$, $u^\sharp(X, \alpha : G \times X \to X) = (X, \alpha(u \times id_X) : G' \times X \to X)$ and, for a G-map $f : X \to Y$, $u^\sharp(f) = f$. Clearly, $u^\sharp$ is faithful, left exact and preserves finite colimits.

Proposition 6.3.24 *The following conditions are equivalent.*

i) $u : G' \to G$ is surjective.
ii) For any connected object X of B_fG, $u^\sharp(X)$ is connected.
iii) $u^\sharp$ is fully faithful.

Proof. $i) \Rightarrow iii)$; Since $u^\sharp$ is faithful, it suffices to show that $u^\sharp$ is full. Let X and Y be objects of B_fG and $\psi : u^\sharp(X) \to u^\sharp(Y)$ a morphism in B_fG'. We denote by $\alpha : G\times X \to X$ and $\beta : G\times Y \to Y$ the left G-actions on X and Y. We claim that ψ is also a morphism of left G-spaces. Since u is surjective, there exists $g' \in G'$ such that $u(g') = g$ for any $g \in G$. Then, for $x \in X$, $\psi\alpha(g,x) = \psi\alpha(u\times id)(g',x) = \beta(u\times id)(id\times\psi)(g',x) = \beta(id\times\psi)(g,x)$. Hence ψ is also a morphism of left G-spaces and it follows that $u^\sharp$ is full.

$ii)$ or $iii) \Rightarrow i)$; Assume that u is not surjective. Since G' is compact, $u(G')$ is a closed subgroup of G such that $u(G') \neq G$. By (6.3.17), there is an open subgroup U containing $u(G')$ such that $U \neq G$. Then, G/U has an element $hU \in G/U$ different from U and $U \in G/U = u^\sharp(G/U)$ is a fixed point of the left G'-action. Hence $u^\sharp(G/U)$ is not connected in B_fG' and this contradicts the condition $ii)$. Let O be the G'-orbit of $u^\sharp(G/U)$ containing hU. Then, $u^\sharp(G/U) = \{U\}\coprod O\coprod(G/U - \{U\} - O)$ as a left G'-space. Define a G'-map $f : u^\sharp(G/U) \to u^\sharp(G/U)$ by $f(x) = U$ if $x \in \{U\}\coprod O$ and $f(x) = x$ otherwise. Since $hf(U) = hU \neq U = f(hU)$, f is not a G-map. It follows that $u^\sharp$ is not full, which contradicts the condition $iii)$.

$i) \Rightarrow ii)$; Let X be a connected object of B_fG. If X is empty, so is $u^\sharp(X)$, hence $u^\sharp(X)$ is also connected. Suppose that X is not empty. Choose $a \in X$ and define a map $\rho : G \to X$ by $\rho(g) = ga$. Then, ρ is surjective and so is $\rho u : G' \to X$. It follows that $u^\sharp(X)$ is connected. □

6.4 Axioms of Galois category

Definition 6.4.1 *A Galois category is a category $\mathcal{C}$ satisfying the following conditions G1)∼G3) such that there exists a functor F from $\mathcal{C}$ to the category of finite sets satisfying the conditions G4)∼G6).*

G1) $\mathcal{C}$ has finite limits.

G2) Finite coproducts exists in $\mathcal{C}$ and, for each object X of $\mathcal{C}$ and a finite group G of automorphisms of X, the quotient object X/G exists in $\mathcal{C}$.

G3) Each morphism $f : X \to Y$ in $\mathcal{C}$ has a factorization $X \xrightarrow{p} Z \xrightarrow{i} Y$ such that p is a regular epimorphism and that there is an isomorphism $s : Z\coprod W \to Y$ for some $W \in \mathrm{Ob}\,\mathcal{C}$ satisfying $i = s\iota_1$, where $\iota_1 : Z \to Z\coprod W$ is the canonical morphism into the first summand.

G4) F is left exact.

G5) F preserves coproducts and regular epimorphisms and also preserves quotients by the finite group of automorphisms.

G6) F reflects isomorphisms.

The above F is called a fundamental functor. We fix a universe $\mathcal{U}$ such that $\mathcal{C}$ is $\mathcal{U}$-small.

Proposition 6.4.2 *Let $\mathcal{C}$ be a Galois category and $G : \mathcal{C} \to \mathcal{U}$-**Ens** a left exact functor.*

1) If G reflects isomorphisms, G preserves and reflects monomorphisms. Moreover, the canonical morphisms $X \to X\coprod Y$, $Y \to X\coprod Y$ are monomorphisms.

2) $\mathcal{C}$ is artinian.

3) G is strictly pro-representable by a $\mathcal{U}$-pro-object.

4) The initial object in $\mathcal{C}$ is strict. If G reflects isomorphisms and preserves initial objects, then G reflects initial objects.

Proof. We fix a fundamental functor F of $\mathcal{C}$.

1) Since G is left exact, it preserves monomorphism (A.3.2). Suppose that $f : X \to Y$ is a morphism in $\mathcal{C}$ such that $G(f)$ is a monomorphism. Consider the kernel pair $Z \underset{h}{\overset{g}{\rightrightarrows}} X$ of f. Since G is left exact, $G(Z) \underset{G(h)}{\overset{G(g)}{\rightrightarrows}} G(X)$ of is a kernel pair of $G(f)$. By the assumption, it follows from (A.3.2) that both $G(g)$ and $G(h)$ are isomorphisms. Hence, by the assumption, g is an isomorphism and f is an monomorphism by (A.3.2). The last assertion follows from $F(X \coprod Y) \cong F(X) \coprod F(Y)$ by G5).

2) For $X \in \mathrm{Ob}\,\mathcal{C}$, let $X_1 \supset X_2 \supset \cdots \supset X_i \supset X_{i+1} \supset \cdots$ be a descending chain of subobjects of X. Since F preserves monomorphisms, we have a chain of monomorphisms $F(X_1) \leftarrowtail F(X_2) \leftarrowtail \cdots \leftarrowtail F(X_i) \leftarrowtail F(X_{i+1}) \leftarrowtail \cdots$. We denote by $\iota_i : X_i \to X_{i+1}$ the inclusion morphism. Since F takes values in finite sets, there exists N such that $F(\iota_i)$ is bijective if $i \geqq N$. By G6), ι_i is an isomorphism if $i \geqq N$.

3) This follows from G1), 2) above and (6.1.16).

4) Let $f : X \to 0$ be a morphism to the initial object. Since $F(0)$ is empty by G5), $F(f) : F(X) \to F(0)$ is an isomorphism to the empty set. Hence f is an isomorphism by G6). Suppose that $G(X)$ is empty. Let $\varphi : 0 \to X$ be the unique morphism. Then, $G(\varphi)$ is an isomorphism thus so is φ. □

Proposition 6.4.3 *Let $\mathcal{C}$ be a Galois category.*

1) Let $f : X \to Y$ be a morphism such that X is not an initial object and Y is connected. Then, f is a regular epimorphism. An endomorphism of a connected object is an automorphism.

2) Let $D : \mathcal{D} \to \mathcal{C}$ be a functor such that $\mathcal{D}$ is a finite discrete category or a finite group (a category with a single object and finite number of invertible morphisms). Then, a colimit of D is universal, that is, finite coproducts and quotients by the finite group of automorphisms are universal. Moreover, finite coproducts in $\mathcal{C}$ are disjoint.

*3) For a functor $G : \mathcal{C} \to \mathcal{U}$-**Ens**, an object (X, ξ) of $\mathcal{C}_G$ is minimal if X is connected. If G is a left exact functor which preserves coproducts of two objects, then the converse holds.*

4) If $f : X \to Y$ is a regular epimorphism and X is connected, then Y is connected.

Proof. Let F be a fundamental functor of $\mathcal{C}$.

1) Consider the factorization $X \xrightarrow{p} Z \xrightarrow{i} Y$ of f as in G3). Since $F(X) \neq \phi$ by 4) of (6.4.2), $F(Z) \neq \phi$ hence Z is not an initial object. It follows from G3) and connectivity of Y that i is an isomorphism. Therefore f is a regular epimorphism. Let $g : X \to X$ be an endomorphism on connected X. If X is an initial object, it is obvious that g is an isomorphism. Otherwise, g is an regular epimorphism. By G5), $F(g) : F(X) \to F(X)$ is a surjection between finite sets of the same cardinalities. Hence $F(g)$ is a bijection and G6) implies that g is an automorphism.

2) Let $(D(i) \xrightarrow{\lambda_i} Y)_{i \in \mathrm{Ob}\,\mathcal{D}}$ be a colimiting cone of D and $f : X \to Y$ a morphism in $\mathcal{C}$. We consider pull-backs of λ_i along f.

$$\begin{tikzcd} D_f(i) \arrow[r, "f_i"] \arrow[d, "\bar{\lambda}_i"] & D(i) \arrow[d, "\lambda_i"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} F(D_f(i)) \arrow[r, "F(f_i)"] \arrow[d, "F(\bar{\lambda}_i)"] & F(D(i)) \arrow[d, "F(\lambda_i)"] \\ F(X) \arrow[r, "F(f)"] & F(Y) \end{tikzcd}$$

For a morphism $\tau : i \to j$ in $\mathcal{D}$, there is a unique morphism $D_f(\tau) : D_f(i) \to D_f(j)$ satisfying $f_j D_f(\tau) = D(\tau) f_i$ and $\bar{\lambda}_j D_f(\tau) = \bar{\lambda}_i$. Thus we have a functor $D_f : \mathcal{D} \to \mathcal{C}$ and a cone $(D_f(i) \xrightarrow{\bar{\lambda}_i} X)_{i \in \mathrm{Ob}\,\mathcal{D}}$. By G2), there exists a colimiting cone $(D_f(i) \xrightarrow{\tilde{\lambda}_i} C)_{i \in \mathrm{Ob}\,\mathcal{D}}$ of D_f. Hence we have a unique morphism $\rho : C \to X$ such that $\rho\tilde{\lambda}_i = \bar{\lambda}_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. Since colimits are universal in the category of sets and the above right diagram is also a pull-back by G4), $(FD_f(i) \xrightarrow{F(\bar{\lambda}_i)} F(X))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of FD_f. On the other hand, $(FD_f(i) \xrightarrow{F(\tilde{\lambda}_i)} F(C))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is also a colimiting cone of FD_f by G5). Thus $F(\rho)$ is an isomorphism. Hence ρ is an isomorphism by G6).

For $Y_1, Y_2 \in \mathrm{Ob}\,\mathcal{C}$, let us denote by $\iota_i : Y_i \to Y_1 \coprod Y_2$ $(i = 1, 2)$ the canonical morphisms. We note that ι_1 and ι_2 are monomorphisms. Consider the pull-back of ι_2 along ι_1 and apply F to it.

$$\begin{tikzcd} Y_0 \arrow[r, "\iota_1'"] \arrow[d, "\iota_2'"] & Y_2 \arrow[d, "\iota_2"] \\ Y_1 \arrow[r, "\iota_1"] & Y_1 \coprod Y_2 \end{tikzcd} \qquad \begin{tikzcd} F(Y_0) \arrow[r, "F(\iota_1')"] \arrow[d, "F(\iota_2')"] & F(Y_2) \arrow[d, "F(\iota_2)"] \\ F(Y_1) \arrow[r, "F(\iota_1)"] & F(Y_1 \coprod Y_2) \end{tikzcd}$$

Since coproducts in the category of sets are disjoint, $F(Y_0)$ is empty by G4) and G5). Hence Y_0 is initial by 4) of (6.4.2).

3) Suppose that X is connected. Let $v : Z \to X$ be a monomorphism and χ an element of $G(Z)$ such that $G(v)(\chi) = \xi$. There is a factorization $Z \xrightarrow{p} Y \xrightarrow{i} X$ of v as in G3). Since X is connected, i is an isomorphism. Hence a regular epimorphism p is also a monomorphism and it is an isomorphism by (A.8.5).

Conversely, suppose that $(X, \xi) \in \operatorname{Ob} \mathcal{C}_G$ is minimal and $X = Y \coprod Z$ for some $Y, Z \in \operatorname{Ob} \mathcal{C}$. Let $\iota_1 : Y \to X$ be the canonical morphism. Since $G(X) \cong G(Y) \coprod G(Z)$, we may assume that there exists $\zeta \in G(Y)$ such that $G(\iota_1)(\zeta) = \xi$. ι_1 is a monomorphism by 1) of (6.4.2) and it follows that ι_1 is an isomorphism. Since finite coproducts in $\mathcal{C}$ is disjoint by 2), the unique morphism $0 \to Z$ is a pull-back of an isomorphism ι_1 along $\iota_2 : Z \to X$. Hence $0 \to Z$ is an isomorphism and Z is an initial object.

4) Suppose that $Y = Y_1 \coprod Y_2$ for some $Y_1, Y_2 \in \operatorname{Ob} \mathcal{C}$ and consider pull-backs of f along the canonical morphisms as above. Then, X is isomorphic to $X_1 \coprod X_2$. By the connectivity of X, we may assume that X_2 is an initial object. Note that $F(f)$ is surjective by G5), hence so is its pull-back $F(f_2)$. Since $F(X_2)$ is empty, so is $F(Y_2)$ and it follows from 4) of (6.4.2) that Y_2 is an initial object. □

Let $\mathcal{C}$ be a Galois category and $G : \mathcal{C} \to \mathcal{U}$-**Ens** a left exact functor. We define $\mathcal{D}_G$ to be the opposite category of a skeleton of the full subcategory of $\mathcal{C}_G$ consisting of minimal objects. $D_G : \mathcal{D}_G^{op} \to \mathcal{C}$ is the restriction of the functor $\mathcal{C}_G \to \mathcal{C}$ given by $(X, \xi) \mapsto X$. Set $I = \operatorname{Ob} \mathcal{D}_G$, $i = (D_{Gi}, \xi_i)$ for $i \in I$ and $\rho_i^j : D_{Gj} \to D_{Gi}$ denotes the transition morphism if $i \leq j$ in I. Then I is a $\mathcal{U}$-small directed set and, for $X \in \operatorname{Ob} \mathcal{C}$, the maps $\lambda_i^X : \mathcal{C}(D_{Gi}, X) \to F(X)$ defined by $\lambda_i^X(u) = F(u)(\xi_i)$ give a colimiting cone $(\mathcal{C}(D_{Gi}, X) \xrightarrow{\lambda_i^X} F(X))_{i \in I}$ of an inductive system $(\mathcal{C}(D_{Gi}, X), (\rho_i^j)^*)_{i,j \in I}$ which is natural in $X \in \operatorname{Ob} \mathcal{C}$. Note that λ_j^X is injective by (6.1.13). We call D_G the pro-object associated with G.

Proposition 6.4.4 *Let $G : \mathcal{C} \to \mathcal{U}$-**Ens** be a left exact functor preserving coproducts of two objects.*

1) D_{Gi} is connected and it is not an initial object for each $i \in \mathcal{D}_G$.

2) The transition morphisms $\rho_i^j : D_{Gj} \to D_{Gi}$ are regular epimorphisms.

3) For a regular epimorphism $\rho : D_{Gj} \to P$, there is an isomorphism $\varphi : D_{Gi} \to P$ for some $i \leq j$ such that $\rho = \varphi \rho_i^j$.

4) The following conditions on D_{Gi} are equivalent.

(i) The map $\lambda_i^{D_{Gi}} : \mathcal{C}(D_{Gi}, D_{Gi}) \to G(D_{Gi})$ is surjective.

(ii) The action of $\operatorname{Aut}(D_{Gi})$ on $G(D_{Gi})$ is transitive.

(iii) The action of $\operatorname{Aut}(D_{Gi})$ on $G(D_{Gi})$ is free and transitive.

Proof. 1) By 3) of (6.4.3), D_{Gi} is connected. If D_{Gi} is an initial object 0 for some $i \in I$, it follows from 4) of (6.4.2) that D_{Gj} is also an initial object for every $j \geq i$. Then, for any $X \in \operatorname{Ob} \mathcal{C}$, $\lambda_j^X : \mathcal{C}(D_{Gj}, X) \to G(X)$ is bijective for large enough j and $\mathcal{C}(D_{Gj}, X)$ consists of a single element. Hence both $G(X \coprod X)$ and $G(X)$ consists of a single element. However, we have $G(X \coprod X) \cong G(X) \coprod G(X)$ by the assumption. This is a contradiction.

2) This follows from 1) of (6.4.3) and 1) above.

3) Set $\xi = G(\rho)(\xi_j)$. By 4) of (6.4.3), P is connected, hence it follows from 3) of (6.4.3) that (P, ξ) is a minimal object in $\mathcal{C}_G$. Since $\{(D_{Gi}, \xi_i) | i \in I\}$ is the set of objects of a skeleton of a full subcategory of $\mathcal{C}_G$ consisting of minimal objects, there is an isomorphism $\varphi : (D_{Gi}, \xi_i) \to (P, \xi)$ in $\mathcal{C}_G$ for some $i \in I$. Recall from (6.1.13) that there is at most one morphism between minimal objects. Since $G(\varphi^{-1}\rho)(\xi_j) = \xi_i$, we have $i \leq j$ and $\varphi^{-1}\rho = \rho_i^j$.

4) By 1) of (6.4.3) and the connectivity of D_{Gi}, we have $\mathcal{C}(D_{Gi}, D_{Gi}) = \operatorname{Aut}(D_{Gi})$. Moreover, the map $\mathcal{C}(D_{Gi}, D_{Gi}) \to G(D_{Gi})$ is injective by (6.1.13). Hence the above three conditions are equivalent. □

The pro-object D_F associated with a fundamental functor F is called the fundamental pro-object. We say that D_{Fi} satisfying the one of the three conditions of 4) above is Galois. More generally, we say that an object X of $\mathcal{C}$ is Galois if X is connected and the action of $\operatorname{Aut}(X)$ on $F(X)$ is free and transitive.

We put $D = D_F$ and $D_i = D_{Fi}$ for short.

Proposition 6.4.5 *1) For $X \in \mathrm{Ob}\,\mathcal{C}$, there exists $i \in I$ such that D_i is Galois and $\lambda_i^X : \mathcal{C}(D_i, X) \to F(X)$ is bijective.*

2) We put $I_{\mathfrak{g}} = \{i \in I \mid D_i \text{ is Galois}\}$. Then $I_{\mathfrak{g}}$ is cofinal in I.

Proof. 1) Since $F(X)$ is a finite set, there exists $j \in I$ such that $\lambda_j^X : \mathcal{C}(D_j, X) \to F(X)$ is bijective. Set $S = F(X)$, $X_s = X$ for $s \in S$ and define a map $f : D_j \to \prod\limits_{s \in S} X_s$ by $\mathrm{pr}_s f = (\lambda_j^X)^{-1}(s)$. By G3), there is a factorization $D_j \xrightarrow{\rho} P \xrightarrow{\iota} \prod\limits_{s\in S} X_s$ such that ρ is a regular epimorphism and ι is a monomorphism. By 3) of (6.4.4), we may assume that $P = D_i$, $\rho = \rho_i^j$ for some $i \leq j$. For any $s \in S = F(X)$, we have $\lambda_i^X(\mathrm{pr}_s\iota) = F(\mathrm{pr}_s\iota)(\xi_i) = F(\mathrm{pr}_s\iota)F(\rho_i^j)(\xi_j) = F(\mathrm{pr}_s f)(\xi_j) = \lambda_j^X((\lambda_j^X)^{-1}(s)) = s$. It follows that $\lambda_i^X : \mathcal{C}(D_i, X) \to F(X)$ is surjective, hence bijective by (6.1.13).

We show that D_i is Galois. Choose $k \geq j$ such that $\lambda_k^{D_i} : \mathcal{C}(D_k, D_i) \to F(D_i)$ is bijective. For $x \in F(D_i)$, set $\varphi = (\lambda_k^{D_i})^{-1}(x) : D_k \to D_i$, then φ is a regular epimorphism by (6.4.3) and (6.4.4). Hence $\varphi^* : \mathcal{C}(D_i, X) \to \mathcal{C}(D_k, X)$ is injective. Define a map $\sigma : S \to S$ to be a composition $S = F(X) \xrightarrow{(\lambda_i^X)^{-1}} \mathcal{C}(D_i, X) \xrightarrow{\varphi^*} \mathcal{C}(D_k, X) \xrightarrow{\lambda_k^X} F(X) = S$. Since σ is a composition of injections and $F(X)$ is a finite set, σ is bijective. Let $\alpha : \prod\limits_{s\in S} X_s \to \prod\limits_{s\in S} X_s$ be the morphism defined by $\mathrm{pr}_s\alpha = \mathrm{pr}_{\sigma(s)}$. For each $s \in S$, $\mathrm{pr}_s\alpha\iota\rho_i^k = \mathrm{pr}_{\sigma(s)}\iota\rho_i^j\rho_j^k = \mathrm{pr}_{\sigma(s)}f\rho_j^k = (\lambda_j^X)^{-1}(\sigma(s))\rho_j^k = (\rho_j^k)^*(\lambda_j^X)^{-1}(\lambda_k^X\varphi^*(\lambda_i^X)^{-1}(s)) = (\lambda_k^X)^{-1}\lambda_k^X((\rho_i^j)^{*-1}(\lambda_j^X)^{-1}(s)\varphi) = (\rho_i^j)^{*-1}(\mathrm{pr}_s f)\varphi = (\rho_i^j)^{*-1}(\mathrm{pr}_s\iota\rho_i^j)\varphi = \mathrm{pr}_s\iota\varphi$. Thus $\alpha\iota\rho_i^k = \iota\varphi$. By (A.8.9), there is an isomorphism $u : D_i \to D_i$ such that $\varphi = u\rho_i^k$. Hence $\lambda_i^{D_i}(u) = \lambda_k^{D_i}(\rho_i^k)^*(u) = \lambda_k^{D_i}(u\rho_i^k) = \lambda_k^{D_i}(\varphi) = x$ and $\lambda_i^{D_i}$ is surjective.

2) For any $i \in I$, there exist $j \in I_{\mathfrak{g}}$ and $f : D_j \to D_i$ such that $\lambda_j^{D_i}(f) = \xi_i$ by 1). Then, $f : (D_j, \xi_j) \to (D_i, \xi_i)$ is a morphism in $\mathcal{C}_F$ and by the definition of the category $\mathcal{D}$, f is a morphism in $\mathcal{D}^{op}$, that is, $f = \rho_i^j$. $\square$

Let π_i be the opposite group of $\mathrm{Aut}(D_i) = \mathcal{C}(D_i, D_i)$. Since $\lambda_i^{D_i} : \mathcal{C}(D_i, D_i) \to F(D_i)$ is injective, π_i is a finite group. If $j \geq i$ and $i \in I_{\mathfrak{g}}$, then $\lambda_i^{D_i} : \mathcal{C}(D_i, D_i) \to F(D_i)$ is bijective and it follows that so are $\lambda_j^{D_i} : \mathcal{C}(D_j, D_i) \to F(D_i)$ and $(\rho_i^j)^* : \mathcal{C}(D_i, D_i) \to \mathcal{C}(D_j, D_i)$. Let $\varrho_i^j : \mathrm{Aut}(D_j) \to \mathrm{Aut}(D_i)$ be a composition $\mathrm{Aut}(D_j) = \mathcal{C}(D_j, D_j) \xrightarrow{(\rho_i^j)_*} \mathcal{C}(D_j, D_i) \xrightarrow{(\rho_i^j)^{*-1}} \mathcal{C}(D_i, D_i) = \mathrm{Aut}(D_i)$. Then, ϱ_i^j is a homomorphism of groups. If $i, j \in I_{\mathfrak{g}}$, ϱ_i^j is surjective. In fact, the following diagram commutes, where the vertical maps are bijective and $F(\rho_i^j)$ is surjective by 2) of (6.4.4) and G5).

$$\begin{tikzcd}
\mathcal{C}(D_j, D_j) \arrow[r, "(\rho_i^j)_*"] \arrow[d, "\lambda_j^{D_j}"] & \mathcal{C}(D_j, D_i) \arrow[d, "\lambda_j^{D_i}"] & \mathcal{C}(D_i, D_i) \arrow[l, "(\rho_i^j)^*"'] \arrow[dl, "\lambda_i^{D_i}"] \\
F(D_j) \arrow[r, "F(\rho_i^j)"] & F(D_i) &
\end{tikzcd}$$

Thus we have a strict projective system $(\pi_i, \varrho_i^j)_{i,j \in I_{\mathfrak{g}}}$ of finite groups and put $\pi = \pi_F = \varprojlim_{i \in I_{\mathfrak{g}}} \pi_i$. Give each π_i the discrete topology. Then, π is a pro-finite group and it is the opposite group of $\mathrm{Pro}(\mathcal{C})(D, D) = \mathrm{Aut}(D)$. In fact, $\mathrm{Pro}(\mathcal{C})(D, D) \cong \varprojlim_i L(D)(D_i)$ and $L(D)(D_i) \cong \mathcal{C}(D_i, D_i) = \mathrm{Aut}(D_i)$ if $i \in I_{\mathfrak{g}}$. Hence π is isomorphic to $\check{\mathcal{C}}(F, F) = \mathrm{Aut}(F)$.

We denote by $p_i : \pi \to \pi_i$ the canonical projection. For $i \in I$, define a map $q_i : \pi \to F(D_i)$ as follows. Choose $j \in I_{\mathfrak{g}}$ such that $j \geq i$. q_i is a composition $\pi \xrightarrow{p_j} \pi_j = \mathcal{C}(D_j, D_j) \xrightarrow{(\rho_i^j)_*} \mathcal{C}(D_j, D_i) \xrightarrow{\lambda_j^{D_i}} F(D_i)$. It is easy to see that q_i does not depend on the choice of j and that $(\pi \xrightarrow{q_i} F(D_i))_{i \in I}$ is a limiting cone of a projective system $(F(D_i), F(\rho_i^j))_{i,j \in I}$.

We note that, for $X \in \mathrm{Ob}\,\mathcal{C}$ and $i \in I$, $\mathcal{C}(D_i, X)$ has a left π_i-action $\bar{\alpha}_i : \pi_i \times \mathcal{C}(D_i, X) \to \mathcal{C}(D_i, X)$ given by $(g, f) \mapsto fg$. Hence if $\lambda_i^X : \mathcal{C}(D_i, X) \to F(X)$ is bijective, $F(X)$ has a left π_i-action $\tilde{\alpha}_i : \pi_i \times F(X) \to F(X)$ defined by

$$\pi_i \times F(X) \xrightarrow{id_{\pi_i} \times (\lambda_i^X)^{-1}} \pi_i \times \mathcal{C}(D_i, X) \xrightarrow{\bar{\alpha}_i} \mathcal{C}(D_i, X) \xrightarrow{\lambda_i^{X}{}''} F(X).$$

Moreover, if both $\lambda_i^X : \mathcal{C}(D_i, X) \to F(X)$ and $\lambda_i^Y : \mathcal{C}(D_i, Y) \to F(Y)$ are bijective, $F(f) : F(X) \to F(Y)$ commutes with the left π_i-actions for a morphism $f : X \to Y$ in $\mathcal{C}$ by the naturality of λ_i^X in X.

For each $X \in \mathrm{Ob}\,\mathcal{C}$, we define a left π action $\alpha_F : \pi \times F(X) \to F(X)$ as follows. Choose $i \in I_{\mathfrak{g}}$ such that $\lambda_i^X : \mathcal{C}(D_i, X) \to F(X)$ is bijective. α_F is a composition

$$\pi \times F(X) \xrightarrow{p_i \times id_{F(X)}} \pi_i \times F(X) \xrightarrow{\tilde{\alpha}_i} F(X).$$

It is easy to verify that α_F does not depend on the choice of i and that, for a morphism $f : X \to Y$, $F(f) : F(X) \to F(Y)$ is π-equivariant. Giving $F(X)$ the discrete topology, this action is continuous. Then F is regarded as a functor $\mathcal{C} \to B_f\pi$. We remark that, if we regard π as $\mathrm{Aut}(F)$, α_F is identified with the map given by $(g, x) \mapsto g_X(x)$.

We construct a left adjoint $G : B_f\pi \to \mathcal{C}$ of $F : \mathcal{C} \to B_f\pi$ below.

Let $\mathcal{A}$ be a category with finite coproducts, H a finite group, Y an object of $\mathcal{A}$ with a homomorphism $\mu : H \to \mathrm{Aut}(Y)^{op}$ and S a finite set with a left H-action. Here we denote by $\mathrm{Aut}(Y)^{op}$ the opposite group of $\mathrm{Aut}(Y)$. Put $Y_s = Y$ $(s \in S)$, $Y \times S = \coprod\limits_{s \in S} Y_s$ and $\iota_s : Y = Y_s \to Y \times S$ denotes the canonical morphism into the s-th summand. For $g \in H$, define $a_g : Y \times S \to Y \times S$ to be the morphism satisfying $a_g\iota_s = \iota_{g(s)}\mu(g^{-1})$. Then, $a_h a_g \iota_s = a_h \iota_{g(s)}\mu(g^{-1}) = \iota_{hg(s)}\mu(h^{-1})\mu(g^{-1}) = \iota_{hg(s)}\mu((hg)^{-1}) = a_{hg}\iota_s$. Hence we have $a_h a_g = a_{hg}$ and H acts on $Y \times S$ on the left. For $X \in \mathrm{Ob}\,\mathcal{A}$, $\mathcal{A}(Y, X)$ is a left H-set by $(g, f) \mapsto f\mu(g)$. Let us denote by $\mathrm{Map}_H(S, \mathcal{A}(Y, X))$ the set of H-equivariant maps.

Lemma 6.4.6 *Suppose that the quotient $Y \times_H S$ of $Y \times S$ by the left H-action exists in $\mathcal{A}$. Define a map $\theta : \mathcal{A}(Y \times_H S, X) \to \mathrm{Map}_H(S, \mathcal{A}(Y, X))$ by $\theta(\psi)(s) = \psi\nu\iota_s$, where $\nu : Y \times S \to Y \times_H S$ is the quotient map. Then, θ is bijective.*

Proof. Let $u : S \to \mathcal{A}(Y, X)$ be an H-equivariant map. Define $\tilde{\psi} : Y \times S \to X$ by $\tilde{\psi}\iota_s = u(s)$. Then, for $g \in H$ and $s \in S$, $\tilde{\psi}a_g\iota_s = \tilde{\psi}\iota_{g(s)}\mu(g^{-1}) = u(g(s))\mu(g^{-1}) = u(s)\mu(g)\mu(g^{-1}) = u(s) = \tilde{\psi}\iota_s$. Hence $\tilde{\psi}a_g = \tilde{\psi}$ for any $g \in H$ and there is a unique morphism $\psi_u : Y \times_H S \to X$ such that $\psi_u\nu = \tilde{\psi}$. The correspondence $u \mapsto \psi_u$ gives the inverse of θ. □

Let $f : Y \to Z$ be a morphism in $\mathcal{A}$ and $u : S \to T$ a map between finite sets. We denote by $f \times u : Y \times S \to Z \times T$ the morphism in $\mathcal{A}$ defined by $(f \times u)\iota_s = \iota_{u(s)}f$.

Consider the case $\mathcal{A} = \mathcal{C}$. $F(Y)$ has a left H-action induced by the left H-action on Y. We denote by $\bar{\iota}_s : F(Y) = F(Y_s) \to F(Y) \times S$ the canonical morphism into the s-th summand. It follows from G5) that the map $\bar{\eta} : F(Y) \times S \to F(Y \times S)$ defined by $\bar{\eta}\bar{\iota}_s = F(\iota_s)$ induces an isomorphism $\tilde{\eta} : F(Y) \times_H S \to F(Y \times_H S)$ such that $\tilde{\eta}\bar{\nu} = F(\nu)\bar{\eta}$, where $\bar{\nu} : F(Y) \times S \to F(Y) \times_H S$ denotes the quotient map. In particular, if the action of H on $F(Y)$ is transitive and free, the map $p_2 : F(Y) \times S \to S$ given by $p_2(F(Y_s)) = \{s\}$ induces a bijection $\bar{p}_2 : F(Y) \times_H S \to S$.

Proposition 6.4.7 *Let S be an object of $B_f\pi$ and choose $i \in I_{\mathfrak{g}}$ such that the left π-action on S factors through $p_i : \pi \to \pi_i$. If $j \in I_{\mathfrak{g}}$ and $j \geq i$, the morphism $\tilde{\rho}_i^j : D_j \times_{\pi_j} S \to D_i \times_{\pi_i} S$ induced by $\rho_i^j \times id_S$ is an isomorphism.*

Proof. Let $\bar{\rho}_i^j : F(D_j) \times_{\pi_j} S \to F(D_i) \times_{\pi_i} S$ be the map induced by $F(\rho_i^j) \times id_S$. Then, the following diagram commutes.

$$\begin{tikzcd}
S & F(D_j) \times_{\pi_j} S \arrow[l, "p_2"', "\cong"] \arrow[r, "\tilde{\eta}", "\cong"'] \arrow[d, "\bar{\rho}_i^j"] & F(D_j \times_{\pi_j} S) \arrow[d, "F(\tilde{\rho}_i^j)"] \\
& F(D_i) \times_{\pi_i} S \arrow[ul, "\cong", "p_2"'] \arrow[r, "\tilde{\eta}", "\cong"'] & F(D_i \times_{\pi_i} S)
\end{tikzcd}$$

Hence $F(\tilde{\rho}_i^j) : F(D_j \times_{\pi_j} S) \to F(D_i \times_{\pi_i} S)$ is bijective and the assertion follows from G6). □

For each object S of $B_f\pi$, choose $i(S) \in I_{\mathfrak{g}}$ such that the left π-action on S factors through $p_{i(S)} : \pi \to \pi_{i(S)}$. We set $G(S) = D_{i(S)} \times_{\pi_{i(S)}} S$. Let $u : S \to T$ be a morphism in $B_f\pi$. We choose $j(u) \in I_{\mathfrak{g}}$ such that $j_u \geq i(S), i(T)$. Then, u is regarded as a $\pi_{j(u)}$-equivariant map and so is the morphism $\bar{u} = id \times u : D_{j(u)} \times S \to D_{j(u)} \times T$. Hence we have a morphism $\tilde{u} : D_{j(u)} \times_{\pi_{j(u)}} S \to D_{j(u)} \times_{\pi_{j(u)}} T$ induced by $\bar{u}$. By the above result, $\tilde{\rho}_{i(S)}^{j(u)} : D_{j(u)} \times_{\pi_{j(u)}} S \to D_{i(S)} \times_{\pi_{i(S)}} S$ and $\tilde{\rho}_{i(T)}^{j(u)} : D_{j(u)} \times_{\pi_{j(u)}} T \to D_{i(T)} \times_{\pi_{i(T)}} T$ are isomorphisms. Define $G(u) : G(S) \to G(T)$ to be the following composition.

$$G(S) = D_{i(S)} \times_{\pi_{i(S)}} S \xrightarrow{(\tilde{\rho}_{i(S)}^{j(u)})^{-1}} D_{j(u)} \times_{\pi_{j(u)}} S \xrightarrow{\tilde{u}} D_{j(u)} \times_{\pi_{j(u)}} T \xrightarrow{\tilde{\rho}_{i(T)}^{j(u)}} D_{i(T)} \times_{\pi_{i(T)}} T = G(T)$$

It is easy to verify that $G(u)$ does not depend on the choice of $j(u)$ and that G is a functor.

Define natural transformations $\eta : id_{B_f\pi} \to FG$ and $\varepsilon : GF \to id_{\mathcal{C}}$ as follows. For $S \in \operatorname{Ob} B_f\pi$, $\eta_S : S \to F(D_{i(S)} \times_{\pi_{i(S)}} S) = FG(S)$ is a composition of isomorphisms $\bar{p}_2^{-1} : S \to F(D_{i(S)}) \times_{\pi_{i(S)}} S$ and $\tilde{\eta} : F(D_{i(S)}) \times_{\pi_{i(S)}} S \to F(D_{i(S)} \times_{\pi_{i(S)}} S)$. Explicitly, η_S is given by $\eta_S(s) = F(\nu\iota_s)(\xi_{i(S)}) = \lambda_{i(S)}^{G(S)}(\nu\iota_s)$. We have to verify that η_S is π-equivariant and natural in S. Choose $j \in I_{\mathfrak{g}}$ such that $j \geq i(S)$ and $\lambda_j^{G(S)} : \mathcal{C}(D_j, G(S)) \to FG(S)$ is bijective. Set $\varphi = (\lambda_j^{G(S)})^{-1}(\eta_S(s))$. Then $\lambda_j^{G(S)}(\varphi) = \eta_S(s) = \lambda_{i(S)}^{G(S)}(\nu\iota_s) = \lambda_j^{G(S)}(\nu\iota_s\rho_{i(S)}^j)$ and it follows that $(\lambda_j^{G(S)})^{-1}(\eta_S(s)) = \varphi = \nu\iota_s\rho_{i(S)}^j$. On the other hand, for $g \in \pi$, it follows from the definition of the $\pi_{i(S)}$-action on $D_{i(S)} \times S$ that $\nu\iota_s\rho_{i(S)}^j p_j(g) = \nu\iota_s p_{i(S)}(g)\rho_{i(S)}^j = \nu a_{p_{i(S)}(g^{-1})}\iota_{p_{i(S)}(g)(s)}\rho_{i(S)}^j = \nu\iota_{p_{i(S)}(g)(s)}\rho_{i(S)}^j$. Hence $g\eta_S(s) = \lambda_j^{G(S)}((\lambda_j^{G(S)})^{-1}(\eta_S(s))p_j(g)) = \lambda_j^{G(S)}(\nu\iota_s\rho_{i(S)}^j p_j(g)) = \lambda_j^{G(S)}(\nu\iota_{p_{i(S)}(g)(s)}\rho_{i(S)}^j)$
$= \lambda_{i(S)}^{G(S)}(\nu\iota_{p_{i(S)}(g)(s)}) = \eta_S(g(s))$. Let $u : S \to T$ be a morphism in $B_f\pi$. By the definition of $G(u)$, $FG(u)\eta_S(s) = FG(u)F(\nu\iota_s)(\xi_{i(S)}) = F(G(u)\nu\iota_s\rho_{i(S)}^{j(u)})(\xi_{j(u)}) = F(G(u)\nu(\rho_{i(S)}^{j(u)} \times id_S)\iota_s)(\xi_{j(u)}) = F(\nu(\rho_{i(T)}^{j(u)} \times u)\iota_s)(\xi_{j(u)}) = F(\nu\iota_{u(s)}\rho_{i(T)}^{j(u)})(\xi_{j(u)}) = F(\nu\iota_{u(s)})(\xi_{i(T)}) = \eta_T(u(s))$.

For $X \in \operatorname{Ob}\mathcal{C}$, we may assume that $\lambda_{i(F(X))}^X : \mathcal{C}(D_{i(F(X))}, X) \to F(X)$ is bijective. Define a morphism $\bar{\varepsilon} : D_{i(F(X))} \times F(X) \to X$ by $\bar{\varepsilon}\iota_s = (\lambda_{i(F(X))}^X)^{-1}(s)$. For $g \in \pi_{i(F(X))}$ and $s \in F(X)$, recall that $g(s) = \lambda_{i(F(X))}^X((\lambda_{i(F(X))}^X)^{-1}(s)\mu(g))$. Hence $\bar{\varepsilon}a_g\iota_s = \bar{\varepsilon}\iota_{g(s)}\mu(g^{-1}) = (\lambda_{i(F(X))}^X)^{-1}(g(s))\mu(g^{-1}) = (\lambda_{i(F(X))}^X)^{-1}(s) = \bar{\varepsilon}\iota_s$. Thus $\bar{\varepsilon}a_g = \bar{\varepsilon}$ for any $g \in \pi_{i(F(X))}$. $\bar{\varepsilon}$ factors through the quotient morphism $\nu : D_{i(F(X))} \times F(X) \to D_{i(F(X))} \times_{\pi_{i(F(X))}} F(X)$ and induces $\varepsilon_X : GF(X) = D_{i(F(X))} \times_{\pi_{i(F(X))}} F(X) \to X$. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. For $s \in F(X)$, set $\varphi = (\lambda_{i(F(X))}^X)^{-1}(s)$ and $\psi = (\lambda_{i(F(Y))}^Y)^{-1}(F(f)(s))$. Then, $\lambda_{j(F(f))}^Y(\psi\rho_{i(F(Y))}^{j(F(f))}) = \lambda_{i(F(Y))}^Y(\psi) = F(f)(s) = F(f)(\lambda_{i(F(X))}^X(\varphi)) = \lambda_{j(F(f))}^X(f\varphi\rho_{i(F(X))}^{j(F(f))})$ and we have $\psi\rho_{i(F(Y))}^{j(F(f))} = f\varphi\rho_{i(F(X))}^{j(F(f))}$. Hence $\varepsilon_Y GF(f)\nu\iota_s\rho_{i(F(X))}^{j(F(f))} = \varepsilon_Y GF(f)\nu(\rho_{i(F(X))}^{j(F(f))} \times id_{F(X)})\iota_s = \varepsilon_Y\nu(\rho_{i(F(Y))}^{j(F(f))} \times F(f))\iota_s = \varepsilon_Y\nu\iota_{F(f)(s)}\rho_{i(F(Y))}^{j(F(f))} = \psi\rho_{i(F(Y))}^{j(F(f))} = f\varphi\rho_{i(F(X))}^{j(F(f))} = f\varepsilon_Y\nu\iota_s\rho_{i(F(X))}^{j(F(f))}$. Since $\rho_{i(F(X))}^{j(F(f))}$ and ν are epimorphisms and s is arbitrary, $\varepsilon_Y GF(f) = f\varepsilon_Y$, thus ε is natural.

Proposition 6.4.8 *1) For any $X \in \operatorname{Ob}\mathcal{C}$ and $S \in \operatorname{Ob} B_f\pi$, the following compositions are identity morphisms.*

$$F(X) \xrightarrow{\eta_{F(X)}} FGF(X) \xrightarrow{F(\varepsilon_X)} F(X) \qquad G(S) \xrightarrow{G(\eta_S)} GFG(S) \xrightarrow{\varepsilon_{G(S)}} G(S)$$

2) η and ε are equivalences of the functors.

Proof. 1) For $s \in F(X)$, $F(\varepsilon_X)\eta_{F(X)}(s) = F(\varepsilon_X)F(\nu\iota_s)(\xi_{i(F(X))}) = F(\bar{\varepsilon}\iota_s)(\xi_{i(F(X))})$
$= F((\lambda_{i(F(X))}^X)^{-1}(s))(\xi_{i(F(X))}) = \lambda_{i(F(X))}^X((\lambda_{i(F(X))}^X)^{-1}(s)) = s$. Thus $F(\varepsilon_X)\eta_{F(X)} = id_{F(X)}$.

Set $\varphi = (\lambda_{i(FG(S))}^{G(S)})^{-1}(\eta_S(s))$. Then, $\lambda_{j(\eta_S)}^{G(S)}(\varphi\rho_{i(FG(S))}^{j(\eta_S)}) = F(\varphi\rho_{i(FG(S))}^{j(\eta_S)})(\xi_{j(\eta_S)})$
$= F(\varphi)(\xi_{i(FG(S))}) = \lambda_{i(FG(S))}^{G(S)}(\varphi) = \eta_S(s) = \lambda_{i(S)}^{G(S)}(\nu\iota_s) = \lambda_{j(\eta_S)}^{G(S)}(\nu\iota_s\rho_{i(S)}^{j(\eta_S)})$. Hence we have $\varphi\rho_{i(FG(S))}^{j(\eta_S)} = \nu\iota_s\rho_{i(S)}^{j(\eta_S)}$. By the definitions of $G(\eta_S)$ and ε, $\varepsilon_{G(S)}G(\eta_S)\nu\iota_s\rho_{i(S)}^{j(\eta_S)} = \varepsilon_{G(S)}G(\eta_S)\nu(\rho_{i(S)}^{j(\eta_S)} \times id_S)\iota_s =$
$\varepsilon_{G(S)}\nu(\rho_{i(S)}^{j(\eta_S)} \times \eta_S)\iota_s = \varepsilon_{G(S)}\nu\iota_{\eta_S(s)}\rho_{i(FG(S))}^{j(\eta_S)} = \varphi\rho_{i(FG(S))}^{j(\eta_S)} = \nu\iota_s\rho_{i(S)}^{j(\eta_S)}$. Since $\rho_{i(S)}^{j(\eta_S)}$ and ν are epimorphisms, we have $\varepsilon_{G(S)}G(\eta_S) = id_{G(S)}$.

2) Clearly, η is an equivalence. Hence by 1), $F(\varepsilon_X)$ is an isomorphism for any $X \in \operatorname{Ob}\mathcal{C}$. Then the ε_X is an isomorphism by G6). □

Summarizing the results so far, we have the following theorem.

Theorem 6.4.9 *Let $\mathcal{C}$ be a Galois category with a fundamental functor F. There exists a pro-finite groups π such that F takes values in the category $B_f\pi$ of finite sets with continuous left π-actions and $F : \mathcal{C} \to B_f\pi$ is an equivalence of categories.*

Remark 6.4.10 *1) The fundamental functor $F : \mathcal{C} \to \mathcal{U}$-**Ens** is regarded as the composition the equivalence $\mathcal{C} \to B_f\pi$ and the functor $B_f\pi \to \mathcal{U}$-**Ens** forgetting the left π-actions. Hence a fundamental functor preserves finite limits and finite colimits.*

2) Since epimorphisms in $B_f\pi$ are regular, epimorphisms in $\mathcal{C}$ are also regular. It follows that a fundamental functor preserves epimorphisms.

3) By (6.3.19) and (6.4.9), the composition $\operatorname{Pro}(\mathcal{C}) \xrightarrow{\operatorname{Pro}(F)} \operatorname{Pro}(B_f\pi) \xrightarrow{\bar{\iota}} B_c\pi$ is an equivalence of categories. Since $B_c\pi$ is $\mathcal{U}$-complete, so is $\operatorname{Pro}(\mathcal{C})$. In particular, $\operatorname{Pro}(\mathcal{C})$ has finite products.

Example 6.4.11 *Let π be a pro-finite group and $F : B_f\pi \to \mathcal{U}$-**Ens** denotes the functor forgetting left π-actions. Then, $(B_f\pi, F)$ is a Galois category. For an object X of $B_f\pi$, there exist open subgroups $H_1, H_2, \ldots, H_n$ of π such that X is isomorphic to $\coprod_{i=1}^{n} \pi/H_i$. It is clear that X is connected if and only if π acts on X transitively, in other words, X is isomorphic to π/H for some open subgroup H.*

Let H and K be open subgroups of π. For objects $(\pi/H, hH)$ and $(\pi/K, kK)$ of $(B_f\pi)_F$, $(B_f\pi)_F((\pi/H, hH), (\pi/K, kK))$ consists of at most one element. It is not empty if and only if $(h^{-1}k)^{-1}Hh^{-1}k$ is contained in K. It follows that $f : (\pi/H, hH) \to (\pi/K, kK)$ is an isomorphism if and only if $(h^{-1}k)^{-1}Hh^{-1}k = K$. Let Σ be the set of open subgroups of π. Define a relation $\equiv$ in Σ by "$H \equiv K \Leftrightarrow g^{-1}Hg = K$ for some $g \in \pi$". Let $\{H_i |\, i \in I\}$ be a set of representatives of $\Sigma/\equiv$. For each $i \in I$ and $j \in \pi/N(H_i)$, we choose a representative $h_{i,j}$ of j and $\mathcal{D}$ denotes the opposite category of the full subcategory of $(B_f\pi)_F$ with the set of objects $\{(\pi/H_i, h_{i,j}H_i) |\, i \in I,\ j \in \pi/N(H_i)\}$. Then, $\mathcal{D}^{op}$ is a skeleton of the full subcategory of $(B_f\pi)_F$ consisting of minimal objects and F is represented by the pro-object $D : \mathcal{D}^{op} \to B_f\pi$ defined by $D(\pi/H_i, h_{i,j}H_i) = \pi/H_i$. It is easy to see that $D(\pi/H_i, h_{i,j}H_i) = \pi/H_i$ is Galois if and only if H_i is a normal subgroup.

Example 6.4.12 *Let **FEt** be a subcategory of the category of schemes consisting of finite etale morphisms. Let X be a locally noetharian scheme and $\bar{x} : \operatorname{Spec} k \to X$ a geometric point of X. Define a functor $F : \mathbf{FEt}/X \to \mathcal{U}$-**Ens** by $F(Y \xrightarrow{p} X) = \mathbf{FEt}/X(\bar{x}, p)$. Then, $\mathbf{FEt}/X$ is a Galois category with fundamental functor F.*

Definition 6.4.13 *Let $\mathcal{C}$ be a Galois category with a fundamental functor F. For a finite group G, if X is a Galois object in $\mathcal{C}$ such that $\operatorname{Aut}(X)$ is isomorphic to G, X is called a G-object in $\mathcal{C}$.*

Let π be the pro-group such that F takes values in $B_f\pi$. For a G-object X, there is a continuous surjective homomorphism $\varphi : \pi \to \operatorname{Aut}(X) \cong G$.

6.5 Properties of Galois categories

Let $\mathcal{C}$ be a Galois category with a fundamental functor F and $D = D_F : \mathcal{D}^{op} \to \mathcal{C}$ the fundamental pro-object. Define a pro-group π as in the previous section so that F takes values in $B_f\pi$.

Proposition 6.5.1 *An object X of $\mathcal{C}$ is connected if and only if π acts on $F(X)$ transitively.*

Proof. Since $F : \mathcal{C} \to B_f\pi$ is an equivalence, X is connected if and only if $F(X)$ is so. It is clear that a finite left π-space is connected if and only if its π-action is transitive. □

Corollary 6.5.2 *For $X \in \operatorname{Ob}\mathcal{C}$, the following conditions are equivalent.*

(i) X is connected and it is not an initial object.
(ii) π acts on $F(X)$ transitively and $F(X)$ is not empty.
(iii) X is isomorphic to some D_i.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows from (6.5.1) and 4) of (6.4.2). $(iii) \Rightarrow (i)$ follows from 1) of (6.4.4). Assume (i). Then $F(X)$ is not empty by (ii) and there exist $j \in \operatorname{Ob}\mathcal{D}$ and a morphism $f : D_i \to X$. By 1) of (6.4.3), f is a regular epimorphism and (iii) follows from 3) of (6.4.4). □

For a closed subgroup H of π, we define a functor $F^H : \mathcal{C} \to \mathcal{U}$-**Ens** by $F^H(X) = F(X)^H =$ (*the set of elements of$F(X)$ fixed by$H$*).

Proposition 6.5.3 *Let $G : \mathcal{C} \to \mathcal{U}$-**Ens** be a left exact functor. The following conditions are equivalent.*

(i) G preserves finite coproducts.
(ii) G preserves coproducts of two objects.
(iii) There is a strict pro-object $E : \mathcal{E}^{op} \to \mathcal{C}$ representing G such that $\mathcal{E}$ is a directed set and that each E_j is connected and it is not an initial object.
(iv) If $E : \mathcal{E}^{op} \to \mathcal{C}$ is a pro-object representing G, then $\operatorname{Pro}(F)(E) : \mathcal{E}^{op} \to B_f\pi$ is isomorphic to P_H (6.3.20) for some closed subgroup H of π.
(v) There exists a closed subgroup H of π such that G is equivalent to the functor F^H.

Proof. $(i) \Rightarrow (ii)$ is obvious. $(ii) \Rightarrow (iii)$ follows from 1), 2) of (6.4.4).

$(iii) \Rightarrow (iv)$; Since the pro-object $E : \mathcal{E}^{op} \to \mathcal{C}$ representing G is unique up to isomorphism in $\mathrm{Pro}(\mathcal{C})$, we may assume that E satisfies the conditions of (ii). By (6.3.16), the limit of $FE : \mathcal{E}^{op} \to B_f\pi$ is not empty. Take $(x_j)_{j \in \mathrm{Ob}\,\mathcal{E}} \in \varprojlim_j \iota F(E_j)$ and define morphisms $p_j : \pi \to \iota F(E_j)$ in $B_c\pi$ by $p_j(g) = gx_j$. Since $F : \mathcal{C} \to B_f\pi$ is an equivalence, $F(E_j)$ is connected and it follows that p_j is surjective. Moreover, $(\pi \xrightarrow{p_j} \iota F(E_j))_{j \in \mathrm{Ob}\,\mathcal{E}}$ is a cone of ιFE. By (6.3.21), the unique morphism $\rho : \pi \to \varprojlim_j \iota F(E_j)$ satisfying $\lambda_j \rho = p_j$ is an epimorphism of left π-spaces, where $\lambda_j : \varprojlim_j \iota F(E_j) \to \iota F(E_j)$ denotes the canonical morphism. Let H be the kernel of ρ. Then, ρ induces an isomorphism $\pi/H \to \varprojlim_j \iota F(E_j) = \bar{\iota}(FE)$. Since $\bar{\iota}(P_H) = \pi/H$ by (6.3.20) and $\bar{\iota}$ is an equivalence by (6.3.19), $FE = \mathrm{Pro}(F)(E)$ is isomorphic to P_H.

$(iv) \Rightarrow (v)$; Suppose that $E : \mathcal{E}^{op} \to \mathcal{C}$ is a pro-object representing G. By the assumption, there is a closed subgroup H of π such that $\mathrm{Pro}(F)(E) : \mathcal{E}^{op} \to B_f\pi$ is isomorphic to P_H. For $X \in \mathrm{Ob}\,\mathcal{C}$, since F is an equivalence, $G(X) = \varinjlim_j \mathcal{C}(E_j, X) \cong \varinjlim_j B_f\pi(F(E_j), F(X)) \cong \varinjlim_{K \in \mathrm{Ob}\,\mathcal{N}_H} B_f\pi(P_H(K), F(X)) \cong \varinjlim_{K \in \mathrm{Ob}\,\mathcal{N}_H} F(X)^K$. In $F(X)$, $F(X)^K \subset F(X)^L$ if $K \supset L$ for $K, L \in \mathrm{Ob}\,\mathcal{N}_H$. Hence $\varinjlim_{K \in \mathrm{Ob}\,\mathcal{N}_H} F(X)^K = F(X)^H$.

$(v) \Rightarrow (i)$; It suffices to verify that F^H preserves coproducts of two objects and initial objects. For $X, Y \in \mathrm{Ob}\,\mathcal{C}$, since F preserves coproducts in $B_f\pi$, we have $F^H(X \coprod Y) = F(X \coprod Y)^H \cong (F(X) \coprod F(Y))^H = F(X)^H \coprod F(Y)^H = F^H(X) \coprod F^H(Y)$ and $F^H(0) = F(0)^H = \phi^H = \phi$. □

Proposition 6.5.4 *Let $G : \mathcal{C} \to \mathcal{U}$-**Ens** be a left exact functor and $D_G : \mathcal{D}_G^{op} \to \mathcal{C}$ the pro-object associated with G. The following conditions are equivalent.*

(i) D_G is isomorphic to the fundamental pro-object D_F.
(ii) G satisfies the conditions G5) and G6) of (6.4.1).
(iii) G preserves coproducts of two objects and reflects initial objects.
(iv) $\mathrm{Pro}(F)(D_G)$ is isomorphic to $P_{\{e\}}$.
(v) Each D_{Gi} is connected and it is not an initial object. If X is a connected object which is not initial, X is isomorphic to some D_{Gi}.

Proof. It is obvious that (i) implies (ii). $(ii) \Rightarrow (iii)$ follows from 4) of (6.4.2).

$(iii) \Rightarrow (iv)$; By (6.5.3), there is a closed subgroup H of π such that $\mathrm{Pro}(F)(D_G) : \mathcal{D}_G^{op} \to B_f\pi$ is isomorphic to P_H. For $Z \in \mathrm{Ob}\,B_f\pi$ such that $L(\mathrm{Pro}(F)(D_G))(Z) = \varinjlim_i B_f\pi(F(D_{Gi}), Z)$ is an initial object, $B_f\pi(F(D_{Gi}), Z)$ is empty for every $i \in \mathrm{Ob}\,\mathcal{D}_G$. Since F is an equivalence, there exists $X \in \mathrm{Ob}\,\mathcal{C}$ such that $F(X) \cong Z$. Hence $\mathcal{C}(D_{Gi}, X)$ is empty for any $i \in \mathrm{Ob}\,\mathcal{D}_G$ and it follows that $G(X)$ is empty. By the assumption, X is an initial object of $\mathcal{C}$ which is preserved by F. Therefore, $L(\mathrm{Pro}(F)(D_G))$ reflects initial objects. By (6.5.3), there is a closed subgroup H of π such that $\mathrm{Pro}(F)(D_G)$ is isomorphic to P_H in $\mathrm{Pro}(B_f\pi)$. Then, $L(P_H)$ reflects initial objects. It follows from (6.3.8) that H is the trivial subgroup $\{e\}$ of π.

$(iv) \Rightarrow (i)$; The fundamental functor F satisfies the conditions of (iii). Hence $\mathrm{Pro}(F)(D_F)$ is isomorphic to $P_{\{e\}}$. Since $\mathrm{Pro}(F) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(B_f\pi)$ is an equivalence, D_F is isomorphic to D_G.

$(iii) \Rightarrow (v)$; It follows from 1) of (6.4.4) that each D_{Gi} is connected and it is not an initial object. Suppose that $X \in \mathrm{Ob}\,\mathcal{C}$ is connected and it is not an initial object. By the assumption, $G(X)$ is not empty. Hence there is a morphism $p : D_{Gj} \to X$ for large enough $j \in \mathrm{Ob}\,\mathcal{D}_G$. By 1) of (6.4.4) and 1) of (6.4.3), p is a regular epimorphism. The (iv) follows from 3) of (6.4.4).

$(v) \Rightarrow (i)$; By (6.5.3), G preserves coproducts of two objects. Then, G is equivalent to F^H for some closed subgroup H of π. Suppose that H is not an trivial subgroup $\{e\}$ of π. There exists an open normal subgroup K of π such that $H \cap K \neq \phi$ by (6.3.8). Let X be an object of $\mathcal{C}$ such that $F(X)$ is isomorphic to π/K. Then, $G(X) \cong F(X)^H \cong (\pi/K)^H = \phi$. Since F is an equivalence and π/K is connected and not initial, X is connected and not initial. Hence X is isomorphic to D_{Gi} for some $i \in \mathrm{Ob}\,\mathcal{D}_G$. It follows that $\mathcal{C}(D_{Gi}, X)$ is not empty and this contradicts $G(X) = \phi$. Therefore $H = \{e\}$ and G is equivalent to F. □

Let $G_s : \mathcal{C} \to \mathcal{U}$-**Ens** $(s = 1, 2)$ be left exact functors preserving finite coproducts. By (6.5.3), there are closed subgroups H_s of π such that G_s are equivalent to F^{H_s}. Since $G_s \cong L(D_{G_s})$, $\mathrm{Pro}(F)(D_{G_s}) \cong P_{H_s}$ and $\bar{\iota}P_{H_s} = \pi/H_s$, $\check{\mathcal{C}}(G_1, G_2) \cong \mathrm{Pro}(\mathcal{C})(D_{G_2}, D_{G_1}) \cong \mathrm{Pro}(B_f\pi)(P_{H_2}, P_{H_1}) \cong B_c\pi(\pi/H_2, \pi/H_1) \cong C(\pi)(H_2, H_1)$. Hence the full subcategory of $\check{\mathcal{C}}$ consisting of left exact functors preserving finite coproducts is equivalent to the opposite category of $C(\pi)$.

Proposition 6.5.5 *Fundamental functors are isomorphic each other. Hence so are fundamental pro-objects. If $\alpha : G \to G'$ is a morphism between fundamental functors, then α is an equivalence of functors.*

Proof. The first assertion is a direct consequence of (6.5.4). Since fundamental pro-objects correspond to the trivial subgroup of π, the second assertion follows from the fact that $\check{\mathcal{C}}(G,G') \cong \mathrm{Pro}(\mathcal{C})(D_{G'},D_G) \cong \mathrm{Pro}(B_f\pi)(P_{\{e\}},P_{\{e\}}) \cong B_c\pi(\pi/\{e\},\pi/\{e\}) \cong C(\pi)(\{e\},\{e\}) = \mathrm{Aut}_{C(\pi)}(\{e\})$ (See (6.3.22)). $\square$

We denote by $\mathcal{T}_{td}$ (resp. $\mathcal{T}_{tdc}$, $\mathcal{T}_f$) the full subcategory of topological spaces consisting of totally disconnected Hausdorff (resp. totally disconnected compact Hausdorff, finite discrete) spaces.

Define a functor $\Psi : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Funct}(\check{\mathcal{C}},\mathcal{T}_{td})$ by $\Psi(D)(F) = \check{\mathcal{C}}(L(D),F)$, $\Psi(D)(f) = f_*$ and $\Psi(g) = L(g)^*$. Since $\Psi(D) : \check{\mathcal{C}}(F,G) \to \mathcal{T}_{td}(\Psi(D)(F),\Psi(D)(G))$ is the adjoint of the composition map $\check{\mathcal{C}}(F,G)\times\check{\mathcal{C}}(L(D),F) \to \check{\mathcal{C}}(L(D),G)$ which is continuous for the natural topologies (6.1.2), $\Psi(D)$ is continuous with respect to the compact-open topology on $\mathcal{T}_{td}(\Psi(D)(F),\Psi(D)(G))$. Let us denote by $\mathrm{Funct}_c(\check{\mathcal{C}},\mathcal{T}_{td})$ the full subcategory of $\mathrm{Funct}(\check{\mathcal{C}},\mathcal{T}_{td})$ consisting of functors $T : \check{\mathcal{C}} \to \mathcal{T}_{td}$ such that $T : \check{\mathcal{C}}(F,G) \to \mathcal{T}_{td}(T(F),T(G))$ is continuous with respect to the compact-open topology on $\mathcal{T}_{td}(T(F),T(G))$. Hence Ψ is regarded as a functor $\mathrm{Pro}(\mathcal{C}) \to \mathrm{Funct}_c(\check{\mathcal{C}},\mathcal{T}_{td})$.

Let $\mathcal{C}$ be a Galois category. We denote by Γ the full subcategory of $\check{\mathcal{C}}$ consisting of fundamental functors on $\mathcal{C}$. By (6.5.5), Γ is a connected groupoid. We call Γ the fundamental groupoid of $\mathcal{C}$. For $D \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$ and $F \in \mathrm{Ob}\,\Gamma$, since $\Psi(D)(F) = \check{\mathcal{C}}(L(D),F) \cong \mathrm{Pro}(\mathcal{C})(D_F,D) \cong \varprojlim_i L(D_F)(D_i)$ and $L(D_F)(D_i) \cong F(D_i)$ is finite and discrete, $\Psi(D)(F)$ is an object of $\mathcal{T}_{tdc}$. Let $\Psi_c : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Funct}_c(\Gamma,\mathcal{T}_{tdc})$ be the functor defined by $\Psi_c(D) = \Psi(D)$. For $X \in \mathrm{Ob}\,\mathcal{C}$ and $F \in \mathrm{Ob}\,\Gamma$, $\Psi_c\kappa(X)(F) = \Psi\kappa(X)(F) = \check{\mathcal{C}}(L\kappa(X),F) = \check{\mathcal{C}}(h^X,F) \cong F(X)$. Hence $\Psi_c\kappa(X)$ is regarded as a functor from Γ to $\mathcal{T}_f$ and there is a functor $\Psi_f : \mathcal{C} \to \mathrm{Funct}_c(\Gamma,\mathcal{T}_f)$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Psi_f"] \arrow[d, "\kappa"] & \mathrm{Funct}_c(\Gamma,\mathcal{T}_f) \arrow[d, "\mathrm{inc}"] \\ \mathrm{Pro}(\mathcal{C}) \arrow[r, "\Psi_c"] & \mathrm{Funct}_c(\Gamma,\mathcal{T}_{tdc}) \end{tikzcd}$$

Proposition 6.5.6 *$\Psi_f : \mathcal{C} \to \mathrm{Funct}_c(\Gamma,\mathcal{T}_f)$ and $\Psi_c : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Funct}_c(\Gamma,\mathcal{T}_{tdc})$ are equivalences of categories.*

Proof. Choose a fundamental functor F and let Γ_F be the full subcategory of Γ consisting of a single object F. Since Γ is a connected groupoid, the inclusion functor $i_F : \Gamma_F \to \Gamma$ is an equivalence of categories. Then, the functors $i_F^* : \mathrm{Funct}_c(\Gamma,\mathcal{T}_{tdc}) \to \mathrm{Funct}_c(\Gamma_F,\mathcal{T}_{tdc})$ and $i_F^* : \mathrm{Funct}_c(\Gamma,\mathcal{T}_f) \to \mathrm{Funct}_c(\Gamma_F,\mathcal{T}_f)$ restricting domains are also equivalences. Note that $\mathrm{Mor}\,\Gamma_F = \mathrm{Aut}_{\check{\mathcal{C}}}(F) = \mathrm{Aut}_{\mathrm{Pro}(\mathcal{C})}(D_F)^{op} = \pi_F$.

For an object α of $\mathrm{Funct}_c(\Gamma_F,\mathcal{T}_{tdc})$, let $\tilde{\alpha} : \pi_F \times \alpha(F) \to \alpha(F)$ be the adjoint of $\alpha : \pi_F \to \mathcal{T}_{tdc}(\alpha(F),\alpha(F))$. Since $\alpha(F)$ is locally compact, $\tilde{\alpha}$ is continuous. A map $\varphi : \alpha(F) \to \beta(F)$ defines a natural transformation of functors $\alpha,\beta : \Gamma_F \to \mathcal{T}_{tdc}$ if and only if $\varphi\tilde{\alpha} = \tilde{\beta}(id_{\pi_F}\times\varphi)$. Thus we have a functor $E_c : \mathrm{Funct}_c(\Gamma_F,\mathcal{T}_{tdc}) \to B_c\pi_F$ given by $E_c(\alpha) = (\alpha(F),\tilde{\alpha})$ and $E_c(\varphi) = \varphi_F$. Clearly, E_c is an isomorphism of categories. Similarly, for an object α of $\mathrm{Funct}_c(\Gamma_F,\mathcal{T}_f)$, let $\hat{\alpha} : \pi_F \times \alpha(F) \to \alpha(F)$ be the adjoint of $\alpha : \pi_F \to \mathcal{T}_f(\alpha(F),\alpha(F))$. Since $\alpha(F)$ is locally compact, $\hat{\alpha}$ is continuous. Hence a functor $E_f : \mathrm{Funct}_c(\Gamma_F,\mathcal{T}_f) \to B_f\pi_F$ defined by $E_f(\alpha) = (\alpha(F),\hat{\alpha})$ and $E_f(\varphi) = \varphi_F$ is an isomorphism of categories.

We show that the composition of functors

$$\mathrm{Pro}(\mathcal{C}) \xrightarrow{\Psi_c} \mathrm{Funct}_c(\Gamma,\mathcal{T}_{tdc}) \xrightarrow{i_F^*} \mathrm{Funct}_c(\Gamma_F,\mathcal{T}_{tdc}) \xrightarrow{E_c} B_c\pi_F$$

is naturally equivalent to the composition

$$\mathrm{Pro}(\mathcal{C}) \xrightarrow{\mathrm{Pro}(F)} \mathrm{Pro}(B_f\pi_F) \xrightarrow{\bar{\iota}} B_c\pi_F.$$

In fact, $E_ci_F^*\Psi_c(D) = (\check{\mathcal{C}}(L(D),F),\alpha_D)$ for $D \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$, where $\alpha_D : \pi_F \times \check{\mathcal{C}}(L(D),F) = \mathrm{Aut}(F) \times \check{\mathcal{C}}(L(D),F) \to \check{\mathcal{C}}(L(D),F)$ is the composition map. Since $L(D) = \varinjlim_i h^{D_i}$ in $\check{\mathcal{C}}$, $\check{\mathcal{C}}(L(D),F)$ is naturally isomorphic to $\varprojlim_i \check{\mathcal{C}}(h^{D_i},F) \cong \varprojlim_i F(D_i)$. On the other hand, we have $\bar{\iota}\mathrm{Pro}(F)(D) = \varprojlim_i \iota F(D_i)$ and the isomorphism $\check{\mathcal{C}}(L(D),F) \to \varprojlim_i \iota F(D_i)$ preserves left π_F-action. Thus we have a natural equivalence $E_ci_F^*\Psi_c \cong \bar{\iota}\mathrm{Pro}(F)$. Since $F : \mathcal{C} \to B_f\pi_F$ and $\bar{\iota} : \mathrm{Pro}(B_f\pi_F) \to B_c\pi_F$ are equivalences of categories, it follows that the above composition is also an equivalence. Therefore, E_c is an equivalence.

Similarly, the composition of functors

$$\mathcal{C} \xrightarrow{\Psi_f} \mathrm{Funct}_c(\Gamma,\mathcal{T}_f) \xrightarrow{i_F^*} \mathrm{Funct}_f(\Gamma_F,\mathcal{T}_f) \xrightarrow{E_f} B_f\pi_F$$

is naturally equivalent to $F : \mathcal{C} \to B_f\pi_F$. In fact, $E_f i_F^*\Psi_f(X) = (\check{\mathcal{C}}(L\kappa(X), \alpha_X)$ for $X \in \mathrm{Ob}\,\mathcal{C}$, where $\alpha_X : \pi_F \times \check{\mathcal{C}}(L\kappa(X), F) = \mathrm{Aut}(F) \times \check{\mathcal{C}}(L\kappa(X), F) \to \check{\mathcal{C}}(L\kappa(X), F)$ is the composition map. Since $L\kappa(X) = h^X$, $\check{\mathcal{C}}(L\kappa(X), F)$ is naturally isomorphic to $F(X)$ and this isomorphism preserves left π_F-action. Thus we have a natural equivalence $E_f i_F^*\Psi_f \cong F$ and it follows that E_f is an equivalence. □

Recall from (6.4.10) that $\mathrm{Pro}(\mathcal{C})$ is $\mathcal{U}$-complete. Clearly, $\mathrm{Funct}_c(\mathit{\Gamma}, \mathcal{T}_{tdc})$ is also $\mathcal{U}$-complete. Hence the equivalence $\Psi_c : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Funct}_c(\mathit{\Gamma}, \mathcal{T}_{tdc})$ preserves $\mathcal{U}$-limits.

Remark 6.5.7 *Let $\varpi : \mathit{\Gamma} \to \mathcal{T}_{tdc}$ be a functor defined by $\varpi(F) = \mathrm{Aut}(F) = \pi_F$ and $\varpi(\rho)(g) = \rho g \rho^{-1}$ for $F \in \mathrm{Ob}\,\mathit{\Gamma}$, $\rho \in \mathit{\Gamma}(F, G)$. Since composition maps in $\mathit{\Gamma}$ are continuous, ϖ is an object of $\mathrm{Funct}_c(\mathit{\Gamma}, \mathcal{T}_{tdc})$. By (6.5.6), there is a pro-object Π in $\mathcal{C}$ such that $\Psi_c(\Pi) \cong \varpi$. Then, there is an isomorphism $\phi_F : \check{\mathcal{C}}(L(\Pi), F) \to \pi_F = \mathrm{Aut}(F)$ which is natural in $F \in \mathrm{Ob}\,\mathit{\Gamma}$. An object α of $\mathrm{Funct}_c(\mathit{\Gamma}, \mathcal{T}_{tdc})$ is a group object if and only if $\alpha(F)$ is a topological group for each $F \in \mathrm{Ob}\,\mathit{\Gamma}$ and $\alpha(f) : \alpha(F) \to \alpha(G)$ is a homomorphism of topological group for any $f \in \mathit{\Gamma}(F, G)$. In particular, ϖ is a group object. Hence Π is a group object in $\mathrm{Pro}(\mathcal{C})$. We call Π the fundamental pro-group of $\mathcal{C}$. For $D \in \mathrm{Pro}(\mathcal{C})$ and $F \in \mathrm{Ob}\,\mathit{\Gamma}$, consider the composition $\Psi_c(\Pi \times D)(F) \cong \Psi_c(\Pi)(F) \times \Psi_c(D)(F) = \check{\mathcal{C}}(L(\Pi), F) \times \check{\mathcal{C}}(L(D), F) \xrightarrow{\phi_F \times id} \mathrm{Aut}(F) \times \check{\mathcal{C}}(L(D), F) \xrightarrow{c} \check{\mathcal{C}}(L(D), F) = \Psi_c(D)(F)$ which is natural in both D and F, where c denotes the composition map. Thus we have a morphism $\beta_D : \Psi_c(\Pi \times D) \to \Psi_c(D)$ in $\mathrm{Funct}_c(\mathit{\Gamma}, \mathcal{T}_{tdc})$ such that $(\beta_D)_F$ is the above composition. Since Ψ_c is fully faithful, there is a unique morphism $\alpha_D : \Pi \times D \to D$ such that $\Psi_c(\alpha_D) = \beta_D$. We note that, by the naturality of β_D in D, α_D is also natural in D. For $\xi, \zeta \in \Psi_c(\Pi)(F)$, we have $\phi_F(\phi_F(\xi)\zeta) = \phi_F(\xi)\phi_F(\zeta)\phi_F(\xi)^{-1}$ by the naturality of ϕ_F in F. It follows that α_Π is the following composition.*

$$\Pi \times \Pi \xrightarrow{\Delta \times id} \Pi \times \Pi \times \Pi \xrightarrow{id \times T} \Pi \times \Pi \times \Pi \xrightarrow{\mu \times \iota} \Pi \times \Pi \xrightarrow{\mu} \Pi$$

Here, $\Delta : \Pi \to \Pi \times \Pi$, $T : \Pi \times \Pi \to \Pi \times \Pi$, $\mu : \Pi \times \Pi \to \Pi$ and $\iota : \Pi \to \Pi$ denote the diagonal morphism, the switching morphism, the multiplication and the inverse of Π, respectively.

Generally, let $\mathcal{C}$ be a category with finite products, G an internal group in $\mathcal{C}$ and X a right G-object with action $\alpha : X \times G \to X$. If the morphism $(\alpha, \mathrm{pr}_1) : X \times G \to X \times X$ is an isomorphism, we call X a formally principal right G-object. Moreover, if the unique morphism $X \to 1$ is a regular epimorphism, X is called a principal right G-object. Since $X \to 1$ is a regular epimorphism if and only if it is an coequalizer of $X \times X \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} X$, X is a principal right G-object if and only if $X \to 1$ is a coequalizer of $X \times G \underset{pr_1}{\overset{\alpha}{\rightrightarrows}} X$.

Remark 6.5.8 *For a Galois category $\mathcal{C}$, define a functor $C : \mathcal{T}_f \to \mathcal{C}$ as follows. For $S \in \mathrm{Ob}\,\mathcal{T}_f$, 1_x $(x \in S)$ denotes a terminal object 1 of $\mathcal{C}$ and set $C(S) = \coprod_{x \in S} 1_x$. For a map $f : S \to T$, $C(f) : C(S) \to C(T)$ is the map induced by $1_x \to 1_{f(x)}$. Let F be a fundamental functor of $\mathcal{C}$ and consider the equivalence $F : \mathcal{C} \to B_f\pi_F$. Since $FC : \mathcal{T}_f \to B_f\pi_F$ preserves finite limits and colimits, so does C. Hence if G is a finite group, $C(G)$ is an internal group in $\mathcal{C}$.*

Suppose that $\mathcal{C} = B_f\pi$ for a pro-finite group π and G is a finite group. Then, $C(G)$ is a trivial left π-set and a right $C(G)$-object X in $B_f\pi$ is a principal right G-object if and only if the right G-action on X is free and transitive. If X is connected, X is isomorphic to π/U as a left π-set for an open subgroup U of π. Hence, for $x, y \in X$, there is at most one π-automorphism $f : X \to X$ which maps x to y. It follows that, if X is a connected principal right G-object in $B_f\pi$, the homomorphism $G \to \mathrm{Aut}_\pi(X)^{op}$ is an isomorphism. In this case, U should be a normal subgroup of π. In fact, if there is an automorphism $f : \pi/U \to \pi/U$ which maps U to gU, g belongs to the normalizer of U. Therefore, by (6.5.2), a connected object X of a general Galois category $\mathcal{C}$ is a principal right $\mathrm{Aut}(X)^{op}$-object if and only if X is isomorphic to some D_i which is Galois (6.4.5).

We fix a fundamental functor F of a Galois category $\mathcal{C}$. Let G be a finite group and $\chi : \pi_F \to G$ a homomorphism. Then, G can be regarded as a finite left π_F-set by $\pi_F \times G \ni (p, g) \mapsto \chi(p)g \in G$ and a right G-set by the right translations. Thus we have a principal right G-object in $B_f\pi_F$ and there exists a principal right G-object X_χ in $\mathcal{C}$ and an isomorphism $\omega_\chi : G \to F(X_\chi)$ in $B_f\pi_F$ of right G-sets. If we specify an element a of $F(X_\chi)$, there is a unique isomorphism from G to $F(X_\chi)$ as a right G-sets that maps the unit e of G to a.

Let $\mathcal{G}_f$ be the category of finite groups, G a finite group and $\mathcal{C}(G)$ a category defined as follows. $\mathrm{Ob}\,\mathcal{C}(G)$ is the set of pairs (X, a) such that X is a principal right G-object in $\mathcal{C}$ and $a \in F(X)$. A morphism $(X, a) \to (Y, b)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that $F(f)(a) = b$ and f preserves the right $C(G)$-actions. For a morphism $\theta : G \to H$ in $\mathcal{G}_f$ and $(X, a) \in \mathcal{C}(G)$, we choose an object $X \times_G H$ of $\mathcal{C}$ and an isomorphism $\theta_{(X,a)} : F(X) \times_G H \to F(X \times_G H)$. If $f : (X, a) \to (Y, b)$ is a morphism in $\mathcal{C}(G)$, $f_* : X \times_G H \to Y \times_G H$ is the morphism defined by

*$F(f_*) = \theta_{(Y,b)}(F(f)\times_G id_H)\theta^{-1}_{(X,a)}$. Let $\theta_* : \mathcal{C}(G) \to \mathcal{C}(H)$ be a functor given by $\theta_*(X,a) = (X\times_G H, \theta_{(X,a)}(a,e))$ and $\theta_*(f) = f_*$. Define a functor $P : \mathcal{G}_f \to \mathcal{U}$-**Ens** by $P(G) =$ (the set of isomorphism classes of$\mathcal{C}(G)$) and $P(\theta) =$ (the map induced by $\theta_* : C(G) \to C(H)$). Then, P is pro-representable, that is, P is equivalent to the functor $h^{\pi_F} : \mathcal{G}_f \to \mathcal{U}$-**Ens** given by $h^{\pi_F}(G) = \mathrm{Hom}(\pi_F, G)$. In fact, for a homomorphism $\chi : \pi_F \to G$, we assign the isomorphism class of $(X_\chi, \omega_\chi(e))$. Conversely, for $(X,a) \in \mathrm{Ob}\,\mathcal{C}(G)$, there is a unique isomorphism $\omega : G \to F(X)$ of right G-sets such that $\omega(e) = a$. Define $\chi_{(X,a)} : \pi_F \to G$ by $\chi_{(X,a)}(g) = \omega^{-1}(ga)$. Then, $\chi_{(X,a)}$ is a homomorphism and if (X,a) is isomorphic to (Y,b), $\chi_{(X.a)} = \chi_{(Y,b)}$. Hence this fact gives a characterization of π_F, namely, π_F is a pro-finite group which (pro-)represents P.*

6.6 Exact functors between Galois categories

Proposition 6.6.1 *Let $\mathcal{C}$, $\mathcal{C}'$ be Galois categories, F' a fundamental functor of $\mathcal{C}$ and $H : \mathcal{C} \to \mathcal{C}'$ a functor. We put $F = F'H$. The following conditions are equivalent.*

(i) H is left exact and preserves finite colimits.
(ii) H is left exact and preserves finite coproducts and epimorphisms.
(iii) F is a fundamental functor of $\mathcal{C}$.

Proof. We first show that *ii)* implies that H reflects initial objects. Suppose that $X \in \mathrm{Ob}\,\mathcal{C}$ is not an initial object. Since $\mathcal{C}$ is equivalent to $B_f\pi$ for some pro-finite group π, the unique morphism $X \to 1_{\mathcal{C}}$ to the terminal object of $\mathcal{C}$ is an epimorphism. Hence $H(X) \to H(1_{\mathcal{C}}) = 1_{\mathcal{C}'}$ is also an epimorphism and it follows that $H(X)$ is not an initial object of $\mathcal{C}'$.

i) $\Rightarrow$ *ii)*; Since epimorphisms in $\mathcal{C}$ are regular (6.4.10), H preserves epimorphisms.

ii) $\Rightarrow$ *iii)*; It is clear that F is left exact and preserves finite coproducts. Since both F' and H reflects initial objects, so does F. Hence F is a fundamental functor of $\mathcal{C}$ by (6.5.4).

iii) $\Rightarrow$ *i)*; Let $\mathcal{D}$ be a finite category and $D : \mathcal{D} \to \mathcal{C}$ a functor. Suppose that $(L \xrightarrow{\lambda_i} D(i))_{i\in\mathrm{Ob}\,\mathcal{D}}$ (resp. $(D(i) \xrightarrow{\iota_i} C)_{i\in\mathrm{Ob}\,\mathcal{D}}$) is a limiting (resp. colimiting) cone of D. Then, $(H(L) \xrightarrow{H(\lambda_i)} HD(i))_{i\in\mathrm{Ob}\,\mathcal{D}}$ (resp. $(HD(i) \xrightarrow{H(\iota_i)} H(C))_{i\in\mathrm{Ob}\,\mathcal{D}}$) is a cone of HD. On the other hand, there is a limiting cone $(L' \xrightarrow{\lambda'_i} HD(i))_{i\in\mathrm{Ob}\,\mathcal{D}}$ (resp. colimiting cone $(HD(i) \xrightarrow{\iota'_i} C'))_{i\in\mathrm{Ob}\,\mathcal{D}}$) of HD. Hence there is a unique morphism $\varphi : H(L) \to L'$ (resp. $\psi : C' \to H(C)$) such that $H(\lambda_i) = \lambda'_i\varphi$ (resp. $H(\iota_i) = \psi\iota'_i$). Since F and F' preserves finite limits (resp. colimits (6.4.10)) $(F(L) \xrightarrow{F(\lambda_i)} FD(i))_{i\in\mathrm{Ob}\,\mathcal{D}}$ and $(F'(L') \xrightarrow{F'(\lambda'_i)} FD(i))_{i\in\mathrm{Ob}\,\mathcal{D}}$ (resp. $(FD(i) \xrightarrow{F(\iota_i)} F(C))_{i\in\mathrm{Ob}\,\mathcal{D}}$ and $(FD(i) \xrightarrow{F'(\iota'_i)} F'(C'))_{i\in\mathrm{Ob}\,\mathcal{D}}$) are limiting (resp. co-limiting) cones of FD. Note that $F'(\varphi)$ (resp. $F'(\psi)$) is the unique morphism satisfying $F(\lambda_i) = F'(\lambda'_i)F'(\varphi)$ (resp. $F(\iota_i) = F'(\psi)F(\iota'_i)$). It follows that $F'(\varphi)$ (resp. $F'(\psi)$) is an isomorphism. Since F' reflects isomorphisms, φ (resp. ψ) is an isomorphism. Therefore H preserves finite limits (resp. colimits). □

Proposition 6.6.2 *If $H : \mathcal{C} \to \mathcal{C}'$ is a functor between Galois categories satisfying the condition of (6.6.1), then H is faithful.*

Proof. Choose a fundamental functor F' of $\mathcal{C}'$. Then, $F = F'H$ is a fundamental functor of $\mathcal{C}$ which is fully faithful, regarded as a functor $\mathcal{C} \to B_f\pi_F$. Hence H is faithful. □

Suppose that $H : \mathcal{C} \to \mathcal{C}'$ is a functor satisfying the conditions of the preceding proposition. Let Γ and Γ' be the fundamental groupoids of $\mathcal{C}$ and $\mathcal{C}'$, respectively. Then, $H^* : \check{\mathcal{C}}' \to \check{\mathcal{C}}$ induces a functor ${}^tH : \Gamma' \to \Gamma$. Since H^* preserves colimits, ${}^tH : \Gamma'(F', G') \to \Gamma(F'H, G'H)$ is continuous for $F', G' \in \mathrm{Ob}\,\Gamma$ by (6.1.2). In particular, if we set $F = {}^tH(F') = F'H$ and $u_H = {}^tH : \pi_{F'} \to \pi_F$, u_H is a continuous homomorphism. Moreover, regarding F, F' as equivalences $\mathcal{C} \to B_f\pi_F$, $\mathcal{C}' \to B_f\pi_{F'}$, $F'H = Fu^\sharp_H$ (See (6.3.24)).

Conversely, for $F \in \mathrm{Ob}\,\Gamma$ and $F' \in \mathrm{Ob}\,\Gamma'$, suppose that a continuous homomorphism $u : \pi_{F'} \to \pi_F$ is given. Since $F : \mathcal{C} \to B_f\pi_F$ and $F' : \mathcal{C}' \to B_f\pi_{F'}$ are equivalences, there is a left exact functor $H : \mathcal{C} \to \mathcal{C}'$ preserving finite colimits such that there is a natural equivalence $\xi : F'H \to u^\sharp F$. Hence, for any $X \in \mathrm{Ob}\,\mathcal{C}$, the following square commutes.

$$\begin{array}{ccc} \pi_{F'} \times F'H(X) & \xrightarrow{u\times\xi_X} & \pi_F \times F(X) \\ \downarrow & & \downarrow \\ F'H(X) & \xrightarrow{\xi_X} & F(X) \end{array}$$

Here the vertical maps are given by $(g', x') \mapsto g'_{H(X)}(x)$ and $(g, x) \mapsto g_X(x)$ for $g' \in \pi_{F'}$, $x' \in F'H(X)$, $g \in \pi_F$, $x \in F(X)$. Forgetting the left $\pi_{F'}$-actions, ξ is regarded as an isomorphism $\xi : {}^tH(F') = F'H \to F$ in Γ. Let $\bar{\xi} : \pi_{{}^tH(F')} \to \pi_F$ be an isomorphism defined by $\bar{\xi}(\omega) = \xi\omega\xi^{-1}$. By the commutativity of the above diagram, $u(g')_X\xi_X = \xi_X g'_{H(X)}$ for any $g \in \pi_{F'}$ and $X \in \mathrm{Ob}\,\mathcal{C}$. Hence $u(g') = \xi g'_H \xi^{-1} = \bar{\xi}{}^tH(g')$ and we have $\bar{\xi}u_H = u$.

Let $U : \Gamma' \to \Gamma$ be a functor such that $U : \Gamma'(F', G') \to \Gamma(U(F'), U(G'))$ is continuous for some pair (F', G') of objects of Γ'. Since Γ' is a connected groupoid and the composition maps in Γ' and Γ are continuous, $U : \Gamma'(F', G') \to \Gamma(U(F'), U(G'))$ is continuous for every pair (F', G') of objects of Γ'. For $F' \in \mathrm{Ob}\,\Gamma$, we denote by $u : \pi_{F'} \to \pi_{U(F')}$ the continuous homomorphism $U : \Gamma'(F', F') \to \Gamma(U(F'), U(F'))$. There is a left exact functor $H : \mathcal{C} \to \mathcal{C}'$ preserving finite colimits with a natural equivalence $\xi_{F'} : F'H \to u^\sharp U(F')$. Let $\gamma_1, \gamma_2 : F' \to G'$ be morphisms in Γ' and set $g' = \gamma_2^{-1}\gamma_1 \in \pi_{F'}$. Since $U(g')\xi_{F'} = u(g')\xi_{F'} = \xi_{F'}{}^tH(g')$ by the above argument, we have $U(\gamma_1)\xi_{F'}{}^tH(\gamma_1)^{-1} = U(\gamma_2)\xi_{F'}{}^tH(\gamma_2)^{-1}$. This implies that, if we define $\xi_{G'} : {}^tH(G') \to U(G')$ by $\xi_{G'} = U(\gamma_1)\xi_{F'}{}^tH(\gamma_1)^{-1}$, $\xi_{G'}$ does not depend on the choice of γ_1. Thus we have a natural equivalence $\xi : {}^tH \to U$.

Summarizing the above arguments, we have the following result.

Proposition 6.6.3 *Let $\mathcal{C}$ and $\mathcal{C}'$ be Galois categories with fundamental groupoids Γ, Γ'.*

1) If $H : \mathcal{C} \to \mathcal{C}'$ is a functor satisfying the conditions of (6.6.1), $H^* : \check{\mathcal{C}}' \to \check{\mathcal{C}}$ induces a functor ${}^tH : \Gamma' \to \Gamma$ such that ${}^tH : \Gamma'(F', G') \to \Gamma(F'H, G'H)$ is continuous for $F', G' \in \mathrm{Ob}\,\Gamma$. In particular, if F' is a fundamental functor of $\mathcal{C}'$ and $F = F'H$, $u_H = {}^tH : \pi_{F'} \to \pi_F$ is a continuous homomorphism such that the following square commutes.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "H"] \arrow[d, "F"'] & \mathcal{C}' \arrow[d, "F'"] \\
B_f\pi_F \arrow[r, "u_H^\sharp"] & B_f\pi_{F'}
\end{tikzcd}$$

2) Suppose that F and F' are fundamental functors of $\mathcal{C}$ and $\mathcal{C}'$, respectively. For a continuous homomorphism $u : \pi_{F'} \to \pi_F$, there exist a functor $H : \mathcal{C} \to \mathcal{C}'$ satisfying the conditions of (6.6.1) and an isomorphism $\bar{\xi} : \pi_{{}^tH(F')} \to \pi_F$ such that $\bar{\xi}u_H = u$ and the following diagram commutes up to natural equivalence.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "H"] \arrow[d, "F"'] & \mathcal{C}' \arrow[d, "F'"] \\
B_f\pi_F \arrow[r, "u^\sharp"] & B_f\pi_{F'}
\end{tikzcd}$$

3) If $U : \Gamma' \to \Gamma$ is a functor such that $U : \Gamma'(F', G') \to \Gamma(U(F'), U(G'))$ is continuous for some pair (F', G') of objects of Γ', there exist a functor $H : \mathcal{C} \to \mathcal{C}'$ satisfying the conditions of (6.6.1) and a natural equivalence $\xi : {}^tH \to U$.

Let us denote by $\mathrm{Ex}(\mathcal{C}, \mathcal{C}')$ the full subcategory of $\mathrm{Funct}(\mathcal{C}, \mathcal{C}')$ consisting of left exact functors preserving finite colimits. We also denote by $\mathrm{Funct}_c(\Gamma, \Gamma')$ the full subcategory of $\mathrm{Funct}(\Gamma, \Gamma')$ consisting of functors $U : \Gamma' \to \Gamma$ such that $U : \Gamma'(F', G') \to \Gamma(U(F'), U(G'))$ is continuous for every pair (F', G') of objects of Γ'. We define a functor $T : \mathrm{Ex}(\mathcal{C}, \mathcal{C}') \to \mathrm{Funct}_c(\Gamma', \Gamma)$ by $T(H) = {}^tH$ and $T(\varphi)_{F'} = F'(\varphi)$ for $\varphi : H \to H'$ and $F' \in \mathrm{Ob}\,\Gamma$.

Proposition 6.6.4 *$T : \mathrm{Ex}(\mathcal{C}, \mathcal{C}') \to \mathrm{Funct}_c(\Gamma', \Gamma)$ is an equivalence of categories.*

Proof. We have already seen that, for any $U \in \mathrm{Ob}\,\mathrm{Funct}_c(\Gamma', \Gamma)$, there exists $H \in \mathrm{Ob}\,\mathrm{Ex}(\mathcal{C}, \mathcal{C}')$ such that $T(H)$ is isomorphic to U. It remains to show that T is fully faithful. Let $\varphi, \psi : H \to H'$ be morphisms in $\mathrm{Ex}(\mathcal{C}, \mathcal{C}')$ such that $T(\varphi) = T(\psi)$. Then, $F'(\varphi_X) = F'(\psi_X) : F'H(X) \to F'H'(X)$ for any $F' \in \mathrm{Ob}\,\Gamma$ and $X \in \mathrm{Ob}\,\mathcal{C}$. Since F' is fully faithful regarded as a functor from $\mathcal{C}'$ to $B_f\pi_{F'}$, we have $\varphi_X = \psi_X$ for any $X \in \mathrm{Ob}\,\mathcal{C}$. Thus T is faithful. Let $\chi : T(H) \to T(H')$ be a morphism in $\mathrm{Funct}_c(\Gamma', \Gamma)$. For $F' \in \Gamma$ and $X \in \mathrm{Ob}\,\mathcal{C}$, $(\chi_{F'})_X : F'H(X) = T(H)(F')(X) \to T(H')(F')(X) = F'H'(X)$ is a morphism of $B_f\pi_{F'}$. In fact, if $g \in \pi_{F'}$, $T(H')(g)\chi_{F'} = \chi_{F'}T(H)(g)$ by the naturality of χ. Hence $(\chi_{F'})_X g_{H(X)} = g_{H'(X)}(\chi_{F'})_X$. Since $F' : \mathcal{C}' \to B_f\pi_{F'}$ is an equivalence, there is a unique morphism $\varphi_X : H(X) \to H'(X)$ in $\mathcal{C}$ such that $F'(\varphi_X) = (\chi_{F'})_X$. For a morphism $f : X \to Y$, we have $F'H'(f)(\chi_{F'})_X = (\chi_{F'})_Y F'H(f)$ by the naturality of $\chi_{F'} : F'H \to F'H'$. Since F' is faithful, $H'(f)\varphi_X = \varphi_Y H(f)$. Thus we have a morphism $\varphi : H \to H'$ in $\mathrm{Ex}(\mathcal{C}, \mathcal{C}')$ such that $T(\varphi)_{F'} = \chi_{F'}$.

Let G' be an object of Γ'. There is an isomorphism $\gamma : F' \to G'$ and $\chi_{G'}\gamma_H = \gamma_{H'}\chi_{F'}$ holds by the naturality of χ. It follows that $\chi_{G'}\gamma_H = \gamma_{H'}\chi_{F'} = \gamma_{H'}T(\varphi)_{F'} = \gamma_{H'}F'(\varphi) = G'(\varphi)\gamma_H = T(\varphi)_{G'}\gamma_H$. Hence $\chi_{G'} = T(\varphi)_{G'}$ and this implies $\chi = T(\varphi)$. □

Proposition 6.6.5 *1) The following square commutes up to a natural equivalence.*

$$\begin{tikzcd} \mathrm{Pro}(\mathcal{C}) \arrow[r, "\Psi_c"] \arrow[d, "\mathrm{Pro}(H)"] & \mathrm{Funct}_c(\Gamma, \mathcal{T}_{tdc}) \arrow[d, "{}^tH^*"] \\ \mathrm{Pro}(\mathcal{C}') \arrow[r, "\Psi_c"] & \mathrm{Funct}_c(\Gamma', \mathcal{T}_{tdc}) \end{tikzcd}$$

2) For a morphism $\rho : F \to F'$ in Γ, regarding F and F' as functors $\mathcal{C} \to B_f\pi_F$ and $\mathcal{C} \to B_f\pi_{F'}$, there is a natural equivalence $\tilde{\rho} : F \to \varpi(\rho)^\sharp F'$.

3) Let $H, H' : \mathcal{C} \to \mathcal{C}'$ be functors satisfying the conditions of (6.6.1) and $\zeta : H \to H'$ a natural transformation. For a fundamental functor F' of $\mathcal{C}'$, $\varpi(F'(\zeta))u_H = u_{H'}$.

Proof. 1) We define $\lambda_{D,F'} : ({}^tH^*\Psi_c(D))(F') \to (\Psi_c\mathrm{Pro}(H)(D))(F')$ for $D \in \mathrm{Ob}\,\mathrm{Pro}(\mathcal{C})$ and $F' \in \mathrm{Ob}\,\Gamma'$ to be the following composition.

$$\check{\mathcal{C}}(L(D), {}^tH(F')) \xrightarrow{\cong} \check{\mathcal{C}}'(H_!L(D), F') \xrightarrow{\cong"} \check{\mathcal{C}}'(L\mathrm{Pro}(H)(D), F')$$

Here, the first bijection comes from the adjointness of $H_!$ and H^* and the second one is induced by the natural equivalence $L\mathrm{Pro}(H) \cong H_!L$ (6.1.6). It is obvious that $\lambda_{D,F'}$ is natural in both D and F'. Thus we have a natural equivalence $\lambda : {}^tH^*\Psi_c \to \Psi_c\mathrm{Pro}(H)$.

2) For $X \in \mathrm{Ob}\,\mathcal{C}$ and $x \in F(X)$, define $\tilde{\rho}$ by $\tilde{\rho}_X(x) = \rho_X(x)$. Then, for $g \in \pi_F$, $\tilde{\rho}_X(gx) = \rho_X(g_X(x)) = \rho_X g_X \rho_X^{-1}\rho_X(x) = \varpi(\rho)(g)_X(\tilde{\rho}_X(x)) = \varpi(\rho)(g)\tilde{\rho}_X(x)$. Hence $\tilde{\rho}_X : F(X) \to \varpi(\rho)^\sharp F'(X)$ is an isomorphism of left π_F-spaces.

3) For $g \in \pi_{F'}$, by the naturality of $g : F' \to F'$, $F'(\zeta)_{X'}g_{H(X')} = g_{H'(X')}F'(\zeta)_{X'}$. Hence $\varpi(F'(\zeta))(g_H) = g_{H'}$. □

Let Π and Π' be the fundamental pro-groups (6.5.7) of $\mathcal{C}$ and $\mathcal{C}'$ with multiplications $\mu : \Pi \times \Pi \to \Pi$ and $\mu' : \Pi' \times \Pi' \to \Pi'$, respectively. For a left exact functor $H : \mathcal{C} \to \mathcal{C}'$ preserving finite colimits, since $\mathrm{Pro}(H) : \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{C}')$ has a left adjoint by (6.1.20), $\mathrm{Pro}(H)$ is left exact. Hence the morphism $\nu : \mathrm{Pro}(H)(\Pi \times \Pi) \to \mathrm{Pro}(H)(\Pi) \times \mathrm{Pro}(H)(\Pi)$ induced by $\mathrm{Pro}(H)(\mathrm{pr}_i) : \mathrm{Pro}(H)(\Pi \times \Pi) \to \mathrm{Pro}(H)(\Pi)$ $(i = 1, 2)$ is an isomorphism and $\mathrm{Pro}(H)(\Pi)$ is a group object in $\mathrm{Pro}(\mathcal{C}')$ with multiplication $\mathrm{Pro}(H)(\mu)\nu^{-1} : \mathrm{Pro}(H)(\Pi) \times \mathrm{Pro}(H)(\Pi) \to \mathrm{Pro}(H)(\Pi)$. For $F' \in \mathrm{Ob}\,\Gamma'$, let $r_{F'} : \Psi_c(\Pi')(F') \to \Psi_c(\mathrm{Pro}(H)(\Pi))(F')$ be the following composition.

$$\Psi_c(\Pi')(F') \xrightarrow[\cong]{\phi_{F'}} \pi_{F'} \xrightarrow{{}^tH} \pi_{{}^tH(F')} \xrightarrow[\cong]{\phi^{-1}_{{}^tH(F')}} {}^tH^*\Psi_c(\Pi)(F') \xrightarrow[\cong]{\lambda_{\Pi,F'}} \Psi_c(\mathrm{Pro}(H)(\Pi))(F')$$

Clearly, $r_{F'}$ is natural in F' and we have a morphism $r : \Psi_c(\Pi') \to \Psi_c(\mathrm{Pro}(H)(\Pi))$ in $\mathrm{Funct}_c(\Gamma, \mathcal{T}_{tdc})$. By (6.5.6), there is a unique morphism $\rho_H : \Pi' \to \mathrm{Pro}(H)(\Pi)$ such that $\Psi_c(\rho_H) = r$. By the naturality of $\lambda : {}^tH^*\Psi_c \to \Psi_c\mathrm{Pro}(H)$, the following diagram commutes.

$$\begin{tikzcd} \Psi_c(\Pi)^tH \times \Psi_c(\Pi)^tH \arrow[r, "\lambda_{\Pi,{}^tH}\times\lambda_{\Pi,{}^tH}"] & \Psi_c(\mathrm{Pro}(H)(\Pi)) \times \Psi_c(\mathrm{Pro}(H)(\Pi)) \\ \Psi_c(\Pi \times \Pi)^tH \arrow[u, "(\Psi_c(\mathrm{pr}_1)_{{}^tH}, \Psi_c(\mathrm{pr}_2)_{{}^tH})"] \arrow[r, "\lambda_{\Pi\times\Pi,{}^tH}"] \arrow[d, "\Psi_c(\mu)_{{}^tH}"] & \Psi_c(\mathrm{Pro}(H)(\Pi \times \Pi)) \arrow[u, "(\Psi_c(\mathrm{Pro}(H)(\mathrm{pr}_1)), \Psi_c(\mathrm{Pro}(H)(\mathrm{pr}_2)))"'] \arrow[d, "\Psi_c(\mathrm{Pro}(\mu))"] \\ \Psi_c(\Pi)^tH \arrow[r, "\lambda_{\Pi,{}^tH}"] & \Psi_c(\mathrm{Pro}(H)(\Pi)) \end{tikzcd}$$

It follows that $\lambda_{\Pi,F'} : \Psi_c(\Pi)^tH(F') = {}^tH^*\Psi_c(\Pi)(F') \to \Psi_c\mathrm{Pro}(H)$ is a homomorphism of groups. Hence so is $r_{F'} : \Psi_c(\Pi')(F') \to \Psi_c(\mathrm{Pro}(H)(\Pi))(F')$ and $\rho_H : \Pi' \to \mathrm{Pro}(H)(\Pi)$ is a morphism of pro-groups in $\mathcal{C}'$. If $\varphi : H \to H'$ is a morphism in $\mathrm{Ex}(\mathcal{C}, \mathcal{C}')$, ${}^tH' : \pi_{F'} \to \pi_{{}^tH'(F')}$ is the composition $\pi_{F'} \xrightarrow{{}^tH} \pi_{{}^tH(F')} \xrightarrow{\varpi(F'(\varphi))} \pi_{{}^tH'(F')}$. Hence, by the naturality of ϕ_F in F (6.5.7), $\rho_{H'} = \mathrm{Pro}(\varphi)_\Pi\rho_H$.

A coproduct of the terminal objects in a category is called a constant object. If π is a pro-group, $X \in B_c\pi$ is a constant object if and only if X is a trivial finite π-space.

Proposition 6.6.6 *Let $H : \mathcal{C} \to \mathcal{C}'$ be a left exact functor preserving finite colimits between Galois categories. We fix a fundamental functor F' of $\mathcal{C}'$ and put $F = F'H$ and $u_H = {}^tH : \pi_{F'} \to \pi_F$. Suppose that (X, a) is an object of $\mathcal{C}_F$ such that X is connected and that U is an open subgroup of π_F such that the map $\rho : \pi_F \to F(X)$ defined by $\rho(g) = ga$ is surjective and $U = \rho^{-1}(a)$.*

1) U contains the image of u_H if and only if there is a morphism $(1_{\mathcal{C}'}, *) \to (H(X), a)$ in $\mathcal{C}'_{F'}$, where $1_{\mathcal{C}'}$ is a terminal object of $\mathcal{C}'$.

2) U contains the closed normal subgroup of π_F generated by the image of u_H if and only if $H(X)$ is a constant object of $\mathcal{C}'$.

Proof. 1) Suppose that U contains the image of u_H, that is, $u_H(g) \in U$ for any $g \in \pi_{F'}$. Hence $ga = u_H(g)\rho(e) = \rho(u_H(g)) = a$ in $F'(H(X)) = F(X)$. This implies that there is a $\pi_{F'}$-map $\epsilon : F'(1_{\mathcal{C}'}) = \{*\} \to F'(H(X))$ which maps $*$ to a. Since $F' : \mathcal{C}' \to B_f\pi_{F'}$ is fully faithful, there is a unique morphism $f : 1_{\mathcal{C}'} \to H(X)$ such that $F'(f) = \epsilon$.

Conversely, suppose that there is a morphism $f : (1_{\mathcal{C}'}, *) \to (H(X), a)$ in $\mathcal{C}'_{F'}$. Then, $a \in F'(H(X))$ is a fixed point of the left $\pi_{F'}$-action. Hence, for any $g \in \pi_{F'}$, $\rho(u_H(g)) = u_H(g)a = ga = a$ in $F(X) = F'(H(X))$. It follows that $u_H(g) \in \rho^{-1}(a) = U$.

2) Suppose that U contains the closed normal subgroup of π_F generated by the image of u_H. For any $x \in F'(H(X)) = F(X)$, there exists $b \in \pi_F$ such that $x = ba$ in $F(X)$. Since $b^{-1}u_H(g)b$ belongs to the normal subgroup of π_F generated by the image of u_H if $g \in \pi_{F'}$, $b^{-1}u_H(g)b \in U$. Then, $gx = u_H(g)ba = b(b^{-1}u_H(g)b)a = b\rho(b^{-1}u_H(g)b) = ba = x$. Hence $F'(H(X))$ is a trivial left $\pi_{F'}$-set, namely, a constant object of $B_f\pi_{F'}$. Since F' preserves terminal objects and finite coproducts, there is a constant object C in $\mathcal{C}'$ such that $F'(C)$ is isomorphic to $F'(H(X))$. Since F' is fully faithful, there is an isomorphism $C \to H(X)$.

Conversely, suppose that $H(X)$ is a constant object of $\mathcal{C}'$. Then, $F'(H(X)) = F(X)$ is a trivial left $\pi_{F'}$-set. For any $g \in \pi_{F'}$ and $b \in \pi_F$, $\rho(b^{-1}u_H(g)b) = b^{-1}u_H(g)ba = b^{-1}(g(ba)) = b^{-1}(ba) = a$. Thus $b^{-1}u_H(g)b \in U$ and this implies that U contains the normal subgroup of π_F generated by the image of u_H. Since U is closed and the closure of a normal subgroup is normal, U contains the closed normal subgroup of π_F generated by the image of u_H. □

Corollary 6.6.7 *$u_H : \pi_{F'} \to \pi_F$ is trivial if and only if $H(X)$ is a constant object of $\mathcal{C}'$ for any $X \in \mathrm{Ob}\,\mathcal{C}$.*

Proof. If $H(X)$ is a constant object of $\mathcal{C}'$ for any $X \in \mathrm{Ob}\,\mathcal{C}$, the image of u_H is contained in every open subgroup of π_F by (6.6.6). It follows from (6.3.7) that the image of u_H is consists of a single element e. The converse is obvious from (6.6.6). □

For a $X \in \mathrm{Ob}\,\mathcal{C}$, a connected component of X is a connected subobject Y of X such that the map $F(Y) \to F(X)$ induced by the inclusion morphism is an isomorphism onto an π_F-orbit of $F(X)$. We note that this definition does not depend on the choice of the fundamental functor F.

Proposition 6.6.8 *Let X' be a connected object of $\mathcal{C}'$ and U' an open subgroup of $\pi_{F'}$ such that the map $\rho' : \pi_{F'} \to F'(X')$ defined by $\rho'(g') = g'a'$ $(a' \in F'(X'))$ satisfies ${\rho'}^{-1}(a') = U'$. U' contains $\mathrm{Ker}\,u_H$ if and only if there exist an object (X, a) of $\mathcal{C}_F$ such that X is connected and a morphism $(X'_0, a) \to (X', a')$ in $\mathcal{C}'_{F'}$, where X'_0 is the connected component of $H(X)$ such that $a \in F'(X'_0)$. If u_H is surjective, $U' \supset \mathrm{Ker}\,u_H$ if and only if there exist an object (X, a) of $\mathcal{C}_F$ and an isomorphism $(H(X), a) \to (X', a')$.*

Proof. Suppose that there exist an object (X, a) of $\mathcal{C}_F$ such that X is connected and a morphism $f : (X'_0, a) \to (X', a')$ in $\mathcal{C}'_{F'}$, where X'_0 is the connected component of $H(X)$ such that $a \in F'(X'_0)$. Define maps $\rho : \pi_F \to F(X)$ and $\rho_0 : \pi_{F'} \to F'(X'_0)$ by $\rho(g) = ga$ and $\rho_0(h) = ha$. Then, ρ, ρ_0 are surjective and $\rho u_H = F'(i)\rho_0$, where $i : X'_0 \to H(X)$ is the inclusion morphism. Since $F'(f)(a) = a'$ and $F'(f)$ is a left $\pi_{F'}$-map, $\rho' = F'(f)\rho_0$. Therefore $U' = {\rho'}^{-1}(a') = \rho_0^{-1}F'(f)^{-1}(a') \supset \rho_0^{-1}(a) = \rho_0^{-1}F'(i)^{-1}(a) = u_H^{-1}\rho^{-1}(a) = u_H^{-1}(U) \supset \mathrm{Ker}\,u_H$.

Suppose that $U' \supset \mathrm{Ker}\,u_H$. It follows from (6.3.10) that there is an open subgroup U of π_F such that $u_H^{-1}(U) \subset U'$. Let X be an object of $\mathcal{C}$ such that $F(X)$ is isomorphic to π_F/U as a left π_F-space. Choose $a \in F(X)$ and define $\rho : \pi_F \to F(X)$ by $\rho(g) = ga$. We note that $F'(H(X))$ is a finite discrete space $F(X)$ having a left $\pi_{F'}$-action $(g', x) \mapsto u_H(g')x$. Hence if X_0 is a connected component of $H(X)$ such that $a \in F'(X'_0)$, the map $\rho_0 : \pi_{F'} \to F'(X_0)$ defined by $\rho_0(g') = u_H(g')a$ is surjective and $\rho_0^{-1}(a) = u_H^{-1}(U) \subset U'$. Thus we have a left $\pi_{F'}$-map $\varphi : F'(X_0) \to F'(X')$ satisfying $\varphi\rho_0 = \rho'$, in particular, $\varphi(a) = a'$. Since $F' : \mathcal{C}' \to B_f\pi_{F'}$ is fully faithful, there is a morphism $f : X_0 \to X'$ such that $F'(f) = \varphi$. Therefore we have a morphism $f : (X'_0, a) \to (X', a')$ in $\mathcal{C}'_{F'}$.

Assume that u_H is surjective. Then, ρu_H is surjective and $F'(H(X)) = F(X)$ is connected as a left $\pi_{F'}$-space. Hence $X'_0 = H(X)$ in this case. Since $\pi_{F'}$ is compact, u_H is a quotient map. If $U' \supset \operatorname{Ker} u_H$, then $u_H^{-1}(u_H(U')) = U'$ and it follows that $u_H(U')$ is an open subgroup of π_F. Hence the above U is defined to be $u_H(U')$. Then, since $u_H^{-1}(U) = U'$, the above $\varphi : F'(H(X)) \to F'(X)$ is an isomorphism. □

Choosing "the base points" a, a' of $F(X)$, $F'(X')$ properly, we have the following result.

Corollary 6.6.9 *Let U' be an open subgroup of $\pi_{F'}$ and X' an object of $\mathcal{C}'$ such that $F(X')$ is isomorphic to $\pi_{F'}/U'$ as a left $\pi_{F'}$-space. U' contains $\operatorname{Ker} u_H$ if and only if there exist an object X of $\mathcal{C}$ and a morphism $X'_0 \to X'$ in $\mathcal{C}'$, where X'_0 is a connected component of $H(X)$. If u_H is surjective, $U' \supset \operatorname{Ker} u_H$ if and only if there exist an object X of $\mathcal{C}$ and an isomorphism $H(X) \to X'$.*

Proof. Suppose that U' contains $\operatorname{Ker} u_H$. Choose $a' \in F'(X')$ and consider a surjection $\rho' : \pi_{F'} \to F'(X')$ defined by $\rho'(g') = g'a'$. By (6.6.8), there exist an object (X, a) of $\mathcal{C}_F$ such that X is connected and a morphism $f : (X'_0, a) \to (X', a')$ in $\mathcal{C}'_{F'}$, where X'_0 is the connected component of $H(X)$ such that $a \in F'(X'_0)$. Thus we have a morphism $f : X'_0 \to X'$.

We show the converse. By the assumption, there exist $X \in \operatorname{Ob}\mathcal{C}$ and a morphism f from a connected component X'_0 of $H(X)$ to X'. Since H preserves finite coproducts, we may assume that X is connected. Choose $a \in F'(X'_0)$ and set $F'(f)(a) = a'$. Hence we have a morphism $f : (X'_0, a) \to (X', a')$ in $\mathcal{C}'_{F'}$. By (6.6.8), we see that U' contains $\operatorname{Ker} u_H$. □

Corollary 6.6.10 *$u_H : \pi_{F'} \to \pi_F$ is injective if and only if, for any $X' \in \operatorname{Ob}\mathcal{C}'$, there exist an object X of $\mathcal{C}$ and a morphism from a connected component of $H(X)$ to X'.*

Proof. Suppose that u is injective. For $X' \in \operatorname{Ob}\mathcal{C}'$, choose a connected component Y' of X' and an element $a' \in F'(Y')$. We denote by $i : Y' \to X'$ the inclusion morphism. By (6.6.9), there exist an object X of $\mathcal{C}$ and a morphism $f : X'_0 \to Y'$ in $\mathcal{C}'$, where X'_0 is a $H(X)$. Thus we have a morphism $if : X'_0 \to X'$.

Since the intersection of all open subgroups of $\pi_{F'}$ is the trivial subgroup $\{e\}$ by (6.3.7), the converse easily follows from (6.6.9). □

Since $F'H = u_H^\sharp F$ and $F : \mathcal{C} \to B_f\pi_F$, $F' : \mathcal{C} \to B_f\pi_{F'}$ are equivalences, H is fully faithful if and only if $u_H^\sharp$ is so and $X \in \operatorname{Ob}\mathcal{C}$ (resp. $X' \in \operatorname{Ob}\mathcal{C}'$) is connected if and only if $F(X) \in \operatorname{Ob} B_f\pi_F$ (resp. $F(X') \in \operatorname{Ob} B_f\pi'_F$) is so. Hence the following result is a direct consequence of (6.3.24).

Proposition 6.6.11 *The following conditions are equivalent.*

i) $u_H : \pi_{F'} \to \pi_F$ is surjective.
ii) For any connected object X of $\mathcal{C}$, $H(X)$ is connected.
iii) H is fully faithful.

Corollary 6.6.12 *The following conditions are equivalent.*

i) $u_H : \pi_{F'} \to \pi_F$ is an isomorphism.
ii) Every $X' \in \operatorname{Ob}\mathcal{C}'$ is isomorphic to $H(X)$ for some $X \in \operatorname{Ob}\mathcal{C}$ and, if $X \in \operatorname{Ob}\mathcal{C}$ is connected, $H(X)$ is connected.
iii) H is an equivalence of categories.

Proof. Since $\pi_{F'}$ is compact and π_F is Hausdorff, u is an isomorphism if and only if it is bijective. It follows from (6.6.11) and (6.6.9) that the above conditions are equivalent. □

Proposition 6.6.13 *$u_H : \pi_{F'} \to \pi_F$ is a split monomorphism if and only if there is a functor $K : \mathcal{C}' \to \mathcal{C}$ satisfying the conditions of (6.6.1) and a natural equivalence $\zeta : HK \to id_{\mathcal{C}'}$. In particular, every $X' \in \operatorname{Ob}\mathcal{C}'$ is isomorphic to $H(X)$ for some $X \in \operatorname{Ob}\mathcal{C}$ in this case.*

Proof. Let $v : \pi_F \to \pi_{F'}$ be a continuous homomorphism such that $vu_H = id_{F'}$. By 2) of (6.6.3), there exist a functor $K : \mathcal{C}' \to \mathcal{C}$ satisfying the conditions of (6.6.1) and a natural equivalence $\xi : FK \to v^\sharp F'$. For $X' \in \operatorname{Ob}\mathcal{C}'$, there is an isomorphism

$$F'HK(X') = u_H^\sharp FK(X') \xrightarrow{u_H^\sharp(\xi_{X'})} u_H^\sharp v^\sharp F'(X') = F'(X)$$

of left $\pi_{F'}$-spaces. Since $F' : \mathcal{C}' \to B_f\pi_{F'}$ is fully faithful, there is an isomorphism $\zeta_{X'} : HK(X') \to X'$ such that $F'(\zeta_{X'}) = u_H^\sharp(\xi_{X'})$. If $f : X' \to Y'$ is a morphism in $\mathcal{C}'$, $\xi_{Y'}FK(f) = v^\sharp F'(f)\xi_{X'}$ by the naturality of ξ. Hence $F'(\zeta_{Y'}HK(f)) = F'(\zeta_{Y'})F'HK(f) = u_H^\sharp(\xi_{Y'})u_H^\sharp FK(f) = u_H^\sharp(\xi_{Y'}FK(f)) = u_H^\sharp(v^\sharp F'(f)\xi_{X'}) = F'(f)u_H^\sharp(\xi_{X'}) = F'(f)F'(\zeta_{X'}) = F'(f\zeta_{X'})$ and it follows that $\zeta_{Y'}HK(f) = f\zeta_{X'}$. Thus we have a natural equivalence $\zeta : HK \to id_{\mathcal{C}'}$.

Conversely, let $K : \mathcal{C}' \to \mathcal{C}$ be a functor satisfying the conditions of (6.6.1) and $\zeta : HK \to id_{\mathcal{C}'}$ a natural equivalence. Then, $\varpi(F'(\zeta))u_K u_H = \varpi(F'(\zeta))u_{HK} = id_{\pi_F}$ by 3) of (6.6.5). Therefore, u_H has a left inverse $\varpi(F'(\zeta))u_K$. □

Proposition 6.6.14 *Let $H : \mathcal{C} \to \mathcal{C}'$ and $H' : \mathcal{C}' \to \mathcal{C}''$ be functors between Galois categories satisfying the conditions of (6.6.1). For a fundamental functor F'' of $\mathcal{C}''$, we put $F' = F''H'$ and $F = F'H$.* $\operatorname{Ker} u_H \supset \operatorname{Im} u_{H'}$ *if and only if $H'H(X)$ is a constant object for any $X \in \operatorname{Ob}\mathcal{C}$.* $\operatorname{Ker} u_H \subset \operatorname{Im} u_{H'}$ *if and only if, for any $(X', a') \in \operatorname{Ob}\mathcal{C}'_{F'}$ such that X' is connected and there is a morphism $(1_{\mathcal{C}''}, *) \to (H'(X'), a')$ in $\mathcal{C}''_{F''}$, there exist an object X of $\mathcal{C}$ and a morphism $X'_0 \to X'$ in $\mathcal{C}'$, where X'_0 is a connected component of $H(X)$.*

Proof. Since $u_H u_{H'} = u_{H'H}$, the former assertion is a direct consequence of (6.6.7).

Suppose that $\operatorname{Ker} u_H \subset \operatorname{Im} u_{H'}$. For $(X', a') \in \operatorname{Ob}\mathcal{C}'_{F'}$ such that X' is connected and there is a morphism $(1_{\mathcal{C}''}, *) \to (H'(X'), a')$ in $\mathcal{C}''_{F''}$. Define a map $\rho' : \pi_{F'} \to F'(X')$ by $\rho'(g') = g'a'$ and set $U' = {\rho'}^{-1}(a')$. Then U' is an open subgroup of $\pi_{F'}$ containing $im\, u_{H'}$ by (6.6.6). Hence $\operatorname{Ker} u_H \subset U'$ and, by (6.6.9), there exist an object X of $\mathcal{C}$ and a morphism $X'_0 \to X'$ in $\mathcal{C}'$, where X'_0 is a connected component of $H(X)$. To show the converse, it suffices to show that each open subgroup U' containing $\operatorname{Im} u_{H'}$ also contains $\operatorname{Ker} u_H$ by (6.3.17). Let X' be an object of $\mathcal{C}'$ such that $F'(X')$ is isomorphic to $\pi_{F'}/U'$ as a left $\pi_{F'}$-space. We choose $a' \in F'(X')$. Since $U' \supset \operatorname{Im} u_{H'}$, it follows from (6.6.6) that there is a morphism $(1_{\mathcal{C}''}, *) \to (H'(X'), a')$ in $\mathcal{C}''_{F''}$. Then, by the assumption, there exist an object X of $\mathcal{C}$ and a morphism $X'_0 \to X'$ in $\mathcal{C}'$, where X'_0 is a connected component of $H(X)$. Thus (6.6.9) implies that U' contains $\operatorname{Ker} u_H$. □

Let $\mathcal{C}$ be a Galois category with a fundamental functor F and (S, a) an object of $\mathcal{C}_F$ such that S is connected. Define a functor $F' : \mathcal{C}/S \to \mathcal{U}$-**Ens** as follows. For $(X \xrightarrow{p} S) \in \operatorname{Ob}\mathcal{C}/S$, set $F'(X \xrightarrow{p} S) = F(p)^{-1}(a)$ and, for $f : (X \xrightarrow{p} S) \to (Y \xrightarrow{q} S)$, $F'(f) : F(p)^{-1}(a) \to F(q)^{-1}(a)$ is the restriction of $F(f) : F(X) \to F(Y)$. Let $\rho : \pi_F \to F(S)$ be the map given by $\rho(g) = ga$. We set $U = \rho^{-1}(U)$ and $H = S^* : \mathcal{C} \to \mathcal{C}/S$ (See (A.3.9)).

Proposition 6.6.15 *1) $\mathcal{C}/S$ is a Galois category with a fundamental functor F'.*

2) H is left exact and preserves finite colimits and there is a natural equivalence $\psi : F'H \to F$.

3) $\varpi(\psi)u_H : \pi_{F'} \to \pi_F$ is an isomorphism onto U.

Proof. 1) Since $\mathcal{C}$ satisfies G1) and G2), so does $\mathcal{C}/S$ by (A.3.11). Let $f : (X \xrightarrow{p} S) \to (Y \xrightarrow{q} S)$ be a morphism in $\mathcal{C}/S$. Then $f = \iota\rho$ for a regular epimorphism $\rho : X \to Z$ and a monomorphism $\iota : Z \to Y$ in $\mathcal{C}$ such that there is an isomorphism $s : Z \coprod W \to Y$ for some $W \in \operatorname{Ob}\mathcal{C}$ satisfying $\iota = s\iota_1$, where $\iota_1 : Z \to Z \coprod W$ is the canonical morphism into the first summand. Let $R \underset{b}{\overset{a}{\rightrightarrows}} X$ be a kernel pair of ρ. By (A.8.14), ρ is a coequalizer of $R \underset{b}{\overset{a}{\rightrightarrows}} X$. Since $pa = qfa = q\iota\rho a = q\iota\rho b = qfb = pb$, there is a unique morphism $r : Z \to S$ such that $r\rho = p$. Then, $q\iota\rho = qf = p = r\rho$ and it follows that $q\iota = r$. Thus we have morphisms $\rho : (X \xrightarrow{p} S) \to (Z \xrightarrow{r} S)$ and $\iota : (Z \xrightarrow{r} S) \to (Y \xrightarrow{q} S)$ in $\mathcal{C}/S$. Set $q' = qs : Z \coprod W \to S$ and $t = q'\iota_2 : W \to S$ where $\iota_2 : W \to Z \coprod W$ is the canonical morphism into the second summand. Note that $\iota_1 : (Z \xrightarrow{r} S) \to (Z \coprod W \xrightarrow{q'} S)$, $\iota_2 : (W \xrightarrow{t} S) \to (Z \coprod W \xrightarrow{q'} S)$ and $s : (Z \coprod W \xrightarrow{q'} S) \to (Y \xrightarrow{q} S)$ are morphisms in $\mathcal{C}/S$. Therefore, $\mathcal{C}/S$ satisfies G3).

Let us denote by $i_0 : \{a\} \to F(S)$ the inclusion map. Then, F' is a composition $\mathcal{C}/S \xrightarrow{F/S} \mathcal{U}\text{-}\mathbf{Ens}/F(S) \xrightarrow{i_0^*} \mathcal{U}\text{-}\mathbf{Ens}/\{a\} \xrightarrow[\cong]{\Sigma_{\{a\}}} \mathcal{U}\text{-}\mathbf{Ens}$. Clearly, F/S preserves terminal objects and $\Sigma_{F(S)}(F/S) = F\Sigma_S$. It follows from (A.3.11) that F/S preserves finite limits, finite coproducts and quotients by a finite group of automorphisms. Moreover, since Σ_S has a right adjoint S^* (A.3.9), it preserves regular epimorphisms. On the other hand, $\Sigma_{F(S)}$ reflects regular epimorphisms (A.3.11). It follows that F/S preserves regular epimorphisms. Since i_0^* has a left adjoint by (A.3.9), it preserves limits. By 2) of (6.4.3) and (A.4.5), i_0^* preserves finite coproducts and quotients by a finite group of automorphisms. It is obvious that i_0^* preserves epimorphisms. Hence F' satisfies G4) and G5). Let $f : (X \xrightarrow{p} S) \to (Y \xrightarrow{q} S)$ be a morphism in $\mathcal{C}/S$ such that $F'(f) : F(p)^{-1}(a) \to F(q)^{-1}(a)$ is a bijection. If $F(f)(x) = F(f)(y)$ for $x, y \in F(X)$, then $F(p)(x) = F(qf)(x) = F(q)F(f)(x) = F(q)F(f)(y) =$

$F(qf)(y) = F(p)(y)$. Since S is connected, there exists $g \in \pi_F$ such that $F(p)(x) = ga$ by (6.5.2). Note that $F(p)$, $F(q)$ and $F(f)$ are left π_F-maps. Hence $g^{-1}x, g^{-1}y \in F(p)^{-1}(a)$ and $F(f)(g^{-1}x) = g^{-1}F(f)(x) = g^{-1}F(f)(y) = F(f)(g^{-1}y)$. By the assumption, we have $g^{-1}x = g^{-1}y$, that is, $x = y$. Thus $F(f)$ is injective. For $z \in F(Y)$, there exists $h \in \pi_F$ such that $F(q)(z) = ha$. Then, $h^{-1}z \in F(q)^{-1}(a)$ and there is a unique $w \in F(p)^{-1}(a)$ such that $F(f)(w) = h^{-1}z$. Hence $F(f)(hw) = z$ and this shows that F is surjective. Therefore f is an isomorphism in $\mathcal{C}$ by G6) for F. It follows that F' satisfies G6).

2) For $X \in \mathrm{Ob}\,\mathcal{C}$, let $\psi_X : F'H(X) \to F(X)$ be the composition of the inclusion map $F'H(X) = F(\mathrm{pr}_2)^{-1}(a) \hookrightarrow F(X \times S)$ and $F(\mathrm{pr}_1) : F(X \times S) \to F(X)$. Obviously, ψ_X is natural in X. Since F preserves products, $(F(\mathrm{pr}_1), F(\mathrm{pr}_2)) : F(X \times S) \to F(X) \times F(S)$ is bijective. Hence, for $x \in F(X)$, there exists a unique $\varphi(x) \in F(X \times S)$ such that $F(\mathrm{pr}_1)(\varphi(x)) = x$ and $F(\mathrm{pr}_2)(\varphi(x)) = a$. Thus the inverse of ψ_X is given by $x \mapsto \varphi(x)$. We have a natural equivalence $\psi : F'H \to F$. It follows that $F'H$ is also a fundamental functor of $\mathcal{C}$. By (6.6.1), H is left exact and preserves finite colimits.

3) We denote by $1_{\mathcal{C}/S}$ the terminal object $(S \xrightarrow{id_S} S)$ of $\mathcal{C}/S$. Let $\Delta : 1_{\mathcal{C}/S} \to (S \times S \xrightarrow{\mathrm{pr}_2} S) = H(S)$ be the diagonal morphism. We note that $F'(1_{\mathcal{C}}/S) = \{a\}$ and $\psi_S(F'(\Delta)(a)) = F(\mathrm{pr}_1)F(\Delta)(a) = a$. For $g' \in \pi_{F'}$, $\rho(\varpi(\psi)u_H(g')) = \rho(\psi g'_H \psi^{-1}) = \psi_S g'_{H(S)}\psi_S^{-1}(a) = \psi_S g'_{H(S)}F'(\Delta)(a) = \psi_S F'(\Delta)g'_{1_{\mathcal{C}/S}}(a) = \psi_S F'(\Delta)(a) = a$ by the naturality of g'. Hence the image of $\varpi(\psi)u_H$ is contained in U. For any $g \in U$ and $(X \xrightarrow{p} S) \in \mathrm{Ob}\,\mathcal{C}/S$, since $F(p) : F(X) \to F(S)$ is a left π_F-map and $g_S : F(S) \to F(S)$ fixes a, $g_X : F(X) \to F(X)$ maps $F(p)^{-1}(a)$ into $F(p)^{-1}(a)$. Let $g' \in \pi_{F'}$ be an element such that $g'_{(X \xrightarrow{p} S)} : F'(X \xrightarrow{p} S) \to F'(X \xrightarrow{p} S)$ is the restriction of g_X. Then, $\varpi(\psi)u_H(g') = g$, that is, $\psi g'_H = g\psi$. In fact, for $x \in F'H(X)$, $\psi_X g'_{H(X)}(x) = F(\mathrm{pr}_1)g_{X\times S}(x) = g_X F(\mathrm{pr}_1)(x) = g_X\psi_X(x)$. Thus $\varpi(\psi)u_H$ is a surjection onto U. Suppose that $\varpi(\psi)u_H(g') = \varpi(\psi)u_H(h')$ for $g', h' \in \pi_{F'}$. Since $\varpi(\psi)$ is bijective, $g'_H = h'_H : F'H \to F'H$. For any $(X \xrightarrow{p} S) \in \mathrm{Ob}\,\mathcal{C}/S$, there is a morphism $(id_X, p) : (X \xrightarrow{p} S) \to (X \times S \xrightarrow{\mathrm{pr}_2} S) = H(X)$ in $\mathcal{C}/S$. We note that, since $\Sigma_S(id_X, p) : X \to X \times S$ has a left inverse $\mathrm{pr}_1 : X \times S \to X$, $F(\Sigma_S(id_X, p)) : F(X) \to F(X \times S)$ is injective. Then, the restriction $F'(id_X, p) : F'(X \xrightarrow{p} S) \to F'H(X)$ is also injective. By the naturality of g' and h', $F'(id_X, p)g'_{(X \xrightarrow{p} S)} = g'_{H(X)}F'(id_X, p) = h'_{H(X)}F'(id_X, p) = F'(id_X, p)h'_{(X \xrightarrow{p} S)}$. Thus we have $g'_{(X \xrightarrow{p} S)} = h'_{(X \xrightarrow{p} S)}$. Hence $\varpi(\psi)u_H$ is injective. $\square$

6.7 Torsers

Let $\mathcal{E}$ be an elemantary topos and G an internal group in $\mathcal{E}$.

Definition 6.7.1 *A flat left G-object $\alpha : G \times X \to X$ is called a torser.*

Since $\mathcal{E}$ is balanced (3.2.2), it follows from (5.4.6) that a left G-object $\alpha : G \times X \to X$ is a torser if and only if $X \to 1$ is an epimorphism and $(\mathrm{pr}_2, \alpha) : G \times X \to X \times X$ is an isomorphism.

6.8 Fundamental group of elementary topos

Definition 6.8.1 *Let $\mathcal{E}$ be an elementary topos with a natural number object. An object X of $\mathcal{E}$ is called a locally constant finite object if there exists an object V such that $V \to 1$ is an epimorphism and V^*X is isomorphic to a finite cardinal in $\mathcal{E}/V$. We denote by $\mathcal{E}_{lcf}$ the full subcategory of $\mathcal{E}$ consisting of locally constant finite objects in $\mathcal{E}$.*

$(\mathcal{U}\text{-}\mathbf{Ens})_{lcf}$ is the category of finite sets.

Definition 6.8.2 *A category $\mathcal{E}$ is called a pretopos if the following conditions holds.*

i) $\mathcal{E}$ has finite limits.
ii) $\mathcal{E}$ has finite coproducts, which are disjoint and universal.
iii) Every equivalence relation in $\mathcal{E}$ is effective and every epimorphism is effective and universal.

To be continued

Chapter 7

Model categories for the working mathematicians

7.1 Definition of model category

Definition 7.1.1 *Let $\mathcal{C}$ be a category together with three classes $\boldsymbol{Cof}(\mathcal{C})$, $\boldsymbol{Fib}(\mathcal{C})$, $\boldsymbol{Weq}(\mathcal{C})$ of morphisms in $\mathcal{C}$. Morphisms in $\boldsymbol{Cof}(\mathcal{C})$, $\boldsymbol{Fib}(\mathcal{C})$ and $\boldsymbol{Weq}(\mathcal{C})$ are called cofibrations, fibrations and weak equivalences, respectively. We call $\mathcal{C}$ a model category if the following conditions are satisfied.*

(M0) $\mathcal{C}$ is closed under finite limits and finite colimits.

(M1) Let $i : A \to B$ is a cofibration and $p : X \to Y$ a fibration. If a diagram

$$\begin{array}{ccc} A & \xrightarrow{\ j\ } & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\ f\ } & Y \end{array}$$

of $\mathcal{C}$ is commutative and i or p is a weak equivalecnce, then there exists a morphism $g : B \to X$ satisfying $gi = j$ and $pg = f$.

(M2) Any morphism f can be factored $f = pi$ where i is a cofibration and weak equivelence and p is a fibration. f also can be factored $f = qj$ where j is a cofibration and q is a fibration and weak equivelence.

(M3) $\boldsymbol{Fib}(\mathcal{C})$ and $\boldsymbol{Cof}(\mathcal{C})$ are closed under compositions. $\boldsymbol{Fib}(\mathcal{C})$ is stable under pull-backs and $\boldsymbol{Cof}(\mathcal{C})$ is stable under push-outs. Any isomorphism is a fibration and cofibration.

(M4) The pull-back of a morphism which is both a fibration and a weak equivalence is a weak equivalence. The push-out of a morphism which is both a cofibration and a weak equivalence is a weak equivalence.

(M5) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{C}$. If two of the morphisms f, g and gf are weak equivalences, so is the third. Any isomorphism is a weak equivalence.

For the rest of this section, $\mathcal{C}$ denotes a fixed model category. By (M0), $\mathcal{C}$ has an initial object and a terminal object. We choose an initial object $\emptyset$ and a terminal object $*$ of $\mathcal{C}$.

Definition 7.1.2 *An object X of $\mathcal{C}$ is said to be cofibrant if the unique morphism $\emptyset \to X$ is a cofibration and fibrant if the unique morphism $X \to *$ is a fibration. A morphism which belongs to $\boldsymbol{Cof}(\mathcal{C}) \cap \boldsymbol{Weq}(\mathcal{C})$ is called a trivial cofibration. A morphism which belongs to $\boldsymbol{Fib}(\mathcal{C}) \cap \boldsymbol{Weq}(\mathcal{C})$ is called a trivial fibration.*

For morphisms $\gamma : X \to Y$ and $\delta : B \to Y$ in $\mathcal{C}$, let

$$\begin{array}{ccc} B \times_Y X & \xrightarrow{\ \mathrm{pr}_2\ } & X \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \gamma} \\ B & \xrightarrow{\ \delta\ } & Y \end{array}$$

be the fibered product of γ and δ. If $\alpha : A \to B$ and $\beta : A \to X$ are morphisms in $\mathcal{C}$ which satisfy $\delta\alpha = \gamma\beta$, we denote by $(\alpha, \beta)_Y : A \to B \times_Y X$ the unique morphism satisfying $\mathrm{pr}_1(\alpha, \beta)_Y = \alpha$ and $\mathrm{pr}_2(\alpha, \beta)_Y = \beta$. If $Y = *$, $B \times_Y X$ is denoted by $B \times X$.

For morphisms $\alpha : A \to B$ and $\beta : A \to X$ in $\mathcal{C}$, let

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\alpha"] & X \arrow[d, "\mathrm{inc}_2"] \\ B \arrow[r, "\mathrm{inc}_1"] & B\coprod_A X \end{tikzcd}$$

be the cofibered product of α and β. If $\gamma : X \to Y$ and $\delta : B \to Y$ are morphisms in $\mathcal{C}$ which satisfy $\delta\alpha = \gamma\beta$, we denote by $\delta +_A \gamma : B\coprod_A X \to Y$ the unique morphism satisfying $(\delta +_A \gamma)\mathrm{inc}_1 = \delta$ and $(\delta +_A \gamma)\mathrm{inc}_2 = \gamma$. If $A = \emptyset$, $B\coprod_A X$ is denoted by $B\coprod X$.

For a morphism $f : X \to Y$, consider the fibered product $X \times_Y X$ and cofibered product $Y\coprod_X Y$ of f and f. We denote $(id_X, id_X)_Y : X \to X \times_Y X$ by Δ_f and call this the diagonal morphism of f. If $Y = *$, Δ_f is denoted by Δ_X. We denote $id_Y +_X id_Y : Y\coprod_X Y \to Y$ by ∇_f and call this the codiagonal morphism of f. If $X = \emptyset$, ∇_f is denoted by ∇_Y.

Definition 7.1.3 *(1) For an object A of $\mathcal{C}$, an object $\widetilde{A}$ with morphisms $\partial_0, \partial_1 : A \to \widetilde{A}$ and a weak equivalence $\sigma : \widetilde{A} \to A$ is called a cylinder object of A if $\sigma(\partial_0 + \partial_1) = \nabla_A$ and $\partial_0 + \partial_1 : A\coprod A \to \widetilde{A}$ is a cofibration. We denote by $A \times I$ a cylinder object of A.*

(2) For an object B of $\mathcal{C}$, an object $\widetilde{B}$ with morphisms $d_0, d_1 : \widetilde{B} \to B$ and a weak equivalence $s : B \to \widetilde{B}$ is called a path object of B if $(d_0, d_1)s = \Delta_B$ and $(d_0, d_1) : \widetilde{B} \to B \times B$ is a fibration. We denote by B^I a path object of A.

Remark 7.1.4 *Suppose that $\mathcal{C}$ is a model category.*

(1) By (M2), the codiagonal morphism $\nabla_A : A\coprod A \to A$ can be factored $\nabla_A = \sigma j$ where $j : A\coprod A \to \widetilde{A}$ is a cofibration and $\sigma : \widetilde{A} \to A$ is a trivial fibration. Set $\partial_i = j\mathrm{inc}_{i+1} : A \to \widetilde{A}$ $(i = 0, 1)$, then we have $j = \partial_0 + \partial_1$. Hence $\widetilde{A}$ is a cylinder object of A.

(2) By (M2), the diagonal morphism $\delta_B : B \to B \times B$ can be factored $\delta_B = ps$ where $p : \widetilde{B} \to B \times B$ is a fibration and $s : B \to \widetilde{B}$ is a trivial cofibration. Set $d_i = \mathrm{pr}_{i+1}p : \widetilde{B} \to B$ $(i = 0, 1)$, then we have $p = (d_0, d_1)$. Hence $\widetilde{B}$ is a path object of B.

Definition 7.1.5 *Let $f, g : A \to B$ be morphisms in $\mathcal{C}$.*

(1) If there exist an object $\widetilde{A}$, morphisms $\partial_0, \partial_1 : A \to \widetilde{A}$, $h : \widetilde{A} \to B$ and a weak equivelence $\sigma : \widetilde{A} \to A$ satisfying $f + g = h(\partial_0 + \partial_1)$ and $\nabla_A = \sigma(\partial_0 + \partial_1)$, then we say that f is left homotopic to g and denote this by $f \overset{l}{\sim} g$. The above morphism h is called a left homotopy from f to g if $\widetilde{A}$ is a cylinder object of A.

(2) If there exist an object $\widetilde{B}$, morphisms $d_0, d_1 : \widetilde{B} \to B$, $k : A \to \widetilde{B}$ and a weak equivelence $s : B \to \widetilde{B}$ satisfying $(f, g) = (d_0, d_1)k$ and $\Delta_B = (d_0, d_1)s$, then we say that f is right homotopic to g and denote this by $f \overset{r}{\sim} g$. The above morphism k is called a right homotopy from f to g if $\widetilde{B}$ is a path object of B.

Lemma 7.1.6 *Let $f, g : A \to B$ be morphisms in $\mathcal{C}$.*

(1) If f is left homotopic to g, there is a left homotopy $h : A \times I \to B$ from f to g.

(2) If f is right homotopic to g, there is a right homotopy $k : A \to B^I$ from f to g.

Proof. (1) There exist an object $\widetilde{A}$, morphisms $\partial_0, \partial_1 : A \to \widetilde{A}$, $h' : \widetilde{A} \to B$ and a weak equivelence $\sigma : \widetilde{A} \to A$ satisfying $f + g = h(\partial_0 + \partial_1)$ and $\nabla_A = \sigma(\partial_0 + \partial_1)$. By (M2), we can factor $\partial_0 + \partial_1 : A\coprod A \to \widetilde{A}$ into $A\coprod A \xrightarrow{\partial_0'+\partial_1'} A' \xrightarrow{\rho} \widetilde{A}$ where $\partial_0' + \partial_1'$ is a cofibration and ρ is a trivial fibration. Then $\sigma\rho : A' \to A$ is a weak equivalence by (M5). Hence A' with $\partial_0, \partial_1 : A \to A'$ and a weak equivalence $\sigma\rho : A' \to A$ is a cylinder object of A and $h\rho : A' \to B$ is a left homotopy from f to g.

(2) There exist an object $\widetilde{B}$, morphisms $d_0, d_1 : \widetilde{B} \to B$, $k : A \to \widetilde{B}$ and a weak equivelence $s : B \to \widetilde{B}$ satisfying $(f, g) = (d_0, d_1)k$ and $\Delta_B = (d_0, d_1)s$. By (M2), we can factor $(d_0, d_1) : \widetilde{B} \to B \times B$ into $\widetilde{B} \xrightarrow{\iota} B' \xrightarrow{(d_0', d_1')} B \times B$ where ι is a trivial cofibration and (d_0', d_1') is a fibration. Then $\iota s : B \to B'$ is a weak equivalence by (M5). Hence B' with $d_0, d_1 : B' \to B$ and a weak equivalence $\iota s : B \to B'$ is a path object of B and $\iota k : A \to B'$ is a right homotopy from f to g. □

Lemma 7.1.7 *(1) Let A be a cofibrant object and let $A \times I$ be a cylinder object for A. Then, $\partial_0 : A \to A \times I$ and $\partial_1 : A \to A \times I$ are trivial cofibrations.*

(2) Let B be a fibrant object and let B^I be a path object for B. Then, $d_0 : B^I \to B$ and $d_1 : B^I \to B$ are trivial fibrations.

Proof. (1) Since $\mathrm{inc}_1 : A \to A \coprod A$ is a push-out of a cofibration $\emptyset \to A$ along $\emptyset \to A$, inc_1 is a cofibration by (M3). Hence $\partial_0 = (\partial_0 + \partial_1)\mathrm{inc}_1$ is a cofibration by the assumption and (M3). Since $\sigma\partial_0 = \sigma(\partial_0 + \partial_1)\mathrm{inc}_1 = \nabla_A \mathrm{inc}_1 = id_A$, it follows from (M5) that ∂_0 is a weak equivalence. Similarly, ∂_1 is a trivial cofibration.

(2) Since $\mathrm{pr}_1 : B \times B \to B$ is a pull-back of a fibration $B \to *$ along $B \to *$, pr_1 is a fibration by (M3). Hence $d_0 = \mathrm{pr}_1(d_0, d_1)$ is a fibration by the assumption and (M3). Since $d_0 s = \mathrm{pr}_1(d_0, d_1)s = \mathrm{pr}_1\Delta_B = id_B$, it follows from (M5) that d_0 is a weak equivalence. Similarly, d_1 is a trivial fibration. □

Proposition 7.1.8 *(Covering homotopy property) Suppose that A is cofibrant and $p : X \to Y$ is a fibration. If diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\partial_0"] & X \arrow[d, "p"] \\ A \times I \arrow[r, "h"] & Y \end{tikzcd}$$

is commutative, then there is a left homotopy $H : A \times I \to X$ satisfying $H\partial_0 = \alpha$ and $pH = h$.

Proof. Since ∂_0 is a trivial cofibration by (7.1.7), the assertion follows from (M1). □

Proposition 7.1.9 *(Homotopy extension property) Suppose that B is fibrant and $i : X \to Y$ is a cofibration. If diagram*

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "i"] & B^I \arrow[d, "d_0"] \\ Y \arrow[r, "\beta"] & B \end{tikzcd}$$

is commutative, then there is a right homotopy $K : Y \to B^I$ satisfying $d_0 K = \beta$ and $Ki = k$.

Proof. Since d_0 is a trivial fibration by (7.1.7), the assertion follows from (M1). □

Lemma 7.1.10 *Suppose that A is cofibrant and let $A \times I$ and $A \times I'$ be cylinder objects for A. Consider the following cocartesian square.*

$$\begin{tikzcd} A \arrow[r, "\partial'_0"] \arrow[d, "\partial_1"] & A \times I' \arrow[d, "\mathrm{inc}_2"] \\ A \times I \arrow[r, "\mathrm{inc}_1"] & \widetilde{A} \end{tikzcd}$$

Let $\sigma'' : \widetilde{A} \to A$ be the morphism satisfying $\sigma''\mathrm{inc}_1 = \sigma$ and $\sigma''\mathrm{inc}_2 = \sigma'$. Then, $\widetilde{A}$ is a cylinder object for A with morphisms $\partial''_0 = \mathrm{inc}_1\partial_0$, $\partial''_1 = \mathrm{inc}_2\partial'_1$ and weak equivalence σ''.

Proof. Since $\sigma(\partial_0 + \partial_1) = \sigma'(\partial'_0 + \partial'_1) = \nabla_A$, we have $\sigma\partial_1 = \sigma(\partial_0 + \partial_1)\mathrm{inc}_2 = \nabla_A\mathrm{inc}_2 = id_A$ and $\sigma'\partial'_0 = \sigma'(\partial'_0 + \partial'_1)\mathrm{inc}_1 = \nabla_A\mathrm{inc}_1 = id_A$. Hence there is a unique morphism $\sigma'' : \widetilde{A} \to A$ satisfying $\sigma''\mathrm{inc}_1 = \sigma$ and $\sigma''\mathrm{inc}_2 = \sigma'$. Since ∂_1 and ∂'_0 are trivial cofibrations by (7.1.7), so are $\mathrm{inc}_1 : A \times I \to \widetilde{A}$ and $\mathrm{inc}_2 : A \times I' \to \widetilde{A}$ by (M4). Since $\sigma''\mathrm{inc}_1 = \sigma$ and σ is a weak equivalence, σ'' is a weak equivalence by (M5).

By the definition of ∂''_0, ∂''_1 and $\sigma(\partial_0 + \partial_1) = \sigma'(\partial'_0 + \partial'_1) = \nabla_A$, we have $\sigma''(\partial''_0 + \partial''_1)\mathrm{inc}_1 = \sigma''\mathrm{inc}_1\partial_0 = \sigma\partial_0 = \sigma(\partial_0 + \partial_1)\mathrm{inc}_1 = \nabla_A\mathrm{inc}_1$, $\sigma''(\partial''_0 + \partial''_1)\mathrm{inc}_2 = \sigma''\mathrm{inc}_2\partial'_1 = \sigma'\partial'_1 = \sigma'(\partial'_0 + \partial'_1)\mathrm{inc}_2 = \nabla_A\mathrm{inc}_2$. Hence $\sigma''(\partial''_0 + \partial''_1) = \nabla_A$ holds. We claim that the following diagrams are cocartesian squares.

$$\begin{tikzcd} A \arrow[r, "\partial_0"] \arrow[d, "\mathrm{inc}_1"] & A \times I \arrow[d, "\mathrm{inc}_1"] \\ A \coprod A \arrow[r, "\partial_0 \coprod id_A"] & (A \times I) \coprod A \end{tikzcd} \qquad \begin{tikzcd} A \coprod A \arrow[r, "\partial'_0 + \partial'_1"] \arrow[d, "\partial_1 \coprod id_A"] & A \times I' \arrow[d, "\mathrm{inc}_2"] \\ (A \times I) \coprod A \arrow[r, "\mathrm{inc}_1 + \partial''_1"] & \widetilde{A} \end{tikzcd}$$

In fact, it is clear that the left diagram is cocartesian. Since $\mathrm{inc}_2(\partial'_0 + \partial'_1)\mathrm{inc}_1 = \mathrm{inc}_2\partial'_0 = \mathrm{inc}_1\partial_1 = (\mathrm{inc}_1 + \partial''_1)\mathrm{inc}_1\partial_1 = (\mathrm{inc}_1 + \partial''_1)(\partial_1 \coprod id_A)\mathrm{inc}_1$ and $\mathrm{inc}_2(\partial'_0 + \partial'_1)\mathrm{inc}_2 = \mathrm{inc}_2\partial'_1 = \partial''_1 = (\mathrm{inc}_1 + \partial''_1)\mathrm{inc}_2 = (\mathrm{inc}_1 + \partial''_1)(\partial_1 \coprod id_A)\mathrm{inc}_2$, the above right diagram commutes. Suppose that there are morphisms $f : (A \times I) \coprod A \to X$ and $g : A \times I' \to X$ satisfying $f(\partial_1 \coprod id_A) = g(\partial'_0 + \partial'_1)$. Then, we have $f\mathrm{inc}_1\partial_1 = f(\partial_1 \coprod id_A)\mathrm{inc}_1 = g(\partial'_0 +$

$\partial_1')\mathrm{inc}_1 = g\partial_0'$ and $f\mathrm{inc}_2 = f(\partial_1 \coprod id_A)\mathrm{inc}_2 = g(\partial_0' + \partial_1')\mathrm{inc}_2 = g\partial_1'$. By the definition of $\widetilde{A}$, there exists a unique morphism $h : \widetilde{A} \to X$ satisfying $h\mathrm{inc}_1 = f\mathrm{inc}_1$ and $h\mathrm{inc}_2 = g$. Then, we have $h(\mathrm{inc}_1 + \partial_1'')\mathrm{inc}_1 = h\mathrm{inc}_1 = f\mathrm{inc}_1$ and $h(\mathrm{inc}_1 + \partial_1'')\mathrm{inc}_2 = h\mathrm{inc}_2\partial_1' = g\partial_1' = f\mathrm{inc}_2$ which imply $h(\mathrm{inc}_1 + \partial_1'') = f$. Thus we see that the above right diagram is also a cocartesian square.

Since ∂_0 and $\partial_0' + \partial_1'$ are cofibrations, so are $\partial_0 \coprod id_A$ and $\mathrm{inc}_1 + \partial_1''$ by (M3). Since $\partial_0'' + \partial_1'' : A \coprod A \to \widetilde{A}$ is composition of $\partial_0 \coprod id_A$ and $\mathrm{inc}_1 + \partial_1''$, it is a cofibration by (M3). $\square$

Lemma 7.1.11 *Suppose that B is fibrant and let B^I and $B^{I'}$ be path objects for B. Consider the following cartesian square.*

$$\begin{tikzcd} \widetilde{B} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & B^{I'} \arrow[d, "d_0'"] \\ B^I \arrow[r, "d_1"] & B \end{tikzcd}$$

Let $s'' : B \to \widetilde{B}$ be the morphism satisfying $\mathrm{pr}_1 s'' = s$ and $\mathrm{pr}_2 s'' = s'$. Then, $\widetilde{B}$ is a path object for B with morphisms $d_0'' = d_0\mathrm{pr}_1$, $d_1'' = d_1'\mathrm{pr}_2$ and weak equivalence s''.

Proof. Since $(d_0, d_1)s = (d_0', d_1')s' = \Delta_B$, we have $d_1 s = \mathrm{pr}_2(d_0, d_1)s = \mathrm{pr}_2\Delta_B = id_B$ and $d_0's' = \mathrm{pr}_1(d_0', d_1')s' = \mathrm{pr}_1\Delta_B = id_B$. Hence there is a unique morphism $s'' : B \to \widetilde{B}$ satisfying $\mathrm{pr}_1 s'' = s$ and $\mathrm{pr}_2 s'' = s'$. Since d_1 and d_0' are trivial fibrations by (7.1.7), so are $\mathrm{pr}_1 : \widetilde{B} \to B^I$ and $\mathrm{pr}_2 : \widetilde{B} \to B^{I'}$ by (M4). Since $\mathrm{pr}_1 s'' = s$ and s is a weak equivalence, s'' is a weak equivalence by (M5).

By the definition of d_0'', d_1'' and $s(d_0, d_1) = s'(d_0', d_1') = \Delta_B$, we have $\mathrm{pr}_1(d_0'', d_1'')s'' = d_0\mathrm{pr}_1 s'' = d_0 s = \mathrm{pr}_1(d_0, d_1)s = \mathrm{pr}_1\Delta_B$, $\mathrm{pr}_2(d_0'', d_1'')s'' = d_1'\mathrm{pr}_2 s'' = d_1's' = \mathrm{pr}_2(d_0', d_1')s' = \mathrm{pr}_2\Delta_B$. Hence $(d_0'', d_1'')s'' = \Delta_B$ holds. We claim that the following diagrams are cartesian squares.

$$\begin{tikzcd} \widetilde{B} \arrow[r, "(\mathrm{pr}_1, d_1'')"] \arrow[d, "\mathrm{pr}_2"] & B^I \times B \arrow[d, "d_1 \times id_B"] \\ B^{I'} \arrow[r, "(d_0', d_1')"] & B \times B \end{tikzcd} \qquad \begin{tikzcd} B^I \times B \arrow[r, "d_0 \times id_B"] \arrow[d, "\mathrm{pr}_1"] & B \times B \arrow[d, "\mathrm{pr}_1"] \\ B^I \arrow[r, "d_0"] & B \end{tikzcd}$$

In fact, it is clear that the right diagram is cartesian. Since $\mathrm{pr}_1(d_0', d_1')\mathrm{pr}_2 = d_0'\mathrm{pr}_2 = d_1\mathrm{pr}_1 = d_1\mathrm{pr}_1(\mathrm{pr}_1, d_1'') = \mathrm{pr}_1(d_1 \times id_B)(\mathrm{pr}_1, d_1'')$ and $\mathrm{pr}_2(d_0', d_1')\mathrm{pr}_2 = d_1'\mathrm{pr}_2 = d_1'' = \mathrm{pr}_2(\mathrm{pr}_1, d_1'') = \mathrm{pr}_2(d_1 \times id_B)(\mathrm{pr}_1, d_1'')$, the above left diagram commutes. Suppose that there are morphisms $f : X \to B^I \times B$ and $g : X \to B^{I'}$ satisfying $(d_1 \times id_B)f = (d_0', d_1')g$. Then, we have $d_1\mathrm{pr}_1 f = \mathrm{pr}_1(d_1 \times id_B)f = \mathrm{pr}_1(d_0', d_1')g = d_0'g$ and $\mathrm{pr}_2 f = \mathrm{pr}_2(d_1 \times id_B)f = \mathrm{pr}_2(d_0', d_1')g = d_1'g$. By the definition of $\widetilde{B}$, there exists a unique morphism $h : X \to \widetilde{B}$ satisfying $\mathrm{pr}_1 h = \mathrm{pr}_1 f$ and $\mathrm{pr}_2 h = g$. Then, we have $\mathrm{pr}_1(\mathrm{pr}_1, d_1'')h = \mathrm{pr}_1 h = \mathrm{pr}_1 f$ and $\mathrm{pr}_2(\mathrm{pr}_1, d_1'')h = d_1'\mathrm{pr}_2 h = d_1'g = \mathrm{pr}_2 f$ which imply $(\mathrm{pr}_1, d_1'')h = f$. Thus we see that the above right diagram is also a cartesian square.

Since d_0 and (d_0', d_1') are fibrations, so are $d_0 \times id_B$ and (pr_1, d_1'') by (M3). Since $d_0'', d_1'' : B \times B \to \widetilde{B}$ is composition of $d_0 \times id_B$ and (pr_1, d_1''), it is a fibration by (M3). $\square$

Lemma 7.1.12 *(1) If A is cofibrant, $\overset{l}{\sim}$ is an equivalence relation on $\mathcal{C}(A, B)$.*

(2) If B is fibrant, $\overset{r}{\sim}$ is an equivalence relation on $\mathcal{C}(A, B)$.

Proof. (1) For $f \in \mathcal{C}(A, B)$, set $\widetilde{A} = A$ and let ∂_0, ∂_1 and σ be the identity morphisms of A. Then, we have $f + f = f\nabla_A = f(\partial_0 + \partial_1)$ and $\nabla_A = \sigma(\partial_0 + \partial_1)$. Hence $f \overset{l}{\sim} f$. Since we can interchange ∂_0 and ∂_1, $f \overset{l}{\sim} g$ implies $g \overset{l}{\sim} f$. Suppose $f_0 \overset{l}{\sim} f_1$ and $f_1 \overset{l}{\sim} f_2$. Then, we have a left homotopy $h : A \times I \to B$ from f_0 to f_1 and a left homotopy $h' : A \times I' \to B$ from f_1 to f_2 by (7.1.6). It follows from (7.1.10) that there exist a unique morphism $h'' : A \times I'' = \widetilde{A} \to B$ satisfying $h''\mathrm{inc}_1 = h$ and $h''\mathrm{inc}_2 = h'$. Since $h''\partial_0'' = h''\mathrm{inc}_1\partial_0 = h\partial_0 = f_0$ and $h''\partial_1'' = h''\mathrm{inc}_2\partial_1' = h'\partial_1' = f_2$, h'' is a left homotopy from f_0 to f_2.

(2) For $f \in \mathcal{C}(A, B)$, set $\widetilde{B} = B$ and let d_0, d_1 and s be the identity morphisms of B. Then, we have $(f, f) = \Delta_A f = (d_0, d_1)f$ and $\Delta_A = (d_0, d_1)s$. Hence $f \overset{r}{\sim} f$. Since we can interchange d_0 and d_1, $f \overset{r}{\sim} g$ implies $g \overset{r}{\sim} f$. Suppose $f_0 \overset{r}{\sim} f_1$ and $f_1 \overset{r}{\sim} f_2$. Then, we have a right homotopy $k : A \to B^I$ from f_0 to f_1 and a right homotopy $k' : A \to B^I$ from f_1 to f_2 by (7.1.6). It follows from (7.1.11) that there exist a unique morphism $k'' : A \to \widetilde{B} = B^{I''}$ satisfying $\mathrm{pr}_1 k'' = k$ and $\mathrm{pr}_2 k'' = k'$. Since $d_0''k'' = d_0\mathrm{pr}_1 k'' = d_0 k = f_0$ and $d_1''k'' = d_1'\mathrm{pr}^2 k'' = d_1'k' = f_2$, k'' is a right homotopy from f_0 to f_2. $\square$

Lemma 7.1.13 *(1) Suppose that $f \overset{l}{\sim} g$ in $\mathcal{C}(A,B)$. f is a weak equivalence if and only if so is g.*

(2) Suppose that $f \overset{r}{\sim} g$ in $\mathcal{C}(A,B)$. f is a weak equivalence if and only if so is g.

Proof. (1) Let $h : A \times I \to B$ be a left homotopy from f to g. Since $f = h\partial_0$ and ∂_0 is a weak equivalence, h is a weak equivalence by (M5) if f is a weak equivalence. Hence g is also a weak equivalnce by (M5).

(2) Let $k : A \to B^I$ be a right homotopy from f to g. Since $f = d_0k$ and d_0 is a weak equivalence, k is a weak equivalence by (M5) if f is a weak equivalence. Hence g is also a weak equivalnce by (M5). □

Lemma 7.1.14 *(1) If $f \overset{l}{\sim} g$ in $\mathcal{C}(A,B)$, then $uf \overset{l}{\sim} ug$ for $u \in \mathcal{C}(B,C)$.*

(2) If $u \overset{r}{\sim} v$ in $\mathcal{C}(B,C)$, then $uf \overset{r}{\sim} vf$ for $f \in \mathcal{C}(A,B)$.

Proof. (1) Let $h : A \times I \to B$ be a left homotopy from f to g, then $fh : A \times I \to C$ is a left homotopy from uf to ug.

(2) Let $k : B \to C^I$ be a right homotopy from u to v, then $kf : A \to C^I$ is a right homotopy from uf to vf. □

Lemma 7.1.15 *Suppose that A is cofibrant and $f, g \in \mathcal{C}(A,B)$.*

(1) $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$.

(2) If $f \overset{r}{\sim} g$, there exists a right homotopy $k : A \to B^I$ from f to g with $s : B \to B^I$ a trivial cofibration.

(3) If $f \overset{r}{\sim} g$, then $uf \overset{r}{\sim} ug$ for $u \in \mathcal{C}(B,C)$.

Proof. (1) We have a left homotopy $h : A \times I \to B$ from f to g by (7.1.6). By factoring $\Delta_B : B \to B \times B$ using (M2), we have a path object B^I for B. Since $(d_0, d_1)sf = \Delta_B f = (f, f) = (f\sigma\partial_0, h\partial_0) = (f\sigma, h)\partial_0$ and $(d_0, d_1) : B^I \to B \times B$ is a fibration, ∂_0 is a trivial cofibration by (7.1.7), there exists a morphism $K : A \times I \to B^I$ satisfying $K\partial_0 = sf$ and $(d_0, d_1)K = (f\sigma, h)$. Set $k = K\partial_1 : A \to B^I$. Then $d_0k = d_0K\partial_1 = f\sigma\partial_1 = f$, $d_1k = d_1K\partial_1 = h\partial_1 = g$, hence k is a right homotopy from f to g.

(2) Let $k' : A \to B^{I'}$ be a right homotopy from f to g and let $B \xrightarrow{s} \widetilde{B} \xrightarrow{\rho} B^{I'}$ be a factorization of the weak equivalence $s' : B \to B^{I'}$ into a trivial cofibration followed by a fibration. Then, ρ is a weak equivalence by (M5). Put $d_0 = d'_0\rho$ and $d_1 = d'_1\rho$. Since $(d_0, d_1) = (d'_0, d'_1)\rho : \widetilde{B} \to B \times B$ is a composition of fibrations, (d_0, d_1) is a fibration by (M3). Thus $\widetilde{B}$ with d_0, d_1 and s is a path object B^I for B. Since A is cofibrant and $\rho : B^I = \widetilde{B} \to B^{I'}$ is a trivial fibration, there exists a morphism $k : A \to B^I$ satisfying $\rho k = k'$. Then, k gives the desired right homotopy from f to g.

(3) Let k be as in (2) and let C^I be a path object for C with morphisms $d''_0, d''_1 : C^I \to C$ and a weak equivalence $s'' : C \to C^I$. Then, we have $(d''_0, d''_1)s''u = \Delta_C u = (u, u) = (u \times u)\Delta_B = (u \times u)(d_0, d_1)s = (ud_0, ud_1)s$. Since s is a trivial cofibration and (d''_0, d''_1) is a fibration, there exists a morphism $\varphi : B^I \to C^I$ satiisfying $\varphi s = s''u$ and $(d''_0, d''_1)\varphi = (ud_0, ud_1)$. Then, $d''_0\varphi k = ud_0k = uf$ and $d''_1\varphi k = ud_1k = ug$ hence φk is a right homotopy from uf to ug. □

Lemma 7.1.16 *Suppose that C is fibrant and $u, v \in \mathcal{C}(B,C)$.*

(1) $u \overset{r}{\sim} v$ implies $u \overset{l}{\sim} v$.

(2) If $u \overset{l}{\sim} v$, there exists a left homotopy $h : B \times I \to C$ from u to v with $\sigma : B \times I \to B$ a trivial fibration.

(3) If $u \overset{l}{\sim} v$, then $uf \overset{l}{\sim} vf$ for $f \in \mathcal{C}(A,B)$.

Proof. (1) We have a right homotopy $k : B \to C^I$ from u to v by (7.1.6). By factoring $\nabla_B : B \coprod B \to B$ using (M2), we have a cylinder object $B \times I$ for B. Since $u\sigma(\partial_0 + \partial_1) = u\nabla_B = u + u = d_0su + d_0k = d_0(su + k)$ and $\partial_0 + \partial_1 : B \coprod B \to B \times I$ is a cofibration, d_0 is a trivial fibration by (7.1.7), there exists a morphism $H : B \times I \to C^I$ satisfying $d_0H = u\sigma$ and $H(\partial_0 + \partial_1) = su + k$. Set $h = d_1H : B \times I \to C$. Then $h\partial_0 = d_1H\partial_0 = d_1su = u$, $h\partial_1 = d_1H\partial_1 = d_1k = v$, hence h is a left homotopy from f to g.

(2) Let $h' : B \times I' \to C$ be a left homotopy from u to v and let $B \times I' \xrightarrow{\iota} \widetilde{B} \xrightarrow{\sigma} B$ be a factorization of the weak equvalence $\sigma' : B \times I' \to B$ into a cofibration followed by a trivial fibration. Then, ι is a weak equivalence by (M5). Put $\partial_0 = \iota\partial'_0$ and $d_1 = \iota\partial'_1$. Since $\partial_0 + \partial_1 = \iota(\partial'_0 + \partial'_1) : B \coprod B \to \widetilde{B}$ is a composition of cofibrations, $\partial_0 + \partial_1$ is a cofibration by (M3). Thus $\widetilde{B}$ with ∂_0, ∂_1 and σ is a cylinder object $B \times I$ for B. Since C is fibrant and $\iota : B \times I' \to \widetilde{B} = B \times I$ is a trivial cofibration, there exists a morphism $h : B \times I \to C$ satisfying $h\iota = h'$. Then, h gives the desired left homotopy from u to v.

(3) Let h be as in (2) and let $A \times I$ be a cylinder object for A with morphisms $\partial''_0, \partial''_1 : A \to A \times I$ and a weak equivalence $\sigma'' : A \times I \to A$. Then, we have $f\sigma''(\partial''_0 + \partial''_1) = f\nabla_A = f + f = \nabla_B(f \coprod f) = \sigma(\partial_0 + \partial_1)(f \coprod f) =$

$\sigma(\partial_0 f, \partial_1 f)$. Since σ is a trivial fibration and $\partial_0'' + \partial_1''$ is a cofibration, there exists a morphism $\psi : A \times I \to B \times I$ satisfying $\sigma\psi = f\sigma''$ and $\psi(\partial_0'' + \partial_1'') = (\partial_0 f, \partial_1 f)$. Then, $h\psi\partial_0'' = h\partial_0 f = uf$ and $h\psi\partial_1'' = h\partial_1 f = vf$ hence $h\psi$ is a left homotopy from uf to vf. □

For objects A and B of $\mathcal{C}$, let us denote by $\pi^r(A, B)$ the quotient set of $\mathcal{C}(A, B)$ by the equivalence relation generated by $\overset{r}{\sim}$. Similarly, we denote by $\pi^l(A, B)$ the quotient set of $\mathcal{C}(A, B)$ by the equivalence relation generated by $\overset{l}{\sim}$.

Suppose that A is cofibrant and B is fibrant. Then, $\overset{r}{\sim}$ and $\overset{l}{\sim}$ are equivalence relations by (7.1.12). Moreover, $f \overset{r}{\sim} g$ if and only if $f \overset{l}{\sim} g$ by (1) of (7.1.15) and (7.1.16). In this case, we denote $f \overset{r}{\sim} g$ by $f \sim g$ and $\pi^r(A, B)$ by $\pi(A, B)$.

Lemma 7.1.17 *(1) If A is cofibrant, composition in $\mathcal{C}$ induces a map $\pi^r(A, B) \times \pi^r(B, C) \to \pi^r(A, C)$.*
(2) If B is fibrant, composition in $\mathcal{C}$ induces a map $\pi^l(A, B) \times \pi^l(B, C) \to \pi^l(A, C)$.

Proof. (1) The assertion follows from (2) of (7.1.14) and (3) of (7.1.15).
(2) The assertion follows from (1) of (7.1.14) and (3) of (7.1.16). □

Lemma 7.1.18 *(1) Suppose that A is cofibrant and $p : X \to Y$ is a trivial fibration. Then, p induces a bijection $p_* : \pi^l(A, X) \to \pi^l(A, Y)$.*
(2) Suppose that X is fibrant and $i : A \to B$ is a trivial cofibration. Then, i induces a bijection $i^* : \pi^r(B, X) \to \pi^r(A, X)$.

Proof. (1) It follows from (2) of (7.1.14) that p_* is well-defined. For $f \in \mathcal{C}(A, Y)$, since A is cofibrant and $p : X \to Y$ is a trivial fibration, there exists a morphism $\bar{f} : A \to X$ satisfying $p\bar{f} = f$ by (M1). Hence p_* is surjective. Suppose that, for $f, g \in \mathcal{C}(A, X)$, pf and pg represent the same element of $\pi^l(A, Y)$. Then, $pf \overset{l}{\sim} pg$ by (1) of (7.1.12) and we ahve a left homotopy $h : A \times I \to Y$ from pf to pg. Since $h(\partial_0 + \partial_1) = pf + pg = p(f + g)$ and p is a trivial fibration, there exists $H : A \times I \to X$ satisfying $H(\partial_0 + \partial_1) = f + g$ and $pH = h$. Hence H is a left homotopy from f to g, which shows p_* is injective.

(2) It follows from (1) of (7.1.14) that i^* is well-defined. For $f \in \mathcal{C}(A, X)$, since X is cofibrant and $i : A \to B$ is a trivial fibration, there exists a morphism $\bar{f} : B \to X$ satisfying $\bar{f}i = f$ by (M1). Hence i^* is surjective. Suppose that, for $f, g \in \mathcal{C}(B, X)$, fi and gi represent the same element of $\pi^r(A, Y)$. Then, $fi \overset{r}{\sim} gi$ by (2) of (7.1.12) and we ahve a right homotopy $k : A \to Y^I$ from fi to gi. Since $(d_0, d_1)k = (fi, gi) = (f, g)i$ and i is a trivial cofibration, there exists $H : B \to Y^I$ satisfying $(d_0 + d_1)H = (f, g)$ and $Hi = h$. Hence H is a right homotopy from f to g, which shows i^* is injective. □

Let $\mathcal{C}_c$, $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ be the full subcategories of $\mathcal{C}$ consisting of the cofibrant, fibrant, anf both fibrant and cofibrant objects respectively. By (1) of (7.1.17), we can define a category $\pi\mathcal{C}_c$ by $\operatorname{Ob}\pi\mathcal{C}_c = \operatorname{Ob}\mathcal{C}_c$ and $\pi\mathcal{C}_c(A, B) = \pi^r(A, B)$ with composition induced from that of $\mathcal{C}$. We denote the right homotopy class of a morphism $f : A \to B$ by $[f]_r$ and define a functor $\mathcal{C}_c \to \pi\mathcal{C}_c$ by $X \mapsto X$ and $f \mapsto [f]_r$. Similarly, by (2) of (7.1.17), we can define a category $\pi\mathcal{C}_f$ by $\operatorname{Ob}\pi\mathcal{C}_f = \operatorname{Ob}\mathcal{C}_f$ and $\pi\mathcal{C}_f(A, B) = \pi^l(A, B)$ with composition induced from that of $\mathcal{C}$. We denote the left homotopy class of a morphism $f : A \to B$ by $[f]_l$ and define a functor $\mathcal{C}_f \to \pi\mathcal{C}_f$ by $X \mapsto X$ and $f \mapsto [f]_l$. Moreover, we define a category $\pi\mathcal{C}_{cf}$ by $\operatorname{Ob}\pi\mathcal{C}_{cf} = \operatorname{Ob}\mathcal{C}_{cf}$ and $\pi\mathcal{C}_{cf}(A, B) = \pi(A, B)$ with composition induced from that of $\mathcal{C}$. We denote the right homotopy class of a morphism $f : A \to B$ by $[f]$ and define a functor $\mathcal{C}_{cf} \to \pi\mathcal{C}_{cf}$ by $X \mapsto X$ and $f \mapsto [f]$.

Lemma 7.1.19 *If a morphism $f : X \to Y$ in $\mathcal{C}_{cf}$ is a weak equivalence, then it is a homotopy equivalence, that is, there exists a morphism $g : Y \to X$ satisfying $gf \sim id_X$ and $fg \sim id_Y$.*

Proof. Suppose that f is a weak equivalence. We can factor f as $f = pi$, where $i : X \to Z$ is a trivial cofibration and $p : Z \to Y$ is a fibration by (M2). Then, i is a weak equivalence by (M5). Since X is fibrant, by applying (M1) to

$$\begin{tikzcd} X \arrow[d, "i"] \arrow[dr] & \\ Z \arrow[r] & * \end{tikzcd}$$

we have a morphism $r : Z \to X$ satisfying $ri = id_X$. Since $i^* : \pi(Z, Z) \to \pi(X, Z)$ is bijective by (2) of (7.1.18) and $i^*([ir]) = [iri] = [i] = i^*([id_Z])$, we have $[ir] = [id_Z]$, namely $ir \sim id_Z$. On the other hand, since Y is cofibrant, by applying (M1) to

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[dr] & Z \arrow[d, "p"] \\ & Y \end{tikzcd}$$

we have a morphism $s : Y \to Z$ satisfying $ps = id_Y$. Since $p_* : \pi(Y, Z) \to \pi(Y, Y)$ is bijective by (2) of (7.1.18) and $p_*([sp]) = [psp] = [p] = p_*([id_Z])$, we have $[sp] = [id_Z]$, namely $sp \sim id_Z$. Set $g = rs$, then $gf = rspi \sim ri = id_X$ and $fg = pirs \sim ps = id_Y$ by (7.1.17). □

Definition 7.1.20 *Let $\mathcal{C}$ be a category and let S be a subset of $\mathrm{Mor}\,\mathcal{C}$. By the localization of $\mathcal{C}$ with respect to S, we mean a category $S^{-1}\mathcal{C}$ with a functor $\gamma : \mathcal{C} \to S^{-1}\mathcal{C}$ having the following property.*

(i) $\gamma(s)$ is an isomorphism if $s \in S$.

(ii) If $F : \mathcal{C} \to \mathcal{D}$ is a functor which maps $f \in S$ to an isomorphism in $\mathcal{D}$, then there exists a unique functor $\overline{F} : S^{-1}\mathcal{C} \to \mathcal{D}$ satisfying $\overline{F}\gamma = F$.

By the definition of $S^{-1}\mathcal{C}$, $S^{-1}\mathcal{C}$ is unique up to isomorphism of categories.

Lemma 7.1.21 *Let $\mathcal{C}$ be a model category.*

(1) Let $F : \mathcal{C}_c \to \mathcal{D}$ be a functor which maps weak equivalences to isomorphisms. If $f \overset{r}{\sim} g$, then $F(f) = F(g)$.

(2) Let $F : \mathcal{C}_f \to \mathcal{D}$ be a functor which maps weak equivalences to isomorphisms. If $f \overset{l}{\sim} g$, then $F(f) = F(g)$.

Proof. (1) Let $k : A \to B^I$ be a right homotopy from f to g. By (2) of (7.1.15), we may assume that $s : B \to B^I$ is a trivial cofibration. Hence s is a morphism in $\mathcal{C}_c$ and $F(s)$ is an isomorphism in $\mathcal{D}$ by the assumption. Since $d_0 s = d_1 s = id_B$, we have $F(d_0)F(s) = F(d_1)F(s)$ which implies $F(d_0) = F(d_1)$. Therefore $F(f) = F(d_0 k) = F(d_0)F(k) = F(d_1)F(k) = F(d_1 k) = F(g)$.

(2) Let $h : A \times I \to B$ be a right homotopy from f to g. By (2) of (7.1.16), we may assume that $\sigma : A \times I \to A$ is a trivial fibration. Hence σ is a morphism in $\mathcal{C}_c$ and $F(\sigma)$ is an isomorphism in $\mathcal{D}$ by the assumption. Since $\sigma\partial_0 = \sigma\partial_1 = id_A$, we have $F(\sigma)F(\partial_0) = F(\sigma)F(\partial_1)$ which implies $F(\partial_0) = F(\partial_1)$. Therefore $F(f) = F(h\partial_0) = F(h)F(\partial_0) = F(h)F(\partial_1) = F(h\partial_1) = F(g)$. □

For an object X of $\mathcal{C}$, we can factor the unique morphism $\emptyset \to X$ as $p_X i$ where i is a cofibration and p_X is a trivial fibration by (M2). We denote by QX the domain of p_X. If X is cofibrant, we choose id_X for p_X hence $QX = X$. Since p_X is a fibration, QX is fibrant if X is fibrant by (M3). For a morphism $f : X \to Y$, there exists a morphism $Qf : QX \to QY$ satisfying $p_Y Qf = fp_X$ by applying (M1) to the following diagram.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & QY \arrow[d, "p_Y"] \\ QX \arrow[r, "fp_X"] & Y \end{tikzcd}$$

Lemma 7.1.22 *(1) If $f_0, f_1 : QX \to QY$ satisfy $p_Y f_0 = p_Y f_1$, then $f_0 \overset{l}{\sim} f_1$ and $f_0 \overset{r}{\sim} f_1$.*

(2) Qf is a weak equivalence if and only if f is a weak equivalence.

(3) If Y is fibrant and $f \overset{l}{\sim} g$, then $Qf \overset{l}{\sim} Qg$ and $Qf \overset{r}{\sim} Qg$.

Proof. (1) Since QX is cofibrant, $p_{Y*} : \pi^l(QX, QY) \to \pi^l(QX, Y)$ is bijective by (1) of (7.1.18). Hence $p_Y f_0 = p_Y f_1$ implies $f_0 \overset{l}{\sim} f_1$ and $f_0 \overset{r}{\sim} f_1$ follows from (1) of (7.1.15).

(2) Since $p_Y Qf = fp_X$ and both p_X and p_Y are weak equivalence, the assertion follows from (M5).

(3) Since Y is fibrant, we have $p_{Y*}([Qf]_l) = [p_Y Qf]_l = [fp_X]_l = [gp_X]_l = [p_Y Qg]_l = p_{Y*}([Qg]_l)$ by (3) of (7.1.16). It follows from (1) of (7.1.18) that $Qf \overset{l}{\sim} Qg$ and $Qf \overset{r}{\sim} Qg$ follows from (1) of (7.1.15). □

For an object X of $\mathcal{C}$, we can factor the unique morphism $X \to *$ as pi_X where i_X is a trivial cofibration and p is a fibration by (M2). We denote by RX the codomain of i_X. If X is fibrant, we choose id_X for i_X hence $RX = X$. Since i_X is a cofibration, RX is cofibrant if X is cofibrant by (M3). For a morphism $f : X \to Y$, there exists a morphism $Rf : RX \to RY$ satisfying $Rfi_X = i_Y f$ by applying (M1) to the following diagram.

$$\begin{tikzcd} X \arrow[r, "i_Y f"] \arrow[d, "i_X"] & RY \arrow[d] \\ RX \arrow[r] & * \end{tikzcd}$$

Lemma 7.1.23 *(1) If $f_0, f_1 : RX \to RY$ satisfy $f_0 i_X = f_1 i_X$, then $f_0 \overset{l}{\sim} f_1$ and $f_0 \overset{r}{\sim} f_1$.*

(2) Rf is a weak equivalence if and only if f is a weak equivalence.

(3) If X is cofibrant and $f \overset{r}{\sim} g$, then $Rf \overset{l}{\sim} Rg$ and $Rf \overset{r}{\sim} Rg$.

Proof. (1) Since RY is fibrant, $i_X^* : \pi^r(RX, RY) \to \pi^r(X, RY)$ is bijective by (2) of (7.1.18). Hence $f_0 i_X = f_1 i_X$ implies $f_0 \overset{r}{\sim} f_1$ and $f_0 \overset{l}{\sim} f_1$ follows from (1) of (7.1.16).

(2) Since $Rfi_X = i_Y f$ and both i_X and i_Y are weak equivalence, the assertion follows from (M5).

(3) Since X is cofibrant, we have $i_X^*([Rf]_r) = [Rfi_Y]_r = [i_Y f]_r = [i_Y g]_r = [Rgi_X]_r = i_X^*([Rg]_r)$ by (3) of (7.1.16). It follows from (1) of (7.1.18) that $Rf \overset{r}{\sim} Rg$ and $Rf \overset{l}{\sim} Rg$ follows from (1) of (7.1.16). □

Definition 7.1.24 *For an object X of $\mathcal{C}$, we call QX the cofibrant replacement of X and call RX the fibrant replacement of X.*

It follows from (1) of (7.1.22) that $Qid_X \overset{r}{\sim} id_{QX}$ and that $Q(gf) \overset{r}{\sim} QgQf$ for $f : X \to Y$ and $g : Y \to Z$. Thus correspondences $X \mapsto QX$, $f \mapsto [Qf]_r$ define a functor $Q : \mathcal{C} \to \pi\mathcal{C}_c$. Moreover, by (3) of (7.1.22), the restriction of Q to $\mathcal{C}_f$ induces a functor $Q' : \pi\mathcal{C}_f \to \pi\mathcal{C}_{cf}$.

Similarly, it follows from (1) of (7.1.23) that $Rid_X \overset{l}{\sim} id_{QX}$ and that $R(gf) \overset{l}{\sim} RgRf$ for $f : X \to Y$ and $g : Y \to Z$. Thus correspondences $X \mapsto RX$, $f \mapsto [Rf]_l$ define a functor $R : \mathcal{C} \to \pi\mathcal{C}_f$. Moreover, by (3) of (7.1.23), the restriction of R to $\mathcal{C}_c$ induces a functor $R' : \pi\mathcal{C}_c \to \pi\mathcal{C}_{cf}$.

Definition 7.1.25 *We define a homotopy category $\mathbf{Ho}(\mathcal{C})$ by $\mathrm{Ob}\,\mathbf{Ho}(\mathcal{C}) = \mathrm{Ob}\,\mathcal{C}$ and*

$$\mathbf{Ho}(\mathcal{C})(X, Y) = \pi\mathcal{C}_{cf}(R'Q(X), R'Q(Y)) = \pi(RQX, RQY)$$

for $X, Y \in \mathrm{Ob}\,\mathcal{C}$. Define a functor $\gamma : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ by $\gamma(X) = X$ and $\gamma(f) = [RQf]$ for $X \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathrm{Mor}\,\mathcal{C}$.

Lemma 7.1.26 *If f is a weak equivalence, then $\gamma(f)$ is an isomorphism in $\mathbf{Ho}(\mathcal{C})$.*

Proof. If $f : X \to Y$ is a weak equivalence, then $RQf : RQX \to RQY$ is a weak equivalence by (2) of (7.1.22) and (7.1.23). It follows from (7.1.19) that RQf is a homotopy equivalence, namely $[RQf]$ is an isomorphism in $\pi\mathcal{C}_{cf}$. Hence $\gamma(f)$ is an isomorphism in $\mathbf{Ho}(\mathcal{C})$. □

By the above result and (7.1.21), $\gamma : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ induces a functor $\bar{\gamma} : \pi\mathcal{C}_{cf} \to \mathbf{Ho}(\mathcal{C})$.

Proposition 7.1.27 *$\bar{\gamma}$ is an equivalence of categories.*

Proof. For $X, Y \in \mathrm{Ob}\,\mathcal{C}_{cf}$, since $QX = X$ and $QY = Y$, we have $Qf \overset{r}{\sim} f$ by (1) of (7.1.22). Then, since $RQX = RX = X$ and $RQY = RY = Y$, we have $RQf \overset{r}{\sim} Qf$ by (1) of (7.1.23). Thus we have $RQf \overset{r}{\sim} f$ by (2) of (7.1.12). It follows that $\bar{\gamma} : \pi\mathcal{C}_{cf}(X, Y) \to \pi(RQX, RQY) = \mathbf{Ho}(\mathcal{C})(X, Y)$ is bijective. For $X \in \mathrm{Ob}\,\mathbf{Ho}(\mathcal{C})$, since $p_X : QX \to X$ and $i_{QX} : QX \to RQX$ are weak equivalences, $\gamma(p_X) : QX \to X$ and $\gamma(i_{QX}) : QX \to RQX$ are isomorphisms by (7.1.26). Hence X is isomorphic to a cofibrant and fibrant object RQX. □

Lemma 7.1.28 *Every morphism in $\mathbf{Ho}(\mathcal{C})$ is a composition of morphisms which are in the image of γ and the inverses of morphisms which are in the image of weak equivalences by γ.*

Proof. As we have seen above, $\gamma(i_{QX})\gamma(p_X)^{-1} : X \to RQX$ is an isomorphism in $\mathbf{Ho}(\mathcal{C})$. For a morphism $f : X \to Y$ in $\mathbf{Ho}(\mathcal{C})$, consider a morphism $\gamma(i_{QY})\gamma(p_Y)^{-1} f \gamma(p_X)\gamma(i_{QX})^{-1} : RQX \to RQY$ in $\mathbf{Ho}(\mathcal{C})$. It follows from (7.1.27) that there exists a morphism $f' : RQX \to RQY$ in $\mathcal{C}$ satisfying $\gamma(f') = \bar{\gamma}([f']) = \gamma(i_{QY})\gamma(p_Y)^{-1} f \gamma(p_X)\gamma(i_{QX})^{-1}$. Hence we have $f = \gamma(p_Y)\gamma(i_{QY})^{-1}\gamma(f')\gamma(i_{QX})\gamma(p_X)^{-1}$. □

Proposition 7.1.29 *Let $F, G : \mathbf{Ho}(\mathcal{C}) \to \mathcal{D}$ be a functor and $\varphi : F\gamma \to G\gamma$ a natural transformation. Then, φ gives a natural transformation $\varphi : F \to G$.*

Proof. For a morphism $h : X \to Y$ in $\mathbf{Ho}(\mathcal{C})$, we have to show that the following diagram $D(h)$ commutes.

$$\begin{tikzcd} F(X) \arrow[r, "F(h)"] \arrow[d, "\varphi_X"] & F(Y) \arrow[d, "\varphi_Y"] & \cdots D(h) \\ G(X) \arrow[r, "G(h)"] & G(Y) & \end{tikzcd}$$

If $h = \gamma(f)$ or $h = \gamma(g)^{-1}$ for a morphism f in $\mathcal{C}$ or a weak equivalence g in $\mathcal{C}$, $D(h)$ commutes by the assumption. If $h = h_1h_2$ and $D(h_1)$, $D(h_2)$ commute, then it is easy to verify that $D(h)$ commute. Hence $D(h)$ commutes for any morphism h of $\mathbf{Ho}(\mathcal{C})$ by (7.1.28). □

Theorem 7.1.30 *$\gamma : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ is a localization of $\mathcal{C}$ with respect to weak equvalences of $\mathcal{C}$.*

Proof. By (7.1.26), γ maps weak equivalences to isomorphisms. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which maps weak equivalences to isomorphisms. We define a functor $F' : \mathbf{Ho}(\mathcal{C}) \to \mathcal{D}$ as follows. Set $F'(X) = F(X)$. For a morphism $f : X \to Y$ in $\mathbf{Ho}(\mathcal{C})$, choose a representative $f' : RQX \to RQY$ of f. It follows from (7.1.21) that $F(f')$ depends only on the homotopy class of f', hence only on f. Define $F'(f)$ by

$$F'(f) = F(p_Y)F(i_{QY})^{-1}F(f')F(i_{QX})F(p_X)^{-1}.$$

If $f = id_X$, we can choose f' as id_{RQX}, hence $F'(id_X) = F(id_X) = id_{F(X)} = id_{F'(X)}$. For a morphism $g : Y \to Z$, choose a representative $g' : RQY \to RQZ$ of g. Then, $g'f' : RQX \to RQZ$ is a representative of gf and we have

$$\begin{aligned} F'(gf) &= F(p_Z)F(i_{QZ})^{-1}F(g'f')F(i_{QX})F(p_X)^{-1} = F(p_Z)F(i_{QZ})^{-1}F(g')F(f')F(i_{QX})F(p_X)^{-1} \\ &= F(p_Z)F(i_{QZ})^{-1}F(g')F(i_{QY})F(p_Y)^{-1}F(p_Y)F(i_{QY})^{-1}F(f')F(i_{QX})F(p_X)^{-1} = F'(g)F'(f). \end{aligned}$$

Therefore F' is a functor. Suppose $f = \gamma(h) = [RQh]$ for some $h : X \to Y$ in $\mathcal{C}$. Applying F to the following commutative diagram, we have $F'\gamma(h) = F(p_Y)F(i_{QY})^{-1}F(RQh)F(i_{QX})F(p_X)^{-1} = F(h)$. Thus $F'\gamma = F$.

$$\begin{tikzcd} X \arrow[d, "h"] & QX \arrow[l, "p_X"'] \arrow[r, "i_{QX}"] \arrow[d, "Qh"] & RQX \arrow[d, "RQh"] \\ Y & QY \arrow[l, "p_Y"'] \arrow[r, "i_{QY}"] & RQY \end{tikzcd}$$

The uniqueness of F' follows from (7.1.28). □

Corollary 7.1.31 *If A is cofibrant and B is fibrant, $\mathbf{Ho}(\mathcal{C})(A, B) \cong \pi(A, B)$.*

Proof. By the assumption, $RQA = RA$ and $RQB = QB$. Hence $\mathbf{Ho}(\mathcal{C})(A, B) = \pi(RQA, RQB) = \pi(RA, QB)$. Since $i_A : A \to RA$ is a trivial cofibration and $p_B : QB \to B$ is a trivial fibration, $i_A^* : \pi(RA, QB) \to \pi(A, QB)$ and $p_{B*} : \pi(A, QB) \to \pi(A, B)$ are bijective by (7.1.18). □

Definition 7.1.32 *We define a homotopy category $\mathbf{Ho}(\mathcal{C}_c)$ and $\mathbf{Ho}(\mathcal{C}_f)$ by $\operatorname{Ob}\mathbf{Ho}(\mathcal{C}_c) = \operatorname{Ob}\mathcal{C}_c$, $\operatorname{Ob}\mathbf{Ho}(\mathcal{C}_f) = \operatorname{Ob}\mathcal{C}_f$ and*

$$\mathbf{Ho}(\mathcal{C}_c)(X, Y) = \pi\mathcal{C}_{cf}(R(X), R(Y)) = \pi(RX, RY), \quad \mathbf{Ho}(\mathcal{C}_f)(X, Y) = \pi\mathcal{C}_{cf}(Q(X), Q(Y)) = \pi(QX, QY).$$

Define functors $\gamma_c : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ and $\gamma_f : \mathcal{C} \to \mathbf{Ho}(\mathcal{C}_f)$ by $\gamma_c(X) = X$ and $\gamma_c(f) = [Rf]$ for $X \in \operatorname{Ob}\mathcal{C}_c$ and $f \in \operatorname{Mor}\mathcal{C}_c$, $\gamma_f(X) = X$ and $\gamma_f(f) = [Qf]$ for $X \in \operatorname{Ob}\mathcal{C}_f$ and $f \in \operatorname{Mor}\mathcal{C}_f$.

The following is clear from (7.1.26).

Lemma 7.1.33 *If f is a weak equivalence, then $\gamma_c(f)$ and $\gamma_f(f)$ are isomorphisms in $\mathbf{Ho}(\mathcal{C}_c)$ and $\mathbf{Ho}(\mathcal{C}_f)$, respectively.*

Let $j_c : \pi\mathcal{C}_{cf} \to \pi\mathcal{C}_c$ and $j_f : \pi\mathcal{C}_{cf} \to \pi\mathcal{C}_f$ be the functors induced by the inclusion functors $\mathcal{C}_{cf} \to \mathcal{C}_c$ and $\mathcal{C}_{cf} \to \mathcal{C}_f$, respectively. By the above result and (7.1.21), $\gamma_c : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ and $\gamma_f : \mathcal{C} \to \mathbf{Ho}(\mathcal{C}_f)$ induce functors $\bar{\gamma}_c : \pi\mathcal{C}_c \to \mathbf{Ho}(\mathcal{C}_c)$ and $\bar{\gamma}_f : \pi\mathcal{C}_f \to \mathbf{Ho}(\mathcal{C}_f)$. The following result can be shown as (7.1.27).

Proposition 7.1.34 *Let be the functors induced by γ_c and γ_f, respectively. Then, $\bar{\gamma}_c j_c : \pi\mathcal{C}_{cf} \to \mathbf{Ho}(\mathcal{C}_c)$ and $\bar{\gamma}_f j_f : \pi\mathcal{C}_{cf} \to \mathbf{Ho}(\mathcal{C}_f)$ are equivalences of categories.*

Let $\iota_c : \mathbf{Ho}(\mathcal{C}_c) \to \mathbf{Ho}(\mathcal{C})$ and $\iota_f : \mathbf{Ho}(\mathcal{C}_f) \to \mathbf{Ho}(\mathcal{C})$ be the functors induced by the inclusion functors $\mathcal{C}_c \to \mathcal{C}$ and $\mathcal{C}_f \to \mathcal{C}$, respectively. Then, the following diagram commutes.

$$\begin{tikzcd}
\pi\mathcal{C}_c \arrow[d, "\bar{\gamma}_c"] & \pi\mathcal{C}_{cf} \arrow[l, "j_c"'] \arrow[r, "j_f"] \arrow[d, "\bar{\gamma}"] & \pi\mathcal{C}_f \arrow[d, "\bar{\gamma}_f"] \\
\mathbf{Ho}(\mathcal{C}_c) \arrow[r, "\iota_c"] & \mathbf{Ho}(\mathcal{C}) & \mathbf{Ho}(\mathcal{C}_f) \arrow[l, "\iota_f"']
\end{tikzcd}$$

Since $\bar{\gamma}_c j_c$, $\bar{\gamma}_f j_f$ and $\bar{\gamma}$ are equivalences, ι_c and ι_f are also equivalences. We can show the following result as (7.1.30).

Theorem 7.1.35 *(1) $\gamma_c : \mathcal{C}_c \to \mathbf{Ho}(\mathcal{C}_c)$ is a localization of $\mathcal{C}_c$ with respect to weak equvalences of $\mathcal{C}_c$.*
(2) $\gamma_f : \mathcal{C}_f \to \mathbf{Ho}(\mathcal{C}_f)$ is a localization of $\mathcal{C}_f$ with respect to weak equvalences of $\mathcal{C}_f$.

Let us denote by $\bar{\gamma}^{-1} : \mathbf{Ho}(\mathcal{C}) \to \pi\mathcal{C}_{cf}$ a quasi-inverse for γ.

Proposition 7.1.36 *(1) $j_c\bar{\gamma}^{-1}\iota_c : \mathbf{Ho}(\mathcal{C}_c) \to \pi\mathcal{C}_c$ is a right adjoint to $\bar{\gamma}_c$.*
(2) $j_f\bar{\gamma}^{-1}\iota_f : \mathbf{Ho}(\mathcal{C}_f) \to \pi\mathcal{C}_f$ is a left adjoint to $\bar{\gamma}_f$.

Proof. (1) Let $\varepsilon : \bar{\gamma}\bar{\gamma}^{-1} \to id_{\mathbf{Ho}(\mathcal{C})}$ be the natural equivalence. Since ι_c is fully faithful, there exists a unique morphism $\varepsilon_{cY} : \bar{\gamma}_c j_c\bar{\gamma}^{-1}\iota_c(Y) \to Y$ satisfying $\iota_c(\varepsilon_{cY}) = \varepsilon_{\iota_c(Y)} : \iota_c\bar{\gamma}_c j_c\bar{\gamma}^{-1}\iota_c(Y) = \bar{\gamma}\bar{\gamma}^{-1}\iota_c(Y) \to \iota_c(Y)$ for $Y \in \operatorname{Ob}\mathbf{Ho}(\mathcal{C}_c)$. Define a map $\alpha_{X,Y} : \pi\mathcal{C}_c(X, j_c\bar{\gamma}^{-1}\iota_c(Y)) \to \mathbf{Ho}(\mathcal{C}_c)(\bar{\gamma}_c(X), Y)$ by $\alpha_{X,Y}(f) = \varepsilon_{cY}\bar{\gamma}_c(f)$. Since $\bar{\gamma}_c$ is fully faithful and ε_{cY} is an isomorphism, $\alpha_{X,Y}$ is bijective.

(2) Let $\eta : id_{\mathbf{Ho}(\mathcal{C})} \to \bar{\gamma}\bar{\gamma}^{-1}$ be the natural equivalence. Since ι_f is fully faithful, there exists a unique morphism $\eta_{fX} : X \to \bar{\gamma}_f j_f\bar{\gamma}^{-1}\iota_f(X)$ satisfying $\iota_f(\eta_{fX}) = \eta_{\iota_f(X)} : \iota_f(X) \to \iota_f\bar{\gamma}_f j_f\bar{\gamma}^{-1}\iota_f(X) = \bar{\gamma}\bar{\gamma}^{-1}\iota_f(X)$ for $X \in \operatorname{Ob}\mathbf{Ho}(\mathcal{C}_f)$. Define a map $\beta_{X,Y} : \pi\mathcal{C}(j_f\bar{\gamma}^{-1}\iota_f(X), Y) \to \mathbf{Ho}(\mathcal{C}_f)(X, \bar{\gamma}_f(Y))$ by $\beta_{X,Y}(f) = \bar{\gamma}_f(f)\eta_{fX}$. Since $\bar{\gamma}_f$ is fully faithful and η_{fX} is an isomorphism, $\beta_{X,Y}$ is bijective. □

Lemma 7.1.37 *Let $\mathcal{C}$ be a model category and $\mathcal{D}$ be a category with a subcategory $\mathcal{W}$ which satisfies the following condition.*

(∗) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{W}$. If two of the morphisms f, g and gf are morphisms in $\mathcal{W}$, so is the third.

(1) Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor which maps trivial cofibrations between cofibrant objects to morphisms in $\mathcal{W}$. Then F maps all weak equivalences between cofibrant objects to morphisms in $\mathcal{W}$.

(2) Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor which maps trivial fibrations between fibrant objects to morphisms in $\mathcal{W}$. Then F maps all weak equivalences between fibrant objects to morphisms in $\mathcal{W}$.

Proof. (1) Let $f : A \to B$ be a weak equivalence between cofibrant objects. We factor the morphism $f + id_B : A\coprod B \to B$ into a cofibration $q : A\coprod B \to C$ followed by a trivial fibration $p : C \to B$. We note that C is cofibrant. Since the unique morphisms $\emptyset \to A$ and $\emptyset \to B$ are cofibrations and the following diagram is a push-out diagram, the canonical morphisms $i_1 : A \to A\coprod B$ and $i_2 : B \to A\coprod B$ are cofibrations by (M3).

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & B \arrow[d, "i_2"] \\
A \arrow[r, "i_1"] & A\coprod B
\end{tikzcd}$$

Hence qi_1 and qi_2 are cofibrations by (M3). Since both $pqi_1 = (f + id_B)i_1 = f$ and p are weak equivalences, qi_1 is also a weak equivalence by (M5). Similarly, since both $pqi_2 = (f + id_B)i_2 = id_B$ and p are weak equivalences, qi_2 is also a weak equivalence by (M5). Thus qi_1 and qi_2 are trivial cofibrations and it follows from the assumption that both $F(qi_1)$ and $F(qi_2)$ are morphisms in $\mathcal{W}$. Since $F(p)F(qi_2) = F(pqi_2) = F(id_B)$ is also a morphism in $\mathcal{W}$, $F(p)$ is a morphism in $\mathcal{W}$ by (∗). Therefore $F(f) = F(pqi_1) = F(p)F(qi_1)$ is a morphism in $\mathcal{W}$.

(2) Let $f : A \to B$ be a weak equivalence between fibrant objects. We factor the morphism $(f, id_A) : A \to A \times B$ into trivial cofibration $q : A \to C$ followed by a fibration $p : C \to A \times B$. We note that C is fibrant. Since the unique morphisms $A \to *$ and $B \to *$ are fibrations and the following diagram is a pull-back diagram, the canonical morphisms $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ are fibrations by (M3).

$$\begin{array}{ccc} A \times B & \xrightarrow{p_2} & B \\ \downarrow{\scriptstyle p_1} & & \downarrow \\ A & \longrightarrow & * \end{array}$$

Hence p_1p and p_2p are fibrations by (M3). Since both $p_1pq = p_1(f, id_A) = f$ and q are weak equivalences, p_1p is also a weak equivalence by (M5). Similarly, since both $p_2pq = p_2(f, id_A) = id_A$ and q are weak equivalences, p_2p is also a weak equivalence by (M5). Thus p_1p and p_2p are trivial cofibrations and it follows from the assumption that both $F(p_1p)$ and $F(p_2p)$ are morphisms in $\mathcal{W}$. Since $F(p_2p)F(q) = F(p_2pq) = F(id_A)$ is also a morphism in $\mathcal{W}$, $F(q)$ is a morphism in $\mathcal{W}$ by $(*)$. Therefore $F(f) = F(p_1pq) = F(p_1p)F(q)$ is a morphism in $\mathcal{W}$. □

7.2 Closed model category

We say that a morphism $f : A \to B$ is a retract of a morphism $g : X \to Y$ if there exist morphisms $i : A \to X$, $p : X \to A$, $j : B \to Y$ and $q : Y \to B$ such that $pi = id_A$ and $qj = id_B$ and that the following diagram commute.

$$\begin{array}{ccccc} A & \xrightarrow{i} & X & \xrightarrow{p} & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{j} & Y & \xrightarrow{q} & B \end{array}$$

Definition 7.2.1 *A closed model structure on a category is three subcategories* $\boldsymbol{Cof}(\mathcal{C})$, $\boldsymbol{Fib}(\mathcal{C})$ *and* $\boldsymbol{Weq}(\mathcal{C})$ *of* $\mathcal{C}$ *which satisfy the conditions below. A morphism in* $\boldsymbol{Cof}(\mathcal{C})$, $\boldsymbol{Fib}(\mathcal{C})$ *or* $\boldsymbol{Weq}(\mathcal{C})$ *is called a cofibration, fibration or weak equivalence, respectively. A morphism which belongs to* $\boldsymbol{Cof}(\mathcal{C}) \cap \boldsymbol{Weq}(\mathcal{C})$ *or* $\boldsymbol{Fib}(\mathcal{C}) \cap \boldsymbol{Weq}(\mathcal{C})$ *is called a trivial cofibration or trivial fibration, respectively.*

(CM1) If f and g are morphisms in $\mathcal{C}$ such that gf is defined and if two of the three morphisms f, g, gf are weak equivalences, then so is the third.

(CM2) If a morphism f is a retract of a morphism in $\boldsymbol{Cof}(\mathcal{C})$, $\boldsymbol{Fib}(\mathcal{C})$ *or* $\boldsymbol{Weq}(\mathcal{C})$, *$f$ is also a morphism in* $\boldsymbol{Cof}(\mathcal{C})$, $\boldsymbol{Fib}(\mathcal{C})$ *or* $\boldsymbol{Weq}(\mathcal{C})$, *respectively.*

(CM3) Suppose that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

If one of the following conditions is satisfied, there exists a morphism $h : B \to Y$ satisfying $hi = f$ and $ph = g$.

(i) i is a cofibration and p is a trivial fibration.

(ii) i is a trivial cofibration and p is a fibration.

(CM4) Each morphism f in $\mathcal{C}$ can be factored in the following two ways:

(i) $f = pi$, where i is a cofibration and p is a trivial fibration.

(ii) $f = pi$, where i is a trivial cofibration and p is a fibration.

Definition 7.2.2 *A finitely complete and finitely cocomplete category with a closed model structure is called a closed model category.*

Definition 7.2.3 *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

be a diagram in a categrm ory $\mathcal{C}$.

(1) We say that a morphism $i : A \to B$ has the left lifting property (LLP) with respect to a morphism $p : X \to Y$ if there exists a morphism $h : B \to X$ satisfying $hi = f$ and $ph = g$ for any morphisms f and g that make the above diagram commute.

(2) We say that a morphism $p : X \to Y$ has the right lifting property (RLP) with respect to a morphism $i : A \to B$ if there exists a morphism $h : B \to X$ satisfying $hi = f$ and $ph = g$ for any morphisms f and g that make the above diagram commute.

Lemma 7.2.4 *Let $f : A \to B$ be a morphism in $\mathcal{C}$. Suppose we have a factorization $f = pi$, where $i : A \to C$, $p : C \to B$.*
(1) If f has LLP with respect to p, then f is a retract of i.
(2) If f has RLP with respect to i, then f is a retract of p.

Proof. (1) Consider the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & C \arrow[d, "p"] \\ & B \end{tikzcd}$$

By the assumption, there is a morphism $r : B \to C$ satisfying $rf = i$ and $pr = id_B$. Hence the following diagram commutes and the assertion follows.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, "i"] \arrow[dr, "f"] & \\ B \arrow[r, "r"] & C \arrow[r, "p"] & B \end{tikzcd}$$

(2) Consider the following commutative diagram.

$$\begin{tikzcd} A \arrow[d, "i"] \arrow[dr, "f"] & \\ C \arrow[r, "p"] & B \end{tikzcd}$$

By the assumption, there is a morphism $r : C \to A$ satisfying $fr = p$ and $ri = id_A$. Hence the following diagram commutes and the assertion follows.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "f"'] & C \arrow[r, "r"] \arrow[d, "p"] & A \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

□

Proposition 7.2.5 *Let $\mathcal{C}$ be a closed model category.*
(1) A morphism i in $\mathcal{C}$ is a cofibration if and only if it has LLP with respect to all trivial fibrations.
(2) A morphism i in $\mathcal{C}$ is a trivial cofibration if and only if it has LLP with respect to all fibrations.
(3) A morphism p in $\mathcal{C}$ is a fibration if and only if it has RLP with respect to all trivial cofibrations.
(4) A morphism p in $\mathcal{C}$ is a trivial fibration if and only if it has RLP with respect to all cofibrations.

Proof. (1) Suppose that i has LLP with respect to all trivial fibrations. We factor i as $i = qj$, where j is a cofibration and q is a trivial fibration. Since i has LLP with respect to q, i is retract of j by (7.2.4). Hence i is a cofibration by (CM2).

(2) Suppose that i has LLP with respect to all fibrations. We factor i as $i = qj$, where j is a trivial cofibration and q is a fibration. Since i has LLP with respect to q, i is retract of j by (7.2.4). Hence i is a trivial cofibration by (CM2).

(3) Suppose that p has RLP with respect to all trivial cofibrations. We factor p as $p = qj$, where j is a trivial cofibration and q is a fibration. Since p has RLP with respect to j, p is retract of q by (7.2.4). Hence p is a fibration by (CM2).

(4) Suppose that p has RLP with respect to all cofibrations. We factor p as $p = qj$, where j is a cofibration and q is a trivial fibration. Since p has RLP with respect to j, p is retract of q by (7.2.4). Hence p is a trivial fibration by (CM2). □

Proposition 7.2.6 *A closed model category is a model category.*

Proof. Suppose that the following diagram commutes and that the right square is cartesian.

$$\begin{tikzcd} Z \arrow[r, "l"] \arrow[d, "j"] & A \arrow[r, "f"] \arrow[d, "q"] & X \arrow[d, "p"] \\ W \arrow[r, "k"] & B \arrow[r, "g"] & Y \end{tikzcd}$$

Assume that p is a fibration (resp. a trivial fibration). If j is a trivial cofibration (resp. a cofibration), there exists a morphism $h : W \to X$ satisfying $hj = fl$ and $ph = gk$ by (ii) (resp. (i)) of (CM3). Hence there exists a unique morphism $s : W \to A$ satisfying $fs = h$ and $qs = k$. Since $fl = hj = f(sj)$ and $ql = kj = q(sj)$ and (f, q) is a monomorphic family, we have $sj = l$. Therefore q has RLP with respect to all trivial cofibrations (resp. all cofibrations). It follows from (3) (resp. (4)) of (7.2.5) that q is a fibration (resp. trivial fibration). Thus $\boldsymbol{Fib}(\mathcal{C})$ (resp. $\boldsymbol{Fib}(\mathcal{C}) \cap \boldsymbol{Weq}(\mathcal{C})$) is stable under pull-backs.

Suppose that the following diagram commutes and that the left square is cocartesian.

$$\begin{tikzcd} Z \arrow[r, "l"] \arrow[d, "j"] & A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ W \arrow[r, "k"] & B \arrow[r, "g"] & Y \end{tikzcd}$$

Assume that j is a cofibration (resp. a trivial cofibration). If p is a trivial fibration (resp. a fibration), there exists a morphism $h : W \to X$ satisfying $hj = fl$ and $ph = gk$ by (i) (resp. (ii)) of (CM3). Hence there exists a unique morphism $t : B \to X$ satisfying $tk = h$ and $ti = f$. Since $gk = ph = (pt)k$ and $gi = pf = (pt)i$ and (i, k) is an epimorphic family, we have $pt = g$. Therefore i has LLP with respect to all trivial fibrations (resp. all fibrations). . It follows from (1) (resp. (2)) of (7.2.5) that i is a cofibration. Thus $\boldsymbol{Cof}(\mathcal{C})$ (resp. $\boldsymbol{Cof}(\mathcal{C}) \cap \boldsymbol{Weq}(\mathcal{C})$) is stable under push-outs. Hence the second condition of (M3) and the condition (M4) of (7.1.1) are satisfied.

If $f : X \to Y$ is an isomorphism, then it is clear that f has LLP with respect to all fibrations and that f has RLP with respect to all cofibrations. Hence f is a both trivial cofibration and a trivial fibration by (7.2.5). Therefore the third condition of (M3) and the second condition of (M5) are satisfied. The conditions (M1), (M2) and the first condition of (M5) are the same conditions as (CM3), (CM4) and (CM1) of (7.2.1), respectively. Since $\boldsymbol{Cof}(\mathcal{C})$ and $\boldsymbol{Fib}(\mathcal{C})$ in (7.2.2) are subcategories, the first condition of (M3) is satisfied. Since we assume finitely completeness in (7.2.2), (M0) is satisfied. □

For a category $\mathcal{C}$, let us denote by $\mathcal{C}^{(2)}$ the category defined as follows. Put $\mathrm{Ob}(\mathcal{C}^{(2)}) = \mathrm{Mor}(\mathcal{C})$. If $f, g \in \mathrm{Ob}(\mathcal{C}^{(2)})$, $\mathcal{C}^{(2)}(f, g)$ is the set of all pairs (s, t) of morphisms such that the following diagram commute.

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"] & Z \arrow[d, "g"] \\ Y \arrow[r, "t"] & W \end{tikzcd}$$

If $(s, t) \in \mathcal{C}^{(2)}(f, g)$ and $(u, v) \in \mathcal{C}^{(2)}(g, h)$, the composition of (s, t) and (u, v) is defined by $(u, v)(s, t) = (us, vt)$.

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"] & Z \arrow[r, "u"] \arrow[d, "g"] & U \arrow[d, "h"] \\ Y \arrow[r, "t"] & W \arrow[r, "v"] & V \end{tikzcd}$$

Then, (id_X, id_Y) is the identity morphism id_f of $f : X \to Y$. Let us define functors $\sigma, \tau : \mathcal{C}^{(2)} \to \mathcal{C}$ by $\sigma(f : X \to Y) = X$, $\tau(f : X \to Y) = Y$ and $\sigma((s, t) : f \to g) = s$, $\tau((s, t) : f \to g) = t$.

Consider the following condition which requires that the factotizations in (CM4) is functorial.

(CM5) There exist functors $\alpha, \beta, \gamma, \delta : \mathcal{C}^{(2)} \to \mathcal{C}^{(2)}$ which satisfy the following conditions.

(i) The following diagrams commute.

$$\begin{tikzcd} \mathcal{C}^{(2)} \arrow[r, "\alpha"] \arrow[dr, "\sigma"'] & \mathcal{C}^{(2)} \arrow[d, "\sigma"] & \mathcal{C}^{(2)} \arrow[l, "\gamma"'] \arrow[dl, "\sigma"] \\ & \mathcal{C} & \end{tikzcd} \quad \begin{tikzcd} \mathcal{C}^{(2)} \arrow[r, "\beta"] \arrow[dr, "\tau"'] & \mathcal{C}^{(2)} \arrow[d, "\tau"] & \mathcal{C}^{(2)} \arrow[l, "\delta"'] \arrow[dl, "\tau"] \\ & \mathcal{C} & \end{tikzcd} \quad \begin{tikzcd} \mathcal{C}^{(2)} \arrow[r, "\alpha"] \arrow[d, "\beta"] & \mathcal{C}^{(2)} \arrow[d, "\tau"] \\ \mathcal{C}^{(2)} \arrow[r, "\sigma"] & \mathcal{C} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{C}^{(2)} \arrow[r, "\gamma"] \arrow[d, "\delta"] & \mathcal{C}^{(2)} \arrow[d, "\tau"] \\ \mathcal{C}^{(2)} \arrow[r, "\sigma"] & \mathcal{C} \end{tikzcd}$$

(*ii*) For any morphism f in $\mathcal{C}$, there are factorizations $f = \beta(f)\alpha(f) = \delta(f)\gamma(f)$ such that $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration and $\delta(f)$ is a fibration.

If we assume that a model category $\mathcal{C}$ satisfies (CM5), then we can choose cofibrant replacements, fibrant replacements, cylinder objects and path objects functorially as follows.

For an object X of $\mathcal{C}$, we choose QX to be $\tau\alpha(\emptyset \to X) = \sigma\beta(\emptyset \to X)$ and $p_X : QX \to X$ to be $\beta(\emptyset \to X)$. If $f : X \to Y$ is a morphism in $\mathcal{C}$, we define $Qf : QX \to QY$ by $Qf = \sigma\beta(id_\emptyset, f) = \tau\alpha(id_\emptyset, f)$. Since QX is cofibrant, it is easy to verify that the correspondences $X \mapsto QX$ and $f \mapsto Qf$ define a functor $Q : \mathcal{C} \to \mathcal{C}_c$. Let us denote by $\iota_c : \mathcal{C}_c \to \mathcal{C}$ the inclusion functor. Then, the correspondence $X \mapsto p_X$ defines natural transformations $p : \iota_c Q \to id_{\mathcal{C}}$ and $p' : Q\iota_c \to id_{\mathcal{C}_c}$.

Dually, for an object X of $\mathcal{C}$, we choose RX to be $\tau\gamma(X \to *) = \sigma\delta(X \to *)$ and $i_X : X \to RX$ to be $\gamma(X \to *)$. If $f : X \to Y$ is a morphism in $\mathcal{C}$, we define $Rf : RX \to RY$ by $Rf = \tau\gamma(f, id_*) = \sigma\delta(f, id_*)$. Since RX is fibrant, it is easy to verify that the correspondences $X \mapsto RX$ and $f \mapsto Rf$ define a functor $R : \mathcal{C} \to \mathcal{C}_f$. Let us denote by $\iota_f : \mathcal{C}_f \to \mathcal{C}$ the inclusion functor. Then, the correspondence $X \mapsto i_X$ defines natural transformations $i : id_{\mathcal{C}} \to \iota_f R$ and $i' : id_{\mathcal{C}_f} \to R\iota_f$.

For an object X of $\mathcal{C}$, we factor the codiagonal morphism $\nabla_X : X \coprod X \to X$ and have $\nabla_X = \beta(\nabla_X)\alpha(\nabla_X)$. We choose $X \times I$ to be $\tau\alpha(\nabla_X) = \sigma\beta(\nabla_X)$ and $\sigma : X \times I \to X$ to be $\beta(\nabla_X)$. If $f : X \to Y$ is a morphism in $\mathcal{C}$, we define $f \times I : X \times I \to Y \times I$ by $f \times I = \sigma\beta\left(f \coprod f, f\right) = \tau\alpha\left(f \coprod f, f\right)$. Thus we have a functor $(-) \times I : \mathcal{C} \to \mathcal{C}$ which maps X to $X \times I$ and $f : X \to Y$ to $f \times I : X \times I \to Y \times I$. Suppose that $X \coprod X \xrightarrow{\partial_0' + \partial_1'} \widetilde{X} \xrightarrow{\sigma'} X$ is also a cylinder object of X. Then the following diagram commutes.

$$\begin{array}{ccc}
X \coprod X & \xrightarrow{\alpha(\nabla_X)} & X \times I \\
\big\downarrow{\scriptstyle \partial_0' + \partial_1'} & & \big\downarrow{\scriptstyle \sigma} \\
\widetilde{X} & \xrightarrow{\sigma'} & X
\end{array}$$

Since $\partial_0' + \partial_1'$ is a cofibration and σ is a trivial fibration, there exists a morphism $f : \widetilde{X} \to X \times I$ satisfying $\sigma f = \sigma'$ and $f(\partial_0' + \partial_1') = \alpha(\nabla_X)$. Since both σ and σ' are weak equivalences, so is f.

For an object X of $\mathcal{C}$, we factor the diagonal morphism $\Delta_X : X \to X \times X$ and have $\Delta_X = \delta(\Delta_X)\gamma(\Delta_X)$. We choose X^I to be $\tau\gamma(\Delta_X) = \sigma\delta(\Delta_X)$ and $s : X \to X^I$ to be $\gamma(\Delta_X)$. If $f : X \to Y$ is a morphism in $\mathcal{C}$, we define $f^I : X^I \to Y^I$ by $f^I = \sigma\delta(f, f \times f) = \tau\gamma(f, f \times f)$. Thus we have a functor $(-)^I : \mathcal{C} \to \mathcal{C}$ which maps X to X^I and $f : X \to Y$ to $f^I : X^I \to Y^I$. Suppose that $X \xrightarrow{s'} \widetilde{X} \xrightarrow{(d_0', d_1')} X \times X$ is also a cylinder object of X. Then the following diagram commutes.

$$\begin{array}{ccc}
X & \xrightarrow{s} & X^I \\
\big\downarrow{\scriptstyle s'} & & \big\downarrow{\scriptstyle \delta(\Delta_X)} \\
\widetilde{X} & \xrightarrow{(d_0', d_1')} & X \times X
\end{array}$$

Since (d_0', d_1') is a fibration and s is a trivial cofibration, there exists a morphism $f : X^I \to \widetilde{X}$ satisfying $fs = s'$ and $(d_0', d_1')f = \delta(\Delta_X)$. Since both s and s' are weak equivalences, so is f.

7.3 Quillen functor

Definition 7.3.1 *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.*

(1) A functor $F : \mathcal{C} \to \mathcal{D}$ is called a left Quillen functor if F has a right adjoint and F preserves cofibrations and trivial cofibrations.

(2) A functor $U : \mathcal{D} \to \mathcal{C}$ is called a right Quillen functor if U has a left adjoint and U preserves fibrations and trivial fibrations.

(3) Suppose that $(F, U, \varphi) : \mathcal{C} \to \mathcal{D}$ is an adjunction, that is, $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$ are functors and $\varphi : \mathcal{D}(F(A), B) \to \mathcal{C}(A, U(B))$ is a natural isomorphism expressing U as a right adjoint of F. (F, U, φ) is called a Quillen adjunction if F is a left Quillen functor.

The following assertion is straightforward from the naturality of φ.

Lemma 7.3.2 *Let $(F, U, \varphi) : \mathcal{C} \to \mathcal{D}$ be an adjunction.*

(1) For a morphism $f : X \to Y$ in $\mathcal{C}$ and a morphism $g : Z \to W$ in $\mathcal{D}$, the following left diagram diagram commutes if and only if the right diagram diagram commutes.

$$\begin{tikzcd} F(X) \arrow[r, "\alpha"] \arrow[d, "F(f)"] & Z \arrow[d, "g"] \\ F(Y) \arrow[r, "\beta"] & W \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\varphi(\alpha)"] \arrow[d, "f"] & U(Z) \arrow[d, "U(g)"] \\ Y \arrow[r, "\varphi(\beta)"] & W \end{tikzcd}$$

(2) Suppose that the above diagrams commute. A morphism $h : F(Y) \to Z$ satisfies $gh = \beta$ and $hF(f) = \alpha$ if and only if $\varphi(h) : Y \to U(Z)$ satisfies $U(g)\varphi(h) = \varphi(\beta)$ and $\varphi(h)f = \varphi(\alpha)$.

Proposition 7.3.3 *Let $(F, U, \varphi) : \mathcal{C} \to \mathcal{D}$ be an adjunction. (F, U, φ) is a Quillen adjunction if and only if U is a right Quillen functor.*

Proof. Suppose that (F, U, φ) is a Quillen adjunction. Let $g : Z \to W$ be a fibration (resp. trivial fibration). It follows from (7.3.2) that $U(g)$ has RLP with respect to all trivial cofibrations (resp. cofibrations). Hence $U(g)$ is a fibration (resp. trivial fibration) by (7.2.5). Thus U is a right Quillen functor.

Suppose that U is a right Quillen functor. Let $f : X \to Y$ be a cofibration (resp. trivial cofibration). It follows from (7.3.2) that $F(f)$ has LLP with respect to all trivial fibrations (resp. fibrations). Hence $F(f)$ is a cofibration (resp. trivial cofibration) by (7.2.5). Thus F is a left Quillen functor. □

The next result follows from (7.1.37).

Proposition 7.3.4 *(1) If F is a left Quillen functor, F preserves weak equivalences between cofibrant objects.*

(2) If F is a right Quillen functor, F preserves weak equivalences between fibrant objects.

Chapter 8

Study on fibered categories

Introduction

The aim of this chapter is to study various notions on fibered categories which are needed to develop a theory of representations of internal category next chapter.

We begin by reviewing the notion of fibered category following [5] and internal category in the first section, we give a detailed description on the relationship between the notions of fibered category and 2-category in section 3, which is originally observed in section 8 of [5]. There, we show that the 2-category of fibered category over a given category $\mathcal{E}$ is equivalent to the 2-category of "lax functors" from the opposite category of $\mathcal{E}$ to the 2-category of categories. Our construction of fibered categories from lax functors allows us to give the notion of fibered categories represented by internal category (8.3.18) and a short definition (8.3.19) of Grothendieck topoi over simplicial object in given site.

8.1 Fibered category

Let $p : \mathcal{F} \to \mathcal{E}$ be a functor. For an object X of $\mathcal{E}$, we denote by $\mathcal{F}_X$ the subcategory of $\mathcal{F}$ consiting of objects M of $\mathcal{F}$ satisfying $p(M) = X$ and morphisms φ satisfying $p(\varphi) = id_X$. For a morphism $f : X \to Y$ in $\mathcal{E}$ and $M \in \operatorname{Ob}\mathcal{F}_X$, $N \in \operatorname{Ob}\mathcal{F}_Y$, we put $\mathcal{F}_f(M, N) = \{\varphi \in \mathcal{F}(M, N) | \, p(\varphi) = f\}$.

Definition 8.1.1 *([5], p.161 Définition 5.1.) Let $\alpha : M \to N$ be a morphism in $\mathcal{F}$ and set $X = p(M)$, $Y = p(N)$, $f = p(\alpha)$. We call α a cartesian morphism if, for any $M' \in \operatorname{Ob}\mathcal{F}_X$, the map $\mathcal{F}_X(M', M) \to \mathcal{F}_f(M', N)$ defined by $\varphi \mapsto \alpha\varphi$ is bijective.*

The following assertion is immediate from the definition.

Proposition 8.1.2 *Let $\alpha_i : M_i \to N_i$ $(i = 1, 2)$ be morphisms in $\mathcal{F}$ such that $p(M_1) = p(M_2)$, $p(N_1) = p(N_2)$, $p(\alpha_1) = p(\alpha_2)$ and $\lambda : N_1 \to N_2$ a morphism in $\mathcal{F}$ such that $p(\lambda) = id_{p(N_1)}$. If α_2 is cartesian, there is a unique morphism $\mu : M_1 \to M_2$ such that $p(\mu) = id_{p(M_1)}$ and $\alpha_2\mu = \lambda\alpha_1$.*

Corollary 8.1.3 *If $\alpha_i : M_i \to N$ $(i = 1, 2)$ are cartesian morphisms in $\mathcal{F}$ such that $p(M_1) = p(M_2)$ and $p(\alpha_1) = p(\alpha_2)$, there is a unique morphism $\mu : M_1 \to M_2$ such that $\alpha_1 = \alpha_2\mu$ and $p(\mu) = id_{p(M_1)}$. Moreover, μ is an isomorphism.*

Definition 8.1.4 *([5], p.162 Définition 5.1.) Let $f : X \to Y$ be a morphism in $\mathcal{E}$ and $N \in \operatorname{Ob}\mathcal{F}_Y$. If there exists a cartesian morphism $\alpha : M \to N$ such that $p(\alpha) = f$, M is called an inverse image of N by f. We denote M by $f^*(N)$ and α by $\alpha_f(N) : f^*(N) \to N$. By (8.1.3), $f^*(N)$ is unique up to isomorphism.*

Remark 8.1.5 *For an identity morphism id_X of $X \in \operatorname{Ob}\mathcal{E}$ and $N \in \operatorname{Ob}\mathcal{F}_X$, the identity morphism id_N of N is obviously cartesian. Hence the inverse image of N by the identity morphism of X always exists and $\alpha_{id_X}(N) : id_X^*(N) \to N$ can be chosen as the identity morphism of N. By the uniqueness of $id_X^*(N)$ up to isomorphism, $\alpha_{id_X}(N) : id_X^*(N) \to N$ is an isomorphism for any choice of $id_X^*(N)$.*

The following assertion is also immediate.

Proposition 8.1.6 *Let $f : X \to Y$ be a morphism in $\mathcal{E}$. If, for any $N \in \mathrm{Ob}\,\mathcal{F}_Y$, there exists a cartesian morphism $\alpha_f(N) : f^*(N) \to N$, $N \mapsto f^*(N)$ defines a functor $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ such that, for any morphism $\varphi : N \to N'$ in $\mathcal{F}_Y$, the following square commutes.*

$$\begin{tikzcd} f^*(N) \arrow[r, "\alpha_f(N)"] \arrow[d, "f^*(\varphi)"] & N \arrow[d, "\varphi"] \\ f^*(N') \arrow[r, "\alpha_f(N')"] & N' \end{tikzcd}$$

Definition 8.1.7 *([5], p.162 Définition 5.1.) If the assumption of (8.1.6) is satisfied, we say that the functor of the inverse image by f exists.*

Definition 8.1.8 *([5], p.164 Définition 6.1.) If a functor $p : \mathcal{F} \to \mathcal{E}$ satisfies the following condition (i), p is called a prefibered category and if p satisfies both (i) and (ii), p is called a fibered category or p is fibrant.*

(i) For any morphism f in $\mathcal{E}$, the functor of the inverse image by f exists.
(ii) The composition of cartesian morphisms is cartesian.

Definition 8.1.9 *([5], p.170 Définition 7.1.) Let $p : \mathcal{F} \to \mathcal{E}$ be a functor. A map*

$$\kappa : \mathrm{Mor}\,\mathcal{E} \longrightarrow \coprod_{X,Y \in \mathrm{Ob}\,\mathcal{E}} \mathrm{Funct}(\mathcal{F}_Y, \mathcal{F}_X)$$

is called a cleavage if $\kappa(f)$ is an inverse image functor $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ for $(f : X \to Y) \in \mathrm{Mor}\,\mathcal{E}$. A cleavage κ is said to be normalized if $\kappa(id_X) = id_{\mathcal{F}_X}$ for any $X \in \mathrm{Ob}\,\mathcal{E}$. A category $\mathcal{F}$ over $\mathcal{E}$ is called a cloven prefibered category (resp. normalized cloven prefibered category) if a cleavage (resp. normalized cleavage) is given.

$p : \mathcal{F} \to \mathcal{E}$ has a cleavage if and only if p is prefibered. If p is prefibered, p has a normalized cleavage by (8.1.5).

Let $f : X \to Y$, $g : Z \to X$ be morphisms in $\mathcal{E}$ and N an object of $\mathcal{F}_Y$. If $p : \mathcal{F} \to \mathcal{E}$ is a prefibered category, there is a unique morphism $c_{f,g}(N) : g^*f^*(N) \to (fg)^*(N)$ such that the following square commutes and $p(c_{f,g}(N)) = id_Z$.

$$\begin{tikzcd} g^*f^*(N) \arrow[r, "\alpha_g(f^*(N))"] \arrow[d, "c_{f,g}(N)"] & f^*(N) \arrow[d, "\alpha_f(N)"] \\ (fg)^*(N) \arrow[r, "\alpha_{fg}(N)"] & N \end{tikzcd}$$

Then, we see the following.

Proposition 8.1.10 *For a morphism $\varphi : M \to N$ in $\mathcal{F}_Y$, the following square commutes.*

$$\begin{tikzcd} g^*f^*(M) \arrow[r, "c_{f,g}(M)"] \arrow[d, "g^*f^*(\varphi)"] & (fg)^*(M) \arrow[d, "(fg)^*(\varphi)"] \\ g^*f^*(N) \arrow[r, "c_{f,g}(N)"] & (fg)^*(N) \end{tikzcd}$$

In other words, $c_{f,g}$ gives a natural transformation $g^*f^* \to (fg)^*$ of functors from $\mathcal{F}_Y$ to $\mathcal{F}_Z$.

Proof. In fact, $\alpha_{fg}(N)(fg)^*(\varphi)c_{f,g}(M) = \varphi\alpha_{fg}(M)c_{f,g}(M) = \varphi\alpha_f(M)\alpha_g(f^*(M)) = \alpha_f(N)f^*(\varphi)\alpha_g(f^*(M)) = \alpha_f(N)\alpha_g(f^*(N))g^*f^*(\varphi) = \alpha_{fg}(N)c_{f,g}(N)g^*f^*(\varphi)$. Since $\alpha_{fg}(N)$ is cartesian and $p((fg)^*(\varphi)c_{f,g}(M)) = p(c_{f,g}(N)g^*f^*(\varphi)) = id_Z$, the assertion follows. □

Proposition 8.1.11 *([5], p.172 Proposition 7.2.) Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven prefibered category. Then, p is a fibered category if and only if $c_{f,g}(N)$ is an isomorphism for any $Z \xrightarrow{g} X \xrightarrow{f} Y$ and $N \in \mathrm{Ob}\,\mathcal{F}_X$.*

Proof. Suppose that p is a fibered category. Then, both $\alpha_{fg}(N)$ and $\alpha_f(N)\alpha_g(f^*(N))$ are cartesian morphisms such that $p(\alpha_{fg}(N)) = p(\alpha_f(N)\alpha_g(f^*(N))) = fg$. Hence by (8.1.3), $c_{f,g}(N)$ is an isomorphism.

Conversely, suppose that $c_{f,g}(N)$ is an isomorphism for any $Z \xrightarrow{g} X \xrightarrow{f} Y$ and $N \in \operatorname{Ob}\mathcal{F}_X$. Let $\alpha : M \to N$ and $\beta : L \to M$ be a cartesian morphisms in $\mathcal{F}$. Put $p(M) = X$, $p(N) = Y$, $p(L) = Z$, $p(\alpha) = f$ and $p(\beta) = g$. There is a unique morphism $\zeta : L \to (fg)^*(N)$ such that $\alpha_{fg}(N)\zeta = \alpha\beta$ and $p(\zeta) = id_Z$. There are isomorphisms $\psi : M \to f^*(N)$ and $\xi : L \to g^*(M)$ such that $\alpha = \alpha_f(N)\psi$, $\beta = \alpha_g(M)\xi$ and $p(\psi) = id_X$, $p(\xi) = id_Z$. By (8.1.6), $\alpha_g(f^*(N))g^*(\psi) = \psi\alpha_g(M)$. Hence $\alpha_{fg}(N)c_{f,g}(N)g^*(\psi)\xi = \alpha_f(N)\alpha_g(f^*(N))g^*(\psi)\xi = \alpha_f(N)\psi\alpha_g(M)\xi = \alpha\beta$ and $p(c_{f,g}(N)g^*(\psi)\xi) = id_Z$. By the uniqueness of ζ, $c_{f,g}(N)g^*(\psi)\xi = \zeta$. Thus ζ is an isomorphism and it follows that $\alpha\beta$ is cartesian. □

Proposition 8.1.12 *([5], p.172 Proposition 7.4.) Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven prefibered category. For a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathcal{E}$ and an object M of $\mathcal{F}_W$, we have*

$$c_{f,id_X}(N) = \alpha_{id_X}(f^*(N)) \qquad c_{id_Y,f}(N) = f^*(\alpha_{id_Y}(N))$$

and the following diagram commutes.

$$\begin{tikzcd}
f^*(g^*h^*)(M) \arrow[r, equal] \arrow[d, "f^*(c_{h,g}(M))"] & (f^*g^*)h^*(M) \arrow[rr, "c_{g,f}(h^*(M))"] & & (gf)^*h^*(M) \arrow[d, "c_{h,gf}(M)"] \\
f^*(hg)^*(M) \arrow[rr, "c_{hg,f}(M)"] & & ((hg)f)^*(M) \arrow[r, equal] & (h(gf))^*(M)
\end{tikzcd}$$

Proof. The following diagrams commute by the definition of $c_{f,id_X}(N)$ and $c_{id_Y,f}(N)$.

$$\begin{tikzcd}
id_X^*f^*(N) \arrow[r, "\alpha_{id_X}(f^*(N))"] \arrow[d, "c_{f,id_X}(N)"] & f^*(N) \arrow[d, "\alpha_f(N)"] \\
f^*(N) \arrow[r, "\alpha_f(N)"] & N
\end{tikzcd} \qquad
\begin{tikzcd}
f^*id_Y^*(N) \arrow[r, "\alpha_f(id_Y^*(N))"] \arrow[d, "c_{id_Y,f}(N)"] & id_Y^*(N) \arrow[d, "\alpha_{id_Y}(N)"] \\
f^*(N) \arrow[r, "\alpha_f(N)"] & N
\end{tikzcd}$$

On the other hand, the following diagrams also commute.

$$\begin{tikzcd}
id_X^*f^*(N) \arrow[r, "\alpha_{id_X}(f^*(N))"] \arrow[d, "\alpha_{id_X}(f^*(N))"] & f^*(N) \arrow[d, "\alpha_f(N)"] \\
f^*(N) \arrow[r, "\alpha_f(N)"] & N
\end{tikzcd} \qquad
\begin{tikzcd}
f^*id_Y^*(N) \arrow[r, "\alpha_f(id_Y^*(N))"] \arrow[d, "f^*(\alpha_{id_Y}(N))"] & id_Y^*(N) \arrow[d, "\alpha_{id_Y}(N)"] \\
f^*(N) \arrow[r, "\alpha_f(N)"] & N
\end{tikzcd}$$

Hence the assertion follows from the uniqueness of $c_{f,id_X}(N)$ and $c_{id_Y,f}(N)$. Similarly, since

$$\begin{aligned}
\alpha_{hgf}(M)c_{h,gf}(M)c_{g,f}(h^*(M)) &= \alpha_h(M)\alpha_{gf}(h^*(M))c_{g,f}(h^*(M)) = \alpha_h(M)\alpha_g(h^*(M))\alpha_f(g^*h^*(M)) \\
&= \alpha_{hg}(M)c_{h,g}(M)\alpha_f(g^*h^*(M)) = \alpha_{hg}(M)\alpha_f((hg)^*(M))f^*(c_{h,g}(M)) \\
&= \alpha_{hgf}(M)c_{hg,f}(M)f^*(c_{h,g}(M)),
\end{aligned}$$

we have $c_{h,gf}(M)c_{g,f}(h^*(M)) = c_{hg,f}(M)f^*(c_{h,g}(M))$. □

Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven fibered category. For morphisms $f : X \to Y$ and $g : X \to Z$ of $\mathcal{E}$, we define a functor $F_{f,g} : \mathcal{F}_Y^{op} \times \mathcal{F}_Z \to \mathcal{S}et$ by $F_{f,g}(M, N) = \mathcal{F}_X(f^*(M), g^*(N))$ for $M \in \operatorname{Ob}\mathcal{F}_Y$, $N \in \operatorname{Ob}\mathcal{F}_Z$ and $F_{f,g}(\varphi, \psi) = f^*(\varphi)^*g^*(\psi)_*$ for $\varphi \in \operatorname{Mor}\mathcal{F}_Y$, $\psi \in \operatorname{Mor}\mathcal{F}_Z$. For a morphism $k : V \to X$ of $\mathcal{E}$, $M \in \operatorname{Ob}\mathcal{F}_Y$ and $N \in \operatorname{Ob}\mathcal{F}_Z$, let $k^\sharp_{M,N} : F_{f,g}(M, N) \to F_{fk,gk}(M, N)$ be the following composition.

$$\begin{aligned}
F_{f,g}(M, N) = \mathcal{F}_X(f^*(M), g^*(N)) &\xrightarrow{k^*} \mathcal{F}_V(k^*(f^*(M)), k^*(g^*(N))) \xrightarrow{(c_{f,k}(M)^{-1})^*} \mathcal{F}_V((fk)^*(M), k^*(g^*(N))) \\
&\xrightarrow{c_{g,k}(N)_*} \mathcal{F}_V((fk)^*(M), (gk)^*(N)) = F_{fk,gk}(M, N)
\end{aligned}$$

Let $\varphi : M \to L$ and $\psi : P \to N$ be morphisms of $\mathcal{F}_Y$ and $\mathcal{F}_Z$, respectively. Since the following diagram is commutative by (8.1.10), $k^\sharp_{M,N}$ is natural in M, N and we have a natural transformation $k^\sharp : F_{f,g} \to F_{fk,gk}$.

$$\begin{array}{ccccc}
\mathcal{F}_X(f^*(L),g^*(P)) & \xrightarrow{k^*} & \mathcal{F}_V(k^*(f^*(L)),k^*(g^*(P))) & \xrightarrow{c_{g,k}(P)_*(c_{f,k}(L)^{-1})^*} & \mathcal{F}_V((fk)^*(L),(gk)^*(P)) \\
\downarrow{\scriptstyle f^*(\varphi)^*g^*(\psi)_*} & & \downarrow{\scriptstyle k^*(f^*(\varphi))^*k^*(g^*(\psi))_*} & & \downarrow{\scriptstyle (fk)^*(\varphi)^*(gk)^*(\psi)_*} \\
\mathcal{F}_X(f^*(M),g^*(N)) & \xrightarrow{k^*} & \mathcal{F}_V(k^*(f^*(M)),k^*(g^*(N))) & \xrightarrow{c_{g,k}(N)_*(c_{f,k}(M)^{-1})^*} & \mathcal{F}_V((fk)^*(M),(gk)^*(N))
\end{array}$$

Proposition 8.1.13 *Let $f : X \to Y$, $g : X \to Z$, $h : X \to W$, $k : V \to X$ be morphisms of $\mathcal{E}$.*

(1) Let L, M, N be objects of $\mathcal{F}_Y$, $\mathcal{F}_Z$, $\mathcal{F}_W$, respectively. For morphisms $\zeta : f^*(L) \to g^*(M)$ and $\xi : g^*(M) \to h^*(N)$ of $\mathcal{F}_X$, we have $k^\sharp_{L,N}(\xi\zeta) = k^\sharp_{M,N}(\xi)k^\sharp_{L,M}(\zeta)$.

(2) For objects M and N of $\mathcal{F}_Y$, a composition

$$\mathcal{F}_Y(M,N) \xrightarrow{f^*} \mathcal{F}_X(f^*(M),f^*(N)) \xrightarrow{k^\sharp_{M,N}} \mathcal{F}_V((fk)^*(M),(fk)^*(N))$$

coincides with $(fk)^* : \mathcal{F}_Y(M,N) \to \mathcal{F}_V((fk)^*(M),(fk)^*(N))$. In particular, $k^\sharp_{M,M} : \mathcal{F}_X(f^*(M),f^*(M)) \to \mathcal{F}_V((fk)^*(M),(fk)^*(M))$ maps the identity morphism of $f^*(M)$ to the identity morphism of $(fk)^*(M)$.

Proof. (1) The assertion follows from

$$\begin{aligned}
k^\sharp_{M,N}(\xi)k^\sharp_{L,M}(\zeta) &= c_{h,k}(N)k^*(\xi)c_{g,k}(M)^{-1}c_{g,k}(M)k^*(\zeta)c_{f,k}(L)^{-1} = c_{h,k}(N)f^*(\xi)f^*(\zeta)c_{f,k}(L)^{-1} \\
&= c_{h,k}(N)f^*(\xi\zeta)c_{f,k}(L)^{-1} = k^\sharp_{L,N}(\xi\zeta).
\end{aligned}$$

(2) The assertion follows from the definition of $k^\sharp$ and (8.1.10). $\square$

Proposition 8.1.14 *For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$ and $j : W \to V$ of $\mathcal{E}$, $(kj)^\sharp = j^\sharp k^\sharp$.*

Proof. For $M \in \operatorname{Ob}\mathcal{F}_Y$, $N \in \operatorname{Ob}\mathcal{F}_Z$ and $\xi \in \mathcal{F}_X(f^*(M),g^*(N))$, it follows from (8.1.10) and (8.1.12) that

$$\begin{aligned}
j^\sharp_{M,N}k^\sharp_{M,N}(\xi) &= c_{gk,j}(N)j^*(c_{g,k}(N)k^*(\xi)c_{f,k}(M)^{-1})c_{fk,j}(M)^{-1} \\
&= c_{gk,j}(N)j^*(c_{g,k}(N))j^*(k^*(\xi))j^*(c_{f,k}(M)^{-1})c_{fk,j}(M)^{-1} \\
&= c_{gk,j}(N)j^*(c_{g,k}(N))c_{k,j}(g^*(N))^{-1}(kj)^*(\xi)c_{k,j}(f^*(M))j^*(c_{f,k}(M)^{-1})c_{fk,j}(M)^{-1} \\
&= c_{gk,j}(N)j^*(c_{g,k}(N))c_{k,j}(g^*(N))^{-1}(kj)^*(\xi)(c_{fk,j}(M)j^*(c_{f,k}(M))c_{k,j}(f^*(M))^{-1})^{-1} \\
&= c_{g,kj}(N)(kj)^*(\xi)c_{f,kj}(M)^{-1} = (kj)^\sharp_{M,N}(\xi).
\end{aligned}$$

Hence we have $j^\sharp_{M,N}k^\sharp_{M,N} = (kj)^\sharp_{M,N}$ for any $M,N \in \operatorname{Ob}\mathcal{F}_Y$. $\square$

For a cloven fibered category $p : \mathcal{F} \to \mathcal{E}$, we define a category $\widetilde{\mathcal{F}}$ as follows. Put

$$\operatorname{Ob}\widetilde{\mathcal{F}} = \{(X,M) \mid X \in \operatorname{Ob}\mathcal{E},\ M \in \operatorname{Ob}\mathcal{F}_X\}.$$

For $(X,M),(Y,N) \in \operatorname{Ob}\widetilde{\mathcal{F}}$, we put

$$\widetilde{\mathcal{F}}((X,M),(Y,N)) = \{(f,\boldsymbol{\varphi}) \mid f \in \mathcal{E}(X,Y),\ \boldsymbol{\varphi} \in \mathcal{F}_X(M,f^*(N))\}.$$

For $(f,\boldsymbol{\varphi}) \in \widetilde{\mathcal{F}}((X,M),(Y,N))$ and $(g,\boldsymbol{\psi}) \in \widetilde{\mathcal{F}}((Y,N),(Z,L))$, define the composition of $(f,\boldsymbol{\varphi})$ and $(g,\boldsymbol{\psi})$ by

$$(g,\boldsymbol{\psi})(f,\boldsymbol{\varphi}) = (gf, c_{g,f}(L)f^*(\boldsymbol{\psi})\boldsymbol{\varphi}).$$

The identity morphism of (X,M) is defined by $id_{(X,M)} = (id_X, \alpha_{id_X}(M)^{-1})$. For $(f,\boldsymbol{\varphi}) \in \widetilde{\mathcal{F}}((X,M),(Y,N))$, $(g,\boldsymbol{\psi}) \in \widetilde{\mathcal{F}}((Y,N),(Z,L))$ and $(h,\boldsymbol{\xi}) \in \widetilde{\mathcal{F}}((Z,L),(W,T))$, it can be verified from (8.1.12) that

$$\begin{aligned}
(f,\boldsymbol{\varphi})(id_X,\alpha_{id_X}(M)^{-1}) &= (f\,id_X, c_{f,id_X}(N)id_X^*(\boldsymbol{\varphi})\alpha_{id_X}(M)^{-1}) = (f, c_{f,id_X}(N)\alpha_{id_X}(f^*(N))^{-1}\boldsymbol{\varphi}) = (f,\boldsymbol{\varphi}) \\
(id_Y,\alpha_{id_Y}(N)^{-1})(f,\boldsymbol{\varphi}) &= (id_Y f, c_{id_Y,f}(N)f^*(\alpha_{id_Y}(N)^{-1})\boldsymbol{\varphi}) = (f,\boldsymbol{\varphi}) \\
(h,\boldsymbol{\xi})((g,\boldsymbol{\psi})(f,\boldsymbol{\varphi})) &= (h,\boldsymbol{\xi})(gf, c_{g,f}(L)f^*(\boldsymbol{\psi})\boldsymbol{\varphi}) = (hgf, c_{h,gf}(T)(gf)^*(\boldsymbol{\xi})c_{g,f}(L)f^*(\boldsymbol{\psi})\boldsymbol{\varphi}) \\
&= (hgf, c_{hg,f}(T)f^*(c_{h,g}(T))f^*(g^*(\boldsymbol{\xi}))f^*(\boldsymbol{\psi})\boldsymbol{\varphi}) = (hgf, c_{hg,f}(T)f^*(c_{h,g}(T)g^*(\boldsymbol{\xi})\boldsymbol{\psi})\boldsymbol{\varphi}) \\
&= (hg, c_{h,g}(T)g^*(\boldsymbol{\xi})\boldsymbol{\psi})(f,\boldsymbol{\varphi}) = ((h,\boldsymbol{\xi})(g,\boldsymbol{\psi}))(f,\boldsymbol{\varphi})
\end{aligned}$$

$$\begin{array}{ccccc}
f^*(g^*(L)) & \xrightarrow{\quad c_{g,f}(L) \quad} & (gf)^*(L) & = & (gf)^*(L) \\
\Big\downarrow{\scriptstyle f^*(g^*(\xi))} & & & & \Big\downarrow{\scriptstyle (gf)^*(\boldsymbol{\xi})} \\
f^*(g^*(h^*(T))) = (f^*g^*)(h^*(T)) & \xrightarrow{\quad c_{g,f}(h^*(T)) \quad} & & & (gf)^*(h^*(T)) \\
\Big\downarrow{\scriptstyle f^*(c_{h,g}(T))} & & & & \Big\downarrow{\scriptstyle c_{h,gf}(T)} \\
f^*((hg)^*(T)) & \xrightarrow{\quad c_{hg,f}(T) \quad} & ((hg)f)^*(T) & = & (h(gf))^*(T)
\end{array}$$

Therefore $\widetilde{\mathcal{F}}$ is a category. We define a functors $\tilde{p} : \widetilde{\mathcal{F}} \to \mathcal{E}$ and $\Phi : \widetilde{\mathcal{F}} \to \mathcal{F}$ by $\tilde{p}(X, M) = X$, $\tilde{p}(f, \boldsymbol{\varphi}) = f$ and $\Phi(X, M) = M$, $\Phi(f, \boldsymbol{\varphi}) = \alpha_f(N)\boldsymbol{\varphi}$ for $(X, M) \in \operatorname{Ob}\widetilde{\mathcal{F}}$ and $(f, \boldsymbol{\varphi}) \in \widetilde{\mathcal{F}}((X, M), (Y, N))$. It is clear that $\tilde{p}$ is a functor and that $p\Phi = \tilde{p}$. Since

$$\begin{aligned}
\Phi(id_X, \alpha_{id_X}(M)^{-1}) &= \alpha_{id_X}(M)\alpha_{id_X}(M)^{-1} = id_M \\
\Phi((g, \boldsymbol{\psi})(f, \boldsymbol{\varphi})) &= \Phi(gf, c_{g,f}(L)f^*(\boldsymbol{\psi})\boldsymbol{\varphi}) = \alpha_{gf}(L)c_{g,f}(L)f^*(\boldsymbol{\psi})\boldsymbol{\varphi} = \alpha_g(L)\alpha_f(g^*(L))f^*(\boldsymbol{\psi})\boldsymbol{\varphi} \\
&= \alpha_g(L)\boldsymbol{\psi}\alpha_f(N)\boldsymbol{\varphi} = \Phi(g, \boldsymbol{\psi})\Phi(f, \boldsymbol{\varphi}),
\end{aligned}$$

Φ is also a functor.

Proposition 8.1.15 *Φ is an isomorphism of categories.*

Proof. Define a functor $\Phi^{-1} : \mathcal{F} \to \widetilde{\mathcal{F}}$ by $\Phi^{-1}(M) = (p(M), M)$ and $\Phi^{-1}(\varphi) = (p(\varphi), \bar{\varphi})$ for $M \in \operatorname{Ob}\mathcal{F}$ and $\varphi \in \mathcal{F}(M, N)$, where $\bar{\varphi} \in \mathcal{F}_{p(M)}(M, p(\varphi)^*(N))$ is unique morphism that is mapped to φ by the bijection $\alpha_{p(\varphi)}(N)_* : \mathcal{F}_{p(M)}(M, p(\varphi)^*(N)) \to \mathcal{F}_{p(\varphi)}(M, N)$. It is clear that Φ^{-1} is the inverse of Φ. $\square$

We give several examples below, some of which will be referred in the later sections.

Example 8.1.16 *Let $\mathcal{E}$ be a category and X an object of $\mathcal{E}$. Define a functor $\Sigma_X : \mathcal{E}/X \to \mathcal{E}$ by $\Sigma_X(Y \xrightarrow{\pi} X) = Y$, $\Sigma_X(f : \pi \to \rho) = (f : Y \to Z)$ for $Y \xrightarrow{\pi} X$ and $Z \xrightarrow{\rho} X$. Obviously, Σ_X is faithful and it is easy to verify that every morphism in $\mathcal{E}/X$ is cartesian. For an object $Y \xrightarrow{\pi} X$ and a morphism $f : Z \to Y$, f gives a cartesian morphism $(Z \xrightarrow{\pi f} X) \to (Y \xrightarrow{\pi} X)$ in $\mathcal{E}/X$. It follows that $\Sigma_X : \mathcal{E}/X \to \mathcal{E}$ is a fibered category.*

Example 8.1.17 *([5], p.182, a)) Let Δ^1 be a category given by $\operatorname{Ob}\Delta^1 = \{0, 1\}$ and $\operatorname{Mor}\Delta^1 = \{id_0, id_1, 0 \to 1\}$. For a category $\mathcal{E}$, we set $\mathcal{E}^{(2)} = \operatorname{Funct}(\Delta^1, \mathcal{E})$. Then, an object of $\mathcal{E}^{(2)}$ is identified with a morphism $(R \xrightarrow{\eta} A)$ in $\mathcal{E}$ and a morphism from $(R \xrightarrow{\eta} A)$ to $(S \xrightarrow{\iota} B)$ in $\mathcal{E}^{(2)}$ is identified with a pair (f, φ) of morphisms $f : R \to S$ and $\varphi : A \to B$ in $\mathcal{E}$ satisfying $\iota f = \varphi\eta$.*

(1) Let $p : \mathcal{E}^{(2)} \to \mathcal{E}$ be the evaluation functor E_0 at 0. For a morphism $f : R \to S$ in $\mathcal{E}$, consider the functor $f^* : \mathcal{E}^{(2)}_S \to \mathcal{E}^{(2)}_R$ given by $f^*(S \xrightarrow{\eta} B) = (R \xrightarrow{\eta f} B)$ and $f^*(id_S, \varphi) = (id_R, \varphi)$. We define a morphism $\alpha_f(S \xrightarrow{\eta} B) : f^*(S \xrightarrow{\eta} B) \to (S \xrightarrow{\eta} B)$ to be (f, id_B). Then, for $(R \xrightarrow{\iota} A) \in \operatorname{Ob}\mathcal{E}^{(2)}_R$, the map $\mathcal{E}^{(2)}_R((R \xrightarrow{\iota} A), f^*(S \xrightarrow{\eta} B)) \to \mathcal{E}^{(2)}_f((R \xrightarrow{\iota} A), (S \xrightarrow{\eta} B))$ given by $(id_R, \varphi) \mapsto \alpha_f(S \xrightarrow{\eta} B)(id_R, \varphi) = (f, \varphi)$ is bijective. Hence $\alpha_f(S \xrightarrow{\eta} B)$ is cartesian. Let $g : Q \to R$ be a morphism in $\mathcal{E}$. Then,

$$\alpha_f(S \xrightarrow{\eta} B)\alpha_g(f^*(S \xrightarrow{\eta} B)) = (f, id_B)(g, id_B) = (fg, id_B) = \alpha_{fg}(S \xrightarrow{\eta} B),$$

hence $c_{f,g}(S \xrightarrow{\eta} B)$ is the identity morphism of $g^*f^*(S \xrightarrow{\eta} B) = (Q \xrightarrow{\eta fg} B) = (fg)^*(S \xrightarrow{\eta} B)$. Thus $p : \mathcal{E}^{(2)} \to \mathcal{E}$ is a fibered category.

(2) Suppose that $\mathcal{E}$ has finite limits. Let $p : \mathcal{E}^{(2)} \to \mathcal{E}$ be the evaluation functor E_1 at 1. For $(f : X \to Y) \in \operatorname{Mor}\mathcal{E}$ and $(N \xrightarrow{\pi} Y) \in \operatorname{Ob}\mathcal{E}^{(2)}_Y$, consider the following cartesian square.

$$\begin{array}{ccc}
N \times_Y X & \xrightarrow{f_\pi} & N \\
\Big\downarrow{\scriptstyle \pi_f} & & \Big\downarrow{\scriptstyle \pi} \\
X & \xrightarrow{f} & Y
\end{array}$$

Then, $(f, f_\pi) : (N \times_Y X \xrightarrow{\pi_f} X) \to (N \xrightarrow{\pi} Y)$ induces a bijection

$$\mathcal{E}^{(2)}_X((M \xrightarrow{\rho} X), (N \times_Y X \xrightarrow{\pi_f} X)) \to \mathcal{E}^{(2)}_f((M \xrightarrow{\rho} X), (N \xrightarrow{\pi} Y)).$$

Hence (f, f_π) is a cartesian morphism and we have a functor $f^* : \mathcal{E}^{(2)}_Y \to \mathcal{E}^{(2)}_X$ which is given by $f^*(N \xrightarrow{\pi} Y) = (N \times_Y X \xrightarrow{\pi_f} X)$ and $f^*(id_Y, \varphi) = (id_X, \varphi \times_Y id_X)$, where $(id_Y, \varphi) : (N \xrightarrow{\pi} Y) \to (N' \xrightarrow{\pi'} Y)$ is a morphism of $\mathcal{E}^{(2)}_Y$ and $\varphi \times_Y id_X : N \times_Y X \to N' \times_Y X$ is the unique morphism that satisfies $\pi'_f(\varphi \times_Y id_X) = \pi_f$ and $f_\pi(\varphi \times_Y id_X) = f_{\pi'}\varphi$. For morphisms $f : X \to Y$, $g : Z \to X$ in $\mathcal{E}$ and an object $N \xrightarrow{\pi} Y$ of $\mathcal{E}^{(2)}$,

$$c_{f,g}(N \xrightarrow{\pi} Y) : (fg)^*(N \xrightarrow{\pi} Y) \to g^* f^*(N \xrightarrow{\pi} Y)$$

is the isomorphism induced by $(id_N \times_Y g, pr_2) : N \times_Y Z \to (N \times_Y X) \times_X Z$. Hence $p : \mathcal{E}^{(2)} \to \mathcal{E}$ is a fibered category.

Remark 8.1.18 *Let $p : \mathcal{E}^{(2)} \to \mathcal{E}$ be the fibered category given in (2) of (8.1.17) and X an object of $\mathcal{E}$. For objects $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ and $\boldsymbol{F} = (F \xrightarrow{\rho} X)$ of $\mathcal{E}^{(2)}_X$, let $E \xleftarrow{\rho_\pi} E \times_X F \xrightarrow{\pi_\rho} F$ be a limit of a diagram $E \xrightarrow{\pi} X \xleftarrow{\rho} F$. Put $\lambda = \pi\rho_\pi : E \times_X F \to X$ and $\boldsymbol{E} \times \boldsymbol{F} = (E \times_X F \xrightarrow{\lambda} X)$ and define $\mathrm{pr}_{\boldsymbol{E}} : \boldsymbol{E} \times \boldsymbol{F} \to \boldsymbol{E}$ and $\mathrm{pr}_{\boldsymbol{F}} : \boldsymbol{E} \times \boldsymbol{F} \to \boldsymbol{F}$ by $\mathrm{pr}_{\boldsymbol{E}} = \langle \rho_\pi, id_X \rangle$ and $\mathrm{pr}_{\boldsymbol{F}} = \langle \pi_\rho, id_X \rangle$, respectively. Then, $\boldsymbol{E} \xleftarrow{\mathrm{pr}_{\boldsymbol{E}}} \boldsymbol{E} \times \boldsymbol{F} \xrightarrow{\mathrm{pr}_{\boldsymbol{F}}} \boldsymbol{F}$ is a product of $\boldsymbol{E}$ and $\boldsymbol{F}$. For morphisms $\boldsymbol{f} = \langle f, id_X \rangle, \boldsymbol{g} = \langle g, id_X \rangle : \boldsymbol{E} \to \boldsymbol{F}$ of $\mathcal{E}^{(2)}_X$, let $e : G \to E$ be an equalizer of $f, g : E \to F$ in $\mathcal{E}$. Put $\boldsymbol{G} = (G \xrightarrow{\pi e} X)$ and $\boldsymbol{e} = \langle e, id_X \rangle : \boldsymbol{G} \to \boldsymbol{E}$. Then, $\boldsymbol{e}$ is an equalizer of $\boldsymbol{f}$ and $\boldsymbol{g}$. Hence $\mathcal{E}^{(2)}_X$ has finite limits.*

Proposition 8.1.19 *Let $\mathcal{E}$ be a category with finite limits and a terminal object 1. Let $p : \mathcal{E}^{(2)} \to \mathcal{E}$ be the fibered category given in (2) of (8.1.17). For objects X and Z of $\mathcal{E}$, define a functor $F_{X,Z} : \mathcal{E}^{(2)op}_1 \to \mathcal{S}et$ by $F_{X,Z}(Y \xrightarrow{o_Y} 1) = \mathcal{E}^{(2)}_X(o^*_X(Y \xrightarrow{o_Y} 1), o^*_X(Z \xrightarrow{o_Z} 1))$ and $F_{X,Z}(f) = (f \times id_X)^*$. Then, $\mathcal{E}$ is cartesian closed if and only if $F_{X,Z}$ is representable for any $X, Z \in \operatorname{Ob}\mathcal{E}$.*

Proof. For $X, Y, Z \in \operatorname{Ob}\mathcal{E}$, let us denote by $q_{Y,X} : Y \times X \to X$, $q_{Z,X} : Z \times X \to X$ and $p_{Z,X} : Z \times X \to Z$ the projections. Since $o^*_X(Y \xrightarrow{o_Y} 1) = (Y \times X \xrightarrow{q_{Y,X}} X)$, we have

$$F_{X,Z}(Y \xrightarrow{o_Y} 1) = \mathcal{E}^{(2)}_X(o^*_X(Y \xrightarrow{o_Y} 1), o^*_X(Z \xrightarrow{o_Z} 1)) = \{f \in \mathcal{E}(Y \times X, Z \times X) \,|\, q_{Z,X} f = q_{Y,X}\}.$$

Define a map $\Phi : \mathcal{E}^{(2)}_X(o^*_X(Y \xrightarrow{o_Y} 1), o^*_X(Z \xrightarrow{o_Z} 1)) \to \mathcal{E}(Y \times X, Z)$ by $\Phi(f) = p_{Z,X} f$. It is clear that Φ is bijective and natural in Y.

If $F_{X,Z}$ is representable for any $X, Z \in \operatorname{Ob}\mathcal{E}$, there exist $(W \xrightarrow{o_W} 1) \in \operatorname{Ob}\mathcal{E}^{(2)}_1$ and a bijection

$$F_{X,Z}(Y \xrightarrow{o_Y} 1) = \mathcal{E}^{(2)}_X(o^*_X(Y \xrightarrow{o_Y} 1), o^*_X(Z \xrightarrow{o_Z} 1)) \to \mathcal{E}^{(2)}_1((Y \xrightarrow{o_Y} 1), (W \xrightarrow{o_W} 1))$$

which is natural in Y. Since $\mathcal{E}^{(2)}_1((Y \xrightarrow{o_Y} 1), (W \xrightarrow{o_W} 1))$ is identified with $\mathcal{E}(Y, W)$, we have a bijection $\mathcal{E}(Y \times X, Z) \to \mathcal{E}(Y, W)$ which is natural in Y. Conversely, assume that $\mathcal{E}$ is cartesian closed. For $X, Z \in \operatorname{Ob}\mathcal{E}$, since $\mathcal{E}^{(2)}_1((Y \xrightarrow{o_Y} 1), (Z^X \xrightarrow{o_{Z^X}} 1))$ is identified with $\mathcal{E}(Y, Z^X)$ and there is a bijection $\mathcal{E}(Y, Z^X) \to \mathcal{E}(Y \times X, Z)$ which is natural in Y, $F_{X,Z}$ is representable. $\square$

Example 8.1.20 *Let $\mathcal{S}ch$ be the category of schemes. We define a category $\mathcal{Q}mod$ as follows.* Ob *$\mathcal{Q}mod$ consists of pairs $(X, \mathcal{M})$ of a scheme X and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$. A morphism $(X, \mathcal{M}) \to (Y, \mathcal{N})$ in $\mathcal{Q}mod$ is a pair (f, φ) of morphisms $f : X \to Y$ in $\mathcal{S}ch$ and $\varphi : \mathcal{N} \to f_*\mathcal{M}$ in the category of $\mathcal{O}_Y$-modules. The composition of morphisms $(f, \varphi) : (X, \mathcal{M}) \to (Y, \mathcal{N})$ and $(g, \psi) : (Z, \mathcal{L}) \to (X, \mathcal{M})$ is defined to be $(fg, f_*(\psi)\varphi) : (Z, \mathcal{L}) \to (Y, \mathcal{N})$. Define a functor $p : \mathcal{Q}mod \to \mathcal{S}ch$ by $p(X, \mathcal{M}) = X$ and $p(f, \varphi) = f$. For a morphism $f : X \to Y$ of schemes and an $\mathcal{O}_Y$-module $\mathcal{N}$, we denote by $\eta_f(\mathcal{N}) : \mathcal{N} \to f_* f^* \mathcal{N}$ the unit of the adjunction of f^* and f_*. Then, $(f, \eta_f(\mathcal{N})) : (X, f^*\mathcal{N}) \to (Y, \mathcal{N})$ is a cartesian morphism. In fact, for $(f, \varphi) \in \mathcal{Q}mod_f((X, \mathcal{M}), (Y, \mathcal{N}))$, $\varphi^a : f^*\mathcal{N} \to \mathcal{M}$ denotes the adjoint of $\varphi : \mathcal{N} \to f_*\mathcal{M}$ then $(id_X, \varphi^a) : (X, \mathcal{M}) \to (X, f^*\mathcal{N})$ is the unique morphism in $\mathcal{Q}mod_X$ such that $(f, \varphi) = (f, \eta_f(\mathcal{N}))(id_X, \varphi^a)$. Thus we define $f^* : \mathcal{Q}mod_Y \to \mathcal{Q}mod_X$ by $f^*(Y, \mathcal{N}) = (X, f^*\mathcal{N})$ and $f^*(id_Y, \varphi) = (id_X, f^*\varphi)$. Moreover, $\alpha_f(Y, \mathcal{N}) : f^*(Y, \mathcal{N}) \to (Y, \mathcal{N})$ is given by $\alpha_f(Y, \mathcal{N}) = (f, \eta_f(\mathcal{N}))$. For morphisms $f : X \to Y$, $g : Z \to X$ in $\mathcal{S}ch$ and an object $(Y, \mathcal{N})$ of $\mathcal{Q}mod$, let $\tilde{c}_{f,g}(\mathcal{N}) : (fg)^*\mathcal{N} \to g^* f^* \mathcal{N}$ be the adjoint of composition*

$$\mathcal{N} \xrightarrow{\eta_f(\mathcal{N})} f_* f^* \mathcal{N} \xrightarrow{f_*(\eta_g(f^*\mathcal{N}))} f_* g_* g^* f^* \mathcal{N} = (fg)_* g^* f^* \mathcal{N}$$

Then, $\tilde{c}_{f,g}(\mathcal{N})$ is an isomorphism of $\mathcal{O}_Z$-modules and $c_{f,g}(Y, \mathcal{N}) = (id_Z, \tilde{c}_{f,g}(\mathcal{N})) : (fg)^*(Y, \mathcal{N}) \to g^* f^*(Y, \mathcal{N})$ is an isomorphism in $\mathcal{Q}mod_Z$. Hence $p : \mathcal{Q}mod \to \mathcal{S}ch$ is a fibered category.

Example 8.1.21 *Define a category $\mathcal{T}op\mathcal{M}od$ as follows.* $\mathrm{Ob}\,\mathcal{T}op\mathcal{M}od$ *consists of pairs (A, M) of a linearly topologized complete Hausdorff commutative ring A and a linearly topologized complete Hausdorff A-module M. A morphism $(A, M) \to (B, N)$ in $\mathcal{T}op\mathcal{M}od$ is a pair (f, φ) of morphisms $f : B \to A$ in $\mathcal{T}op\mathcal{A}lg^c$ and $\varphi : N \to f_*M$ in $\mathcal{T}op\mathcal{M}od^c(B)$, where f_*M is a topological B-module given by $f_*M = M$ as a topological abelian group and $bm = f(b)m$ for $b \in B$ and $m \in M$. The composition of morphisms $(f, \varphi) : (A, M) \to (B, N)$ and $(g, \psi) : (C, L) \to (A, M)$ is defined to be $(gf, \psi\varphi) : (C, L) \to (B, N)$. Define a functor $p : \mathcal{T}op\mathcal{M}od^c \to (\mathcal{T}op\mathcal{A}lg^c)^{op}$ by $p(A, M) = A$ and $p(f, \varphi) = f$. For a morphism $f : B \to A$ in $\mathcal{T}op\mathcal{A}lg^c$ and a B-module N, we denote by $\eta_f(N) : N \to f_*f^*N$ the unit of the adjunction of f^* and f_*. Then, $(f, \eta_f(N)) : (A, f^*N) \to (B, N)$ is a cartesian morphism. In fact, for $(f, \varphi) \in \mathcal{T}op\mathcal{M}od^c_f((A, M), (B, N))$, $\varphi^a : f^*N \to M$ denotes the adjoint of $\varphi : N \to f_*M$ then $(id_A, \varphi^a) : (A, M) \to (A, f^*N)$ is the unique morphism in $\mathcal{T}op\mathcal{M}od^c_A$ such that $(f, \varphi) = (f, \eta_f(N))(id_A, \varphi^a)$. For morphisms $f : A \to B$, $g : C \to A$ in $(\mathcal{T}op\mathcal{A}lg^c)^{op}$ and an object (B, N) of $\mathcal{T}op\mathcal{M}od^c$, let $\tilde{c}_{f,g}(N) : (fg)^*N \to g^*f^*N$ be the adjoint of composition*

$$N \xrightarrow{\eta_f(N)} f_*f^*N \xrightarrow{f_*(\eta_g(f^*N))} f_*g_*g^*f^*N = (fg)_*g^*f^*N.$$

Then, $\tilde{c}_{f,g}(N)$ is an isomorphism of C-modules and $c_{f,g}(B, N) = (id_C, \tilde{c}_{f,g}(N)) : (fg)^*(B, N) \to g^*f^*(B, N)$ is an isomorphism in $\mathcal{T}op\mathcal{M}od^c_C$. Hence $p : \mathcal{T}op\mathcal{M}od^c \to (\mathcal{T}op\mathcal{A}lg^c)^{op}$ is a fibered category.

Definition 8.1.22 *([5], p.152) Let $p : \mathcal{F} \to \mathcal{E}$ and $F : \mathcal{D} \to \mathcal{E}$ be functors. Define a subcategory $\mathcal{D} \times_{\mathcal{E}} \mathcal{F}$ of $\mathcal{D} \times \mathcal{F}$ as follows. An object (X, M) of $\mathcal{D} \times \mathcal{F}$ belongs to $\mathcal{D} \times_{\mathcal{E}} \mathcal{F}$ if and only if $F(X) = p(M)$. A morphism $(f, \varphi) : (X, M) \to (Y, N)$ in $\mathcal{D} \times \mathcal{F}$ belongs to $\mathcal{D} \times_{\mathcal{E}} \mathcal{F}$ if and only if $F(f) = p(\varphi)$. Define functors $p_F : \mathcal{D} \times_{\mathcal{E}} \mathcal{F} \to \mathcal{D}$ and $\widetilde{F} : \mathcal{D} \times_{\mathcal{E}} \mathcal{F} \to \mathcal{F}$ by $p_F(X, M) = X$, $p_F(f, \varphi) = f$ and $\widetilde{F}(X, M) = M$, $\widetilde{F}(f, \varphi) = \varphi$. We call $p_F : \mathcal{D} \times_{\mathcal{E}} \mathcal{F} \to \mathcal{D}$ the pull-back of p along F.*

Proposition 8.1.23 *([5], p.167 Proposition 6.6.) A morphism $(f, \alpha) : (X, M) \to (Y, N)$ in $\mathcal{D} \times_{\mathcal{E}} \mathcal{F}$ is cartesian if and only if $\alpha : M \to N$ is cartesian.*

Proof. Since $(\mathcal{D} \times_{\mathcal{E}} \mathcal{F})_X((X, M'), (X, M)) = \{(id_X, \varphi) |\, \alpha \in \mathcal{F}_{F(X)}(M', M)\}$ and $(\mathcal{D} \times_{\mathcal{E}} \mathcal{F})_f((X, M'), (Y, N)) = \{(f, \psi) |\, \psi \in \mathcal{F}_{F(f)}(M', N)\}$ for $(X, M') \in \mathrm{Ob}(\mathcal{D} \times_{\mathcal{E}} \mathcal{F})_X$, it is clear that the map $(\mathcal{D} \times_{\mathcal{E}} \mathcal{F})_X((X, M'), (X, M)) \to (\mathcal{D} \times_{\mathcal{E}} \mathcal{F})_f((X, M'), (Y, N))$ given by $(id_X, \varphi) \mapsto (f, \alpha)(id_X, \varphi) = (f, \alpha\varphi)$ is bijective if and only if α is cartesian. □

Proposition 8.1.24 *([5], p.168 Corollaire 6.9.) If $p : \mathcal{F} \to \mathcal{E}$ is a prefibered (resp. fibered) category, so is the pull-back $p_F : \mathcal{D} \times_{\mathcal{E}} \mathcal{F} \to \mathcal{D}$ of p along $F : \mathcal{D} \to \mathcal{E}$. Moreover, $\widetilde{F} : \mathcal{D} \times_{\mathcal{E}} \mathcal{F} \to \mathcal{F}$ maps cartesian morphisms to cartesian morphisms.*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{D}$ and (Y, N) an object of $(\mathcal{D} \times_{\mathcal{E}} \mathcal{F})_Y$. We put $f^*(Y, N) = (X, F(f)^*(N))$ and define $\alpha_f(Y, N) : f^*(Y, N) \to (Y, N)$ by $\alpha_f(Y, N) = (f, \alpha_{F(f)}(N))$. It follows from (8.1.23) that $\alpha_f(Y, N)$ is cartesian. Hence p_F is a prefibered category.

Suppose that p is a fibered category. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{D}$. For $(Z, L) \in \mathrm{Ob}\,(\mathcal{D} \times_{\mathcal{E}} \mathcal{F})_Z$, define $c_{g,f}(Z, L) : f^*g^*(Z, L) = (X, F(f)^*F(g)^*(L)) \to (X, F(gf)^*(L)) = (gf)^*(Z, L)$ by $c_{g,f}(Z, L) = (id_X, c_{F(g),F(f)}(L))$. Since p is a fibered category, $c_{F(g),F(f)}(L) : F(f)^*F(g)^*(L) \to F(gf)^*(L)$ is an isomorphism by (8.1.11). Hence $c_{g,f}(Z, L)$ is an isomorphism and p_F is a fibered category by (8.1.11).

By the definition of $\alpha_f(Y, N)$ above, we have $\widetilde{F}(\alpha_f(Y, N)) = \alpha_{F(f)}(N)$. Since every cartesian morphism in $\mathcal{D} \times_{\mathcal{E}} \mathcal{F}$ is a composition of an isomorphism and a cartesian morphism of the form $\alpha_f(Y, N)$, $\widetilde{F}$ maps cartesian morphisms to cartesian morphisms. □

We need to introduce the notion of "cartesian section" in order to define the notion of trivial representation.

Definition 8.1.25 *([5], p.164 Définition 5.5.) Let $p : \mathcal{F} \to \mathcal{E}$ be a functor. We call a functor $s : \mathcal{E} \to \mathcal{F}$ a cartesian section if $ps = id_{\mathcal{E}}$ and $s(f)$ is cartesian for any $f \in \mathrm{Mor}\,\mathcal{E}$. The subcategory of $\mathrm{Funct}(\mathcal{E}, \mathcal{F})$ consisting of cartesian sections and morphisms $\varphi : s \to s'$ satisfying $p(\varphi_X) = id_X$ for any $X \in \mathrm{Ob}\,\mathcal{E}$ is denoted by $\varprojlim(\mathcal{F}/\mathcal{E})$.*

Proposition 8.1.26 *([3], Lemme 5.7) If $\mathcal{E}$ has a terminal object 1, then the functor $e : \varprojlim(\mathcal{F}/\mathcal{E}) \to \mathcal{F}_1$ given by $e(s) = s(1)$ and $e(\varphi) = \varphi_1$ is fully faithful. Moreover, if $p : \mathcal{F} \to \mathcal{E}$ is a fibered category, e is an equivalence of categories.*

Proof. Let s and s' be cartesian sections of $p : \mathcal{F} \to \mathcal{E}$. For $X \in \operatorname{Ob}\mathcal{E}$, we denote by $o_X : X \to 1$ by the unique morphism. Take a morphism $\theta : s(1) \to s'(1)$. Since $s'(o_X) : s'(X) \to s'(1)$ is cartesian and $p(\theta s(o_X)) = p(\theta)p(s(o_X)) = id_1 o_X = o_X = p(s'(o_X))$, there is a unique morphism $\varphi_X : s(X) \to s'(X)$ such that $p(\varphi_X) = id_X$ and $\theta s(o_X) = s'(o_X)\varphi_X$. If $X = 1$, then $o_X = id_1$, hence we have $\varphi_1 = s'(id_1)\varphi_X\theta = \theta s(id_1) = \theta$. Since $s'(o_Y) : s'(Y) \to s'(1)$ is cartesian and $p(\varphi_Y s(f)) = f = p(s'(f)\varphi_X)$, it follows from $s'(o_Y)\varphi_Y s(f) = \theta s(o_Y)s(f) = \theta s(o_Y f) = \theta s(o_X) = s'(o_X)\varphi_X = s'(o_Y f)\varphi_X = s'(o_Y)s'(f)\varphi_X$, that $\varphi_Y s(f) = s'(f)\varphi_X$. Thus we have a morphism $\varphi : s \to s'$ in $\varprojlim(\mathcal{F}/\mathcal{E})$ such that $e(\varphi) = \theta$.

Let $\varphi, \varphi' : s \to s'$ be morphisms in $\varprojlim(\mathcal{F}/\mathcal{E})$ such that $\varphi_1 = \varphi'_1$. For any $X \in \operatorname{Ob}\mathcal{E}$, $s'(o_X)\varphi_X = \varphi_1 s(o_X) = \varphi'_1 s(o_X) = s'(o_X)\varphi'_X$. Since $s'(o_X)$ is cartesian and $p(\varphi_X) = p(\varphi'_X) = id_X$, we have $\varphi_X = \varphi'_X$.

Suppose that $p : \mathcal{F} \to \mathcal{E}$ is a fibered category. We give $p : \mathcal{F} \to \mathcal{E}$ a normalized cleavage. For $T \in \operatorname{Ob}\mathcal{F}_1$, define $s_T : \mathcal{E} \to \mathcal{F}$ as follows. Set $s_T(X) = o_X^*(T)$ for $X \in \operatorname{Ob}\mathcal{E}$. If $f : X \to Y$ is a morphism in $\mathcal{E}$, $s_T(f) : s_T(X) \to s_T(Y)$ is defined to be the composition

$$s_T(X) = o_X^*(T) = (o_Y f)^*(T) \xrightarrow{c_{o_Y,f}(T)^{-1}} f^* o_Y^*(T) \xrightarrow{\alpha_f(o_Y^*(T))} o_Y^*(T) = s_T(Y).$$

Since we take a normalized cleavage, $id_X^* o_X^*(T) = o_X^*(T) = s_T(X)$ and $c_{o_X,id_X}(T) = \alpha_{id_X}(T) = id_{s_T(X)}$. Hence we have $s_T(id_X) = id_{s_T(X)}$. For morphisms $f : X \to Y$, $g : Y \to Z$ in $\mathcal{E}$,

$$\begin{aligned} s_T(gf)c_{o_Z,gf}(T)c_{g,f}(o_Z^*(T)) &= \alpha_{gf}(o_Z^*(T))c_{g,f}(o_Z^*(T)) = \alpha_g(o_Z^*(T))\alpha_f(g^*o_Z^*(T)) = s_T(g)c_{o_Z,g}(T)\alpha_f(g^*o_Z^*(T)) \\ &= s_T(g)\alpha_f((o_Z g)^*(T))f^*(c_{o_Z,g}(T)) = s_T(g)\alpha_f(o_Y^*(T))f^*(c_{o_Z,g}(T)) \\ &= s_T(g)s_T(f)c_{o_Y,f}(T)f^*(c_{o_Z,g}(T)) = s_T(g)s_T(f)c_{o_Z g,f}(T)f^*(c_{o_Z,g}(T)) \\ &= s_T(g)s_T(f)c_{o_Z,gf}(T)c_{g,f}(o_Z^*(T)). \end{aligned}$$

Thus we have $s_T(gf) = s_T(g)s_T(f)$. It is clear from the definition that s_T is a cartesian section and $s_T(1) = id_1^*(T) = T$. □

Remark 8.1.27 *For a cartesian section $s : \mathcal{E} \to \mathcal{F}$ of a fibered category $p : \mathcal{F} \to \mathcal{E}$ and a morphism $f : X \to Y$ of $\mathcal{E}$ and , let us denote by $s_f : s(X) \to f^*(s(Y))$ the unique morphism of $\mathcal{F}_X$ satisfying $\alpha_f(s(Y))s_f = s(f)$. We note that if $s = s_T$ for $T \in \operatorname{Ob}\mathcal{F}_1$, $s_f = c_{o_Y,f}(T)^{-1}$ by the definition of $s_T(f)$ above. Since both $s(f)$ and $\alpha_f(s(Y))$ are cartesian morphisms, s_f is necessarily an isomorphism. Hence, for morphisms $f : X \to Y$ and $g : X \to Z$ of $\mathcal{E}$, we define $s_{f,g} : f^*(s(Y)) \to g^*(s(Z))$ by $s_{f,g} = s_g s_f^{-1}$.*

8.2 Bifibered category

We briefly review the notion of bifibered category following section 10 of [5].

Definition 8.2.1 *Let $p : \mathcal{F} \to \mathcal{E}$ be a functor and $\alpha : M \to N$ a morphism in $\mathcal{F}$. Set $X = p(M)$, $Y = p(N)$, $f = p(\alpha)$. We call α a cocartesian morphism if, for any $N' \in \operatorname{Ob}\mathcal{F}_Y$, the map $\mathcal{F}_X(N, N') \to \mathcal{F}_f(M, N')$ defined by $\varphi \mapsto \varphi\alpha$ is bijective.*

The following assertion is the dual of (8.1.2).

Proposition 8.2.2 *If $\alpha_i : M \to N_i$ $(i = 1, 2)$ are cocartesian morphisms in $\mathcal{F}$ such that $p(N_1) = p(N_2)$ and $p(\alpha_1) = p(\alpha_2)$, there is a unique morphism $\psi : N_1 \to N_2$ such that $\alpha_1 = \alpha_2\psi$ and $p(\psi) = id_{p(N_1)}$. Moreover, ψ is an isomorphism.*

Definition 8.2.3 *Let $f : X \to Y$ be a morphism in $\mathcal{E}$ and $M \in \operatorname{Ob}\mathcal{F}_X$. If there exists a cocartesian morphism $\alpha : M \to N$ such that $p(\alpha) = f$, N is called a direct image of M by f. We denote M by $f_*(N)$ and α by $\alpha^f(M) : M \to f_*(M)$. By (8.2.2), $f_*(N)$ is unique up to isomorphism.*

Proposition 8.2.4 *Let $\alpha : M \to N$, $\alpha' : M' \to N'$ be morphisms in $\mathcal{F}$ such that $p(M) = p(M')$, $p(N) = p(N')$, $p(\alpha) = p(\alpha')(= f)$ and $\lambda : M \to M'$ a morphism in $\mathcal{F}$ such that $p(\lambda) = id_{p(M)}$. If α' is cocartesian, there is a unique morphism $\mu : N \to N'$ such that $p(\mu) = id_{p(N)}$ and $\alpha'\mu = \lambda\alpha$.*

Corollary 8.2.5 *Let $f : X \to Y$ be a morphism in $\mathcal{E}$. If, for any $M \in \operatorname{Ob}\mathcal{F}_X$, there exists a cocartesian morphism $\alpha^f(M) : M \to f_*(M)$, $M \mapsto f_*(M)$ defines a functor $f_* : \mathcal{F}_X \to \mathcal{F}_Y$.*

Definition 8.2.6 *If the assumption of (8.2.5) is satisfied, we say that the functor of the direct image by f exists.*

Definition 8.2.7 *If a functor $p : \mathcal{F} \to \mathcal{E}$ sadisfies the following condition (i), p is called a precofibered category and if p satisfies both (i) and (ii), p is called a cofibered category or p is cofibrant.*

(i) For any morphism f in $\mathcal{E}$, the functor of the direct image by f exists.
(ii) The composition of cocartesian morphisms is cocartesian.

In other words, $p : \mathcal{F} \to \mathcal{E}$ is a precofibered (resp. cofibered) category if and only if $p : \mathcal{F}^{op} \to \mathcal{E}^{op}$ is a prefibered (resp. fibered) category.

Let $p : \mathcal{F} \to \mathcal{E}$ be a functor. A map $\kappa : \operatorname{Mor}\mathcal{E} \to \coprod_{X,Y \in \operatorname{Ob}\mathcal{E}} \operatorname{Funct}(\mathcal{F}_X, \mathcal{F}_Y)$ is called a cocleavage if $\kappa(f)$ is a direct image functor $f_* : \mathcal{F}_X \to \mathcal{F}_Y$ for $(f : X \to Y) \in \operatorname{Mor}\mathcal{E}$. A cocleavage κ is said to be normalized if $\kappa(id_X) = id_{\mathcal{F}_X}$ for any $X \in \operatorname{Ob}\mathcal{E}$. A category $\mathcal{F}$ over $\mathcal{E}$ is called a cloven precofibered category (resp. normalized cloven precofibered category) if a cocleavage (resp. normalized cocleavage) is given.

$p : \mathcal{F} \to \mathcal{E}$ has a cocleavage if and only if p is precofibered. If p is precofibered, p has a normalized cocleavage.

Let $f : X \to Y$, $g : Z \to X$ be morphisms in $\mathcal{E}$ and M an object of $\mathcal{F}_Z$. If $p : \mathcal{F} \to \mathcal{E}$ is a precofibered category, there is a unique morphism $c^{f,g}(M) : (fg)_*(M) \to f_*g_*(M)$ such that the following square commutes and $p(c_{f,g}(M)) = id_Z$.

$$\begin{tikzcd}
M \arrow[r, "\alpha^{fg}(M)"] \arrow[d, "\alpha^g(M)"] & (fg)_*(M) \arrow[d, "c^{f,g}(M)"] \\
g_*(M) \arrow[r, "\alpha^f(g_*(M))"] & f_*g_*(M)
\end{tikzcd}$$

The following is the dual of (8.1.9).

Proposition 8.2.8 *Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven precofibered category. Then, p is a cofibered category if and only if $c^{f,g}(M)$ is an isomorphism for any $Z \xrightarrow{g} X \xrightarrow{f} Y$ and $M \in \operatorname{Ob}\mathcal{F}_Z$.*

Proposition 8.2.9 *Let $p : \mathcal{F} \to \mathcal{E}$ be a functor and $f : X \to Y$ a morphism in $\mathcal{E}$.*

(1) Suppose that the functor of the inverse image by f exists. Then, the inverse image $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ by f has a left adjoint if and only if the functor of the direct image by f exists.

(2) Suppose that the functor of the direct image by f exists. Then, the direct image $f_* : \mathcal{F}_X \to \mathcal{F}_Y$ by f has a right adjoint if and only if the functor of the inverse image by f exists.

Proof. (1) Suppose that the functor of the inverse image by f exists and that it has a left adjoint $f_* : \mathcal{F}_X \to \mathcal{F}_Y$. We denote by $\eta : id_{\mathcal{F}_X} \to f^*f_*$ the unit of the adjunction $f_* \dashv f^*$. For $M \in \operatorname{Ob}\mathcal{F}_X$, set $\alpha^f(M) = \alpha_f(f_*(M))\eta_M : M \to f_*(M)$. By the assumption, the following composition is bijective for any $M \in \operatorname{Ob}\mathcal{F}_X$, $N \in \operatorname{Ob}\mathcal{F}_Y$.

$$\mathcal{F}_Y(f_*(M), N) \xrightarrow{f^*} \mathcal{F}_X(f^*f_*(M), f^*(N)) \xrightarrow{\eta_M^*} \mathcal{F}_X(M, f^*(N)) \xrightarrow{\alpha_f(N)_*} \mathcal{F}_f(M, N)$$

We note that, since $\alpha_f(N)f^*(\varphi) = \varphi\alpha_f(f_*(M))$ for $\varphi \in \mathcal{F}_Y(f_*(M), N)$, the above composition coincides with the map $\alpha^f(M)^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_f(M, N)$ induced by $\alpha^f(M)$. This shows that the functor of the direct image by f exists.

Conversely, assume that the functor of the direct image by f exists. For $M \in \operatorname{Ob}\mathcal{F}_X$, let us denote by $\alpha^f(M) : M \to f_*(M)$ a cocartesian morphism. Then, we have bijections $\alpha^f(M)^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_f(M, N)$ and $\alpha_f(M)_* : \mathcal{F}_X(M, f^*(N)) \to \mathcal{F}_f(M, N)$ given by $\psi \mapsto \psi\alpha^f(M)$ and $\varphi \mapsto \alpha_f(M)\varphi$, which are natural in $M \in \operatorname{Ob}\mathcal{F}_X$ and $N \in \operatorname{Ob}\mathcal{F}_Y$. Thus we have a natural bijection $\mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$.

(2) Suppose that the functor of the direct image by f exists and that it has a right adjoint $f^* : \mathcal{F}_Y \to \mathcal{F}_X$. We denote by $\varepsilon : f_*f^* \to id_{\mathcal{F}_Y}$ the counit of the adjunction $f_* \dashv f^*$. For $N \in \operatorname{Ob}\mathcal{F}_Y$, set $\alpha_f(N) = \varepsilon_N\alpha^f(f^*(N)) : f^*(N) \to N$. By the assumption, the following composition is bijective for any $M \in \operatorname{Ob}\mathcal{F}_X$, $N \in \operatorname{Ob}\mathcal{F}_Y$.

$$\mathcal{F}_X(M, f^*(N)) \xrightarrow{f_*} \mathcal{F}_Y(f_*(M), f_*f^*(N)) \xrightarrow{\varepsilon_{N*}} \mathcal{F}_Y(f_*(M), N) \xrightarrow{\alpha^f(M)^*} \mathcal{F}_f(M, N)$$

We note that, since $f_*(\varphi)\alpha^f(M) = \alpha^f(f^*(N))\varphi$ for $\varphi \in \mathcal{F}_X(M, f^*(N))$, the above composition coincides with the map $\alpha_f(N)_* : \mathcal{F}_X(M, f^*(N)) \to \mathcal{F}_f(M, N)$ induced by $\alpha_f(N)$. This shows that the functor of the inverse image by f exists.

Conversely, assume that the functor of the inverse image by f exists. For $N \in \mathrm{Ob}\,\mathcal{F}_Y$, let us denote by $\alpha_f(N) : f^*(N) \to N$ a cartesian morphism. Then, we have bijections $\alpha_f(N)_* : \mathcal{F}_X(M, f^*(N)) \to \mathcal{F}_f(M, N)$ and $\alpha^f(M)^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_f(M, N)$ given by $\varphi \mapsto \alpha_f(N)\varphi$ and $\psi \mapsto \psi\alpha^f(M)\varphi$, which are natural in $M \in \mathrm{Ob}\,\mathcal{F}_X$ and $N \in \mathrm{Ob}\,\mathcal{F}_Y$. Thus we have a natural bijection $\mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$. □

Remark 8.2.10 *Let $p : \mathcal{F} \to \mathcal{E}$ be a functor and $f : X \to Y$ a morphism in $\mathcal{E}$ such that the functors of the inverse and direct images by f exist. For $M \in \mathrm{Ob}\,\mathcal{F}_X$ and $N \in \mathcal{F}_Y$, since there exist a cartesian morphism $\alpha_f(N) : f^*(N) \to N$ and a cocartesian morphism $\alpha^f(M) : M \to f_*(M)$, there is a bijection $ad_f(M, N) : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$ which satisfies $\alpha_f(N)ad_f(M, N)(\varphi) = \varphi\alpha^f(M)$ for any $\varphi \in \mathcal{F}_Y(f_*(M), N)$. Hence the unit $\eta : id_{\mathcal{F}_X} \to f^*f_*$ of the adjunction $f_* \dashv f^*$ is the unique natural transformation satisfying $\alpha_f(f_*(M))\eta_M = \alpha^f(M)$ for any $M \in \mathrm{Ob}\,\mathcal{F}_X$. Dually, the counit $\varepsilon : f_*f^* \to id_{\mathcal{F}_Y}$ is the unique natural transformation satisfying $\varepsilon_N\alpha^f(f^*(N)) = \alpha_f(N)$ for any $N \in \mathrm{Ob}\,\mathcal{F}_Y$.*

Proposition 8.2.11 *([5], p.182 Proposition 10.1.) Let $p : \mathcal{E} \to \mathcal{F}$ be a prefibered and precofibered category. Then, it is a fibered category if and only if it is a cofibered category.*

Proof. For a morphism $f : X \to Y$ in $\mathcal{E}$, we denote by $\eta^f : id_{\mathcal{F}_X} \to f^*f_*$ the unit of the adjunction $f_* \dashv f^*$. Let $f : X \to Y$, $g : Z \to X$ be morphisms in $\mathcal{E}$. For $M \in \mathrm{Ob}\,\mathcal{F}_Z$ and $N \in \mathrm{Ob}\,\mathcal{F}_Y$, we claim that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}_X(f^*f_*g_*(M), f^*(N)) \arrow[d, "\eta^{f*}_{g_*(M)}"] & \mathcal{F}_Y(f_*g_*(M), N) \arrow[l, "f^*"'] \arrow[r, "c^{f,g}(M)^*"] & \mathcal{F}_Y((fg)_*(M), N) \arrow[d, "(fg)^*"] \\
\mathcal{F}_X(g_*(M), f^*(N)) \arrow[d, "g^*"] & & \mathcal{F}_Z((fg)^*(fg)_*(M), (fg)^*(N)) \arrow[d, "\eta^{fg*}_M"] \\
\mathcal{F}_Z(g^*g_*(M), g^*f^*(N)) \arrow[r, "\eta^{g*}_M"] & \mathcal{F}_Z(M, g^*f^*(N)) \arrow[r, "c_{f,g}(M)_*"] & \mathcal{F}_Z(M, (fg)^*(N))
\end{tikzcd}$$

Let $\psi : f_*g_*(M) \to N$ be a morphism in $\mathcal{F}_Y$. Then we have

$$\begin{aligned}
\alpha_{fg}(N)\eta^{fg*}_M(fg)^*c^{f,g}(M)^*(\psi) &= \alpha_{fg}(N)(fg)^*(\psi)(fg)^*(c^{f,g}(M))\eta^{fg}_M = \psi\alpha_{fg}(f_*g_*(M))(fg)^*(c^{f,g}(M))\eta^{fg}_M \\
&= \psi c^{f,g}(M)\alpha_{fg}((fg)_*(M))\eta^{fg}_M = \psi c^{f,g}(M)\alpha^{fg}(M) = \psi\alpha^f(g_*(M))\alpha^g(M) \\
&= \psi\alpha_f(f_*g_*(M))\eta^f_{g_*(M)}\alpha_g(g_*(M))\eta^g_M \\
&= \alpha_f(N)f^*(\psi)\alpha_g(f^*f_*g_*(M))g^*(\eta^f_{g_*(M)})\eta^g_M \\
&= \alpha_f(N)\alpha_g(f^*(N))g^*f^*(\psi)g^*(\eta^f_{g_*(M)})\eta^g_M \\
&= \alpha_{fg}(N)c_{f,g}(N)g^*f^*(\psi)g^*(\eta^f_{g_*(M)})\eta^g_M = \alpha_{fg}(N)c_{f,g}(N)_*\eta^{g*}_M g^*\eta^{f*}_{g_*(M)}(\psi).
\end{aligned}$$

Since $\alpha_{fg}(N) : (fg)^*(N) \to N$ is cartesian and both $\eta^{fg*}_M(fg)^*c^{f,g}(M)^*(\psi)$ and $c_{f,g}(N)_*\eta^{g*}_M g^*\eta^{f*}_{g_*(M)}(\psi)$ are morphisms in $\mathcal{F}_Y$, we see that the above diagram commutes. Note that the compositions $\eta^{f*}_M f^* : \mathcal{F}_Y(f_*(M), N) \to \mathcal{F}_X(M, f^*(N))$, $\eta^{g*}_M g^* : \mathcal{F}_X(g_*(M), N) \to \mathcal{F}_Z(M, g^*(N))$ and $\eta^{fg*}_M(fg)^* : \mathcal{F}_Y((fg)_*(M), N) \to \mathcal{F}_Z(M, (fg)^*(N))$ are bijective. Hence, by the commutativity of the above diagram, $c_{f,g}(N)_*$ is bijective if and only if $c^{f,g}(M)^*$ is so. Then the assertion follows from (8.1.9) and (8.2.8). □

Definition 8.2.12 *We call a functor $p : \mathcal{F} \to \mathcal{E}$ a bifibered category if it is a fibered and cofibered category.*

Proposition 8.2.13 *The fibered category $p : \mathcal{E}^{(2)} \to \mathcal{E}$ given in (2) of (8.1.17) is a bifibered category.*

Proof. For a morphism $f : X \to Y$ of $\mathcal{E}$, define a functor $f_* : \mathcal{E}^{(2)}_X \to \mathcal{E}^{(2)}_Y$ by $f_*(\boldsymbol{E}) = (E \xrightarrow{f\pi} Y)$ for $\boldsymbol{E} = (E \xrightarrow{\pi} X) \in \mathrm{Ob}\,\mathcal{E}^{(2)}_X$ and $f_*(\langle\varphi, id_X\rangle) = \langle\varphi, id_Y\rangle$ for a morphism $\langle\varphi, id_X\rangle : \boldsymbol{E} \to \boldsymbol{F}$ of $\mathcal{E}^{(2)}_X$.

For $\boldsymbol{F} = (F \xrightarrow{\rho} Y) \in \mathrm{Ob}\,\mathcal{E}^{(2)}_Y$, let $F \xleftarrow{f_\rho} F \times_Y X \xrightarrow{\rho_f} X$ be a limit of a diagram $F \xrightarrow{\rho} Y \xleftarrow{f} X$. Then, for an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathcal{E}^{(2)}$, we have

$$\mathcal{E}^{(2)}_Y(f_*(\boldsymbol{E}), \boldsymbol{F}) = \{\langle\varphi, id_Y\rangle \,|\, \varphi \in \mathcal{E}(E, F),\ \rho\varphi = f\pi\},\ \ \mathcal{E}^{(2)}_X(\boldsymbol{E}, f^*(\boldsymbol{F})) = \{\langle\psi, id_X\rangle \,|\, \psi \in \mathcal{E}(E, F \times_Y X),\ \rho_f\psi = \pi\}$$

and define a map $\Psi : \mathcal{E}^{(2)}_X(\boldsymbol{E}, f^*(\boldsymbol{F})) \to \mathcal{E}^{(2)}_Y(f_*(\boldsymbol{E}), \boldsymbol{F})$ by $\Psi(\langle\psi, id_X\rangle) = \langle f_\rho\psi, id_Y\rangle$. Since the inverse of Ψ is given by $\Psi^{-1}(\langle\varphi, id_Y\rangle) = \langle(\varphi, \pi), id_X\rangle$, Ψ is bijective and f_* is a left adjoint of f^*. □

Remark 8.2.14 *The counit $\varepsilon_f : f_*f^* \to id_{\mathcal{E}^{(2)}_Y}$ of the above adjunction is given by $(\varepsilon_f)_{\boldsymbol{F}} = \langle f_\rho, id_Y\rangle : f_*f^*(\boldsymbol{F}) = (F \times_Y X \xrightarrow{f\rho_f} Y) \to \boldsymbol{F}$ for an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}_Y$. The unit $\eta_f : id_{\mathcal{E}^{(2)}_X} \to f^*f_*$ is given as follows. For an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathcal{E}^{(2)}_X$, let $E \xleftarrow{f_{f\pi}} E \times_Y X \xrightarrow{(f\pi)_f} X$ be a limit of $E \xrightarrow{f\pi} Y \xleftarrow{f} X$. Then, $(\eta_f)_{\boldsymbol{E}} = \langle (id_E, \pi), id_X\rangle : \boldsymbol{E} \to (E \times_Y X \xrightarrow{\pi_f} X) = f^*f_*(\boldsymbol{E})$.*

$$\begin{tikzcd}
E \arrow[dr, "(id_E,\pi)"] \arrow[drr, bend left, "id_E"] \arrow[ddr, bend right, "\pi"'] & & \\
& E \times_Y X \arrow[r, "f_{f\pi}"] \arrow[d, "(f\pi)_f"] & E \arrow[d, "f\pi"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven fibered category. Suppose that morphisms $f, g : X \to Y$ and $h : Y \to Z$ of $\mathcal{E}$ satisfy $hf = hg$ and that functors $f^*, g^* : \mathcal{F}_Y \to \mathcal{F}_X$ and $h^* : \mathcal{F}_Z \to \mathcal{F}_Y$ have left adjoints $f_*, g_* : \mathcal{F}_X \to \mathcal{F}_Y$ and $h_* : \mathcal{F}_Y \to \mathcal{F}_Z$, respectively. We denote by $ad_f(M,N) : \mathcal{F}_Y(f_*(M),N) \to \mathcal{F}_X(M, f^*(N))$, $ad_g(M,N) : \mathcal{F}_Y(g_*(M),N) \to \mathcal{F}_X(M, g^*(N))$, $ad_h(N,L) : \mathcal{F}_Z(h_*(N),L) \to \mathcal{F}_Y(N, h^*(L))$ the natural bijections for $M \in \operatorname{Ob}\mathcal{F}_X$, $N \in \operatorname{Ob}\mathcal{F}_Y$, $L \in \operatorname{Ob}\mathcal{F}_Z$. Let $\Phi_{M,L}$ be the following composition.

$$\begin{aligned}
&\mathcal{F}_Z(h_*(f_*(M)),L) \xrightarrow{ad_h(f_*(M),L)} \mathcal{F}_Y(f_*(M),h^*(L)) \xrightarrow{ad_f(M,h^*(L))} \mathcal{F}_X(M, f^*(h^*(L))) \xrightarrow{c_{h,f}(L)_*} \\
&\qquad \mathcal{F}_X(M,(hf)^*(L)) = \mathcal{F}_X(M,(hg)^*(L)) \xrightarrow{c_{h,g}(L)^{-1}} \mathcal{F}_X(M, g^*(h^*(L))) \xrightarrow{ad_g(M,h^*(L))^{-1}} \\
&\qquad \mathcal{F}_Y(g_*(M),h^*(L)) \xrightarrow{ad_h(g_*(M),L)^{-1}} \mathcal{F}_Z(h_*(g_*(M)),L)
\end{aligned}$$

Then, $\Phi_{M,L}$ is a natural bijection. We put $\xi_M = \Phi_{M,h_*(f_*(M))}(id_{h_*(f_*(M))}) : h_*(g_*(M)) \to h_*(f_*(M))$. Then, ξ_M gives a natural equivalence $\xi : h_*g_* \to h_*f_*$. For $\varphi \in \mathcal{F}_Z(h_*(f_*(M)),L)$, the following diagram commutes by the naturality of $\Phi_{M,L}$.

$$\begin{tikzcd}
\mathcal{F}_Z(h_*(f_*(M)),h_*(f_*(M))) \arrow[r, "\varphi_*"] \arrow[d, "\Phi_{M,h_*(f_*(M))}"] & \mathcal{F}_Z(h_*(f_*(M)),L) \arrow[d, "\Phi_{M,L}"] \\
\mathcal{F}_Z(h_*(g_*(M)),h_*(f_*(M))) \arrow[r, "\varphi_*"] & \mathcal{F}_Z(h_*(g_*(M)),L)
\end{tikzcd}$$

Thus we have $\Phi_{M,L}(\varphi) = \varphi\xi_M = \xi_M^*(\varphi)$, in other words, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}_Z(h_*(f_*(M)),L) \arrow[r, "ad_h(f_*(M),L)"] \arrow[d, "\xi_M^*"] & \mathcal{F}_Y(f_*(M),h^*(L)) \arrow[r, "ad_f(M,h^*(L))"] & \mathcal{F}_X(M, f^*(h^*(L))) \arrow[d, "c_{h,f}(L)_*"] \\
\mathcal{F}_Z(h_*(g_*(M)),L) \arrow[d, "ad_h(g_*(M),L)"] & & \mathcal{F}_X(M,(hf)^*(L)) \arrow[d, equal] \\
\mathcal{F}_Y(g_*(M),h^*(L)) \arrow[r, "ad_g(M,h^*(L))"] & \mathcal{F}_X(M, g^*(h^*(L))) \arrow[r, "c_{h,g}(L)_*"] & \mathcal{F}_X(M,(hg)^*(L))
\end{tikzcd}$$

Proposition 8.2.15 *Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven bifibered category. Suppose that a pair of morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ of $\mathcal{E}$ has a coequalizer $h : Y \to Z$. Let $\varphi, \psi : M \to N$ be morphisms of $\mathcal{F}$ satisfying $p(\varphi) = f$ and $p(\psi) = g$. Let $\tilde{\varphi} : M \to f^*(N)$ and $\tilde{\psi} : M \to g^*(N)$ be unique morphisms of $\mathcal{F}_X$ that satisfy $\alpha_f(N)\tilde{\varphi} = \varphi$ and $\alpha_g(N)\tilde{\psi} = \psi$. We put ${}^t\tilde{\varphi} = ad_f(M,N)^{-1}(\tilde{\varphi}) : f_*(M) \to N$ and ${}^t\tilde{\psi} = ad_g(M,N)^{-1}(\tilde{\psi}) : g_*(M) \to N$. Suppose that there exists a coequalizer $\pi : h_*(N) \to L$ of morphisms $h_*({}^t\tilde{\varphi})\xi_M : h_*(g_*(M)) \to h_*(N)$ and $h_*({}^t\tilde{\psi}) : h_*(g_*(M)) \to h_*(N)$ of $\mathcal{F}_Z$. Then a composition $N \xrightarrow{ad_h(N,L)(\pi)} h^*(L) \xrightarrow{\alpha_h(L)} L$ is a coequalizer of $M \underset{\psi}{\overset{\varphi}{\rightrightarrows}} N$.*

Proof. Since $\pi h_*({}^t\tilde{\psi}) = \pi h_*({}^t\tilde{\varphi})\xi_M = \xi_M^*(\pi h_*({}^t\tilde{\varphi})) = \Phi_{M,L}(\pi h_*({}^t\tilde{\varphi}))$, we have the following equality.

$$c_{h,g}(L)ad_g(M,h^*(L))(ad_h(g_*(M),L)(\pi h_*({}^t\tilde{\psi}))) = c_{h,f}(L)ad_f(M,h^*(L))(ad_h(f_*(M),L)(\pi h_*({}^t\tilde{\varphi}))) \cdots (i)$$

We put $\pi^a = ad_h(N,L)(\pi) : N \to h^*(L)$. Then, by the naturality of ad_f, ad_g, ad_h we have

$$\text{(the left hand side of } (i)) = c_{h,g}(L)ad_g(M,h^*(L))(\pi^{a\,t}\tilde{\psi}) = c_{h,g}(L)g^*(\pi^a)ad_f(M,N)({}^t\tilde{\psi}) = c_{h,g}(L)g^*(\pi^a)\tilde{\psi}$$
$$\text{(the right hand side of } (i)) = c_{h,f}(L)ad_f(M,h^*(L))(\pi^{a\,t}\tilde{\varphi}) = c_{h,f}(L)f^*(\pi^a)ad_f(M,N)({}^t\tilde{\varphi}) = c_{h,f}(L)f^*(\pi^a)\tilde{\varphi}$$

and since the following diagrams commutes, it follows $\alpha_h(L)\pi^a\varphi = \alpha_h(L)\pi^a\psi$.

$$\begin{tikzcd}
(hg)^*(L) \arrow[d, "\alpha_{hg}(L)"] & g^*(h^*(L)) \arrow[l, "c_{h,g}(L)"'] \arrow[d, "\alpha_g(h^*(L))"] & g^*(N) \arrow[l, "g^*(\pi^a)"'] \arrow[d, "\alpha_g(N)"] & M \arrow[l, "\tilde{\psi}"'] \arrow[r, "\tilde{\varphi}"] \arrow[d, equal] & f^*(N) \arrow[r, "f^*(\pi^a)"] \arrow[d, "\alpha_f(N)"] & f^*(h^*(L)) \arrow[r, "c_{h,f}(L)"] \arrow[d, "\alpha_f(h^*(L))"] & (hf)^*(L) \arrow[d, "\alpha_{hf}(L)"] \\
L & h^*(L) \arrow[l, "\alpha_h(L)"'] & N \arrow[l, "\pi^a"'] & M \arrow[l, "\psi"'] \arrow[r, "\varphi"] & N \arrow[r, "\pi^a"] & h^*(L) \arrow[r, "\alpha_h(L)"] & L
\end{tikzcd}$$

Let $\rho : N \to P$ be a morphism of $\mathcal{F}$ which satisfies $\rho\varphi = \rho\psi$. Then $p(\rho)f = p(\rho)g$ and there exists unique morphism $k : Z \to p(P)$ that satisfies $kh = p(\rho)$. Let $\tilde{\rho} : N \to p(\rho)^*(P) = (kh)^*(P)$ the unique morphism of $\mathcal{F}_Y$ that satisfies $\alpha_{kh}(P)\tilde{\rho} = \rho$. Then, $\alpha_{kh}(P)\tilde{\rho}\alpha_f(N)\tilde{\varphi} = \alpha_{kh}(P)\tilde{\rho}\alpha_g(N)\tilde{\psi}$ and this implies the following.

$$\alpha_{khf}(P)c_{kh,f}(P)f^*(\tilde{\rho})\tilde{\varphi} = \alpha_{kh}(P)\alpha_f((kh)^*(P))f^*(\tilde{\rho})\tilde{\varphi} = \alpha_{kh}(P)\alpha_g((kh)^*(P))g^*(\tilde{\rho})\tilde{\psi} = \alpha_{khg}(P)c_{kh,g}(P)g^*(\tilde{\rho})\tilde{\psi}$$

Since $hf = hg$ and $\alpha_{khf}(P)$ is a cartesian morphism, we have $c_{kh,f}(P)f^*(\tilde{\rho})\tilde{\varphi} = c_{kh,g}(P)g^*(\tilde{\rho})\tilde{\psi}$. On the other hand, it follows from (8.1.12) that there are the following equalities.

$$\begin{aligned}
c_{h,f}(k^*(P))^{-1}c_{k,hf}(P)^{-1}c_{kh,f}(P)f^*(\tilde{\rho})\tilde{\varphi} &= (c_{k,hf}(P)c_{h,f}(k^*(P)))^{-1}c_{kh,f}(P)f^*(\tilde{\rho})\tilde{\varphi} \\
&= (c_{kh,f}(P)f^*(c_{k,h}(P)))^{-1}c_{kh,f}(P)f^*(\tilde{\rho})\tilde{\varphi} \\
&= f^*(c_{k,h}(P)^{-1})f^*(\tilde{\rho})\tilde{\varphi} = f^*(c_{k,h}(P)^{-1}\tilde{\rho})\tilde{\varphi} \\
c_{h,g}(k^*(P))^{-1}c_{k,hg}(P)^{-1}c_{kh,g}(P)g^*(\tilde{\rho})\tilde{\psi} &= (c_{k,hg}(P)c_{h,g}(k^*(P)))^{-1}c_{kh,g}(P)g^*(\tilde{\rho})\tilde{\psi} \\
&= (c_{kh,g}(P)g^*(c_{k,h}(P)))^{-1}c_{kh,g}(P)g^*(\tilde{\rho})\tilde{\psi} \\
&= g^*(c_{k,h}(P)^{-1})g^*(\tilde{\rho})\tilde{\psi} = g^*(c_{k,h}(P)^{-1}\tilde{\rho})\tilde{\psi}
\end{aligned}$$

Put $\check{\rho} = c_{k,h}(P)^{-1}\tilde{\rho} : N \to h^*(k^*(P))$ and ${}^t\check{\rho} = ad_h(N,k^*(P))^{-1}(\check{\rho}) : h_*(N) \to k^*(P)$. Then, the above equalities imply the following.

$$c_{h,f}(k^*(P))f^*(\check{\rho})\tilde{\varphi} = c_{h,g}(k^*(P))g^*(\check{\rho})\tilde{\psi} \cdots (ii)$$

Since the following diagrams commute by the naturality of ad_f and ad_g, we have

$$f^*(\check{\rho})\tilde{\varphi} = ad_f(M,h^*(k^*(P)))(\check{\rho}\,{}^t\tilde{\varphi}), \qquad g^*(\check{\rho})\tilde{\psi} = ad_g(M,h^*(k^*(P)))(\check{\rho}\,{}^t\tilde{\psi}) \cdots (iii).$$

$$\begin{tikzcd}
\mathcal{F}_Y(f_*(M),N) \arrow[r, "ad_f(M,N)"] \arrow[d, "\check{\rho}_*"] & \mathcal{F}_X(M,f^*(N)) \arrow[d, "f^*(\check{\rho})_*"] \\
\mathcal{F}_Y(f_*(M),h^*(k^*(P))) \arrow[r, "ad_f(M,h^*(k^*(P)))"] & \mathcal{F}_X(M,f^*(h^*(k^*(P))))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_Y(g_*(M),N) \arrow[r, "ad_g(M,N)"] \arrow[d, "\tilde{\rho}_*"] & \mathcal{F}_X(M,g^*(N)) \arrow[d, "g^*(\tilde{\rho})_*"] \\
\mathcal{F}_Y(g_*(M),(kh)^*(P)) \arrow[r, "ad_g(M,(kh)^*(P))"] \arrow[d, "c_{k,h}(P)_*^{-1}"] & \mathcal{F}_X(M,g^*((kh)^*(P))) \arrow[d, "g^*(c_{k,h}(P)^{-1})_*"] \\
\mathcal{F}_Y(g_*(M),h^*(k^*(P))) \arrow[r, "ad_g(M,h^*(k^*(P)))"] & \mathcal{F}_X(M,g^*(h^*(k^*(P))))
\end{tikzcd}$$

Moreover, the following diagrams commute by the naturality of ad_h, we have

$$\check{\rho}\,{}^t\tilde{\varphi} = ad_h(f_*(M),k^*(P))({}^t\check{\rho}h_*({}^t\tilde{\varphi})), \qquad \check{\rho}\,{}^t\tilde{\psi} = ad_h(g_*(M),k^*(P))({}^t\check{\rho}h_*({}^t\tilde{\psi})) \cdots (iv).$$

$$\begin{CD}
\mathcal{F}_Z(h_*(N), k^*(P)) @>{ad_h(N,k^*(P))}>> \mathcal{F}_Y(N, h^*(k^*(P))) \\
@V{h_*({}^t\tilde{\varphi})^*}VV @VV{{}^t\tilde{\varphi}^*}V \\
\mathcal{F}_Z(h_*(f_*(M)), k^*(P)) @>{ad_h(f_*(M),k^*(P))}>> \mathcal{F}_Y(f_*(M), h^*(k^*(P)))
\end{CD}$$

$$\begin{CD}
\mathcal{F}_Z(h_*(N), k^*(P)) @>{ad_h(N,k^*(P))}>> \mathcal{F}_Y(N, h^*(k^*(P))) \\
@V{h_*({}^t\tilde{\psi})^*}VV @VV{{}^t\tilde{\psi}^*}V \\
\mathcal{F}_Z(h_*(g_*(M)), k^*(P)) @>{ad_h(g_*(M),k^*(P))}>> \mathcal{F}_Y(g_*(M), h^*(k^*(P)))
\end{CD}$$

Since the following diagram commutes, it follows from (ii), (iii) and (iv) that ${}^t\check{\rho}h_*({}^t\tilde{\varphi})\xi_M = {}^t\check{\rho}h_*({}^t\tilde{\psi})$.

$$\begin{tikzcd}
\mathcal{F}_Z(h_*(f_*(M)), k^*(P)) \arrow[r, "ad_h(f_*(M){,}k^*(P))"] \arrow[d, "\xi_M^*"] & \mathcal{F}_Y(f_*(M), h^*(k^*(P))) \arrow[r, "ad_f(M{,}h^*(k^*(P)))"] & \mathcal{F}_X(M, f^*(h^*(k^*(P)))) \arrow[d, "c_{h,f}(k^*(P))_*"] \\
\mathcal{F}_Z(h_*(g_*(M)), k^*(P)) \arrow[d, "ad_h(g_*(M){,}k^*(P))"] & & \mathcal{F}_X(M, (hf)^*(k^*(P))) \arrow[d, equal] \\
\mathcal{F}_Y(g_*(M), h^*(k^*(P))) \arrow[r, "ad_g(M{,}h^*(k^*(P)))"] & \mathcal{F}_X(M, g^*(h^*(k^*(P)))) \arrow[r, "c_{h,g}(k^*(P))_*"] & \mathcal{F}_X(M, (hg)^*(k^*(P)))
\end{tikzcd}$$

Hence there exists unique morphism $\bar{\rho} : L \to k^*(P)$ of $\mathcal{F}_Z$ that satisfies $\bar{\rho}\pi = {}^t\check{\rho}$. By the naturality of ad_h, the following diagram commutes.

$$\begin{CD}
\mathcal{F}_Z(h_*(N), L) @>{ad_h(N,L)}>> \mathcal{F}_Y(N, h^*(L)) \\
@V{\bar{\rho}_*}VV @VV{h^*(\bar{\rho})_*}V \\
\mathcal{F}_Z(h_*(N), k^*(P)) @>{ad_h(N,k^*(P))}>> \mathcal{F}_Y(N, h^*(k^*(P)))
\end{CD}$$

Thus $h^*(\bar{\rho})\pi^a = ad_h(N, k^*(P))(\bar{\rho}\pi) = ad_h(N, k^*(P))({}^t\check{\rho}) = \check{\rho} = c_{k,h}(P)^{-1}\tilde{\rho}$, which implies $c_{k,h}(P)h^*(\bar{\rho})\pi^a = \tilde{\rho}$. Therefore we have $\alpha_k(P)\bar{\rho}\alpha_h(L)\pi^a = \alpha_k(P)\alpha_h(k^*(P))h^*(\bar{\rho})\pi^a = \alpha_{kh}(P)c_{k,h}(P)h^*(\bar{\rho})\pi^a = \alpha_{kh}(P)\tilde{\rho} = \rho$.

It remains to show that $\alpha_h(L)\pi^a : N \to L$ is an epimorphism in $\mathcal{F}$. Suppose that morphisms $\beta, \gamma : L \to Q$ of $\mathcal{F}$ satisfy $\beta\alpha_h(L)\pi^a = \gamma\alpha_h(L)\pi^a$. Then, we have $p(\beta)h = p(\gamma)h$ which implies $p(\beta) = p(\gamma)$ since h is an epimorphism. We put $q = p(\beta) = p(\gamma) : Z \to p(Q)$. Let $\tilde{\beta}, \tilde{\gamma} : L \to q^*(Q)$ be the unique morphisms of $\mathcal{F}_Z$ that satisfy $\alpha_q(Q)\tilde{\beta} = \beta$ and $\alpha_q(Q)\tilde{\gamma} = \gamma$, respectively. Then,

$$\begin{aligned}
\alpha_{qh}(Q)c_{q,h}(Q)h^*(\tilde{\beta})\pi^a &= \alpha_q(Q)\alpha_h(q^*(Q))h^*(\tilde{\beta})\pi^a = \alpha_q(Q)\tilde{\beta}\alpha_h(L)\pi^a = \alpha_q(Q)\tilde{\gamma}\alpha_h(L)\pi^a \\
&= \alpha_q(Q)\alpha_h(q^*(Q))h^*(\tilde{\gamma})\pi^a = \alpha_{qh}(Q)c_{q,h}(Q)h^*(\tilde{\gamma})\pi^a
\end{aligned}$$

and it follows $h^*(\tilde{\beta})\pi^a = h^*(\tilde{\gamma})\pi^a \in \mathcal{F}_Y(N, h^*(q^*(Q)))$. By the naturality of ad_h,

$$ad_h(N, q^*(Q))^{-1} : \mathcal{F}_Y(N, h^*(q^*(Q))) \to \mathcal{F}_Z(h_*(N), q^*(Q))$$

maps $h^*(\tilde{\beta})\pi^a$ and $h^*(\tilde{\gamma})\pi^a$ to $\tilde{\beta}\pi$ and $\tilde{\gamma}\pi$, respectively and we see $\tilde{\beta}\pi = \tilde{\gamma}\pi$. Since π is an epimorphism, it follows $\tilde{\beta} = \tilde{\gamma}$ which implies $\beta = \gamma$. □

8.3 2-categories and lax functors

First we give the definitions of 2-category and lax functor (See [1], [10], [11]).

Definition 8.3.1 *A 2-category $\mathfrak{C}$ is determined by the following data:*

(1) A set $\mathrm{Ob}\,\mathfrak{C}$ *called set of objects.*

(2) For each pair of objects (X, Y), a category $\mathfrak{C}(X, Y)$. An object of $\mathfrak{C}(X, Y)$ is called a 1-arrow and a morphism of $\mathfrak{C}(X, Y)$ is called a 2-arrow. The composition of 2-arrows $S \xrightarrow{f} T$, $T \xrightarrow{g} U$ in $\mathfrak{C}(X, Y)$ is denoted by gf.

(3) For each triple (X,Y,Z) of objects of $\mathfrak{C}$, a functor $\mu_{X,Y,Z} : \mathfrak{C}(X,Y)\times\mathfrak{C}(Y,Z) \to \mathfrak{C}(X,Z)$ called composition functor (Here $\mathfrak{C}(X,Y)\times\mathfrak{C}(Y,Z)$ is the product category of $\mathfrak{C}(X,Y)$ and $\mathfrak{C}(Y,Z)$.) such that the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{C}(X,Y)\times\mathfrak{C}(Y,Z)\times\mathfrak{C}(Z,W) & \xrightarrow{\mu_{X,Y,Z}\times 1} & \mathfrak{C}(X,Z)\times\mathfrak{C}(Z,W) \\ \big\downarrow{\scriptstyle 1\times\mu_{Y,Z,W}} & & \big\downarrow{\scriptstyle \mu_{X,Z,W}} \\ \mathfrak{C}(X,Y)\times\mathfrak{C}(Y,W) & \xrightarrow{\mu_{X,Y,W}} & \mathfrak{C}(X,W) \end{array}$$

We usually denote $\mu_{X,Y,Z}(S,T)$ by $T\circ S$ if $(S,T)\in \mathrm{Ob}\,\mathfrak{C}(X,Y)\times\mathfrak{C}(Y,Z)$ and denote $\mu_{X,Y,Z}(f,g)$ by $g{*}f$ if $(f,g)\in \mathrm{Mor}\,\mathfrak{C}(X,Y)\times\mathfrak{C}(Y,Z)$.

(4) For each object X of $\mathfrak{C}$, a 1-arrow 1_X in $\mathfrak{C}(X,X)$ called an identity 1-arrow of X satisfying $S\circ 1_X = S$ and $1_X\circ T = T$ for any $S\in\mathrm{Ob}\,\mathfrak{C}(X,Y)$ and $T\in\mathrm{Ob}\,\mathfrak{C}(Y,X)$.

Example 8.3.2 *(1) Let $\mathcal{C}$ be a category. For any pair of objects (X,Y) of $\mathcal{C}$, we regard the set $\mathcal{C}(X,Y)$ of morphisms from X to Y as a discrete category. The composition of morphisms $\mu_{X,Y,Z} : \mathcal{C}(X,Y)\times\mathcal{C}(Y,Z)\to\mathcal{C}(X,Z)$ gives a composition functor; that is, $id_f{*}id_g = id_{gf}$. Hence $\mathcal{C}$ can be regarded as a 2-category.*

*(2) Let us denote by **cat** the category of categories. For a pair $(\mathcal{C},\mathcal{D})$ of categories, $\boldsymbol{cat}(\mathcal{C},\mathcal{D})$ is the functor category $\mathrm{Funct}(\mathcal{C},\mathcal{D})$. Define the composition functor $\mu_{\mathcal{C},\mathcal{D},\mathcal{E}} : \boldsymbol{cat}(\mathcal{C},\mathcal{D})\times\boldsymbol{cat}(\mathcal{D},\mathcal{E})\to\boldsymbol{cat}(\mathcal{C},\mathcal{E})$ by $\mu_{\mathcal{C},\mathcal{D},\mathcal{E}}(F,G) = G\circ F$ (the composition of functors), $\mu_{\mathcal{C},\mathcal{D},\mathcal{E}}(f,g)_X = g_{F'(X)}G(f_X) = G'(f_X)g_{F(X)}$ for an object $X\in\mathrm{Ob}\,\mathcal{C}$ and natural transformations $f : F\to F'$, $g : G\to G'$. For a category $\mathcal{C}$, the identity functor of $\mathcal{C}$ is the identity 1-arrow of $\mathcal{C}$. Hence **cat** has a structure of 2-category.*

(3) Let us denote by $\boldsymbol{pfib}(\mathcal{E})$ the 2-category of cloven prefibered category over a category $\mathcal{E}$ defined as follows. Objects of $\boldsymbol{pfib}(\mathcal{E})$ are cloven prefibered category over $\mathcal{E}$. For cloven prefibered categories $p:\mathcal{F}\to\mathcal{E}$ and $q:\mathcal{D}\to\mathcal{E}$, a 1-arrow of $\boldsymbol{pfib}(\mathcal{E})(p,q)$ is a functor $F:\mathcal{F}\to\mathcal{D}$ satisfying $qF=p$. A 2-arrow $\varphi : F\to G$ in $\boldsymbol{pfib}(\mathcal{E})(p,q)$ is a natural transformation of functors such that, for each $M\in\mathrm{Ob}\,\mathcal{F}$, $q(\varphi_M) = id_{p(M)}$. Let $r:\mathcal{C}\to\mathcal{E}$ be another cloven prefibered category. The composition functor $\mu_{p,q,r} : \boldsymbol{pfib}(\mathcal{E})(p,q)\times\boldsymbol{pfib}(\mathcal{E})(q,r)\to\boldsymbol{pfib}(\mathcal{E})(p,r)$ is defined in similar way as the above example. That is, the composition of 1-arrows is just the composition of functors and composition of 2-arrows is given by $\mu_{p,q,r}(f,g)_M = g_{F'(M)}G(f_M) = G'(f_M)g_{F(M)}$ for an object $M\in\mathrm{Ob}\,\mathcal{F}$ and natural transformations $f:F\to F'$, $g:G\to G'$ of functors $F,F':\mathcal{F}\to\mathcal{D}$, $G,G':\mathcal{D}\to\mathcal{C}$.

(4) We define 2-categories $\boldsymbol{fib}(\mathcal{E})$, $\boldsymbol{pfib}^c(\mathcal{E})$, $\boldsymbol{fib}^c(\mathcal{E})$ as follows. $\boldsymbol{fib}(\mathcal{E})$ is a full subcategory of $\boldsymbol{pfib}(\mathcal{E})$ consisting of fibered categories. $\boldsymbol{pfib}^c(\mathcal{E})$ is a subcategory of $\boldsymbol{pfib}(\mathcal{E})$ having the same objects as those of $\boldsymbol{pfib}(\mathcal{E})$ and morphisms which maps catesian morphisms to cartesian morphisms. $\boldsymbol{fib}^c(\mathcal{E})$ is a full subcategory of $\boldsymbol{pfib}^c(\mathcal{E})$ consisting of fibered categories.

(5) Let $\mathcal{E}$ be a category with finite limits and $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$ internal categories in $\mathcal{E}$. Let $f,f' : \boldsymbol{C}\to\boldsymbol{D}$, $g,g' : \boldsymbol{D}\to\boldsymbol{E}$ be internal functors and $\varphi : f\to f'$, $\psi : g\to g'$ internal natural transformations. Then, $\tau g_1\varphi = \sigma\psi f'_0$ and there exists a morphism $(g_1\varphi,\psi f'_0) : C_0\to E_1\times_{E_0}E_1$ satisfying $\mathrm{pr}_1(g_1\varphi,\psi f'_0) = g_1\varphi$ and $\mathrm{pr}_2(g_1\varphi,\psi f'_0) = \psi f'_0$. We put $\varphi{*}\psi = \mu(g_1\varphi,\psi f'_0)$. It is a routine to check that $\varphi{*}\psi$ is an internal natural transformation from gf to $g'f'$. By using this composition of internal natural transformations, the category of internal category $\boldsymbol{cat}(\mathcal{E})$ has a structure of 2-category whose 2-arrows are internal natural transformations.

Definition 8.3.3 *Let $\mathfrak{D}$ and $\mathfrak{C}$ be 2-categories. A lax functor $(\Gamma,\gamma):\mathfrak{C}\to\mathfrak{D}$ consists of the following data:*

(1) A map $\Gamma : \mathrm{Ob}\,\mathfrak{C}\to\mathrm{Ob}\,\mathfrak{D}$.

(2) For each pair (X,Y) of objects of $\mathfrak{C}$, a functor $\Gamma_{X,Y} : \mathfrak{C}(X,Y)\to\mathfrak{D}(\Gamma(X),\Gamma(Y))$.

(3) For each object X of $\mathfrak{C}$, a 2-arrow $\gamma_X : 1_{\Gamma(X)}\to\Gamma_{X,X}(1_X)$.

(4) For each triple (X,Y,Z) of objects of $\mathfrak{C}$, there is a natural transformation

$$\gamma_{X,Y,Z} : \mu_{\Gamma(X),\Gamma(Y),\Gamma(Z)}(\Gamma_{X,Y}\times\Gamma_{Y,Z}) \longrightarrow \Gamma_{X,Z}\mu_{X,Y,Z}$$

(namely, there is a 2-arrow $(\gamma_{X,Y,Z})_{(S,T)} : \Gamma_{Y,Z}(T)\circ\Gamma_{X,Y}(S)\to\Gamma_{X,Z}(T\circ S)$ in $\mathfrak{D}$ for composable 1-arrows $S:X\to Y$, $T:Y\to Z$) making the following diagrams in $\mathfrak{D}(\Gamma(X),\Gamma(Y))$ and $\mathfrak{D}(\Gamma(X),\Gamma(W))$, respectively, commute for 1-arrows $S:X\to Y$, $T:Y\to Z$ and $U:Z\to W$ in $\mathfrak{C}$.

$$\begin{array}{ccccc} \Gamma_{X,Y}(S)\circ 1_{\Gamma(X)} & = & \Gamma_{X,Y}(S) & = & 1_{\Gamma(Y)}\circ\Gamma_{X,Y}(S) \\ \big\downarrow{\scriptstyle 1_{\Gamma(S)_{X,Y}}*\gamma_X} & & \big\downarrow{\scriptstyle 1_{\Gamma_{X,Y}(S)}} & & \big\downarrow{\scriptstyle \gamma_Y*1_{\Gamma_{X,Y}(S)}} \\ \Gamma_{X,Y}(S)\circ\Gamma_{X,X}(1_X) & \xrightarrow{(\gamma_{X,X,Y})_{(1_X,S)}} & \Gamma_{X,Y}(S) & \xleftarrow[(\gamma_{X,Y,Y})_{(S,1_Y)}]{} & \Gamma_{Y,Y}(1_Y)\circ\Gamma_{X,Y}(S) \end{array}$$

$$\begin{array}{ccc} \Gamma_{Z,W}(U)\circ\Gamma_{Y,Z}(T)\circ\Gamma_{X,Y}(S) & \xrightarrow{1_{\Gamma_{Z,W}(U)}*(\gamma_{X,Y,Z})_{(S,T)}} & \Gamma_{Z,W}(U)\circ\Gamma_{X,Z}(T\circ S) \\ \big\downarrow{\scriptstyle (\gamma_{Y,Z,W})_{(T,U)}*1_{\Gamma_{X,Y}(S)}} & & \big\downarrow{\scriptstyle (\gamma_{X,Z,W})_{(T\circ S,U)}} \\ \Gamma_{Y,W}(U\circ T)\circ\Gamma_{X,Y}(S) & \xrightarrow{(\gamma_{X,Y,W})_{(S,U\circ T)}} & \Gamma_{X,W}(U\circ T\circ S) \end{array}$$

A lax functor $(\Gamma,\gamma):\mathfrak{C}\to\mathfrak{D}$ is a functor if, for any $X,Y,Z\in\operatorname{Ob}\mathfrak{C}$, γ_X is the identity 2-arrow in $\mathfrak{D}$ and $\gamma_{X,Y,Z}$ is the identity natural transformation.

Definition 8.3.4 *Let $(\Gamma,\gamma):\mathfrak{C}\to\mathfrak{D}$ and $(\Delta,\delta):\mathfrak{D}\to\mathfrak{E}$ be lax functors. We define a lax functor $(\Pi,\pi):\mathfrak{C}\to\mathfrak{E}$ as follows. Put $\Pi(X)=\Delta(\Gamma(X))$ and $\Pi_{X,Y}=\Delta_{\Gamma(X),\Gamma(Y)}\Gamma_{X,Y}:\mathfrak{C}(X,Y)\to\mathfrak{D}(\Delta(\Gamma(X)),\Delta(\Gamma(Y)))$ for $X,Y\in\operatorname{Ob}\mathfrak{C}$. $\pi_X:1_{\Pi(X)}\to\Pi_{X,X}(1_X)$ is a composition*

$$1_{\Pi(X)}=1_{\Delta(\Gamma(X))}\xrightarrow{\delta_{\Gamma(X)}}\Delta_{\Gamma(X),\Gamma(X)}(1_{\Gamma(X)})\xrightarrow{\Delta_{\Gamma(X),\Gamma(X)}(\gamma_X)}\Delta_{\Gamma(X),\Gamma(X)}\Gamma_{X,X}(1_X)=\Pi_{X,X}(1_X)$$

For 1-arrows $S:X\to Y$, $T:Y\to Z$ in $\mathfrak{C}$,

$$(\pi_{X,Y,Z})_{(S,T)}:\mu_{\Pi(X),\Pi(Y),\Pi(Z)}(\Pi_{X,Y}\times\Pi_{Y,Z})(S,T)\to\Pi_{X,Z}\mu_{X,Y,Z}(S,T)$$

is defined to be the composition below.

$$\mu_{\Delta(\Gamma(X)),\Delta(\Gamma(Y)),\Delta(\Gamma)(Z)}(\Delta_{\Gamma(X),\Gamma(Y)}\Gamma_{X,Y}\times\Delta_{\Gamma(Y),\Gamma(Z)}\Gamma_{Y,Z})(S,T)\xrightarrow{(\delta_{\Gamma(X),\Gamma(Y),\Gamma(Z)})_{(\Gamma_{X,Y}(S),\Gamma_{Y,Z}(T))}}$$

$$\Delta_{\Gamma(X),\Gamma(Z)}\mu_{\Gamma(X),\Gamma(Y),\Gamma(Z)}(\Gamma_{X,Y}\times\Gamma_{Y,Z})(S,T)\xrightarrow{\Delta_{\Gamma(X),\Gamma(Z)}((\gamma_{X,Y,Z})_{(S,T)})}\Delta_{\Gamma(X),\Gamma(Z)}\Gamma_{X,Z}\mu_{X,Y,Z}(S,T).$$

We call (Π,π) the composition of (Γ,γ) and (Δ,δ).

We denote by $I_{\mathfrak{C}}=(I,\iota):\mathfrak{C}\to\mathfrak{C}$ the identity lax functor, that is, I is the identity map of $\operatorname{Ob}\mathfrak{C}$, $I_{X,Y}:\mathfrak{C}(X,Y)\to\mathfrak{C}(I(X),I(Y))$ is the identity functor, $\iota_X:1_{I(X)}\to I_{X,X}(1_X)$ is the identity 2-arrow and $\iota_{X,Y,Z}:\mu_{I(X),I(Y),I(Z)}(I_{X,Y}\times I_{Y,Z})\to I_{X,Z}\mu_{X,Y,Z}$ is the identity natural transformation.

Definition 8.3.5 *Let $\mathfrak{D}$ and $\mathfrak{C}$ be 2-categories.*

(1) A lax functor $(\Gamma,\gamma):\mathfrak{C}\to\mathfrak{D}$ is called a 2-functor if the 2-arrow $\gamma_X:1_{\Gamma(X)}\to\Gamma_{X,X}(1_X)$ is an isomorphism for every $X\in\operatorname{Ob}\mathfrak{C}$ and $\gamma_{X,Y,Z}:\mu_{\Gamma(X),\Gamma(Y),\Gamma(Z)}(\Gamma_{X,Y}\times\Gamma_{Y,Z})\to\Gamma_{X,Z}\mu_{X,Y,Z}$ is a natural equivalence for every $X,Y,Z\in\operatorname{Ob}\mathfrak{C}$.

(2) If $\mathcal{C}$ is a category regarded as a 2-category as in (1) of (8.3.2), we call a lax functor $(\Gamma,\gamma):\mathcal{C}\to\mathfrak{D}$ a lax diagram.

(3) A lax diagram which is also a 2-functor is called a pseudo-functor.

Example 8.3.6 *For a functor $F:\mathcal{D}\to\mathcal{E}$, we define a 2-functor $\boldsymbol{pfib}(F)=(F^*,\gamma_F):\boldsymbol{pfib}(\mathcal{E})\to\boldsymbol{pfib}(\mathcal{D})$ as follows. For an object $p:\mathcal{F}\to\mathcal{E}$ of $\boldsymbol{pfib}(\mathcal{E})$, let $F^*(p)=p_F:\mathcal{D}\times_{\mathcal{E}}\mathcal{F}\to\mathcal{D}$ be the pull-back of p along F. If κ is a cleavege of p, the cleavege κ_F of p_F is given by $(\kappa_F(f))(Y,N)=(X,\kappa(F(f))(N))$ and $(\kappa_F(f))(id_Y,\varphi)=(id_X,\kappa(F(f))(\varphi))$ for a morphism $f:X\to Y$ in $\mathcal{D}$ and $N\in\operatorname{Ob}\mathcal{F}_{F(Y)}$, $\varphi\in\operatorname{Mor}\mathcal{F}_{F(Y)}$. For a 1-arrow $\varphi:p\to q$ from an object $p:\mathcal{F}\to\mathcal{E}$ to an object $q:\mathcal{C}\to\mathcal{E}$ of $\boldsymbol{pfib}(\mathcal{E})$, let $F^*_{p,q}(\varphi):p_F\to q_F$ be the 1-arrow in $\boldsymbol{pfib}(\mathcal{D})$ induced by $id_{\mathcal{D}}\times\varphi:\mathcal{D}\times\mathcal{F}\to\mathcal{D}\times\mathcal{C}$. It follows from (8.1.23) that if φ is a 1-arrow in $\boldsymbol{pfib}^c(\mathcal{E})$, $F^*_{p,q}(\varphi)$ is a 1-arrow in $\boldsymbol{pfib}^c(\mathcal{D})$. Let $\varphi,\psi:p\to q$ be 1-arrows in $\boldsymbol{pfib}(\mathcal{E})$ and $\chi:\varphi\to\psi$ a 2-arrow. Define a 2-arrow $F^*_{p,q}(\chi):F^*_{p,q}(\varphi)\to F^*_{p,q}(\psi)$ by $F^*_{p,q}(\chi)_{(X,M)}=(id_X,\chi_M):F^*_{p,q}(\varphi)(X,M)\to F^*_{p,q}(\psi)(X,M)$ for $(X,M)\in\operatorname{Ob}(\mathcal{D}\times_{\mathcal{E}}\mathcal{F})$. Thus we have a functor $F^*_{p,q}:\boldsymbol{pfib}(\mathcal{E})(p,q)\to\boldsymbol{pfib}(\mathcal{D})(F(p),F(q))$. For $p\in\operatorname{Ob}\boldsymbol{pfib}(\mathcal{E})$, since $1_{F^*(p)}=id_{p_F}=F^*_{p,p}(1_p):p_F\to p_F$, let $(\gamma_F)_p:1_{F^*(p)}\to F^*_{p,p}(1_p)$ be the identity 2-arrow. For 1-arrows $\varphi:p\to q$, $\psi:q\to r$ in $\boldsymbol{pfib}(\mathcal{E})$, composition of 1-arrows $F^*_{p,q}(\varphi):p_F\to q_F$ and $F^*_{q,r}(\psi):q_F\to r_F$ coincides with $F^*_{p,r}(\psi\varphi)$. We define a 2-arrow $((\gamma_F)_{p,q,r})_{(\varphi,\psi)}:F^*_{q,r}(\psi)F^*_{p,q}(\varphi)\to F^*_{p,r}(\psi\varphi)$ in $\boldsymbol{pfib}(\mathcal{D})$ as the identity 2-arrow. In fact, $\boldsymbol{pfib}(F)$ is a functor.*

Definition 8.3.7 *For 2-categories $\mathfrak{C}$ and $\mathfrak{D}$, we define 2-category $\operatorname{Lax}(\mathfrak{C},\mathfrak{D})$ of lax functors as follows. Objects of $\operatorname{Lax}(\mathfrak{C},\mathfrak{D})$ are lax functors from $\mathfrak{C}$ to $\mathfrak{D}$. Let $(\Gamma,\gamma),(\Delta,\delta):\mathfrak{C}\to\mathfrak{D}$ be lax functors. A 1-arrow $(\Lambda,\lambda):(\Gamma,\gamma)\to(\Delta,\delta)$ consists of the following data.*

(1) For each $X\in\operatorname{Ob}\mathfrak{C}$, a 1-arrow $\Lambda_X:\Gamma(X)\to\Delta(X)$ in $\mathfrak{D}$.

(2) For each 1-arrow $S:X\to Y$ in $\mathfrak{C}$, a 2-arrow $\lambda_S:\Lambda_Y\circ\Gamma_{X,Y}(S)\to\Delta_{X,Y}(S)\circ\Lambda_X$ in $\mathfrak{D}$, making the following diagrams commute for every $X\in\operatorname{Ob}\mathfrak{C}$ and 1-arrows $S:X\to Y$, $T:Y\to Z$ in $\mathfrak{C}$.

$$\begin{tikzcd}
\Lambda_X \circ 1_{\Gamma(X)} \arrow[r, "id_{\Lambda_X}"] \arrow[d, "id_{\Lambda_X} * \gamma_X"] & 1_{\Delta(X)} \circ \Lambda_X \arrow[d, "\delta_X * id_{\Lambda_X}"] \\
\Lambda_X \circ \Gamma_{X,X}(1_X) \arrow[r, "\lambda_{1_X}"] & \Delta_{X,X}(1_X) \circ \Lambda_X
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda_Z \circ \Gamma_{Y,Z}(T) \circ \Gamma_{X,Y}(S) \arrow[r, "id_{\Lambda_Z} * (\gamma_{X,Y,Z})_{(S,T)}"] \arrow[d, "\lambda_T * id_{\Gamma_{X,Y}(S)}"] & \Lambda_Z \circ \Gamma_{X,Z}(T \circ S) \arrow[d, "\lambda_{T \circ S}"] \\
\Delta_{Y,Z}(T) \circ \Lambda_Y \circ \Gamma_{X,Y}(S) \arrow[d, "id_{\Gamma_{Y,Z}(T)} * \lambda_S"] & \Delta_{X,Z}(T \circ S) \circ \Lambda_X \\
\Delta_{Y,Z}(T) \circ \Delta_{X,Y}(S) \circ \Lambda_X \arrow[ur, "(\delta_{X,Y,Z})_{(S,T)} * id_{\Lambda_X}"'] &
\end{tikzcd}$$

For 1-arrows $(\Lambda, \lambda) : (\Gamma, \gamma) \to (\Delta, \delta)$ and $(\Phi, \varphi) : (\Delta, \delta) \to (E, \epsilon)$, define a 1-arrow $(\Psi, \psi) : (\Gamma, \gamma) \to (E, \epsilon)$ by $\Psi_X = \Phi_X \circ \Lambda_X : \Gamma(X) \to E(X)$, $\psi_S = (\varphi_S * id_{\Lambda_X})(id_{\Phi_Y} * \lambda_S) : \Phi_Y \circ \Lambda_Y \circ \Gamma_{X,Y}(S) \to E_{X,Y}(S) \circ \Phi_X \circ \Lambda_X$ for $X \in \mathrm{Ob}\,\mathfrak{C}$ and a 1-arrow $S : X \to Y$ in $\mathfrak{C}$. Composition $(\Phi, \varphi) \circ (\Lambda, \lambda)$ is defined to be (Ψ, ψ). The identity 1-arrow $1_{(\Gamma,\gamma)}$ of $(\Gamma, \gamma) \in \mathrm{Ob}\,\mathrm{Lax}(\mathfrak{C}, \mathfrak{D})$ is a pair (I, ι) such that I_X is the identity 1-arrow of $\Gamma(X)$ for $X \in \mathrm{Ob}\,\mathfrak{C}$ and ι_S is the identity 2-arrow of $\Gamma_{X,Y}(S)$ for a 1-arrow $S : X \to Y$ in $\mathfrak{C}$.

Let $(\Lambda, \lambda), (\Phi, \varphi) : (\Gamma, \gamma) \to (\Delta, \delta)$ be 1-arrows in $\mathrm{Lax}(\mathfrak{C}, \mathfrak{D})$. A 2-arrow $\chi : (\Lambda, \lambda) \to (\Phi, \varphi)$ consists of 2-arrows $\chi_X : \Lambda_X \to \Phi_X$ in $\mathfrak{D}$ for $X \in \mathrm{Ob}\,\mathfrak{C}$ such that, for every 1-arrow $S : X \to Y$ in $\mathfrak{C}$, the following diagram commutes.

$$\begin{tikzcd}
\Lambda_Y \circ \Gamma_{X,Y}(S) \arrow[r, "\lambda_S"] \arrow[d, "\chi_Y * id_{\Gamma_{X,Y}(S)}"] & \Delta_{X,Y}(S) \circ \Lambda_X \arrow[d, "id_{\Delta_{X,Y}(S)} * \chi_X"] \\
\Phi_Y \circ \Gamma_{X,Y}(S) \arrow[r, "\varphi_S"] & \Delta_{X,Y}(S) \circ \Phi_X
\end{tikzcd}$$

For 2-arrows $\chi : (\Lambda, \lambda) \to (\Phi, \varphi)$ and $\omega : (\Phi, \varphi) \to (\Psi, \psi)$ ($(\Lambda, \lambda), (\Phi, \varphi), (\Psi, \psi) : (\Gamma, \gamma) \to (\Delta, \delta)$), composition $\omega\chi : (\Lambda, \lambda) \to (\Psi, \psi)$ in $\mathrm{Lax}(\mathfrak{C}, \mathfrak{D})((\Gamma, \gamma), (\Delta, \delta))$ is defined by $(\omega\chi)_X = \omega_X \chi_X$. If $\chi : (\Lambda, \lambda) \to (\Phi, \varphi)$ and $\omega : (\Psi, \psi) \to (\Upsilon, \upsilon)$ ($(\Lambda, \lambda), (\Phi, \varphi) : (\Gamma, \gamma) \to (\Delta, \delta)$, $(\Psi, \psi), (\Upsilon, \upsilon) : (\Delta, \delta) \to (E, \epsilon)$) are 2-arrows, composition $\omega * \chi : (\Psi, \psi) \circ (\Lambda, \lambda) \to (\Upsilon, \upsilon) \circ (\Phi, \varphi)$ is defined by $(\omega * \chi)_X = \omega_X * \chi_X : \Psi_X \circ \Lambda_X \to \Upsilon_X \circ \Phi_X$. The identity 2-arrow $\iota_{(\Lambda,\lambda)}$ of a 1-arrrow (Λ, λ) in $\mathrm{Lax}(\mathfrak{C}, \mathfrak{D})$ is given by $(\iota_{(\Lambda,\lambda)})_X =$ (the identity 2-arrow of Λ_X) for any $X \in \mathrm{Ob}\,\mathfrak{C}$.

Definition 8.3.8 *For later use, we denote by $\mathrm{Lax}^s(\mathfrak{C}, \mathfrak{D})$ the full subcategory of $\mathrm{Lax}(\mathfrak{C}, \mathfrak{D})$ consisting of lax functors $(\Gamma, \gamma) : \mathfrak{C} \to \mathfrak{D}$ such that $\gamma_X : 1_{\Gamma(X)} \to \Gamma_{X,X}(1_X)$ is an isomorphism for each $X \in \mathrm{Ob}\,\mathfrak{C}$. Moreover, $2\text{-Funct}(\mathfrak{C}, \mathfrak{D})$ denotes the full subcategory of $\mathrm{Lax}(\mathfrak{C}, \mathfrak{D})$ consisting of 2-functors. We also consider a subcategory $\mathrm{Lax}^c(\mathfrak{C}, \mathfrak{D})$ of $\mathrm{Lax}^s(\mathfrak{C}, \mathfrak{D})$ and a subcategory $2\text{-Funct}^c(\mathfrak{C}, \mathfrak{D})$ of $2\text{-Funct}(\mathfrak{C}, \mathfrak{D})$ given as follows. $\mathrm{Lax}^c(\mathfrak{C}, \mathfrak{D})$ (resp. $2\text{-Funct}^c(\mathfrak{C}, \mathfrak{D})$) has the same objects as $\mathrm{Lax}^s(\mathfrak{C}, \mathfrak{D})$ (resp. $2\text{-Funct}(\mathfrak{C}, \mathfrak{D})$). A 1-arrow (Λ, λ) in $\mathrm{Lax}^s(\mathfrak{C}, \mathfrak{D})$ (resp. $2\text{-Funct}(\mathfrak{C}, \mathfrak{D})$) belongs to $\mathrm{Lax}^c(\mathfrak{C}, \mathfrak{D})$ (resp. $2\text{-Funct}^c(\mathfrak{C}, \mathfrak{D})$) if and only if a 2-arrow λ_S in $\mathfrak{D}$ is an isomorphism for every 1-arrow S in $\mathfrak{C}$. A 2-arrow in $\mathrm{Lax}^s(\mathfrak{C}, \mathfrak{D})$ (resp. $2\text{-Funct}(\mathfrak{C}, \mathfrak{D})$) belongs to $\mathrm{Lax}^c(\mathfrak{C}, \mathfrak{D})$ (resp. $2\text{-Funct}^c(\mathfrak{C}, \mathfrak{D})$) if and only if its domain and codomain are 1-arrows in $\mathrm{Lax}^c(\mathfrak{C}, \mathfrak{D})$ (resp. $2\text{-Funct}^c(\mathfrak{C}, \mathfrak{D})$).*

Note that $2\text{-Funct}(\mathfrak{C}, \mathfrak{D})$ is also a full subcategory of $\mathrm{Lax}^s(\mathfrak{C}, \mathfrak{D})$.

Example 8.3.9 *Let $(A, \alpha) : \mathfrak{C}' \to \mathfrak{C}$ be a lax functor. Define a lax functor $(A, \alpha)^* : \mathrm{Lax}(\mathfrak{C}, \mathfrak{D}) \to \mathrm{Lax}(\mathfrak{C}', \mathfrak{D})$ as follows. Put $(A, \alpha)^* = (A^*, \gamma_A)$. For a lax functor $(\Gamma, \gamma) : \mathfrak{C} \to \mathfrak{D}$, we set $A^*((\Gamma, \gamma)) = (\Gamma, \gamma) \circ (A, \alpha)$. For a 1-arrow $(\Lambda, \lambda) : (\Gamma, \gamma) \to (\Delta, \delta)$ in $\mathrm{Lax}(\mathfrak{C}, \mathfrak{D})$, let us define a 1-arrow $(\Lambda_A, \lambda_A) : (\Gamma, \gamma) \circ (A, \alpha) \to (\Delta, \delta) \circ (A, \alpha)$ in $\mathrm{Lax}(\mathfrak{C}', \mathfrak{D})$ by $(\Lambda_A)_X = \Lambda_{A(X)}$ for $X \in \mathrm{Ob}\mathfrak{C}'$ and*

$$(\lambda_A)_S = \lambda_{A(S)} : \Lambda_{A(Y)} \circ \Gamma_{A(X),A(Y)}(A_{X,Y}(S)) \longrightarrow \Delta_{A(X),A(Y)}(A_{X,Y}(S)) \circ \Lambda_{A(X)}$$

for a 1-arrow $S : X \to Y$ in $\mathfrak{C}'$. We set $A^*_{(\Gamma,\gamma),(\Delta,\delta)}((\Lambda, \lambda)) = (\Lambda_A, \lambda_A)$. For a 2-arrow $\chi : (\Lambda, \lambda) \to (\Phi, \varphi)$ in $\mathrm{Lax}(\mathfrak{C}, \mathfrak{D})$, let $\chi_A : (\Lambda_A, \lambda_A) \to (\Phi_A, \varphi_A)$ be the 2-arrow given by $(\chi_A)_X = \chi_{A(X)} : (\Lambda_A)_X = \Lambda_{A(X)} \to \Phi_{A(X)} = (\Phi_A)_X$ for $X \in \mathrm{Ob}\mathfrak{C}'$. We set $A^*_{(\Gamma,\gamma),(\Delta,\delta)}(\chi) = \chi_A$.

For a lax functor $(\Gamma, \gamma) : \mathfrak{C} \to \mathfrak{D}$, since $A^*_{(\Gamma,\gamma),(\Gamma,\gamma)}(1_{(\Gamma,\gamma)})$ is th identity 1-arrow of $1_{A^*((\Gamma,\gamma))}$, let $(\gamma_A)_{(\Gamma,\gamma)} : 1_{A^*((\Gamma,\gamma))} \to A^*_{(\Gamma,\gamma),(\Gamma,\gamma)}(1_{(\Gamma,\gamma)})$ be the identity 2-arrow of $1_{A^*((\Gamma,\gamma))}$. For lax functors $(\Gamma, \gamma), (\Delta, \delta), (E, \epsilon) : \mathfrak{C} \to \mathfrak{D}$, let $(\gamma_A)_{(\Gamma,\gamma),(\Delta,\delta),(E,\epsilon)}$ be the identity natural transformation

$$\mu_{A^*((\Gamma,\gamma)),A^*((\Delta,\delta)),A^*((E,\epsilon))} \left(A^*_{(\Gamma,\gamma),(\Delta,\delta)} \times A^*_{(\Delta,\delta),(E,\epsilon)} \right) \longrightarrow A^*_{(\Gamma,\gamma),(E,\epsilon)} \mu_{(\Gamma,\gamma),(\Delta,\delta),(E,\epsilon)}.$$

In fact, for composable 1-arrows $(\Lambda,\lambda):(\Gamma,\gamma)\to(\Delta,\delta)$ *and* $(\Phi,\varphi):(\Delta,\delta)\to(E,\epsilon)$ *in* $\mathrm{Lax}(\mathfrak{C},\mathfrak{D})$*, it is easy to verfy* $A^*_{(\Delta,\delta),(E,\epsilon)}((\Phi,\varphi))\circ A^*_{(\Gamma,\gamma),(\Delta,\delta)}((\Lambda,\lambda)) = A^*_{(\Gamma,\gamma),(E,\epsilon)}((\Phi,\varphi)\circ(\Lambda,\lambda))$*, and let*

$$((\gamma_A)_{(\Gamma,\gamma),(\Delta,\delta),(E,\epsilon)})_{((\Lambda,\lambda),(\Phi,\varphi))}: A^*_{(\Delta,\delta),(E,\epsilon)}((\Phi,\varphi))\circ A^*_{(\Gamma,\gamma),(\Delta,\delta)}((\Lambda,\lambda))\to A^*_{(\Gamma,\gamma),(E,\epsilon)}((\Phi,\varphi)\circ(\Lambda,\lambda))$$

be the indentity 2-arrow in $\mathrm{Lax}(\mathfrak{C},\mathfrak{D})$*. It is a routine to verify that* $(A,\alpha)^*:\mathrm{Lax}(\mathfrak{C},\mathfrak{D})\to\mathrm{Lax}(\mathfrak{C}',\mathfrak{D})$ *is a lax functor.*

Remark 8.3.10 *We note that the lax functor* $(A,\alpha)^*:\mathrm{Lax}(\mathfrak{C},\mathfrak{D})\to\mathrm{Lax}(\mathfrak{C}',\mathfrak{D})$ *defined above maps* $\mathrm{Lax}^s(\mathfrak{C},\mathfrak{D})$ *to* $\mathrm{Lax}^s(\mathfrak{C}',\mathfrak{D})$*,* $2\text{-Funct}(\mathfrak{C},\mathfrak{D})$ *to* $2\text{-Funct}(\mathfrak{C}',\mathfrak{D})$ *and* $\mathrm{Lax}^c(\mathfrak{C},\mathfrak{D})$ *to* $\mathrm{Lax}^c(\mathfrak{C}',\mathfrak{D})$*. Therefore we have lax functors*

$$\begin{array}{ll}(A,\alpha)^*:\mathrm{Lax}^s(\mathfrak{C},\mathfrak{D})\to\mathrm{Lax}^s(\mathfrak{C}',\mathfrak{D}), & (A,\alpha)^*:\mathrm{Lax}^c(\mathfrak{C},\mathfrak{D})\to\mathrm{Lax}^c(\mathfrak{C}',\mathfrak{D}),\\ (A,\alpha)^*:2\text{-Funct}(\mathfrak{C},\mathfrak{D})\to 2\text{-Funct}(\mathfrak{C}',\mathfrak{D}), & (A,\alpha)^*:2\text{-Funct}^c(\mathfrak{C},\mathfrak{D})\to 2\text{-Funct}^c(\mathfrak{C}',\mathfrak{D})\end{array}$$

Example 8.3.11 *Let* $(B,\beta):\mathfrak{D}\to\mathfrak{D}'$ *be a 2-functor. Define a lax functor* $(B,\beta)_*:\mathrm{Lax}(\mathfrak{C},\mathfrak{D})\to\mathrm{Lax}(\mathfrak{C},\mathfrak{D}')$ *as follows. Put* $(B,\beta)_*=(B_*,\gamma_B)$*. For a lax functor* $(\Gamma,\gamma):\mathfrak{C}\to\mathfrak{D}$*, we set* $B_*((\Gamma,\gamma))=(B,\beta)\circ(\Gamma,\gamma)$*. For a 1-arrow* $(\Lambda,\lambda):(\Gamma,\gamma)\to(\Delta,\delta)$ *in* $\mathrm{Lax}(\mathfrak{C},\mathfrak{D})$*, let us define a 1-arrow* $(\Lambda_B,\lambda_{(B,\beta)}):(B,\beta)\circ(\Gamma,\gamma)\to(B,\beta)\circ(\Delta,\delta)$ *in* $\mathrm{Lax}(\mathfrak{C},\mathfrak{D}')$ *by* $(\Lambda_B)_X=B_{\Gamma(X),\Delta(X)}(\Lambda_X)$ *for* $X\in\mathrm{Ob}\mathfrak{C}$ *and* $(\lambda_{(B,\beta)})_S$ *is given by the following composition for a 1-arrow* $S:X\to Y$ *in* $\mathfrak{C}$*.*

$$B_{\Gamma(Y),\Delta(Y)}(\Lambda_Y)\circ B_{\Gamma(X),\Gamma(Y)}(\Gamma_{X,Y}(S))\xrightarrow{(\beta_{\Gamma(X),\Gamma(Y),\Delta(Y)})_{(\Lambda_Y,\Gamma_{X,Y}(S))}} B_{\Gamma(X),\Delta(Y)}(\Lambda_Y\circ\Gamma_{X,Y}(S))\xrightarrow{B_{\Gamma(X),\Delta(Y)}(\lambda_S)}$$
$$B_{\Gamma(X),\Delta(Y)}(\Delta_{X,Y}(S)\circ\Lambda_X)\xrightarrow{(\beta_{\Gamma(X),\Delta(X),\Delta(Y)})^{-1}_{(\Delta_{X,Y}(S),\Lambda_X)}} B_{\Delta(X),\Delta(Y)}(\Delta_{X,Y}(S))\circ B_{\Gamma(X),\Delta(X)}(\Lambda_X)$$

We set $(B_*)_{(\Gamma,\gamma),(\Delta,\delta)}((\Lambda,\lambda))=(\Lambda_B,\lambda_{(B,\beta)})$*. For a 2-arrow* $\chi:(\Lambda,\lambda)\to(\Phi,\varphi)$ *in* $\mathrm{Lax}(\mathfrak{C},\mathfrak{D})$*, let* $\chi_B:(\Lambda_B,\lambda_{(B,\beta)})\to(\Phi_B,\varphi_{(B,\beta)})$ *be the 2-arrow given by* $(\chi_B)_X=B_{\Gamma(X),\Delta(X)}(\chi_X):(\Lambda_B)_X=B_{\Gamma(X),\Delta(X)}(\Lambda_X)\to B_{\Gamma(X),\Delta(X)}(\Phi_X)=(\Phi_B)_X$ *for* $X\in\mathrm{Ob}\mathfrak{C}$*. We set* $(B_*)_{(\Gamma,\gamma),(\Delta,\delta)}(\chi)=\chi_B$*.*

For a lax functor $(\Gamma,\gamma):\mathfrak{C}\to\mathfrak{D}$*, let* $(\gamma_B)_{(\Gamma,\gamma)}:1_{B_*(\Gamma,\gamma)}\to(B_*)_{(\Gamma,\gamma),(\Gamma,\gamma)}(1_{(\Gamma,\gamma)})$ *be the 2-arrow in* $\mathrm{Lax}(\mathfrak{C},\mathfrak{D}')$ *given by* $((\gamma_B)_{(\Gamma,\gamma)})_X=\beta_{\Gamma(X)}$ *for each* $X\in\mathrm{Ob}\,\mathfrak{C}$*. For lax functors* $(\Gamma,\gamma),(\Delta,\delta),(E,\epsilon):\mathfrak{C}\to\mathfrak{D}$*, let* $(\gamma_B)_{(\Gamma,\gamma),(\Delta,\delta),(E,\epsilon)}$ *be the natural transformation*

$$\mu_{B_*((\Gamma,\gamma)),B_*((\Delta,\delta)),B_*((E,\epsilon))}\left((B_*)_{(\Gamma,\gamma),(\Delta,\delta)}\times(B_*)_{(\Delta,\delta),(E,\epsilon)}\right)\longrightarrow(B_*)_{(\Gamma,\gamma),(E,\epsilon)}\mu_{(\Gamma,\gamma),(\Delta,\delta),(E,\epsilon)}$$

defined as follows. For composable 1-arrows $(\Lambda,\lambda):(\Gamma,\gamma)\to(\Delta,\delta)$ *and* $(\Phi,\varphi):(\Delta,\delta)\to(E,\epsilon)$ *in* $\mathrm{Lax}(\mathfrak{C},\mathfrak{D})$*, let* $((\gamma_B)_{(\Gamma,\gamma),(\Delta,\delta),(E,\epsilon)})_{((\Lambda,\lambda),(\Phi,\varphi))}:(B_*)_{(\Delta,\delta),(E,\epsilon)}((\Phi,\varphi))\circ(B_*)_{(\Gamma,\gamma),(\Delta,\delta)}((\Lambda,\lambda))\to(B_*)_{(\Gamma,\gamma),(E,\epsilon)}((\Phi,\varphi)\circ(\Lambda,\lambda))$ *be the 2-arrow in* $\mathrm{Lax}(\mathfrak{C},\mathfrak{D}')$ *given by*

$$\left(((\gamma_B)_{(\Gamma,\gamma),(\Delta,\delta),(E,\epsilon)})_{((\Lambda,\lambda),(\Phi,\varphi))}\right)_X=(\beta_{\Gamma(X),\Delta(X),E(X)})_{(\Lambda_X,\Phi_X)}$$

for $X\in\mathrm{Ob}\,fC$*. Again, it is a routine to verify that* $(B,\beta)_*:\mathrm{Lax}(\mathfrak{C},\mathfrak{D})\to\mathrm{Lax}(\mathfrak{C},\mathfrak{D}')$ *is a lax functor.*

Remark 8.3.12 *The lax functor* $(B,\beta)_*:\mathrm{Lax}(\mathfrak{C},\mathfrak{D})\to\mathrm{Lax}(\mathfrak{C},\mathfrak{D}')$ *defined above maps* $\mathrm{Lax}^s(\mathfrak{C},\mathfrak{D})$ *to* $\mathrm{Lax}^s(\mathfrak{C},\mathfrak{D}')$*,* $2\text{-Funct}(\mathfrak{C},\mathfrak{D})$ *to* $2\text{-Funct}(\mathfrak{C},\mathfrak{D}')$ *and* $\mathrm{Lax}^c(\mathfrak{C},\mathfrak{D})$ *to* $\mathrm{Lax}^c(\mathfrak{C},\mathfrak{D}')$*. Therefore we have following lax functors.*

$$\begin{array}{ll}(B,\beta)_*:\mathrm{Lax}^s(\mathfrak{C},\mathfrak{D})\to\mathrm{Lax}^s(\mathfrak{C},\mathfrak{D}'), & (B,\beta)_*:\mathrm{Lax}^c(\mathfrak{C},\mathfrak{D})\to\mathrm{Lax}^c(\mathfrak{C},\mathfrak{D}'),\\ (B,\beta)_*:2\text{-Funct}(\mathfrak{C},\mathfrak{D})\to 2\text{-Funct}(\mathfrak{C},\mathfrak{D}'), & (B,\beta)_*:2\text{-Funct}^c(\mathfrak{C},\mathfrak{D})\to 2\text{-Funct}^c(\mathfrak{C},\mathfrak{D}')\end{array}$$

Let $p:\mathcal{F}\to\mathcal{E}$ be a cloven prefibered category with cleavage κ. We associate a lax diagram $(\Gamma(p),\gamma(p)):\mathcal{E}^{op}\to\boldsymbol{cat}$ as follows.

Construction 8.3.13 *We set* $\Gamma(p)(X)=\mathcal{F}_X$ *for* $X\in\mathrm{Ob}\,\mathcal{E}$*. For a morphism* $f:X\to Y$ *in* $\mathcal{E}$*, we define* $\Gamma(p)_{Y,X}(f):\mathcal{F}_Y\to\mathcal{F}_X$ *to be the inverse image functor* $f^*=\kappa(f):\mathcal{F}_Y\to\mathcal{F}_X$*. For* $X\in\mathrm{Ob}\,\mathcal{E}$*, since* $\alpha_{id_X}(N):id_X^*(N)\to N$ *is an isomorphism by (8.1.5), a natural transformation* $\gamma(p)_X:1_{\mathcal{F}_X}\to id_X^*$ *is given by* $(\gamma(p)_X)_N=\alpha_{id_X}(N)^{-1}:N\to id_X^*(N)$ *(*$N\in\mathrm{Ob}\,\mathcal{F}_X$*). For each pair* $(g,f)\in\mathcal{E}(Z,X)\times\mathcal{E}(X,Y)$*, a 2-arrow* $(\gamma(p)_{Y,X,Z})_{(f,g)}:g^*f^*\to(fg)^*$ *is defined to be* $c_{f,g}$*. It follows from (8.1.12) that* $(\Gamma(p),\gamma(p))$ *is a lax diagram. We call* $(\Gamma(p),\gamma(p))$ *the lax diagram associated with* $p:\mathcal{F}\to\mathcal{E}$*. Moreover,* $(\Gamma(p),\gamma(p))$ *is a pseudo-functor if and only if* $p:\mathcal{F}\to\mathcal{E}$ *is a fibered category.*

Let $p:\mathcal{F}\to\mathcal{E}$*,* $q:\mathcal{D}\to\mathcal{E}$ *be objects of* $\boldsymbol{pfib}(\mathcal{E})$ *and* $F:p\to q$ *a morphism in* $\boldsymbol{pfib}(\mathcal{E})$*. We define a morphism* $(\Lambda(F),\lambda(F)):(\Gamma(p),\gamma(p))\to(\Gamma(q),\gamma(q))$ *as follows.*

For $X \in \operatorname{Ob}\mathcal{E}$, $\Lambda(F)_X : \mathcal{F}_X \to \mathcal{D}_X$ is the restriction F_X of F. For a morphism $f : X \to Y$ in $\mathcal{E}$ and an object N of $\mathcal{F}_Y$, let $(\lambda(F)_f)_N : F_X f^*(N) \to f^*(F_Y(N))$ be the unique morphism in $\mathcal{D}_X$ such that $\alpha_f(F_Y(N))(\lambda(F)_f)_N = F(\alpha_f(N))$. Then, $(\lambda(F)_f)_N$ is natural in N and we have a natural transformation $\lambda(F)_f : \Lambda(F)_X \circ \Gamma(p)_{Y,X}(f) \to \Gamma(q)_{Y,X}(f) \circ \Lambda(F)_Y$. We note that, if F is the identity morphism of p, $\Lambda(F)_X$ is the identity functor of $\mathcal{F}_X$ for every $X \in \operatorname{Ob}\mathcal{E}$ and $\lambda(F)_f = id_{f^*}$ for every morphism f in $\mathcal{E}$. Also note that $\lambda(F)_f$ is an equivalence for every morphism f if and only if F preserves cartesian morphisms.

Let $F, G : p \to q$ be a morphisms in $\boldsymbol{pfib}(\mathcal{E})$ and $\varphi : F \to G$ a 2-arrow in $\boldsymbol{pfib}(\mathcal{E})$. For each object X of $\mathcal{E}$, the natural transformation $\varphi_X : F_X \to G_X$ induced by φ defines a 2-arrow $\chi(\varphi) : (\Lambda(F), \lambda(F)) \to (\Lambda(G), \lambda(G))$ by $\chi(\varphi)_X = \varphi_X : F_X \to G_X$.

For a category $\mathcal{E}$, define a functor $\Theta = \Theta_{\mathcal{E}} : \boldsymbol{pfib}(\mathcal{E}) \to \operatorname{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat})$ as follows. Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven prefbered category with cleavage κ. Put $\Theta(p) = (\Gamma(p), \gamma(p))$ (8.3.13). If $F : p \to q$ ($p : \mathcal{F} \to \mathcal{E}$, $q : \mathcal{D} \to \mathcal{E}$) is a morphism in $\boldsymbol{pfib}(\mathcal{E})$, put $\Theta(F) = (\Lambda(F), \lambda(F))$. Let $G : q \to r$ ($r : \mathcal{C} \to \mathcal{E}$) be a morphism in $\boldsymbol{pfib}(\mathcal{E})$. Then, $\Lambda(GF)_X = G_X F_X = \Lambda(G)_X \Lambda(F)_X$ and $\alpha_f(G_Y F_Y(N))(\lambda(G)_f)_{F_Y(N)} G_X((\lambda(F)_f)_N) = G(\alpha_f(F_Y(N))) G_X((\lambda(F)_f)_N) = G(\alpha_f(F_Y(N))(\lambda(F)_f)_N) = GF(\alpha_f(N))$. It follows that $(\lambda(GF)_f)_N = (\lambda(G)_f)_{F_Y(N)} G_X((\lambda(F)_f)_N) = ((\lambda(G)_f * id_{\Lambda(F)_Y}) \circ (id_{\Lambda(G)_X} * \lambda(F)_f))_N$. Hence $(\Lambda(GF), \lambda(GF)) = (\Lambda(G), \lambda(G)) \circ (\Lambda(F), \lambda(F))$. For a 2-arrow $\varphi : F \to G$ ($F, G : p \to q$) in $\boldsymbol{pfib}(\mathcal{E})$, we set $\Theta(\varphi) = \chi(\varphi)$ (8.3.13), namely $\Theta(\varphi)_X = \varphi_X : \mathcal{F}_X \to \mathcal{D}_X$ for $X \in \operatorname{Ob}\mathcal{E}$. If $\xi : G \to H$ ($H : p \to q$) and $\psi : H \to K$ ($H, K : q \to r$) are 2-arrows in $\boldsymbol{pfib}(\mathcal{E})$, then it is straightforward to verify $\Theta(\psi\varphi) = \Theta(\psi)\Theta(\varphi)$ and $\Theta(\psi * \varphi) = \Theta(\psi) * \Theta(\varphi)$. Thus Θ is a functor. In other words, we have a lax functor $(\Theta_{\mathcal{E}}, \theta_{\mathcal{E}}) : \boldsymbol{pfib}(\mathcal{E}) \to \operatorname{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat})$, where $(\theta_{\mathcal{E}})_p$ is the identity 2-arrow of $1_{\Theta(p)}$ and $(\theta_{\mathcal{E}})_{p,q,r}$ is the identity natural transformation for any object p, q, r of $\boldsymbol{pfib}(\mathcal{E})$.

Let $(\Gamma, \gamma) : \mathfrak{C} \to \mathfrak{D}$ be an object of $\operatorname{Lax}^s(\mathfrak{C}, \mathfrak{D})$. For a 1-arrow $S : X \to Y$ in $\mathfrak{C}$, we put $R_\gamma(S) = (\gamma_{X,Y,Y})_{(S,1_Y)} : \Gamma_{Y,Y}(1_Y) \circ \Gamma_{X,Y}(S) \to \Gamma_{X,Y}(S)$, $L_\gamma(S) = (\gamma_{X,X,Y})_{(1_X,S)} : \Gamma_{X,Y}(S) \circ \Gamma_{X,X}(1_X) \to \Gamma_{X,Y}(S)$. Since $\gamma_Y : 1_{\Gamma(Y)} \to \Gamma_{Y,Y}(1_Y)$ is an isomorphism, the commutativity of the upper diagram of (4) in (8.3.3) implies the following assertion.

Proposition 8.3.14 *2-arrows $R_\gamma(f)$ and $L_\gamma(f)$ in $\mathfrak{D}$ are isomorphisms.*

Let $(\Gamma, \gamma) : \mathcal{E}^{op} \to \boldsymbol{cat}$ be an object of $\operatorname{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat})$. We construct a cloven prefibered category $p(\Gamma) : \mathcal{F}(\Gamma) \to \mathcal{E}$ as follows.

Construction 8.3.15 *Set $\operatorname{Ob}\mathcal{F}(\Gamma) = \{(X, x) | \, X \in \operatorname{Ob}\mathcal{E},\ x \in \operatorname{Ob}\Gamma(X)\}$. For $(X, x), (Y, y) \in \operatorname{Ob}\mathcal{F}(\Gamma)$, we put $\mathcal{F}(\Gamma)((X, x), (Y, y)) = \{(f, u) | \, f \in \mathcal{E}(X, Y), u \in \Gamma(X)(x, \Gamma_{Y,X}(f)(y))\}$. Composition of morphisms $(f, u) : (X, x) \to (Y, y)$ and $(g, v) : (Y, y) \to (Z, z)$ is defined to be $(gf, ((\gamma_{Z,Y,X})_{(g,f)})_z \Gamma_{Y,X}(f)(v)u)$. Note that $(id_X, (\gamma_X)_x)$ is the identity morphism of (X, x). Define a functor $p(\Gamma) : \mathcal{F}(\Gamma) \to \mathcal{E}$ by $p(\Gamma)(X, x) = X$, $p(\Gamma)(f, u) = f$. For each $X \in \operatorname{Ob}\mathcal{E}$, there is an isomorphism $\mathfrak{f}_X : \mathcal{F}(\Gamma)_X \to \Gamma(X)$ of categories given by $\mathfrak{f}_X(X, x) = x$ and $\mathfrak{f}_X(id_X, u) = (\gamma_X)_y^{-1} u$ ($(id_X, u) : (X, x) \to (X, y)$).*

We claim that a morphism $(f, u) : (X, x) \to (Y, y)$ is cartesian if and only if $u : x \to \Gamma_{Y,X}(f)(y)$ is an isomorphism in $\Gamma(X)$. In fact, for $z \in \operatorname{Ob}\Gamma(X)$, since $(f, u)(id_X, v) = (f, R_\gamma(f)_y \Gamma_{X,X}(id_X)(u)v) = (f, (R_\gamma(f))_y (\gamma_X)_{\Gamma_{Y,X}(y)} u (\gamma_X)_x^{-1} v) = (f, u(\gamma_X)_x^{-1} v)$ by the assumption and the commutativity of the upper diagram of (4) of (8.3.3), the map $\mathcal{F}(\Gamma)_X((X, z), (X, x)) \to \mathcal{F}(\Gamma)_f((X, z), (Y, y))$ given by $(id_X, v) \mapsto (f, u)(id_X, v)$ is bijective for every $z \in \operatorname{Ob}\Gamma(X)$ if and only if u is an isomorphism.

In particular, $(id_X, u) : (X, x) \to (X, y)$ is an isomorphism if and only if u is an isomorphism. The inverse of (id_X, u) is given by $(id_X, (\gamma_X)_x u^{-1} (\gamma_X)_y)$.

For a morphism $f : X \to Y$ in $\mathcal{E}$, set $f^*(Y, y) = (X, \Gamma_{Y,X}(f)(y))$ and the canonical morphism $\alpha_f(Y, y) : f^*(Y, y) \to (Y, y)$ is defined to be $(f, id_{\Gamma_{Y,X}(f)(y)})$. Then $\alpha_f(Y, y)$ is catesian by the above fact, hence the inverse image functor $f^* : \mathcal{F}(\Gamma)_Y \to \mathcal{F}(\Gamma)_X$ of f is given by $f^*(Y, y) = (X, \Gamma_{Y,X}(f)(y))$ and $f^*(id_Y, v) = (id_X, R_\gamma(f)_z^{-1} L_\gamma(f)_z \Gamma_{Y,X}(f)(v))$. Note that, for a morphism $f : X \to Y$ in $\mathcal{E}$,

$$\begin{tikzcd}
\mathcal{F}(\Gamma)_X \arrow[r, "f^*"] \arrow[d, "\mathfrak{f}_X"] & \mathcal{F}(\Gamma)_Y \arrow[d, "\mathfrak{f}_Y"] \\
\Gamma(X) \arrow[r, "\Gamma_{Y,X}(f)"] & \Gamma(Y)
\end{tikzcd}$$

commutes. For morphisms $f : X \to Y$ and $g : Z \to X$ in $\mathcal{E}$ and $(Y, y) \in \operatorname{Ob}\mathcal{F}(\Gamma)$, define $c_{f,g}(Y, y) : g^* f^*(Y, y) \to (fg)^*(Y, y)$ by $c_{f,g}(Y, y) = (id_Z, (R_\gamma(fg))_y^{-1}((\gamma_{Y,X,Z})_{(f,g)})_y)$. Then, the following square commutes.

$$\begin{tikzcd}
g^*f^*(Y,y) \arrow[r, "\alpha_g(f^*(Y,y))"] \arrow[d, "c_{f,g}(Y,y)"] & f^*(Y,y) \arrow[d, "\alpha_f(Y,y)"] \\
(fg)^*(Y,y) \arrow[r, "\alpha_{fg}(Y,y)"] & (Y,y)
\end{tikzcd}$$

It follows from (8.1.11) that $p(\Gamma):\mathcal{F}(\Gamma)\to\mathcal{E}$ is a fibered category if and only if (Γ,γ) is a pseudo-functor.

Let $(\Gamma,\gamma),(\Delta,\delta):\mathcal{E}^{op}\to\boldsymbol{cat}$ be objects of $\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$. For 1-arrow $(\Lambda,\lambda):(\Gamma,\gamma)\to(\Delta,\delta)$ of lax diagrams, we construct a functor $F_\Lambda:\mathcal{F}(\Gamma)\to\mathcal{F}(\Delta)$ as follows. For $(X,x)\in\mathrm{Ob}\,\mathcal{F}(\Gamma)$, $F_\Lambda(X,x)=(X,\Lambda_X(x))$ and, for a morphism $(f,u):(X,x)\to(Y,y)$ in $\mathcal{F}(\Gamma)$, $F_\Lambda(f,u)=(f,(\lambda_f)_y\Lambda_X(u))$. It is clear that F_Λ preserves fibers. Since $F_\Lambda(\alpha_f(Y,y))=F_\Lambda(f,id_{\Gamma_{Y,X}(f)(y)})=(f,(\lambda_f)_y\Lambda_X(id_{\Gamma_{Y,X}(f)(y)}))=(f,(\lambda_f)_y)$, F_Λ preserves cartesian morphism if and only if $\lambda_f:\Lambda_X\Gamma_{Y,X}(f)\to\Delta_{Y,X}(f)\Lambda_Y$ is a natural equivalence of functors from $\Gamma(Y)$ to $\Delta(X)$ for every morphism $f:X\to Y$ in $\mathcal{E}$.

Let $(\Lambda,\lambda),(\Phi,\varphi):(\Gamma,\gamma)\to(\Delta,\delta)$ be 1-arrows in $\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$. For a 2-arrow $\chi:(\Lambda,\lambda)\to(\Phi,\varphi)$, we define a natural transformation $\tilde{\chi}:F_\Lambda\to F_\Phi$ by $\tilde{\chi}_{(X,x)}=(id_X,(\delta_X)_{\Phi_X(x)}(\chi_X)_x)$ for $(X,x)\in\mathrm{Ob}\,\mathcal{F}(\Gamma)$.

Remark 8.3.16 *Let (Γ,γ) be an object of $\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$ and $f:X\to Y$ a morphism in $\mathcal{E}$. Then, the pull-back functor $f^*:\mathcal{F}(\Gamma)_Y\to\mathcal{F}(\Gamma)_X$ has a left adjoint if and only if $\Gamma_{Y,X}(f):\Gamma(Y)\to\Gamma(X)$ has a left adjoint. If $(\Gamma,\gamma):\mathcal{E}^{op}\to\boldsymbol{cat}$ is a 2-functor, the fibered category $p(\Gamma):\mathcal{F}(\Gamma)\to\mathcal{E}$ is a bifibered category if and only if $\Gamma_{Y,X}(f):\Gamma(Y)\to\Gamma(X)$ has a left adjoint for every morphism $f:X\to Y$ in $\mathcal{E}$.*

The following examples are applications of the above construction.

Example 8.3.17 *Let $\mathcal{C}$ be a category and F a presheaf of sets on $\mathcal{C}$. $\mathcal{C}_F$ denotes a category with objects (X,x) for $X\in\mathrm{Ob}\,\mathcal{C}$, $x\in F(X)$ and morphisms $\mathcal{C}_F((X,x),(Y,y))=\{\alpha\in\mathcal{C}(X,Y)\,|\,F(\alpha)(y)=x\}$. We call $\mathcal{C}_F$ the category of F-models. Note that there is a functor $U_F:\mathcal{C}_F\to\mathcal{C}$ given by $U_F(X,x)=X$.*

For a morphism $u:F\to G$ of presheaves, define a functor $u_\sharp:\mathcal{C}_F\to\mathcal{C}_G$ by $u_\sharp(X,x)=(X,u_X(x))$ and $u_\sharp(\alpha)=\alpha$. Let us denote by $\widehat{\mathcal{C}}$ the category of presheaves of sets on $\mathcal{C}$. Then, $u_\sharp$ induces a functor $u^*:\widehat{\mathcal{C}_G}\to\widehat{\mathcal{C}_F}$ by $u^*(S)=Su_\sharp$. We note that, for morphisms $u:F\to G$, $v:E\to F$ of presheaves, since $(uv)_\sharp=u_\sharp v_\sharp$, we have $(uv)^*=v^*u^*$. Define a functor $\Gamma:\widehat{\mathcal{C}}^{op}\to\boldsymbol{cat}$ by $\Gamma(F)=\widehat{\mathcal{C}_F}^{op}$ and $\Gamma(u)=u^*$. The fibered category $p(\Gamma):\mathcal{F}(\Gamma)\to\widehat{\mathcal{C}}$ associated with Γ is called the fibered category of models on $\mathcal{C}$.

Example 8.3.18 *Let $\mathcal{E}$ be a category with finite limits and $\boldsymbol{C}=(C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ an internal category in $\mathcal{E}$. We denote by $\Gamma_{\boldsymbol{C}}:\mathcal{E}^{op}\to\boldsymbol{cat}$ the functor represented by $\boldsymbol{C}$. That is, $\Gamma_{\boldsymbol{C}}$ is described as follows. For $X\in\mathrm{Ob}\,\mathcal{E}$, put $\mathrm{Ob}\,\Gamma_{\boldsymbol{C}}(X)=\mathcal{E}(X,C_0)$ and $\Gamma_{\boldsymbol{C}}(X)(u,v)=\{\varphi\in\mathcal{E}(X,C_1)\,|\,\sigma\varphi=u,\ \tau\varphi=v\}$. The composition of morphisms $\varphi:u\to v$ and $\psi:v\to w$ is defined to be a composition $X\xrightarrow{(\varphi,\psi)}C_1\times_{C_0}C_1\xrightarrow{\mu}C_1$. For an object u of $\Gamma_{\boldsymbol{C}}(X)$, $\varepsilon u:X\to C_1$ is the identity morphism $1_u:u\to u$. For a morphism $f:X\to Y$ in $\mathcal{E}$, $\Gamma_{\boldsymbol{C}}(f):\Gamma_{\boldsymbol{C}}(Y)\to\Gamma_{\boldsymbol{C}}(X)$ is defined by $\Gamma_{\boldsymbol{C}}(f)(u)=uf$ for an object $u:Y\to C_0$ of $\Gamma_{\boldsymbol{C}}(Y)$ and $\Gamma_{\boldsymbol{C}}(f)(\varphi)=\varphi f$ for a morphism $\varphi:Y\to C_1$ in $\Gamma_{\boldsymbol{C}}(Y)$. We call $p_{\Gamma_{\boldsymbol{C}}}:\mathcal{F}(\Gamma_{\boldsymbol{C}})\to\mathcal{E}$ the fibered category represented by $\boldsymbol{C}$ and we simply denote this by $p_{\boldsymbol{C}}:\mathcal{F}(\boldsymbol{C})\to\mathcal{E}$.*

Example 8.3.19 *Let $(\mathcal{E},J)$ be a site. For each object X of $\mathcal{E}$, we give $\mathcal{E}/X$ the topology induced by $\Sigma_X:\mathcal{E}/X\to\mathcal{E}$. If $f:X\to Y$ is a morphism in $\mathcal{E}$, then $\Sigma_f:\mathcal{E}/X\to\mathcal{E}/Y$ is continuous and cocontinuous by (2.13.3). Then, $\Sigma_f^*:\widehat{\mathcal{E}/Y}\to\widehat{\mathcal{E}/X}$ induces $\widetilde{\Sigma}_f^*:\widetilde{\mathcal{E}/Y}\to\widetilde{\mathcal{E}/X}$ which is naturally equivalent to a composition $\widetilde{\mathcal{E}/Y}\xrightarrow{i}\widehat{\mathcal{E}/Y}\xrightarrow{\Sigma_f^*}\widehat{\mathcal{E}/X}\xrightarrow{a}\widetilde{\mathcal{E}/X}$. It follows from (2.15.9) that $\widetilde{\Sigma}_f^*$ is left exact and it has a right adjoint $\tilde{f}_*:\widetilde{\mathcal{E}/X}\to\widetilde{\mathcal{E}/Y}$. Thus $(\tilde{f}_*,\widetilde{\Sigma}_f^*):\widetilde{\mathcal{E}/X}\to\widetilde{\mathcal{E}/Y}$ is a geomtric morphism of Grothendieck topoi. Define a functor $\Gamma:\mathcal{E}^{op}\to\boldsymbol{cat}$ by $\Gamma(X)=\widetilde{\mathcal{E}/X}$ and $\Gamma(f)=\widetilde{\Sigma}_f^*$. Applying the construction given in (8.3.15) to Γ, we have a bifibered category $p(\Gamma):\mathcal{F}(\Gamma)\to\mathcal{E}$ which is an example of fibered topos ([8], Définition 7.1.1.).*

Let $X.$ be a simplicial object in $\mathcal{E}$, namely a functor $\Delta^{op}\to\mathcal{E}$. Consider the pull-back $p_{X.}:\mathcal{F}(X.)\to\Delta^{op}$ of $p(\Gamma):\mathcal{F}(\Gamma)\to\mathcal{E}$ along $X.$. This fibered category is called the Grothendieck topos over $X.$. In the case $\mathcal{E}=\mathcal{S}ch$ and J is the etale topology, $X.$ is the category of sheaves on simplicial scheme $X.$ (Compare [2]).

We define a functor $\Xi=\Xi_{\mathcal{E}}:\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})\to\boldsymbol{pfib}(\mathcal{E})$ as follows. For an object $(\Gamma,\gamma):\mathcal{E}^{op}\to\boldsymbol{cat}$ of $\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$, put $\Xi(\Gamma,\gamma)=(p(\Gamma):\mathcal{F}(\Gamma)\to\mathcal{E})$ (8.3.15). If $(\Lambda,\lambda):(\Gamma,\gamma)\to(\Delta,\delta)$ is a 1-arrow in $\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$, we put $\Xi(\Lambda,\lambda)=(F_\Lambda:\mathcal{F}(\Gamma)\to\mathcal{F}(\Delta))$. For 1-arrows $(\Lambda,\lambda):(\Gamma,\gamma)\to(\Delta,\delta)$ and $(\Phi,\varphi):(\Delta,\delta)\to(\Pi,\pi)$, put $(\Psi,\psi)=(\Phi,\varphi)\circ(\Lambda,\lambda):(\Gamma,\gamma)\to(\Pi,\pi)$. Let (X,x) be an object of $\mathcal{F}(\Gamma)$ and (f,u) :

$(X,x) \to (Y,y)$ a morphism in $\mathcal{F}(\Gamma)$. Then, $F_\Phi F_\Lambda(X,x) = F_\Phi(X,\Lambda_X(x)) = (X,\Phi_X\Lambda_X(x)) = F_\Psi(X,x)$, $F_\Phi F_\Lambda(f,u) = F_\Phi(f,(\lambda_f)_y\Lambda_X(u)) = (f,(\varphi_f)_{\Lambda_Y(y)}\Phi_X((\lambda_f)_y\Lambda_X(u))) = (f,(\varphi_f)_{\Lambda_Y(y)}\Phi_X((\lambda_f)_y)\Phi_X(\Lambda_X(u))) = (f,(\psi_f)_y\Psi_X(u)) = F_\Psi(f,u)$. It follows that $\Xi((\Phi,\varphi)\circ(\Lambda,\lambda)) = \Xi(\Phi,\varphi)\circ\Xi(\Lambda,\lambda)$. Let $(\Lambda,\lambda),(\Phi,\varphi) : (\Gamma,\gamma) \to (\Delta,\delta)$ be 1-arrows in $\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$. For a 2-arrow $\chi : (\Lambda,\lambda) \to (\Phi,\varphi)$, we set $\Xi(\chi) = \tilde{\chi}$. If $\omega : (\Phi,\varphi) \to (\Psi,\psi)$ is a 2-arrow in $\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$, then

$$
\begin{aligned}
(\Xi(\omega)\Xi(\chi))_{(X,x)} &= \Xi(\omega)_{(X,x)}\Xi(\chi)_{(X,x)} = (id_X,(\delta_X)_{\Psi_X(x)}(\omega_X)_x)(id_X,(\delta_X)_{\Phi_X(x)}(\chi_X)_x)\\
&= (id_X,((\delta_{X,X,X})_{(id_X,id_X)})_{\Psi_X(x)}\Delta_{X,X}(id_X)((\delta_X)_{\Psi_X(x)}(\omega_X)_x)(\delta_X)_{\Phi_X(x)}(\chi_X)_x)\\
&= (id_X,((\delta_{X,X,X})_{(id_X,id_X)})_{\Psi_X(x)}\Delta_{X,X}(id_X)((\delta_X)_{\Psi_X(x)})\Delta_{X,X}(id_X)((\omega_X)_x)(\delta_X)_{\Phi_X(x)}(\chi_X)_x)\\
&= (id_X,\Delta_{X,X}(id_X)((\omega_X)_x)(\delta_X)_{\Phi_X(x)}(\chi_X)_x) = (id_X,(\delta_X)_{\Psi_X(x)}(\omega_X)_x(\chi_X)_x)\\
&= (id_X,(\delta_X)_{\Psi_X(x)}(\omega_X\chi_X)_x) = \widetilde{(\omega\chi)}_{(X,x)} = \Xi(\omega\chi)_{(X,x)}.
\end{aligned}
$$

Thus we have $\Xi(\omega\chi) = \Xi(\omega)\Xi(\chi)$. Let $(\Lambda,\lambda),(\Phi,\varphi) : (\Gamma,\gamma) \to (\Delta,\delta)$ and $(\Psi,\psi),(\Upsilon,\upsilon) : (\Delta,\delta) \to (\Pi,\pi)$ be 1-arrows in $\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$. If $\chi : (\Lambda,\lambda) \to (\Phi,\varphi)$ and $\omega : (\Phi,\varphi) \to (\Psi,\psi)$ are 2-arrows in $\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$,

$$
\begin{aligned}
&(\Xi(\omega)*\Xi(\chi))_{(X,x)} = (\tilde{\omega}*\tilde{\chi})_{(X,x)} = \tilde{\omega}_{F_\Phi(X,x)}F_\Psi(\tilde{\chi}_{(X,x)}) = \tilde{\omega}_{(X,\Phi_X(x))}F_\Psi(id_X,(\delta_X)_{\Phi_X(x)}(\chi_X)_x)\\
&= (id_X,(\pi_X)_{\Upsilon_X\Phi_X(x)}(\omega_X)_{\Phi_X(x)})(id_X,(\psi_{id_X})_{\Phi_X(x)}\Psi_X((\delta_X)_{\Phi_X(x)}(\chi_X)_x))\\
&= (id_X,((\pi_{X,X,X})_{(id_X,id_X)})_{\Upsilon_X\Phi_X(x)}\Pi_{X,X}(id_X)((\pi_X)_{\Upsilon_X\Phi_X(x)}(\omega_X)_{\Phi_X(x)})(\psi_{id_X})_{\Phi_X(x)}\Psi_X((\delta_X)_{\Phi_X(x)})\Psi_X((\chi_X)_x))\\
&= (id_X,\Pi_{X,X}(id_X)((\omega_X)_{\Phi_X(x)})(\psi_{id_X})_{\Phi_X(x)}\Psi_X((\delta_X)_{\Phi_X(x)})\Psi_X((\chi_X)_x))\\
&= (id_X,(\upsilon_{id_X})_{\Phi_X(x)}(\omega_X)_{\Delta_X(id_X)\Phi_X(x)}\Psi_X((\delta_X)_{\Phi_X(x)})\Psi_X((\chi_X)_x))\\
&= (id_X,(\upsilon_{id_X})_{\Phi_X(x)}\Upsilon_X((\delta_X)_{\Phi_X(x)})(\omega_X)_{\Phi_X(x)}\Psi_X((\chi_X)_x)) = (id_X,(\pi_X)_{\Upsilon_X\Phi_X(x)}\Upsilon_X((\chi_X)_x)(\omega_X)_{\Lambda_X(x)})\\
&= (id_X,(\pi_X)_{\Upsilon_X\Phi_X(x)}((\omega*\chi)_X)_x) = \widetilde{(\omega*\chi)}_{(X,x)} = \Xi(\omega*\chi)_{(X,x)}.
\end{aligned}
$$

Hence $\Xi(\omega*\chi) = \Xi(\omega)*\Xi(\chi)$. Thus we have a lax functor $(\Xi_{\mathcal{E}},\xi_{\mathcal{E}}) : \mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat}) \to \boldsymbol{pfib}(\mathcal{E})$, where $(\xi_{\mathcal{E}})_{(\Gamma,\gamma)}$ is the identity 2-arrow of $1_{\Xi(\Gamma,\gamma)}$ and $\xi_{(\Gamma,\gamma),(\Delta,\delta),(\Phi.\varphi)}$ is the identity natural transformation for any object (Γ,γ), (Δ,δ), (Φ,φ) of $\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$.

Theorem 8.3.20 *There is an equivalence $\Theta : \boldsymbol{pfib}(\mathcal{E}) \to \mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$ of 2-categories. This induces the following equivalences.*

$$\boldsymbol{pfib}^c(\mathcal{E}) \to \mathrm{Lax}^c(\mathcal{E}^{op},\boldsymbol{cat}),\quad \boldsymbol{fib}(\mathcal{E}) \to \text{2-Funct}(\mathcal{E}^{op},\boldsymbol{cat}),\quad \boldsymbol{fib}^c(\mathcal{E}) \to \text{2-Funct}^c(\mathcal{E}^{op},\boldsymbol{cat})$$

Proof. Put $(Z,\zeta) = (\Xi,\xi)(\Theta,\theta) : \boldsymbol{pfib}(\mathcal{E}) \to \boldsymbol{pfib}(\mathcal{E})$. For an object $p : \mathcal{F} \to \mathcal{E}$ of $\boldsymbol{pfib}(\mathcal{E})$, $Z(p) = \Xi(\Theta(p)) : \mathcal{F}(\Gamma(p)) \to \mathcal{E}$ is a cloven prefibered category such that $\mathrm{Ob}\,\mathcal{F}(\Gamma(p)) = \{(X,M)|\, X \in \mathrm{Ob}\,\mathcal{E},\ M \in \mathrm{Ob}\,\mathcal{F}_X\}$ and $\mathcal{F}(\Gamma(p))((X,M),(Y,N)) = \{(f,u)|\, f \in \mathcal{E}(X,Y),\ u \in \mathcal{F}(M,f^*(N))\}$. Define a 1-arrow $E_p : Z(p) \to p$ in $\boldsymbol{pfib}(\mathcal{E})$ as follows.

If $(X,M),(X,N) \in \mathrm{Ob}\,\mathcal{F}(\Gamma(p))$ and $(f,u) : (X,M) \to (X,N)$ is a morphism in $\mathcal{F}(\Gamma(p))$, we put $E_p(X,M) = M$ and $E_p(f,u) = \alpha_f(N)u$. Then, E_p is an isomorphism and its inverse $E_p^{-1} : p \to \Xi\Theta(p)$ is given by $E_p^{-1}(M) = (p(M),M)$ and $E_p^{-1}(\rho) = (p(\rho),u)$, where $u : M \to p(\rho)^*(N)$ is the unique morphism satisfying $\alpha_{p(\rho)}(N)u = \rho$.

If $F : p \to q$ is a 1-arrow in $\boldsymbol{pfib}(\mathcal{E})$, for $(X,M) \in \mathrm{Ob}\,\mathcal{F}(\Gamma(p))$ and a morphism $(f,u) : (X,M) \to (Y,N)$ in $\mathcal{F}(\Gamma(p))$, then $Z(F)(X,M) = (\Xi\Theta(F))(X,M) = (\Xi(\Lambda(F),\lambda(F)))(X,M) = F_{\Lambda(F)}(X,M) = (X,\Lambda(F)_X(M)) = (X,F(M))$. Hence $E_qZ(F)(X,M) = E_q(X,F(M)) = F(M) = FE_p(X,M)$ and $\epsilon_F : E_qZ(F) \to FE_p$ is defined to be the identity 2-arrow in $\boldsymbol{pfib}(\mathcal{E})$. Therefore we have a 1-arrow $(E,\epsilon) : (Z,\zeta) \to I_{\boldsymbol{pfib}(\mathcal{E})}$ in $\mathrm{Lax}(\boldsymbol{pfib}(\mathcal{E}),\boldsymbol{pfib}(\mathcal{E}))$ which is an isomorphism.

Put $(\Omega,\omega) = (\Theta,\theta)(\Xi,\xi) : \mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat}) \to \mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$ and let (Γ,γ) be an object of $\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})$. By the definitions of Θ and Ξ, we have $\Omega(\Gamma,\gamma) = \Theta(p(\Gamma) : \mathcal{F}(\Gamma) \to \mathcal{E}) = (\Gamma(p(\Gamma)),\gamma(p(\Gamma)))$ and $\mathrm{Ob}\,\Gamma(p(\Gamma))(X) = \mathrm{Ob}\,\mathcal{F}(\Gamma)_X = \{(X,x)|\, x \in \mathrm{Ob}\,\Gamma(X)\}$, $\mathrm{Mor}\,\Gamma(p(\Gamma))(X) = \mathrm{Mor}\,\mathcal{F}(\Gamma)_X = \{(id_X,u)|\, u \in \mathrm{Mor}\,\Gamma(X)\}$.

For $X \in \mathrm{Ob}\,\mathcal{E}$, let $H(\Gamma,\gamma)_X : \Gamma(X) \to \Gamma(p(\Gamma))(X)$ be a functor given by $H(\Gamma,\gamma)_X(x) = (X,x)$ and $H(\Gamma,\gamma)_X(u) = (id_X,(\gamma_X)_yu)$, where $x,y \in \mathrm{Ob}\,\Gamma(X)$, $(u : x \to y) \in \mathrm{Mor}\,\Gamma(X)$. Then, $H(\Gamma,\gamma)_X$ is an isomorphism. In fact, the inverse $H(\Gamma,\gamma)_X^{-1} : \Gamma(p(\Gamma))(X) \to \Gamma(X)$ is given by $H(\Gamma,\gamma)_X^{-1}(X,x) = x$, $H(\Gamma,\gamma)_X^{-1}(id_X,v) = (\gamma_X)_y^{-1}v$ for $x,y \in \mathrm{Ob}\,\Gamma(X)$ and $(v : x \to \Gamma_{X,X}(id_X)(y)) \in \mathrm{Mor}\,\Gamma(X)$.

For a morphism $f : X \to Y$ in $\mathcal{E}$ and $y \in \mathrm{Ob}\,\Gamma(Y)$, $\Gamma(p(\Gamma))(f)_{Y,X}H(\Gamma,\gamma)_Y(y) = \Gamma(p(\Gamma))_{Y,X}(f)(Y,y) = (X,\Gamma_{Y,X}(f)(y)) = H(\Gamma,\gamma)_X\Gamma_{Y,X}(f)(y)$, thus $H(\Gamma,\gamma)_X$ is natural in X. Let $\eta(\Gamma,\gamma)_f : H(\Gamma,\gamma)_X\Gamma_{Y,X}(f) \to$

$\Gamma(p(\Gamma))_{Y,X}(f)H(\Gamma,\gamma)_Y$ be the identity natural transformation. Define a 1-arrow $H_{(\Gamma,\gamma)} : (\Gamma,\gamma) \to \Omega(\Gamma,\gamma)$ in $\mathrm{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat})$ by $H_{(\Gamma,\gamma)} = (H(\Gamma,\gamma), \eta(\Gamma,\gamma))$. For a 1-arrow $(\Lambda,\lambda) : (\Gamma,\gamma) \to (\Delta,\delta)$ in $\mathrm{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat})$ and $x, y \in \mathrm{Ob}\,\Gamma(X)$, $(u : x \to y) \in \mathrm{Mor}\,\Gamma(X)$, we have

$$\begin{aligned}
\Lambda(F_\Lambda)_X H(\Gamma,\gamma)_X(x) &= \Lambda(F_\Lambda)_X(X,x) = (X, \Lambda_X(x)) = H(\Delta,\delta)_X\Lambda_X(x),\\
\Lambda(F_\Lambda)_X H(\Gamma,\gamma)_X(u) &= \Lambda(F_\Lambda)_X(id_X, (\gamma_X)_y u) = F_\Lambda(id_X, (\gamma_X)_y u) = (id_X, (\lambda_{id_X})_y \Lambda_X((\gamma_X)_y u))\\
&= (id_X, (\lambda_{id_X})_y \Lambda_X((\gamma_X)_y)\Lambda_X(u)) = (id_X, (\delta_X)_y\Lambda_X(u)) = H(\Delta,\delta)_X\Lambda_X(u),
\end{aligned}$$

which show the naturality of $H(\Gamma,\gamma)$ in (Γ,γ). We denote by $\eta_{(\Lambda,\lambda)} : H_{(\Delta,\delta)}(\Lambda,\lambda) \to \Omega(\Lambda,\lambda)H_{(\Gamma,\gamma)}$ the identity 2-arrow in $\mathrm{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat})$. Now we have a 1-arrow $(H,\eta) : I_{\mathrm{Lax}^s(\mathcal{E}^{op},\boldsymbol{cat})} \to \Omega$ in $\mathrm{Lax}(\mathrm{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat}), \mathrm{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat}))$ which is an isomorphism. □

For a functor $F : \mathcal{D} \to \mathcal{E}$, we denote by $F^{op} : \mathcal{D}^{op} \to \mathcal{E}^{op}$ the functor induced by F. Regarding (F^{op}, id) as a lax functor, we have a lax functor $(F^{op}, id)^* : \mathrm{Lax}(\mathcal{E}^{op}, \boldsymbol{cat}) \to \mathrm{Lax}(\mathcal{D}^{op}, \boldsymbol{cat})$.

Proposition 8.3.21 *The following diagrams commutes up to natural equivalence.*

$$\begin{tikzcd}
\boldsymbol{pfib}(\mathcal{E}) \arrow[r, "\boldsymbol{pfib}(F)"] \arrow[d, "(\Theta_{\mathcal{E}},\theta_{\mathcal{E}})"] & \boldsymbol{pfib}(\mathcal{D}) \arrow[d, "(\Theta_{\mathcal{D}},\theta_{\mathcal{D}})"] \\
\mathrm{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat}) \arrow[r, "(F^{op},id)^*"] & \mathrm{Lax}^s(\mathcal{D}^{op}, \boldsymbol{cat})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat}) \arrow[r, "(F^{op},id)^*"] \arrow[d, "(\Xi_{\mathcal{E}},\xi_{\mathcal{E}})"] & \mathrm{Lax}^s(\mathcal{D}^{op}, \boldsymbol{cat}) \arrow[d, "(\Xi_{\mathcal{D}},\xi_{\mathcal{D}})"] \\
\boldsymbol{pfib}(\mathcal{E}) \arrow[r, "\boldsymbol{pfib}(F)"] & \boldsymbol{pfib}(\mathcal{D})
\end{tikzcd}$$

Proof. Let $(\Gamma', \gamma') : \boldsymbol{pfib}(\mathcal{E}) \to \mathrm{Lax}^s(\mathcal{D}^{op}, \boldsymbol{cat})$ be the composition of $\boldsymbol{pfib}(F) : \boldsymbol{pfib}(\mathcal{E}) \to \boldsymbol{pfib}(\mathcal{D})$ and $(\Theta_{\mathcal{D}}, \theta_{\mathcal{D}}) : \boldsymbol{pfib}(\mathcal{D}) \to \mathrm{Lax}^s(\mathcal{D}^{op}, \boldsymbol{cat})$. For an object $p : \mathcal{F} \to \mathcal{E}$ of $\boldsymbol{pfib}(\mathcal{E})$, $\Gamma'(p) : \mathcal{D}^{op} \to \boldsymbol{cat}$ is given by $\Gamma'(p)(X) = (\mathcal{D} \times_{\mathcal{E}} \mathcal{F})_X$ for $X \in \mathrm{Ob}\,\mathcal{D}$ and $\Gamma'_{X,Y}(f) = \kappa_F(f)$ for $(f : X \to Y) \in \mathrm{Mor}\,\mathcal{D}$. On the other hand, let $(\Delta', \delta') : \boldsymbol{pfib}(\mathcal{E}) \to \mathrm{Lax}^s(\mathcal{D}^{op}, \boldsymbol{cat})$ be the composition of $(\Theta_{\mathcal{E}}, \theta_{\mathcal{E}}) : \boldsymbol{pfib}(\mathcal{E}) \to \mathrm{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat})$ and $(F^{op}, id)^* : \mathrm{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat}) \to \mathrm{Lax}^s(\mathcal{D}^{op}, \boldsymbol{cat})$. Then, for an object $p : \mathcal{F} \to \mathcal{E}$ of $\boldsymbol{pfib}(\mathcal{E})$, $\Delta'(p) : \mathcal{D}^{op} \to \boldsymbol{cat}$ is given by $\Delta'(p)(X) = \mathcal{F}_{F(X)}$ for $X \in \mathrm{Ob}\,\mathcal{D}$ and $\Delta'_{X,Y}(f) = \kappa(F(f))$ for $(f : X \to Y) \in \mathrm{Mor}\,\mathcal{D}$. Since the projection functor $\widetilde{F} : \mathcal{D} \times_{\mathcal{E}} \mathcal{F} \to \mathcal{F}$ induces an isomorphism from $(\mathcal{D} \times_{\mathcal{E}} \mathcal{F})_X$ to $\mathcal{F}_{F(X)}$ for each object X of $\mathcal{D}$, Γ' and Δ' are naturally equivalent. This shows the commutativity of the first diagram. The commutativity of the second diagram can be verified similarly. □

If we identify $\mathrm{Lax}^s(\mathcal{E}^{op}, \boldsymbol{cat})$ with $\mathrm{Lax}^s(\mathcal{E}, \boldsymbol{cat}^{op})$, then we can say that the functor "$\boldsymbol{pfib}$" from $\boldsymbol{cat}$ to the category of prefibered categories is "represented" by $\boldsymbol{cat}^{op}$ by (8.3.20) and above result.

8.4 Fibered category with products

Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category. For morphisms $f : X \to Y$, $g : X \to Z$ of $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, we define a presheaf $F_{f,g,M} : \mathcal{F}_Z \to \mathcal{S}et$ on $\mathcal{F}_Z^{op}$ by $F_{f,g,M}(N) = F_{f,g}(M,N) = \mathcal{F}_X(f^*(M), g^*(N))$ for $N \in \mathrm{Ob}\,\mathcal{F}_Z$ and $F_{f,g,M}(\psi) = F_{f,g}(id_M, \psi) = g^*(\psi)_*$ for $\psi \in \mathrm{Mor}\,\mathcal{F}_Z$.

Suppose that $F_{f,g,M}$ is representable. We choose an object $M_{[f,g]}$ of $\mathcal{F}_Z$ such that there exists a natural equivalence $P_{f,g}(M) : F_{f,g,M} \to \hat{h}_{M_{[f,g]}}$, where $\hat{h}_{M_{[f,g]}}$ is the presheaf on $\mathcal{F}_Z^{op}$ represented by $M_{[f,g]}$. If $X = Z$ and g is the identity morphism of Z, we take $f^*(M)$ as $M_{[f,id_X]}$. Hence $P_{f,id_X}(M)_N$ is the identity map of $\mathcal{F}_X(f^*(M), N)$. Let us denote by $\iota_{f,g}(M) : f^*(M) \to g^*(M_{[f,g]})$ the morphism of $\mathcal{F}_X$ which is mapped to the identity morphism of $M_{[f,g]}$ by $P_{f,g}(M)_{M_{[f,g]}} : \mathcal{F}_X(f^*(M), g^*(M_{[f,g]})) \to \mathcal{F}_Z(M_{[f,g]}, M_{[f,g]})$.

Remark 8.4.1 *If $g^* : \mathcal{F}_Y \to \mathcal{F}_X$ has a left adjoint $g_* : \mathcal{F}_X \to \mathcal{F}_Y$, $F_{f,g,M} : \mathcal{F}_Y \to \mathcal{S}et$ is representable for any object M of $\mathcal{F}_Y$. In fact, $M_{[f,g]}$ is defined to be $g_*f^*(M)$ in this case. If we denote by $(\mathrm{ad}_g)_{P,N} : \mathcal{F}_Y(g_*(P), N) \to \mathcal{F}_X(P, g^*(N))$ the bijection which is natural in $P \in \mathrm{Ob}\,\mathcal{F}_X$ and $N \in \mathrm{Ob}\,\mathcal{F}_Y$, we have $P_{f,g}(M)_N = (\mathrm{ad}_g)^{-1}_{f^*(M),N} : \mathcal{F}_X(f^*(M), g^*(N)) \to \mathcal{F}_Y(g_*f^*(M), N)$. Let us denote by $\eta_g : id_{\mathcal{F}_X} \to g^*g_*$ the unit of the adjunction $g_* \dashv g^*$. We have $\iota_{f,g}(M) = (\eta_g)_{f^*(M)} : f^*(M) \to g^*g_*f^*(M) = g^*(M_{[f,g]})$.*

Proposition 8.4.2 *The inverse of $P_{f,g}(M)_N : \mathcal{F}_X(f^*(M), g^*(N)) \to \mathcal{F}_Z(M_{[f,g]}, N)$ is given by the map defined by $\varphi \mapsto g^*(\varphi)\iota_{f,g}(M)$.*

Proof. For $\varphi \in \mathcal{F}_Y(M_{[f,g]}, N)$, the following diagram commutes by naturality of $P_{f,g}(M)$.

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(M_{[f,g]})) \arrow[r, "g^*(\varphi)_*"] \arrow[d, "P_{f,g}(M)_{M_{[f,g]}}"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "P_{f,g}(M)_N"] \\
\mathcal{F}_Z(M_{[f,g]}, M_{[f,g]}) \arrow[r, "\varphi_*"] & \mathcal{F}_Z(M_{[f,g]}, N)
\end{tikzcd}$$

It follows that $P_{f,g}(M)_N$ maps $g^*(\varphi)\iota_X(M)$ to φ. □

For a morphism $\varphi : L \to M$ of $\mathcal{F}_Y$, define a natural transformation $F_{f,g,\varphi} : F_{f,g,M} \to F_{f,g,L}$ by

$$(F_{f,g,\varphi})_N = f^*(\varphi)^* : F_{f,g,M}(N) = \mathcal{F}_X(f^*(M), g^*(N)) \to \mathcal{F}_X(f^*(L), g^*(N)) = F_{f,g,L}(N).$$

It is clear that $F_{f,g,\psi\varphi} = F_{f,g,\varphi}F_{f,g,\psi}$ for morphisms $\psi : M \to P$ and $\varphi : L \to M$ of $\mathcal{F}_Y$. If $F_{f,g,L}$ is also representable, we define a morphism $\varphi_{[f,g]} : L_{[f,g]} \to M_{[f,g]}$ of $\mathcal{F}_Z$ by

$$\varphi_{[f,g]} = P_{f,g}(L)_{M_{[f,g]}}((F_{f,g,\varphi})_{M_{[f,g]}}(\iota_{f,g}(M))) = P_{f,g}(L)_{M_{[f,g]}}(\iota_{f,g}(M)f^*(\varphi)) \in \hat{h}_{L_{[f,g]}}(M_{[f,g]}).$$

Proposition 8.4.3 *Let $\varphi : L \to M$ be a morphism of $\mathcal{F}_Y$.*

(1) The following diagrams commute for any $N \in \operatorname{Ob}\mathcal{F}_Z$.

$$\begin{tikzcd}
f^*(L) \arrow[r, "f^*(\varphi)"] \arrow[d, "\iota_{f,g}(L)"] & f^*(M) \arrow[d, "\iota_{f,g}(M)"] \\
g^*(L_{[f,g]}) \arrow[r, "g^*(\varphi_{[f,g]})"] & g^*(M_{[f,g]})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "f^*(\varphi)^*"] \arrow[d, "P_{f,g}(M)_N"] & \mathcal{F}_X(f^*(L), g^*(N)) \arrow[d, "P_{f,g}(L)_N"] \\
\mathcal{F}_Z(M_{[f,g]}, N) \arrow[r, "\varphi^*_{[f,g]}"] & \mathcal{F}_Z(L_{[f,g]}, N)
\end{tikzcd}$$

(2) For morphisms $\psi : M \to K$ and $\varphi : L \to M$ of $\mathcal{F}_Y$, we have $(\psi\varphi)_{[f,g]} = \psi_{[f,g]}\varphi_{[f,g]}$.

(3) If $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ preserves epimorphisms (f^* has a right adjoint, for example) and $\varphi : L \to M$ is an epimorphism, so is $\varphi_{[f,g]} : L_{[f,g]} \to M_{[f,g]}$.

Proof. (1) We have $P_{f,g}(L)_{M_{[f,g]}}(\iota_{f,g}(M)f^*(\varphi)) = \varphi_{[f,g]}$ by the definition of $\varphi_{[f,g]}$. On the other hand, $P_{f,g}(L)_{M_{[f,g]}}(g^*(\varphi_{[f,g]})\iota_{f,g}(L)) = \varphi_{[f,g]}$ by (8.4.2). Since $P_{f,g}(L)_{M_{[f,g]}}$ is bijective, the left diagram commutes.

For $\psi \in \mathcal{F}_Z(M_{[f,g]}, N)$, it follows from (8.4.2) and commutativity of the left diagram that we have

$$\begin{aligned}
f^*(\varphi)^*P_{f,g}(M)_N^{-1}(\psi) &= g^*(\psi)\iota_{f,g}(M)f^*(\varphi) = g^*(\psi)g^*(\varphi_{[f,g]})\iota_{f,g}(L) = g^*(\psi\varphi_{[f,g]})\iota_{f,g}(L) \\
&= P_{f,g}(L)_N^{-1}(\psi\varphi_{[f,g]}) = P_{f,g}(L)_N^{-1}\varphi^*_{[f,g]}(\psi).
\end{aligned}$$

Hence the right diagram commutes.

(2) The following diagram commutes by (1).

$$\begin{tikzcd}
\mathcal{F}_X(f^*(K), g^*(K_{[f,g]})) \arrow[r, "f^*(\psi)^*"] \arrow[d, "P_{f,g}(K)_{K_{[f,g]}}"] & \mathcal{F}_X(f^*(M), g^*(K_{[f,g]})) \arrow[r, "f^*(\varphi)^*"] \arrow[d, "P_{f,g}(M)_{K_{[f,g]}}"] & \mathcal{F}_X(f^*(L), g^*(K_{[f,g]}))) \arrow[d, "P_{f,g}(L)_{K_{[f,g]}}"] \\
\mathcal{F}_Z(K_{[f,g]}, K_{[f,g]}) \arrow[r, "\psi^*_{[f,g]}"] & \mathcal{F}_Z(M_{[f,g]}, K_{[f,g]}) \arrow[r, "\varphi^*_{[f,g]}"] & \mathcal{F}_Z(L_{[f,g]}, K_{[f,g]})
\end{tikzcd}$$

Hence, by the definition of $(\psi\varphi)_{[f,g]}$ we have

$$\begin{aligned}
\psi_{[f,g]}\varphi_{[f,g]} &= \varphi^*_{[f,g]}\psi^*_{[f,g]}(id_{K_{[f,g]}}) = \varphi^*_{[f,g]}\psi^*_{[f,g]}P_{f,g}(K)_{K_{[f,g]}}(\iota_{f,g}(K)) = P_{f,g}(L)_{K_{[f,g]}}f^*(\varphi)^*f^*(\psi)^*(\iota_{f,g}(K)) \\
&= P_{f,g}(L)_{K_{[f,g]}}(\iota_{f,g}(K)f^*(\varphi\psi)) = (\psi\varphi)_{[f,g]}.
\end{aligned}$$

(3) is a direct consequence of (1). □

Remark 8.4.4 *If $g^* : \mathcal{F}_Z \to \mathcal{F}_X$ has a left adjoint $g_* : \mathcal{F}_X \to \mathcal{F}_Z$, for a morphism $\varphi : L \to M$ of $\mathcal{F}_Y$, we have $\varphi_{[f,g]} = g_*f^*(\varphi) : L_{[f,g]} = g_*f^*(L) \to g_*f^*(M) = M_{[f,g]}$. In fact, if we denote by $\varepsilon_g : g^*g_* \to id_{\mathcal{F}_X}$ the counit of the adjunction $g_* \dashv g^*$, we have $\varphi_{[f,g]} = P_{f,g}(L)_{M_{[f,g]}}(\iota_{f,g}(M)f^*(\varphi)) = (\mathrm{ad}_g)^{-1}_{f^*(L),M_{[f,g]}}((\eta_g)_{f^*(M)}f^*(\varphi)) = (\varepsilon_g)_{g_*f^*(M)}g_*((\eta_g)_{f^*(M)})g_*f^*(\varphi) = g_*f^*(\varphi)$.*

Lemma 8.4.5 *Let $\xi : f^*(L) \to g^*(M)$ and $\zeta : f^*(N) \to g^*(K)$ be morphisms of $\mathcal{F}_X$ for morphisms $\varphi : L \to N$ and $\psi : M \to K$ of $\mathcal{F}_Y$ and $\mathcal{F}_Z$, respectively. We put $\hat{\xi} = P_{f,g}(L)_M(\xi)$ and $\hat{\zeta} = P_{f,g}(N)_K(\zeta)$. The following left diagram commutes if and only if the right one commutes.*

$$\begin{tikzcd} f^*(L) \arrow[r, "\xi"] \arrow[d, "f^*(\varphi)"] & g^*(M) \arrow[d, "g^*(\psi)"] \\ f^*(N) \arrow[r, "\zeta"] & g^*(K) \end{tikzcd} \qquad \begin{tikzcd} L_{[f,g]} \arrow[r, "\hat{\xi}"] \arrow[d, "\varphi_{[f,g]}"] & M \arrow[d, "\psi"] \\ N_{[f,g]} \arrow[r, "\hat{\zeta}"] & K \end{tikzcd}$$

Proof. The following diagram is commutative by (8.4.3).

$$\begin{tikzcd} \mathcal{F}_X(f^*(L), g^*(M)) \arrow[r, "g^*(\psi)_*"] \arrow[d, "P_{f,g}(L)_M"] & \mathcal{F}_X(f^*(L), g^*(K)) \arrow[d, "P_{f,g}(L)_K"] & \mathcal{F}_X(f^*(N), g^*(K)) \arrow[l, "f^*(\varphi)^*"'] \arrow[d, "P_{f,g}(N)_K"] \\ \mathcal{F}_Z(L_{[f,g]}, M) \arrow[r, "\psi_*"] & \mathcal{F}_Z(L_{[f,g]}, K) & \mathcal{F}_Z(N_{[f,g]}, K) \arrow[l, "\varphi^*_{[f,g]}"'] \end{tikzcd}$$

Since $\hat{\xi} = P_{f,g}(L)_M(\xi)$, $\hat{\zeta} = P_{f,g}(N)_K(\zeta)$ and $P_{f,g}(L)_K$ is bijective, $g^*(\psi)\xi = g^*(\psi)_*(\xi) = f^*(\varphi)^*(\zeta) = \zeta f^*(\varphi)$ if and only if $\psi\hat{\xi} = \psi_*(\hat{\xi}) = \varphi^*_{[f,g]}(\hat{\zeta}) = \hat{\zeta}\varphi_{[f,g]}$. □

For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$ of $\mathcal{E}$ and $M \in \mathrm{Ob}\,\mathcal{F}_Y$, suppose that $F_{f,g,M}$ and $F_{fk,gk,M}$ are representable. We define a morphism $M_k : M_{[fk,gk]} \to M_{[f,g]}$ of $\mathcal{F}_Z$ by

$$M_k = P_{fk,gk}(M)_{M_{[f,g]}}(k^\sharp_{M,M_{[f,g]}}(\iota_{f,g}(M))).$$

Proposition 8.4.6 *(1) The following diagrams commute for any $N \in \mathrm{Ob}\,\mathcal{F}_Z$.*

$$\begin{tikzcd} \mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "k^\sharp_{M,N}"] \arrow[d, "P_{f,g}(M)_N"] & \mathcal{F}_V((fk)^*(M), (gk)^*(N)) \arrow[d, "P_{fk,gk}(M)_N"] \\ \mathcal{F}_Z(M_{[f,g]}, N) \arrow[r, "M_k^*"] & \mathcal{F}_Z(M_{[fk,gk]}, N) \end{tikzcd} \qquad \begin{tikzcd} (fk)^*(M) \arrow[rr, "k^\sharp_{M,M_{[f,g]}}(\iota_{f,g}(M))"] \arrow[dr, "\iota_{fk,gk}(M)"'] & & (gk)^*(M_{[f,g]}) \\ & (gk)^*(M_{[fk,gk]}) \arrow[ur, "(gk)^*(M_k)"'] & \end{tikzcd}$$

(2) For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$, $h : U \to V$ and $M \in \mathrm{Ob}\,\mathcal{F}_Y$, suppose that $F_{f,g,M}$, $F_{fk,gk,M}$ and $F_{fkh,gkh,M}$ are representable. Then, we have $M_{kh} = M_k M_h$.

(3) The image of the identity morphism of $k^*(M)$ by $P_{k,k}(M)_M$ is $M_k : M_{[k,k]} \to M_{[id_X,id_X]} = M$ if $X = Y$.

(4) A composition $k^*(M) \xrightarrow{\iota_{k,k}(M)} k^*(M_{[k,k]}) \xrightarrow{k^*(M_k)} k^*(M_{[id_X,id_X]}) = k^*(M)$ is the identity morphism of $k^*(M)$ if $X = Y$.

Proof. (1) For $\varphi \in \mathcal{F}_Z(M_{[f,g]}, N)$, it follows from the naturality of $k^\sharp_{M,N}$ and (8.4.2) that we have

$$\begin{aligned} k^\sharp_{M,N}P_{f,g}(M)^{-1}_N(\varphi) &= k^\sharp_{M,N}(g^*(\varphi)\iota_{f,g}(M)) = k^\sharp_{M,N}g^*(\varphi)_*(\iota_{f,g}(M)) = (gk)^*(\varphi)_*k^\sharp_{M,M_{[f,g]}}(\iota_{f,g}(M)) \\ &= (gk)^*(\varphi)_*P_{fk,gk}(M)^{-1}_{M_{[f,g]}}(M_k) = (gk)^*(\varphi)(gk)^*(M_k)\iota_{fk,gk}(M) = (gk)^*(\varphi M_k)\iota_{fk,gk}(M) \\ &= (gk)^*(M_k^*(\varphi))\iota_{fk,gk}(M) = P_{fk,gk}(M)^{-1}_N M_k^*(\varphi). \end{aligned}$$

The commutativity of the right diagram follows from (8.4.2) and the commutativity of the left diagram for the case $N = M_{[f,g]}$.

(2) The following diagram commutes by (1). Hence the assertion follows from (8.1.14).

$$\begin{tikzcd} \mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "k^\sharp_{M,N}"] \arrow[d, "P_{f,g}(M)_N"] & \mathcal{F}_V((fk)^*(M), (gk)^*(N)) \arrow[r, "h^\sharp_{M,N}"] \arrow[d, "P_{fk,gk}(M)_N"] & \mathcal{F}_U((fkh)^*(M), (gkh)^*(N)) \arrow[d, "P_{fkh,gkh}(M)_N"] \\ \mathcal{F}_Z(M_{[f,g]}, N) \arrow[r, "M_k^*"] & \mathcal{F}_Z(M_{[fk,gk]}, N) \arrow[r, "M_h^*"] & \mathcal{F}_Z(M_{[fkh,gkh]}, N) \end{tikzcd}$$

(3) Apply (1) for $N = M$, $Z = Y = X$ and $f = g = id_X$.

(4) It follows from (8.4.2) that $P_{k,k}(M)_M : \mathcal{F}_V(k^*(M), k^*(M)) \to \mathcal{F}_X(M_{[k,k]}, M)$ maps $k^*(M_k)\iota_{k,k}(M)$ to $M_k : M_{[k,k]} \to M$. Thus the assertion follows from (3). □

Remark 8.4.7 *Suppose that the inverse image functors $g^* : \mathcal{F}_Z \to \mathcal{F}_X$ and $(gk)^* : \mathcal{F}_Z \to \mathcal{F}_V$ have left adjoints $g_* : \mathcal{F}_X \to \mathcal{F}_Z$ and $(gk)_* : \mathcal{F}_V \to \mathcal{F}_Z$, respectively.*

(1) Since $k^\sharp_{M,M_{[f,g]}}(\iota_{f,g}(M)) = c_{g,k}(M_{[f,g]})k^*\big((\eta_g)_{f^*(M)}\big)c_{f,k}(M)^{-1}$ by (8.4.1) and

$$P_{fk,gk}(M)_{M_{[f,g]}} = (\mathrm{ad}_{gk})^{-1}_{(fk)^*(M),M_{[f,g]}} : \mathcal{F}_V((fk)^*(M),(gk)^*(M_{[f,g]})) \to \mathcal{F}_Z(M_{[fk,gk]},M_{[f,g]})$$

maps $\varphi \in \mathcal{F}_V((fk)^*(M),(gk)^*(M_{[f,g]}))$ to $(\varepsilon_{gk})_{M_{[f,g]}}(gk)_*(\varphi)$, $M_k : M_{[fk,gk]} \to M_{[f,g]}$ coincides with the following composition.

$$\begin{aligned} M_{[fk,gk]} = (gk)_*(fk)^*(M) &\xrightarrow{(gk)_*(c_{f,k}(M))^{-1}} (gk)_*k^*f^*(M) \xrightarrow{(gk)_*k^*\left((\eta_g)_{f^*(M)}\right)} (gk)_*k^*g^*g_*f^*(M) \\ &= (gk)_*k^*g^*(M_{[f,g]}) \xrightarrow{(gk)_*(c_{g,k}(M_{[f,g]}))} (gk)_*(gk)^*(M_{[f,g]}) \xrightarrow{(\varepsilon_{gk})_{M_{[f,g]}}} M_{[f,g]} \end{aligned}$$

(2) The following diagram commutes by (8.4.6) if $X = Y = Z$ and $f = g = id_X$.

$$\begin{array}{ccc} \mathcal{F}_X(M_{[id_X,id_X]},M) & \xrightarrow{M_k^*} & \mathcal{F}_X(k_*(k^*(M)),M) \\ \big\downarrow{\scriptstyle (\mathrm{ad}_{id_X})_{id_X^*(M),M}} & & \big\downarrow{\scriptstyle (\mathrm{ad}_k)_{k^*(M),M}} \\ \mathcal{F}_X(id_X^*(M),id_X^*(M)) & \xrightarrow{k^\sharp_{M,M}} & \mathcal{F}_V(k^*(M),k^*(M)) \end{array}$$

Since id_X^* is the identity functor of $\mathcal{F}_X$, so is id_{X*}. Hence $M_{[k,k]} : k_*k^*(M) = M_{[k,k]} \to M_{[id_X,id_X]} = M$ is identified with the counit $(\varepsilon_k)_M : k_*k^*(M) \to M$ of the adjunction $k_* \dashv k^*$ by the above diagram.

Proposition 8.4.8 *For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$ of $\mathcal{E}$ and a morphism $\varphi : L \to M$ of $\mathcal{F}_Y$, the following diagram commutes.*

$$\begin{array}{ccc} L_{[fk,gk]} & \xrightarrow{L_k} & L_{[f,g]} \\ \big\downarrow{\scriptstyle \varphi_{[fk,gk]}} & & \big\downarrow{\scriptstyle \varphi_{[f,g]}} \\ M_{[fk,gk]} & \xrightarrow{M_k} & M_{[f,g]} \end{array}$$

Proof. The following diagram commutes by the naturality of $k^\sharp$.

$$\begin{array}{ccc} \mathcal{F}_X(f^*(M),g^*(N)) & \xrightarrow{k^\sharp_{M,N}} & \mathcal{F}_V((fk)^*(M),(gk)^*(N)) \\ \big\downarrow{\scriptstyle f^*(\varphi)^*} & & \big\downarrow{\scriptstyle (fk)^*(\varphi)^*} \\ \mathcal{F}_X(f^*(L),g^*(N)) & \xrightarrow{k^\sharp_{L,N}} & \mathcal{F}_V((fk)^*(L),(fk)^*(N)) \end{array}$$

Then, it follows from the commutativity of four diagrams

$$\begin{array}{ccc} \mathcal{F}_X(f^*(M),g^*(N)) & \xrightarrow{P_{f,g}(M)_N} & \mathcal{F}_Z(M_{[f,g]},N) \\ \big\downarrow{\scriptstyle f^*(\varphi)^*} & & \big\downarrow{\scriptstyle (\varphi_{[f,g]})^*} \\ \mathcal{F}_X(f^*(L),g^*(N)) & \xrightarrow{P_{f,g}(L)_N} & \mathcal{F}_Z(L_{[f,g]},N) \end{array} \qquad \begin{array}{ccc} \mathcal{F}_V((fk)^*(M),(gk)^*(N)) & \xrightarrow{P_{fk,gk}(M)_N} & \mathcal{F}_Z(M_{[fk,gk]},N) \\ \big\downarrow{\scriptstyle (fk)^*(\varphi)^*} & & \big\downarrow{\scriptstyle (\varphi_{[fk,gk]})^*} \\ \mathcal{F}_V((fk)^*(L),(gk)^*(N)) & \xrightarrow{P_{fk,gk}(L)_N} & \mathcal{F}_Z(L_{[fk,gk]},N) \end{array}$$

$$\begin{array}{ccc} \mathcal{F}_X(f^*(M),g^*(N)) & \xrightarrow{P_{f,g}(M)_N} & \mathcal{F}_Z(M_{[f,g]},N) \\ \big\downarrow{\scriptstyle k^\sharp_{M,N}} & & \big\downarrow{\scriptstyle M_k^*} \\ \mathcal{F}_V((fk)^*(M),(gk)^*(N)) & \xrightarrow{P_{fk,gk}(M)_N} & \mathcal{F}_Z(M_{[fk,gk]},N) \end{array} \qquad \begin{array}{ccc} \mathcal{F}_X(f^*(L),g^*(N)) & \xrightarrow{P_{f,g}(L)_N} & \mathcal{F}_Z(L_{[f,g]},N) \\ \big\downarrow{\scriptstyle k^\sharp_{L,N}} & & \big\downarrow{\scriptstyle L_k^*} \\ \mathcal{F}_V((fk)^*(L),(gk)^*(N)) & \xrightarrow{P_{fk,gk}(L)_N} & \mathcal{F}_Z(L_{[fk,gk]},N) \end{array}$$

and the fact that $P_{f,g}(M)_N : \mathcal{F}_X(f^*(M),g^*(N)) \to \mathcal{F}_Z(M_{[f,g]},N)$ is bijective that the following diagram commutes for any $N \in \mathrm{Ob}\,\mathcal{F}_1$.

$$\begin{array}{ccc} \mathcal{F}_Z(M_{[f,g]},N) & \xrightarrow{M_k^*} & \mathcal{F}_Z(M_{[fk,gk]},N) \\ \big\downarrow{\scriptstyle \varphi^*_{[f,g]}} & & \big\downarrow{\scriptstyle \varphi^*_{[fk,gk]}} \\ \mathcal{F}_Z(L_{[f,g]},N) & \xrightarrow{L_k^*} & \mathcal{F}_Z(L_{[fk,gk]},N) \end{array}$$

Thus the assertion follows. □

Remark 8.4.9 *We denote by $\varphi_{[f,g],k} : L_{[fk,gk]} \to M_{[f,g]}$ the composition $M_k\varphi_{[fk,gk]} = \varphi_{[f,g]}L_k$. For morphisms $i : W \to Z$, $j : W \to T$, $h : U \to W$ of $\mathcal{E}$, it follows from (8.4.8) that the following diagram commutes.*

$$\begin{tikzcd}
(M_{[fk,gk]})_{[ih,jh]} \arrow[r, "(M_{[fk,gk]})_h"] \arrow[d, "(M_k)_{[ik,jk]}"] & (M_{[fk,gk]})_{[i,j]} \arrow[d, "(M_k)_{[i,j]}"] \\
(M_{[f,g]})_{[ih,jh]} \arrow[r, "(M_{[f,g]})_h"] & (M_{[f,g]})_{[i,j]}
\end{tikzcd}$$

Namely, we have $(M_k)_{[i,j],h} = (M_{[f,g]})_h(M_k)_{[ih,jh]} = (M_k)_{[i,j]}(M_{[fk,gk]})_h$ which we denote by $(M_k)_h$ for short.

For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$ of $\mathcal{E}$ and $M \in \operatorname{Ob}\mathcal{F}_Y$, we define a morphism $\delta_{f,g,h,M} : M_{[f,h]} \to (M_{[f,g]})_{[g,h]}$ of $\mathcal{F}_W$ to be the image of $\iota_{g,h}(M_{[f,g]})\iota_{f,g}(M) \in \mathcal{F}_X(f^*(M), h^*((M_{[f,g]})_{[g,h]}))$ by

$$P_{f,h}(M)_{(M_{[f,g]})_{[g,h]}} : \mathcal{F}_X(f^*(M), h^*((M_{[f,g]})_{[g,h]})) \to \mathcal{F}_W(M_{[f,h]}, (M_{[f,g]})_{[g,h]}).$$

Proposition 8.4.10 *The following diagram commutes for any $N \in \operatorname{Ob}\mathcal{F}_W$.*

$$\begin{tikzcd}
\mathcal{F}_X(g^*(M_{[f,g]}), h^*(N)) \arrow[r, "\iota_{f,g}(M)^*"] \arrow[d, "P_{g,h}(M_{[f,g]})_N"] & \mathcal{F}_X(f^*(M), h^*(N)) \arrow[d, "P_{f,h}(M)_N"] \\
\mathcal{F}_W((M_{[f,g]})_{[g,h]}, N) \arrow[r, "\delta^*_{f,g,h,M}"] & \mathcal{F}_W(M_{[f,h]}, N)
\end{tikzcd}$$

Proof. For $\varphi \in \mathcal{F}_W((M_{[f,g]})_{[g,h]}, N)$, by the definition of $\delta_{f,g,h,M}$ and the naturality of $P_X(M)$, we have

$$\begin{aligned}
\iota_{f,g}(M)^*P_{g,h}(M_{[f,g]})_N^{-1}(\varphi) &= h^*(\varphi)\iota_{g,h}(M_{[f,g]})\iota_{f,g}(M) = h^*(\varphi)_*P_{f,h}(M)^{-1}_{(M_{[f,g]})_{[g,h]}}(\delta_{f,g,h,M}) \\
&= P_{f,h}(M)_N^{-1}\varphi_*(\delta_{f,g,h,M}) = P_{f,h}(M)_N^{-1}\delta^*_{f,g,h,M}(\varphi).
\end{aligned}$$

□

We note that $\delta_{f,g,h,M} : M_{[f,h]} \to (M_{[f,g]})_{[g,h]}$ is the unique morphism that makes the diagram of (8.4.10) commute for any $N \in \operatorname{Ob}\mathcal{F}_W$.

Remark 8.4.11 *If $g^* : \mathcal{F}_Z \to \mathcal{F}_X$ and $h^* : \mathcal{F}_W \to \mathcal{F}_X$ have left adjoints $g_* : \mathcal{F}_X \to \mathcal{F}_Z$ and $h_* : \mathcal{F}_X \to \mathcal{F}_W$, the following diagram is commutative for any $N \in \operatorname{Ob}\mathcal{F}_W$ by the naturality of ad_h.*

$$\begin{tikzcd}
\mathcal{F}_X(g^*g_*f^*(M), h^*(N)) \arrow[r, "(\eta_g)^*_{f^*(M)}"] \arrow[d, "(\mathrm{ad}_h)^{-1}_{g^*g_*f^*(M),N}"] & \mathcal{F}_X(f^*(M), h^*(N)) \arrow[d, "(\mathrm{ad}_h)^{-1}_{f^*(M),N}"] \\
\mathcal{F}_W(h_*g^*g_*f^*(M), N) \arrow[r, "h_*\left((\eta_g)_{f^*(M)}\right)^*"] & \mathcal{F}_W(h_*f^*(M), N)
\end{tikzcd}$$

It follows that $\delta_{f,g,h,M} = h_*\big((\eta_g)_{f^*(M)}\big)$.

Proposition 8.4.12 *For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$, $k : V \to X$ of $\mathcal{E}$ and a morphism $\varphi : L \to M$ of $\mathcal{F}_Y$, the following diagrams are commutative.*

$$\begin{tikzcd}
L_{[f,h]} \arrow[r, "\delta_{f,g,h,L}"] \arrow[d, "\varphi_{[f,h]}"] & (L_{[f,g]})_{[g,h]} \arrow[d, "(\varphi_{[f,g]})_{[g,h]}"] \\
M_{[f,h]} \arrow[r, "\delta_{f,g,h,M}"] & (M_{[f,g]})_{[g,h]}
\end{tikzcd}
\qquad
\begin{tikzcd}
M_{[fk,hk]} \arrow[r, "\delta_{fk,gk,hk,M}"] \arrow[d, "M_k"] & (M_{[fk,gk]})_{[gk,hk]} \arrow[d, "(M_k)_k"] \\
M_{[f,h]} \arrow[r, "\delta_{f,g,h,M}"] & (M_{[f,g]})_{[g,h]}
\end{tikzcd}$$

Proof. The following diagram is commutative for any $N \in \operatorname{Ob}\mathcal{F}_W$ by (1) of (8.4.3).

$$\begin{tikzcd}
\mathcal{F}_X(g^*((M_{[f,g]}), h^*(N)) \arrow[r, "\iota_{f,g}(M)^*"] \arrow[d, "g^*(\varphi_{[f,g]})^*"] & \mathcal{F}_X(f^*(M), h^*(N)) \arrow[d, "f^*(\varphi)^*"] \\
\mathcal{F}_X(g^*((L_{[f,g]}), h^*(N)) \arrow[r, "\iota_{f,g}(L)^*"] & \mathcal{F}_X(f^*(L), h^*(N))
\end{tikzcd}$$

Hence the following diagram commutes by (8.4.10) and (1) of (8.4.3).

$$\begin{tikzcd} \mathcal{F}_W((M_{[f,g]})_{[g,h]}, N) \arrow[r, "\delta^*_{f,g,h,M}"] \arrow[d, "(\varphi_{[f,g]})^*_{[g,h]}"] & \mathcal{F}_1(M_{[f,h]}, N) \arrow[d, "\varphi^*_{[f,h]}"] \\ \mathcal{F}_W((L_{[f,g]})_{[g,h]}, N) \arrow[r, "\delta^*_{f,g,h,L}"] & \mathcal{F}_W(L_{[f,h]}, N) \end{tikzcd}$$

Thus the left diagram is commutative.

For $N \in \mathrm{Ob}\,\mathcal{F}_W$ and $\xi \in \mathcal{F}_X(g^*(M_{[f,g]}), h^*(N))$, it follows from (8.4.6) and (8.1.13) that we have

$$k^\sharp_{M_{[f,g]},N}(\xi)(gk)^*(M_k)\iota_{fk,gk}(M) = k^\sharp_{M_{[f,g]},N}(\xi)k^\sharp_{M,M_{[f,g]}}(\iota_{f,g}(M)) = k^\sharp_{M,N}(\xi\iota_{f,g}(M)).$$

This shows that the following diagram commutes.

$$\begin{tikzcd} \mathcal{F}_X(g^*(M_{[f,g]}), h^*(N)) \arrow[r, "\iota_{f,g}(M)^*"] \arrow[d, "(gk)^*(M_k)^* k^\sharp_{M_{[f,g]},N}"] & \mathcal{F}_X(f^*(M), h^*(N)) \arrow[d, "k^\sharp_{M,N}"] \\ \mathcal{F}_V((gk)^*(M_{[fk,gk]}), (hk)^*(N)) \arrow[r, "\iota_{fk,gk}(M)^*"] & \mathcal{F}_V((fk)^*(M), (hk)^*(N)) \end{tikzcd}$$

The following diagram commutes by (1) of (8.4.3) and (8.4.6).

$$\begin{tikzcd} \mathcal{F}_X(g^*(M_{[f,g]}), h^*(N)) \arrow[r, "k^\sharp_{M_{[f,g]},N}"] \arrow[d, "P_{g,h}(M_{[f,g]})_N"] & \mathcal{F}_V((gk)^*(M_{[f,g]}), (hk)^*(N)) \arrow[r, "(gk)^*(M_k)^*"] \arrow[d, "P_{gk,hk}(M_{[f,g]})_N"] & \mathcal{F}_V((gk)^*(M_{[fk,gk]}), (hk)^*(N)) \arrow[d, "P_{gk,hk}(M_{[fk,gk]})_N"] \\ \mathcal{F}_W((M_{[f,g]})_{[g,h]}, N) \arrow[r, "(M_{[f,g]})^*_k"] & \mathcal{F}_W((M_{[f,g]})_{[gk,hk]}, N) \arrow[r, "(M_k)^*_{[gk,hk]}"] & \mathcal{F}_W((M_{[fk,gk]})_{[gk,hk]}, N) \end{tikzcd}$$

Since $(M_k)_k = (M_{[f,g]})_h(M_k)_{[gk,hk]}$, it follows from (8.4.10) and (1) of (8.4.6) that the following diagram commutes for any $N \in \mathrm{Ob}\,\mathcal{F}_W$.

$$\begin{tikzcd} \mathcal{F}_W((M_{[f,g]})_{[g,h]}, N) \arrow[r, "\delta^*_{f,g,h,M}"] \arrow[d, "(M_k)^*_k"] & \mathcal{F}_W(M_{[f,h]}, N) \arrow[d, "M^*_k"] \\ \mathcal{F}_W((M_{[fk,gk]})_{[gk,hk]}, N) \arrow[r, "\delta^*_{fk,gk,hk,M}"] & \mathcal{F}_W(M_{[fk,hk]}, N) \end{tikzcd}$$

Thus the right diagram is also commutative. □

Proposition 8.4.13 *For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$, $i : X \to V$ of $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, the following diagrams are commutative.*

$$\begin{tikzcd} f^*(M) \arrow[r, "\iota_{f,g}(M)"] \arrow[d, "\iota_{f,h}(M)"] & g^*(M_{[f,g]}) \arrow[d, "\iota_{g,h}(M_{[f,g]})"] \\ h^*(M_{[f,h]}) \arrow[r, "h^*(\delta_{f,g,h,M})"] & h^*((M_{[f,g]})_{[g,h]}) \end{tikzcd} \qquad \begin{tikzcd} M_{[f,i]} \arrow[r, "\delta_{f,g,i,M}"] \arrow[d, "\delta_{f,h,i,M}"] & (M_{[f,g]})_{[g,i]} \arrow[d, "\delta_{g,h,i,M_{[f,g]}}"] \\ (M_{[f,h]})_{[h,i]} \arrow[r, "(\delta_{f,g,h,M})_{[h,i]}"] & ((M_{[f,g]})_{[g,h]})_{[h,i]} \end{tikzcd}$$

Proof. It follows from the definition of $\delta_{f,g,h,M}$ and (8.4.2) that

$$\iota_{g,h}(M_{[f,g]})\iota_{f,g}(M) = P_{f,h}(M)^{-1}_{(M_{[f,g]})_{[g,h]}}(\delta_{f,g,h,M}) = h^*(\delta_{f,g,h,M})\iota_{f,h}(M).$$

Hence the following diagram commutes for $N \in \mathrm{Ob}\,\mathcal{F}_V$.

$$\begin{tikzcd} \mathcal{F}_X(h^*((M_{[f,g]})_{[g,h]}), i^*(N)) \arrow[r, "h^*(\delta_{f,g,h,M})^*"] \arrow[d, "\iota_{g,h}(M_{[f,g]})^*"] & \mathcal{F}_X(h^*(M_{[f,h]}), i^*(N)) \arrow[d, "\iota_{f,h}(M)^*"] \\ \mathcal{F}_X(g^*(M_{[f,g]}), i^*(N)) \arrow[r, "\iota_{f,g}(M)^*"] & \mathcal{F}_X(f^*(M), i^*(N)) \end{tikzcd}$$

Therefore the following diagram commutes by (8.4.10) and (1) of (8.4.3).

$$\begin{tikzcd}
\mathcal{F}_V(((M_{[f,g]})_{[g,h]})_{[h,i]}, N) \arrow[r, "(\delta_{f,g,h,M})^*_{[h,i]}"] \arrow[d, "\delta^*_{g,h,i,M_{[f,g]}}"] & \mathcal{F}_V((M_{[f,h]})_{[h,i]}, N) \arrow[d, "\delta^*_{f,h,i,M}"] \\
\mathcal{F}_V((M_{[f,g]})_{[g,i]}, N) \arrow[r, "\delta^*_{f,g,i,M}"] & \mathcal{F}_V(M_{[f,i]}, N)
\end{tikzcd}$$

□

Proposition 8.4.14 *For morphisms $f : X \to Y$, $g : X \to Z$ of $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, the following compositions coincide with the identity morphism of $M_{[f,g]}$.*

$$M_{[f,g]} \xrightarrow{\delta_{f,g,g,M}} (M_{[f,g]})_{[g,g]} \xrightarrow{(M_{[f,g]})_g} (M_{[f,g]})_{[id_Z, id_Z]} = M_{[f,g]}$$

$$M_{[f,g]} \xrightarrow{\delta_{f,f,g,M}} (M_{[f,f]})_{[f,g]} \xrightarrow{(M_f)_{[f,g]}} (M_{[id_Y, id_Y]})_{[f,g]} = M_{[f,g]}$$

Proof. The following diagram commutes for any $N \in \operatorname{Ob} \mathcal{F}_Z$ by (1) of (8.4.6) and (8.4.10).

$$\begin{tikzcd}
\mathcal{F}_Z(id_Z^*(M_{[f,g]}), id_Z^*(N)) \arrow[r, "g^\sharp_{M_{[f,g]},N}"] \arrow[d, "P_{id_Z,id_Z}(M_{[f,g]})_N"] & \mathcal{F}_X(g^*(M_{[f,g]}), g^*(N)) \arrow[r, "\iota_{f,g}(M)^*"] \arrow[d, "P_{g,g}(M_{[f,g]})_N"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "P_{f,g}(M)_N"] \\
\mathcal{F}_Z((M_{[f,g]})_{[id_Z,id_Z]}, N) \arrow[r, "(M_{[f,g]})_g^*"] & \mathcal{F}_Z((M_{[f,g]})_{[g,g]}, N) \arrow[r, "\delta^*_{f,g,g,M}"] & \mathcal{F}_Z(M_{[f,g]}, N)
\end{tikzcd}$$

It follows from (8.4.2) that $\delta^*_{f,g,g,M}(M_{[f,g]})_g^* : \mathcal{F}_Z(M_{[f,g]}, N) = \mathcal{F}_Z((M_{[f,g]})_{[id_Z,id_Z]}, N) \to \mathcal{F}_Z(M_{[f,g]}, N)$ is the identity map of $\mathcal{F}_Z(M_{[f,g]}, N)$.

The following diagram commutes for any $N \in \operatorname{Ob} \mathcal{F}_Z$ by (1) of (8.4.3) and and (8.4.10).

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M_{[id_Y,id_Y]}), g^*(N)) \arrow[r, "f^*(M_f)^*"] \arrow[d, "P_{f,g}(M_{[id_Y,id_Y]})_N"] & \mathcal{F}_X(f^*(M_{[f,f]}), g^*(N)) \arrow[r, "\iota_{f,f}(M)^*"] \arrow[d, "P_{f,g}(M_{[f,f]})_N"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "P_{f,g}(M)_N"] \\
\mathcal{F}_Z((M_{[id_Y,id_Y]})_{[f,g]}, N) \arrow[r, "(M_f)^*_{[f,g]}"] & \mathcal{F}_Z((M_{[f,f]})_{[f,g]}, N) \arrow[r, "\delta^*_{f,f,g,M}"] & \mathcal{F}_Z(M_{[f,g]}, N)
\end{tikzcd}$$

Since the composition of the upper horizontal maps of the above diagram coincides with the identity map of $\mathcal{F}_X(o_X^*(M), o_X^*(N))$ by (4) of (8.4.6), the composition of the lower horizontal maps of the above diagram is the identity map of $\mathcal{F}_Z(M_{[f,g]}, N)$. □

Let $f : X \to Y$, $g : X \to Z$, $h : X \to W$ be morphisms of $\mathcal{E}$ and L, M, N objects of $\mathcal{F}_Y$, $\mathcal{F}_Z$, $\mathcal{F}_W$, respectively. We define a map

$$\gamma^{f,g,h}_{L,M,N} : \mathcal{F}_Z(L_{[f,g]}, M) \times \mathcal{F}_W(M_{[g,h]}, N) \to \mathcal{F}_W(L_{[f,h]}, N)$$

as follows. For $\varphi \in \mathcal{F}_Z(L_{[f,g]}, M)$ and $\psi \in \mathcal{F}_W(M_{[g,h]}, N)$, let $\gamma^{f,g,h}_{L,M,N}(\varphi, \psi)$ be the following composition.

$$L_{[f,h]} \xrightarrow{\delta_{f,g,h,L}} (L_{[f,g]})_{[g,h]} \xrightarrow{\varphi_{[g,h]}} M_{[g,h]} \xrightarrow{\psi} N$$

Proposition 8.4.15 *The following diagram is commutative.*

$$\begin{tikzcd}
\mathcal{F}_X(f^*(L), g^*(M)) \times \mathcal{F}_X(g^*(M), h^*(N)) \arrow[r, "composition"] \arrow[d, "P_{f,g}(L)_M \times P_{g,h}(M)_N"] & \mathcal{F}_X(f^*(L), h^*(N)) \arrow[d, "P_{f,h}(L)_N"] \\
\mathcal{F}_Z(L_{[f,g]}, M) \times \mathcal{F}_W(M_{[g,h]}, N) \arrow[r, "\gamma^{f,g,h}_{L,M,N}"] & \mathcal{F}_W(L_{[f,h]}, N)
\end{tikzcd}$$

Proof. For $\zeta \in \mathcal{F}_X(f^*(L), g^*(M))$ and $\xi \in \mathcal{F}_X(g^*(M), h^*(N))$, we put $\varphi = P_{f,g}(L)_M(\zeta)$ and $\psi = P_{g,h}(M)_N(\xi)$. Then, we have $\psi\varphi_{[g,h]} = P_{[g,h]}(L_{[f,g]})_N(\xi g^*(\varphi))$ by (8.4.3). It follows from (8.4.10) and (8.4.2) that

$$\psi\varphi_{[g,h]}\delta_{f,g,h,L} = \delta^*_{f,g,h,L} P_{g,h}(L_{[f,g]})_N(\xi g^*(\varphi)) = P_{f,h}(L)_N(\xi g^*(\varphi)\iota_{f,g}(L)) = P_{f,h}(L)_N(\xi\zeta).$$

Thus the result follows. □

We define a poset $\mathcal{P}$ as follows. Set $\operatorname{Ob}\mathcal{P} = \{0,1,2,3,4,5\}$ and $\mathcal{P}(i,j)$ is not an empty set if and only if $i=j$ or $i=0$ or $(i,j) = (1,3),(1,4),(2,4),(2,5)$. We put $\mathcal{P}(i,j) = \{\tau_{ij}\}$ if $\mathcal{P}(i,j)$ is not empty. For a functor $D : \mathcal{P} \to \mathcal{E}$ and an object M of $\mathcal{F}_{D(3)}$, we put $D(\tau_{ij}) = f_{ij}$ and define a morphism

$$\theta_D(M) : M_{[f_{13}f_{01},f_{25}f_{02}]} \to (M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}$$

of $\mathcal{F}_{D(5)}$ to be the following composition.

$$M_{[f_{13}f_{01},f_{25}f_{02}]} \xrightarrow{\delta_{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02},M}} (M_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24}f_{02},f_{25}f_{02}]} \xrightarrow{(M_{f_{01}})_{f_{02}}} (M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}$$

Proposition 8.4.16 *The following diagram is commutative.*

$$\begin{tikzcd}
(f_{13}f_{01})^*(M) \arrow[r, "f_{01}^\sharp(\iota_{f_{13},f_{14}}(M))"] \arrow[d, "\iota_{f_{13}f_{01},f_{25}f_{02}}(M)"] & (f_{14}f_{01})^*(M_{[f_{13},f_{14}]}) \arrow[r, equal] & (f_{24}f_{02})^*(M_{[f_{13},f_{14}]}) \arrow[d, "f_{02}^\sharp(\iota_{f_{24},f_{25}}(M_{[f_{13},f_{14}]}))"] \\
(f_{25}f_{02})^*(M_{[f_{13}f_{01},f_{25}f_{02}]}) \arrow[rr, "(f_{25}f_{02})^*(\theta_D(M))"] & & (f_{25}f_{02})^*((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})
\end{tikzcd}$$

Proof. By the naturality of $P_{f_{13}f_{01},f_{25}f_{02}}(M)$, $\theta_D(M)$ is the image of

$$(f_{25}f_{02})^*((M_{f_{01}})_{f_{02}})\iota_{f_{14}f_{01},f_{25}f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]})\iota_{f_{13}f_{01},f_{14}f_{01}}(M) : (f_{13}f_{01})^*(M) \to (f_{25}f_{02})^*((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})$$

by $P_{f_{13}f_{01},f_{25}f_{02}}(M)_{(M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}}$. Hence the following equality holds by (8.4.2).

$$(f_{25}f_{02})^*(\theta_D(M))\iota_{f_{13}f_{01},f_{25}f_{02}}(M) = (f_{25}f_{02})^*((M_{f_{01}})_{f_{02}})\iota_{f_{14}f_{01},f_{25}f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]})\iota_{f_{13}f_{01},f_{14}f_{01}}(M) \quad \cdots (*)$$

It follows from (8.4.6), (8.1.10) and (8.4.3) that we have

$$\begin{aligned}
&(f_{25}f_{02})^*((M_{f_{01}})_{f_{02}})\iota_{f_{24}f_{02},f_{25}f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]}) \\
&=(f_{25}f_{02})^*((M_{f_{01}})_{[f_{24},f_{25}]})(f_{25}f_{02})^*((M_{[f_{13}f_{01},f_{14}f_{01}]})_{f_{02}})\iota_{f_{24}f_{02},f_{25}f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]}) \\
&=(f_{25}f_{02})^*((M_{f_{01}})_{[f_{24},f_{25}]})f_{02}^\sharp(\iota_{f_{24},f_{25}}(M_{[f_{13}f_{01},f_{14}f_{01}]})) \\
&=(f_{25}f_{02})^*((M_{f_{01}})_{[f_{24},f_{25}]})c_{f_{25},f_{02}}((M_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24},f_{25}]})f_{02}^*(\iota_{f_{24},f_{25}}(M_{[f_{13}f_{01},f_{14}f_{01}]}))c_{f_{24},f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]})^{-1} \\
&=c_{f_{25},f_{02}}((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})f_{02}^*(f_{25}^*((M_{f_{01}})_{[f_{24},f_{25}]}))f_{02}^*(\iota_{f_{24},f_{25}}(M_{[f_{13}f_{01},f_{14}f_{01}]}))c_{f_{24},f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]})^{-1} \\
&=c_{f_{25},f_{02}}((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})f_{02}^*(\iota_{f_{24},f_{25}}(M_{[f_{13},f_{14}]}))f_{02}^*(f_{24}^*(M_{f_{01}}))c_{f_{24},f_{02}}(M_{[f_{13}f_{01},f_{14}f_{01}]})^{-1} \\
&=c_{f_{25},f_{02}}((M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})f_{02}^*(\iota_{f_{24},f_{25}}(M_{[f_{13},f_{14}]}))c_{f_{24},f_{02}}(M_{[f_{13},f_{14}]})^{-1}(f_{24}f_{02})^*(M_{f_{01}}) \\
&=f_{02}^\sharp(\iota_{f_{24},f_{25}}(M_{[f_{13},f_{14}]}))(f_{24}f_{02})^*(M_{f_{01}})
\end{aligned}$$

Therefore we have

$$(*) = f_{02}^\sharp(\iota_{f_{24},f_{25}}(M_{[f_{13},f_{14}]}))(f_{24}f_{02})^*(M_{f_{01}})\iota_{f_{13}f_{01},f_{14}f_{01}}(M) = f_{02}^\sharp(\iota_{f_{24},f_{25}}(M_{[f_{13},f_{14}]}))f_{01}^\sharp(\iota_{f_{13},f_{14}}(M))$$

which implies the assertion. □

Proposition 8.4.17 *For a morphism $\varphi : L \to M$ of $\mathcal{F}_Y$, the following diagram commutes.*

$$\begin{tikzcd}
L_{[f_{13}f_{01},f_{25}f_{02}]} \arrow[r, "\theta_D(L)"] \arrow[d, "\varphi_{[f_{13}f_{01},f_{25}f_{02}]}"] & (L_{[f_{13},f_{14}]})_{[f_{24},f_{25}]} \arrow[d, "(\varphi_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}"] \\
M_{[f_{13}f_{01},f_{25}f_{02}]} \arrow[r, "\theta_D(M)"] & (M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}
\end{tikzcd}$$

Proof. The following diagram commutes by (8.4.12), (8.4.8), (8.4.3) and (8.4.6).

$$\begin{tikzcd}
L_{[f_{13}f_{01},f_{25}f_{02}]} \arrow[r, "\delta_{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02},L}"] \arrow[d, "\varphi_{[f_{13}f_{01},f_{25}f_{02}]}"] & (L_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24}f_{02},f_{25}f_{02}]} \arrow[r, "(L_{f_{01}})_{f_{02}}"] \arrow[d, "(\varphi_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24}f_{02},f_{25}f_{02}]}"] & (L_{[f_{13},f_{14}]})_{[f_{24},f_{25}]} \arrow[d, "(\varphi_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}"] \\
M_{[f_{13}f_{01},f_{25}f_{02}]} \arrow[r, "\delta_{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02},M}"] & (M_{[f_{13}f_{01},f_{14}f_{01}]})_{[f_{24}f_{02},f_{25}f_{02}]} \arrow[r, "(M_{f_{01}})_{f_{02}}"] & (M_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}
\end{tikzcd}$$

Hence the assertion follows. $\square$

Proposition 8.4.18 *Let $E : \mathcal{P} \to \mathcal{E}$ be a functor which satisfies $E(i) = D(i)$ for $i = 3, 4, 5$ and a natural transformation $\lambda : D \to E$ which satisfies $\lambda_i = id_{D(i)}$ for $i = 3, 4, 5$. We put $E(\tau_{ij}) = g_{ij}$, then the following diagram commutes.*

$$
\begin{tikzcd}
M_{[f_{13}f_{01}, f_{25}f_{02}]} \arrow[r, "\theta_D(M)"] \arrow[d, "M_{\lambda_0}"] & (M_{[f_{13}, f_{14}]})_{[f_{24}, f_{25}]} \arrow[d, "(M_{\lambda_1})_{\lambda_2}"] \\
M_{[g_{13}g_{01}, g_{25}g_{02}]} \arrow[r, "\theta_E(M)"] & (M_{[g_{13}, g_{14}]})_{[g_{24}, g_{25}]}
\end{tikzcd}
\qquad
\begin{tikzcd}
D(i) \arrow[r, "f_{ij}"] \arrow[d, "\lambda_i"] & D(j) \arrow[d, "\lambda_j"] \\
E(i) \arrow[r, "g_{ij}"] & E(j)
\end{tikzcd}
$$

Proof. Since $f_{ij} = g_{ij}\lambda_i$ for $i = 1, 2$, we have $f_{13}f_{01} = g_{13}\lambda_1 f_{01} = g_{13}g_{01}\lambda_0$, $f_{14}f_{01} = g_{14}\lambda_1 f_{01} = g_{14}g_{01}\lambda_0$ and $f_{25}f_{02} = g_{25}\lambda_2 f_{02} = g_{25}g_{02}\lambda_0$. It follows from (8.4.6), (8.4.8) and (8.4.12) that

$$
\begin{tikzcd}
M_{[f_{13}f_{01}, f_{25}f_{02}]} \arrow[r, "\delta_{f_{13}f_{01}, f_{14}f_{01}, f_{25}f_{02}, M}"] \arrow[d, "M_{\lambda_0}"] & (M_{[f_{13}f_{01}, f_{14}f_{01}]})_{[f_{24}f_{02}, f_{25}f_{02}]} \arrow[r, "(M_{f_{01}})_{f_{02}}"] \arrow[d, "(M_{\lambda_0})_{\lambda_0}"] & (M_{[f_{13}, f_{14}]})_{[f_{24}, f_{25}]} \arrow[d, "(M_{\lambda_1})_{\lambda_2}"] \\
M_{[g_{13}g_{01}, g_{25}g_{02}]} \arrow[r, "\delta_{g_{13}g_{01}, g_{14}g_{01}, g_{25}g_{02}, M}"] & (M_{[g_{13}g_{01}, g_{14}g_{01}]})_{[g_{24}g_{02}, g_{25}g_{02}]} \arrow[r, "(M_{g_{01}})_{g_{02}}"] & (M_{[g_{13}, g_{14}]})_{[g_{24}, g_{25}]}
\end{tikzcd}
$$

is commutative. $\square$

For morphisms $f : X \to Y$, $g : X \to Z$, $h : V \to Z$, $i : V \to W$ of $\mathcal{E}$, let $X \xleftarrow{\mathrm{pr}_X} X \times_Z V \xrightarrow{\mathrm{pr}_V} V$ be a limit of a diagram $X \xrightarrow{g} Z \xleftarrow{h} V$. We define a functor $D_{f,g,h,i} : \mathcal{P} \to \mathcal{E}$ by $D_{f,g,h,i}(0) = X \times_Z V$, $D_{f,g,h,i}(1) = X$, $D_{f,g,h,i}(2) = V$, $D_{f,g,h,i}(3) = Y$, $D_{f,g,h,i}(4) = Z$, $D_{f,g,h,i}(5) = W$ and $D_{f,g,h,i}(\tau_{01}) = \mathrm{pr}_X$, $D_{f,g,h,i}(\tau_{02}) = \mathrm{pr}_V$, $D_{f,g,h,i}(\tau_{13}) = f$, $D_{f,g,h,i}(\tau_{14}) = g$, $D_{f,g,h,i}(\tau_{24}) = h$, $D_{f,g,h,i}(\tau_{25}) = i$. For an object M of $\mathcal{F}_Y$, we denote $\theta_{D_{f,g,h,i}}(M)$ by $\theta_{f,g,h,i}(M)$. The following facts are special cases of (8.4.17) and (8.4.18).

Proposition 8.4.19 *Let $f : X \to Y$, $g : X \to Z$, $h : V \to Z$, $i : V \to W$, $j : S \to X$, $k : T \to V$ be morphisms of $\mathcal{E}$ and $\varphi : L \to M$ a morphism of $\mathcal{F}_Y$. The following diagrams are commutative.*

$$
\begin{tikzcd}
L_{[f\mathrm{pr}_X,\, i\mathrm{pr}_V]} \arrow[r, "\theta_{f,g,h,i}(L)"] \arrow[d, "\varphi_{[f\mathrm{pr}_X,\, i\mathrm{pr}_V]}"] & (L_{[f,g]})_{[h,i]} \arrow[d, "(\varphi_{[f,g]})_{[h,i]}"] \\
M_{[f\mathrm{pr}_X,\, i\mathrm{pr}_V]} \arrow[r, "\theta_{f,g,h,i}(M)"] & (M_{[f,g]})_{[h,i]}
\end{tikzcd}
\qquad
\begin{tikzcd}
M_{[fj\mathrm{pr}_S,\, ik\mathrm{pr}_T]} \arrow[r, "\theta_{fj,gj,hk,ik}(M)"] \arrow[d, "M_{j \times_Z k}"] & (M_{[fj,gj]})_{[hk,\, ik]} \arrow[d, "(M_j)_k"] \\
M_{[f\mathrm{pr}_X,\, i\mathrm{pr}_V]} \arrow[r, "\theta_{f,g,h,i}(M)"] & (M_{[f,g]})_{[h,i]}
\end{tikzcd}
$$

Remark 8.4.20 *If $X \xleftarrow{\mathrm{pr}'_X} X \times'_Z V \xrightarrow{\mathrm{pr}'_V} V$ is another limit of a diagram $X \xrightarrow{g} Z \xleftarrow{h} V$, there exists unique isomorphism $l : X \times'_Z V \to X \times_Z V$ that satisfies $\mathrm{pr}'_X = \mathrm{pr}_X l$ and $\mathrm{pr}'_V = \mathrm{pr}_V l$. We denote by $\theta'_{f,g,h,i}(M) : M_{[f\mathrm{pr}'_X, i\mathrm{pr}'_V]} \to (M_{[f,g]})_{[h,i]}$ the morphism of $\mathcal{F}_W$ obtained from $X \xleftarrow{\mathrm{pr}'_X} X \times'_Z V \xrightarrow{\mathrm{pr}'_V} V$. Then, $M_l : M_{[f\mathrm{pr}'_X, i\mathrm{pr}'_V]} \to M_{[f\mathrm{pr}_X,\, i\mathrm{pr}_V]}$ is an isomorphism and (8.4.18) implies $\theta'_{f,g,h,i}(M) = \theta_{f,g,h,i}(M)M_l$.*

Proposition 8.4.21 *Suppose that the following diagram in $\mathcal{E}$ is commutative.*

$$
\begin{tikzcd}
& & & Q \arrow[dl, "v"'] \arrow[dr, "w"] & & & \\
& & R \arrow[dl, "r"'] \arrow[dr, "s"] & & S \arrow[dl, "t"'] \arrow[dr, "u"] & & \\
& X \arrow[dl, "f"'] \arrow[dr, "g"] & & V \arrow[dl, "h"'] \arrow[dr, "i"] & & T \arrow[dl, "j"'] \arrow[dr, "k"] & \\
Y & & Z & & W & & U
\end{tikzcd}
$$

Define functors $D_l : \mathcal{P} \to \mathcal{E}$ for $l = 1, 2, 3, 4$ as follows.

$D_1(0) = S$	$D_1(1) = V$	$D_1(2) = T$	$D_1(3) = Z$	$D_1(4) = W$	$D_1(5) = U$
$D_1(\tau_{01}) = t$	$D_1(\tau_{02}) = u$	$D_1(\tau_{13}) = h$	$D_1(\tau_{14}) = i$	$D_1(\tau_{24}) = j$	$D_1(\tau_{25}) = k$
$D_2(0) = Q$	$D_2(1) = R$	$D_2(2) = T$	$D_2(3) = Y$	$D_2(4) = W$	$D_2(5) = U$
$D_2(\tau_{01}) = v$	$D_2(\tau_{02}) = uw$	$D_2(\tau_{13}) = fr$	$D_2(\tau_{14}) = is$	$D_2(\tau_{24}) = j$	$D_2(\tau_{25}) = k$
$D_3(0) = Q$	$D_3(1) = X$	$D_3(2) = S$	$D_3(3) = Y$	$D_3(4) = Z$	$D_3(5) = U$
$D_3(\tau_{01}) = rv$	$D_3(\tau_{02}) = w$	$D_3(\tau_{13}) = f$	$D_3(\tau_{14}) = g$	$D_3(\tau_{24}) = ht$	$D_3(\tau_{25}) = ku$
$D_4(0) = R$	$D_4(1) = X$	$D_4(2) = V$	$D_4(3) = Y$	$D_4(4) = Z$	$D_4(5) = W$
$D_4(\tau_{01}) = r$	$D_4(\tau_{02}) = s$	$D_4(\tau_{13}) = f$	$D_4(\tau_{14}) = g$	$D_4(\tau_{24}) = h$	$D_4(\tau_{25}) = i$

Then, the following diagram is commutative.

$$\begin{tikzcd}
M_{[frv,kuw]} \arrow[r, "\theta_{D_3}(M)"] \arrow[d, "\theta_{D_2}(M)"] & (M_{[f,g]})_{[ht,ku]} \arrow[d, "\theta_{D_1}(M_{[f,g]})"] \\
(M_{[fr,is]})_{[j,k]} \arrow[r, "\theta_{D_4}(M)_{[j,k]}"] & ((M_{[f,g]})_{[h,i]})_{[j,k]}
\end{tikzcd}$$

Proof. The following diagrams are commutative by (8.4.13), (8.4.12), (8.4.8), (8.4.3) and (8.4.6).

$$\begin{tikzcd}
M_{[frv,kuw]} \arrow[r, "\delta_{frv,htw,kuw,M}"] \arrow[d, "\delta_{frv,isv,kuw,M}"] & (M_{[frv,grv]})_{[htw,kuw]} \arrow[r, "(M_{[frv,grv]})_w"] \arrow[d, "\delta_{htw,isv,kuw,M_{[frv,grv]}}"'] & (M_{[frv,grv]})_{[ht,ku]} \arrow[d, "\delta_{ht,it,ku,M_{[frv,grv]}}"'] \\
(M_{[frv,isv]})_{[juw,kuw]} \arrow[r, "(\delta_{frv,grv,isv,M})_{[juw,kuw]}"] \arrow[d, "(M_v)_{[juw,kuw]}"] & ((M_{[frv,grv]})_{[grv,isv]})_{[juw,kuw]} \arrow[r, "((M_{[frv,grv]})_w)_w"] \arrow[d, "((M_v)_v)_{[juw,kuw]}"'] & ((M_{[frv,grv]})_{[ht,it]})_{[ju,ku]} \arrow[d, "((M_{rv})_t)_{[ju,ku]}"'] \\
(M_{[fr,is]})_{[juw,kuw]} \arrow[r, "(\delta_{fr,gr,is,M})_{[juw,kuw]}"] \arrow[d, "(M_{[fr,is]})_{uw}"] & ((M_{[fr,gr]})_{[hs,is]})_{[juw,kuw]} \arrow[r, "((M_r)_s)_w"] \arrow[d, "((M_{[fr,gr]})_{[hs,is]})_{uw}"'] & ((M_{[f,g]})_{[h,i]})_{[ju,ku]} \arrow[d, "((M_{[f,g]})_{[h,i]})_u"'] \\
(M_{[fr,is]})_{[j,k]} \arrow[r, "(\delta_{fr,gr,is,M})_{[j,k]}"] & ((M_{[fr,gr]})_{[hs,is]})_{[j,k]} \arrow[r, "((M_r)_s)_{[j,k]}"] & ((M_{[f,g]})_{[h,i]})_{[j,k]}
\end{tikzcd}$$

$$\begin{tikzcd}
(M_{[frv,grv]})_{[ht,ku]} \arrow[r, "(M_{rv})_{[ht,ku]}"] \arrow[d, "\delta_{ht,it,ku,M_{[frv,grv]}}"] & (M_{[f,g]})_{[ht,ku]} \arrow[d, "\delta_{ht,it,ku,M_{[f,g]}}"] \\
((M_{[frv,grv]})_{[ht,it]})_{[ju,ku]} \arrow[r, "((M_{rv})_{[ht,it]})_{[ju,ku]}"] \arrow[d, "((M_{rv})_t)_{[ju,ku]}"] & ((M_{[f,g]})_{[ht,it]})_{[ju,ku]} \arrow[dl, "((M_{[f,g]})_t)_{[ju,ku]}"] \\
((M_{[f,g]})_{[h,i]})_{[ju,ku]} &
\end{tikzcd}$$

Hence the assertion follows from the definition of $\theta_{D_l}(M)$. □

For morphisms $g : X \to Z$, $h : V \to Z$, $i : V \to W$, $j : T \to W$ of $\mathcal{E}$, let $X \xleftarrow{\mathrm{pr}_X} X \times_Z V \xrightarrow{\mathrm{pr}_{2V}} V$ and $V \xleftarrow{\mathrm{pr}_{1V}} V \times_W T \xrightarrow{\mathrm{pr}_T} T$ be limits of diagrams $X \xrightarrow{g} Z \xleftarrow{h} V$ and $V \xrightarrow{i} W \xleftarrow{j} T$, respectively. We also assume that a limit $X \times_Z V \xleftarrow{\mathrm{pr}_{X \times_Z V}} X \times_Z V \times_W T \xrightarrow{\mathrm{pr}_{V \times_W T}} V \times_W T$ of a diagram $X \times_Z V \xrightarrow{\mathrm{pr}_{2V}} V \xleftarrow{\mathrm{pr}_{1V}} V \times_W T$ exists. Then, $X \xleftarrow{\mathrm{pr}_X \mathrm{pr}_{X \times_Z V}} X \times_Z V \times_W T \xrightarrow{\mathrm{pr}_{V \times_W T}} V \times_W T$ and $X \times_Z V \xleftarrow{\mathrm{pr}_{X \times_Z V}} X \times_Z V \times_W T \xrightarrow{\mathrm{pr}_{V \times_W T} \mathrm{pr}_T} T$ are limits of diagrams $X \xrightarrow{g} Z \xleftarrow{h\mathrm{pr}_{1V}} V \times_W T$ and $X \times_Z V \xrightarrow{i\mathrm{pr}_{2V}} W \xleftarrow{j} T$, respectively.

Corollary 8.4.22 *Let $f : X \to Y$, $g : X \to Z$, $h : V \to Z$, $i : V \to W$, $j : T \to W$, $k : T \to U$ be morphisms of $\mathcal{E}$ and M an object of $\mathcal{F}_Y$. The following diagram is commutative.*

$$\begin{tikzcd}
M_{[f\mathrm{pr}_X \mathrm{pr}_{X \times_Z V},\, k\mathrm{pr}_T \mathrm{pr}_{V \times_W T}]} \arrow[r, "\theta_{f,g,h\mathrm{pr}_{1V},k\mathrm{pr}_T}(M)"] \arrow[d, "\theta_{f\mathrm{pr}_X,i\mathrm{pr}_{2V},j,k}(M)"] & (M_{[f,g]})_{[h\mathrm{pr}_{1V},\, k\mathrm{pr}_T]} \arrow[d, "\theta_{h,i,j,k}(M_{[f,g]})"] \\
(M_{[f\mathrm{pr}_X,\, i\mathrm{pr}_{2V}]})_{[j,k]} \arrow[r, "\theta_{f,g,h,i}(M)_{[j,k]}"] & ((M_{[f,g]})_{[h,i]})_{[j,k]}
\end{tikzcd}$$

Proof. The assertion follows by applying the result of (8.4.21) to the following diagram.

$$\begin{tikzcd}
 & & & & X\times_Z V\times_W T \arrow[dll, "\mathrm{pr}_{X\times_Z V}"'] \arrow[drr, "\mathrm{pr}_{V\times_W T}"] & & & & \\
 & & X\times_Z V \arrow[dl, "\mathrm{pr}_X"'] \arrow[drr, "\mathrm{pr}_{2V}"] & & & & V\times_W T \arrow[dll, "\mathrm{pr}_{1V}"'] \arrow[dr, "\mathrm{pr}_T"] & & \\
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & & & V \arrow[dll, "h"'] \arrow[drr, "i"] & & & T \arrow[dl, "j"'] \arrow[dr, "k"] & \\
Y & & Z & & & & W & & U
\end{tikzcd}$$

□

Proposition 8.4.23 *For morphisms $f : X \to Y$, $g : X \to Z$ of $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, the following morphisims of $\mathcal{F}_Z$ are identified with the identity morphism of $M_{[f,\,g]}$.*

$$\theta_{f,g,id_Z,id_Z}(M) : M_{[f\,id_X,\,id_Z g]} \to (M_{[f,\,g]})_{[id_Z,\,id_Z]}, \qquad \theta_{id_Y,id_Y,f,g}(M) : M_{[id_Y f,\,g\,id_X]} \to (M_{[id_Y,\,id_Y]})_{[f,\,g]}$$

Proof. Since $\theta_{f,g,id_Z,id_Z}(M)$ is a composition

$$M_{[f,\,g]} = M_{[f\,id_X,\,id_Z g]} \xrightarrow{\delta_{f\,id_X,\,g\,id_X,\,id_Z g,\,M}} (M_{[f\,id_X,\,g\,id_X]})_{[id_Z g,\,id_Z g]} \xrightarrow{(M_{[f,\,g]})_g} (M_{[f,\,g]})_{[id_Z,\,id_Z]} = M_{[f,\,g]}$$

and $\theta_{id_Y,id_Y,f,g}(M)$ is a composition

$$M_{[f,\,g]} = M_{[id_Y f,\,g\,id_X]} \xrightarrow{\delta_{id_Y f,\,f\,id_X,\,g\,id_X,\,M}} (M_{[id_Y f,\,id_Y f]})_{[f\,id_X,\,g\,id_X]} \xrightarrow{(M_f)_{[f,\,g]}} (M_{[id_Y,\,id_Y]})_{[f,\,g]} = M_{[f,\,g]},$$

the assertion is a direct consequence of (8.4.14). □

Lemma 8.4.24 *For a functor $D : \mathcal{P} \to \mathcal{E}$, we put $D(\tau_{01}) = j$, $D(\tau_{02}) = k$, $D(\tau_{13}) = f$, $D(\tau_{14}) = g$, $D(\tau_{24}) = h$, $D(\tau_{25}) = i$. For an object M of $\mathcal{F}_{D(3)}$, the following diagram is commutative.*

$$\begin{tikzcd}
(fj)^*(M) \arrow[r, "\iota_{fj,ik}(M)"] \arrow[d, "j^\sharp(\iota_{f,g}(M))"] & (ik)^*(M_{[fj,ik]}) \arrow[d, "(ik)^*(\theta_D(M))"] \\
(gj)^*(M_{[f,g]}) \arrow[r, "k^\sharp(\iota_{h,i}(M_{[f,g]}))"] & (ik)^*((M_{[f,g]})_{[h,i]})
\end{tikzcd}$$

Proof. It follows from (8.4.6) and (1) of (8.4.3) that we have

$$\begin{aligned}
k^\sharp(\iota_{h,i}(M_{[f,g]}))j^\sharp(\iota_{f,g}(M)) &= (ik)^*((M_{[f,g]})_k)\iota_{hk,ik}(M_{[f,g]})(gj)^*(M_j)\iota_{fj,gj}(M) \\
&= (ik)^*((M_{[f,g]})_k)(ik)^*((M_j)_{[hk,ik]})\iota_{hk,ik}(M_{[fj,gj]})\iota_{fj,gj}(M) \\
&= (ik)^*((M_j)_k)\iota_{hk,ik}(M_{[fj,gj]})\iota_{fj,gj}(M)
\end{aligned}$$

By the naturality of $P_{fj,ik}(M)$ and the definition of $\delta_{fj,gj,ik,M}$, the above equality implies that

$$P_{fj,ik}(M)_{(M_{[f,g]})_{[h,i]}} : \mathcal{F}_{D(0)}((fj)^*(M), (ik)^*((M_{[f,g]})_{[h,i]}) \to \mathcal{F}_{D(5)}(M_{[fj,ik]}, (M_{[f,g]})_{[h,i]})$$

maps $k^\sharp(\iota_{h,i}(M_{[f,g]}))j^\sharp(\iota_{f,g}(M))$ to $(M_j)_k\delta_{fj,gj,ik,M} = \theta_D(M)$. On the other hand, it follows from (8.4.2) that $P_{fj,ik}(M)_{(M_{[f,g]})_{[h,i]}}$ also maps $(ik)^*(\theta_D(M))\iota_{fj,ik}(M)$ to $\theta_D(M)$. □

For a morphism $g : X \to Z$, let $X \xleftarrow{\mathrm{pr}_{1X}} X\times_Z X \xrightarrow{\mathrm{pr}_{2X}} X$ be a limit of a diagram $X \xrightarrow{g} Z \xleftarrow{g} X$. We denote by $\Delta_g : X \to X\times_Z X$ the diagonal morphism, that is, the unique morphism that satisfies $\mathrm{pr}_{1X}\Delta_g = \mathrm{pr}_{2X}\Delta_g = id_X$.

Proposition 8.4.25 *For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$ of $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, $\delta_{f,g,h,M} : M_{[f,h]} \to (M_{[f,g]})_{[g,h]}$ coincides with the following composition.*

$$M_{[f,h]} = M_{[f\mathrm{pr}_{1X}\Delta_g,h\mathrm{pr}_{2X}\Delta_g]} \xrightarrow{M_{\Delta_g}} M_{[f\mathrm{pr}_{1X},h\mathrm{pr}_{2X}]} \xrightarrow{\theta_{f,g,g,h}(M)} (M_{[f,g]})_{[g,h]}$$

Proof. Define a functor $E : \mathcal{P} \to \mathcal{E}$ by $E(i) = X$ for $i = 0, 1, 2$, $E(i) = D_{f,g,g,h}(i)$ for $i = 3, 4, 5$ and $E(\tau_{01}) = E(\tau_{02}) = id_X$, $E(\tau_{ij}) = D_{f,g,g,h}(\tau_{ij})$ if $i \neq 0$. Then, $\theta_E(M) = \delta_{f,g,h,M} : M_{[f,h]} \to (M_{[f,g]})_{[g,h]}$ and we have a natural transformation $\lambda : E \to D$ defined by $\lambda_0 = \Delta_g$ and $\lambda_i = id_{E(i)}$ if $i \geqq 1$. It follows from (8.4.18) that $\theta_{f,g,g,h}(M)M_{\Delta_g} = \theta_E(M) = \delta_{f,g,h,M}$. □

Let $\mathcal{Q}$ be a subposet of $\mathcal{P}$ given by $\mathrm{Ob}\,\mathcal{Q} = \{0, 1, 2\}$. Let $D, E : \mathcal{Q} \to \mathcal{E}$ be functors and M an object of $\mathcal{F}_{E(1)}$. We put $D(\tau_{0j}) = f_j$ and $E(\tau_{0j}) = g_j$ for $j = 1, 2$. For a natural transformation $\omega : D \to E$, let $\omega_M : \omega_1^*(M)_{[f_1,f_2]} \to \omega_2^*(M_{[g_1,g_2]})$ be the image of $\iota_{g_1,g_2}(M) \in \mathcal{F}_{E(0)}(g_1^*(M), g_2^*(M_{[g_1,g_2]}))$ by the following composition of maps.

$$\begin{aligned}\mathcal{F}_{E(0)}(g_1^*(M), g_2^*(M_{[g_1,g_2]})) &\xrightarrow{\omega_0^\sharp} \mathcal{F}_{D(0)}((g_1\omega_0)^*(M), (g_2\omega_0)^*(M_{[g_1,g_2]})) = \mathcal{F}_{D(0)}((\omega_1 f_1)^*(M), (\omega_2 f_2)^*(M_{[g_1,g_2]}))\\ &\xrightarrow{c_{\omega_1,f_1}(M)^* c_{\omega_2,f_2}(M_{[g_1,g_2]})_*^{-1}} \mathcal{F}_{D(0)}(f_1^*(\omega_1^*(M)), f_2^*(\omega_2^*(M_{[g_1,g_2]})))\\ &\xrightarrow{P_{f_1,f_2}(\omega_1^*(M))_{\omega_2^*(M_{[g_1,g_2]})}} \mathcal{F}_{D(2)}(\omega_1^*(M)_{[f_1,f_2]}, \omega_2^*(M_{[g_1,g_2]}))\end{aligned}$$

Remark 8.4.26 *(1) If $D(i) = E(i)$ and ω_i is the identity morphism of $D(i)$ for $i = 1, 2$, then ω_M coincides with $M_{\omega_0} : M_{[f_1,f_2]} = M_{[g_1\omega_0,g_2\omega_0]} \to M_{[g_1,g_2]}$.*

(2) It follows from (8.4.2) and the definition of ω_M that the following diagram is commutative.

$$\begin{tikzcd} f_1^*(\omega_1^*(M)) \arrow[r, "c_{\omega_1,f_1}(M)"] \arrow[d, "\iota_{f_1,f_2}(\omega_1^*(M))"] & (\omega_1 f_1)^*(M) = (g_1\omega_0)^*(M) \arrow[r, "\omega_0^\sharp(\iota_{g_1,g_2}(M))"] & (g_2\omega_0)^*(M_{[g_1,g_2]}) \arrow[d, equal] \\ f_2^*(\omega_1^*(M)_{[f_1,f_2]}) \arrow[r, "f_2^*(\omega_M)"] & f_2^*(\omega_2^*(M_{[g_1,g_2]})) \arrow[r, "c_{\omega_2,f_2}(M_{[g_1,g_2]})"] & (\omega_2 f_2)^*(M_{[g_1,g_2]}) \end{tikzcd}$$

Proposition 8.4.27 *Assume that $D(0) = E(0)$ and ω_0 is the identity morphism of $D(0)$. For an object N of $\mathcal{F}_{E(2)}$, the following diagram is commutative.*

$$\begin{tikzcd} \mathcal{F}_{D(0)}(g_1^*(M), g_2^*(N)) \arrow[r, "c_{\omega_2,f_2}(N)_*^{-1}"] \arrow[d, "P_{g_1,g_2}(M)_N"] & \mathcal{F}_{D(0)}(g_1^*(M), f_2^*(\omega_2^*(N))) \arrow[r, "c_{\omega_1,f_1}(M)^*"] & \mathcal{F}_{D(0)}(f_1^*(\omega_1^*(M)), f_2^*(\omega_2^*(N))) \arrow[d, "P_{f_1,f_2}(\omega_1^*(M))_{\omega_2^*(N)}"] \\ \mathcal{F}_{E(2)}(M_{[g_1,g_2]}, N) \arrow[r, "\omega_2^*"] & \mathcal{F}_{D(2)}(\omega_2^*(M_{[g_1,g_2]}), \omega_2^*(N)) \arrow[r, "\omega_M^*"] & \mathcal{F}_{D(2)}(\omega_1^*(M)_{[f_1,f_2]}, \omega_2^*(N)) \end{tikzcd}$$

Proof. First we note that $g_i = \omega_i f_i$ for $i = 1, 2$. It follows from (8.4.26) and the definition of ω_M that we have $f_2^*(\omega_M)\iota_{f_1,f_2}(\omega_1^*(M)) = c_{\omega_2,f_2}(M_{[g_1,g_2]})^{-1}\iota_{g_1,g_2}(M)c_{\omega_1,f_1}(M)$. (8.4.2) and (8.1.10) imply

$$\begin{aligned} c_{\omega_2,f_2}(N)^{-1}P_{g_1,g_2}(M)_N^{-1}(\varphi)c_{\omega_1,f_1}(M) &= c_{\omega_2,f_2}(N)^{-1}g_2^*(\varphi)\iota_{g_1,g_2}(M)c_{\omega_1,f_1}(M)\\ &= f_2^*\omega_2^*(\varphi)c_{\omega_2,f_2}(M_{[g_1,g_2]})^{-1}\iota_{g_1,g_2}(M)c_{\omega_1,f_1}(M)\\ &= f_2^*\omega_2^*(\varphi)f_2^*(\omega_M)\iota_{f_1,f_2}(\omega_1^*(M)) = f_2^*(\omega_2^*(\varphi)\omega_M)\iota_{f_1,f_2}(\omega_1^*(M))\\ &= P_{f_1,f_2}(\omega_1^*(M))_{\omega_2^*(N)}^{-1}(\omega_2^*(\varphi)\omega_M) \end{aligned}$$

for $\varphi \in \mathcal{F}_{E(2)}(M_{[g_1,g_2]}, N)$, which shows that the above diagram is commutative. □

Proposition 8.4.28 *For a morphism $\varphi : M \to N$ of $\mathcal{F}_{E(1)}$, the following diagram is commutative.*

$$\begin{tikzcd} \omega_1^*(M)_{[f_1,f_2]} \arrow[r, "\omega_M"] \arrow[d, "\omega_1^*(\varphi)_{[f_1,f_2]}"] & \omega_2^*(M_{[g_1,g_2]}) \arrow[d, "\omega_2^*(\varphi_{[g_1,g_2]})"] \\ \omega_1^*(N)_{[f_1,f_2]} \arrow[r, "\omega_N"] & \omega_2^*(N_{[g_1,g_2]}) \end{tikzcd}$$

Proof. It follows from (8.1.10), (1) of (8.4.3) and (8.1.13) that the following diagrams are commutative.

$$\begin{tikzcd}
f_1^*\omega_1^*(M) \arrow[r, "c_{\omega_1,f_1}(M)"] \arrow[d, "f_1^*\omega_1^*(\varphi)"] & (\omega_1 f_1)^*(M) = (g_1\omega_0)^*(M) \arrow[r, "\omega_0^\sharp(\iota_{g_1,g_2}(M))"] \arrow[d, "(g_1\omega_0)^*(\varphi)"] & (g_2\omega_0)^*(M_{[g_1,g_2]}) = (\omega_2 f_2)^*(M_{[g_1,g_2]}) \arrow[d, "(g_2\omega_0)^*(\varphi_{[g_1,g_2]})"] \\
f_1^*\omega_1^*(N) \arrow[r, "c_{\omega_1,f_1}(N)"] & (\omega_1 f_1)^*(N) = (g_1\omega_0)^*(N) \arrow[r, "\omega_0^\sharp(\iota_{g_1,g_2}(N))"] & (g_2\omega_0)^*(N_{[g_1,g_2]}) = (\omega_2 f_2)^*(N_{[g_1,g_2]})
\end{tikzcd}$$

$$\begin{tikzcd}
(\omega_2 f_2)^*(M_{[g_1,g_2]}) \arrow[r, "c_{\omega_2,f_2}(M_{[g_1,g_2]})^{-1}"] \arrow[d, "(\omega_2 f_2)^*(\varphi_{[g_1,g_2]})"] & f_2^*\omega_2^*(M_{[g_1,g_2]}) \arrow[d, "f_2^*\omega_2^*(\varphi_{[g_1,g_2]})"] \\
(\omega_2 f_2)^*(N_{[g_1,g_2]}) \arrow[r, "c_{\omega_2,f_2}(N_{[g_1,g_2]})^{-1}"] & f_2^*\omega_2^*(N_{[g_1,g_2]})
\end{tikzcd}$$

By applying (8.4.5) to the following commutative diagram,

$$\begin{tikzcd}
f_1^*\omega_1^*(M) \arrow[r, "c_{\omega_2,f_2}(M_{[g_1,g_2]})^{-1}\omega_0^\sharp(\iota_{g_1,g_2}(M))c_{\omega_1,f_1}(M)"] \arrow[d, "f_1^*\omega_1^*(\varphi)"] & f_2^*\omega_2^*(M_{[g_1,g_2]}) \arrow[d, "f_2^*\omega_2^*(\varphi_{[g_1,g_2]})"] \\
f_1^*\omega_1^*(N) \arrow[r, "c_{\omega_2,f_2}(N_{[g_1,g_2]})^{-1}\omega_0^\sharp(\iota_{g_1,g_2}(N))c_{\omega_1,f_1}(N)"] & f_2^*\omega_2^*(N_{[g_1,g_2]})
\end{tikzcd}$$

the assertion follows. □

Lemma 8.4.29 *Let $D, E, F : \mathcal{Q} \to \mathcal{E}$ be functors and $\omega : D \to E$, $\chi : E \to F$ natural transformations. We put $D(\tau_{0j}) = f_j$, $E(\tau_{0j}) = g_j$ and $F(\tau_{0j}) = h_j$ for $j = 1, 2$. For $M \in \operatorname{Ob}\mathcal{F}_{F(1)}$, $N \in \operatorname{Ob}\mathcal{F}_{F(2)}$ and a morphism $\varphi : h_1^*(M) \to h_2^*(N)$ of $\mathcal{F}_{F(0)}$, the following diagram is commutative.*

$$\begin{tikzcd}
\omega_0^*((\chi_1 g_1)^*(M))) \arrow[r, "c_{\chi_1 g_1,\omega_0}(M)"] \arrow[d, equal] & (\chi_1 g_1\omega_0)^*(M) = (h_1\chi_0\omega_0)^*(M) \arrow[r, "(\chi_0\omega_0)^\sharp(\varphi)"] & (h_2\chi_0\omega_0)^*(N) \arrow[d, equal] \\
\omega_0^*((h_1\chi_0)^*(M)) \arrow[r, "\omega_0^*(\chi_0^\sharp(\varphi))"] & \omega_0^*((h_2\chi_0)^*(N)) = \omega_0^*((\chi_2 g_2)^*(N)) \arrow[r, "c_{\chi_2 g_2,\omega_0}(N)"] & (\chi_2 g_2\omega_0)^*(N)
\end{tikzcd}$$

Proof. The following diagram is commutative by (8.1.12), (8.1.14) and the definition of $\omega_0^\sharp$.

$$\begin{tikzcd}
\omega_0^*((\chi_1 g_1)^*(M))) \arrow[r, "c_{\chi_1 g_1,\omega_0}(M)"] \arrow[d, equal] & (\chi_1 g_1\omega_0)^*(M) \arrow[r, equal] \arrow[d, equal] & (h_1\chi_0\omega_0)^*(M) \arrow[d, "(\chi_0\omega_0)^\sharp(\varphi)"] \\
\omega_0^*((h_1\chi_0)^*(M)) \arrow[r, "c_{h_1\chi_0,\omega_0}(M)"] \arrow[d, "\omega_0^*(\chi_0^\sharp(\varphi))"] & (h_1\chi_0\omega_0)^*(M) \arrow[r, "\omega_0^\sharp(\chi_0^\sharp(\varphi))"] & (h_2\chi_0\omega_0)^*(N) \arrow[d, equal] \\
\omega_0^*((h_2\chi_0)^*(N)) \arrow[r, equal] \arrow[urr, "c_{h_2\chi_0,\omega_0}(N)"] & \omega_0^*((\chi_2 g_2)^*(N)) \arrow[r, "c_{\chi_2 g_2,\omega_0}(N)"] & (\chi_2 g_2\omega_0)^*(N)
\end{tikzcd}$$

□

Proposition 8.4.30 *Let $D, E, F : \mathcal{Q} \to \mathcal{E}$ be functors and M an object of $\mathcal{F}_{F(1)}$. We put $D(\tau_{0j}) = f_j$, $E(\tau_{0j}) = g_j$ and $F(\tau_{0j}) = h_j$ for $j = 1, 2$. For natural transformations $\omega : D \to E$ and $\chi : E \to F$, the following diagram is commutative.*

$$\begin{tikzcd}
\omega_1^*(\chi_1^*(M))_{[f_1,f_2]} \arrow[r, "\omega_{\chi_1^*(M)}"] \arrow[d, "c_{\chi_1,\omega_1}(M)_{[f_1,f_2]}"] & \omega_2^*(\chi_1^*(M)_{[g_1,g_2]}) \arrow[r, "\omega_2^*(\chi_M)"] & \omega_2^*(\chi_2^*(M_{[h_1,h_2]})) \arrow[d, "c_{\chi_2,\omega_2}(M_{[h_1,h_2]})"] \\
(\chi_1\omega_1)^*(M)_{[f_1,f_2]} \arrow[rr, "(\chi\omega)_M"] & & (\chi_2\omega_2)^*(M_{[h_1,h_2]})
\end{tikzcd}$$

Proof. It follows from (8.4.2) and (8.4.26) that we have

$$\begin{aligned}
P_{f_1,f_2}(\omega_1^*(\chi_1^*(M)))^{-1}_{\omega_2^*(\chi_2^*(M_{[h_1,h_2]}))}(\omega_2^*(\chi_M)\omega_{\chi_1^*(M)}) &= f_2^*(\omega_2^*(\chi_M)\omega_{\chi_1^*(M)})\iota_{f_1,f_2}(\omega_1^*(\chi_1^*(M))) \\
&= f_2^*(\omega_2^*(\chi_M))f_2^*(\omega_{\chi_1^*(M)})\iota_{f_1,f_2}(\omega_1^*(\chi_1^*(M))) \\
&= f_2^*(\omega_2^*(\chi_M))c_{\omega_2,f_2}(\chi_1^*(M)_{[g_1,g_2]})^{-1}\omega_0^\sharp(\iota_{g_1,g_2}(\chi_1^*(M)))c_{\omega_1,f_1}(\chi_1^*(M))
\end{aligned}$$

Hence it suffices to show that the following diagram is commutative by (8.4.5).

$$
\begin{tikzcd}
f_1^*(\omega_1^*(\chi_1^*(M))) \arrow[rr, "f_2^*(\omega_2^*(\chi_M))c_{\omega_2,f_2}(\chi_1^*(M)_{[g_1,g_2]})^{-1}\omega_0^\sharp(\iota_{g_1,g_2}(\chi_1^*(M)))c_{\omega_1,f_1}(\chi_1^*(M))"] \arrow[d, "f_1^*(c_{\chi_1,\omega_1}(M))"] & & f_2^*(\omega_2^*(\chi_2^*(M_{[h_1,h_2]}))) \arrow[d, "f_2^*(c_{\chi_2,\omega_2}(M_{[h_1,h_2]}))"] \\
f_1^*(\chi_1\omega_1)^*(M) \arrow[rr, "c_{\chi_2\omega_2,f_2}(M_{[h_1,h_2]})^{-1}(\chi_0\omega_0)^\sharp(\iota_{h_1,h_2}(M))c_{\chi_1\omega_1,f_1}(M)"] & & f_2^*(\chi_2\omega_2)^*(M_{[h_1,h_2]})
\end{tikzcd}
$$

It follows from (8.1.10) and (8.1.12) that we have

$$
\begin{aligned}
&f_2^*(\omega_2^*(\chi_M))c_{\omega_2,f_2}(\chi_1^*(M)_{[g_1,g_2]})^{-1} = c_{\omega_2,f_2}(\chi_2^*(M_{[h_1,h_2]}))^{-1}(\omega_2 f_2)^*(\chi_M) = c_{\omega_2,f_2}(\chi_2^*(M_{[h_1,h_2]}))^{-1}(g_2\omega_0)^*(\chi_M) \\
&c_{\chi_1\omega_1,f_1}(M)f_1^*(c_{\chi_1,\omega_1}(M))c_{\omega_1,f_1}(\chi_1^*(M))^{-1} = c_{\chi_1,\omega_1 f_1}(M) = c_{\chi_1,g_1\omega_0}(M) \\
&c_{\chi_2\omega_2,f_2}(M_{[h_1,h_2]})f_2^*(c_{\chi_2,\omega_2}(M_{[h_1,h_2]}))c_{\omega_2,f_2}(\chi_2^*(M_{[h_1,h_2]}))^{-1} = c_{\chi_2,\omega_2 f_2}(M_{[h_1,h_2]}) = c_{\chi_2,g_2\omega_0}(M_{[h_1,h_2]}).
\end{aligned}
$$

Hence the commutativity of the above diagram is equivalent to the following equality.

$$
c_{\chi_2,g_2\omega_0}(M_{[h_1,h_2]})(g_2\omega_0)^*(\chi_M)\omega_0^\sharp(\iota_{g_1,g_2}(\chi_1^*(M))) = (\chi_0\omega_0)^\sharp(\iota_{h_1,h_2}(M))c_{\chi_1,g_1\omega_0}(M) \ \cdots (*)
$$

The following diagram is commutative by (8.1.10) and (8.4.26).

$$
\begin{tikzcd}
\omega_0^*((h_1\chi_0)^*(M)) \arrow[rr, "\omega_0^*(\chi_0^\sharp(\iota_{h_1,h_2}(M)))"] \arrow[d, equal] & & \omega_0^*((h_2\chi_0)^*(M_{[h_1,h_2]})) \arrow[d, equal] \\
\omega_0^*((\chi_1 g_1)^*(M)) & & \omega_0^*((\chi_2 g_2)^*(M_{[h_1,h_2]})) \\
\omega_0^*(g_1^*(\chi_1^*(M))) \arrow[u, "\omega_0^*(c_{\chi_1,g_1}(M))"] \arrow[r, "\omega_0^*(\iota_{g_1,g_2}(\chi_1^*(M)))"] \arrow[d, "c_{g_1,\omega_0}(\chi_1^*(M))"] & \omega_0^*(g_2^*(\chi_1^*(M)_{[g_1,g_2]})) \arrow[r, "\omega_0^*(g_2^*(\chi_M))"] \arrow[d, "c_{g_2,\omega_0}(\chi_1^*(M)_{[g_1,g_2]})"] & \omega_0^*(g_2^*(\chi_2^*(M_{[h_1,h_2]}))) \arrow[u, "\omega_0^*(c_{\chi_2,g_2}(M_{[h_1,h_2]}))"] \arrow[d, "c_{g_2,\omega_0}(\chi_2^*(M_{[h_1,h_2]}))"] \\
(g_1\omega_0)^*(\chi_1^*(M)) \arrow[r, "\omega_0^\sharp(\iota_{g_1,g_2}(\chi_1^*(M)))"] & (g_2\omega_0)^*(\chi_1^*(M)_{[g_1,g_2]}) \arrow[r, "(g_2\omega_0)^*(\chi_M)"] & (g_2\omega_0)^*(\chi_2^*(M_{[h_1,h_2]}))
\end{tikzcd}
$$

Hence the left hand side of $(*)$ equals

$$
\begin{aligned}
&c_{\chi_2,g_2\omega_0}(M_{[h_1,h_2]})c_{g_2,\omega_0}(\chi_2^*(M_{[h_1,h_2]}))\omega_0^*(c_{\chi_2,g_2}(M_{[h_1,h_2]}))^{-1}\omega_0^*(\chi_0^\sharp(\iota_{h_1,h_2}(M)))\omega_0^*(c_{\chi_1,g_1}(M))c_{g_1,\omega_0}(\chi_1^*(M))^{-1} \\
&\qquad = c_{\chi_2 g_2,\omega_0}(M_{[h_1,h_2]})\omega_0^*(\chi_0^\sharp(\iota_{h_1,h_2}(M)))c_{\chi_1,g_1\omega_0}(M)^{-1}c_{\chi_1,g_1\omega_0}(M) \\
&\qquad = (\chi_0\omega_0)^\sharp(\iota_{h_1,h_2}(M))c_{\chi_1,g_1\omega_0}(M)
\end{aligned}
$$

by (8.1.12) and (8.4.29) for $N = M_{[h_1,h_2]}$ and $\varphi = \iota_{h_1,h_2}(M)$. □

Proposition 8.4.31 *For functors $D, E : \mathcal{P} \to \mathcal{E}$, we put $D(\tau_{ij}) = f_{ij}$ and $E(\tau_{ij}) = g_{ij}$ and define functors $D_i, E_i : \mathcal{Q} \to \mathcal{E}$ for $i = 0, 1, 2$ as follows.*

$$
\begin{array}{lllll}
D_0(0) = D(0) & D_0(1) = D(3) & D_0(2) = D(5) & D_0(\tau_{01}) = f_{13}f_{01} & D_0(\tau_{02}) = f_{25}f_{02} \\
E_0(0) = E(0) & E_0(1) = E(3) & E_0(2) = E(5) & E_0(\tau_{01}) = g_{13}g_{01} & E_0(\tau_{02}) = g_{25}g_{02} \\
D_1(0) = D(1) & D_1(1) = D(3) & D_1(2) = D(4) & D_1(\tau_{01}) = f_{13} & D_1(\tau_{02}) = f_{14} \\
E_1(0) = E(1) & E_1(1) = E(3) & E_1(2) = E(4) & E_1(\tau_{01}) = g_{13} & E_1(\tau_{02}) = g_{14} \\
D_2(0) = D(2) & D_2(1) = D(4) & D_2(2) = D(5) & D_2(\tau_{01}) = f_{24} & D_2(\tau_{02}) = f_{25} \\
E_2(0) = E(2) & E_2(1) = E(4) & E_2(2) = E(5) & E_2(\tau_{01}) = g_{24} & E_2(\tau_{02}) = g_{25}
\end{array}
$$

For a natural transformation $\gamma : D \to E$, we define a natural transformations $\gamma^i : D_i \to E_i$ $(i = 0, 1, 2)$ by

$$
\gamma^0_0 = \gamma_0 \quad \gamma^0_1 = \gamma_3 \quad \gamma^0_2 = \gamma_5 \quad \gamma^1_0 = \gamma_1 \quad \gamma^1_1 = \gamma_3 \quad \gamma^1_2 = \gamma_4 \quad \gamma^2_0 = \gamma_2 \quad \gamma^2_1 = \gamma_4 \quad \gamma^2_2 = \gamma_5
$$

For an object M of $\mathcal{F}_{E_0(1)} = \mathcal{F}_{E(3)}$, the following diagram is commutative.

$$
\begin{tikzcd}
\gamma_3^*(M)_{[f_{13}f_{01},f_{25}f_{02}]} \arrow[rr, "\gamma^0_M"] \arrow[d, "\theta_D(\gamma_3^*(M))"] & & \gamma_5^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, "\gamma_5^*(\theta_E(M))"] \\
(\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]} \arrow[r, "(\gamma^1_M)_{[f_{24},f_{25}]}"] & (\gamma_4^*(M_{[g_{13},g_{14}]}))_{[f_{24},f_{25}]} \arrow[r, "\gamma^2_{M_{[g_{13},g_{14}]}}"] & \gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})
\end{tikzcd}
$$

Proof. By the naturality of $P_{f_{13}f_{01},f_{25}f_{02}}(\gamma_3^*(M))$ and the definition of γ_M^0, $\gamma_5^*(\theta_E(M))\gamma_M^0$ is the image of the following composition by $P_{f_{13}f_{01},f_{25}f_{02}}(\gamma_3^*(M))_{\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})}$.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*(M)) \xrightarrow{c_{\gamma_3,f_{13}f_{01}}(M)} (\gamma_3 f_{13}f_{01})^*(M) = (g_{13}g_{01}\gamma_0)^*(M) \xrightarrow{\gamma_0^\sharp(\iota_{g_{13}g_{01},g_{25}g_{02}}(M))} \\
&\qquad (g_{25}g_{02}\gamma_0)^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) = (\gamma_5 f_{25}f_{02})^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) \xrightarrow{c_{\gamma_5,f_{25}f_{02}}(M_{[g_{13}g_{01},g_{25}g_{02}]})^{-1}} \\
&\qquad (f_{25}f_{02})^*(\gamma_5^*(M_{[g_{13}g_{01},g_{25}g_{02}]})) \xrightarrow{(f_{25}f_{02})^*(\gamma_5^*(\theta_E(M)))} (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{aligned}$$

On the other hand, $\gamma^2_{M_{[g_{13},g_{14}]}}(\gamma_M^1)_{[f_{24},f_{25}]}\theta_D(\gamma_3^*(M)))$ is the image of the following composition.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*(M)) \xrightarrow{\iota_{f_{13}f_{01},f_{25}f_{02}}(\gamma_3^*(M))} (f_{25}f_{02})^*(\gamma_3^*(M)_{[f_{13}f_{01},f_{25}f_{02}]}) \xrightarrow{(f_{25}f_{02})^*(\theta_D(\gamma_3^*(M)))} \\
&\qquad (f_{25}f_{02})^*((\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}) \xrightarrow{(f_{25}f_{02})^*((\gamma_M^1)_{[f_{24},f_{25}]})} (f_{25}f_{02})^*((\gamma_4^*(M_{[g_{13},g_{14}]}))_{[f_{24},f_{25}]}) \\
&\qquad \xrightarrow{(f_{25}f_{02})^*(\gamma^2_{M_{[g_{13},g_{14}]}})} (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{aligned}$$

We see that $\gamma^2_{M_{[g_{13},g_{14}]}}(\gamma_M^1)_{[f_{24},f_{25}]}\theta_D(\gamma_3^*(M)))$ is the image of the following composition by applying (8.4.16) to the first two morphisms of the above diagram.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*(M)) \xrightarrow{f_{01}^\sharp(\iota_{f_{13},f_{14}}(\gamma_3^*(M))} (f_{14}f_{01})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) = (f_{24}f_{02})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \\
&\qquad \xrightarrow{f_{02}^\sharp(\iota_{f_{24},f_{25}}(\gamma_3^*(M)_{[f_{13},f_{14}]}))} (f_{25}f_{02})^*((\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}) \xrightarrow{(f_{25}f_{02})^*((\gamma_M^1)_{[f_{24},f_{25}]})} \\
&\qquad (f_{25}f_{02})^*((\gamma_4^*(M_{[g_{13},g_{14}]}))_{[f_{24},f_{25}]}) \xrightarrow{(f_{25}f_{02})^*(\gamma^2_{M_{[g_{13},g_{14}]}})} (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{aligned}$$

Hence it suffices to show that the following diagram (i) is commutative.

$$\begin{tikzcd}
(f_{13}f_{01})^*(\gamma_3^*(M)) \arrow[r, "c_{\gamma_3,f_{13}f_{01}}(M)"] \arrow[d, "f_{01}^\sharp(\iota_{f_{13},f_{14}}(\gamma_3^*(M)))"] & (\gamma_3 f_{13}f_{01})^*(M) \arrow[r, equal] & (g_{13}g_{01}\gamma_0)^*(M) \arrow[d, "\gamma_0^\sharp(\iota_{g_{13}g_{01},g_{25}g_{02}}(M))"] \\
(f_{14}f_{01})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \arrow[d, equal] & & (g_{25}g_{02}\gamma_0)^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, equal] \\
(f_{24}f_{02})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \arrow[d, "f_{02}^\sharp(\iota_{f_{24},f_{25}}(\gamma_3^*(M)_{[f_{13},f_{14}]}))"] & & (\gamma_5 f_{25}f_{02})^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, "c_{\gamma_5,f_{25}f_{02}}(M_{[g_{13}g_{01},g_{25}g_{02}]})^{-1}"] \\
(f_{25}f_{02})^*((\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}) \arrow[d, "(f_{25}f_{02})^*((\gamma_M^1)_{[f_{24},f_{25}]})"] & & (f_{25}f_{02})^*(\gamma_5^*(M_{[g_{13}g_{01},g_{25}g_{02}]})) \arrow[d, "(f_{25}f_{02})^*(\gamma_5^*(\theta_E(M)))"] \\
(f_{25}f_{02})^*((\gamma_4^*(M_{[g_{13},g_{14}]}))_{[f_{24},f_{25}]}) \arrow[rr, "(f_{25}f_{02})^*(\gamma^2_{M_{[g_{13},g_{14}]}})"] & & (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{tikzcd}$$

diagram (i)

The following diagram (ii) is commutative by (8.1.10) and the definition of $f_{02}^\sharp$.

$$\begin{tikzcd}
(f_{24}f_{02})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \arrow[r, "c_{f_{24},f_{02}}(\gamma_3^*(M)_{[f_{13},f_{14}]})^{-1}"] \arrow[d, "f_{02}^\sharp(\iota_{f_{24},f_{25}}(\gamma_3^*(M)_{[f_{13},f_{14}]}))"] & f_{02}^*(f_{24}^*(\gamma_3^*(M)_{[f_{13},f_{14}]})) \arrow[d, "f_{02}^*(\iota_{f_{24},f_{25}}(\gamma_3^*(M)_{[f_{13},f_{14}]}))"] \\
(f_{25}f_{02})^*((\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]}) \arrow[d, "(f_{25}f_{02})^*((\gamma_M^1)_{[f_{24},f_{25}]})"] & f_{02}^*(f_{25}^*((\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})) \arrow[l, "c_{f_{25},f_{02}}((\gamma_3^*(M)_{[f_{13},f_{14}]})_{[f_{24},f_{25}]})"'] \arrow[d, "f_{02}^*(f_{25}^*((\gamma_M^1)_{[f_{24},f_{25}]}))"] \\
(f_{25}f_{02})^*((\gamma_4^*(M_{[g_{13},g_{14}]}))_{[f_{24},f_{25}]}) \arrow[d, "(f_{25}f_{02})^*(\gamma^2_{M_{[g_{13},g_{14}]}})"] & f_{02}^*(f_{25}^*(\gamma_4^*(M_{[g_{13},g_{14}]})_{[f_{24},f_{25}]}) \arrow[l, "c_{f_{25},f_{02}}(\gamma_4^*(M_{[g_{13},g_{14}]})_{[f_{24},f_{25}]})"'] \arrow[d, "f_{02}^*(f_{25}^*(\gamma^2_{M_{[g_{13},g_{14}]}}))"] \\
(f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})) & f_{02}^*(f_{25}^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})) \arrow[l, "c_{f_{25},f_{02}}(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))"']
\end{tikzcd}$$

diagram (ii)

It follows from (8.4.3), (8.4.2) and the definition of $\gamma^2_{M_{[g_{13},g_{14}]}}$ that the following equalities hold.

$$f_{25}^*((\gamma^1_M)_{[f_{24},f_{25}]})\iota_{f_{24},f_{25}}(\gamma_3^*(M)_{[f_{13},f_{14}]}) = \iota_{f_{24},f_{25}}(\gamma_4^*(M_{[g_{13},g_{14}]}))f_{24}^*(\gamma^1_M)$$
$$f_{25}^*(\gamma^2_{M_{[g_{13},g_{14}]}})\iota_{f_{24},f_{25}}(\gamma_4^*(M_{[g_{13},g_{14}]})) = c_{\gamma_5,f_{25}}((M_{[g_{13},g_{14}]})_{[g_{13},g_{14}]})^{-1}\gamma_2^\sharp(\iota_{g_{24},g_{25}}(M_{[g_{13},g_{14}]}))c_{\gamma_4,f_{24}}(M_{[g_{13},g_{14}]})$$

Hence the composition of the right vertical morphisms of diagram (ii) coincides with the following.

$$\begin{aligned}
&f_{02}^*(f_{25}^*(\gamma^2_{M_{[g_{13},g_{14}]}}))f_{02}^*(f_{25}^*((\gamma^1_M)_{[f_{24},f_{25}]}))f_{02}^*(\iota_{f_{24},f_{25}}(\gamma_3^*(M)_{[f_{13},f_{14}]}))\\
&\quad= f_{02}^*(f_{25}^*(\gamma^2_{M_{[g_{13},g_{14}]}}))f_{02}^*(\iota_{f_{24},f_{25}}(\gamma_4^*(M_{[g_{13},g_{14}]})))f_{02}^*(f_{24}^*(\gamma^1_M))\\
&\quad= f_{02}^*(c_{\gamma_5,f_{25}}((M_{[g_{13},g_{14}]})_{[g_{13},g_{14}]})^{-1})f_{02}^*(\gamma_2^\sharp(\iota_{g_{24},g_{25}}(M_{[g_{13},g_{14}]})))f_{02}^*(c_{\gamma_4,f_{24}}(M_{[g_{13},g_{14}]}))f_{02}^*(f_{24}^*(\gamma^1_M))
\end{aligned}$$

Since $f_{02}^*(f_{24}^*(\gamma^1_M))c_{f_{24},f_{02}}(\gamma_3^*(M)_{[f_{13},f_{14}]})^{-1} = c_{f_{24},f_{02}}(\gamma_4^*(M_{[g_{13},g_{14}]}))^{-1}(f_{24}f_{02})^*(\gamma^1_M)$ by (8.1.10), the commutativity of diagram (ii) implies that the composition of the right vertical morphisms and the lower horizontal morphism of diagram (i) coincides with the following composition.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*(M)) \xrightarrow{f_{01}^\sharp(\iota_{f_{13},f_{14}}(\gamma_3^*(M)))} (f_{14}f_{01})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \xrightarrow{(f_{14}f_{01})^*(\gamma^1_M)} (f_{14}f_{01})^*(\gamma_4^*(M_{[g_{13},g_{14}]})) =\\
&(f_{24}f_{02})^*(\gamma_4^*(M_{[g_{13},g_{14}]})) \xrightarrow{c_{f_{24},f_{02}}(\gamma_4^*(M_{[g_{13},g_{14}]}))^{-1}} f_{02}^*(f_{24}^*(\gamma_4^*(M_{[g_{13},g_{14}]}))) \xrightarrow{f_{02}^*(c_{\gamma_4,f_{24}}(M_{[g_{13},g_{14}]}))}\\
&f_{02}^*((\gamma_4f_{24})^*(M_{[g_{13},g_{14}]})) = f_{02}^*((g_{24}\gamma_2)^*(M_{[g_{13},g_{14}]})) \xrightarrow{f_{02}^*(\gamma_2^\sharp(\iota_{g_{24},g_{25}}(M_{[g_{13},g_{14}]})))}\\
&f_{02}^*((g_{25}\gamma_2)^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})) = f_{02}^*((\gamma_5f_{25})^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})) \xrightarrow{f_{02}^*(c_{\gamma_5,f_{25}}((M_{[g_{13},g_{14}]})_{[g_{13},g_{14}]})^{-1})}\\
&f_{02}^*(f_{25}^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))) \xrightarrow{c_{f_{25},f_{02}}(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))} (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{aligned}$$

diagram (iii)

Next, we consider the composition of the upper horizontal morphism and the right vertical morphisms of diagram (i). It follows from (8.1.10) and (8.4.16) that the following diagram is commutative.

$$\begin{tikzcd}
\gamma_0^*((g_{13}g_{01})^*(M)) \arrow[r, "\gamma_0^*(g_{01}^\sharp(\iota_{g_{13},g_{14}}(M)))"] \arrow[d, "\gamma_0^*(\iota_{g_{13}g_{01},g_{25}g_{02}}(M))"] & \gamma_0^*((g_{14}g_{01})^*(M_{[g_{13},g_{14}]})) \arrow[r, equal] & \gamma_0^*((g_{24}g_{02})^*(M_{[g_{13},g_{14}]})) \arrow[d, "\gamma_0^*(g_{02}^\sharp(\iota_{g_{24},g_{25}}(M_{[g_{13},g_{14}]})))"'] \\
\gamma_0^*((g_{25}g_{02})^*(M_{[g_{13}g_{01},g_{25}g_{02}]})) \arrow[rr, "\gamma_0^*((g_{25}g_{02})^*(\theta_E(M)))"] \arrow[d, "c_{g_{25}g_{02},\gamma_0}(M_{[g_{13}g_{01},g_{25}g_{02}]})"] & & \gamma_0^*((g_{25}g_{02})^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})) \arrow[d, "c_{g_{25}g_{02},\gamma_0}((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})"'] \\
(g_{25}g_{02}\gamma_0)^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[rr, "(g_{25}g_{02}\gamma_0)^*(\theta_E(M))"] \arrow[d, equal] & & (g_{25}g_{02}\gamma_0)^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}) \arrow[d, equal] \\
(\gamma_5f_{25}f_{02})^*(M_{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[rr, "(\gamma_5f_{25}f_{02})^*(\theta_E(M))"] \arrow[d, "c_{\gamma_5,f_{25}f_{02}}(M_{[g_{13}g_{01},g_{25}g_{02}]})^{-1}"] & & (\gamma_5f_{25}f_{02})^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}) \arrow[d, "c_{\gamma_5,f_{25}f_{02}}((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})^{-1}"'] \\
(f_{25}f_{02})^*(\gamma_5^*(M_{[g_{13}g_{01},g_{25}g_{02}]})) \arrow[rr, "(f_{25}f_{02})^*(\gamma_5^*(\theta_E(M)))"] & & (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{tikzcd}$$

Since $\gamma_0^\sharp(\iota_{g_{13}g_{01},g_{25}g_{02}}(M)) = c_{g_{25}g_{02},\gamma_0}(M_{[g_{13}g_{01},g_{25}g_{02}]})\gamma_0^*(\iota_{g_{13}g_{01},g_{25}g_{02}}(M))c_{g_{13}g_{01},\gamma_0}(M)^{-1}$, it follows from the above diagram that the composition of the upper horizontal morphism and the right vertical morphisms of diagram (i) coincides with the following composition.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*(M)) \xrightarrow{c_{\gamma_3,f_{13}f_{01}}(M)} (\gamma_3f_{13}f_{01})^*(M) = (g_{13}g_{01}\gamma_0)^*(M) \xrightarrow{c_{g_{13}g_{01},\gamma_0}(M)^{-1}} \gamma_0^*((g_{13}g_{01})^*(M))\\
&\xrightarrow{\gamma_0^*(g_{01}^\sharp(\iota_{g_{13},g_{14}}(M)))} \gamma_0^*((g_{14}g_{01})^*(M_{[g_{13},g_{14}]})) = \gamma_0^*((g_{24}g_{02})^*(M_{[g_{13},g_{14}]})) \xrightarrow{\gamma_0^*(g_{02}^\sharp(\iota_{g_{24},g_{25}}(M_{[g_{13},g_{14}]})))}\\
&\gamma_0^*((g_{25}g_{02})^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})) \xrightarrow{c_{g_{25}g_{02},\gamma_0}((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})} (g_{25}g_{02}\gamma_0)^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}) =\\
&(\gamma_5f_{25}g_{02})^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}) \xrightarrow{c_{\gamma_5,f_{25}f_{02}}((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]})^{-1}} (f_{25}f_{02})^*(\gamma_5^*((M_{[g_{13},g_{14}]})_{[g_{24},g_{25}]}))
\end{aligned}$$

diagram (iv)

The following diagram is commutative by (8.1.10), (8.1.12) and (8.4.26).

$$
\begin{tikzcd}
(f_{13}f_{01})^*(\gamma_3^*(M)) \arrow[r, "c_{\gamma_3,f_{13}f_{01}}(M)"] \arrow[d, "c_{f_{13},f_{01}}(\gamma_3^*(M))^{-1}"] & (\gamma_3 f_{13}f_{01})^*(M) \arrow[r, equal] \arrow[d, "c_{\gamma_3 f_{13},f_{01}}(M)^{-1}"'] & (g_{13}\gamma_1 f_{01})^*(M) \arrow[d, "c_{g_{13}\gamma_1,f_{01}}(M)^{-1}"'] \\
f_{01}^*(f_{13}^*(\gamma_3^*(M))) \arrow[r, "f_{01}^*(c_{\gamma_3,f_{13}}(M))"] \arrow[dd, "f_{01}^*(\iota_{f_{13},f_{14}}(\gamma_3^*(M)))"] & f_{01}^*((\gamma_3 f_{13})^*(M)) \arrow[r, equal] & f_{01}^*((g_{13}\gamma_1)^*(M)) \arrow[d, "f_{01}^*(\gamma_1^\sharp(\iota_{g_{13},g_{14}}(M)))"'] \\
 & f_{01}^*((\gamma_4 f_{14})^*(M_{[g_{13},g_{14}]})) \arrow[r, equal] & f_{01}^*((g_{14}\gamma_1)^*(M_{[g_{13},g_{14}]})) \arrow[d, "c_{g_{14}\gamma_1,f_{01}}(M_{[g_{13},g_{14}]})"'] \\
f_{01}^*(f_{14}^*(\gamma_3^*(M)_{[f_{13},f_{14}]})) \arrow[r, "f_{01}^*(f_{14}^*(\gamma_M^1))"] \arrow[d, "c_{f_{14},f_{01}}(\gamma_3^*(M)_{[f_{13},f_{14}]})"] & f_{01}^*(f_{14}^*(\gamma_4^*(M_{[g_{13},g_{14}]}))) \arrow[u, "f_{01}^*(c_{\gamma_4,f_{14}}(M_{[g_{13},g_{14}]}))"] \arrow[d, "c_{f_{14},f_{01}}(\gamma_4^*(M_{[g_{13},g_{14}]}))"] & (g_{14}\gamma_1 f_{01})^*(M_{[g_{13},g_{14}]}) \arrow[d, equal] \\
(f_{14}f_{01})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \arrow[r, "(f_{14}f_{01})^*(\gamma_M^1)"] & (f_{14}f_{01})^*(\gamma_4^*(M_{[g_{13},g_{14}]})) \arrow[r, "c_{\gamma_4,f_{14}f_{01}}(M_{[g_{13},g_{14}]})"] & (\gamma_4 f_{14}f_{01})^*(M_{[g_{13},g_{14}]})
\end{tikzcd}
$$

Hence the following diagram is commutative by (8.1.12) and (8.1.14). Here we put $N = M_{[g_{13},g_{14}]}$ below.

$$
\begin{tikzcd}
(f_{13}f_{01})^*(\gamma_3^*(M)) \arrow[r, "c_{\gamma_3,f_{13}f_{01}}(M)"] \arrow[d, "f_{01}^\sharp(\iota_{f_{13},f_{14}}(\gamma_3^*(M)))"] & (\gamma_3 f_{13}f_{01})^*(M) \arrow[r, equal] \arrow[d, equal] & (g_{13}g_{01}\gamma_0)^*(M) \arrow[d, "c_{g_{13}g_{01},\gamma_0}(M)^{-1}"] \arrow[ddl, "\gamma_0^\sharp(g_{01}^\sharp(\iota_{g_{13},g_{14}}(M)))"'] \\
(f_{14}f_{01})^*(\gamma_3^*(M)_{[f_{13},f_{14}]}) \arrow[d, "(f_{14}f_{01})^*(\gamma_M^1)"] & (g_{13}\gamma_1 f_{01})^*(M) \arrow[d, "f_{01}^\sharp(\gamma_1^\sharp(\iota_{g_{13},g_{14}}(M)))"'] & \gamma_0^*((g_{13}g_{01})^*(M)) \arrow[d, "\gamma_0^*(g_{01}^\sharp(\iota_{g_{13},g_{14}}(M)))"'] \\
(f_{14}f_{01})^*(\gamma_4^*(N)) \arrow[d, equal] \arrow[ddr, "c_{\gamma_4,f_{14}f_{01}}(N)"] & (g_{14}g_{01}\gamma_0)^*(N) \arrow[d, equal] & \gamma_0^*((g_{14}g_{01})^*(N)) \arrow[l, "c_{g_{14}g_{01},\gamma_0}(N)"'] \arrow[d, equal] \\
(f_{24}f_{02})^*(\gamma_4^*(N)) \arrow[d, "c_{f_{24},f_{02}}(\gamma_4^*(N))^{-1}"] & (g_{24}g_{02}\gamma_0)^*(N) \arrow[d, equal] \arrow[ddr, bend right, "\gamma_0^\sharp(g_{02}^\sharp(\iota_{g_{24},g_{25}}(N)))"] & \gamma_0^*((g_{24}g_{02})^*(N)) \arrow[l, "c_{g_{24}g_{02},\gamma_0}(N)"'] \arrow[d, "\gamma_0^*(g_{02}^\sharp(\iota_{g_{24},g_{25}}(N)))"'] \\
f_{02}^*(f_{24}^*(\gamma_4^*(N))) \arrow[d, "f_{02}^*(c_{\gamma_4,f_{24}}(N))"] & (\gamma_4 f_{14}f_{01})^*(N) \arrow[d, equal] & \gamma_0^*((g_{25}g_{02})^*(N_{[g_{24},g_{25}]})) \arrow[d, "c_{g_{25}g_{02},\gamma_0}(N_{[g_{24},g_{25}]})"'] \\
f_{02}^*((\gamma_4 f_{24})^*(N)) \arrow[d, equal] & (\gamma_4 f_{24}f_{02})^*(N) \arrow[l, "c_{\gamma_4 f_{24},f_{02}}(N)"'] \arrow[d, equal] & (g_{25}g_{02}\gamma_0)^*(N_{[g_{24},g_{25}]}) \arrow[d, equal] \\
f_{02}^*((g_{24}\gamma_2)^*(N)) \arrow[d, "f_{02}^*(\gamma_2^\sharp(\iota_{g_{24},g_{25}}(N)))"] & (g_{24}\gamma_2 f_{02})^*(N) \arrow[l, "c_{g_{24}\gamma_2,f_{02}}(N)"'] \arrow[r, "f_{02}^\sharp(\gamma_2^\sharp(\iota_{g_{24},g_{25}}(N)))"] & (\gamma_5 f_{25}f_{02})^*(N_{[g_{24},g_{25}]}) \\
f_{02}^*((g_{25}\gamma_2)^*(N_{[g_{24},g_{25}]})) \arrow[d, equal] \arrow[urr, "c_{f_{25}\gamma_2,f_{02}}(N_{[g_{24},g_{25}]})"] & & (f_{25}f_{02})^*(\gamma_5^*(N_{[g_{24},g_{25}]})) \arrow[u, "c_{\gamma_5,f_{25}f_{02}}(N_{[g_{24},g_{25}]})"'] \\
f_{02}^*((\gamma_5 f_{25})^*(N_{[g_{24},g_{25}]})) \arrow[rr, "f_{02}^*(c_{\gamma_5,f_{25}}(N_{[g_{24},g_{25}]})^{-1})"] \arrow[uurr, "c_{\gamma_5 f_{25},f_{02}}(N_{[g_{24},g_{25}]})"'] & & f_{02}^*(f_{25}^*(\gamma_5^*(N_{[g_{24},g_{25}]}))) \arrow[u, "c_{f_{25},f_{02}}(\gamma_5^*(N_{[g_{24},g_{25}]}))"']
\end{tikzcd}
$$

We see that the compositions of diagram (iii) and the compositions of diagram (iv) coincide, which implies the assertion. □

8.5 Fibered category with exponents

Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category. For morphisms $f : X \to Y$, $g : X \to Z$ of $\mathcal{E}$ and an object N of $\mathcal{F}_Z$, we define a presheaf $F_N^{f,g} : \mathcal{F}_Y^{op} \to \mathcal{S}et$ on $\mathcal{F}_Y$ by $F_N^{f,g}(M) = F_{f,g}(M,N) = \mathcal{F}_X(f^*(M), g^*(N))$ for $M \in \operatorname{Ob}\mathcal{F}_Y$ and $F_N^{f,g}(\varphi) = F_{f,g}(\varphi, id_N) = f^*(\varphi)^*$ for $\varphi \in \operatorname{Mor}\mathcal{F}_Y$.

Suppose that $F_N^{f,g}$ is representable. We choose an object $N^{[f,g]}$ of $\mathcal{F}_Y$ such that there exists a natural equivalence $E_{f,g}(N) : F_N^{f,g} \to h_{N^{[f,g]}}$, where $h_{N^{[f,g]}}$ is the presheaf on $\mathcal{F}_Y$ represented by $N^{[f,g]}$. If $X = Y$ and f is the identity morphism of X, we take $g^*(N)$ as $N^{[id_X,g]}$. Hence $E_{id_X,g}(N)_M$ is the identity map of

$\mathcal{F}_X(M, g^*(N))$. Let us denote by $\pi_{f,g}(N) : f^*(N^{[f,g]}) \to g^*(N)$ the morphism of $\mathcal{F}_X$ which is mapped to the identity morphism of $N^{[f,g]}$ by $E_{f,g}(N)_{N^{[f,g]}} : \mathcal{F}_X(f^*(N^{[f,g]}), g^*(N)) \to \mathcal{F}_Y(N^{[f,g]}, N^{[f,g]})$.

Remark 8.5.1 *If $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ has a right adjoint $f_! : \mathcal{F}_X \to \mathcal{F}_Y$, $F_N^{f,g} : \mathcal{F}_Y^{op} \to \mathcal{S}et$ is representable for any object N of $\mathcal{F}_Z$. In fact, $N^{[f,g]}$ is defined to be $f_!g^*(N)$ in this case. If we denote by $\mathrm{ad}^f_{M,P} : \mathcal{F}_X(f^*(M), P) \to \mathcal{F}_Y(M, f_!(P))$ the bijection which is natural in $M \in \mathrm{Ob}\,\mathcal{F}_Y$ and $P \in \mathrm{Ob}\,\mathcal{F}_X$, we have $E_{f,g}(N)_M = \mathrm{ad}^f_{M,g^*(N)} : \mathcal{F}_X(f^*(M), g^*(N)) \to \mathcal{F}_Y(M, f_!g^*(N))$. Let us denote by $\varepsilon^f : f^*f_! \to id_{\mathcal{F}_X}$ the counit of the adjunction $f^* \dashv f_!$. We have $\pi_{f,g}(N) = \varepsilon^f_{g^*(N)} : f^*(N^{[f,g]}) = f^*f_!g^*(N) \to g^*(N)$.*

Proposition 8.5.2 *The inverse of $E_{f,g}(N)_M : \mathcal{F}_X(f^*(M), g^*(N)) \to \mathcal{F}_Y(M, N^{[f,g]})$ is given by the map defined by $\varphi \mapsto \pi_{f,g}(N)f^*(\varphi)$.*

Proof. For $\varphi \in \mathcal{F}_Y(M, N^{[f,g]})$, the following diagram commutes by naturality of $E_{f,g}(N)$.

$$\begin{tikzcd}
\mathcal{F}_X(f^*(N^{[f,g]}), g^*(N)) \arrow[r, "f^*(\varphi)^*"] \arrow[d, "E_{f,g}(N)_{N^{[f,g]}}"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "E_{f,g}(N)_M"] \\
\mathcal{F}_Y(N^{[f,g]}, N^{[f,g]}) \arrow[r, "\varphi^*"] & \mathcal{F}_Y(M, N^{[f,g]})
\end{tikzcd}$$

It follows that $E_{f,g}(N)_M$ maps $\pi_{f,g}(N)f^*(\varphi)$ to φ. □

For a morphism $\varphi : L \to N$ of $\mathcal{F}_Z$, define a natural transformation $F_\varphi^{f,g} : F_L^{f,g} \to F_N^{f,g}$ by

$$(F_\varphi^{f,g})_M = g^*(\varphi)_* : F_L^{f,g}(M) = \mathcal{F}_X(f^*(M), g^*(L)) \to \mathcal{F}_X(f^*(M), g^*(N)) = F_N^{f,g}(M).$$

It is clear that $F_{\psi\varphi}^{f,g} = F_\psi^{f,g}F_\varphi^{f,g}$ for morphisms $\psi : N \to P$ and $\varphi : L \to N$ of $\mathcal{F}_Z$. We define $\varphi^{[f,g]} : L^{[f,g]} \to N^{[f,g]}$ by $\varphi^{[f,g]} = E_{f,g}(N)_{L^{[f,g]}}((F_\varphi^{f,g})_{L^{[f,g]}}(\pi_{f,g}(L))) = E_{f,g}(N)_{L^{[f,g]}}(g^*(\varphi)\pi_{f,g}(L)) \in h_{N^{[f,g]}}(L^{[f,g]})$.

Proposition 8.5.3 *(1) The following diagrams commute for any $M \in \mathrm{Ob}\,\mathcal{F}_Y$.*

$$\begin{tikzcd}
f^*(L^{[f,g]}) \arrow[r, "f^*(\varphi^{[f,g]})"] \arrow[d, "\pi_{f,g}(L)"] & f^*(N^{[f,g]}) \arrow[d, "\pi_{f,g}(N)"] \\
g^*(L) \arrow[r, "g^*(\varphi)"] & g^*(N)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(L)) \arrow[r, "g^*(\varphi)_*"] \arrow[d, "E_{f,g}(L)_M"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "E_{f,g}(N)_M"] \\
\mathcal{F}_Y(M, L^{[f,g]}) \arrow[r, "\varphi^{[f,g]}_*"] & \mathcal{F}_Y(M, N^{[f,g]})
\end{tikzcd}$$

(2) For morphisms $\psi : N \to P$ and $\varphi : L \to N$ of $\mathcal{F}_Z$, we have $(\psi\varphi)^{[f,g]} = \psi^{[f,g]}\varphi^{[f,g]}$.

(3) If $g^* : \mathcal{F}_Z \to \mathcal{F}_X$ preserves monomorphisms (g^* has a left adjoint, for example) and $\varphi : L \to N$ is a monomorphism, so is $\varphi^{[f,g]} : L^{[f,g]} \to N^{[f,g]}$.

Proof. (1) We have $E_{f,g}(N)_{L^{[f,g]}}(g^*(\varphi)\pi_{f,g}(L)) = \varphi^{[f,g]}$ by the definition of $\varphi^{[f,g]}$. On the other hand, it follows from (8.5.2) that $E_{f,g}(N)_{L^{[f,g]}}(\pi_{f,g}(N)f^*(\varphi^{[f,g]})) = \varphi^{[f,g]}$. Since $E_{f,g}(N)_{L^{[f,g]}}$ is bijective, the left diagram commutes.

For $\psi \in \mathcal{F}_Y(M, L^{[f,g]})$, it follows from 8.5.2 and commutativity of the left diagram that we have

$$\begin{aligned}
g^*(\varphi)_*E_{f,g}(L)_M^{-1}(\psi) &= g^*(\varphi)\pi_{f,g}(L)f^*(\psi) = \pi_{f,g}(N)f^*(\varphi^{[f,g]})f^*(\psi) = \pi_{f,g}(N)f^*(\varphi^{[f,g]}\psi) \\
&= E_{f,g}(N)_M^{-1}(\varphi^{[f,g]}\psi) = E_{f,g}(N)_M^{-1}\varphi^{[f,g]}_*(\psi).
\end{aligned}$$

Hence the right diagram commutes.

(2) The following diagram commutes by (1).

$$\begin{tikzcd}
\mathcal{F}_X(f^*(L^{[f,g]}), g^*(L)) \arrow[r, "g^*(\varphi)_*"] \arrow[d, "E_{f,g}(L)_{L^{[f,g]}}"] & \mathcal{F}_X(f^*(L^{[f,g]}), g^*(N)) \arrow[r, "g^*(\psi)_*"] \arrow[d, "E_{f,g}(N)_{L^{[f,g]}}"] & \mathcal{F}_X(f^*(L^{[f,g]}), g^*(P))) \arrow[d, "E_{f,g}(P)_{L^{[f,g]}}"] \\
\mathcal{F}_Y(L^{[f,g]}, L^{[f,g]}) \arrow[r, "\varphi^{[f,g]}_*"] & \mathcal{F}_Y(L^{[f,g]}, N^{[f,g]}) \arrow[r, "\psi^{[f,g]}_*"] & \mathcal{F}_Y(L^{[f,g]}, P^{[f,g]})
\end{tikzcd}$$

Hence $\psi^{[f,g]}\varphi^{[f,g]} = \psi_*^{[f,g]}\varphi_*^{[f,g]}(id_{L^{[f,g]}}) = E_{f,g}(P)_{L^{[f,g]}}(g^*(\psi)g^*(\varphi)\pi_{f,g}(L)) = E_{f,g}(P)_{L^{[f,g]}}(g^*(\psi\varphi)\pi_{f,g}(L)) = (\psi\varphi)^{[f,g]}$.

(3) is a direct consequence of (1). □

Remark 8.5.4 *Suppose that $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ has a right adjoint $f_! : \mathcal{F}_X \to \mathcal{F}_Y$. For a morphism $\varphi : L \to N$ of $\mathcal{F}_Z$, we have $\varphi^{[f,g]} = f_!g^*(\varphi) : L^{[f,g]} = f_!g^*(L) \to f_!g^*(N) = N^{[f,g]}$. In fact, if we denote by $\eta^f : id_{\mathcal{F}_X} \to f_!f^*$ the unit of the adjunction $f^* \dashv f_!$, we have $\varphi^{[f,g]} = E_X(N)_{L^{[f,g]}}(g^*(\varphi)\pi_{f,g}(L)) = \mathrm{ad}^f_{L^{[f,g]},g^*(N)}\big(g^*(\varphi)\varepsilon^f_{f^*(L)}\big) = f_!g^*(\varphi)f_!\big(\varepsilon^f_{g^*(L)}\big)\eta^f_{f_!g^*(L)} = f_!g^*(\varphi)$.*

Lemma 8.5.5 *Let $\xi : f^*(L) \to g^*(M)$, $\zeta : f^*(N) \to g^*(K)$ be morphisms of $\mathcal{F}_X$ for $L, N \in \mathrm{Ob}\,\mathcal{F}_Y$, $M, K \in \mathrm{Ob}\,\mathcal{F}_Z$. Let $\varphi : L \to N$ be a morphism of $\mathcal{F}_Y$ and $\psi : M \to K$ a morphism of $\mathcal{F}_Z$. We put $\check{\xi} = E_{f,g}(L)_M(\xi)$ and $\check{\zeta} = E_{f,g}(K)_N(\zeta)$. The following left diagram commutes if and only if the right one commutes.*

$$\begin{tikzcd} f^*(L) \arrow[r, "\xi"] \arrow[d, "f^*(\varphi)"] & g^*(M) \arrow[d, "g^*(\psi)"] \\ f^*(N) \arrow[r, "\zeta"] & g^*(K) \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, "\check{\xi}"] \arrow[d, "\varphi"] & M^{[f,g]} \arrow[d, "\psi^{[f,g]}"] \\ N \arrow[r, "\check{\zeta}"] & K^{[f,g]} \end{tikzcd}$$

Proof. The following diagram is commutative by (8.5.3) and the naturality of $E_{f,g}(K)$.

$$\begin{tikzcd} \mathcal{F}_X(f^*(L), g^*(M)) \arrow[r, "g^*(\psi)_*"] \arrow[d, "E_{f,g}(M)_L"] & \mathcal{F}_X(f^*(L), g^*(K)) \arrow[d, "E_{f,g}(K)_L"] & \mathcal{F}_X(f^*(N), g^*(K)) \arrow[l, "f^*(\varphi)^*"'] \arrow[d, "E_{f,g}(K)_N"] \\ \mathcal{F}_Y(L, M^{[f,g]}) \arrow[r, "\psi_*^{[f,g]}"] & \mathcal{F}_Y(L, K^{[f,g]}) & \mathcal{F}_Y(N, K^{[f,g]}) \arrow[l, "\varphi^*"'] \end{tikzcd}$$

Since $\check{\xi} = E_{f,g}(L)_M(\xi)$, $\check{\zeta} = E_{f,g}(K)_N(\zeta)$ and $E_{f,g}(K)_L$ is bijective, $g^*(\psi)\xi = g^*(\psi)_*(\xi) = f^*(\varphi)^*(\zeta) = \zeta f^*(\varphi)$ if and only if $\psi^{[f,g]}\check{\xi} = \psi_*^{[f,g]}(\check{\xi}) = \varphi^*(\check{\zeta}) = \check{\zeta}\varphi$. □

For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$ of $\mathcal{E}$ and $N \in \mathrm{Ob}\,\mathcal{F}_Z$, suppose that $F_N^{f,g}$ and $F_N^{fk,gk}$ are representable. We define a morphism $N^k : N^{[f,g]} \to N^{[fk,gk]}$ of $\mathcal{F}_Y$ by

$$N^k = E_{fk,gk}(N)_{N^{[f,g]}}(k^\sharp_{N^{[f,g]},N}(\pi_{f,g}(N))) \in \mathcal{F}_Y(N^{[f,g]}, N^{[fk,gk]}).$$

Proposition 8.5.6 *(1) The following diagram commutes for any $M \in \mathrm{Ob}\,\mathcal{F}_Y$.*

$$\begin{tikzcd} \mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "k^\sharp_{M,N}"] \arrow[d, "E_{f,g}(N)_M"] & \mathcal{F}_V((fk)^*(M), (gk)^*(N)) \arrow[d, "E_{fk,gk}(N)_M"] \\ \mathcal{F}_Y(M, N^{[f,g]}) \arrow[r, "N^k_*"] & \mathcal{F}_Y(M, N^{[fk,gk]}) \end{tikzcd} \qquad \begin{tikzcd} (fk)^*(N^{[f,g]}) \arrow[rr, "k^\sharp_{N^{[f,g]},N}(\pi_{f,g}(N))"] \arrow[dr, "(fk)^*(N^k)"'] & & (gk)^*(N) \\ & (fk)^*(N^{[fk,gk]}) \arrow[ur, "\pi_{fk,gk}(N)"'] & \end{tikzcd}$$

(2) For morphisms $k : V \to X$ and $l : U \to V$ of $\mathcal{E}$, $N^{kl} = N^l N^k$.

(3) The image of the identity morphism of $k^*(N)$ by $E_{k,k}(N)_N$ is $N^k : N = N^{[id_X,id_X]} \to N^{[k,k]}$ if $X = Z$.

(4) A composition $k^*(N) = k^*(N^{[id_X,id_X]}) \xrightarrow{k^*(N^k)} k^*(N^{[k,k]}) \xrightarrow{\pi_{k,k}(N)} k^*(N)$ is the identity morphism of $k^*(N)$ if $X = Z$.

Proof. (1) For $\varphi \in \mathcal{F}_Y(M, N^{[f,g]})$, it follows from the naturality of $k^\sharp_{M,N}$ and (8.5.2) that we have

$$\begin{aligned} k^\sharp_{M,N}E_{f,g}(N)_M^{-1}(\varphi) &= k^\sharp_{M,N}(\pi_{f,g}(N)f^*(\varphi)) = k^\sharp_{M,N}f^*(\varphi)^*(\pi_Y(N)) = f^*(\varphi)^*k^\sharp_{N^{[f,g]},N}(\pi_{f,g}(N)) \\ &= f^*(\varphi)^*E_{fk,gk}(N)^{-1}_{N^{[f,g]}}(N^k) = \pi_{fk,gk}(N)f^*(N^k)f^*(\varphi) = \pi_{fk,gk}(N)f^*(N^k\varphi) \\ &= \pi_{fk,gk}(N)f^*((N^k)_*(\varphi)) = E_{fk,gk}(N)^{-1}_M(N^k)_*(\varphi). \end{aligned}$$

The commutativity of the right diagram follows from (8.5.2) and the commutativity of the left diagram for the case $M = N^{[f,g]}$.

(2) The following diagram commutes by (1).

$$\begin{array}{ccccc}
\mathcal{F}_X(f^*(N^{[f,g]}), g^*(N)) & \xrightarrow{k^\sharp_{N^{[f,g]},N}} & \mathcal{F}_V((fk)^*(N^{[f,g]}), (gk)^*(N)) & \xrightarrow{l^\sharp_{N^{[f,g]},N}} & \mathcal{F}_U((fkl)^*(N^{[f,g]}), (gkl)^*(N)) \\
\downarrow E_{f,g}(N)_{N^{[f,g]}} & & \downarrow E_{fk,gk}(N)_{N^{[f,g]}} & & \downarrow E_{fkl,gkl}(N)_{N^{[f,g]}} \\
\mathcal{F}_Y(N^{[f,g]}, N^{[f,g]}) & \xrightarrow{N^k_*} & \mathcal{F}_Y(N^{[f,g]}, N^{[fk,gk]}) & \xrightarrow{N^l_*} & \mathcal{F}_Y(N^{[f,g]}, N^{[fkl,gkl]})
\end{array}$$

It follows the above diagram and (8.1.14) that

$$\begin{aligned}
N^l N^k &= N^l_* N^k_*(id_{N^{[f,g]}}) = E_{fkl,gkl}(N)_{N^{[f,g]}}(l^\sharp_{N^{[f,g]},N} k^\sharp_{N^{[f,g]},N}(\pi_{f,g}(N))) \\
&= E_{fkl,gkl}(N)_{N^{[f,g]}}((kl)^\sharp_{N^{[f,g]},N}(\pi_{f,g}(N))) = N^{kl}.
\end{aligned}$$

(3) Apply (1) for $M = N$, $Z = Y = X$ and $f = g = id_X$.

(4) It follows from (8.5.2) that $E_{k,k}(N)_N : \mathcal{F}_X(k^*(N), k^*(N)) \to \mathcal{F}_1(N, N^{[k,k]})$ maps $\pi_{k,k}(N)k^*(N^k)$ to $N^k : N \to N^{[k,k]}$. Thus the assertion follows from (3). □

Remark 8.5.7 *Suppose that the inverse image functors $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ and $(fk)^* : \mathcal{F}_Y \to \mathcal{F}_V$ have right adjoints $f_! : \mathcal{F}_X \to \mathcal{F}_Y$ and $(fk)_! : \mathcal{F}_V \to \mathcal{F}_Y$, respectively.*

(1) Since $k^\sharp_{N^{[f,g]},N}(\pi_{f,g}(N)) = c_{g,k}(N)k^*(\varepsilon^f_{g^*(N)})c_{f,k}(N^{[f,g]})^{-1}$ by (8.5.1) and

$$E_{fk,gk}(N)_{N^{[f,g]}} = \mathrm{ad}^{fk}_{N^{[f,g]},g^*(N)} : \mathcal{F}_V((fk)^*(N^{[f,g]}), (gk)^*(N)) \to \mathcal{F}_Y(N^{[f,g]}, N^{[fk,gk]})$$

maps $\varphi \in \mathcal{F}_X((fk)^*(N^{[f,g]}), (gk)^*(N))$ to $(fk)_!(\varphi)\eta^{fk}_{N^{[f,g]}}$, $N^k : N^{[f,g]} \to N^{[fk,gk]}$ coincides with the following composition.

$$\begin{aligned}
N^{[f,g]} &\xrightarrow{\eta^{fk}_{N^{[f,g]}}} (fk)_!(fk)^*(N^{[f,g]}) \xrightarrow{(fk)_!(c_{f,k}(N^{[f,g]}))^{-1}} (fk)_!k^*f^*(N^{[f,g]}) = (fk)_!k^*f^*f_!g^*(N) \\
&\xrightarrow{(fk)_!k^*\left(\varepsilon^f_{g^*(N)}\right)} (fk)_!k^*g^*(N) \xrightarrow{(fk)_!(c_{g,k}(N))} (fk)_!(gk)^*(N) = N^{[fk,gk]}
\end{aligned}$$

(2) The following diagram commutes by (8.5.6) if $X = Y = Z$ and $f = g = id_X$.

$$\begin{array}{ccc}
\mathcal{F}_X(N, N^{[id_X,id_X]}) & \xrightarrow{N^k_*} & \mathcal{F}_1(N, N^{[k,k]}) \\
\downarrow (\mathrm{ad}^{id_X}_{N,id^*_X(N)})^{-1} & & \downarrow (\mathrm{ad}^k_{N,k^*(N)})^{-1} \\
\mathcal{F}_X(id^*_X(N), id^*_X(N)) & \xrightarrow{(k^\sharp)_{N,N}} & \mathcal{F}_V(k^*(N), k^*(N))
\end{array}$$

Since id^*_X is the identity functor of $\mathcal{F}_X$, so is $id_{X!}$. Hence $N^k : N = N^{[id_X,id_X]} \to N^{[k,k]} = k_!k^*(N)$ is identified with the unit $\eta^k_N : N \to k_!k^*(N)$ of the adjunction $k^* \dashv k_!$ by the above diagram.

Proposition 8.5.8 *For morphisms $f : X \to Y$, $g : X \to Z$, $k : V \to X$ of $\mathcal{E}$ and a morphism $\varphi : L \to N$ of $\mathcal{F}_Z$, the following diagram commutes.*

$$\begin{array}{ccc}
L^{[f,g]} & \xrightarrow{\varphi^{[f,g]}} & N^{[f,g]} \\
\downarrow L^k & & \downarrow N^k \\
L^{[fk,gk]} & \xrightarrow{\varphi^{[fk,gk]}} & N^{[fk,gk]}
\end{array}$$

Proof. The following diagram commutes by the naturality of $k^\sharp$.

$$\begin{array}{ccc}
\mathcal{F}_X(f^*(M), g^*(L)) & \xrightarrow{k^\sharp_{M,L}} & \mathcal{F}_V((fk)^*(M), (gk)^*(L)) \\
\downarrow g^*(\varphi)_* & & \downarrow (gk)^*(\varphi)_* \\
\mathcal{F}_X(f^*(M), g^*(N)) & \xrightarrow{k^\sharp_{M,N}} & \mathcal{F}_V((fk)^*(M), (gk)^*(N))
\end{array}$$

Then, it follows from the commutativity of four diagrams

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(L)) \arrow[r, "E_{f,g}(L)_M"] \arrow[d, "g^*(\varphi)_*"] & \mathcal{F}_Y(M, L^{[f,g]}) \arrow[d, "\varphi_*^{[f,g]}"] \\
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "E_{f,g}(N)_M"] & \mathcal{F}_Y(M, N^{[f,g]})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_Y((fk)^*(M), (gk)^*(L)) \arrow[r, "E_{fk,gk}(L)_M"] \arrow[d, "(gk)^*(\varphi)_*"] & \mathcal{F}_Y(M, L^{[fk,gk]}) \arrow[d, "\varphi_*^{[fk,gk]}"] \\
\mathcal{F}_Y((fk)^*(M), (gk)^*(N)) \arrow[r, "E_{fk,gk}(N)_M"] & \mathcal{F}_Y(M, N^{[fk,gk]})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(L)) \arrow[r, "E_{f,g}(L)_M"] \arrow[d, "k^\sharp_{M,L}"] & \mathcal{F}_Y(M, L^{[f,g]}) \arrow[d, "L^k_*"] \\
\mathcal{F}_Y((fk)^*(M), (gk)^*(L)) \arrow[r, "E_{fk,gk}(L)_M"] & \mathcal{F}_Y(M, L^{[fk,gk]})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N)) \arrow[r, "E_{f,g}(N)_M"] \arrow[d, "k^\sharp_{M,N}"] & \mathcal{F}_Y(M, N^{[f,g]}) \arrow[d, "N^k_*"] \\
\mathcal{F}_Y((fk)^*(M), (gk)^*(N)) \arrow[r, "E_{fk,gk}(N)_M"] & \mathcal{F}_Y(M, N^{[fk,gk]})
\end{tikzcd}$$

and the fact that $E_{f,g}(L)_M : \mathcal{F}_X(f^*(M), g^*(L)) \to \mathcal{F}_Y(M, L^{[f,g]})$ is bijective that the following diagram commutes for any $M \in \operatorname{Ob}\mathcal{F}_Y$.

$$\begin{tikzcd}
\mathcal{F}_Y(M, L^{[f,g]}) \arrow[r, "\varphi_*^{[f,g]}"] \arrow[d, "L^k_*"] & \mathcal{F}_Y(M, N^{[f,g]}) \arrow[d, "N^k_*"] \\
\mathcal{F}_Y(M, L^{[fk,gk]}) \arrow[r, "\varphi_*^{[fk,gk]}"] & \mathcal{F}_Y(M, N^{[fk,gk]})
\end{tikzcd}$$

Thus the assertion follows. □

Remark 8.5.9 *We denote by $\varphi^k : L^{[f,g]} \to N^{[fk,gk]}$ the composition $N^k\varphi^{[f,g]} = \varphi^{[fk,gk]}L^k$. For morphisms $i : W \to Z$, $j : W \to T$, $h : U \to W$ of $\mathcal{E}$, it follows from (8.5.8) that the following diagram commutes.*

$$\begin{tikzcd}
(N^{[f,g]})^{[i,j]} \arrow[r, "(N^k)^{[i,j]}"] \arrow[d, "(N^{[f,g]})^h"] & (N^{[fk,gk]})^{[i,j]} \arrow[d, "(N^{[fk,gk]})^h"] \\
(N^{[f,g]})^{[ih,jh]} \arrow[r, "(N^k)^{[ih,jh]}"] & (N^{[fk,gk]})^{[ih,jh]}
\end{tikzcd}$$

Namely, we have $(N^k)^h = (N^{[fk,gk]})^h(N^k)^{[i,j]} = (N^k)^{[ih,jh]}(N^{[f,g]})^h$.

For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$ of $\mathcal{E}$ and $N \in \operatorname{Ob}\mathcal{F}_W$, we define a morphism $\epsilon_N^{f,g,h} : (N^{[g,h]})^{[f,g]} \to N^{[f,h]}$ of $\mathcal{F}_Y$ to be the image of $\pi_{g,h}(N)\pi_{f,g}(N^{[g,h]}) \in \mathcal{F}_X(f^*((N^{[g,h]})^{[f,g]}), h^*(N))$ by

$$E_{f,h}(N)_{(N^{[g,h]})^{[f,g]}} : \mathcal{F}_X(f^*((N^{[g,h]})^{[f,g]}), h^*(N)) \to \mathcal{F}_Y((N^{[g,h]})^{[f,g]}, N^{[f,h]}).$$

Proposition 8.5.10 *The following diagram commutes for any $M \in \operatorname{Ob}\mathcal{F}_Z$.*

$$\begin{tikzcd}
\mathcal{F}_X(f^*(M), g^*(N^{[g,h]})) \arrow[r, "\pi_{g,h}(N)_*"] \arrow[d, "E_{f,g}(N^{[g,h]})_M"] & \mathcal{F}_X(f^*(M), h^*(N)) \arrow[d, "E_{f,h}(N)_M"] \\
\mathcal{F}_Y(M, (N^{[g,h]})^{[f,g]}) \arrow[r, "\epsilon^{f,g,h}_{N*}"] & \mathcal{F}_Y(M, N^{[f,h]})
\end{tikzcd}$$

Proof. For $\varphi \in \mathcal{F}_Y(M, (N^{[g,h]})^{[f,g]})$, by the definition of $\epsilon_N^{f,g,h}$ and the naturality of $E_{f,h}(N)$, we have

$$\begin{aligned}
\pi_{g,h}(N)_*E_{f,g}(N^{[g,h]})^{-1}_M(\varphi) &= \pi_{g,h}(N)\pi_{f,g}(N^{[g,h]})f^*(\varphi) = f^*(\varphi)^*E_{f,h}(N)^{-1}_{(N^{[g,h]})^{[f,g]}}(\epsilon_N^{f,g,h}) \\
&= E_{f,h}(N)^{-1}_M\varphi^*(\epsilon_N^{f,g,h}) = E_{f,h}(N)^{-1}_M\epsilon^{f,g,h}_{N*}(\varphi).
\end{aligned}$$

□

We note that $\epsilon_N^{f,g,h} : (N^{[g,h]})^{[f,g]} \to N^{[f,h]}$ is the unique morphism that makes the diagram of (8.5.10) commute for any $M \in \operatorname{Ob}\mathcal{F}_W$.

Remark 8.5.11 *If $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ and $g^* : \mathcal{F}_Z \to \mathcal{F}_X$ have right adjoints $f_! : \mathcal{F}_X \to \mathcal{F}_Y$ and $g_! : \mathcal{F}_X \to \mathcal{F}_Z$, the following diagram is commutative for any $M \in \operatorname{Ob}\mathcal{F}_Y$ by the naturality of ad^f.*

$$\begin{CD}
\mathcal{F}_X(f^*(M), g^*g_!h^*(N)) @>{\varepsilon^g_{h^*(N)*}}>> \mathcal{F}_X(f^*(M), h^*(N))\\
@VV{\mathrm{ad}^f_{M,g^*g_!h^*(N)}}V @VV{\mathrm{ad}^f_{M,h^*(N)}}V\\
\mathcal{F}_Y(M, f_!g^*g_!h^*(N)) @>{f_!\left(\varepsilon^g_{h^*(N)}\right)_*}>> \mathcal{F}_1(M, f_!h^*(N))
\end{CD}$$

It follows that $\epsilon^{f,g,h}_N = f_!\big(\varepsilon^X_{o^*_X(N)}\big)$.

Lemma 8.5.12 *For morphisms* $f : X \to Y$, $g : X \to Z$, $h : X \to W$, $k : V \to X$ *of* $\mathcal{E}$ *and a morphism* $\varphi : M \to N$ *of* $\mathcal{F}_W$, *the following diagrams are commutative.*

$$\begin{CD}
(M^{[g,h]})^{[f,g]} @>{\epsilon^{f,g,h}_M}>> M^{[f,h]}\\
@VV{(\varphi^{[g,h]})^{[f,g]}}V @VV{\varphi^{[f,h]}}V\\
(N^{[g,h]})^{[f,g]} @>{\epsilon^{f,g,h}_N}>> N^{[f,h]}
\end{CD}
\qquad
\begin{CD}
(N^{[g,h]})^{[f,g]} @>{\epsilon^{f,g,h}_N}>> N^{[f,h]}\\
@VV{(N^k)^k}V @VV{N^k}V\\
(N^{[gk,hk]})^{[fk,gk]} @>{\epsilon^{fk,gk,hk}_N}>> N^{[fk,hk]}
\end{CD}$$

Proof. The following diagram is commutative by (1) of (8.5.3) for any $L \in \operatorname{Ob}\mathcal{F}_Y$.

$$\begin{CD}
\mathcal{F}_X(f^*(L), g^*(M^{[g,h]})) @>{\pi_{g,h}(M)_*}>> \mathcal{F}_X(f^*(L), h^*(M))\\
@VV{g^*(\varphi^{[g,h]})_*}V @VV{h^*(\varphi)_*}V\\
\mathcal{F}_X(f^*(L), g^*(N^{[g,h]})) @>{\pi_{g,h}(N)_*}>> \mathcal{F}_X(f^*(L), h^*(N))
\end{CD}$$

Hence the following diagram commutes by (8.5.10) and (1) of (8.5.3).

$$\begin{CD}
\mathcal{F}_Y(L, (M^{[g,h]})^{[f,g]}) @>{\epsilon^{f,g,h}_{M*}}>> \mathcal{F}_Y(L, M^{[f,h]})\\
@VV{(\varphi^{[g,h]})^{[f,g]}_*}V @VV{\varphi^{[f,h]}_*}V\\
\mathcal{F}_Y(L, (N^{[g,h]})^{[f,g]}) @>{\epsilon^{f,g,h}_{N*}}>> \mathcal{F}_Y(L, N^{[f,h]})
\end{CD}$$

Thus the left diagram is commutative.

For $M \in \operatorname{Ob}\mathcal{F}_Y$ and $\xi \in \mathcal{F}_X(f^*(M), g^*(N^{[g,h]}))$, it follows from (8.5.6) and (8.1.13) that we have

$$\pi_{gk,hk}(N)(gk)^*(N^k)k^\sharp_{M,N^{[g,h]}}(\xi) = k^\sharp_{N^{[g,h]},N}(\pi_{g,h}(N))k^\sharp_{M,N^{[g,h]}}(\xi) = k^\sharp_{M,N}(\pi_{g,h}(N)\xi).$$

This shows that the following diagram commutes.

$$\begin{CD}
\mathcal{F}_X(f^*(M), g^*(N^{[g,h]})) @>{\pi_{g,h}(N)_*}>> \mathcal{F}_X(f^*(M), g^*(N))\\
@VV{(gk)^*(N^k)_*k^\sharp_{M,N^{[g,h]}}}V @VV{k^\sharp_{M,N}}V\\
\mathcal{F}_V((fk)^*(M), (gk)^*(N^{[gk,hk]})) @>{\pi_{gk,hk}(N)_*}>> \mathcal{F}_Y((fk)^*(M), (hk)^*(N))
\end{CD}$$

The following diagram commutes by (1) of (8.5.3) and (8.5.6).

$$\begin{CD}
\mathcal{F}_X(f^*(M), g^*(N^{[g,h]})) @>{k^\sharp_{M,N^{[g,h]}}}>> \mathcal{F}_Y((fk)^*(M), (gk)^*(N^{[g,h]})) @>{(gk)^*(N^k)_*}>> \mathcal{F}_Y((fk)^*(M), (gk)^*(N^{[gk,hk]}))\\
@VV{E_{f,g}(N^{[g,h]})_M}V @VV{E_{fk,gk}(N^{[g,h]})_M}V @VV{E_{fk,gk}(N)_M}V\\
\mathcal{F}_Y(M, (N^{[g,h]})^{[f,g]}) @>{(N^{[g,h]})^k_*}>> \mathcal{F}_Y(M, (N^{[g,h]})^{[fk,gk]}) @>{(N^k)^{[fk,gk]}_*}>> cf_V(M, (N^{[gk,hk]})^{[fk,gk]})
\end{CD}$$

Since $(N^k)^k = (N^k)^{[fk,gk]}(N^{[g,h]})^k$, it follows from (8.5.10) and (1) of (8.5.6) that the following diagram commutes for any $M \in \operatorname{Ob}\mathcal{F}_Y$.

$$\begin{CD}
\mathcal{F}_Y(M, (N^{[g,h]})^{[f,g]}) @>{\epsilon^{f,g,h}_{N*}}>> \mathcal{F}_Y(M, N^{[f,h]})\\
@VV{(N^k)^k_*}V @VV{N^k_*}V\\
\mathcal{F}_Y(M, (N^{[gk,hk]})^{[fk,gk]}) @>{\epsilon^{fk,gk,hk}_{N*}}>> \mathcal{F}_Y(M, N^{[fk,hk]})
\end{CD}$$

Thus the right diagram is also commutative. □

Proposition 8.5.13 *For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$, $i : X \to V$ of $\mathcal{E}$ and an object N of $\mathcal{F}_V$, the following diagrams are commutative.*

$$\begin{tikzcd} g^*((N^{[h,i]})^{[g,h]}) \arrow[r, "g^*(\epsilon_N^{g,h,i})"] \arrow[d, "\pi_{g,h}(N^{[h,i]})"] & g^*(N^{[g,i]}) \arrow[d, "\pi_{g,i}(N)"] \\ h^*(N^{[h,i]}) \arrow[r, "\pi_{h,i}(N)"] & i^*(N) \end{tikzcd} \qquad \begin{tikzcd} ((N^{[h,i]})^{[g,h]})^{[f,g]} \arrow[r, "(\epsilon_N^{g,h,i})^{[f,g]}"] \arrow[d, "\epsilon^{f,g,h}_{N^{[h,i]}}"] & (N^{[g,i]})^{[f,g]} \arrow[d, "\epsilon_N^{f,g,i}"] \\ (N^{[h,i]})^{[f,h]} \arrow[r, "\epsilon_N^{f,h,i}"] & N^{[f,i]} \end{tikzcd}$$

Proof. It follows from the definition of $\epsilon_N^{g,h,i}$ and (8.5.2) that

$$\pi_{h,i}(N)\pi_{g,h}(N^{[h,i]}) = E_{g,i}(N)^{-1}_{(N^{[h,i]})^{[g,h]}}(\epsilon_N^{g,h,i}) = \pi_{g,i}(N)g^*(\epsilon_N^{g,h,i}).$$

Hence the following diagram commutes for $M \in \operatorname{Ob}\mathcal{F}_Y$.

$$\begin{tikzcd} \mathcal{F}_X(f^*(M), g^*((N^{[h,i]})^{[g,h]})) \arrow[r, "g^*(\epsilon_N^{g,h,i})_*"] \arrow[d, "\pi_{g,h}(N^{[h,i]})_*"] & \mathcal{F}_X(f^*(M), g^*(N^{[g,i]})) \arrow[d, "\pi_{g,i}(N)_*"] \\ \mathcal{F}_X(f^*(M), h^*(N^{[h,i]})) \arrow[r, "\pi_{h,i}(N)_*"] & \mathcal{F}_X(f^*(M), i^*(N)) \end{tikzcd}$$

Therefore the following diagram commutes by (8.5.10) and (1) of (8.5.3).

$$\begin{tikzcd} \mathcal{F}_Y(M, ((N^{[h,i]})^{[g,h]})^{[f,g]}) \arrow[r, "(\epsilon_N^{g,h,i})^{[f,g]}_*"] \arrow[d, "\epsilon^{f,g,h}_{N^{[h,i]}*}"] & \mathcal{F}_Y(M, (N^{[g,i]})^{[f,g]}) \arrow[d, "\epsilon^{f,g,i}_{N*}"] \\ \mathcal{F}_Y(M, (N^{[h,i]})^{[f,h]}) \arrow[r, "\epsilon^{f,h,i}_{N*}"] & \mathcal{F}_Y(M, N^{[f,i]}) \end{tikzcd}$$

□

Proposition 8.5.14 *For morphisms $f : X \to Y$, $g : X \to Z$ of $\mathcal{E}$ and an object N of $\mathcal{F}_Z$, the following compositions coincide with the identity morphism of $N^{[f,g]}$.*

$$N^{[f,g]} = (N^{[f,g]})^{[id_Y, id_Y]} \xrightarrow{(N^{[f,g]})^f} (N^{[f,g]})^{[f,f]} \xrightarrow{\epsilon_N^{f,f,g}} N^{[f,g]}$$

$$N^{[f,g]} = (N^{[id_Z, id_Z]})^{[f,g]} \xrightarrow{(N^g)^{[f,g]}} (N^{[g,g]})^{[f,g]} \xrightarrow{\epsilon_N^{f,g,g}} N^{[f,g]}$$

Proof. The following diagram commutes for any $M \in \operatorname{Ob}\mathcal{F}_Y$ by (1) of (8.5.6) and (8.5.10).

$$\begin{tikzcd} \mathcal{F}_Y(id_Y^*(M), id_Y^*(N^{[f,g]})) \arrow[r, "f^\sharp_{M,N^{[f,g]}}"] \arrow[d, "E_{id_Y,id_Y}(N^{[f,g]})_M"] & \mathcal{F}_X(f^*(M), f^*(N^{[f,g]})) \arrow[r, "\pi_{f,g}(N)_*"] \arrow[d, "E_{f,f}(N^{[f,g]})_N"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "E_{f,g}(N)_M"] \\ \mathcal{F}_Y(M, (N^{[f,g]})^{[id_Y,id_Y]}) \arrow[r, "(N^{[f,g]})^f_*"] & \mathcal{F}_Y(M, (N^{[f,g]})^{[f,f]}) \arrow[r, "\epsilon^{f,f,g}_{N*}"] & \mathcal{F}_Y(M, N^{[f,g]}) \end{tikzcd}$$

It follows from (8.5.2) that $\epsilon^{f,f,g}_{N*}(N^{[f,g]})^f_* : \mathcal{F}_Y(M, N^{[f,g]}) = \mathcal{F}_Y(M, (N^{[f,g]})^{[id_Y,id_Y]}) \to \mathcal{F}_Y(M, N^{[f,g]})$ is the identity map of $\mathcal{F}_Y(M, N^{[f,g]})$.

The following diagram commutes for any $M \in \operatorname{Ob}\mathcal{F}_Y$ by (1) of (8.5.3) and (8.5.10).

$$\begin{tikzcd} \mathcal{F}_X(f^*(M), g^*(N^{[id_Y,id_Y]})) \arrow[r, "g^*(N^g)_*"] \arrow[d, "E_{f,g}(N^{[id_Y,id_Y]})_M"] & \mathcal{F}_X(f^*(M), g^*(N^{[g,g]})) \arrow[r, "\pi_{g,g}(N)_*"] \arrow[d, "E_{f,g}(N^{[g,g]})_M"] & \mathcal{F}_X(f^*(M), g^*(N)) \arrow[d, "E_{f,g}(N)_M"] \\ \mathcal{F}_Y(M, (N^{[id_Y,id_Y]})^{[f,g]}) \arrow[r, "(N^g)^{[f,g]}_*"] & \mathcal{F}_Y(M, (N^{[g,g]})^{[f,g]}) \arrow[r, "\epsilon^{f,g,g}_{N*}"] & \mathcal{F}_Y(M, N^{[f,g]}) \end{tikzcd}$$

Since the composition of the upper horizontal maps of the above diagram coincides with the identity map of $\mathcal{F}_X(f^*(M), g^*(N))$ by (4) of (8.5.6), the composition of the lower horizontal maps of the above diagram is the identity map of $\mathcal{F}_Y(M, N^{[f,g]})$. $\square$

Let $f : X \to Y$, $g : X \to Z$, $h : X \to W$ be morphisms of $\mathcal{E}$ and L, M, N objects of $\mathcal{F}_Y$, $\mathcal{F}_Z$, $\mathcal{F}_W$, respectively. We define a map

$$\chi^{f,g,h}_{L,M,N} : \mathcal{F}_Y(L, M^{[f,g]}) \times \mathcal{F}_Z(M, N^{[g,h]}) \to \mathcal{F}_Y(L, N^{[f,h]})$$

as follows. For $\varphi \in \mathcal{F}_Y(L, M^{[f,g]})$ and $\psi \in \mathcal{F}_Z(M, N^{[g,h]})$, let $\chi^{f,g,h}_{L,M,N}(\varphi, \psi)$ be the following composition.

$$L \xrightarrow{\varphi} M^{[f,g]} \xrightarrow{\psi^{[f,g]}} (N^{[g,h]})^{[f,g]} \xrightarrow{\epsilon^{f,g,h}_N} N^{[f,h]}$$

Proposition 8.5.15 *The following diagram is commutative.*

$$\begin{array}{ccc}
\mathcal{F}_X(f^*(L), g^*(M)) \times \mathcal{F}_X(g^*(M), h^*(N)) & \xrightarrow{\text{composition}} & \mathcal{F}_X(f^*(L), h^*(N)) \\
\downarrow{\scriptstyle E_{f,g}(M)_L \times E_{g,h}(N)_M} & & \downarrow{\scriptstyle E_{f,h}(N)_L} \\
\mathcal{F}_Y(L, M^{[f,g]}) \times \mathcal{F}_Z(M, N^{[g,h]}) & \xrightarrow{\chi^{f,g,h}_{L,M,N}} & \mathcal{F}_Y(L, N^{[f,h]})
\end{array}$$

Proof. For $\zeta \in \mathcal{F}_X(f^*(L), g^*(M))$ and $\xi \in \mathcal{F}_X(g^*(M), h^*(N))$, we put $\varphi = E_{f,g}(M)_L(\zeta)$ and $\psi = E_{g,h}(N)_M(\xi)$. Then, we have $\psi^{[f,g]}\varphi = E_{f,g}(N^{[g,h]})_L(g^*(\psi)\zeta)$ by (8.5.3). It follows from (8.5.10) and (8.5.2) that

$$\epsilon^{f,g,h}_N \psi^{[f,g]}\varphi = \epsilon^{f,g,h}_{N*} E_{f,g}(N^{[g,h]})_L(g^*(\psi)\zeta) = E_{f,h}(N)_L(\pi_{g,h}(N)g^*(\psi)\zeta) = E_{f,h}(N)_L(\xi\zeta).$$

Thus the result follows. $\square$

For a functor $D : \mathcal{P} \to \mathcal{E}$ and an object N of $\mathcal{F}_{D(5)}$, we put $D(\tau_{ij}) = f_{ij}$ and define a morphism

$$\theta^D(N) : (N^{[f_{24}, f_{25}]})^{[f_{13}, f_{14}]} \to N^{[f_{13}f_{01}, f_{25}f_{02}]}$$

of $\mathcal{F}_{D(3)}$ to be the following composition.

$$(N^{[f_{24}, f_{25}]})^{[f_{13}, f_{14}]} \xrightarrow{(N^{f_{02}})^{f_{01}}} (N^{[f_{24}f_{02}, f_{25}f_{02}]})^{[f_{13}f_{01}, f_{14}f_{01}]} \xrightarrow{\epsilon^{f_{13}f_{01}, f_{14}f_{01}, f_{25}f_{02}}_N} N^{[f_{13}f_{01}, f_{25}f_{02}]}$$

Proposition 8.5.16 *The following diagram is commutative.*

$$\begin{array}{ccccc}
(f_{13}f_{01})^*((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) & & \xrightarrow{(f_{13}f_{01})^*(\theta^D(N))} & & (f_{13}f_{01})^*(N^{[f_{13}f_{01}, f_{25}f_{02}]}) \\
\downarrow{\scriptstyle f^\sharp_{01}(\pi_{f_{13},f_{14}}(N^{[f_{24},f_{25}]}))} & & & & \downarrow{\scriptstyle \pi_{f_{13}f_{01}, f_{25}f_{02}}(N)} \\
(f_{14}f_{01})^*(N^{[f_{24},f_{25}]}) & = & (f_{24}f_{02})^*(N^{[f_{24},f_{25}]}) & \xrightarrow{f^\sharp_{02}(\pi_{f_{24},f_{25}}(N))} & (f_{25}f_{02})^*(N)
\end{array}$$

Proof. By the naturality of $E_{f_{13}f_{01}, f_{25}f_{02}}(N)$, $\theta^D(N)$ is the image of

$$\pi_{f_{24}f_{02}, f_{25}f_{02}}(N)\pi_{f_{13}f_{01}, f_{14}f_{01}}(N^{[f_{24}f_{02}, f_{25}f_{02}]})(f_{13}f_{01})^*((N^{f_{02}})^{f_{01}}) : (f_{13}f_{01})^*((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \to (f_{25}f_{02})^*(N)$$

by $E_{f_{13}f_{01}, f_{25}f_{02}}(N)_{(N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}}$. Hence the following equality holds by (8.5.2).

$$\pi_{f_{13}f_{01}, f_{25}f_{02}}(N)(f_{13}f_{01})^*(\theta^D(N)) = \pi_{f_{24}f_{02}, f_{25}f_{02}}(N)\pi_{f_{13}f_{01}, f_{14}f_{01}}(N^{[f_{24}f_{02}, f_{25}f_{02}]})(f_{13}f_{01})^*((N^{f_{02}})^{f_{01}}) \quad \cdots (*)$$

It follows from (8.5.6), (8.1.10) and (8.5.3) that we have

$$\begin{aligned}
&\pi_{f_{13}f_{01}, f_{14}f_{01}}(N^{[f_{24}f_{02}, f_{25}f_{02}]})(f_{13}f_{01})^*((N^{f_{02}})^{f_{01}}) \\
&= \pi_{f_{13}f_{01}, f_{14}f_{01}}(N^{[f_{24}f_{02}, f_{25}f_{02}]})(f_{13}f_{01})^*((N^{[f_{24}f_{02}, f_{25}f_{02}]})^{f_{01}})(f_{13}f_{01})^*((N^{f_{02}})^{[f_{13},f_{14}]}) \\
&= f^\sharp_{01}(\pi_{f_{13},f_{14}}(N^{[f_{24}f_{02}, f_{25}f_{02}]}))(f_{13}f_{01})^*((N^{f_{02}})^{[f_{13},f_{14}]}) \\
&= c_{f_{14},f_{01}}(N^{[f_{24}f_{02}, f_{25}f_{02}]})f^*_{01}(\pi_{f_{13},f_{14}}(N^{[f_{24}f_{02}, f_{25}f_{02}]}))c_{f_{13},f_{01}}((N^{[f_{24}f_{02}, f_{25}f_{02}]})^{[f_{13},f_{14}]})^{-1}(f_{13}f_{01})^*((N^{f_{02}})^{[f_{13},f_{14}]}) \\
&= c_{f_{14},f_{01}}(N^{[f_{24}f_{02}, f_{25}f_{02}]})f^*_{01}(\pi_{f_{13},f_{14}}(N^{[f_{24}f_{02}, f_{25}f_{02}]}))f^*_{01}(f^*_{13}((N^{f_{02}})^{[f_{13},f_{14}]}))c_{f_{13},f_{01}}((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})^{-1} \\
&= c_{f_{14},f_{01}}(N^{[f_{24}f_{02}, f_{25}f_{02}]})f^*_{01}(f^*_{14}(N^{f_{02}}))f^*_{01}(\pi_{f_{13},f_{14}}(N^{[f_{24},f_{25}]}))c_{f_{13},f_{01}}((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})^{-1} \\
&= (f_{14}f_{01})^*(N^{f_{02}})c_{f_{14},f_{01}}(N^{[f_{24},f_{25}]})f^*_{01}(\pi_{f_{13},f_{14}}(N^{[f_{24},f_{25}]}))c_{f_{13},f_{01}}((N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})^{-1} \\
&= (f_{24}f_{02})^*(N^{f_{02}})f^\sharp_{01}(\pi_{f_{13},f_{14}}(N^{[f_{24},f_{25}]}))
\end{aligned}$$

Therefore we have

$$(*) = \pi_{f_{24}f_{02},f_{25}f_{02}}(N)(f_{24}f_{02})^*(N^{f_{02}})f_{01}^\sharp(\pi_{f_{13},f_{14}}(N^{[f_{24},f_{25}]})) = f_{02}^\sharp(\pi_{f_{24},f_{25}}(N))f_{01}^\sharp(\pi_{f_{13},f_{14}}(N^{[f_{24},f_{25}]}))$$

which implies the assertion. $\square$

Proposition 8.5.17 *For a morphism $\varphi : N \to N$ of $\mathcal{F}_Z$, the following diagram commutes.*

$$\begin{tikzcd}
(M^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \arrow[r, "\theta^D(M)"] \arrow[d, "(\varphi^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}"] & M^{[f_{13}f_{01},f_{25}f_{02}]} \arrow[d, "\varphi^{[f_{13}f_{01},f_{25}f_{02}]}"] \\
(N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \arrow[r, "\theta^D(N)"] & N^{[f_{13}f_{01},f_{25}f_{02}]}
\end{tikzcd}$$

Proof. The following diagram commutes by (8.5.12), (8.5.8), (8.5.3) and (8.5.6).

$$\begin{tikzcd}
(M^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \arrow[r, "(M^{f_{02}})^{f_{01}}"] \arrow[d, "(\varphi^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}"] & (M^{[f_{13}f_{01},f_{14}f_{01}]})^{[f_{24}f_{02},f_{25}f_{02}]} \arrow[r, "\epsilon_M^{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02}}"] \arrow[d, "(\varphi^{[f_{13}f_{01},f_{14}f_{01}]})^{[f_{24}f_{02},f_{25}f_{02}]}"] & M^{[f_{13}f_{01},f_{25}f_{02}]} \arrow[d, "\varphi^{[f_{13}f_{01},f_{25}f_{02}]}"] \\
(N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \arrow[r, "(N^{f_{02}})^{f_{01}}"] & (N^{[f_{13}f_{01},f_{14}f_{01}]})^{[f_{24}f_{02},f_{25}f_{02}]} \arrow[r, "\epsilon_N^{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02}}"] & N^{[f_{13}f_{01},f_{25}f_{02}]}
\end{tikzcd}$$

Hence the assertion follows. $\square$

Proposition 8.5.18 *Let $E : \mathcal{P} \to \mathcal{E}$ be a functor which satisfies $E(i) = D(i)$ for $i = 3, 4, 5$ and a natural transformation $\lambda : D \to E$ which satisfies $\lambda_i = id_{D(i)}$ for $i = 3, 4, 5$. We put $E(\tau_{ij}) = g_{ij}$, then the following diagram commutes.*

$$\begin{tikzcd}
(N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]} \arrow[r, "\theta^E(N)"] \arrow[d, "(N^{\lambda_2})^{\lambda_1}"] & N^{[g_{13}g_{01},g_{25}g_{02}]} \arrow[d, "N^{\lambda_0}"] \\
(N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \arrow[r, "\theta^D(N)"] & N^{[f_{13}f_{01},f_{25}f_{02}]}
\end{tikzcd}$$

Proof. Since $f_{ij} = g_{ij}\lambda_i$ for $i = 1, 2$, we have $f_{13}f_{01} = g_{13}\lambda_1 f_{01} = g_{13}g_{01}\lambda_0$, $f_{14}f_{01} = g_{14}\lambda_1 f_{01} = g_{14}g_{01}\lambda_0$ and $f_{25}f_{02} = g_{25}\lambda_2 f_{02} = g_{25}g_{02}\lambda_0$. It follows from (8.5.6), (8.5.8) and (8.5.12) that

$$\begin{tikzcd}
(N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]} \arrow[r, "(N^{g_{02}})^{g_{01}}"] \arrow[d, "(N^{\lambda_2})^{\lambda_1}"] & (N^{[g_{24}g_{02},g_{25}g_{02}]})^{[g_{13}g_{01},g_{14}g_{01}]} \arrow[r, "\epsilon_N^{g_{13}g_{01},g_{14}g_{01},g_{25}g_{02}}"] \arrow[d, "(N^{\lambda_0})^{\lambda_0}"] & N^{[g_{13}g_{01},g_{25}g_{02}]} \arrow[d, "N^{\lambda_0}"] \\
(N^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \arrow[r, "(N^{f_{02}})^{f_{01}}"] & (N^{[f_{24}f_{02},f_{25}f_{02}]})^{[f_{13}f_{01},f_{14}f_{01}]} \arrow[r, "\epsilon_N^{f_{13}f_{01},f_{14}f_{01},f_{25}f_{02}}"] & N^{[f_{13}f_{01},f_{25}f_{02}]}
\end{tikzcd}$$

is commutative. $\square$

For morphisms $f : X \to Y$, $g : X \to Z$, $h : V \to Z$, $i : V \to W$ of $\mathcal{E}$, let $X \xleftarrow{\mathrm{pr}_X} X \times_Z V \xrightarrow{\mathrm{pr}_V} V$ be a limit of a diagram $X \xrightarrow{g} Z \xleftarrow{h} V$. We define a functor $D_{f,g,h,i} : \mathcal{P} \to \mathcal{E}$ by $D_{f,g,h,i}(0) = X \times_Z V$, $D_{f,g,h,i}(1) = X$, $D_{f,g,h,i}(2) = V$, $D_{f,g,h,i}(3) = Y$, $D_{f,g,h,i}(4) = Z$, $D_{f,g,h,i}(5) = W$ and $D_{f,g,h,i}(\tau_{01}) = \mathrm{pr}_X$, $D_{f,g,h,i}(\tau_{02}) = \mathrm{pr}_V$, $D_{f,g,h,i}(\tau_{13}) = f$, $D_{f,g,h,i}(\tau_{14}) = g$, $D_{f,g,h,i}(\tau_{24}) = h$, $D_{f,g,h,i}(\tau_{25}) = i$. For an object N of $\mathcal{F}_W$, we denote $\theta^{D_{f,g,h,i}}(N)$ by $\theta^{f,g,h,i}(N)$. The following facts are special cases of (8.5.17) and (8.5.18).

Proposition 8.5.19 *Let $f : X \to Y$, $g : X \to Z$, $h : V \to Z$, $i : V \to W$, $j : S \to X$, $k : T \to V$ be morphisms of $\mathcal{E}$ and $\varphi : M \to N$ a morphism of $\mathcal{F}_Z$. The following diagrams are commutative.*

$$\begin{tikzcd}
(M^{[h,i]})^{[f,g]} \arrow[r, "\theta^{f,g,h,i}(M)"] \arrow[d, "(\varphi^{[h,i]})^{[f,g]}"] & M^{[f\mathrm{pr}_X,i\mathrm{pr}_V]} \arrow[d, "\varphi^{[f\mathrm{pr}_X,i\mathrm{pr}_V]}"] \\
(N^{[h,i]})^{[f,g]} \arrow[r, "\theta^{f,g,h,i}(N)"] & N^{[f\mathrm{pr}_X,i\mathrm{pr}_V]}
\end{tikzcd}
\qquad
\begin{tikzcd}
(N^{[h,i]})^{[f,g]} \arrow[r, "\theta^{f,g,h,i}(N)"] \arrow[d, "(N^k)^j"] & N^{[f\mathrm{pr}_X,i\mathrm{pr}_V]} \arrow[d, "N^{j\times_Z k}"] \\
(N^{[hk,ik]})^{[fj,gj]} \arrow[r, "\theta^{fj,gj,hk,ik}(N)"] & N^{[fj\mathrm{pr}_S,ik\mathrm{pr}_T]}
\end{tikzcd}$$

Remark 8.5.20 *If $X \xleftarrow{\mathrm{pr}'_X} X \times'_Z V \xrightarrow{\mathrm{pr}'_V} V$ is another limit of a diagram $X \xrightarrow{g} Z \xleftarrow{h} V$, there exists unique isomorphism $l : X \times'_Z V \to X \times_Z V$ that satisfies $\mathrm{pr}'_X = \mathrm{pr}_X l$ and $\mathrm{pr}'_V = \mathrm{pr}_V l$. We denote by $\theta'^{f,g,h,i}(N) : (N^{[f,g]})^{[h,i]} \to N^{[f\mathrm{pr}'_X, i\mathrm{pr}'_V]}$ the morphism of $\mathcal{F}_W$ obtained from $X \xleftarrow{\mathrm{pr}'_X} X \times'_Z V \xrightarrow{\mathrm{pr}'_V} V$. Then, $N^l : N^{[f\mathrm{pr}_X, i\mathrm{pr}_V]} \to N^{[f\mathrm{pr}'_X, i\mathrm{pr}'_V]}$ is an isomorphism and (8.5.18) implies $\theta'^{f,g,h,i}(N) = N^l\theta^{f,g,h,i}(N)$.*

Proposition 8.5.21 *Under the assumption of (8.4.21), the following diagram is commutative.*

$$\begin{tikzcd}
((N^{[j,k]})^{[h,i]})^{[f,g]} \arrow[r, "\theta^{D_1}(N)^{[f,g]}"] \arrow[d, "\theta^{D_4}(N^{[j,k]})"] & (N^{[ht,ku]})^{[f,g]} \arrow[d, "\theta^{D_3}(N)"] \\
(N^{[j,k]})^{[fr,is]} \arrow[r, "\theta^{D_2}(N)"] & N^{[frv,kuw]}
\end{tikzcd}$$

Proof. The following diagrams are commutative by (8.5.13), (8.5.12), (8.5.8), (8.5.3) and (8.5.6).

$$\begin{tikzcd}
 & ((N^{[j,k]})^{[h,i]})^{[fr,gr]} \arrow[dl, "((N^{[j,k]})^{s})^{[fr,gr]}"'] \arrow[d, "((N^{uw})^{s})^{[fr,gr]}"] \\
((N^{[j,k]})^{[hs,is]})^{[fr,gr]} \arrow[r, "((N^{uw})^{[hs,is]})^{[fr,gr]}"] \arrow[d, "\epsilon^{fr,gr,is}_{N^{[j,k]}}"] & ((N^{[juw,kuw]})^{[htw,itw]})^{[frv,grv]} \arrow[d, "\epsilon^{fr,gr,is}_{N^{[juw,kuw]}}"] \\
(N^{[j,k]})^{[fr,is]} \arrow[r, "(N^{uw})^{[fr,is]}"] & (N^{[juw,kuw]})^{[fr,is]}
\end{tikzcd}$$

$$\begin{tikzcd}
((N^{[j,k]})^{[h,i]})^{[f,g]} \arrow[r, "((N^{u})^{t})^{[f,g]}"] \arrow[d, "((N^{[j,k]})^{[h,i]})^{r}"] & ((N^{[ju,ku]})^{[ht,it]})^{[f,g]} \arrow[r, "(\epsilon^{ht,it,ku}_{N})^{[f,g]}"] \arrow[d, "((N^{[ju,ku]})^{[ht,it]})^{rv}"] & (N^{[ht,ku]})^{[f,g]} \arrow[d, "(N^{[ht,ku]})^{rv}"'] \\
((N^{[j,k]})^{[h,i]})^{[fr,gr]} \arrow[r, "((N^{u})^{t})^{v}"] \arrow[d, "((N^{uw})^{s})^{[fr,gr]}"] & ((N^{[ju,ku]})^{[ht,it]})^{[frv,grv]} \arrow[r, "(\epsilon^{ht,it,ku}_{N})^{[frv,grv]}"] \arrow[d, "((N^{w})^{w})^{[frv,grv]}"] & (N^{[ht,ku]})^{[frv,grv]} \arrow[d, "(N^{w})^{[frv,grv]}"'] \\
((N^{[juw,kuw]})^{[hs,is]})^{[fr,gr]} \arrow[r, "((N^{[juw,kuw]})^{v})^{v}"] \arrow[d, "\epsilon^{fr,gr,is}_{N^{[juw,kuw]}}"] & ((N^{[juw,kuw]})^{[htw,itw]})^{[frv,grv]} \arrow[r, "(\epsilon^{htw,itw,kuw}_{N})^{[frv,grv]}"] \arrow[d, "\epsilon^{frv,grv,itw}_{N^{[juw,kuw]}}"] & (N^{[htw,kuw]})^{[frv,grv]} \arrow[d, "\epsilon^{frv,grv,kuw}_{N}"'] \\
(N^{[juw,kuw]})^{[fr,is]} \arrow[r, "(N^{[juw,kuw]})^{v}"] & (N^{[juw,kuw]})^{[frv,itw]} \arrow[r, "\epsilon^{frv,itw,kuw}_{N}"] & N^{[frv,kuw]}
\end{tikzcd}$$

Hence the asserion follows from the definition of $\theta^{D_l}(N)$. □

For morphisms $g : X \to Z$, $h : V \to Z$, $i : V \to W$, $j : T \to W$ of $\mathcal{E}$, let $X \xleftarrow{\mathrm{pr}_X} X \times_Z V \xrightarrow{\mathrm{pr}_{2V}} V$ and $V \xleftarrow{\mathrm{pr}_{1V}} V \times_W T \xrightarrow{\mathrm{pr}_T} T$ be limits of diagrams $X \xrightarrow{g} Z \xleftarrow{h} V$ and $V \xrightarrow{i} W \xleftarrow{j} T$, respectively. We also assume that a limit $X \times_Z V \xleftarrow{\mathrm{pr}_{X\times_Z V}} X \times_Z V \times_W T \xrightarrow{\mathrm{pr}_{V\times_W T}} V \times_W T$ of a diagram $X \times_Z V \xrightarrow{\mathrm{pr}_{2V}} V \xleftarrow{\mathrm{pr}_{1V}} V \times_W T$ exists. Then, $X \xleftarrow{\mathrm{pr}_X\mathrm{pr}_{X\times_Z V}} X \times_Z V \times_W T \xrightarrow{\mathrm{pr}_{V\times_W T}} V \times_W T$ and $X \times_Z V \xleftarrow{\mathrm{pr}_{X\times_Z V}} X \times_Z V \times_W T \xrightarrow{\mathrm{pr}_{V\times_W T}\mathrm{pr}_T} T$ are limits of diagrams $X \xrightarrow{g} Z \xleftarrow{h\mathrm{pr}_{1V}} V \times_W T$ and $X \times_Z V \xrightarrow{i\mathrm{pr}_{2V}} W \xleftarrow{j} T$, respectively.

Corollary 8.5.22 *Let $f : X \to Y$, $g : X \to Z$, $h : V \to Z$, $i : V \to W$, $j : T \to W$, $k : T \to U$ be morphisms of $\mathcal{E}$ and N an object of $\mathcal{F}_U$. The following diagram is commutative.*

$$\begin{tikzcd}
((N^{[j,k]})^{[h,i]})^{[f,g]} \arrow[r, "\theta^{h,i,j,k}(N)^{[f,g]}"] \arrow[d, "\theta^{f,g,h,i}(N^{[j,k]})"] & (N^{[h\mathrm{pr}_{1V},k\mathrm{pr}_T]})^{[f,g]} \arrow[d, "\theta^{f,g,h\mathrm{pr}_{1V},k\mathrm{pr}_T}(N)"] \\
(N^{[j,k]})^{[f\mathrm{pr}_X,i\mathrm{pr}_{2V}]} \arrow[r, "\theta^{f\mathrm{pr}_X,i\mathrm{pr}_{2V},j,k}(N)"] & N^{[f\mathrm{pr}_X pr_{X\times_Z V},k\mathrm{pr}_T\mathrm{pr}_{V\times_W T}]}
\end{tikzcd}$$

Proof. The assertion follows by applying the result of (8.5.21) to the following diagram.

$$\begin{tikzcd}
 & & & X \times_Z V \times_W T \arrow[dl, "\mathrm{pr}_{X\times_Z V}"'] \arrow[dr, "\mathrm{pr}_{V\times_W T}"] & & & \\
 & & X \times_Z V \arrow[dl, "\mathrm{pr}_X"'] \arrow[dr, "\mathrm{pr}_{2V}"] & & V \times_W T \arrow[dl, "\mathrm{pr}_{1V}"'] \arrow[dr, "\mathrm{pr}_T"] & & \\
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & & V \arrow[dl, "h"'] \arrow[dr, "i"] & & T \arrow[dl, "j"'] \arrow[dr, "k"] & \\
Y & & Z & & W & & U
\end{tikzcd}$$

□

Proposition 8.5.23 *For morphisms $f : X \to Y$, $g : X \to Z$ of $\mathcal{E}$ and an object N of $\mathcal{F}_Z$, the following morphisims of $\mathcal{F}_Y$ are identified with the identity morphism of $N^{[f,\,g]}$.*

$$\theta^{f,g,id_Z,id_Z}(N) : (N^{[id_Z,\,id_Z]})^{[f,\,g]} \to N^{[f\,id_X,\,id_Z g]}, \qquad \theta^{id_Y,id_Y,f,g}(N) : (N^{[f,\,g]})^{[id_Y,\,id_Y]} \to N^{[id_Y f,\,g\,id_X]}$$

Proof. Since $\theta^{f,g,id_Z,id_Z}(N)$ is a composition

$$N^{[f,\,g]} = (N^{[id_Z,\,id_Z]})^{[f,\,g]} \xrightarrow{(N^g)^{[f,\,g]}} (N^{[id_Z g,\,id_Z g]})^{[f\,id_X,\,g\,id_X]} \xrightarrow{\epsilon_N^{f\,id_X,\,g\,id_X,\,id_Z g}} N^{[id_Y f,\,id_Z\,g]} = N^{[f,\,g]}$$

and $\theta^{id_Y,id_Y,f,g}(N)$ is a composition

$$N^{[f,\,g]} = (N^{[f,\,g]})^{[id_Y,\,id_Y]} \xrightarrow{(N^{[f,\,g]})^f} (N^{[f\,id_X,\,g\,id_X]})^{[id_Y f,\,id_Y f]} \xrightarrow{\epsilon^{id_Y f,\,f\,id_X,\,g\,id_X,\,N}} N^{[id_Y f,\,g\,id_X]} = N^{[f,\,g]},$$

the assertion is a direct consequence of (8.5.14). □

Lemma 8.5.24 *For a functor $D : \mathcal{P} \to \mathcal{E}$, we put $D(\tau_{01}) = j$, $D(\tau_{02}) = k$, $D(\tau_{13}) = f$, $D(\tau_{14}) = g$, $D(\tau_{24}) = h$, $D(\tau_{25}) = i$. For an object N of $\mathcal{F}_{D(5)}$, the following diagram is commutative.*

$$\begin{tikzcd}
(fj)^*((N^{[h,i]})^{[f,g]}) \arrow[r, "j^\sharp(\pi_{f,g}(N^{[h,i]}))"] \arrow[d, "(fj)^*(\theta^D(N))"] & (gj)^*(N^{[h,i]}) \arrow[d, "k^\sharp(\pi_{h,i}(N))"] \\
(fj)^*(N^{[fj,ik]}) \arrow[r, "\pi_{fj,ik}(N)"] & (ik)^*(N)
\end{tikzcd}$$

Proof. It follows from (8.5.6) and (1) of (8.5.3) that we have

$$\begin{aligned}
k^\sharp(\pi_{h,i}(N))j^\sharp(\pi_{f,g}(N^{[h,i]})) &= \pi_{hk,ik}(N)(hk)^*(N^k)\pi_{fj,gj}(N^{[h,i]})(fj)^*((N^{[h,i]})^j) \\
&= \pi_{hk,ik}(N)\pi_{fj,gj}(N^{[hk,ik]})(fj)^*((N^k)^{[fj,gj]})(fj)^*((N^{[h,i]})^j) \\
&= \pi_{hk,ik}(N)\pi_{fj,gj}(N^{[hk,ik]})(fj)^*((N^k)^j)
\end{aligned}$$

By the naturality of $E_{fj,ik}(N)$ and the definition of $\epsilon_N^{fj,gj,ik}$,

$$E_{fj,ik}(N)_{(N^{[h,i]})^{[f,g]}} : \mathcal{F}_{D(0)}((fj)^*((N^{[h,i]})^{[f,g]}), (ik)^*(N)) \to \mathcal{F}_{D(3)}((N^{[h,i]})^{[f,g]}, N^{[fj,ik]})$$

maps $k^\sharp(\pi_{h,i}(N))j^\sharp(\pi_{f,g}(N^{[h,i]}))$ to $\epsilon_N^{fj,gj,ik}(N^k)^j = \theta^D(N)$. On the other hand, it follows from (8.5.2) that $E_{fj,ik}(N)_{(N^{[h,i]})^{[f,g]}}$ also maps $\pi_{fj,ik}(N)(fj)^*(\theta^D(N))$ to $\theta^D(N)$. □

For a morphism $g : X \to Z$, let $X \xleftarrow{\mathrm{pr}_{1X}} X \times_Z X \xrightarrow{\mathrm{pr}_{2X}} X$ be a limit of a diagram $X \xrightarrow{g} Z \xleftarrow{g} X$. We denote by $\Delta_g : X \to X \times_Z X$ the diagonal morphism, that is, the unique morphism that satisfies $\mathrm{pr}_{1X}\Delta_g = \mathrm{pr}_{2X}\Delta_g = id_X$.

Proposition 8.5.25 *For morphisms $f : X \to Y$, $g : X \to Z$, $h : X \to W$ of $\mathcal{E}$ and an object N of $\mathcal{F}_W$, $\epsilon_N^{f,g,h} : (N^{[g,h]})^{[f,g]} \to N^{[f,h]}$ coincides with the following composition.*

$$(N^{[g,h]})^{[f,g]} \xrightarrow{\theta^{f,g,g,h}(N)} N^{[f\mathrm{pr}_{1X},h\mathrm{pr}_{2X}]} \xrightarrow{N^{\Delta_g}} N^{[f\mathrm{pr}_{1X}\Delta_g,h\mathrm{pr}_{2X}\Delta_g]} = N^{[f,h]}$$

Proof. Define a functor $E : \mathcal{P} \to \mathcal{E}$ by $E(i) = X$ for $i = 0,1,2$, $E(i) = D_{f,g,g,h}(i)$ for $i = 3,4,5$ and $E(\tau_{01}) = E(\tau_{02}) = id_X$, $E(\tau_{ij}) = D_{f,g,g,h}(\tau_{ij})$ if $i \neq 0$. Then, $\theta^E(N) = \epsilon_N^{f,g,h} : (N^{[g,h]})^{[f,g]} \to N^{[f,h]}$ and we have a natural transformation $\lambda : E \to D$ defined by $\lambda_0 = \Delta_g$ and $\lambda_i = id_{E(i)}$ if $i \geqq 1$. It follows from (8.5.18) that $N^{\Delta_g}\theta^{f,g,g,h}(N) = \theta^E(N) = \epsilon_N^{f,g,h}$. □

Let $D, E : \mathcal{Q} \to \mathcal{E}$ be functors and N an object of $\mathcal{F}_{E(2)}$. We put $D(\tau_{0j}) = f_j$ and $E(\tau_{0j}) = g_j$ for $j = 1,2$. For a natural transformation $\omega : D \to E$, let $\omega^N : \omega_1^*(N^{[g_1,g_2]}) \to \omega_2^*(N)^{[f_1,f_2]}$ be the image of $\pi_{g_1,g_2}(N) \in \mathcal{F}_{E(0)}(g_1^*(N^{[g_1,g_2]}), g_2^*(N))$ by the following composition of maps.

$$\begin{aligned}
\mathcal{F}_{E(0)}(g_1^*(N^{[g_1,g_2]}), g_2^*(N)) &\xrightarrow{\omega_0^\sharp} \mathcal{F}_{D(0)}((g_1\omega_0)^*(N^{[g_1,g_2]}), (g_2\omega_0)^*(N)) = \mathcal{F}_{D(0)}((\omega_1 f_1)^*(N^{[g_1,g_2]}), (\omega_2 f_2)^*(N)) \\
&\xrightarrow{c_{\omega_1,f_1}(N^{[g_1,g_2]})^* c_{\omega_2,f_2}(N)_*^{-1}} \mathcal{F}_{D(0)}(f_1^*(\omega_1^*(N^{[g_1,g_2]})), f_2^*(\omega_2^*(N))) \\
&\xrightarrow{E_{f_1,f_2}(\omega_2^*(N))_{\omega_1^*(N^{[g_1,g_2]})}} \mathcal{F}_{D(2)}(\omega_1^*(N^{[g_1,g_2]}), \omega_2^*(N)^{[f_1,f_2]})
\end{aligned}$$

Remark 8.5.26 *(1) If $D(i) = E(i)$ and ω_i is the identity morphism of $D(i)$ for $i = 1,2$, then ω^N coincides with $N^{\omega_0} : N^{[g_1,g_2]} \to N^{[g_1\omega_0, g_2\omega_0]} = N^{[f_1,f_2]}$.*

(2) It follows from (8.5.2) and the definition of ω^N that the following diagram is commutative.

$$\begin{CD}
(\omega_1 f_1)^*(N^{[g_1,g_2]}) @>{c_{\omega_1,f_1}(N^{[g_1,g_2]})^{-1}}>> f_1^*(\omega_1^*(N^{[g_1,g_2]})) @>{f_1^*(\omega^N)}>> f_1^*(\omega_2^*(N)^{[f_1,f_2]}) \\
@| @. @VV{\pi_{f_1,f_2}(\omega_2^*(N))}V \\
(g_1\omega_0)^*(N^{[g_1,g_2]}) @>{\omega_0^\sharp(\pi_{g_1,g_2}(N))}>> (g_2\omega_0)^*(N) = (\omega_2 f_2)^*(N) @>{c_{\omega_2,f_2}(N)^{-1}}>> f_2^*(\omega_2^*(N))
\end{CD}$$

Proposition 8.5.27 *Assume that $D(0) = E(0)$ and ω_0 is the identity morphism of $D(0)$. For an object M of $\mathcal{F}_{E(1)}$, the following diagram is commutative.*

$$\begin{CD}
\mathcal{F}_{D(0)}(g_1^*(M), g_2^*(N)) @>{c_{\omega_2,f_2}(N)_*^{-1}}>> \mathcal{F}_{D(0)}(g_1^*(M), f_2^*(\omega_2^*(N))) @>{c_{\omega_1,f_1}(M)^*}>> \mathcal{F}_{D(0)}(f_1^*(\omega_1^*(M)), f_2^*(\omega_2^*(N))) \\
@VV{E_{g_1,g_2}(N)_M}V @. @VV{E_{f_1,f_2}(\omega_2^*(N))_{\omega_1^*(M)}}V \\
\mathcal{F}_{E(1)}(M, N^{[g_1,g_2]}) @>{\omega_1^*}>> \mathcal{F}_{D(1)}(\omega_1^*(M), \omega_1^*(N^{[g_1,g_2]})) @>{\omega_*^N}>> \mathcal{F}_{D(1)}(\omega_1^*(M), \omega_2^*(N)^{[f_1,f_2]})
\end{CD}$$

Proof. First we note that $g_i = \omega_i f_i$ for $i = 1,2$. It follows from (8.5.26) and the definition of ω^N that we have $\pi_{f_1,f_2}(\omega_2^*(N)) f_1^*(\omega^N) = c_{\omega_2,f_2}(N)^{-1}\pi_{g_1,g_2}(N)c_{\omega_1,f_1}(N^{[g_1,g_2]})$. (8.5.2) and (8.1.10) imply

$$\begin{aligned}
c_{\omega_2,f_2}(N)^{-1}E_{g_1,g_2}(N)_M^{-1}(\varphi)c_{\omega_1,f_1}(M) &= c_{\omega_2,f_2}(N)^{-1}\pi_{g_1,g_2}(N)g_1^*(\varphi)c_{\omega_1,f_1}(M) \\
&= c_{\omega_2,f_2}(N)^{-1}\pi_{g_1,g_2}(N)c_{\omega_2,f_2}(N^{[g_1,g_2]})f_1^*\omega_1^*(\varphi) \\
&= \pi_{f_1,f_2}(\omega_2^*(N))f_1^*(\omega^N)f_1^*\omega_1^*(\varphi) = \pi_{f_1,f_2}(\omega_2^*(N))f_1^*(\omega^N\omega_1^*(\varphi)) \\
&= E_{f_1,f_2}(\omega_2^*(N))_{\omega_1^*(M)}^{-1}(\omega^N\omega_1^*(\varphi))
\end{aligned}$$

for $\varphi \in \mathcal{F}_{E(1)}(M, N^{[g_1,g_2]})$, which shows that the above diagram is commutative. □

Proposition 8.5.28 *For a morphism $\varphi : M \to N$ of $\mathcal{F}_{E(2)}$, the following diagram is commutative.*

$$\begin{CD}
\omega_1^*(M^{[g_1,g_2]}) @>{\omega^M}>> \omega_2^*(M)^{[f_1,f_2]} \\
@VV{\omega_1^*(\varphi^{[g_1,g_2]})}V @VV{\omega_2^*(\varphi)^{[f_1,f_2]}}V \\
\omega_1^*(N^{[g_1,g_2]}) @>{\omega^N}>> \omega_2^*(N)^{[f_1,f_2]}
\end{CD}$$

Proof. It follows from (8.1.10), (1) of (8.5.3) and (8.1.13) that the following diagrams are commutative.

$$\begin{tikzcd}
f_1^*\omega_1^*(M^{[g_1,g_2]}) \arrow[r, "c_{\omega_1,f_1}(M^{[g_1,g_2]})"] \arrow[d, "f_1^*\omega_1^*(\varphi^{[g_1,g_2]})"] & (\omega_1 f_1)^*(M^{[g_1,g_2]}) = (g_1\omega_0)^*(M^{[g_1,g_2]}) \arrow[r, "\omega_0^\sharp(\pi_{g_1,g_2}(M))"] \arrow[d, "(g_1\omega_0)^*(\varphi^{[g_1,g_2]})"] & (g_2\omega_0)^*(M) \arrow[d, "(f_2\omega_0)^*(\varphi)"] \\
f_1^*\omega_1^*(N^{[g_1,g_2]}) \arrow[r, "c_{\omega_1,g_1}(N^{[g_1,g_2]})"] & (\omega_1 f_1)^*(N^{[g_1,g_2]}) = (g_1\omega_0)^*(N^{[g_1,g_2]}) \arrow[r, "\omega_0^\sharp(\pi_{g_1,g_2}(N))"] & (g_2\omega_0)^*(N)
\end{tikzcd}$$

$$\begin{tikzcd}
(g_2\omega_0)^*(M) = (\omega_2 f_2)^*(M) \arrow[r, "c_{\omega_2,f_2}(M)^{-1}"] \arrow[d, "(\omega_2 f_2)^*(\varphi)"] & f_2^*\omega_2^*(M) \arrow[d, "f_2^*\omega_2^*(\varphi)"] \\
(g_2\omega_0)^*(N) = (\omega_2 f_2)^*(N) \arrow[r, "c_{\omega_2,f_2}(N)^{-1}"] & f_2^*\omega_2^*(N)
\end{tikzcd}$$

By applying (8.5.5) to the following commutative diagram,

$$\begin{tikzcd}
f_1^*\omega_1^*(M^{[g_1,g_2]}) \arrow[r, "c_{\omega_2,f_2}(M)^{-1}\omega_0^\sharp(\pi_{g_1,g_2}(M))c_{\omega_1,f_1}(M^{[g_1,g_2]})"] \arrow[d, "f_1^*\omega_1^*(\varphi^{[g_1,g_2]})"] & f_2^*\omega_2^*(M) \arrow[d, "f_2^*\omega_2^*(\varphi)"] \\
f_1^*\omega_1^*(N^{[g_1,g_2]}) \arrow[r, "c_{\omega_2,f_2}(N)^{-1}\omega_0^\sharp(\pi_{g_1,g_2}(N))c_{\omega_1,f_1}(N^{[g_1,g_2]})"] & f_2^*\omega_2^*(N)
\end{tikzcd}$$

the assertion follows. □

Proposition 8.5.29 *Let $D, E, F : \mathcal{Q} \to \mathcal{E}$ be functors and M an object of $\mathcal{F}_{F(1)}$. We put $D(\tau_{0j}) = f_j$, $E(\tau_{0j}) = g_j$ and $F(\tau_{0j}) = h_j$ for $j = 1, 2$. For natural transformations $\omega : D \to E$ and $\chi : E \to F$, the following diagram is commutative.*

$$\begin{tikzcd}
\omega_1^*(\chi_1^*(N^{[h_1,h_2]})) \arrow[r, "\omega_1^*(\chi^N)"] \arrow[d, "c_{\chi_1,\omega_1}(N^{[h_1,h_2]})"] & \omega_1^*(\chi_2^*(N)^{[g_1,g_2]}) \arrow[r, "\omega^{\chi_2^*(N)}"] & \omega_2^*(\chi_2^*(N))^{[f_1,f_2]} \arrow[d, "c_{\chi_2,\omega_2}(N)^{[f_1,f_2]}"] \\
(\chi_1\omega_1)^*(N^{[h_1,h_2]}) \arrow[rr, "(\chi\omega)^N"] & & (\chi_2\omega_2)^*(N)^{[f_1,f_2]}
\end{tikzcd}$$

Proof. It follows from (8.5.2) and (8.5.26) that we have

$$\begin{aligned}
E_{f_1,f_2}(\omega_2^*(\chi_2^*(N)))^{-1}_{\omega_1^*(\chi_1^*(N^{[h_1,h_2]}))}(\omega^{\chi_2^*(N)}\omega_1^*(\chi^N)) &= \pi_{f_1,f_2}(\omega_2^*(\chi_2^*(N)))f_1^*(\omega^{\chi_2^*(N)}\omega_1^*(\chi^N)) \\
&= \pi_{f_1,f_2}(\omega_2^*(\chi_2^*(N)))f_1^*(\omega^{\chi_2^*(N)})f_1^*(\omega_1^*(\chi^N)) \\
&= c_{\omega_2,f_2}(\chi_2^*(N))^{-1}\omega_0^\sharp(\pi_{g_1,g_2}(\chi_2^*(N)))c_{\omega_1,f_1}(\chi_2^*(N)^{[g_1,g_2]})f_1^*(\omega_1^*(\chi^N))
\end{aligned}$$

Hence it suffices to show that the following diagram is commutative by (8.5.5).

$$\begin{tikzcd}
f_1^*(\omega_1^*(\chi_1^*(N^{[h_1,h_2]}))) \arrow[r, "c_{\omega_2,f_2}(\chi_2^*(N))^{-1}\omega_0^\sharp(\pi_{g_1,g_2}(\chi_2^*(N)))c_{\omega_1,f_1}(\chi_2^*(N)^{[g_1,g_2]})f_1^*(\omega_1^*(\chi^N))"] \arrow[d, "f_1^*(c_{\chi_1,\omega_1}(N^{[h_1,h_2]}))"] & f_2^*(\omega_2^*(\chi_2^*(N))) \arrow[d, "f_2^*(c_{\chi_2,\omega_2}(N))"] \\
f_1^*(\chi_1\omega_1)^*(N^{[h_1,h_2]}) \arrow[r, "c_{\chi_2\omega_2,f_2}(N)^{-1}(\chi_0\omega_0)^\sharp(\pi_{h_1,h_2}(N))c_{\chi_1\omega_1,f_1}(N^{[h_1,h_2]})"] & f_2^*(\chi_2\omega_2)^*(N)
\end{tikzcd}$$

It follows from (8.1.10) and (8.1.12) that we have

$$\begin{aligned}
&c_{\omega_1,f_1}(\chi_2^*(N^{[g_1,g_2]}))f_1^*(\omega_1^*(\chi^N)) = (\omega_1 f_1)^*(\chi^N)c_{\omega_1,f_1}(\chi_1^*(N^{[h_1,h_2]})) = (g_1\omega_0)^*(\chi^N)c_{\omega_1,f_1}(\chi_1^*(N^{[h_1,h_2]})) \\
&c_{\chi_2\omega_2,f_2}(N)f_2^*(c_{\chi_2,\omega_2}(N))c_{\omega_2,f_2}(\chi_2^*(N))^{-1} = c_{\chi_2,\omega_2 f_2}(N) = c_{\chi_2,g_2\omega_0}(N) \\
&c_{\chi_1\omega_1,f_1}(N^{[h_1,h_2]})f_1^*(c_{\chi_1,\omega_1}(N^{[h_1,h_2]}))c_{\omega_1,f_1}(\chi_1^*(N^{[h_1,h_2]}))^{-1} = c_{\chi_1,\omega_1 f_1}(N^{[h_1,h_2]}) = c_{\chi_1,g_1\omega_0}(N^{[h_1,h_2]}).
\end{aligned}$$

Hence the commutativity of the above diagram is equivalent to the following equality.

$$c_{\chi_2,g_2\omega_0}(N)\omega_0^\sharp(\pi_{g_1,g_2}(\chi_2^*(N)))(g_1\omega_0)^*(\chi^N) = (\chi_0\omega_0)^\sharp(\pi_{h_1,h_2}(N))c_{\chi_1,g_1\omega_0}(N^{[h_1,h_2]}) \ \cdots (*)$$

The following diagram is commutative by (8.1.10) and (8.4.26).

$$\begin{tikzcd}
\omega_0^*((h_1\chi_0)^*(N^{[h_1,h_2]})) \arrow[rr, "\omega_0^*(\chi_0^\sharp(\pi_{h_1,h_2}(N)))"] \arrow[d, equal] & & \omega_0^*((h_2\chi_0)^*(N)) \arrow[d, equal] \\
\omega_0^*((\chi_1 g_1)^*(N^{[h_1,h_2]})) & & \omega_0^*((\chi_2 g_2)^*(N)) \\
\omega_0^*(g_1^*(\chi_1^*(N^{[h_1,h_2]}))) \arrow[u, "\omega_0^*(c_{\chi_1,g_1}(N^{[h_1,h_2]}))"'] \arrow[r, "\omega_0^*(g_1^*(\chi_N))"] \arrow[d, "c_{g_1,\omega_0}(\chi_1^*(N^{[h_1,h_2]}))"] & \omega_0^*(g_1^*(\chi_2^*(N)^{[g_1,g_2]})) \arrow[r, "\omega_0^*(\pi_{g_1,g_2}(\chi_2^*(N)))"] \arrow[d, "c_{g_1,\omega_0}(\chi_2^*(N)^{[g_1,g_2]})"] & \omega_0^*(g_2^*(\chi_2^*(N))) \arrow[u, "\omega_0^*(c_{\chi_2,g_2}(N))"'] \arrow[d, "c_{g_2,\omega_0}(\chi_2^*(N))"] \\
(g_1\omega_0)^*(\chi_1^*(N^{[h_1,h_2]})) \arrow[r, "(g_1\omega_0)^*(\chi^N)"] & (g_1\omega_0)^*(\chi_2^*(N)^{[g_1,g_2]}) \arrow[r, "\omega_0^\sharp(\pi_{g_1,g_2}(\chi_2^*(N)))"] & (g_2\omega_0)^*(\chi_2^*(N))
\end{tikzcd}$$

Hence the left hand side of $(*)$ equals

$$\begin{aligned}
&c_{\chi_2,g_2\omega_0}(N)c_{g_2,\omega_0}(\chi_2^*(N))\omega_0^*(c_{\chi_2,g_2}(N))^{-1}\omega_0^*(\chi_0^\sharp(\pi_{h_1,h_2}(N)))\omega_0^*(c_{\chi_1,g_1}(N^{[h_1,h_2]}))c_{g_1,\omega_0}(\chi_1^*(N^{[h_1,h_2]}))^{-1}\\
&\quad = c_{\chi_2 g_2,\omega_0}(N)\omega_0^*(\chi_0^\sharp(\pi_{h_1,h_2}(N)))c_{\chi_1 g_1,\omega_0}(N^{[h_1,h_2]})^{-1}c_{\chi_1,g_1\omega_0}(N^{[h_1,h_2]})\\
&\quad = (\chi_0\omega_0)^\sharp(\pi_{h_1,h_2}(N))c_{\chi_1,g_1\omega_0}(N^{[h_1,h_2]})
\end{aligned}$$

by (8.1.12) and (8.4.29) for $M = N^{[h_1,h_2]}$ and $\varphi = \pi_{h_1,h_2}(N)$. □

Proposition 8.5.30 *For functors $D, E : \mathcal{P} \to \mathcal{E}$, we put $D(\tau_{ij}) = f_{ij}$ and $E(\tau_{ij}) = g_{ij}$ and define functors $D_i, E_i : \mathcal{Q} \to \mathcal{E}$ for $i = 0, 1, 2$ as follows.*

$$\begin{array}{lllll}
D_0(0) = D(0) & D_0(1) = D(3) & D_0(2) = D(5) & D_0(\tau_{01}) = f_{13}f_{01} & D_0(\tau_{02}) = f_{25}f_{02}\\
E_0(0) = E(0) & E_0(1) = E(3) & E_0(2) = E(5) & E_0(\tau_{01}) = g_{13}g_{01} & E_0(\tau_{02}) = g_{25}g_{02}\\
D_1(0) = D(1) & D_1(1) = D(3) & D_1(2) = D(4) & D_1(\tau_{01}) = f_{13} & D_1(\tau_{02}) = f_{14}\\
E_1(0) = E(1) & E_1(1) = E(3) & E_1(2) = E(4) & E_1(\tau_{01}) = g_{13} & E_1(\tau_{02}) = g_{14}\\
D_2(0) = D(2) & D_2(1) = D(4) & D_2(2) = D(5) & D_2(\tau_{01}) = f_{24} & D_2(\tau_{02}) = f_{25}\\
E_2(0) = E(2) & E_2(1) = E(4) & E_2(2) = E(5) & E_2(\tau_{01}) = g_{24} & E_2(\tau_{02}) = g_{25}
\end{array}$$

For a natural transformation $\gamma : D \to E$, we define a natural transformations $\gamma^i : D_i \to E_i$ $(i = 0, 1, 2)$ by

$$\gamma^0_0 = \gamma_0 \quad \gamma^0_1 = \gamma_3 \quad \gamma^0_2 = \gamma_5 \quad \gamma^1_0 = \gamma_1 \quad \gamma^1_1 = \gamma_3 \quad \gamma^1_2 = \gamma_4 \quad \gamma^2_0 = \gamma_2 \quad \gamma^2_1 = \gamma_4 \quad \gamma^2_2 = \gamma_5$$

For an object N of $\mathcal{F}_{E_0(2)} = \mathcal{F}_{E(5)}$, the following diagram is commutative.

$$\begin{tikzcd}
\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}) \arrow[r, "\gamma^{1N^{[g_{24},g_{25}]}}"] \arrow[d, "\gamma_3^*(\theta^D(N))"] & (\gamma_4^*(N^{[g_{24},g_{25}]}))^{[f_{13},f_{14}]} \arrow[r, "(\gamma^{2N})^{[f_{13},f_{14}]}"] & (\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]} \arrow[d, "\theta^E(\gamma_5^*(N))"] \\
\gamma_3^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[rr, "\gamma^{0N}"] & & \gamma_5^*(N)^{[f_{13}f_{01},f_{25}f_{02}]}
\end{tikzcd}$$

Proof. By the naturality of $E_{f_{13}f_{01},f_{25}f_{02}}(\gamma_5^*(N))$ and the definition of γ^{0N}, $\gamma^{0N}\gamma_3^*(\theta^D(N))$ is the image of the following composition by $E_{f_{13}f_{01},f_{25}f_{02}}(\gamma_5^*(N))_{\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})}$.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \xrightarrow{(f_{13}f_{01})^*(\gamma_3^*(\theta^D(N)))} (f_{13}f_{01})^*(\gamma_3^*(N^{[g_{13}g_{01},g_{25}g_{02}]})) \xrightarrow{c_{\gamma_3,f_{13}f_{01}}(N^{[g_{13}g_{01},g_{25}g_{02}]})}\\
&\qquad (\gamma_3 f_{13}f_{01})^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) = (g_{13}g_{01}\gamma_0)^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) \xrightarrow{\gamma_0^\sharp(\pi_{g_{13}g_{01},g_{25}g_{02}}(N))} (g_{25}g_{02}\gamma_0)^*(N)\\
&\qquad = (\gamma_5 f_{25}f_{02})^*(N) \xrightarrow{c_{\gamma_5,f_{25}f_{02}}(N)^{-1}} (f_{25}f_{02})^*(\gamma_5^*(N))
\end{aligned}$$

On the other hand, $\theta^E(\gamma_5^*(N))(\gamma^{2N})^{[f_{13},f_{14}]}\gamma^{1N^{[g_{24},g_{25}]}}$ is the image of the following composition.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \xrightarrow{(f_{13}f_{01})^*(\gamma^{1N^{[g_{24},g_{25}]}})} (f_{13}f_{01})^*((\gamma_4^*(N^{[g_{24},g_{25}]}))^{[f_{13},f_{14}]}) \xrightarrow{(f_{13}f_{01})^*((\gamma^{2N})^{[f_{13},f_{14}]})}\\
&\qquad (f_{13}f_{01})^*((\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \xrightarrow{(f_{13}f_{01})^*(\theta^E(\gamma_5^*(N)))} (f_{13}f_{01})^*(\gamma_5^*(N)^{[f_{13}f_{01},f_{25}f_{02}]})\\
&\qquad \xrightarrow{\pi_{f_{13}f_{01},f_{25}f_{02}}(\gamma_5^*(N))} (f_{25}f_{02})^*(\gamma_5^*(N))
\end{aligned}$$

We see that $\theta^E(\gamma_5^*(N))(\gamma^{2N})^{[f_{13},f_{14}]}\gamma^{1N^{[g_{24},g_{25}]}}$ is the image of the following composition by applying (8.5.16) to the last two morphisms of the above diagram.

$$
\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \xrightarrow{(f_{13}f_{01})^*(\gamma^{1N^{[g_{24},g_{25}]}})} (f_{13}f_{01})^*((\gamma_4^*(N^{[g_{24},g_{25}]}))^{[f_{13},f_{14}]}) \xrightarrow{(f_{13}f_{01})^*((\gamma^{2N})^{[f_{13},f_{14}]})}\\
&\qquad (f_{13}f_{01})^*((\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \xrightarrow{f_{01}^\sharp(\pi_{f_{13},f_{14}}((\gamma_5^*(N)^{[f_{24},f_{25}]}))} (f_{14}f_{01})^*(\gamma_5^*(N)^{[f_{24},f_{25}]})\\
&\qquad = (f_{24}f_{02})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \xrightarrow{f_{02}^\sharp(\pi_{f_{24},f_{25}}(\gamma_5^*(N)))} (f_{25}f_{02})^*(\gamma_5^*(N))
\end{aligned}
$$

Hence it suffices to show that the following diagram (i) is commutative.

$$
\begin{tikzcd}
(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \arrow[rr, "(f_{13}f_{01})^*(\gamma^{1N^{[g_{24},g_{25}]}})"] \arrow[d, "(f_{13}f_{01})^*(\gamma_3^*(\theta^D(N)))"] & & (f_{13}f_{01})^*((\gamma_4^*(N^{[g_{24},g_{25}]}))^{[f_{13},f_{14}]}) \arrow[d, "(f_{13}f_{01})^*((\gamma^{2N})^{[f_{13},f_{14}]})"] \\
(f_{13}f_{01})^*(\gamma_3^*(N^{[g_{13}g_{01},g_{25}g_{02}]})) \arrow[d, "c_{\gamma_3,f_{13}f_{01}}(N^{[g_{13}g_{01},g_{25}g_{02}]})"] & & (f_{13}f_{01})^*((\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \arrow[d, "f_{01}^\sharp(\pi_{f_{13},f_{14}}((\gamma_5^*(N)^{[f_{24},f_{25}]}))"] \\
(\gamma_3f_{13}f_{01})^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, equal] & \text{diagram } (i) & (f_{14}f_{01})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \arrow[d, equal] \\
(g_{13}g_{01}\gamma_0)^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, "\gamma_0^\sharp(\pi_{g_{13}g_{01},g_{25}g_{02}}(N))"] & & (f_{24}f_{02})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \arrow[d, "f_{02}^\sharp(\pi_{f_{24},f_{25}}(\gamma_5^*(N)))"] \\
(g_{25}g_{02}\gamma_0)^*(N) \arrow[r, equal] & (\gamma_5f_{25}f_{02})^*(N) \arrow[r, "c_{\gamma_5,f_{25}f_{02}}(N)^{-1}"] & (f_{25}f_{02})^*(\gamma_5^*(N))
\end{tikzcd}
$$

The following diagram (ii) is commutative by (8.1.10) and the definition of $f_{01}^\sharp$.

$$
\begin{tikzcd}
f_{01}^*(f_{13}^*(\gamma_3^*(N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \arrow[r, "c_{f_{13},f_{01}}(\gamma_3^*(N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})"] \arrow[d, "f_{01}^*(f_{13}^*(\gamma^{1N^{[g_{24},g_{25}]}}))"] & (f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \arrow[d, "(f_{13}f_{01})^*(\gamma^{1N^{[g_{24},g_{25}]}})"] \\
f_{01}^*(f_{13}^*(\gamma_4^*(N^{[g_{24},g_{25}]})^{[f_{13},f_{14}]})) \arrow[r, "c_{f_{13},f_{01}}(\gamma_4^*(N^{[g_{24},g_{25}]})^{[f_{13},f_{14}]}))"] \arrow[d, "f_{01}^*(f_{13}^*((\gamma^{2N})^{[f_{13},f_{14}]}))"] & (f_{13}f_{01})^*(\gamma_4^*(N^{[g_{24},g_{25}]})^{[f_{13},f_{14}]}) \arrow[d, "(f_{13}f_{01})^*((\gamma^{2N})^{[f_{13},f_{14}]})"] \\
f_{01}^*(f_{13}^*((\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})) \arrow[r, "c_{f_{13},f_{01}}((\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]})"] \arrow[d, "f_{01}^*(\pi_{f_{13},f_{14}}(\gamma_5^*(N)^{[f_{24},f_{25}]}))"] & (f_{13}f_{01})^*((\gamma_5^*(N)^{[f_{24},f_{25}]})^{[f_{13},f_{14}]}) \arrow[d, "f_{01}^\sharp(\pi_{f_{13},f_{14}}(\gamma_5^*(N)^{[f_{24},f_{25}]}))"] \\
f_{01}^*(f_{14}^*(\gamma_5^*(N)^{[f_{24},f_{25}]})) \arrow[r, "c_{f_{14},f_{01}}(\gamma_5^*(N)^{[f_{24},f_{25}]})"] & (f_{14}f_{01})^*(\gamma_5^*(N)^{[f_{24},f_{25}]})
\end{tikzcd}
$$

diagram (ii)

It follows from (8.5.3), (8.5.2) and the definition of $\gamma^{1N^{[g_{24},g_{25}]}}$ that the following equalities hold.

$$
\begin{aligned}
&\pi_{f_{13},f_{14}}(\gamma_5^*(N)^{[f_{24},f_{25}]})f_{13}^*((\gamma^{2N})^{[f_{13},f_{14}]}) = f_{14}^*(\gamma^{2N})\pi_{f_{13},f_{14}}(\gamma_4^*(N^{[g_{24},g_{25}]}))\\
&\pi_{f_{13},f_{14}}(\gamma_4^*(N^{[g_{24},g_{25}]}))f_{13}^*(\gamma^{1N^{[g_{24},g_{25}]}}) = c_{\gamma_4,f_{14}}(N^{[g_{24},g_{25}]})^{-1}\gamma_1^\sharp(\pi_{g_{13},g_{14}}(N^{[g_{24},g_{25}]}))c_{\gamma_3,f_{13}}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})
\end{aligned}
$$

Hence the composition of the left vertical morphisms of diagram (ii) coincides with the following.

$$
\begin{aligned}
&f_{01}^*(\pi_{f_{13},f_{14}}(\gamma_5^*(N)^{[f_{24},f_{25}]}))f_{01}^*(f_{13}^*((\gamma^{2N})^{[f_{13},f_{14}]}))f_{01}^*(f_{13}^*(\gamma^{1N^{[g_{24},g_{25}]}}))\\
&\quad = f_{01}^*(f_{14}^*(\gamma^{2N}))f_{01}^*(\pi_{f_{13},f_{14}}(\gamma_4^*(N^{[g_{24},g_{25}]})))f_{01}^*(f_{13}^*(\gamma^{1N^{[g_{24},g_{25}]}}))\\
&\quad = f_{01}^*(f_{14}^*(\gamma^{2N}))f_{01}^*(c_{\gamma_4,f_{14}}(N^{[g_{24},g_{25}]})^{-1})f_{01}^*(\gamma_1^\sharp(\pi_{g_{13},g_{14}}(N^{[g_{24},g_{25}]})))f_{01}^*(c_{\gamma_3,f_{13}}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}))
\end{aligned}
$$

Since $c_{f_{14},f_{01}}(\gamma_5^*(N)^{[f_{24},f_{25}]})f_{01}^*(f_{14}^*(\gamma^{2N})) = (f_{14}f_{01})^*(\gamma^{2N})c_{f_{14},f_{01}}(\gamma_4^*(N^{[g_{24},g_{25}]}))$ by (8.1.10), the commutativity of diagram (ii) implies that the composition of the upper horizontal morphism and the right vertical morphisms of diagram (i) coincides with the following composition.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \xrightarrow{c_{f_{13},f_{01}}(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}))^{-1}} f_{01}^*(f_{13}^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})))\\
&\xrightarrow{f_{01}^*(c_{\gamma_3,f_{13}}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}))} f_{01}^*((\gamma_3 f_{13})^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) = f_{01}^*((g_{13}\gamma_1)^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}))\\
&\xrightarrow{f_{01}^*(\gamma_1^\sharp(\pi_{g_{13},g_{14}}(N^{[g_{24},g_{25}]})))} f_{01}^*((g_{14}\gamma_1)^*(N^{[g_{24},g_{25}]})) = f_{01}^*((\gamma_4 f_{14})^*(N^{[g_{24},g_{25}]})) \xrightarrow{f_{01}^*(c_{\gamma_4,f_{14}}(N^{[g_{24},g_{25}]})^{-1})}\\
&f_{01}^*(f_{14}^*(\gamma_4^*(N^{[g_{24},g_{25}]}))) \xrightarrow{c_{f_{14},f_{01}}(\gamma_4^*(N^{[g_{24},g_{25}]}))} (f_{14}f_{01})^*(\gamma_4^*(N^{[g_{24},g_{25}]})) \xrightarrow{(f_{14}f_{01})^*(\gamma^{2N})}\\
&(f_{14}f_{01})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) = (f_{24}f_{02})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \xrightarrow{f_{02}^\sharp(\pi_{f_{24},f_{25}}(\gamma_5^*(N)))} (f_{25}f_{02})^*(\gamma_5^*(N))
\end{aligned}$$

diagram (iii)

Next, we consider the composition of the left vertical morphisms and the lower horizontal morphism of diagram (i). It follows from (8.1.10) and (8.5.16) that the following diagram is commutative.

$$\begin{tikzcd}
(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \arrow[rr, "(f_{13}f_{01})^*(\gamma_3^*(\theta^D(N)))"] \arrow[d, "c_{\gamma_3,f_{13}f_{01}}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})"] & & (f_{13}f_{01})^*(\gamma_3^*(N^{[g_{13}g_{01},g_{25}g_{02}]})) \arrow[d, "c_{\gamma_3,f_{13}f_{01}}(N^{[g_{13}g_{01},g_{25}g_{02}]})"'] \\
(\gamma_3 f_{13}f_{01})^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}) \arrow[rr, "(\gamma_3 f_{13}f_{01})^*(\theta^D(N))"] \arrow[d, equal] & & (\gamma_3 f_{13}f_{01})^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, equal] \\
(g_{13}g_{01}\gamma_0)^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}) \arrow[rr, "(g_{13}g_{01}\gamma_0)^*(\theta^D(N))"] \arrow[d, "c_{g_{13}g_{01},\gamma_0}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})"] & & (g_{13}g_{01}\gamma_0)^*(N^{[g_{13}g_{01},g_{25}g_{02}]}) \arrow[d, "c_{g_{13}g_{01},\gamma_0}(N^{[g_{13}g_{01},g_{25}g_{02}]})^{-1}"'] \\
\gamma_0^*((g_{13}g_{01})^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \arrow[rr, "\gamma_0^*((g_{13}g_{01})^*(\theta^D(N)))"] \arrow[d, "\gamma_0^*(g_{01}^\sharp(\pi_{g_{13},g_{14}}(N^{[g_{24},g_{25}]})))"] & & \gamma_0^*((g_{13}g_{01})^*(N^{[g_{13}g_{01},g_{25}g_{02}]})) \arrow[d, "\gamma_0^*(\pi_{g_{13}g_{01},g_{25}g_{02}}(N))"'] \\
\gamma_0^*((g_{14}g_{01})^*(N^{[g_{24},g_{25}]})) \arrow[r, equal] & \gamma_0^*((g_{24}g_{02})^*(N^{[g_{24},g_{25}]})) \arrow[r, "\gamma_0^*(g_{02}^\sharp(\pi_{g_{24},g_{25}}(N)))"] & \gamma_0^*((g_{25}g_{02})^*(N))
\end{tikzcd}$$

Since $\gamma_0^\sharp(\pi_{g_{13}g_{01},g_{25}g_{02}}(N)) = c_{g_{25}g_{02},\gamma_0}(N)\gamma_0^*(\pi_{g_{13}g_{01},g_{25}g_{02}}(N))c_{g_{13}g_{01},\gamma_0}(N^{[g_{13}g_{01},g_{25}g_{02}]})^{-1}$, it follows from the above diagram that the composition of the left vertical morphisms and the lower horizontal morphism of diagram (i) coincides with the following composition.

$$\begin{aligned}
&(f_{13}f_{01})^*(\gamma_3^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})) \xrightarrow{c_{\gamma_3,f_{13}f_{01}}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})} (\gamma_3 f_{13}f_{01})^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})\\
&= (g_{13}g_{01}\gamma_0)^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}) \xrightarrow{c_{g_{13}g_{01},\gamma_0}((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]})} \gamma_0^*((g_{13}g_{01})^*((N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}))\\
&\xrightarrow{\gamma_0^*(g_{01}^\sharp(\pi_{g_{13},g_{14}}(N^{[g_{24},g_{25}]})))} \gamma_0^*((g_{14}g_{01})^*(N^{[g_{24},g_{25}]})) = \gamma_0^*((g_{24}g_{02})^*(N^{[g_{24},g_{25}]})) \xrightarrow{\gamma_0^*(g_{02}^\sharp(\pi_{g_{24},g_{25}}(N)))}\\
&\gamma_0^*((g_{25}g_{02})^*(N)) \xrightarrow{c_{g_{25}g_{02},\gamma_0}(N)} (g_{25}g_{02}\gamma_0)^*(N) = (\gamma_5 f_{25}f_{02})^*(N) \xrightarrow{c_{\gamma_5,f_{25}f_{02}}(N)^{-1}} (f_{25}f_{02})^*(\gamma_5^*(N))
\end{aligned}$$

diagram (iv)

The following diagram is commutative by (8.1.10), (8.1.12) and (8.5.26).

$$\begin{tikzcd}
f_{02}^*((g_{24}\gamma_2)^*(N^{[g_{24},g_{25}]})) \arrow[r, "c_{g_{24}\gamma_2,f_{02}}(N^{[g_{24},g_{25}]})"] \arrow[d, equal] \arrow[dr, "f_{02}^*(\gamma_2^\sharp(\pi_{g_{24},g_{25}}(N)))"] & (g_{24}\gamma_2 f_{02})^*(N^{[g_{24},g_{25}]}) \arrow[r, "f_{02}^\sharp(\gamma_2^\sharp(\pi_{g_{24},g_{25}}(N)))"] & (\gamma_5 f_{25}f_{02})^*(N) \\
f_{02}^*((\gamma_4 f_{24})^*(N^{[g_{24},g_{25}]})) & f_{02}^*((g_{25}\gamma_2)^*(N)) \arrow[r, equal] & f_{02}^*((\gamma_5 f_{25})^*(N)) \arrow[u, "c_{\gamma_5 f_{25},f_{02}}(N)"'] \\
f_{02}^*(f_{24}^*(\gamma_4^*(N^{[g_{24},g_{25}]}))) \arrow[r, "f_{02}^*(f_{24}^*(\gamma^{2N}))"] \arrow[u, "f_{02}^*(c_{\gamma_4,f_{24}}(N^{[g_{24},g_{25}]}))"] & f_{02}^*(f_{25}^*(\gamma_5^*(N)^{[f_{24},f_{25}]})) \arrow[r, "f_{02}^*(\pi_{f_{24},f_{25}}(\gamma_5^*(N)))"] & f_{02}^*(f_{25}^*(\gamma_5^*(N))) \arrow[u, "f_{02}^*(c_{\gamma_5,f_{25}}(N))"'] \\
(f_{24}f_{02})^*(\gamma_4^*(N^{[g_{24},g_{25}]})) \arrow[r, "(f_{24}f_{02})^*(\gamma^{2N})"] \arrow[u, "c_{f_{24},f_{02}}(\gamma_4^*(N^{[g_{24},g_{25}]}))^{-1}"] & (f_{25}f_{02})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \arrow[r, "f_{02}^\sharp(\pi_{f_{24},f_{25}}(\gamma_5^*(N)))"] \arrow[u, "c_{f_{25},f_{02}}(\gamma_5^*(N)^{[f_{24},f_{25}]})^{-1}"] & (f_{25}f_{02})^*(\gamma_5^*(N)) \arrow[u, "c_{f_{25},f_{02}}(\gamma_5^*(N))^{-1}"'] \arrow[uuu, bend right, "c_{\gamma_5,f_{25}f_{02}}(N)"']
\end{tikzcd}$$

We note that, by (8.1.12), $c_{\gamma_4, f_{24}f_{02}}(M) : (f_{24}f_{02})^*(\gamma_4^*(M)) \to (\gamma_4 f_{24} f_{02})^*(M)$ coincides with a commposition $c_{g_{24}\gamma_2, f_{02}}(N^{[g_{24},g_{25}]})c_{f_{25},f_{02}}(\gamma_5^*(N)^{[f_{24},f_{25}]})c_{f_{25},f_{02}}(\gamma_5^*(N))^{-1}$. Hence the following diagram is commutative by (8.1.12) and (8.1.14). Here we put $M = N^{[g_{24},g_{25}]}$ and $L = (N^{[g_{24},g_{25}]})^{[g_{13},g_{14}]}$ below.

$$\begin{tikzcd}
(f_{13}f_{01})^*(\gamma_3^*(L)) \arrow[r, "c_{\gamma_3, f_{13}f_{01}}(L)"] \arrow[d, "c_{f_{13},f_{01}}(\gamma_3^*(L))^{-1}"] & (\gamma_3 f_{13} f_{01})^*(L) \arrow[r, equal] \arrow[d, equal] & (g_{13}g_{01}\gamma_0)^*(L) \arrow[d, "c_{g_{13}g_{01},\gamma_0}(L)"] \arrow[ddl, "\gamma_0^\sharp(g_{01}^\sharp(\pi_{g_{13},g_{14}}(M))"'] \\
f_{01}^*(f_{13}^*(\gamma_3^*(L))) \arrow[d, "f_{01}^*(c_{\gamma_3,f_{13}}(L))"] & (g_{13}\gamma_1 f_{01})^*(L) \arrow[d, "f_{01}^\sharp(\gamma_1^\sharp(\pi_{g_{13},g_{14}}(M))"'] & \gamma_0^*((g_{13}g_{01})^*(L)) \arrow[d, "\gamma_0^*(g_{01}^\sharp(\pi_{g_{13},g_{14}}(M))"'] \\
f_{01}^*((\gamma_3 f_{13})^*(L)) \arrow[d, equal] & (g_{14}g_{01}\gamma_0)^*(M) \arrow[d, equal] & \gamma_0^*((g_{14}g_{01})^*(M)) \arrow[l, "c_{g_{14}g_{01},\gamma_0}(M)"'] \arrow[d, equal] \\
f_{01}^*((g_{13}\gamma_1)^*(L)) \arrow[uur, bend left, "c_{g_{13}\gamma_1,f_{01}}(L)"] \arrow[d, "f_{01}^*(\gamma_1^\sharp(\pi_{g_{13},g_{14}}(M)))"] & (g_{24}g_{02}\gamma_0)^*(M) \arrow[d, equal] \arrow[ddr, "\gamma_0^\sharp(g_{02}^\sharp(\pi_{g_{24},g_{25}}(N)))"] & \gamma_0^*((g_{24}g_{02})^*(M)) \arrow[l, "c_{g_{24}g_{02},\gamma_0}(M)"'] \arrow[d, "\gamma_0^*(g_{02}^\sharp(\pi_{g_{24},g_{25}}(N)))"'] \\
f_{01}^*((g_{14}\gamma_1)^*(M)) \arrow[d, equal] \arrow[uur, bend right, "c_{g_{14}\gamma_1,f_{01}}(M)"'] & (\gamma_4 f_{14} f_{01})^*(M) \arrow[d, equal] & \gamma_0^*((g_{25}g_{02})^*(N)) \arrow[d, "c_{g_{25}g_{02},\gamma_0}(N)"] \\
f_{01}^*((\gamma_4 f_{14})^*(M)) \arrow[d, "f_{01}^*(c_{\gamma_4,f_{14}}(M)^{-1})"] \arrow[uur, "c_{\gamma_4 f_{14},f_{01}}(M)"'] & (\gamma_4 f_{24} f_{02})^*(M) & (g_{25}g_{02}\gamma_0)^*(N) \arrow[d, equal] \\
f_{01}^*(f_{14}^*(\gamma_4^*(M))) \arrow[d, "c_{f_{14},f_{01}}(\gamma_4^*(M))"'] & & (\gamma_5 f_{25} f_{02})^*(N) \\
(f_{14}f_{01})^*(\gamma_4^*(M)) \arrow[r, equal] \arrow[d, "(f_{14}f_{01})^*(\gamma^{2N})"] \arrow[uuur, "c_{\gamma_4,f_{14}f_{01}}(M)"'] & (f_{24}f_{02})^*(\gamma_4^*(M)) \arrow[uu, "c_{\gamma_4,f_{24}f_{02}}(M)"'] \arrow[d, "(f_{24}f_{02})^*(\gamma^{2N})"] & \\
(f_{14}f_{01})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \arrow[r, equal] & (f_{24}f_{02})^*(\gamma_5^*(N)^{[f_{24},f_{25}]}) \arrow[r, "f_{02}^\sharp(\pi_{f_{24},f_{25}}(\gamma_5^*(N)))"] & (f_{25}f_{02})^*(\gamma_5^*(N)) \arrow[uu, "c_{\gamma_5,f_{25}f_{02}}(N)"]
\end{tikzcd}$$

We see that the compositions of diagram (iii) and the compositions of diagram (iv) coincide, which implies the assertion. □

8.6 Cartesian closed fibered category

Proposition 8.6.1 *Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category and $f : X \to Y$, $g : X \to Z$ morphisms of $\mathcal{E}$.*

(1) Suppose that the presheaf $F_K^{f,g}$ on $\mathcal{F}_Y$ is representable for any $K \in \mathrm{Ob}\,\mathcal{F}_Z$. If a morphism $\varphi : M \to N$ of $\mathcal{F}_Y$ is an epimorphism and the presheaves $F_{f,g,M}$ and $F_{f,g,N}$ on $\mathcal{F}_Z^{op}$ are representable, then $\varphi_{[f,g]} : M_{[f,g]} \to N_{[f,g]}$ is an epimorphism of $\mathcal{F}_Z$.

(2) Suppose that the presheaf $F_{f,g,K}$ on $\mathcal{F}_Z^{op}$ is representable for any $K \in \mathrm{Ob}\,\mathcal{F}_Y$. If a morphism $\varphi : M \to N$ of $\mathcal{F}_Z$ is a monomorphism and the presheaves $F_M^{f,g}$ and $F_N^{f,g}$ on $\mathcal{F}_Y$ are representable, then $\varphi^{[f,g]} : M^{[f,g]} \to N^{[f,g]}$ is a monomorphism of $\mathcal{F}_Y$.

Proof. (1) The following diagram commutes by (8.4.3) and the naturality of $E_{f,g}(K)$.

$$\begin{tikzcd}
\mathcal{F}_Z(N_{[f,g]}, K) \arrow[d, "\varphi^{[f,g]*}"] & \mathcal{F}_X(f^*(N), g^*(K)) \arrow[l, "P_{f,g}(N)_K"'] \arrow[r, "E_{f,g}(K)_N"] \arrow[d, "f^*(\varphi)^*"] & \mathcal{F}_Y(N, K^{[f,g]}) \arrow[d, "\varphi^*"] \\
\mathcal{F}_Z(M_{[f,g]}, K) & \mathcal{F}_X(f^*(M), f^*(K)) \arrow[l, "P_{f,g}(M)_K"'] \arrow[r, "E_{f,g}(K)_M"] & \mathcal{F}_Y(M, K^{[f,g]})
\end{tikzcd}$$

Since $\varphi^* : \mathcal{F}_Y(N, K^{[f,g]}) \to \mathcal{F}_Y(M, K^{[f,g]})$ is injective by the assumption, it follows from the above diagram that $\varphi^{[f,g]*} : \mathcal{F}_Z(N_{[f,g]}, K) \to \mathcal{F}_Z(M_{[f,g]}, K)$ is also injective.

(2) The following diagrams commute by (8.5.3) and the naturality of $P_{f,g}(K)$.

$$\begin{tikzcd}
\mathcal{F}_Y(K, M^{[f,g]}) \arrow[d, "\varphi_*^{[f,g]}"] & \mathcal{F}_X(f^*(K), g^*(M)) \arrow[l, "E_{f,g}(M)_K"'] \arrow[r, "P_{f,g}(K)_M"] \arrow[d, "g^*(\varphi)_*"] & \mathcal{F}_Z(K_{[f,g]}, M) \arrow[d, "\varphi_*"] \\
\mathcal{F}_Y(K, N^{[f,g]}) & \mathcal{F}_{f,g}(f^*(K), g^*(N)) \arrow[l, "E_{f,g}(N)_K"'] \arrow[r, "P_{f,g}(K)_N"] & \mathcal{F}_Z(K_{[f,g]}, N)
\end{tikzcd}$$

Since $\varphi_* : \mathcal{F}_1(K_{[f,g]}, M) \to \mathcal{F}_1(K_{[f,g]}, N)$ is injective by the assumption, it follows from the above diagram that $\varphi^{[f,g]} : \mathcal{F}_1(K, M^{[f,g]}) \to \mathcal{F}_1(K, N^{[f,g]})$ is also injective. □

Proposition 8.6.2 *Let $p : \mathcal{F} \to \mathcal{T}$ be a normalized cloven fibered category and $f : X \to Y$, $g : X \to Z$ morphisms of $\mathcal{E}$.*

(1) Suppose that the presheaf $F_K^{f,g}$ on $\mathcal{F}_Y$ is representable for any $K \in \operatorname{Ob}\mathcal{F}_Z$ and that the presheaves $F_{f,g,L}$, $F_{f,g,M}$, $F_{f,g,N}$ on $\mathcal{F}_Z^{op}$ are representable for objects L, M, N of $\mathcal{F}_Y$. If $\lambda : N \to L$ is a coequalizer of morphisms $\varphi, \psi : M \to N$ of $\mathcal{F}_Y$, then $\lambda_{[f,g]} : N_{[f,g]} \to L_{[f,g]}$ is a coequalizer of morphisms $\varphi_{[f,g]}, \psi_{[f,g]} : M_{[f,g]} \to N_{[f,g]}$.

(2) Suppose that the presheaf $F_{X,K}$ on $\mathcal{F}_Z^{op}$ is representable for any $K \in \operatorname{Ob}\mathcal{F}_Y$ and that the presheaves $F_L^{f,g}$, $F_M^{f,g}$, $F_N^{f,g}$ on $\mathcal{F}_Y$ are representable for objects L, M, N of $\mathcal{F}_Z$. If $\lambda : L \to M$ is an equalizer of morphisms $\varphi, \psi : M \to N$ of $\mathcal{F}_Z$, then $\lambda^{[f,g]} : L^{[f,g]} \to M^{[f,g]}$ is an equalizer of morphisms $\varphi^{[f,g]}, \psi^{[f,g]} : M^{[f,g]} \to N^{[f,g]}$.

Proof. (1) The following diagrams commute by (8.4.3) and the naturality of $E_{f,g}(K)$.

$$\begin{tikzcd}
\mathcal{F}_Z(N_{[f,g]}, K) \arrow[d, "(\varphi_{[f,g]})^*"] & \mathcal{F}_X(f^*(N), g^*(K)) \arrow[l, "P_{f,g}(N)_K"'] \arrow[r, "E_{f,g}(K)_N"] \arrow[d, "f^*(\varphi)^*"] & \mathcal{F}_Y(N, K^{[f,g]}) \arrow[d, "\varphi^*"] \\
\mathcal{F}_Z(M_{[f,g]}, K) & \mathcal{F}_X(f^*(M), g^*(K)) \arrow[l, "P_{f,g}(M)_K"'] \arrow[r, "E_{f,g}(K)_M"] & \mathcal{F}_Y(M, K^{[f,g]})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_Z(N_{[f,g]}, K) \arrow[d, "(\psi_{[f,g]})^*"] & \mathcal{F}_X(f^*(N), g^*(K)) \arrow[l, "P_{f,g}(N)_K"'] \arrow[r, "E_{f,g}(K)_N"] \arrow[d, "f^*(\psi)^*"] & \mathcal{F}_Y(N, K^{[f,g]}) \arrow[d, "\psi^*"] \\
\mathcal{F}_Z(M_{[f,g]}, K) & \mathcal{F}_X(f^*(M), g^*(K)) \arrow[l, "P_{f,g}(M)_K"'] \arrow[r, "E_{f,g}(K)_M"] & \mathcal{F}_Y(M, K^{[f,g]})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_Z(L_{[f,g]}, K) \arrow[d, "(\lambda_{[f,g]})^*"] & \mathcal{F}_X(f^*(L), g^*(K)) \arrow[l, "P_{f,g}(L)_K"'] \arrow[r, "E_{f,g}(K)_L"] \arrow[d, "f^*(\lambda)^*"] & \mathcal{F}_Y(L, K^{[f,g]}) \arrow[d, "\lambda^*"] \\
\mathcal{F}_Z(N_{[f,g]}, K) & \mathcal{F}_X(f^*(N), g^*(K)) \arrow[l, "P_{f,g}(N)_K"'] \arrow[r, "E_{f,g}(K)_N"] & \mathcal{F}_Y(N, K^{[f,g]})
\end{tikzcd}$$

Since $\lambda^* : \mathcal{F}_Y(L, K^{[f,g]}) \to \mathcal{F}_Y(N, K^{[f,g]})$ is an equalizer of maps $\varphi^*, \psi^* : \mathcal{F}_Y(N, K^{[f,g]}) \to \mathcal{F}_Y(M, K^{[f,g]})$, it follows from the above diagrams that $(\lambda_{[f,g]})^* : \mathcal{F}_Z(L_{[f,g]}, K) \to \mathcal{F}_Z(N_{[f,g]}, K)$ is an equalizer of maps $(\varphi_{[f,g]})^*, (\psi_{[f,g]})^* : \mathcal{F}_Z(N_{[f,g]}, K) \to \mathcal{F}_Z(M_{[f,g]}, K)$.

(2) The following diagrams commute by (8.5.3) and the naturality of $P_{f,g}(K)$.

$$\begin{tikzcd}
\mathcal{F}_Y(K, M^{[f,g]}) \arrow[d, "\varphi_*^{[f,g]}"] & \mathcal{F}_X(f^*(K), g^*(M)) \arrow[l, "E_{f,g}(M)_K"'] \arrow[r, "P_{f,g}(K)_M"] \arrow[d, "g^*(\varphi)_*"] & \mathcal{F}_Z(K_{[f,g]}, M) \arrow[d, "\varphi_*"] \\
\mathcal{F}_Y(K, N^{[f,g]}) & \mathcal{F}_X(f^*(K), g^*(N)) \arrow[l, "E_{f,g}(N)_K"'] \arrow[r, "P_{f,g}(K)_N"] & \mathcal{F}_Z(K_{[f,g]}, N)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_Y(K, M^{[f,g]}) \arrow[d, "\psi_*^{[f,g]}"] & \mathcal{F}_X(f^*(K), g^*(M)) \arrow[l, "E_{f,g}(M)_K"'] \arrow[r, "P_{f,g}(K)_M"] \arrow[d, "g^*(\psi)_*"] & \mathcal{F}_Z(K_{[f,g]}, M) \arrow[d, "\psi_*"] \\
\mathcal{F}_Y(K, N^{[f,g]}) & \mathcal{F}_X(f^*(K), g^*(N)) \arrow[l, "E_{f,g}(N)_K"'] \arrow[r, "P_{f,g}(K)_N"] & \mathcal{F}_Z(K_{[f,g]}, N)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}_Y(K, M^{[f,g]}) \arrow[d, "\lambda_*^{[f,g]}"] & \mathcal{F}_X(f^*(K), g^*(M)) \arrow[l, "E_{f,g}(M)_K"'] \arrow[r, "P_{f,g}(K)_M"] \arrow[d, "g^*(\lambda)_*"] & \mathcal{F}_Z(K_{[f,g]}, M) \arrow[d, "\lambda_*"] \\
\mathcal{F}_Y(K, N^{[f,g]}) & \mathcal{F}_X(f^*(K), g^*(N)) \arrow[l, "E_{f,g}(N)_K"'] \arrow[r, "P_{f,g}(K)_N"] & \mathcal{F}_Z(K_{[f,g]}, N)
\end{tikzcd}$$

Since $\lambda_* : \mathcal{F}_Z(K_{[f,g]}, L) \to \mathcal{F}_Z(K_{[f,g]}, M)$ is an equalizer of maps $\varphi_*, \psi_* : \mathcal{F}_Z(K_{[f,g]}, M) \to \mathcal{F}_Z(K_{[f,g]}, N)$, it follows from the above diagrams that $\lambda_* : \mathcal{F}_Y(K, L^{[f,g]}) \to \mathcal{F}_Y(K, M^{[f,g]})$ is an equalizer of maps $\varphi_*^{[f,g]}, \psi_*^{[f,g]} : \mathcal{F}_Y(K, M^{[f,g]}) \to \mathcal{F}_Y(K, N^{[f,g]})$. □

Proposition 8.6.3 *For a functor $D : \mathcal{P} \to \mathcal{E}$, we put $D(\tau_{01}) = j$, $D(\tau_{02}) = k$, $D(\tau_{13}) = f$, $D(\tau_{14}) = g$, $D(\tau_{24}) = h$, $D(\tau_{25}) = i$. For objects M of $\mathcal{F}_{D(3)}$ and N of $\mathcal{F}_{D(5)}$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathcal{F}_{D(5)}((M_{[f,g]})_{[h,i]}, N) \arrow[r, "\theta_D(M)^*"] \arrow[d, "P_{h,i}(M_{[f,g]})_N^{-1}"] & \mathcal{F}_{D(5)}(M_{[fj,ik]}, N) \arrow[r, "P_{fj,ik}(M)_N^{-1}"] & \mathcal{F}_{D(0)}((fj)^*(M), (ik)^*(N)) \arrow[d, "E_{fj,ik}(N)_M"] \\
\mathcal{F}_{D(2)}(h^*(M_{[f,g]}), i^*(N)) \arrow[d, "E_{h,i}(N)_{M_{[f,g]}}"] & & \mathcal{F}_{D(3)}(M, N^{[fj,ik]}) \\
\mathcal{F}_{D(4)}(M_{[f,g]}, N^{[h,i]}) \arrow[r, "P_{f,g}(M)^{-1}_{N^{[h,i]}}"] & \mathcal{F}_{D(1)}(f^*(M), g^*(N^{[h,i]})) \arrow[r, "E_{f,g}(N^{[h,i]})_M"] & \mathcal{F}_{D(3)}(M, (N^{[h,i]})^{[f,g]}) \arrow[u, "\theta^D(N)_*"]
\end{tikzcd}$$

Proof. For $\varphi \in \mathcal{F}_{D(5)}((M_{[f,g]})_{[h,i]}, N)$, we put $\psi = E_{h,i}(N)_{M_{[f,g]}} P_{h,i}(M_{[f,g]})_N^{-1}(\varphi) : M_{[f,g]} \to N^{[h,i]}$ and $\xi = E_{f,g}(N^{[h,i]})_M P_{f,g}(M)^{-1}_{N^{[h,i]}}(\psi) : M \to (N^{[h,i]})^{[f,g]}$. It follows from (8.4.2) and (8.5.2) that the following diagrams commute.

$$\begin{tikzcd}
f^*(M) \arrow[r, "\iota_{f,g}(M)"] \arrow[d, "f^*(\xi)"] & g^*(M_{[f,g]}) \arrow[d, "g^*(\psi)"] \\
f^*((N^{[h,i]})^{[f,g]}) \arrow[r, "\pi_{f,g}(N^{[h,i]})"] & g^*(N^{[h,i]})
\end{tikzcd}
\qquad
\begin{tikzcd}
h^*(M_{[f,g]}) \arrow[r, "\iota_{h,i}(M_{[f,g]})"] \arrow[d, "h^*(\psi)"] & i^*((M_{[f,g]})_{[h,i]}) \arrow[d, "i^*(\varphi)"] \\
h^*(N^{[h,i]}) \arrow[r, "\pi_{h,i}(N)"] & i^*(N)
\end{tikzcd}$$

By applying $j^\sharp$ to the above left diagram and $k^\sharp$ to the right one, we have the following commutative diagram by (8.1.13).

$$\begin{tikzcd}
(fj)^*(M) \arrow[r, "j^\sharp(\iota_{f,g}(M))"] \arrow[d, "(fj)^*(\xi)"] & (gj)^*(M_{[f,g]}) \arrow[r, equal] \arrow[d, "(gj)^*(\psi)"] & (hk)^*(M_{[f,g]}) \arrow[r, "k^\sharp(\iota_{h,i}(M_{[f,g]}))"] \arrow[d, "(hk)^*(\psi)"] & (ik)^*((M_{[f,g]})_{[h,i]}) \arrow[d, "(ik)^*(\varphi)"] \\
(fj)^*((N^{[h,i]})^{[f,g]}) \arrow[r, "j^\sharp(\pi_{f,g}(N^{[h,i]}))"] & (gj)^*(N^{[h,i]}) \arrow[r, equal] & (hk)^*(N^{[h,i]}) \arrow[r, "k^\sharp(\pi_{h,i}(N))"] & (ik)^*(N)
\end{tikzcd}$$

Hence, by (8.4.24) and (8.5.24), the following diagram commutes.

$$\begin{tikzcd}
(fj)^*(M) \arrow[r, "(ik)^*(\theta_D(M))\iota_{fj,ik}(M)"] \arrow[d, "(fj)^*(\xi)"] & (ik)^*((M_{[f,g]})_{[h,i]}) \arrow[d, "(ik)^*(\varphi)"] \\
(fj)^*((N^{[h,i]})^{[f,g]}) \arrow[r, "\pi_{fj,ik}(N)(fj)^*(\theta^D(N))"] & (ik)^*(N)
\end{tikzcd}$$

By (8.4.2) and (8.5.2), we have

$$P_{fj,ik}(M)_N((ik)^*(\varphi)(ik)^*(\theta_D(M))\iota_{fj,ik}(M)) = P_{fj,ik}(M)_N((ik)^*(\varphi\,\theta_D(M))\iota_{fj,ik}(M)) = \varphi\,\theta_D(N)$$
$$E_{fj,ik}(N)_M(\pi_{fj,ik}(N)(fj)^*(\theta^D(N))(fj)^*(\xi)) = E_{fj,ik}(N)_M(\pi_{fj,ik}(N)(fj)^*(\theta^D(N)\,\xi)) = \theta^D(N)\,\xi.$$

This shows that $P_{fj,ik}(M)_N^{-1}(\varphi\,\theta_D(N)) = E_{fj,ik}(N)_M^{-1}(\theta^D(N)\,\xi)$, which implies the result. □

Remark 8.6.4 *The above result implies that $\theta_D(M) : M_{[fj,ik]} \to (M_{[f,g]})_{[h,i]}$ is an isomorphism for all object M of $\mathcal{F}_{D(3)}$ if and only if $\theta^D(N) : (N^{[h,i]})^{[f,g]} \to N^{[fj,ik]}$ is an isomorphism for all object N of $N\mathcal{F}_{D(5)}$.*

8.7 Fibered category of modules

Let K_* be a graded commutative algebra. We denote by $\mathcal{A}lg_{K_*}$ the category of graded K_*-algebras and homomorphisms between them. We also denote by $\mathcal{M}od_{K_*}$ the category of graded left K_*-modules and homomorphisms which preserve degrees. For an object R_* of $\mathcal{A}lg_{K_*}$, we denote by $\eta_{R_*} : K_* \to R_*$ the unit of R_* and by $\mu_{R_*} : R_* \otimes_{K_*} R_* \to R_*$ is the map induced by the product of R_*.

Let $\mathcal{C}$ be a subcategory of $\mathcal{A}lg_{K_*}$ and $\mathcal{M}$ a subcategory of $\mathcal{M}od_{K_*}$.

Condition 8.7.1 *We assume $\mathcal{M}$ satisfies the following conditions.*

(∗) *If a morphism $S_* \to R_*$ of $\mathcal{C}$ and a right S_* module structure on $M_* \in \operatorname{Ob}\mathcal{M}$ are given, then $M_*\otimes_{S_*}R_*$ is an object of $\mathcal{M}$.*

We define a category $\mathcal{M}od(\mathcal{C},\mathcal{M})$ as follows. $\operatorname{Ob}\mathcal{M}od(\mathcal{C},\mathcal{M})$ consists of triples (R_*, M_*, α) where $R_* \in \operatorname{Ob}\mathcal{C}$, $M_* \in \operatorname{Ob}\mathcal{M}$ and $\alpha : M_*\otimes_{K_*}R_* \to M_*$ is a right R_*-module structure of M_*. A morphism from (R_*, M_*, α) to (S_*, N_*, β) is a pair (λ, φ) of morphisms $\lambda \in \mathcal{C}(R_*, S_*)$ and $\varphi \in \mathcal{M}(M_*, N_*)$ such that the following diagram commutes.

$$\begin{tikzcd} M_*\otimes_{K_*}R_* \arrow[r, "\alpha"] \arrow[d, "\varphi\otimes_{K_*}\lambda"] & M_* \arrow[d, "\varphi"] \\ N_*\otimes_{K_*}R_* \arrow[r, "\beta"] & N_* \end{tikzcd}$$

Composition of $(\lambda, \varphi) : (R_*, M_*, \alpha) \to (S_*, N_*, \beta)$ and $(\nu, \psi) : (S_*, N_*, \beta) \to (T_*, L_*, \gamma)$ is defined to be $(\nu\lambda, \psi\varphi)$.

Define functors $p_{\mathcal{C}} : \mathcal{M}od(\mathcal{C},\mathcal{M}) \to \mathcal{C}$ and $p_{\mathcal{M}} : \mathcal{M}od(\mathcal{C},\mathcal{M}) \to \mathcal{M}$ by $p_{\mathcal{C}}(R_*, M_*, \alpha) = R_*$, $p_{\mathcal{C}}(\lambda, \varphi) = \lambda$ and $p_{\mathcal{M}}(R_*, M_*, \alpha) = M_*$, $p_{\mathcal{M}}(\lambda, \varphi) = \varphi$.

For $R_* \in \operatorname{Ob}\mathcal{C}$, we denote by $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$ a subcategory of $\mathcal{M}od(\mathcal{C},\mathcal{M})$ consisting of objects which map to R_* by $p_{\mathcal{C}}$ and morphisms which map the identity morphism of R_* by $p_{\mathcal{C}}$. Hence $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$ is a subcategory of the category of right R_*-modules.

Proposition 8.7.2 *If $\mathcal{C}$ and $\mathcal{M}$ are complete, so is $\mathcal{M}od(\mathcal{C},\mathcal{M})$.*

Proof. For a functor $D : \mathcal{I} \to \mathcal{M}od(\mathcal{C},\mathcal{M})$, we assume that limits of $p_{\mathcal{C}}D : \mathcal{I} \to \mathcal{C}$ and $p_{\mathcal{M}}D : \mathcal{I} \to \mathcal{M}$ exist. Let $\left(A_* \xrightarrow{\rho_i} p_{\mathcal{C}}D(i)\right)_{i\in\operatorname{Ob}\mathcal{I}}$ be a limiting cone of $p_{\mathcal{C}}D : \mathcal{I} \to \mathcal{C}$ and $\left(L_* \xrightarrow{\pi_i} p_{\mathcal{M}}D(i)\right)_{i\in\operatorname{Ob}\mathcal{I}}$ a limiting cone of $p_{\mathcal{M}}D : \mathcal{I} \to \mathcal{M}$. For $i \in \operatorname{Ob}\mathcal{I}$ and $(\tau : i \to j) \in \operatorname{Mor}\mathcal{I}$, we put $D(i) = (R_{i*}, M_{i*}, \alpha_i)$ and $D(\tau) = (\lambda_\tau, \varphi_\tau)$. Since the following diagram commutes for any $(\tau : i \to j) \in \operatorname{Mor}\mathcal{I}$, there exists unique morphism $\lambda : L_*\otimes_{K_*}A_* \to L_*$ satisfying $\pi_i\lambda = \alpha_i(\pi_i \otimes_{K_*} \rho_i)$ for any $i \in \operatorname{Ob}\mathcal{I}$.

$$\begin{tikzcd} L_*\otimes_{K_*}A_* \arrow[r, "\pi_i\otimes_{K_*}\rho_i"] \arrow[dr, "\pi_j\otimes_{K_*}\rho_j"'] & M_{i*}\otimes_{K_*}R_{i*} \arrow[r, "\alpha_i"] \arrow[d, "\varphi_\tau\otimes_{K_*}\lambda_\tau"] & M_{i*} \arrow[d, "\varphi_\tau"] \\ & M_{j*}\otimes_{K_*}R_{j*} \arrow[r, "\alpha_j"] & M_j \end{tikzcd}$$

It can be verified that (A_*, L_*, λ) is an object of $\mathcal{M}od(\mathcal{C},\mathcal{M})$ and that $\left((A_*, L_*, \lambda) \xrightarrow{(\rho_i,\pi_i)} D(i)\right)_{i\in\operatorname{Ob}\mathcal{I}}$ is a limiting cone of D. □

Proposition 8.7.3 *$p_{\mathcal{C}}^{op} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \to \mathcal{C}^{op}$ is a fibered category.*

Proof. For a morphism $\lambda : S_* \to R_*$ of $\mathcal{C}$ and $\boldsymbol{N} = (S_*, N_*, \beta) \in \operatorname{Ob}\mathcal{M}od(\mathcal{C},\mathcal{M})$, let $i_\lambda(\boldsymbol{N}) : N_* \to N_*\otimes_{S_*}R_*$ be a map defined by $i_\lambda(\boldsymbol{N})(x) = x \otimes 1$ and $\beta_\lambda : (N_*\otimes_{S_*}R_*)\otimes_{K_*}R_* \to R_*\otimes_{S_*}N_*$ the following composition.

$$(N_*\otimes_{S_*}R_*)\otimes_{K_*}R_* \xrightarrow{\cong} N_*\otimes_{S_*}(R_*\otimes_{K_*}R_*) \xrightarrow{id_{N_*}\otimes_{S_*}\mu_{R_*}} N_*\otimes_{S_*}R_*$$

Since the following diagram commutes, $(\lambda,i_\lambda(\boldsymbol{N})) : (S_*, N_*, \beta) \to (R_*, N_*\otimes_{S_*}R_*, \beta_\lambda)$ is a morphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})$.

$$\begin{tikzcd} N_*\otimes_{K_*}S_* \arrow[r, "\beta"] \arrow[d, "i_\lambda(\boldsymbol{N})\otimes_{K_*}\lambda"] & N_* \arrow[d, "i_\lambda(\boldsymbol{N})"] \\ (N_*\otimes_{S_*}R_*)\otimes_{K_*}R_* \arrow[r, "\beta_\lambda"] & N_*\otimes_{S_*}R_* \end{tikzcd}$$

A map $(\lambda, i_\lambda(\boldsymbol{N}))_* : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{R_*}((R_*, M_*, \alpha), (R_*, N_*\otimes_{S_*}R_*, \beta_\lambda)) \to \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{\lambda}((R_*, M_*, \alpha), (S_*, N_*, \beta))$ given by $(\lambda, i_\lambda(\boldsymbol{N}))_*((id_{R_*}, \varphi)) = (\lambda, \varphi i_\lambda(\boldsymbol{N}))$ is bijective. In fact, if $(\lambda, \psi) : (S_*, N_*, \beta) \to (R_*, M_*, \alpha)$ is an element of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{\lambda}((R_*, M_*, \alpha), (S_*, N_*, \beta))$, since $\psi\beta = \alpha(\psi \otimes_{K_*} \lambda) : N_*\otimes_{K_*}S_* \to M_*$, we have

$$\begin{aligned} \alpha(\psi \otimes_{K_*} id_{R_*})(z \otimes \lambda(y)x) &= \alpha(\psi(z) \otimes \lambda(y)x) = \alpha(\alpha(\psi(z) \otimes \lambda(y)) \otimes x) \\ &= \alpha(\psi\beta(y \otimes z) \otimes x) = \alpha(\psi \otimes_{K_*} id_{R_*})(\beta(z \otimes y) \otimes x) \end{aligned}$$

for $x \in R_*$, $y \in S_*$ and $z \in N_*$. Hence there exists unique morphism $\tilde{\psi} : N_*\otimes_{S_*}R_* \to M_*$ that makes the following diagram commute. Here, $\otimes_\lambda : N_*\otimes_{K_*}R_* \to N_*\otimes_{S_*}R_*$ denotes the quotient map.

$$\begin{array}{ccc} N_*\otimes_{K_*} R_* & \xrightarrow{\psi\otimes_{K_*} id_{R_*}} & M_*\otimes_{K_*} R_* \\ \downarrow{\scriptstyle \otimes_\lambda} & & \downarrow{\scriptstyle \alpha} \\ N_*\otimes_{S_*} R_* & \xrightarrow{\tilde{\psi}} & M_* \end{array}$$

Then, a correspondence $(\lambda,\psi)\mapsto(id_{R_*},\tilde{\psi})$ gives the inverse of $(\lambda, i_\lambda(\boldsymbol{N}))_*$. In fact, since

$$\begin{tikzcd} N_*\otimes_{K_*} R_* \arrow[r, "i_\lambda(\boldsymbol{N})\otimes_{K_*} id_{R_*}"] \arrow[dr, "\otimes_\lambda"'] & N_*\otimes_{K_*} R_*\otimes_{S_*} R_* \arrow[r, "\varphi\otimes_{K_*} id_{R_*}"] \arrow[d, "\beta_\lambda"] & M_*\otimes_{K_*} R_* \arrow[d, "\alpha"] \\ & N_*\otimes_{S_*} R_* \arrow[r, "\varphi"] & M_* \end{tikzcd}$$

commutes for $(id_{R_*},\varphi)\in \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{R_*}((R_*,M_*,\alpha),(R_*,N_*\otimes_{S_*}R_*,\beta_\lambda))$, the correspondence $(\lambda,\psi)\mapsto(id_{R_*},\tilde{\psi})$ is a left inverse of $(\lambda,i_\lambda(\boldsymbol{N}))_*$. For $(\lambda,\psi)\in\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_\lambda((R_*,M_*,\alpha),(S_*,N_*,\beta))$ and $x\in N_*$, since

$$\tilde{\psi}i_\lambda(\boldsymbol{N})(x)=\tilde{\psi}(x\otimes_{S_*}1)=\tilde{\psi}\otimes_\lambda(x\otimes_{K_*}1)=\alpha(\psi\otimes_{K_*}id_{R_*})(x\otimes_{K_*}1)=\psi(x),$$

it follows that the correspondence $(\lambda,\psi)\mapsto(id_{R_*},\tilde{\psi})$ is a right inverse of $(\lambda,i_\lambda(\boldsymbol{N}))_*$. Thus $(\lambda,i_\lambda(\boldsymbol{N}))$ is a cartesian morphism and $p^{op}_{\mathcal{C}}:\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}\to\mathcal{C}^{op}$ is a prefibered category. We set $\lambda^*(\boldsymbol{N})=(R_*,N_*\otimes_{S_*}R_*,\beta_\lambda)$ and $\boldsymbol{\alpha}_\lambda(\boldsymbol{N})=(\lambda,i_\lambda(\boldsymbol{N})):\lambda^*(\boldsymbol{N})\to\boldsymbol{N}$ in $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}$.

For morphisms $\lambda:S_*\to R_*$, $\nu:T_*\to S_*$ of $\mathcal{C}$ and $\boldsymbol{L}=(T_*,L_*,\gamma)\in\operatorname{Ob}\mathcal{M}od(\mathcal{C},\mathcal{M})$, there is an isomorphism $c_{\nu,\lambda}(\boldsymbol{N}):L_*\otimes_{T_*}R_*\to(L_*\otimes_{T_*}S_*)\otimes_{S_*}R_*$ given by $c_{\nu,\lambda}(\boldsymbol{N})(w\otimes x)=w\otimes1\otimes x$. We put $\boldsymbol{c}_{\nu,\lambda}(\boldsymbol{N})=(id_{R_*},c_{\nu,\lambda}(\boldsymbol{N}))$. Then, $\boldsymbol{c}_{\nu,\lambda}(\boldsymbol{N}):\lambda^*\nu^*(\boldsymbol{N})\to(\lambda\nu)^*(\boldsymbol{N})$ is an isomorphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{R_*}$ and the following diagram commutes.

$$\begin{array}{ccc} \lambda^*\nu^*(\boldsymbol{N}) & \xrightarrow{\boldsymbol{\alpha}_\lambda(\nu^*(\boldsymbol{N}))} & \nu^*(\boldsymbol{N}) \\ \downarrow{\scriptstyle \boldsymbol{c}_{\nu,\lambda}(\boldsymbol{N})} & & \downarrow{\scriptstyle \boldsymbol{\alpha}_\nu(\boldsymbol{N})} \\ (\lambda\nu)^*(\boldsymbol{N}) & \xrightarrow{\boldsymbol{\alpha}_{\lambda\nu}(\boldsymbol{N})} & \boldsymbol{N} \end{array}$$

Therefore $p^{op}_{\mathcal{C}}:\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}\to\mathcal{C}^{op}$ is a fibered category. □

Proposition 8.7.4 *For a morphism $\lambda:S_*\to R_*$ of $\mathcal{C}$, $\lambda^*:\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{S_*}\to\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{R_*}$ has a left adjoint.*

Proof. Define a functor $\lambda_*:\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}\to\mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}$ as follows. For $(R_*,M_*,\alpha)\in\operatorname{Ob}\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$, set $\lambda_*(R_*,M_*,\alpha)=(S_*,M_*,\alpha(id_{M_*}\otimes_{K_*}\lambda))$. For $(id_{R_*},\psi)\in\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}((R_*,L_*,\gamma),(R_*,M_*,\alpha))$, we set $\lambda_*(id_{R_*},\psi)=(id_{S_*},\psi)$. It is clear that $(id_{S_*},\varphi)\in\mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}((S_*,N_*,\beta),\lambda_*(R_*,M_*,\alpha))$ if and only if $(\lambda,\varphi)\in\mathcal{M}od(\mathcal{C},\mathcal{M})_\lambda((S_*,N_*,\beta),(R_*,M_*,\alpha))$. It follows from the proof of (8.7.3) that we have a natural bijection $(\lambda,i_\lambda(\boldsymbol{N}))^*:\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}(\lambda^*(S_*,N_*,\beta),(R_*,M_*,\alpha))\to\mathcal{M}od(\mathcal{C},\mathcal{M})_\lambda((S_*,N_*,\beta),(R_*,M_*,\alpha))$. Thus a correspondence $(id_{R_*},\varphi)\mapsto(id_{S_*},\varphi\, i_\lambda(\boldsymbol{N}))$ gives a bijection

$$\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}(\lambda^*(S_*,N_*,\beta),(R_*,M_*,\alpha))\to\mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}((S_*,N_*,\beta),\lambda_*(R_*,M_*,\alpha))$$

which is natural. Hence λ_* is a right adjoint of $\lambda^*:\mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}\to\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$. □

Remark 8.7.5 *Let $\lambda:S_*\to R_*$ be a morphism of $\mathcal{C}$.*

(1) The unit $\varepsilon(\lambda):id_{\mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}}\to\lambda_*\lambda^*$ is given as follows. For an object $\boldsymbol{N}=(S_*,N_*,\beta)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}$, $\varepsilon(\lambda)_{\boldsymbol{N}}:\boldsymbol{N}\to\lambda_*\lambda^*(\boldsymbol{N})$ is defined to be

$$(id_{S_*},i_\lambda(\boldsymbol{N})):(S_*,N_*,\beta)\to(S_*,N_*\otimes_{S_*}R_*,\beta_\lambda(id_{N_*\otimes_{S_*}R_*}\otimes_{K_*}\lambda))$$

(2) The counit $\eta(\lambda):\lambda^*\lambda_*\to id_{\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}}$ is given as follows. For an object $\boldsymbol{M}=(R_*,M_*,\alpha)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$, we put $\alpha'=\alpha(id_{M_*}\otimes_{K_*}\lambda)$. Then, we have $\lambda^*(\lambda_*(\boldsymbol{M}))=(R_*,M_*\otimes_{S_*}R_*,\alpha'_\lambda)$. Let us denote by $\bar{\alpha}:M_*\otimes_{R_*}R_*\to M_*$ the isomorphism induced by α. $\eta(\lambda)_{\boldsymbol{M}}:\lambda^*(\lambda_*(\boldsymbol{M}))\to\boldsymbol{M}$ is defined to be

$$(id_{R_*},\bar{\alpha}\otimes_\lambda):(R_*,M_*\otimes_{S_*}R_*,\alpha'_\lambda)\to(R_*,M_*,\alpha).$$

We assume that K_* is an object of $\mathcal{C}$ in the following proposition. Then, K_* is an initial object of $\mathcal{C}$.

Proposition 8.7.6 *Let $\boldsymbol{M} = (K_*, M_*, \alpha)$ be an object of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{K_*}$*

(1) The cartesian section $s_{\boldsymbol{M}} : \mathcal{C}^{op} \to \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}$ of $p_{\mathcal{C}}^{op} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \to \mathcal{C}^{op}$ associated with $\boldsymbol{M}$ is given as follows. Put $s_{\boldsymbol{M}}(R_*) = \eta_{R_*}^*(\boldsymbol{M}) = (R_*, M_* \otimes_{K_*} R_*, \alpha_{\eta_{R_*}})$ for $R_* \in \mathrm{Ob}\,\mathcal{C}$. For a morphism $\lambda : S_* \to R_*$ of $\mathcal{C}^{op}$, $s_{\boldsymbol{M}}(\lambda) \in \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{\lambda}(s_{\boldsymbol{M}}(S_*), s_{\boldsymbol{M}}(R_*))$ is defined by

$$s_{\boldsymbol{M}}(\lambda) = (\lambda, id_{M_*} \otimes_{K_*} \lambda) : (S_*, M_* \otimes_{K_*} S_*, \alpha_{\eta_{S_*}}) \to (R_*, M_* \otimes_{K_*} R_*, \alpha_{\eta_{R_*}}).$$

(2) For a morphism $\lambda : S_* \to R_*$ of $\mathcal{C}^{op}$, Then, the morphism

$$(s_{\boldsymbol{M}})_{\lambda} : s_{\boldsymbol{M}}(S_*) = (S_*, M_* \otimes_{K_*} S_*, \alpha_{\eta_{S_*}}) \to (S_*, (M_* \otimes_{K_*} R_*) \otimes_{R_*} S_*, (\alpha_{\eta_{R_*}})_{\lambda}) = \lambda^*(s_{\boldsymbol{M}}(R_*))$$

of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{S_*}$ coincides with $(id_{S_*}, c_{\eta_{R_*},\lambda}(\boldsymbol{M})^{-1})$. Here, $c_{\eta_{R_*},\lambda}(\boldsymbol{M})^{-1} : (M_* \otimes_{K_*} R_*) \otimes_{R_*} S_* \to M_* \otimes_{K_*} S_*$ is given by $c_{\eta_{R_*},\lambda}(\boldsymbol{M})^{-1}(x \otimes r \otimes s) = x \otimes \lambda(r)s$.

(3) For morphisms $\lambda : S_* \to R_*$ and $\nu : S_* \to T_*$ of $\mathcal{C}^{op}$, the morphism $(s_{\boldsymbol{M}})_{\lambda,\nu} : \lambda^*(s_{\boldsymbol{M}}(R_*)) \to \nu^*(s_{\boldsymbol{M}}(T_*))$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{S_*}$ is given by $(id_{S_*}, c_{\eta_{T_*},\nu}(\boldsymbol{M})^{-1} c_{\eta_{R_*},\lambda}(\boldsymbol{M}))$.

Proof. The assertions follow from (8.1.26), (8.1.27) and the definition of $p_{\mathcal{C}}^{op} : \mathcal{M}od(\mathcal{C},\mathcal{M})^{op} \to \mathcal{C}^{op}$. □

Proposition 8.7.7 *Let $\lambda : R_* \to S_*$ and $\nu : T_* \to S_*$ be morphisms of $\mathcal{C}$.*

(1) *For an object $\boldsymbol{M} = (R_*, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$, $\boldsymbol{M}_{[\lambda,\nu]}$ is given by*

$$\boldsymbol{M}_{[\lambda,\nu]} = \nu_*(\lambda^*(\boldsymbol{M})) = (T_*, M_* \otimes_{R_*} S_*, \alpha_{\lambda}(id_{M_* \otimes_{R_*} S_*} \otimes_{K_*} \nu)).$$

(2) *For an object $\boldsymbol{M} = (R_*, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$, we define $i_{\lambda,\nu}(\boldsymbol{M}) : (M_* \otimes_{R_*} S_*) \otimes_{T_*} S_* \to M_* \otimes_{R_*} S_*$ by $i_{\lambda,\nu}(\boldsymbol{M})(x \otimes s \otimes t) = x \otimes st$. Then, F*

$$\iota_{\lambda,\nu}(\boldsymbol{M}) : \nu^*(\boldsymbol{M}_{[\lambda,\nu]}) = (S_*, (M_* \otimes_{R_*} S_*) \otimes_{T_*} S_*, \beta_{\nu}) \to (S_*, M_* \otimes_{R_*} S_*, \alpha_{\lambda}) = \lambda^*(\boldsymbol{M})$$

is given by $\iota_{\lambda,\nu}(\boldsymbol{M}) = (id_{S_*}, i_{\lambda,\nu}(\boldsymbol{M}))$. Here we put $\beta = \alpha_{\lambda}(id_{M_* \otimes_{R_*} S_*} \otimes_{K_*} \nu) : (M_* \otimes_{R_*} S_*) \otimes_{K_*} T_* \to M_* \otimes_{R_*} S_*$.

(3) *For an object $\boldsymbol{M}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$ and an object $\boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{T_*}$,*

$$P_{\lambda,\nu}(\boldsymbol{M})_{\boldsymbol{N}} : \mathcal{M}od(\mathcal{C},\mathcal{M})_{S_*}(\nu^*(\boldsymbol{N}), \lambda^*(\boldsymbol{M})) \to \mathcal{M}od(\mathcal{C},\mathcal{M})_{T_*}(\boldsymbol{N}, \boldsymbol{M}_{[\lambda,\nu]})$$

maps (id_{S_*}, φ) to $(id_{T_*}, \varphi i_{\nu}(\boldsymbol{N}))$.

(4) *For a morphism $\boldsymbol{\varphi} = (id_{R_*}, \varphi) : \boldsymbol{M} \to \boldsymbol{N}$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$, $\boldsymbol{\varphi}_{[\lambda,\nu]} : \boldsymbol{M}_{[\lambda,\nu]} \to \boldsymbol{N}_{[\lambda,\nu]}$ is given by $\nu_*(\lambda^*(\boldsymbol{\varphi})) = (id_{T_*}, \varphi \otimes_{R_*} id_{S_*})$.*

(5) *For a morphisms $\gamma : S_* \to A_*$ of $\mathcal{C}$,*

$$\boldsymbol{M}_{\gamma} : \boldsymbol{M}_{[\lambda,\nu]} = (T_*, M_* \otimes_{R_*} S_*, \alpha_{\lambda}(id_{M_* \otimes_{R_*} S_*} \otimes_{K_*} \nu)) \to (T_*, M_* \otimes_{R_*} A_*, \alpha_{\gamma\lambda}(id_{M_* \otimes_{R_*} A_*} \otimes_{K_*} \gamma\nu)) = \boldsymbol{M}_{[\gamma\lambda,\gamma\nu]}$$

is given by $\boldsymbol{M}_{\gamma} = (id_{T_*}, id_{M_*} \otimes_{R_*} \gamma)$.

Proof. (1) The assertion follows from (8.7.3), (8.7.4) and (8.4.1).

(2) Since $\iota_{\lambda,\nu}(\boldsymbol{M}) = (\eta_{\nu})_{\lambda^*(\boldsymbol{M})}$ by (8.4.1), the assertion follows from and (8.7.5).

(3) The assertion follows from (8.4.1) and (8.7.4).

(4) This is a direct consequence of (8.4.4).

(5) The assertion can be verified from (8.4.7) and (8.7.5). □

Proposition 8.7.8 *For morphisms $\lambda : R_* \to S_*$, $\nu : T_* \to S_*$, $\gamma : A_* \to S_*$ of $\mathcal{C}$ and an object $\boldsymbol{M} = (R_*, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_*}$, define a map $\tilde{\delta}_{\lambda,\nu,\gamma,\boldsymbol{M}} : (M_* \otimes_{R_*} S_*) \otimes_{T_*} S_* \to M_* \otimes_{R_*} S_*$ by $\tilde{\delta}_{\lambda,\nu,\gamma,\boldsymbol{M}}(x \otimes s \otimes t) = x \otimes st$. Then, $\delta_{\lambda,\nu,\gamma,\boldsymbol{M}} : (\boldsymbol{M}_{[\lambda,\nu]})_{[\nu,\gamma]} \to \boldsymbol{M}_{[\lambda,\gamma]}$ is given by $\delta_{\lambda,\nu,\gamma,\boldsymbol{M}} = (id_{A_*}, \tilde{\delta}_{\lambda,\nu,\gamma,\boldsymbol{M}})$.*

Proof. First we note that it follows from (1) of (8.7.7) that $(\boldsymbol{M}_{[\lambda,\nu]})_{[\nu,\gamma]}$ is given as follows.

$$(\boldsymbol{M}_{[\lambda,\nu]})_{[\nu,\gamma]} = (T_*, M_* \otimes_{R_*} S_*, \tilde{\alpha})_{[\nu,\gamma]} = (A_*, (M_* \otimes_{R_*} S_*) \otimes_{T_*} S_*, \tilde{\alpha}_{\nu}(id_{(M_* \otimes_{R_*} S_*) \otimes_{T_*} S_*} \otimes_{K_*} \gamma))$$

Here we put $\tilde{\alpha} = \alpha_{\lambda}(id_{M_* \otimes_{R_*} S_*} \otimes_{K_*} \nu)$. Since $\delta_{\lambda,\nu,\gamma,\boldsymbol{M}} = \gamma_*\big(\eta(\nu)_{\lambda^*(\boldsymbol{M})}\big)$ by (8.4.11), the assertion follows from (2) of (8.7.5). □

Proposition 8.7.9 *For a functor $D : \mathcal{P} \to \mathcal{C}^{op}$, we put $D(i) = R_{i*}$ $(i = 0,1,2,3,4,5)$, $D(\tau_{ij}) = \lambda_{ij}$ $((i,j) = (0,1),(0,2),(1,3),(1,4),(2,4),(2,5))$. For an object $\boldsymbol{M} = (R_{3*}, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_{3*}}$, we define*

$$\tilde{\theta}_D(\boldsymbol{M}) : (M_* \otimes_{R_{3*}} R_{1*}) \otimes_{R_{4*}} R_{2*} \to M_* \otimes_{R_{3*}} R_{0*}$$

by $\tilde{\theta}_D(\boldsymbol{M})(x \otimes s \otimes t) = x \otimes \lambda_{01}(s)\lambda_{02}(t)$. Then, $\theta_D(\boldsymbol{M}) : (\boldsymbol{M}_{[\lambda_{13},\lambda_{14}]})_{[\lambda_{24},\lambda_{25}]} \to \boldsymbol{M}_{[\lambda_{01}\lambda_{13},\lambda_{02}\lambda_{25}]}$ is given by $\theta_D(\boldsymbol{M}) = (id_{R_{5*}}, \tilde{\theta}_D(\boldsymbol{M}))$. Hence if $R_{0*} = R_{1*} \otimes_{R_{4*}} R_{2*}$ and $\lambda_{01} : R_{1*} \to R_{0*}$, $\lambda_{02} : R_{2*} \to R_{0*}$ are given by $\lambda_{01}(s) = s \otimes 1$, $\lambda_{02}(t) = 1 \otimes t$, then $\theta_D(\boldsymbol{M})$ is an isomorphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{R_{5*}}$.

Proof. Put $\tilde{\alpha} = \alpha_{\lambda_{13}}(id_{R_{1*}\otimes_{R_{3*}} M_*} \otimes_{K_*} \lambda_{14})$ and $\hat{\alpha} = \alpha_{\lambda_{01}\lambda_{13}}(id_{M_*\otimes_{R_{3*}} R_{0*}} \otimes_{K_*} \lambda_{01}\lambda_{14})$. Then, we have the following equalities by (1) of (8.7.7).

$$\begin{aligned}
(\boldsymbol{M}_{[\lambda_{13},\lambda_{14}]})_{[\lambda_{24},\lambda_{25}]} &= (R_{5*}, (M_* \otimes_{R_{3*}} R_{1*}) \otimes_{R_{4*}} R_{2*}, \tilde{\alpha}_{\lambda_{24}}(id_{(M_*\otimes_{R_{3*}} R_{1*})\otimes_{R_{4*}} R_{2*}} \otimes_{K_*} \lambda_{25})) \\
(\boldsymbol{M}_{[\lambda_{01}\lambda_{13},\lambda_{01}\lambda_{14}]})_{[\lambda_{02}\lambda_{24},\lambda_{02}\lambda_{25}]} &= (R_{5*}, (M_* \otimes_{R_{3*}} R_{0*}) \otimes_{R_{4*}} R_{0*}, \hat{\alpha}_{\lambda_{02}\lambda_{24}}(id_{(M_*\otimes_{R_{3*}} R_{1*})\otimes_{R_{4*}} R_{2*}} \otimes_{K_*} \lambda_{25})) \\
\boldsymbol{M}_{[\lambda_{01}\lambda_{13},\lambda_{02}\lambda_{25}]} &= (R_{5*}, M_* \otimes_{R_{3*}} R_{0*}, \alpha_{\lambda_{01}\lambda_{13}}(id_{M_*\otimes_{R_{3*}} R_{0*}} \otimes_{K_*} \lambda_{02}\lambda_{25}))
\end{aligned}$$

Since $\theta_D(\boldsymbol{M})$ is defined to be a composition

$$(\boldsymbol{M}_{[\lambda_{13},\lambda_{14}]})_{[\lambda_{24},\lambda_{25}]} \xrightarrow{(\boldsymbol{M}_{\lambda_{01}})_{\lambda_{02}}} (\boldsymbol{M}_{[\lambda_{01}\lambda_{13},\lambda_{01}\lambda_{14}]})_{[\lambda_{02}\lambda_{24},\lambda_{02}\lambda_{25}]} \xrightarrow{\delta_{\lambda_{01}\lambda_{13},\lambda_{01}\lambda_{14},\lambda_{02}\lambda_{25},\boldsymbol{M}}} \boldsymbol{M}_{[\lambda_{01}\lambda_{13},\lambda_{02}\lambda_{25}]},$$

the assertion follows from (3) of (8.7.4) and (8.7.8). □

Remark 8.7.10 *For morphisms $\lambda : R_* \to S_*$, $\nu : T_* \to S_*$, $\kappa : T_* \to A_*$, $\rho : B_* \to A_*$ of $\mathcal{C}$, assume that maps $\iota_1 : S_* \to S_* \otimes_{T_*} A_*$ and $\iota_2 : A_* \to S_* \otimes_{T_*} A_*$ defined by $\iota_1(s) = s \otimes 1$, $\iota_2(a) = 1 \otimes a$ are morphisms of $\mathcal{C}$. Then, if we define $\tilde{\theta}_{\lambda,\nu,\kappa,\rho}(\boldsymbol{M}) : (M_* \otimes_{R_*} S_*) \otimes_{T_*} A_* \to M_* \otimes_{R_*} (S_* \otimes_{T_*} A_*)$ by $\tilde{\theta}_{\lambda,\nu,\kappa,\rho}(\boldsymbol{M}) = (x \otimes s) \otimes t = x \otimes (s \otimes t)$, $\theta_{\lambda,\nu,\kappa,\rho}(\boldsymbol{M}) = (id_{B_*}, \tilde{\theta}_{\lambda,\nu,\kappa,\rho}(\boldsymbol{M}))$ is an isomorphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{A_*}$.*

Proposition 8.7.11 *For functor $D, E : \mathcal{Q} \to \mathcal{C}^{op}$ and a natural transformation $\omega : D \to E$, we put $D(i) = R_{i*}$, $E(i) = S_{i*}$ $(i = 0,1,2)$, $D(\tau_{0i}) = \lambda_i$, $E(\tau_{0i}) = \nu_i$ $(i = 1,2)$. For an object $\boldsymbol{M} = (S_{1*}, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{S_{1*}}$, define a map $\tilde{\omega}_{\boldsymbol{M}} : (M_* \otimes_{S_{1*}} S_{0*}) \otimes_{S_{2*}} R_{2*} \to (M_* \otimes_{S_{1*}} R_{1*}) \otimes_{R_{1*}} R_{0*}$ by $\tilde{\omega}_{\boldsymbol{M}}(x \otimes s \otimes r) = x \otimes 1 \otimes \omega_0(s)\lambda_2(r)$. Then, $\omega_{\boldsymbol{M}} : \omega_2^*(\boldsymbol{M}_{[\nu_1,\nu_2]}) \to \omega_1^*(\boldsymbol{M})_{[\lambda_1,\lambda_2]}$ is given by $\omega_{\boldsymbol{M}} = (id_{R_{2*}}, \tilde{\omega}_{\boldsymbol{M}})$.*

Proof. Put $\tilde{\alpha} = \alpha_{\nu_1}(id_{S_{0*}\otimes_{S_{1*}} M_*} \otimes_{K_*} \nu_2)$. It follows from (1) of (8.7.7) that we have

$$\begin{aligned}
\omega_2^*(\boldsymbol{M}_{[\nu_1,\nu_2]}) &= \omega_2^*(S_{2*}, M_* \otimes_{S_{1*}} S_{0*}, \tilde{\alpha}) = (R_{2*}, (M_* \otimes_{S_{1*}} S_{0*}) \otimes_{S_{1*}} R_{2*}, \tilde{\alpha}_{\omega_2}) \\
\omega_1^*(\boldsymbol{M})_{[\lambda_1,\lambda_2]} &= (R_{1*}, M_* \otimes_{S_{1*}} R_{1*}, \alpha_{\omega_1})_{[\lambda_1,\lambda_2]} \\
&= (R_{2*}, (M_* \otimes_{S_{1*}} R_{1*}) \otimes_{R_{1*}} R_{0*}, (\alpha_{\omega_1})_{\lambda_1}(id_{M_*\otimes_{S_{1*}} R_{1*})\otimes_{R_{1*}} R_{0*}} \otimes_{K_*} \lambda_2).
\end{aligned}$$

Define $i_{\nu_1,\nu_2,\omega_0}(\boldsymbol{M}) : (M_* \otimes_{S_{1*}} S_{0*}) \otimes_{S_{2*}} R_{0*} \to M_* \otimes_{S_{1*}} R_{0*}$ by $i_{\nu_1,\nu_2,\omega_0}(\boldsymbol{M})(x \otimes s \otimes r) = x \otimes \omega_0(s)r$. It follows from (2) of (8.7.7) that $\omega_0^\sharp(\iota_{\nu_1,\nu_2}(\boldsymbol{M})) : (\lambda_2\omega_2)^*(\boldsymbol{M}_{[\nu_1,\nu_2]}) = (\omega_0\nu_2)^*(\boldsymbol{M}_{[\nu_1,\nu_2]}) \to (\omega_0\nu_1)^*(\boldsymbol{M}) = (\lambda_1\omega_1)^*(\boldsymbol{M})$ is given by $\omega_0^\sharp(\iota_{\nu_1,\nu_2}(\boldsymbol{M})) = (id_{R_{0*}}, i_{\nu_1,\nu_2,\omega_0}(\boldsymbol{M}))$. Hence

$$\boldsymbol{c}_{\omega_1,\lambda_1}(\boldsymbol{M})\omega_0^\sharp(\iota_{\nu_1,\nu_2}(\boldsymbol{M}))\boldsymbol{c}_{\omega_2,\lambda_2}(\boldsymbol{M}_{[\nu_1,\nu_2]})^{-1} : \lambda_2^*(\omega_2^*(\boldsymbol{M}_{[\nu_1,\nu_2]})) \to \lambda_1^*(\omega_1^*(\boldsymbol{M}))$$

is equal to $(id_{R_{0*}}, c_{\omega_1,\lambda_1}(\boldsymbol{M})i_{\nu_1,\nu_2,\omega_0}(\boldsymbol{M})c_{\omega_2,\lambda_2}(\boldsymbol{M}_{[\nu_1,\nu_2]})^{-1})$. Thus, by the definition of $\omega_{\boldsymbol{M}}$, we have

$$\omega_{\boldsymbol{M}} = (id_{R_{2*}}, c_{\omega_1,\lambda_1}(\boldsymbol{M})i_{\nu_1,\nu_2,\omega_0}(\boldsymbol{M})c_{\omega_2,\lambda_2}(\boldsymbol{M}_{[\nu_1,\nu_2]})^{-1}i_{\lambda_2}(\omega_2^*(\boldsymbol{M}_{[\nu_1,\nu_2]})))$$

and it can be verified that

$$c_{\omega_1,\lambda_1}(\boldsymbol{M})i_{\nu_1,\nu_2,\omega_0}(\boldsymbol{M})c_{\omega_2,\lambda_2}(\boldsymbol{M}_{[\nu_1,\nu_2]})^{-1}i_{\lambda_2}(\omega_2^*(\boldsymbol{M}_{[\nu_1,\nu_2]})) : (M_* \otimes_{S_{1*}} S_{0*}) \otimes_{S_{1*}} R_{2*} \to (M_* \otimes_{S_{1*}} R_{1*}) \otimes_{R_{1*}} R_{0*}$$

maps $x \otimes s \otimes r$ to $x \otimes 1 \otimes \omega_0(s)\lambda_2(r)$. □

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Chapter 9

Representations of internal categories

Introduction

In [1], J.F.Adams generalized the notion of Hopf algebras which are obtained from generalized homology theories satisfying certain conditions and showed that such a generalized homology theory, say E_*, takes values in the category of comodules over the "generalized Hopf algebra" associated with E_*. The notion introduced by Adams is now called Hopf algebroid which represents a functor taking values in the category of groupoids. A comodule over a Hopf algebroid Γ can be regarded as a representation of the groupoid represented by Γ. The aim of this note is to set a categorical foundation of representations of an internal category which is a category object in a given category.

By making use of the notion of fibered category, we give a definition of the representations of internal categories in section 1 which generalizes the definition given by P. Deligne in [2]. We give the definition of "trivial representation" and several examples of representations and show that the category of representations of an internal category $\boldsymbol{G}$ on objects of a fibered category represented by an internal category $\boldsymbol{C}$ is isomorphic to the category of internal functors from $\boldsymbol{G}$ to $\boldsymbol{C}$ and internal natural transformations between them (9.1.18). In section 2, we reformulate the notion of descent theory ([4]) in terms of representations of special groupoids, namely equivalence relations. We construct restrictions of representations and give a definition of regular representations in section 3.

9.1 Representations of internal categories

Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category over $\mathcal{E}$ and $f : X \to Y$, $g : X \to Z$, $k : V \to X$ morphisms of $\mathcal{E}$. For objects M of $\mathcal{F}_Y$, N of $\mathcal{F}_Z$ and a morphism $\xi : f^*(M) \to g^*(N)$ of $\mathcal{F}_X$, we denote $k^\sharp_{M,N}(\xi)$ by ξ_k for short. That is, ξ_k is the following composition.

$$(fk)^*(M) \xrightarrow{c_{f,k}(M)^{-1}} k^*f^*(M) \xrightarrow{k^*(\xi)} k^*g^*(N) \xrightarrow{c_{g,k}(N)} (gk)^*(N)$$

Definition 9.1.1 *Suppose that $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category and that $\mathcal{E}$ is a category with finite limits. Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$. A pair (M, ξ) of an object M of $\mathcal{F}_{C_0}$ and a morphism $\xi : \sigma^*(M) \to \tau^*(M)$ of $\mathcal{F}_{C_1}$ is called a representation of $\boldsymbol{C}$ on M if the following conditions are satisfied.*

(A) *Let $C_1 \xleftarrow{\mathrm{pr}_1} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1$ be a limit of diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma} C_1$. $\xi_\mu : (\sigma\mu)^*(M) \to (\tau\mu)^*(M)$ coincides with the following composition.*

$$(\sigma\mu)^*(M) = (\sigma\mathrm{pr}_1)^*(M) \xrightarrow{\xi_{\mathrm{pr}_1}} (\tau\mathrm{pr}_1)^*(M) = (\sigma\mathrm{pr}_2)^*(M) \xrightarrow{\xi_{\mathrm{pr}_2}} (\tau\mathrm{pr}_2)^*(M) = (\tau\mu)^*(M)$$

(U) *$\xi_\varepsilon : M = (\sigma\varepsilon)^*(M) \to (\tau\varepsilon)^*(M) = M$ coincides with the identity morphism of M.*

Let (M, ξ) and (N, ζ) be representations of $\boldsymbol{C}$ on M and N, respectively. A morphism $\varphi : M \to N$ in $\mathcal{F}_{C_0}$ is called a morphism of representations of $\boldsymbol{C}$ if φ makes the following diagram commute.

$$\begin{array}{ccc} \sigma^*(M) & \xrightarrow{\xi} & \tau^*(M) \\ \downarrow{\scriptstyle \sigma^*(\varphi)} & & \downarrow{\scriptstyle \tau^*(\varphi)} \\ \sigma^*(N) & \xrightarrow{\xi} & \tau^*(N) \end{array}$$

We denote by $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ *the category of the representations of* $\boldsymbol{C}$*.*

We denote by $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ the forgetful functor which assigns $(M,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ to $M \in \mathrm{Ob}\,\mathcal{F}_{C_0}$ and $(\varphi : (M,\xi) \to (N,\zeta)) \in \mathrm{Mor}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ to $\varphi : M \to N$.

Definition 9.1.2 *Let* $\varphi : (M,\xi) \to (N,\zeta)$ *be a morphism of* $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$.
(1) *If* $\mathscr{F}_{\boldsymbol{C}}(\varphi) : M \to N$ *is a monomorphism of* $\mathcal{F}_{C_0}$*, we call* (M,ξ) *a subrepresentation of* (N,ζ)*.*
(2) *If* $\mathscr{F}_{\boldsymbol{C}}(\varphi) : M \to N$ *is an epimorphism of* $\mathcal{F}_{C_0}$*, we call* (N,ζ) *a quotient representation of* (M,ξ)*.*

Proposition 9.1.3 *Let* $\varphi : (M,\xi) \to (N,\zeta)$ *be a morphism of representations of an internal category* $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ *in* $\mathcal{E}$*.*

(1) *Suppose that* $\mathscr{F}_{\boldsymbol{C}}(\varphi) : M \to N$ *is a monomorphism of* $\mathcal{F}_{C_1}$*. For a representation* (M,ξ') *of* $\mathcal{C}$ *and a morphism* $\varphi' : (M,\xi') \to (N,\zeta)$ *of representations such that* $\mathscr{F}_{\boldsymbol{C}}(\varphi) = \mathscr{F}_{\boldsymbol{C}}(\varphi')$*, if one of the following conditions is satisfied, we have* $\xi' = \xi$*.*

(*i*) $\tau^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ *preserves monomorphisms.* (*ii*) *The presheaf* $F_{\sigma,\tau,M}$ *on* $\mathcal{F}_{C_0}^{op}$ *is representable.*

(2) *Suppose that* $\mathscr{F}_{\boldsymbol{C}}(\varphi) : M \to N$ *is an epimorphism of* $\mathcal{F}_{C_1}$*. For a representation* (N,ζ') *of* $\mathcal{C}$ *and a morphism* $\varphi' : (M,\xi) \to (N,\zeta')$ *of representations such that* $\mathscr{F}_{\boldsymbol{C}}(\varphi) = \mathscr{F}_{\boldsymbol{C}}(\varphi')$*, if one of the following conditions is satisfied, we have* $\zeta' = \zeta$*.*

(*i*) $\sigma^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ *preserves epimorphisms.* (*ii*) *The presheaf* $F_N^{\sigma,\tau}$ *on* $\mathcal{F}_{C_0}$ *is representable.*

Proof. (1) Since $\tau^*(\varphi)\xi' = \zeta\sigma^*(\varphi) = \tau^*(\varphi)\xi$ by the assumption, it suffices to show that

$$\tau^*(\varphi)_* : \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M)) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(N))$$

is injective. If (i) is satisfied, then $\tau^*(\varphi)$ is a monomorphism, hence $\tau^*(\varphi)_*$ is injective.

Suppose that (ii) is satisfied. Then the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M)) & \xrightarrow{P_{\sigma,\tau}(M)_M} & \mathcal{F}_{C_0}(M_{[\sigma,\tau]}, M) \\ \downarrow{\scriptstyle \tau^*(\varphi)_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(N)) & \xrightarrow{P_{\sigma,\tau}(M)_N} & \mathcal{F}_{C_0}(M_{[\sigma,\tau]}, N) \end{array}$$

Since both φ_* and $P_{\sigma,\tau}(M)_M$ are injective, so is $\tau^*(\varphi)_*$.

(2) Since $\zeta'\sigma^*(\varphi) = \tau^*(\varphi)\xi = \zeta\sigma^*(\varphi)$ by the assumption, it suffices to show that

$$\sigma^*(\varphi)^* : \mathcal{F}_{C_1}(\sigma^*(N), \tau^*(N)) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(N))$$

is injective. If (i) is satisfied, then $\sigma^*(\varphi)$ is an epimorphism, hence $\sigma^*(\varphi)_*$ is injective.

Suppose that (ii) is satisfied. Then the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{F}_{C_1}(\sigma^*(N), \tau^*(N)) & \xrightarrow{E_{\sigma,\tau}(N)_N} & \mathcal{F}_{C_0}(N, N^{[\sigma,\tau]}) \\ \downarrow{\scriptstyle \sigma^*(\varphi)^*} & & \downarrow{\scriptstyle \varphi^*} \\ \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(N)) & \xrightarrow{E_{\sigma,\tau}(N)_M} & \mathcal{F}_{C_0}(M, N^{[\sigma,\tau]}) \end{array}$$

Since both φ^* and $E_{\sigma,\tau}(N)_N$ are injective, so is $\sigma^*(\varphi)^*$. □

Proposition 9.1.4 *Let* M*,* N *be objects of* $\mathcal{F}_{C_0}$ *and* $\xi : \sigma^*(M) \to \tau^*(M)$*,* $\zeta : \sigma^*(N) \to \tau^*(N)$ *morphisms of* $\mathcal{F}_{C_1}$*. We assume that a morphism* $\varphi : M \to N$ *of* $\mathcal{F}_{C_0}$ *makes the following diagram commute.*

$$\begin{array}{ccc} \sigma^*(M) & \xrightarrow{\xi} & \tau^*(M) \\ \downarrow{\scriptstyle \sigma^*(\varphi)} & & \downarrow{\scriptstyle \tau^*(\varphi)} \\ \sigma^*(N) & \xrightarrow{\zeta} & \tau^*(N) \end{array}$$

(1) *Suppose that* ζ *is a representation of* $\boldsymbol{C}$ *on* N *and that* $\varphi : M \to N$ *is an monomorphism. If one of the following conditions is satisfied,* ξ *is a representation of* $\boldsymbol{C}$ *on* M.

(*i*) $(\tau\mu)^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1 \times_{C_0} C_1}$ *preserves monomorphisms.* (*ii*) *The presheaf* $F_{\sigma\mu,\tau\mu,M}$ *on* $\mathcal{F}_{C_0}^{op}$ *is representable.*

(2) *Suppose that* ξ *is a representation of* $\boldsymbol{C}$ *on* M *and that* $\varphi : M \to N$ *is an epimorphism. If one of the following conditions is satisfied,* ζ *is a representation of* $\boldsymbol{C}$ *on* N.

(*i*) $(\sigma\mu)^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1 \times_{C_0} C_1}$ *preserves epimorphisms.* (*ii*) *The presheaf* $F^N_{\sigma\mu,\tau\mu}$ *on* $\mathcal{F}_{C_0}$ *is representable.*

Proof. The following diagram commutes by the assumption and (8.1.13).

$$\begin{array}{ccccccc}
(\sigma\mathrm{pr}_1)^*(M) & \xrightarrow{\xi_{\mathrm{pr}_1}} & (\tau\mathrm{pr}_1)^*(M) & = & (\sigma\mathrm{pr}_2)^*(M) & \xrightarrow{\xi_{\mathrm{pr}_2}} & (\tau\mathrm{pr}_2)^*(M) \\
\downarrow{\scriptstyle (\sigma\mathrm{pr}_1)^*(\varphi)} & & \downarrow{\scriptstyle (\tau\mathrm{pr}_1)^*(\varphi)} & & \downarrow{\scriptstyle (\sigma\mathrm{pr}_2)^*(\varphi)} & & \downarrow{\scriptstyle (\tau\mathrm{pr}_2)^*(\varphi)} \\
(\sigma\mathrm{pr}_1)^*(N) & \xrightarrow{\zeta_{\mathrm{pr}_1}} & (\tau\mathrm{pr}_1)^*(N) & = & (\sigma\mathrm{pr}_2)^*(N) & \xrightarrow{\zeta_{\mathrm{pr}_2}} & (\tau\mathrm{pr}_2)^*(N)
\end{array}$$

$$\begin{array}{ccccccc}
(\sigma\mu)^*(M) & \xrightarrow{\xi_\mu} & (\tau\mu)^*(M) & & (\sigma\varepsilon)^*(M) & \xrightarrow{\xi_\varepsilon} & (\tau\varepsilon)^*(M) \\
\downarrow{\scriptstyle (\sigma\mu)^*(\varphi)} & & \downarrow{\scriptstyle (\tau\mu)^*(\varphi)} & & \downarrow{\scriptstyle (\sigma\varepsilon)^*(\varphi)=\varphi} & & \downarrow{\scriptstyle (\tau\varepsilon)^*(\varphi)=\varphi} \\
(\sigma\mu)^*(N) & \xrightarrow{\zeta_\mu} & (\tau\mu)^*(N) & & (\sigma\varepsilon)^*(N) & \xrightarrow{\zeta_\varepsilon} & (\tau\varepsilon)^*(N)
\end{array}$$

(1) It follows from the commutativity of the above diagrams that we have

$$(\tau\mu)^*(\varphi)\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1} = (\tau\mathrm{pr}_2)^*(\varphi)\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1} = \zeta_{\mathrm{pr}_1}\zeta_{\mathrm{pr}_2}(\sigma\mathrm{pr}_1)^*(\varphi) = \zeta_\mu(\sigma\mu)^*(\varphi) = (\tau\mu)^*(\varphi)\xi_\mu \text{ and } \varphi\xi_\varepsilon = \zeta_\varepsilon\varphi = \varphi.$$

Hence $\xi_\varepsilon = id_M$ and it suffices to show that $(\tau\mu)^*(\varphi)_* : \mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^*(M), (\tau\mu)^*(M)) \to \mathcal{F}_{C_0}(M^{[\sigma\mu,\tau\mu]}, N)$ is injective. If (*i*) is satisfied, $(\tau\mu)^*(\varphi)$ is a monomorphism. Assume that (*ii*) is satisfied. Then, we have the following commutative diagram by the assumption.

$$\begin{array}{ccc}
\mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^*(M), (\tau\mu)^*(M)) & \xrightarrow{P_{\sigma\mu,\tau\mu}(M)_M} & \mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]}, M) \\
\downarrow{\scriptstyle (\tau\mu)^*(\varphi)_*} & & \downarrow{\scriptstyle \varphi_*} \\
\mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^*(M), (\tau\mu)^*(N)) & \xrightarrow{P_{\sigma\mu,\tau\mu}(M)_N} & \mathcal{F}_{C_0}(M^{[\sigma\mu,\tau\mu]}, N)
\end{array}$$

Since both φ_* and $P_{\sigma\mu,\tau\mu}(M)_M$ are injective, so is $(\tau\mu)^*(\varphi)_*$.

(2) It follows from the commutativity of the above diagrams that we have

$$\zeta_{\mathrm{pr}_2}\zeta_{\mathrm{pr}_1}(\sigma\mu)^*(\varphi) = \zeta_{\mathrm{pr}_2}\zeta_{\mathrm{pr}_1}(\sigma\mathrm{pr}_1)^*(\varphi) = (\tau\mathrm{pr}_2)^*(\varphi)\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1} = (\tau\mu)^*(\varphi)\xi_\mu = \zeta_\mu(\sigma\mu)^*(\varphi) \text{ and } \zeta_\varepsilon\varphi = \varphi\xi_\varepsilon = \varphi.$$

Hence $\zeta_\varepsilon = id_N$ and it suffices to show that $(\sigma\mu)^*(\varphi)^* : \mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^*(N), (\tau\mu)^*(N)) \to \mathcal{F}_{C_0}(M^{[\sigma\mu,\tau\mu]}, N)$ is injective. If (*i*) is satisfied, $(\sigma\mu)^*(\varphi)$ is an epimorphism. Assume that (*ii*) is satisfied. Then, we have the following commutative diagram by the assumption.

$$\begin{array}{ccc}
\mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^*(N), (\tau\mu)^*(N)) & \xrightarrow{E_{\sigma\mu,\tau\mu}(N)_N} & \mathcal{F}_{C_0}(N, N^{[\sigma\mu,\tau\mu]}) \\
\downarrow{\scriptstyle (\sigma\mu)^*(\varphi)^*} & & \downarrow{\scriptstyle \varphi^*} \\
\mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^*(M), (\tau\mu)^*(N)) & \xrightarrow{E_{\sigma\mu,\tau\mu}(N)_M} & \mathcal{F}_{C_0}(M, N^{[\sigma\mu,\tau\mu]})
\end{array}$$

Since both φ^* and $E_{\sigma\mu,\tau\mu}(N)_N$ are injective, so is $(\sigma\mu)^*(\varphi)^*$. □

Proposition 9.1.5 *Let* $D : \mathcal{D} \to \mathrm{Rep}(\boldsymbol{C}; \mathcal{F})$ *be a functor.*

(1) *Let* $(\pi_i : M \to \mathscr{F}_{\boldsymbol{C}} D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$ *be a limiting cone of* $\mathscr{F}_{\boldsymbol{C}} D : D \to \mathcal{F}_{C_0}$. *Assume that*

$$\left(\tau^*(\pi_i)_* : \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M)) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^*\mathscr{F}_{\boldsymbol{C}} D(i))\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$$

is a limiting cone of a functor $\mathcal{D} \to \mathcal{S}et$ *which assigns* $i \in \mathrm{Ob}\,\mathcal{D}$ *to* $\mathcal{F}_{C_1}(\sigma^*(M), \tau^*\mathscr{F}_{\boldsymbol{C}} D(i))$ *and* $\alpha \in \mathcal{D}(i,j)$ *to* $\tau^*\mathscr{F}_{\boldsymbol{C}} D(\alpha)_* : \mathcal{F}_{C_1}(\sigma^*(M), \tau^*\mathscr{F}_{\boldsymbol{C}} D(i))) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^*\mathscr{F}_{\boldsymbol{C}} D(j)))$. *We also assume that*

$$\left((\tau\mu)^*(\pi_i)_* : \mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^*(M), (\tau\mu)^*(M)) \to \mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^*(M), (\tau\mu)^*\mathscr{F}_{\boldsymbol{C}} D(i))\right)_{i \in \mathrm{Ob}\,\mathcal{D}}$$

is a monomorphic family. Then, there exists a unique morphism $\xi : \sigma^*(M) \to \tau^*(M)$ *such that* (M, ξ) *is a representation of* $\boldsymbol{C}$ *on* M *and* $(\pi_i : (M, \xi) \to D(i))_{i \in \operatorname{Ob} \mathcal{D}}$ *is a limiting cone of* D.

(2) *Let* $(\iota_i : \mathscr{F}_{\boldsymbol{C}} D(i) \to M)_{i \in \operatorname{Ob} \mathcal{D}}$ *be a colimiting cone of* $\mathscr{F}_{\boldsymbol{C}} D : D \to \mathcal{F}_{C_0}$. *Asuume that*

$$\left(\sigma^*(\iota_i)^* : \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M)) \to \mathcal{F}_{C_1}(\sigma^* \mathscr{F}_{\boldsymbol{C}} D(i), \tau^*(M))\right)_{i \in \operatorname{Ob} \mathcal{D}}$$

is a limiting cone of a functor $\mathcal{D}^{op} \to \mathcal{S}et$ *which assigns* $i \in \operatorname{Ob} \mathcal{D}$ *to* $\mathcal{F}_{C_1}(\sigma^* \mathscr{F}_{\boldsymbol{C}} D(i), \tau^*(M))$ *and* $\alpha \in \mathcal{D}(i, j)$ *to* $\tau^* \mathscr{F}_{\boldsymbol{C}} D(\alpha)^* : \mathcal{F}_{C_1}(\sigma^* \mathscr{F}_{\boldsymbol{C}} D(j), \tau^*(M)) \to \mathcal{F}_{C_1}(\sigma^* \mathscr{F}_{\boldsymbol{C}} D(i), \tau^*(M))$. *We also assume that*

$$\left((\sigma\mu)^*(\iota_i)^* : \mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^*(M), (\tau\mu)^*(M)) \to \mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^* \mathscr{F}_{\boldsymbol{C}} D(i), (\tau\mu)^*(M)\right)_{i \in \operatorname{Ob} \mathcal{D}}$$

is a monomorphic family. Then, there exists a unique morphism $\xi : \sigma^*(M) \to \tau^*(M)$ *such that* (M, ξ) *is a representation of* $\boldsymbol{C}$ *on* M *and* $(\iota_i : D(i) \to (M, \xi))_{i \in \operatorname{Ob} \mathcal{D}}$ *is a colimiting cone of* D.

Proof. For $i \in \operatorname{Ob} \mathcal{D}$, we denote by $\xi_i : \sigma^* \mathscr{F}_{\boldsymbol{C}} D(i) \to \tau^* \mathscr{F}_{\boldsymbol{C}} D(i)$ the structure morphism of the representation of $\boldsymbol{C}$ on $\mathscr{F}_{\boldsymbol{C}} D(i)$.

(1) Since $\xi_j \sigma^* D(\alpha) = \tau^* D(\alpha) \xi_i$ for any morphism $\alpha : i \to j$ of $\mathcal{D}$,

$$\left(\xi_{i*} \sigma^*(\pi_i)_* : \mathcal{F}_{C_1}(\sigma^*(M), \sigma^*(M)) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^* \mathscr{F}_{\boldsymbol{C}} D(i))\right)_{i \in \operatorname{Ob} \mathcal{D}}$$

is a cone of a functor $\mathcal{D} \to \mathcal{S}et$ which assigns $i \in \operatorname{Ob} \mathcal{D}$ to $\mathcal{F}_{C_1}(\sigma^*(M), \sigma^* \mathscr{F}_{\boldsymbol{C}} D(i))$. Hence there exists a unique map $\chi : \mathcal{F}_{C_1}(\sigma^*(M), \sigma^*(M)) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M))$ satisfying $\tau^*(\pi_i)_* \chi = \xi_{i*} \sigma^*(\pi_i)_*$ for every $i \in \operatorname{Ob} \mathcal{D}$. Put $\xi = \chi(id_{\sigma^*(M)})$, then we have $\tau^*(\pi_i)\xi = \xi_i \sigma^*(\pi_i)$ and

$$\begin{aligned} f^\sharp_{\sigma^* \mathscr{F}_{\boldsymbol{C}} D(i), \tau^* \mathscr{F}_{\boldsymbol{C}} D(i)}(\xi_i) f^\sharp_{\sigma^*(M), \sigma^* \mathscr{F}_{\boldsymbol{C}} D(i)}(\sigma^*(\pi_i)) &= f^\sharp_{\sigma^*(M), \tau^* \mathscr{F}_{\boldsymbol{C}} D(i)}(\xi_i \sigma^*(\pi_i)) = f^\sharp_{\sigma^*(M), \tau^* \mathscr{F}_{\boldsymbol{C}} D(i)}(\tau^*(\pi_i)\xi) \\ &= f^\sharp_{\tau^*(M), \tau^* \mathscr{F}_{\boldsymbol{C}} D(i)}(\tau^*(\pi_i)) f^\sharp_{\sigma^*(M), \tau^*(M)}(\xi) \end{aligned}$$

for $f = \mathrm{pr}_1, \mathrm{pr}_2, \mu : C_1 \times_{C_0} C_1 \to C_1$. We note that $\mu^\sharp(\tau^*(\pi_i)) = (\tau\mu)^*(\pi_i) = (\tau \mathrm{pr}_2)^*(\pi_i) = \mathrm{pr}_2^\sharp(\tau^*(\pi_i))$, $\mathrm{pr}_1^\sharp(\tau^*(\pi_i)) = (\tau \mathrm{pr}_1)^*(\pi_i) = (\sigma \mathrm{pr}_2)^*(\pi_i)) = \mathrm{pr}_2^\sharp(\sigma^*(\pi_i))$ and $\mu^\sharp(\sigma^*(\pi_i)) = (\sigma\mu)^*(\pi_i) = (\sigma \mathrm{pr}_1)^*(\pi_i) = \mathrm{pr}_1^\sharp(\sigma^*(\pi_i))$. Since ξ_i satisfies (A) of (9.1.1), we have

$$\begin{aligned} \mu^\sharp(\tau^*(\pi_i))\mu^\sharp(\xi) &= \mu^\sharp(\xi_i)\mu^\sharp(\sigma^*(\pi_i)) = \mathrm{pr}_2^\sharp(\xi_i)\mathrm{pr}_1^\sharp(\xi_i)\mathrm{pr}_1^\sharp(\sigma^*(\pi_i)) = \mathrm{pr}_2^\sharp(\xi_i)\mathrm{pr}_1^\sharp(\xi_i\sigma^*(\pi_i)) = \mathrm{pr}_2^\sharp(\xi_i)\mathrm{pr}_1^\sharp(\tau^*(\pi_i)\xi) \\ &= \mathrm{pr}_2^\sharp(\xi_i)\mathrm{pr}_1^\sharp(\tau^*(\pi_i))\mathrm{pr}_1^\sharp(\xi) = \mathrm{pr}_2^\sharp(\xi_i)\mathrm{pr}_2^\sharp(\sigma^*(\pi_i))\mathrm{pr}_1^\sharp(\xi) = \mathrm{pr}_2^\sharp(\xi_i\sigma^*(\pi_i))\mathrm{pr}_1^\sharp(\xi) \\ &= \mathrm{pr}_2^\sharp(\tau^*(\pi_i)\xi)\mathrm{pr}_1^\sharp(\xi) = \mathrm{pr}_2^\sharp(\tau^*(\pi_i))\mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi) = \mu^\sharp(\tau^*(\pi_i))\mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi) \end{aligned}$$

for any $i \in \operatorname{Ob} \mathcal{D}$. Since $\mu^\sharp(\xi), \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi) \in \mathcal{F}_{C_1 \times_{C_0} C_1}((\sigma\mu)^*(M), (\tau\mu)^*(M))$, the second assumption implies that ξ satisfies (A) of (9.1.1). Since $\varepsilon^\sharp(\xi_i)$ is the identity morphism of $\mathscr{F}_{\boldsymbol{C}} D(i)$, we have

$$\begin{aligned} \pi_i \varepsilon^\sharp(\xi) &= (\tau\varepsilon)^*(\pi_i)\varepsilon^\sharp(\xi) = \varepsilon^\sharp(\tau^*(\pi_i))\varepsilon^\sharp(\xi) = \varepsilon^\sharp(\tau^*(\pi_i)\xi) = \varepsilon^\sharp(\xi_i\sigma^*(\pi_i)) \\ &= \varepsilon^\sharp(\xi_i)\varepsilon^\sharp(\sigma^*(\pi_i)) = \varepsilon^\sharp(\sigma^*(\pi_i)) = (\sigma\varepsilon)^*(\pi_i) = \pi_i \end{aligned}$$

for any $i \in \operatorname{Ob} \mathcal{D}$. Since $(\pi_i : M \to \mathscr{F}_{\boldsymbol{C}} D(i))_{i \in \operatorname{Ob} \mathcal{D}}$ is a monomorphic family, ξ satisfies (U) of (9.1.1).

(2) Since $\xi_j \sigma^* D(\alpha) = \tau^* D(\alpha) \xi_i$ for any morphism $\alpha : i \to j$ of $\mathcal{D}$,

$$\left(\xi_i^* \tau^*(\iota_i)^* : \mathcal{F}_{C_1}(\tau^*(M), \tau^*(M)) \to \mathcal{F}_{C_1}(\sigma^* \mathscr{F}_{\boldsymbol{C}} D(i), \tau^*(M))\right)_{i \in \operatorname{Ob} \mathcal{D}}$$

is a cone of a functor $\mathcal{D}^{op} \to \mathcal{S}et$ which assigns $i \in \operatorname{Ob} \mathcal{D}$ to $\mathcal{F}_{C_1}(\sigma^* \mathscr{F}_{\boldsymbol{C}} D(i), \tau^*(M))$. Hence there exists a unique map $\chi : \mathcal{F}_{C_1}(\tau^*(M), \tau^*(M)) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M))$ satisfying $\sigma^*(\iota_i)^* \chi = \xi_i^* \tau^*(\iota_i)^*$ for every $i \in \operatorname{Ob} \mathcal{D}$. Put $\xi = \chi(id_{\tau^*(M)})$, then we have $\xi \sigma^*(\iota_i) = \tau^*(\iota_i)\xi_i$ and

$$\begin{aligned} f^\sharp_{\tau^* \mathscr{F}_{\boldsymbol{C}} D(i), \tau^*(M)}(\tau^*(\iota_i)) f^\sharp_{\sigma^* \mathscr{F}_{\boldsymbol{C}} D(i), \tau^* \mathscr{F}_{\boldsymbol{C}} D(i)}(\xi_i) &= f^\sharp_{\sigma^* \mathscr{F}_{\boldsymbol{C}} D(i), \tau^*(M)}(\tau^*(\iota_i)\xi_i) = f^\sharp_{\sigma^* \mathscr{F}_{\boldsymbol{C}} D(i), \tau^*(M)}(\xi\sigma^*(\iota_i)) \\ &= f^\sharp_{\sigma^*(M), \tau^*(M)}(\xi) f^\sharp_{\sigma^* \mathscr{F}_{\boldsymbol{C}} D(i), \sigma^*(M)}(\sigma^*(\iota_i)) \end{aligned}$$

for $f = \mathrm{pr}_1, \mathrm{pr}_2, \mu : C_1 \times_{C_0} C_1 \to C_1$. We note that $\mu^\sharp(\tau^*(\iota_i)) = (\tau\mu)^*(\iota_i) = (\tau \mathrm{pr}_2)^*(\iota_i) = \mathrm{pr}_2^\sharp(\tau^*(\iota_i))$, $\mathrm{pr}_2^\sharp(\sigma^*(\iota_i)) = (\sigma \mathrm{pr}_2)^*(\iota_i)) = (\tau \mathrm{pr}_1)^*(\iota_i) = \mathrm{pr}_1^\sharp(\tau^*(\iota_i))$ and $\mathrm{pr}_1^\sharp(\sigma^*(\iota_i)) = (\sigma \mathrm{pr}_1)^*(\iota_i) = (\sigma\mu)^*(\iota_i) = \mu^\sharp(\sigma^*(\iota_i))$. Since ξ_i satisfies (A) of (9.1.1), we have

$$\begin{aligned} \mu^\sharp(\xi)\mu^\sharp(\sigma^*(\iota_i)) &= \mu^\sharp(\tau^*(\iota_i))\mu^\sharp(\xi_i) = \mathrm{pr}_2^\sharp(\tau^*(\iota_i))\mathrm{pr}_2^\sharp(\xi_i)\mathrm{pr}_1^\sharp(\xi_i) = \mathrm{pr}_2^\sharp(\tau^*(\iota_i)\xi_i)\mathrm{pr}_1^\sharp(\xi_i) = \mathrm{pr}_2^\sharp(\xi\sigma^*(\iota_i))\mathrm{pr}_1^\sharp(\xi_i) \\ &= \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_2^\sharp(\sigma^*(\iota_i))\mathrm{pr}_1^\sharp(\xi_i) = \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\tau^*(\iota_i))\mathrm{pr}_1^\sharp(\xi_i) = \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\tau^*(\iota_i)\xi_i) \\ &= \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi\sigma^*(\iota_i)) = \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi)\mathrm{pr}_1^\sharp(\sigma^*(\iota_i)) = \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi)\mu^\sharp(\sigma^*(\iota_i)) \end{aligned}$$

for any $i \in \mathrm{Ob}\,\mathcal{D}$. Since $\mu^\sharp(\xi), \mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi) \in \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M),(\tau\mu)^*(M))$, the second assumption implies that ξ satisfies (A) of (9.1.1). Since $\varepsilon^\sharp(\xi_i)$ is the identity morphism of $\mathscr{F}_{\boldsymbol{C}}D(i)$, we have

$$\begin{aligned}\varepsilon^\sharp(\xi)\iota_i &= \varepsilon^\sharp(\xi)(\sigma\varepsilon)^*(\iota_i) = \varepsilon^\sharp(\xi)\varepsilon^\sharp(\sigma^*(\iota_i)) = \varepsilon^\sharp(\xi\sigma^*(\iota_i)) = \varepsilon^\sharp(\tau^*(\iota_i)\xi_i)\\ &= \varepsilon^\sharp(\tau^*(\iota_i))\varepsilon^\sharp(\xi_i) = \varepsilon^\sharp(\tau^*(\iota_i)) = (\tau\varepsilon)^*(\iota_i) = \iota_i\end{aligned}$$

for any $i \in \mathrm{Ob}\,\mathcal{D}$. Since $(\iota_i : \mathscr{F}_{\boldsymbol{C}}D(i) \to M)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is an epimorphic family, ξ satisfies (U) of (9.1.1). $\square$

Remark 9.1.6 (1) *If $\tau^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ preserves limits and $\mu^* : \mathcal{F}_{C_1} \to \mathcal{F}_{C_1\times_{C_0}C_1}$ preserves monomorphic families, the assumptions of* (1) *of* (9.1.5) *are satisfied for any functor $D : \mathcal{D} \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ such that $\mathscr{F}_{\boldsymbol{C}}D : \mathcal{D} \to \mathcal{F}_{C_0}$ has a limit. This case, $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ creates limits in the sense of Mac Lane ([12], chapter V). In particular, if $p : \mathcal{F} \to \mathcal{E}$ is a bifibered category, $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ creates limits.*

(2) *If $\sigma^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ preserves colimits and $\mu^* : \mathcal{F}_{C_1} \to \mathcal{F}_{C_1\times_{C_0}C_1}$ preserves epimorphic families, the assumptions of* (2) *of* (9.1.5) *are satisfied for any functor $D : \mathcal{D} \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ such that $\mathscr{F}_{\boldsymbol{C}}D : \mathcal{D} \to \mathcal{F}_{C_0}$ has a colimit. This case, $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ creates colimits.*

(3) *If the presheaf $F_{\sigma,\tau,M}$ on $\mathcal{F}_{C_1}^{op}$ is representable, then the first assumption of* (1) *of* (9.1.5) *is satisfied. In fact, $\left(\pi_{i*} : \mathcal{F}_{C_0}(M_{[\sigma,\tau]}, M) \to \mathcal{F}_{C_0}(M_{[\sigma,\tau]}, \mathscr{F}_{\boldsymbol{C}}D(i))\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of a functor $\mathcal{D} \to \mathcal{S}et$ which assigns $i \in \mathrm{Ob}\,\mathcal{D}$ to $\mathcal{F}_{C_0}(M_{[\sigma,\tau]}, \mathscr{F}_{\boldsymbol{C}}D(i))$, $\alpha \in \mathcal{D}(i,j)$ to $\mathscr{F}_{\boldsymbol{C}}D(\alpha)_*$ and the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M)) \arrow[r, "\tau^*(\pi_i)_*"] \arrow[d, "P_{\sigma,\tau}(M)_M"] & \mathcal{F}_{C_1}(\sigma^*(M), \tau^*\mathscr{F}_{\boldsymbol{C}}D(i)) \arrow[d, "P_{\sigma,\tau}(M)_{\mathscr{F}_{\boldsymbol{C}}D(i)}"] \\
\mathcal{F}_{C_0}(M_{[\sigma,\tau]}, M) \arrow[r, "\pi_{i*}"] & \mathcal{F}_{C_0}(M_{[\sigma,\tau]}, \mathscr{F}_{\boldsymbol{C}}D(i))
\end{tikzcd}$$

Similarly, if the presheaf $F_{\mu\sigma,\mu\tau,M}$ on $\mathcal{F}_{C_1\times_{C_0}C_1}^{op}$ is representable, then the second assumption of (1) *of* (9.1.5) *is satisfied. In fact, $(\pi_{i*})_{i\in\mathrm{Ob}\,\mathcal{D}} : \mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]}, M) \to \prod_{i\in\mathrm{Ob}\,\mathcal{D}} \mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]}, \mathscr{F}_{\boldsymbol{C}}D(i))$ is injective and the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M), (\tau\mu)^*(M)) \arrow[r, "((\tau\mu)^*(\pi_i)_*)_{i\in\mathrm{Ob}\,\mathcal{D}}"] \arrow[d, "P_{\sigma\mu,\tau\mu}(M)_M"] & \prod\limits_{i\in\mathrm{Ob}\,\mathcal{D}} \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M), (\tau\mu)^*\mathscr{F}_{\boldsymbol{C}}D(i)) \arrow[d, "\prod_{i\in\mathrm{Ob}\,\mathcal{D}} P_{\sigma\mu,\tau\mu}(M)_{\mathscr{F}_{\boldsymbol{C}}D(i)}"] \\
\mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]}, M) \arrow[r, "(\pi_{i*})_{i\in\mathrm{Ob}\,\mathcal{D}}"] & \prod\limits_{i\in\mathrm{Ob}\,\mathcal{D}} \mathcal{F}_{C_0}(M_{[\sigma\mu,\tau\mu]}, \mathscr{F}_{\boldsymbol{C}}D(i))
\end{tikzcd}$$

(4) *If the presheaf $F^M_{\sigma,\tau}$ on $\mathcal{F}_{C_1}$ is representable, then the first assumption of* (2) *of* (9.1.5) *is satisfied. In fact, $\left(\iota_i^* : \mathcal{F}_{C_0}(M, M^{[\sigma,\tau]}) \to \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}D(i), M^{[\sigma,\tau]})\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of a functor $\mathcal{D}^{op} \to \mathcal{S}et$ which assigns $i \in \mathrm{Ob}\,\mathcal{D}$ to $\mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}D(i), M^{[\sigma,\tau]})$, $\alpha \in \mathcal{D}(i,j)$ to $\mathscr{F}_{\boldsymbol{C}}D(\alpha)^*$ and the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M)) \arrow[r, "\sigma^*(\iota_i)^*"] \arrow[d, "E_{\sigma,\tau}(M)_M"] & \mathcal{F}_{C_1}(\sigma^*\mathscr{F}_{\boldsymbol{C}}D(i), \tau^*(M)) \arrow[d, "E_{\sigma,\tau}(M)_{\mathscr{F}_{\boldsymbol{C}}D(i)}"] \\
\mathcal{F}_{C_0}(M, M^{[\sigma,\tau]}) \arrow[r, "\iota_i^*"] & \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}D(i), M^{[\sigma,\tau]})
\end{tikzcd}$$

Similarly, if the presheaf $F^M_{\mu\sigma,\mu\tau}$ on $\mathcal{F}_{C_1\times_{C_0}C_1}$ is representable, then the second assumption of (2) *of* (9.1.5) *is satisfied. In fact, $(\iota_i^*)_{i\in\mathrm{Ob}\,\mathcal{D}} : \mathcal{F}_{C_0}(M, M^{[\sigma\mu,\tau\mu]}) \to \prod_{i\in\mathrm{Ob}\,\mathcal{D}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}D(i), M^{[\sigma\mu,\tau\mu]})$ is injective and the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M), (\tau\mu)^*(M)) \arrow[r, "((\sigma\mu)^*(\iota_i)^*)_{i\in\mathrm{Ob}\,\mathcal{D}}"] \arrow[d, "E_{\sigma\mu,\tau\mu}(M)_M"] & \prod\limits_{i\in\mathrm{Ob}\,\mathcal{D}} \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*\mathscr{F}_{\boldsymbol{C}}D(i), (\tau\mu)^*(M)) \arrow[d, "\prod_{i\in\mathrm{Ob}\,\mathcal{D}} E_{\sigma\mu,\tau\mu}(M)_{\mathscr{F}_{\boldsymbol{C}}D(i)}"] \\
\mathcal{F}_{C_0}(M, M^{[\sigma\mu,\tau\mu]}) \arrow[r, "(\iota_i^*)_{i\in\mathrm{Ob}\,\mathcal{D}}"] & \prod\limits_{i\in\mathrm{Ob}\,\mathcal{D}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}D(i), M^{[\sigma\mu,\tau\mu]})
\end{tikzcd}$$

Proposition 9.1.7 *The forgetful functor $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ reflects isomorphisms.*

Proof. Let $\varphi : \xi \to \zeta$ be a morphism of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ such that $\mathscr{F}_{\boldsymbol{C}}(\varphi)$ is an isomorphism. Since $\tau^*(\varphi^{-1})\zeta = \tau^*(\varphi^{-1})\zeta\sigma^*(\varphi)\sigma^*(\varphi^{-1}) = \tau^*(\varphi^{-1})\tau^*(\varphi)\xi\sigma^*(\varphi^{-1}) = \xi\sigma^*(\varphi^{-1})$, φ^{-1} is also a morphism in $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$. Hence φ is an isomorphism in $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$. □

Proposition 9.1.8 *Let $\xi : \sigma^*(M) \to \tau^*(M)$ be a morphism of $\mathcal{F}_{C_1}$.*

(1) *If ξ is a monomorphism or epimorphism which satisfies (A) of (9.1.1), then ξ satisfies (U) of (9.1.1).*

(2) *If $\boldsymbol{C}$ is an internal groupoid in $\mathcal{E}$ and ξ satisfies (A) and (U) of (9.1.1), then ξ is an isomorphism.*

Proof. (1) We put $\varepsilon_1 = (id_{C_1}, \varepsilon\tau), \varepsilon_2 = (\varepsilon\sigma, id_{C_1}) : C_1 \to C_1 \times_{C_0} C_1$. Since $\mu\varepsilon_1 = \mu\varepsilon_2 = id_{C_1}$, we have maps

$$\varepsilon_i^\sharp : \mathcal{F}_{C_1\times_{C_0}C_1}((\sigma\mu)^*(M), (\tau\mu)^*(M)) \to \mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M))$$

for $i = 1, 2$. Then, we have the following by (8.1.13) and (8.1.14).

$$\xi = (\mu\varepsilon_i)^\sharp(\xi) = \varepsilon_i^\sharp(\mu^\sharp(\xi)) = \varepsilon_i^\sharp(\mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi)) = \varepsilon_i^\sharp(\mathrm{pr}_2^\sharp(\xi))\varepsilon_i^\sharp(\mathrm{pr}_1^\sharp(\xi)) = (\mathrm{pr}_2\varepsilon_i)^\sharp(\xi)(\mathrm{pr}_1\varepsilon_i)^\sharp(\xi) = \begin{cases} (\varepsilon\tau)^\sharp(\xi)\xi & i=1 \\ \xi(\varepsilon\sigma)^\sharp(\xi) & i=2 \end{cases}$$

Hence $(\varepsilon\tau)^\sharp(\xi)\xi = \xi(\varepsilon\sigma)^\sharp(\xi) = \xi$ which implies $(\varepsilon\tau)^\sharp(\xi) = id_{\tau^*(M)}$ if ξ is an epimorphism, $(\varepsilon\sigma)^\sharp(\xi) = id_{\sigma^*(M)}$ if ξ is a monomorphism. In the former case, since $\varepsilon^\sharp : \mathcal{F}_{C_1}(\tau^*(M), \tau^*(M)) \to \mathcal{F}_{C_0}(M, M)$ maps $id_{\tau^*(M)}$ and $(\varepsilon\tau)^\sharp(\xi)$ to id_M and $(\varepsilon\tau\varepsilon)^\sharp(\xi) = \varepsilon^\sharp(\xi) = \xi_\varepsilon$ respectively, ξ satisfies (U) of (9.1.1). In the latter case, since $\varepsilon^\sharp : \mathcal{F}_{C_1}(\sigma^*(M), \sigma^*(M)) \to \mathcal{F}_{C_0}(M, M)$ maps $id_{\sigma^*(M)}$ and $(\varepsilon\sigma)^\sharp(\xi)$ to id_M and $(\varepsilon\sigma\varepsilon)^\sharp(\xi) = \varepsilon^\sharp(\xi) = \xi_\varepsilon$ respectively, ξ satisfies (U) of (9.1.1).

(2) Let us denote by $\iota : C_1 \to C_1$ the inverse of $\boldsymbol{C}$. Since $\sigma\iota = \tau$ and $\tau\iota = \sigma$, we have a morphism $\xi_\iota = \iota^\sharp(\xi) : \tau^*(M) \to \sigma^*(M)$ $\mathcal{F}_{C_1}$ and morphisms $\iota_1 = (id_{C_1}, \iota), \iota_2 = (\iota, id_{C_1}) : C_1 \to C_1 \times_{C_0} C_1$ of $\mathcal{E}$. Since $(\mathrm{pr}_2\iota_i)^\sharp(\xi)(\mathrm{pr}_1\iota_i)^\sharp(\xi) = \iota_i^\sharp(\mathrm{pr}_2^\sharp(\xi))\iota_i^\sharp(\mathrm{pr}_1^\sharp(\xi)) = \iota_i^\sharp(\mathrm{pr}_2^\sharp(\xi)\mathrm{pr}_1^\sharp(\xi)) = \iota_i^\sharp(\mu^\sharp(\xi)) = (\mu\iota_i)^\sharp(\xi)$ for $i = 1, 2$ and $\mu\iota_1 = \varepsilon\sigma$, $\mu\iota_2 = \varepsilon\tau$, we have $\xi_\iota\xi = \iota^\sharp(\xi)\xi = (\mathrm{pr}_2\iota_1)^\sharp(\xi)(\mathrm{pr}_1\iota_1)^\sharp(\xi) = (\mu\iota_1)^\sharp(\xi) = (\varepsilon\sigma)^\sharp(\xi) = \sigma^\sharp(\varepsilon^\sharp(\xi)) = \sigma^\sharp(id_M) = id_{\sigma^*(M)}$ and $\xi\xi_\iota = \xi\iota^\sharp(\xi) = (\mathrm{pr}_2\iota_2)^\sharp(\xi)(\mathrm{pr}_1\iota_2)^\sharp(\xi) = (\mu\iota_2)^\sharp(\xi) = (\varepsilon\tau)^\sharp(\xi) = \tau^\sharp(\varepsilon^\sharp(\xi)) = \tau^\sharp(id_M) = id_{\tau^*(M)}$. □

The above notion of the representations of internal categories generalizes as follows.

Definition 9.1.9 *Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$, $(\pi : X \to C_0, \alpha)$ an internal diagram on $\boldsymbol{C}$ and M an object of $\mathcal{F}_X$. A morphism $\xi : p_X^*(M) \to \alpha^*(M)$ ($p_X : X \times_{C_0} C_1 \to X$ is the projection) in $\mathcal{F}_{X\times_{C_0}C_1}$ is called a representation of $\boldsymbol{C}$ on M over (X, α) if the following conditions are satisfied.*

(A) $(id_X \times \mu)^\sharp_{M,M}(\xi) : (p_X(id_X \times \mu))^*(M) \to (\alpha(id_X \times \mu))^*(M)$ coincides with the following composition. Here, $p_{12} : X \times_{C_0} C_1 \times_{C_0} C_1 \to X \times_{C_0} C_1$ is the projection.

$$(p_X(id_X \times \mu))^*(M) = (p_X p_{12})^*(M) \xrightarrow{(p_{12}^\sharp)_{M,M}(\xi)} (\alpha p_{12})^*(M) = (p_X(\alpha \times id_{C_1}))^*(M) \xrightarrow{(\alpha\times id_{C_1})^\sharp_{M,M}(\xi)} (\alpha(\alpha \times id_{C_1}))^*(M) = (\alpha(id_X \times \mu))^*(M)$$

(U) $(id_X, \varepsilon\pi)^\sharp_{M,M}(\xi) : M = (p_X(id_X, \varepsilon\pi))^*(M) \to (\alpha(id_X, \varepsilon\pi))^*(M) = M$ is the identity morphism of M.

Let $\xi : p_X^*(M) \to \alpha^*(M)$ and $\zeta : p_X^*(N) \to \alpha^*(N)$ be representations of $\boldsymbol{C}$ on M and N over (X, α), respectively. A morphism $\varphi : M \to N$ in $\mathcal{F}_X$ is called a morphism of representations of $\boldsymbol{C}$ over (X, α) if $\alpha^*(\varphi)\xi = \zeta p_X^*(\varphi)$. We denote by $\mathrm{Rep}(\boldsymbol{C}, X\,;\mathcal{F})$ the category of the representations of $\boldsymbol{C}$ over (X, α). We denote an object $\xi : p_X^*(M) \to \alpha^*(M)$ of $\mathrm{Rep}(\boldsymbol{C}, X\,;\mathcal{F})$ by (M, ξ).

For an internal diagram $(\pi : X \to C_0, \alpha)$ on $\boldsymbol{C}$, we define an internal category $\boldsymbol{C}_\alpha = (X, C_\alpha; \sigma_\alpha, \tau_\alpha, \varepsilon_\alpha, \mu_\alpha)$ associated with (X, α) by $C_\alpha = X \times_{C_0} C_1$, $\sigma_\alpha = p_X, \tau_\alpha = \alpha : C_\alpha \to X$, $\varepsilon_\alpha = (id_X, \varepsilon\pi) : X \to C_\alpha$ and $\mu_\alpha = (id_X \times \mu)(id_{C_\alpha} \times p_{C_1}) : C_\alpha \times_X C_\alpha \to C_\alpha \times_X C_1 \to C_\alpha$. Here $p_{C_1} : C_\alpha = X \times_{C_0} C_1 \to C_1$ denote the projection.

Let M be an object of $\mathcal{F}_X$ and $\xi : p_X^*(M) \to \alpha^*(M)$ a morphism in $\mathcal{F}_{C_\alpha}$. Then, ξ is a representation of $\boldsymbol{C}_\alpha$ if and only if it is a representation of $\boldsymbol{C}$ over (X, α). Thus we see the following result.

Proposition 9.1.10 *Let $\boldsymbol{C}$ is an internal category and (X, α) an internal diagram on $\boldsymbol{C}$. Then, the category $\mathrm{Rep}(\boldsymbol{C}, X\,;\mathcal{F})$ is isomorphic to $\mathrm{Rep}(\boldsymbol{C}_\alpha\,;\mathcal{F})$.*

Example 9.1.11 *Let $p : \mathcal{Q}mod \to \mathcal{S}ch$ be the fibered category given in (8.1.20) and $\boldsymbol{C}$ an internal category in $\mathcal{S}ch$. For an object $(C_0, \mathcal{M})$ of $\mathcal{Q}mod_{C_0}$, a morphism $(id_{C_1}, \xi) : \sigma^*(C_0, \mathcal{M}) = (C_1, \sigma^*\mathcal{M}) \to (C_1, \tau^*\mathcal{M}) = \tau^*(C_0, \mathcal{M})$ is an representation of $\boldsymbol{C}$ on $(C_0, \mathcal{M})$ if and only if a morphism $\xi : \tau^*\mathcal{M} \to \sigma^*\mathcal{M}$ of $\mathcal{O}_{C_1}$-modules satisfies*

$$\tilde{c}_{\sigma,p_1}(\mathcal{M})^{-1}p_1^*(\xi)\tilde{c}_{\tau,p_1}(\mathcal{M})\tilde{c}_{\sigma,p_2}(\mathcal{M})^{-1}p_2^*(\xi)\tilde{c}_{\tau,p_2}(\mathcal{M}) = \tilde{c}_{\sigma,\mu}(\mathcal{M})^{-1}\mu^*(\xi)\tilde{c}_{\tau,\mu}(\mathcal{M})$$

and $\varepsilon^*(\xi)\tilde{c}_{\sigma,\varepsilon}(\mathcal{M}) = \tilde{c}_{\tau,\varepsilon}(\mathcal{M})$. Suppose that $\boldsymbol{C}$ is the internal category in $\mathcal{S}ch$ assosiated with a Hopf algebroid (A, H), that is $C_0 = \operatorname{Spec} A$, $C_1 = \operatorname{Spec} H$, and that $\mathcal{M}$ is the quasi-coherent $\mathcal{O}_{C_0}$-module associated with an A-module M. There is a natural bijection $\Phi : \operatorname{Hom}_{\mathcal{O}_{C_1}}(\tau^*\mathcal{M}, \sigma^*\mathcal{M}) \to \operatorname{Hom}_A(M, M \otimes_A H)$, where H is regarded as a left A-module by the left unit $\eta_L : A \to H$ inducing σ. An $\mathcal{O}_{C_1}$-module homomorphism $\xi : \tau^*(M) \to \sigma^*(M)$ defines an representation of $\boldsymbol{C}$ on $(C_0, \mathcal{M})$ if and only if $\Phi(\xi)$ is a structure map of H-comodule.

Proposition 9.1.12 *Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$ and $s : \mathcal{E} \to \mathcal{F}$ a cartesian section. Then, $s_{\sigma,\tau} : \sigma^*s(C_0) \to \tau^*s(C_0)$ defined in (8.1.27) is a representation of $\boldsymbol{C}$ on $s(C_0)$.*

Proof. By (9.1.8), we only have to verify the condition (A) of (9.1.1). Since we assumed that $\mathcal{E}$ has finite limits, we may assume that $s = s_T$ for some $T \in \operatorname{Ob}\mathcal{F}_1$ by (8.1.26), here o_{C_0} denotes the unique morphism $C_0 \to 1$. Then, $s_\sigma = c_{o_{C_0},\sigma}(T)^{-1}$, $s_\tau = c_{o_{C_0},\tau}(T)^{-1}$ and we have the following equalities by (8.1.12) for $f = \mu, \mathrm{pr}_1, \mathrm{pr}_2$.

$$c_{\tau,f}(s(C_0))f^*(s_\tau) = c_{\tau,f}(o_{C_0}^*(T))f^*(c_{o_{C_0},\tau}(T)^{-1}) = c_{o_{C_0},\tau f}(T)^{-1}c_{o_{C_0}\tau,f}(T) = c_{o_{C_0},\tau f}(T)^{-1}c_{o_{C_1},f}(T)$$
$$f^*(s_\sigma^{-1})c_{\sigma,f}(s(C_0))^{-1} = f^*(c_{o_{C_0},\sigma}(T))c_{\sigma,f}(o_{C_0}^*(T))^{-1} = c_{o_{C_0}\sigma,f}(T)^{-1}c_{o_{C_0},\sigma f}(T) = c_{o_{C_1},f}(T)^{-1}c_{o_{C_0},\sigma f}(T)$$

Hence we have $f^\sharp(s_{\sigma,\tau}) = c_{\tau,f}(s(C_0))f^*(s_\tau)f^*(s_\sigma^{-1})c_{\sigma,f}(s(C_0))^{-1} = c_{o_{C_0},\tau f}(T)^{-1}c_{o_{C_0},\sigma f}(T)$. Since $\tau\mathrm{pr}_2 = \tau\mu$, $\sigma\mathrm{pr}_2 = \tau\mathrm{pr}_1$ and $\sigma\mathrm{pr}_1 = \sigma\mu$, above equality implies

$$\mathrm{pr}_2^\sharp(s_{\sigma,\tau})\mathrm{pr}_1^\sharp(s_{\sigma,\tau}) = c_{o_{C_0},\tau\mathrm{pr}_2}(T)^{-1}c_{o_{C_0},\sigma\mathrm{pr}_2}(T)c_{o_{C_0},\tau\mathrm{pr}_1}(T)^{-1}c_{o_{C_0},\sigma\mathrm{pr}_1}(T) = c_{o_{C_0},\tau\mu}(T)^{-1}c_{o_{C_0},\sigma\mu}(T) = \mu^\sharp(s_{\sigma,\tau}).$$

Thus $s_{\sigma,\tau}$ satisfies the condition (A) of (9.1.1). □

Definition 9.1.13 *Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$ and $s : \mathcal{E} \to \mathcal{F}$ a cartesian section.*

(1) *We set $s_{\boldsymbol{C}} = s_{\sigma,\tau}$ and call $(s(C_0), s_{\boldsymbol{C}})$ the trivial representation associated with s. In the case $s = s_T$ for some $T \in \operatorname{Ob}\mathcal{F}_1$, we also call $(s(C_0), (s_T)_{\boldsymbol{C}})$ the trivial representation associated with T.*

(2) *Let $\xi : \sigma^*(M) \to \tau^*(M)$ be a representation of $\boldsymbol{C}$ on M and T an object of $\mathcal{F}_1$. We call a morphism $\varphi : (M, \xi) \to (s(C_0), (s_T)_{\boldsymbol{C}})$ a primitive element of (M, ξ) with respect to T.*

Example 9.1.14 *Let $p : \mathcal{E}^{(2)} \to \mathcal{E}$ be the fibered category given in (2) of (8.1.17) and $\boldsymbol{C}$ an internal category in $\mathcal{E}$. We note that $\mathcal{E}_1^{(2)}$ is identified with $\mathcal{E}$ by an isomorphism of categories $\mathcal{E}_1^{(2)} \to \mathcal{E}$ $(X \to 1) \mapsto X$. For an object T of $\mathcal{E}_1^{(2)} = \mathcal{E}$, the cartesian section $s_T : \mathcal{E} \to \mathcal{E}^{(2)}$ associated with T is given by $s_T(X) = (pr_1 : X \times T \to X)$. $\sigma^*s_T(C_0)$ and $\tau^*s_T(C_0)$ are both identified with $(pr_1 : C_1 \times T \to C_1)$. Hence the trivial representation $(s_T)_{\boldsymbol{C}} : \sigma^*s_T(C_0) \to \tau^*s_T(C_0)$ associated with T can be regarded as the identity morphism of $C_1 \times T$.*

Example 9.1.15 *Let $p : \mathcal{Q}mod \to \mathcal{S}ch$ be the fibered category given in (8.1.20) and $\boldsymbol{C}$ an internal category in $\mathcal{S}ch$. In this case, since the terminal object 1 in $\mathcal{S}ch$ is $\operatorname{Spec}\boldsymbol{Z}$, $\mathcal{Q}mod_1$ is identified with the category of abelian groups. For an abelian group G, the cartesian section $s_G : \mathcal{S}ch \to \mathcal{Q}mod$ associated with G is given by $s_G(X) = (X, o_X^*\widetilde{G})$ $(o_X : X \to 1)$. The isomorphisms $\tilde{c}_{\sigma,o_{C_0}} : o_{C_1}^*\widetilde{G} \to \sigma^*o_{C_0}^*\widetilde{G}$, $\tilde{c}_{\tau,o_{C_0}} : o_{C_1}^*\widetilde{G} \to \tau^*o_{C_0}^*\widetilde{G}$ define isomorphisms $c_{\sigma,o_{C_0}} : \sigma^*o_{C_0}^*(1, \widetilde{G}) \to o_{C_1}^*(1, \widetilde{G})$, $c_{\tau,o_{C_0}} : \tau^*o_{C_0}^*(1, \widetilde{G}) \to o_{C_1}^*(1, \widetilde{G})$ in $\mathcal{Q}mod_{C_0}$. The trivial representation $(s_G)_{\boldsymbol{C}} : \sigma^*s_G(C_0) \to \tau^*s_G(C_0)$ associated with G is $c_{\sigma,o_{C_0}}c_{\tau,o_{C_0}}^{-1}$.*

We describe the notion of representation of internal categories in terms of 2-categories and lax diagrams.

Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ be an internal category in $\mathcal{E}$ and M an object of $\Gamma(C_0)$. Recall that $\sigma^*(C_0, M) = (C_1, \Gamma_{C_0,C_1}(\sigma)(M))$, $\tau^*(C_0, M) = (C_1, \Gamma_{C_0,C_1}(\tau)(M))$. For a morphism $\zeta : \Gamma_{C_0,C_1}(\sigma)(M) \to \Gamma_{C_0,C_1}(\tau)(M)$ in $\Gamma(C_1)$, define a morphism $\xi(\zeta) : \sigma^*(C_0, M) \to \tau^*(C_0, M)$ in $\mathcal{F}(\Gamma)_{C_1}$ by $\xi(\zeta) = (id_{C_1}, R_\gamma(\tau)_M^{-1}\zeta)$.

Proposition 9.1.16 (1) *$\xi(\zeta)$ satisfies (A) of (9.1.1) if and only if ζ satisfies the following equality.*

$$\begin{aligned}&((\gamma_{C_0,C_1,C_1\times_{C_0}C_1})_{(\tau,p_2)})_M\Gamma_{C_1,C_1\times_{C_0}C_1}(p_2)(\zeta)((\gamma_{C_0,C_1,C_1\times_{C_0}C_1})_{(\sigma,p_2)})_M^{-1}\\ &((\gamma_{C_0,C_1,C_1\times_{C_0}C_1})_{(\tau,p_1)})_M\Gamma_{C_1,C_1\times_{C_0}C_1}(p_1)(\zeta)((\gamma_{C_0,C_1,C_1\times_{C_0}C_1})_{(\sigma,p_1)})_M^{-1}\\ &= ((\gamma_{C_0,C_1,C_1\times_{C_0}C_1})_{(\tau,\mu)})_M\Gamma_{C_1,C_1\times_{C_0}C_1}(\mu)(\zeta)((\gamma_{C_0,C_1,C_1\times_{C_0}C_1})_{(\sigma,\mu)})_M^{-1}\end{aligned}$$

(2) *$\xi(\zeta)$ satisfies (U) of (9.1.1) if and only if ζ satisfies $((\gamma_{C_0,C_1,C_0})_{(\tau,\varepsilon)})_M\Gamma_{C_1,C_0}(\varepsilon)(\zeta) = ((\gamma_{C_0,C_1,C_0})_{(\sigma,\varepsilon)})_M$.*

Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{G} = (G_0, G_1; \sigma', \tau', \varepsilon', \mu')$ be internal categories in $\mathcal{E}$. Consider the fibered category $p_{\boldsymbol{C}} : \mathcal{F}(\boldsymbol{C}) \to \mathcal{E}$ represented by $\boldsymbol{C}$ given in (8.3.18). The following is immedeiate from definitions.

Lemma 9.1.17 (1) *Let $f_0 : G_0 \to C_0$ and $f_1 : G_1 \to C_1$ be morphisms in $\mathcal{E}$. $(f_0, f_1) : \boldsymbol{G} \to \boldsymbol{C}$ is an internal functor if and only if a morphism $(id_{G_1}, f_1) : \sigma^*(G_0, f_0) = (G_1, f_0\sigma) \to (G_1, f_0\tau) = \tau^*(G_0, f_0)$ is a representation of $\boldsymbol{G}$ on $(G_0, f_0) \in \operatorname{Ob}\mathcal{F}(\boldsymbol{C})_{G_0}$.*

(2) *Let $\boldsymbol{f} = (f_0, f_1), \boldsymbol{g} = (g_0, g_1) : \boldsymbol{G} \to \boldsymbol{C}$ be internal functors and $\varphi : G_0 \to C_1$ a morphism in $\mathcal{E}$. φ is an internal natural transformation from $\boldsymbol{f}$ to $\boldsymbol{g}$ if and only if (id_{G_0}, φ) is a morphism of representations from $(id_{G_1}, f_1) : \sigma^*(G_0, f_0) \to \tau^*(G_0, f_0)$ to $(id_{G_1}, g_1) : \sigma^*(G_0, g_0) \to \tau^*(G_0, g_0)$.*

Thus we have the following result.

Theorem 9.1.18 *Define a functor $F : \boldsymbol{cat}(\mathcal{E})(\boldsymbol{G}, \boldsymbol{C}) \to \operatorname{Rep}(\boldsymbol{G}\,; \mathcal{F}(\boldsymbol{C}))$ by $F(\boldsymbol{f}) = ((G_0, f_0), (id_{G_1}, f_1))$ for an internal functor $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{G} \to \boldsymbol{C}$ and $F(\varphi) = (id_{G_0}, \varphi)$. Then, F is an isomorphism of categories.*

Remark 9.1.19 *We note that a composition $\boldsymbol{cat}(\mathcal{E})(\boldsymbol{G}, \boldsymbol{C}) \xrightarrow{F} \operatorname{Rep}(\boldsymbol{G}\,; \mathcal{F}(\boldsymbol{C})) \xrightarrow{\mathscr{F}_{\boldsymbol{G}}} \mathcal{F}(\boldsymbol{C})_{G_0}$ maps an internal functor $(f_0, f_1) : \boldsymbol{G} \to \boldsymbol{C}$ to an object (G_0, f_0) of $\mathcal{F}(\boldsymbol{C})_{G_0}$ and an internal natural transformation $\varphi : (f_0, f_1) \to (g_0, g_1)$ to a morphism $\varphi : (G_0, f_0) \to (G_0, g_0)$ of $\mathcal{F}(\boldsymbol{C})_{G_0}$.*

9.2 Descent formalism

Definition 9.2.1 *([4], Définition 1.3.) Let $p : \mathcal{F} \to \mathcal{E}$ be a cloven fibered category. We say that a diagram $R \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \xrightarrow{f} Y$ in $\mathcal{E}$ is $\mathcal{F}$-exact if $fp_1 = fp_2$ and, for any $M, N \in \operatorname{Ob}\mathcal{F}_Y$, the following diagram is an equalizer, where we put $g = fp_1 = fp_2$.*

$$\mathcal{F}_Y(M, N) \xrightarrow{f^*} \mathcal{F}_X(f^*(M), f^*(N)) \overset{p_1^\sharp}{\underset{p_2^\sharp}{\rightrightarrows}} \mathcal{F}_R(g^*(M), g^*(N)) \quad \cdots (*)$$

Example 9.2.2 *A diagram $R \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \xrightarrow{f} Y$ in $\mathcal{E}$ is $\mathcal{E}^{(2)}$-exact if and only if, for any $\pi : M \to Y$, $id_M \times f : M \times_Y X \to M \times_Y Y = M$ is a coequalizer of $id_M \times p_1, id_M \times p_2 : M \times_Y R \to M \times_Y X$, in other words, f is a universal strict epimorphism.*

Definition 9.2.3 *([4], Définition 1.4.) Let $p_1, p_2 : R \to X$ be morphisms in $\mathcal{E}$ and M is an objct of $\mathcal{F}_X$. An isomorphism $\xi : p_1^*(M) \to p_2^*(M)$ is called a glueing morphism on M with respect to a pair (p_1, p_2). If $\xi : p_1^*(M) \to p_2^*(M)$ and $\zeta : p_1^*(N) \to p_2^*(N)$ are glueing morphisms on $M, N \in \operatorname{Ob}\mathcal{F}_X$, a morphism $\varphi : M \to N$ in $\mathcal{F}_X$ is said to be compatible with ξ and ζ if the following square commutes.*

$$\begin{tikzcd} p_1^*(M) \arrow[r, "\xi"] \arrow[d, "p_1^*(\varphi)"] & p_2^*(M) \arrow[d, "p_2^*(\varphi)"] \\ p_1^*(N) \arrow[r, "\zeta"] & p_2^*(N) \end{tikzcd}$$

Thus we can consider the category of glueing morphisms.

Definition 9.2.4 *([4], Définition 1.5.) Let $R \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \xrightarrow{f} Y$ be a diagram in $\mathcal{E}$ such that $fp_1 = fp_2$. We say that a glueing morphism $\xi : p_1^*(M) \to p_2^*(M)$ on $M \in \operatorname{Ob}\mathcal{F}_X$ is effective with respect to f if there exists an isomorphism $\kappa : M \to f^*(N)$ in $\mathcal{F}_X$ for some $N \in \operatorname{Ob}\mathcal{F}_Y$ such that the following diagram commutes.*

$$\begin{tikzcd} p_1^*(M) \arrow[r, "p_1^*(\kappa)"] \arrow[d, "\xi"] & p_1^*f^*(N) \arrow[r, "c_{f,p_1}(N)"] & (fp_1)^*(N) \arrow[d, equal] \\ p_2^*(M) \arrow[r, "p_2^*(\kappa)"] & p_2^*f^*(N) \arrow[r, "c_{f,p_2}(N)"] & (fp_2)^*(N) \end{tikzcd}$$

Assume that $\mathcal{E}$ is a category with finite limits.

An internal groupoid $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ in $\mathcal{E}$ which is an internal poset, that is, $(\sigma, \tau) : C_1 \to C_0 \times C_0$ is a monomorphism, is called an equivalence relation on C_0.

For a morphism $f : X \to Y$ in $\mathcal{E}$, the kernel pair $X \times_Y X \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$ of f is an equivalence relation on X with the following structure maps; The domain $\sigma = p_1$, the codomain $\tau = p_2$, the identity $\varepsilon = \Delta$ (the diagonal morphism), the composition $\mu = p_1 \times p_2 : (X \times_Y X) \times_X (X \times_Y X) \to X \times_Y X$ and the inverse $\iota = (p_2, p_1) : X \times_Y X \to X \times_Y X$. We denote this internal groupoid by $\boldsymbol{E}_f = (X \times_Y X, X; p_1, p_2, \Delta, p_1 \times p_2)$. The notion of descent data is given in terms of representation of groupoids as follows.

Definition 9.2.5 *([4], Définition 1.6.) For an object M of $\mathcal{F}_X$, a representation $\xi : p_1^*(M) \to p_2^*(M)$ of $\boldsymbol{E}_f$ on M is called a descent data on M for a morphism $f : X \to Y$ in $\mathcal{E}$.*

Let $f : X \to Y$ be a morphism in $\mathcal{E}$. We denote by $p_i : X \times_Y X \to X$ $(i = 1, 2)$, $q_i : X \times_Y X \times_Y X \to X$ $(i = 1, 2, 3)$ the projections onto the i-th component. $\Delta : X \to X \times_Y X$ denotes the diagonal morphism. Define $p_{ij} : X \times_Y X \times_Y X \to X \times_Y X$ $(1 \leq i < j \leq 3)$ by $p_{ij} = (q_i, q_j)$. We note that $p_1 p_{12} = p_1 p_{13} = q_1$, $p_1 p_{23} = p_2 p_{12} = q_2$, $p_2 p_{13} = p_2 p_{23} = q_3$.

The category of glueing morphisms $p_1^*(M) \to p_2^*(M)$ is denoted by $\mathrm{Glue}(\mathcal{F}/\mathcal{E}, f)$. The following assertion is immediate from the definition.

Proposition 9.2.6 *Let $\mathcal{E}$ be a category with finite limits and $p : \mathcal{F} \to \mathcal{E}$ a cloven fibered category. A glueing morphism $\xi : p_1^*(M) \to p_2^*(M)$ on $M \in \mathrm{Ob}\,\mathcal{F}_X$ with respect to a pair (p_1, p_2) is a descent data on M for a morphism $f : X \to Y$ in $\mathcal{E}$ if and only if ξ satisfies the following equalities.*

$$c_{p_1,\Delta}(M)\Delta^*(\xi) = c_{p_2,\Delta}(M)$$
$$c_{p_2,p_{13}}(M)p_{13}^*(\xi)c_{p_1,p_{13}}(M)^{-1} = c_{p_2,p_{23}}(M)p_{23}^*(\xi)c_{p_1,p_{23}}(M)^{-1}c_{p_2,p_{12}}(M)p_{12}^*(\xi)c_{p_1,p_{12}}(M)^{-1}$$

Definition 9.2.7 *([4], Définition 1.7.) A morphism $f : X \to Y$ in $\mathcal{E}$ is called a morphism of $\mathcal{F}$-descent if*

$$X \times_Y X \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \xrightarrow{f} Y$$

is $\mathcal{F}$-exact. Moreover, if every descent data on arbitrary object of $\mathcal{F}_X$ is effective, we say that f is a morphism of effective $\mathcal{F}$-descent.

We set $\mathrm{Desc}(\mathcal{F}/\mathcal{E}, f) = \mathrm{Rep}(\boldsymbol{E}_f; \mathcal{F})$ and regard this as a full subcategory of $\mathrm{Glue}(\mathcal{F}/\mathcal{E}, f)$. We define a functor $\bar{D}_f : \mathcal{F}_Y \to \mathrm{Glue}(\mathcal{F}/\mathcal{E}, f)$ as follows. For $N \in \mathcal{F}_Y$, let $\bar{D}_f(N) : p_1^* f^*(N) \to p_2^* f^*(N)$ be the composition

$$p_1^* f^*(N) \xrightarrow{c_{f,p_1}(N)} (fp_1)^*(N) = (fp_2)^*(N) \xrightarrow{c_{f,p_2}(N)^{-1}} p_2^* f^*(N).$$

For a morphism $\varphi : N \to N'$, $\bar{D}_f(\varphi) = f^*(\varphi)$. Then, $\bar{D}_f$ factors through the inclusion functor $\mathrm{Desc}(\mathcal{F}/\mathcal{E}, f) \to \mathrm{Glue}(\mathcal{F}/\mathcal{E}, f)$ and we have a functor $D_f : \mathcal{F}_Y \to \mathrm{Desc}(\mathcal{F}/\mathcal{E}, f)$. Moreover, a glueing morphism $\xi : p_1^*(M) \to p_2^*(M)$ on $M \in \mathrm{Ob}\,\mathcal{F}_X$ is effective with respect to $f : X \to Y$ if and only if ξ is isomorphic to an object in the image of $D_f : \mathcal{F}_Y \to \mathrm{Desc}(\mathcal{F}/\mathcal{E}, f)$.

The following fact is also immediate.

Proposition 9.2.8 *A morphism $f : X \to Y$ is $\mathcal{F}$-descent (resp. effective $\mathcal{F}$-descent) if and only if $D_f : \mathcal{F}_Y \to \mathrm{Desc}(\mathcal{F}/\mathcal{E}, f)$ is fully faithful (resp. an equivalence).*

Example 9.2.9 *Let $\mathcal{T}op$ be the category of topological spaces and continuous maps. Consider the fibered category $p : \mathcal{T}op^{(2)} \to \mathcal{T}op$ (8.1.17 (2)). For a topological space B, suppose that an open covering $(U_i)_{i \in I}$ of B is given. Put $X = \coprod_{i \in I} U_i$ and let $f : X \to B$ be the map induced by the inclusion maps $U_i \hookrightarrow B$. Then, $X \times_B X = \coprod_{i,j \in I} U_i \cap U_j$ and the following diagrams commute.*

$$\begin{tikzcd} U_i \arrow[d, "\iota_i"] & U_i \cap U_j \arrow[l, "inc"'] \arrow[r, "inc"] \arrow[d, "\iota_{ij}"] & U_j \arrow[d, "\iota_j"] \\ X & X \times_B X \arrow[l, "p_1"'] \arrow[r, "p_2"] & X \end{tikzcd} \qquad \begin{tikzcd} U_i \arrow[r, equal] \arrow[d, "\iota_i"] & U_i \cap U_i \arrow[d, "\iota_{ii}"] \\ X \arrow[r, "\Delta"] & X \times_B X \end{tikzcd}$$

$$\begin{array}{ccc}
U_i\cap U_j\cap U_k \xrightarrow{inc} U_i\cap U_j & U_i\cap U_j\cap U_k \xrightarrow{inc} U_j\cap U_k & U_i\cap U_j\cap U_k \xrightarrow{inc} U_i\cap U_k \\
\downarrow\iota_{ijk} \qquad\quad \downarrow\iota_{ij} & \downarrow\iota_{ijk} \qquad\quad \downarrow\iota_{jk} & \downarrow\iota_{ijk} \qquad\quad \downarrow\iota_{ik} \\
X\times_B X\times_B X \xrightarrow{p_{12}} X\times_B X & X\times_B X\times_B X \xrightarrow{p_{23}} X\times_B X & X\times_B X\times_B X \xrightarrow{p_{13}} X\times_B X
\end{array}$$

For a topological space F, consider an object $pr_1 : X\times F\to X$ of $\mathcal{T}op_X^{(2)} = \mathcal{T}op/X$. Then, the pull-back of $pr_1 : X\times F\to X$ along $p_i : X\times_B X\to X$ $(i=1,2)$ is the map $q : (X\times_B X)\times F\to X\times_B X$ given by $q(x,y)=x$ $(x\in X\times_B X,\ y\in F)$. For a map $\xi : (X\times_B X)\times F = \coprod_{i,j\in I}(U_i\cap U_j)\times F \to \coprod_{i,j\in I}(U_i\cap U_j)\times F = (X\times_B X)\times F$ making the following diagram commute, we denote by $\xi_{ij} : (U_i\cap U_j)\times F\to (U_i\cap U_j)\times F$ the restriction of ξ.

$$\begin{array}{ccc}
\coprod_{i,j\in I}(U_i\cap U_j)\times F & \xrightarrow{\quad\xi\quad} & \coprod_{i,j\in I}(U_i\cap U_j)\times F \\
\Big\downarrow \coprod_{i,j\in I} pr_1 & & \Big\downarrow \coprod_{i,j\in I} pr_1 \\
\coprod_{i,j\in I} U_i\cap U_j & =\!=\!=\!= & \coprod_{i,j\in I} U_i\cap U_j
\end{array}$$

We also denote by $\xi_{ij}^k : (U_i\cap U_j\cap U_k)\times F\to (U_i\cap U_j\cap U_k)\times F$ the restriction of ξ_{ij}. Then, a descent data ξ of $(X\times_B X, X; p_1, p_2, \Delta, p_1\times p_2)$ on $pr_1 : X\times F\to X$ is a homeomorphism $\xi : \coprod_{i,j\in I}(U_i\cap U_j)\times F\to \coprod_{i,j\in I}(U_i\cap U_j)\times F$ which makes the above diagram commute and satisfies $\xi_{jk}^i\xi_{ij}^k = \xi_{ik}^j$.

9.3 Restrictions, regular representations

Let $\boldsymbol{C} = (C_0, C_1; \sigma,\tau,\varepsilon,\mu)$ and $\boldsymbol{D} = (D_0, D_1; \sigma',\tau',\varepsilon',\mu')$ be internal categories in $\mathcal{E}$, $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{D}\to\boldsymbol{C}$ an internal functor and $p : \mathcal{F}\to\mathcal{E}$ a cloven fibered category. Suppose that a representation (M,ξ) of $\boldsymbol{C}$ on $M\in\operatorname{Ob}\mathcal{F}_{C_0}$ is given. We denote by $\xi_{\boldsymbol{f}} : \sigma'^*(f_0^*(M))\to\tau'^*(f_0^*(M))$ the following composition.

$$\sigma'^*(f_0^*(M)) \xrightarrow{c_{f_0,\sigma'}(M)} (f_0\sigma')^*(M) = (\sigma f_1)^*(M) \xrightarrow{(f_1^\sharp)_{M,M}(\xi)} (\tau f_1)^*(M) = (f_0\tau')^*(M) \xrightarrow{c_{f_0,\tau'}(M)^{-1}} \tau'^*(f_0^*(M))$$

Proposition 9.3.1 *$(f_0^*(M), \xi_{\boldsymbol{f}})$ is a representation of $\boldsymbol{D}$ on $f_0^*(M)\in\operatorname{Ob}\mathcal{F}_{D_0}$.*

Proof. $(\mathrm{pr}_i^\sharp)_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ is the following composition for $i=1,2$.

$$\begin{aligned}
(\sigma'\mathrm{pr}_i)^*(f_0^*(M)) &\xrightarrow{c_{\sigma',\mathrm{pr}_i}(f_0^*(M))^{-1}} \mathrm{pr}_i^*\sigma'^*(f_0^*(M)) \xrightarrow{\mathrm{pr}_i^*(c_{f_0,\sigma'}(M))} \mathrm{pr}_i^*(f_0\sigma')^*(M) = \mathrm{pr}_i^*(\sigma f_1)^*(M) \xrightarrow{\mathrm{pr}_i^*((f_1^\sharp)_{M,M}(\xi))} \\
&\mathrm{pr}_i^*(\tau f_1)^*(M) = \mathrm{pr}_i^*(f_0\tau')^*(M) \xrightarrow{\mathrm{pr}_i^*(c_{f_0,\tau'}(M)^{-1})} \mathrm{pr}_i^*\tau'^*(f_0^*(M)) \xrightarrow{c_{\tau',\mathrm{pr}_i}(f_0^*(M))} (\tau'\mathrm{pr}_i)^*(f_0^*(M))
\end{aligned}$$

It follows from (8.1.12) and $f_0\sigma' = \sigma f_1$, $f_0\tau' = \tau f_1$ that $(\mathrm{pr}_i^\sharp)_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ is the following composition.

$$\begin{aligned}
(\sigma'\mathrm{pr}_i)^*(f_0^*(M)) &\xrightarrow{c_{f_0,\sigma'\mathrm{pr}_i}(M)} (f_0\sigma'\mathrm{pr}_i)^*(M) = (\sigma f_1\mathrm{pr}_i)^*(M) \xrightarrow{(\mathrm{pr}_i^\sharp)_{M,M}((f_1^\sharp)_{M,M}(\xi))} (\tau f_1\mathrm{pr}_i)^*(M) \\
&= (f_0\tau'\mathrm{pr}_i)^*(M) \xrightarrow{c_{f_0,\tau'\mathrm{pr}_i}(M)^{-1}} (\tau'\mathrm{pr}_i)^*(f_0^*(M))
\end{aligned}$$

Moreover, since $(\mathrm{pr}_i^\sharp)_{M,M}((f_1^\sharp)_{M,M}(\xi)) = (f_1\mathrm{pr}_i)^\sharp_{M,M}(\xi) = (\mathrm{pr}_i(f_1\times_{C_0}f_1))^\sharp_{M,M}(\xi) = (f_1\times_{C_0}f_1)^\sharp_{M,M}((\mathrm{pr}_i)^\sharp_{M,M}(\xi))$ by (8.1.14), $(\mathrm{pr}_i^\sharp)_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ is the following composition.

$$\begin{aligned}
(\sigma'\mathrm{pr}_i)^*(f_0^*(M)) &\xrightarrow{c_{f_0,\sigma'\mathrm{pr}_i}(M)} (f_0\sigma'\mathrm{pr}_i)^*(M) = (\sigma\mathrm{pr}_i(f_1\times_{C_0}f_1))^*(M) \xrightarrow{(f_1\times_{C_0}f_1)^\sharp_{M,M}((\mathrm{pr}_i)^\sharp_{M,M}(\xi))} \\
&(\tau\mathrm{pr}_i(f_1\times_{C_0}f_1))^*(M) = (f_0\tau'\mathrm{pr}_i)^*(M) \xrightarrow{c_{f_0,\tau'\mathrm{pr}_i}(M)^{-1}} (\tau'\mathrm{pr}_i)^*(f_0^*(M))
\end{aligned}$$

Hence the composition

$$\begin{aligned}
(\sigma'\mu')^*(f_0^*(M)) = (\sigma'\mathrm{pr}_1)^*(f_0^*(M)) &\xrightarrow{(\mathrm{pr}_1^\sharp)_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})} (\tau'\mathrm{pr}_1)^*(f_0^*(M)) = (\sigma'\mathrm{pr}_2)^*(f_0^*(M)) \\
&\xrightarrow{(\mathrm{pr}_2^\sharp)_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})} (\tau'\mathrm{pr}_2)^*(f_0^*(M)) = (\tau'\mu')^*(f_0^*(M)) \ \cdots(*)
\end{aligned}$$

coincides with the following composition since $\sigma'\mathrm{pr}_1 = \sigma'\mu'$, $\tau'\mathrm{pr}_2 = \tau'\mu'$.

$$\begin{aligned}(\sigma'\mu')^*(f_0^*(M)) &\xrightarrow{c_{f_0,\sigma'\mu'}(M)} (f_0\sigma'\mu')^*(M) = (\sigma\mathrm{pr}_1(f_1 \times_{C_0} f_1))^*(M) \xrightarrow{(f_1\times_{C_0}f_1)^\sharp_{M,M}((\mathrm{pr}_1)^\sharp_{M,M}(\xi))} \\ &(\tau\mathrm{pr}_1(f_1 \times_{C_0} f_1))^*(M) = (\sigma\mathrm{pr}_2(f_1 \times_{C_0} f_1))^*(M) \xrightarrow{(f_1\times_{C_0}f_1)^\sharp_{M,M}((\mathrm{pr}_2)^\sharp_{M,M}(\xi))} \\ &(\tau\mathrm{pr}_2(f_1 \times_{C_0} f_1))^*(M) = (f_0\tau'\mu')^*(M) \xrightarrow{c_{f_0,\tau'\mu'}(M)^{-1}} (\tau'\mu')^*(f_0^*(M))\end{aligned}$$

Since ξ satisfies (A) of (9.1.1), it follows from (8.1.13) that we have

$$\begin{aligned}(f_1 \times_{C_0} f_1)^\sharp_{M,M}((\mathrm{pr}_2)^\sharp_{M,M}(\xi))(f_1 \times_{C_0} f_1)^\sharp_{M,M}((\mathrm{pr}_1)^\sharp_{M,M}(\xi)) &= (f_1 \times_{C_0} f_1)^\sharp_{M,M}((\mathrm{pr}_2)^\sharp_{M,M}(\xi)(\mathrm{pr}_1)^\sharp_{M,M}(\xi)) \\ &= (f_1 \times_{C_0} f_1)^\sharp_{M,M}(\mu^\sharp_{M,M}(\xi)).\end{aligned}$$

Therefore the above composition $(*)$ coincides with the following composition.

$$\begin{aligned}(\sigma'\mu')^*(f_0^*(M)) &\xrightarrow{c_{f_0,\sigma'\mu'}(M)} (f_0\sigma'\mu')^*(M) = (\sigma\mu(f_1 \times_{C_0} f_1))^*(M) \xrightarrow{(f_1\times_{C_0}f_1)^\sharp_{M,M}(\mu^\sharp_{M,M}(\xi))} (\tau\mu(f_1 \times_{C_0} f_1))^*(M) \\ &= (f_0\tau'\mu')^*(M) \xrightarrow{c_{f_0,\tau'\mu'}(M)^{-1}} (\tau'\mu')^*(f_0^*(M))\end{aligned}$$

On the other hand, ${\mu'}^\sharp_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ is the following composition.

$$\begin{aligned}(\sigma'\mu')^*(f_0^*(M)) &\xrightarrow{c_{\sigma',\mu'}(f_0^*(M))^{-1}} \mu'^*\sigma'^*(f_0^*(M)) \xrightarrow{\mu'^*(c_{f_0,\sigma'}(M))} \mu'^*(f_0\sigma')^*(M) = \mu'^*(\sigma f_1)^*(M) \xrightarrow{\mu'^*((f_1^\sharp)_{M,M}(\xi))} \\ &\mu'^*(\tau f_1)^*(M) = \mu'^*(f_0\tau')^*(M) \xrightarrow{\mu'^*(c_{f_0,\tau'}(M)^{-1})} \mu'^*\tau'^*(f_0^*(M)) \xrightarrow{c_{\tau',\mu'}(f_0^*(M))} (\tau'\mu')^*(f_0^*(M))\end{aligned}$$

It follows from (8.1.12) and $f_0\sigma' = \sigma f_1$, $f_0\tau' = \tau f_1$ that ${\mu'}^\sharp_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ is the following composition.

$$\begin{aligned}(\sigma'\mu')^*(f_0^*(M)) &\xrightarrow{c_{f_0,\sigma'\mu'}(M)} (f_0\sigma'\mu')^*(M) = (\sigma f_1\mu')^*(M) \xrightarrow{{\mu'}^\sharp_{M,M}((f_1^\sharp)_{M,M}(\xi))} (\tau f_1\mu')(M) = (f_0\tau'\mu')^*(M) \\ &\xrightarrow{c_{f_0,\tau'\mu'}(M)^{-1}} (\tau'\mu')^*(f_0^*(M))\end{aligned}$$

By (8.1.14), ${\mu'}^\sharp_{M,M}((f_1^\sharp)_{M,M}(\xi)) : (\sigma\mu(f_1 \times_{C_0} f_1))^*(M) = (\sigma f_1\mu')^*(M) \to (\tau f_1\mu')^*(M) = (\tau\mu(f_1 \times_{C_0} f_1))^*(M)$ coincides with

$$(f_1\mu')^\sharp_{M,M}(\xi) = (\mu(f_1 \times_{C_0} f_1))^\sharp_{M,M}(\xi) = (f_1 \times_{C_0} f_1)^\sharp_{M,M}(\mu^\sharp_{M,M}(\xi)) : (\sigma\mu(f_1 \times_{C_0} f_1))^*(M) \to (\tau\mu(f_1 \times_{C_0} f_1))^*(M).$$

Thus we have verified that $\xi_{\boldsymbol{f}}$ satisfies (A) of (9.1.1).

${\varepsilon'}^\sharp_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}}) : f_0^*(M) = (\sigma'\varepsilon')^*(f_0^*(M)) \to (\tau'\varepsilon')^*(f_0^*(M)) = f_0^*(M)$ is the following composition.

$$\begin{aligned}(\sigma'\varepsilon')^*(f_0^*(M)) &\xrightarrow{c_{\sigma',\varepsilon'}(f_0^*(M))^{-1}} \varepsilon'^*\sigma'^*(f_0^*(M)) \xrightarrow{\varepsilon'^*(c_{f_0,\sigma'}(M))} \varepsilon'^*(f_0\sigma')^*(M) = \varepsilon'^*(\sigma f_1)^*(M) \xrightarrow{\varepsilon'^*((f_1^\sharp)_{M,M}(\xi))} \\ &\varepsilon'^*(\tau f_1)^*(M) = \varepsilon'^*(f_0\tau')^*(M) \xrightarrow{\varepsilon'^*(c_{f_0,\tau'}(M)^{-1})} \varepsilon'^*\tau'^*(f_0^*(M)) \xrightarrow{c_{\tau',\varepsilon'}(f_0^*(M))} (\tau'\varepsilon')^*(f_0^*(M))\end{aligned}$$

It follows from (8.1.12) and $f_0\sigma' = \sigma f_1$, $f_0\tau' = \tau f_1$ that ${\varepsilon'}^\sharp_{f_0^*(M),f_0^*(M)}(\xi_{\boldsymbol{f}})$ is the following composition.

$$\begin{aligned}(\sigma'\varepsilon')^*(f_0^*(M)) &\xrightarrow{c_{f_0,\sigma'\varepsilon'}(M)} (f_0\sigma'\varepsilon')^*(M) = (\sigma f_1\varepsilon')^*(M) \xrightarrow{{\varepsilon'}^\sharp_{M,M}((f_1^\sharp)_{M,M}(\xi))} (\tau f_1\varepsilon')^*(M) = (f_0\tau'\varepsilon')^*(M) \\ &\xrightarrow{c_{f_0,\tau'\varepsilon'}(M)^{-1}} (\tau'\varepsilon')^*(f_0^*(M))\end{aligned}$$

Since ${\varepsilon'}^\sharp_{M,M}((f_1^\sharp)_{M,M}(\xi)) = (f_1\varepsilon')^\sharp_{M,M}(\xi) = (\varepsilon f_0)^\sharp_{M,M}(\xi) = (f_0^\sharp)_{M,M}(\varepsilon^\sharp_{M,M}(\xi)) = (f_0^\sharp)_{M,M}(id_M) = id_{f_0^*(M)}$ by (8.1.13) and (8.1.14), the above composition is the identity morphism of $f_0^*(M)$. □

Definition 9.3.2 *We call $(f_0^*(M), \xi_{\boldsymbol{f}})$ the restriction of (M, ξ) along $\boldsymbol{f}$.*

Let (M,ξ) and (N,ζ) be representations of $\boldsymbol{C}$ and $\boldsymbol{f}:\boldsymbol{D}\to\boldsymbol{C}$ an internal functor. For a morphism of representations $\varphi:(M,\xi)\to(N,\zeta)$ of $\boldsymbol{C}$, we have $(\tau f_1)^*(\varphi)f_1^\sharp(\xi)=f_1^\sharp(\tau^*(\varphi)\xi)=f_1^\sharp(\zeta\sigma^*(\varphi))=f_1^\sharp(\zeta)(\sigma f_1)^*(\varphi)$ by the naturality of $f_1^\sharp$. Then, the following diagram commute.

$$\begin{tikzcd}
\sigma'^*f_0^*(M) \arrow[r, "c_{f_0,\sigma'}(M)"] \arrow[d, "\sigma'^*f_0^*(\varphi)"] & (f_0\sigma')^*(M) \arrow[r, equal] \arrow[d, "(f_0\sigma')^*(\varphi)"] & (\sigma f_1)^*(M) \arrow[r, "f_1^\sharp(\xi)"] \arrow[d, "(\sigma f_1)^*(\varphi)"] & (\tau f_1)^*(M) \arrow[r, equal] \arrow[d, "(\tau f_1)^*(\varphi)"] & (f_0\tau')^*(M) \arrow[r, "c_{f_0,\tau'}(M)^{-1}"] \arrow[d, "(f_0\tau')^*(\varphi)"] & \tau'^*f_0^*(M) \arrow[d, "\tau'^*f_0^*(\varphi)"] \\
\sigma'^*f_0^*(N) \arrow[r, "c_{f_0,\sigma'}(N)"] & (f_0\sigma')^*(N) \arrow[r, equal] & (\sigma f_1)^*(N) \arrow[r, "f_1^\sharp(\zeta)"] & (\tau f_1)^*(N) \arrow[r, equal] & (f_0\tau')^*(N) \arrow[r, "c_{f_0,\tau'}(N)^{-1}"] & \tau'^*f_0^*(N)
\end{tikzcd}$$

Hence $f_0^*(\varphi):f_0^*(M)\to f_0^*(N)$ defines a morphism $f_0^*(\varphi):(f_0^*(M),\xi_{\boldsymbol{f}})\to(f_0^*(N),\zeta_{\boldsymbol{f}})$ of representations and we have a functor $\boldsymbol{f}^\bullet:\mathrm{Rep}(\boldsymbol{C};\mathcal{F})\to\mathrm{Rep}(\boldsymbol{D};\mathcal{F})$ given by $\boldsymbol{f}^\bullet(M,\xi)=(f_0^*(M),\xi_{\boldsymbol{f}})$ for an object (M,ξ) of $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$ and $\boldsymbol{f}^\bullet(\varphi)=f_0^*(\varphi)$ for a morphism φ of $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$.

If $g=(g_0,g_1):\boldsymbol{D}\to\boldsymbol{C}$ is an internal functor and χ is an internal natural transformation from f to g, let us define a morphism $\chi_{(M,\xi)}:f_0^*(M)\to g_0^*(M)$ of $\mathcal{F}_{D_0}$ to be $\chi^\sharp_{M,M}(\xi):f_0^*(M)=(\sigma\chi)^*(M)\to(\tau\chi)^*(M)=g_0^*(M)$.

Proposition 9.3.3 *$\chi_{(M,\xi)}$ is a morphism of representations from $(f_0^*(M),\xi_{\boldsymbol{f}})$ to $(g_0^*(M),\xi_{\boldsymbol{g}})$ and the following diagram in $\mathrm{Rep}(\boldsymbol{D};\mathcal{F})$ commutes for a morphism $\varphi:(M,\xi)\to(N,\zeta)$ of representations of $\boldsymbol{C}$.*

$$\begin{tikzcd}
(f_0^*(M),\xi_{\boldsymbol{f}}) \arrow[r, "f^*(\varphi)"] \arrow[d, "\chi_{(M,\xi)}"] & (f_0^*(N),\zeta_{\boldsymbol{f}}) \arrow[d, "\chi_{(N,\zeta)}"] \\
(g_0^*(M),\xi_{\boldsymbol{g}}) \arrow[r, "g^*(\varphi)"] & (g_0^*(N),\zeta_{\boldsymbol{g}})
\end{tikzcd}$$

Proof. Since ξ satisfies the condition (A) of (9.1.1), it follows from (8.1.13) and (8.1.14) that we have

$$\begin{aligned}
(\chi\tau')^\sharp(\xi)(f_1)^\sharp(\xi)&=(\mathrm{pr}_2(f_1,\chi\tau'))^\sharp(\xi)(\mathrm{pr}_1(f_1,\chi\tau'))^\sharp(\xi)=(f_1,\chi\tau')^\sharp((\mathrm{pr}_2)^\sharp(\xi))(f_1,\chi\tau')^\sharp((\mathrm{pr}_1)^\sharp(\xi))\\
&=(f_1,\chi\tau')^\sharp((\mathrm{pr}_2)^\sharp(\xi)(\mathrm{pr}_1)^\sharp(\xi))=(f_1,\chi\tau')^\sharp(\mu^\sharp(\xi))=(\mu(f_1,\chi\tau'))^\sharp(\xi)=(\mu(\chi\sigma',g_1))^\sharp(\xi)\\
&=(\chi\sigma',g_1)^\sharp(\mu^\sharp(\xi))=(\chi\sigma',g_1)^\sharp((\mathrm{pr}_2)^\sharp(\xi)(\mathrm{pr}_1)^\sharp(\xi))=(\chi\sigma',g_1)^\sharp((\mathrm{pr}_2)^\sharp(\xi))(\chi\sigma',g_1)^\sharp((\mathrm{pr}_1)^\sharp(\xi))\\
&=(\mathrm{pr}_2(\chi\sigma',g_1))^\sharp(\xi)(\mathrm{pr}_1(\chi\sigma',g_1))^\sharp(\xi)=(g_1)^\sharp(\xi)(\chi\sigma')^\sharp(\xi).
\end{aligned}$$

Hence the middle rectangle of the following diagram is commutative.

$$\begin{tikzcd}
\sigma'^*f_0^*(M) \arrow[r, equal] \arrow[d, "c_{f_0,\sigma'}(M)"] & \sigma'^*(\sigma\chi)^*(M) \arrow[r, "\sigma'^*(\chi^\sharp_{M,M}(\xi))"] \arrow[d, "c_{\sigma\chi,\sigma'}(M)"] & \sigma'^*(\tau\chi)^*(M) \arrow[r, equal] \arrow[d, "c_{\tau\chi,\sigma'}(M)"] & \sigma'^*g_0^*(M) \arrow[d, "c_{g_0,\sigma'}(M)"] \\
(f_0\sigma')^*(M) \arrow[r, equal] \arrow[d, equal] & (\sigma\chi\sigma')^*(M) \arrow[r, "(\chi\sigma')^\sharp_{M,M}(\xi)"] & (\tau\chi\sigma')^*(M) \arrow[r, equal] & (g_0\sigma')^*(M) \arrow[d, equal] \\
(\sigma f_1)^*(M) \arrow[d, "(f_1)^\sharp_{M,M}(\xi)"] & & & (\sigma g_1)^*(M) \arrow[d, "(g_1)^\sharp_{M,M}(\xi)"] \\
(\tau f_1)^*(M) \arrow[d, equal] & & & (\tau g_1)^*(M) \arrow[d, equal] \\
(f_0\tau')^*(M) \arrow[r, equal] \arrow[d, "c_{f_0,\tau'}(M)^{-1}"] & (\sigma\chi\tau')^*(M) \arrow[r, "(\chi\tau')^\sharp_{M,M}(\xi)"] \arrow[d, "c_{\sigma\chi,\tau'}(M)^{-1}"] & (\tau\chi\tau')^*(M) \arrow[r, equal] \arrow[d, "c_{\tau\chi,\tau'}(M)"] & (g_0\tau')^*(M) \arrow[d, "c_{g_0,\tau'}(M)"] \\
\tau'^*f_0^*(M) \arrow[r, equal] & \tau'^*(\sigma\chi)^*(M) \arrow[r, "\tau'^*(\chi^\sharp_{M,M}(\xi))"] & \tau'^*(\tau\chi)^*(M) \arrow[r, equal] & \tau'^*g_0^*(M)
\end{tikzcd}$$

Since the upper and lower middle small rectangles of the above diagram also commutes by (8.1.14) the outer rectangle of the above diagram is commutative. Since the left (resp. right) vertical composition of the above is $\xi_{\boldsymbol{f}}$ (resp. $\xi_{\boldsymbol{g}}$), we see that $\chi_{(M,\xi)}$ is a morphism of representations from $(f_0^*(M),\xi_{\boldsymbol{f}})$ to $(g_0^*(M),\xi_{\boldsymbol{g}})$.

The following diagram commutes by by (8.1.10) and (8.1.12).

$$\begin{tikzcd}
\sigma'^* f_0^*(M) \arrow[r, "c_{f_0,\sigma'}(M)"] \arrow[d, equal] & (f_0\sigma')^*(M) \arrow[r, "(f_0\sigma')^*(\varphi)"] \arrow[d, equal] & (f_0\sigma')^*(N) \arrow[r, "c_{f_0,\sigma'}(N)^{-1}"] \arrow[d, equal] & \sigma'^* f_0^*(N) \arrow[d, equal] \\
\sigma'^*(\sigma\chi)^*(M) \arrow[r, "c_{\sigma\chi,\sigma'}(M)"] \arrow[d, "\sigma'^*(c_{\sigma,\chi}(M))^{-1}"] & (\sigma\chi\sigma')^*(M) \arrow[r, "(\sigma\chi\sigma')^*(\varphi)"] \arrow[d, "c_{\sigma,\chi\sigma'}(M)^{-1}"] & (\sigma\chi\sigma')^*(N) \arrow[r, "c_{\sigma\chi,\sigma'}(N)^{-1}"] \arrow[d, "c_{\sigma,\chi\sigma'}(N)^{-1}"] & \sigma'^*(\sigma\chi)^*(N) \arrow[d, "\sigma'^*(c_{\sigma,\chi}(N))^{-1}"] \\
\sigma'^*\chi^*\sigma^*(M) \arrow[r, "c_{\chi,\sigma'}(\sigma^*(M))"] \arrow[d, "\sigma'^*\chi^*(\xi)"] & (\chi\sigma')^*\sigma^*(M) \arrow[r, "(\chi\sigma')^*\sigma^*(\varphi)"] \arrow[d, "(\chi\sigma')^*(\xi)"] & (\chi\sigma')^*\sigma^*(N) \arrow[r, "c_{\chi,\sigma'}(\sigma^*(N))^{-1}"] \arrow[d, "(\chi\sigma')^*(\zeta)"] & \sigma'^*\chi^*\sigma^*(N) \arrow[d, "\sigma'^*\chi^*(\zeta)"] \\
\sigma'^*\chi^*\tau^*(M) \arrow[r, "c_{\chi,\sigma'}(\tau^*(M))"] \arrow[d, "\sigma'^*(c_{\tau,\chi}(M))"] & (\chi\sigma')^*\tau^*(M) \arrow[r, "(\chi\sigma')^*\tau^*(\varphi)"] \arrow[d, "c_{\tau,\chi\sigma'}(M)"] & (\chi\sigma')^*\tau^*(N) \arrow[r, "c_{\chi,\sigma'}(\tau^*(N))^{-1}"] \arrow[d, "c_{\tau,\chi\sigma'}(N)"] & \sigma'^*\chi^*\tau^*(N) \arrow[d, "\sigma'^*(c_{\tau,\chi}(N))"] \\
\sigma'^*(\tau\chi)^*(M) \arrow[r, "c_{\tau\chi,\sigma'}(M)"] \arrow[d, equal] & (\tau\chi\sigma')^*(M) \arrow[r, "(\tau\chi\sigma')^*(\varphi)"] \arrow[d, equal] & (\tau\chi\sigma')^*(N) \arrow[r, "c_{\tau\chi,\sigma'}(N)^{-1}"] \arrow[d, equal] & \sigma'^*(\tau\chi)^*(N) \arrow[d, equal] \\
\sigma'^* g_0^*(M) \arrow[r, "c_{g_0,\sigma'}(M)"] & (g_0\sigma')^*(M) \arrow[r, "(g_0\sigma')^*(\varphi)"] & (g_0\sigma')^*(N) \arrow[r, "c_{g_0,\sigma'}(N)^{-1}"] & \sigma'^* g_0^*(N)
\end{tikzcd}$$

The composition of the left (resp. right) vertical morphisms of the above diagram is $\chi_{(M,\xi)}$ (resp. $\chi_{(N,\zeta)}$) and the composition of the upper (resp. lower) horizontal morphisms is $f_0^*(\varphi)$ (resp. $g_0^*(\varphi)$). Thus the second assertion follows. □

Define a functor $\mathrm{Res} : \boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{C})\times\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})$ by $\mathrm{Res}(\boldsymbol{f},\xi) = \xi_{\boldsymbol{f}}$ for $\boldsymbol{f} \in \mathrm{Ob}\,\boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{C})$, $(M,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ and $\mathrm{Res}(\chi,\varphi) = g^*(\varphi)\chi_{(M,\xi)} = \chi_{(N,\zeta)}f^*(\varphi)$ for $\chi \in \boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{C})(\boldsymbol{f},\boldsymbol{g})$ and $\varphi \in \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),(N,\zeta))$. If $\mathcal{F} = \mathcal{F}(\boldsymbol{G})$ for an internal category $\boldsymbol{G}$, we remark that Res is identified with the composition of internal functors by the isomorphism of (9.1.18), that is, the following diagram commutes.

$$\begin{tikzcd}
\boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{C}) \times \boldsymbol{cat}(\mathcal{E})(\boldsymbol{C},\boldsymbol{G}) \arrow[r, "\text{composition}"] \arrow[d, "id\times F"] & \boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{G}) \arrow[d, "F"] \\
\boldsymbol{cat}(\mathcal{E})(\boldsymbol{D},\boldsymbol{C}) \times \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}(\boldsymbol{G})) \arrow[r, "\mathrm{Res}"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F}(\boldsymbol{G}))
\end{tikzcd}$$

Definition 9.3.4 *Let (M,ρ) be a representation of $\boldsymbol{C}$ on $M \in \mathrm{Ob}\,\mathcal{F}_{C_0}$.*

(1) *(M,ρ) is called a left regular representation if there exist an object L of $\mathcal{F}_{C_0}$ and a bijection*

$$\mathscr{A}^l_{(N,\xi)} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\rho),(N,\xi)) \to \mathcal{F}_{C_0}(L, \mathscr{F}_{\boldsymbol{C}}(N,\xi))$$

for each $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ which is natural in (N,ξ).

(2) *(M,ρ) is called a right regular representation if there exist an object R of $\mathcal{F}_{C_0}$ and a bijection*

$$\mathscr{A}^r_{(N,\xi)} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((N,\xi),(M,\rho)) \to \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N,\xi), R)$$

for each $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ which is natural in (N,ξ).

Proposition 9.3.5 *Let (M,ρ) be a representation of $\boldsymbol{C}$ on $M \in \mathcal{F}_{C_0}$.*

(1) *(M,ρ) is a left regular representation if and only if there exists a morphism $\eta : L \to \mathscr{F}_{\boldsymbol{C}}(M,\rho)$ of $\mathcal{F}_{C_0}$ such that, for any $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$, the following composition is bijective.*

$$\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\rho),(N,\xi)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(M,\rho), \mathscr{F}_{\boldsymbol{C}}(N,\xi)) \xrightarrow{\eta^*} \mathcal{F}_{C_0}(L, \mathscr{F}_{\boldsymbol{C}}(N,\xi))$$

(2) *(M,ρ) is a right regular representation if and only if there exists a morphism $\varepsilon : \mathscr{F}_{\boldsymbol{C}}(M,\rho) \to R$ of $\mathcal{F}_{C_0}$ such that, for any $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$, the following composition is bijective.*

$$\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((N,\xi),(M,\rho)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N,\xi), \mathscr{F}_{\boldsymbol{C}}(M,\rho)) \xrightarrow{\varepsilon_*} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N,\xi), R)$$

Proof. (1) Suppose that (M,ρ) is a left regular representation. We take $L \in \mathrm{Ob}\,\mathcal{F}_{C_0}$ and a natural bijection $\mathscr{A}^l_{(N,\xi)}$ as in (1) of (9.3.4). Put $\eta = \mathscr{A}^l_{(M,\rho)}(id_{(M,\rho)}) : L \to \mathscr{F}_{\boldsymbol{C}}(M,\rho)$. For $f \in \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\rho),(N,\xi))$, the naturality of $\mathscr{A}^l$ implies $\mathscr{F}_{\boldsymbol{C}}(f)\eta = \mathscr{F}_{\boldsymbol{C}}(f)\mathscr{A}^l_{(M,\rho)}(id_{(M,\rho)}) = \mathscr{A}^l_{(N,\xi)}(f)$. Hence the composition $\eta^*\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\rho),(N,\xi)) \to \mathcal{F}_{C_0}(L, \mathscr{F}_{\boldsymbol{C}}(N,\xi))$ coincides with $\mathscr{A}^l_{(N,\xi)}$. The converse is obvious.

(2) Suppose that (M,ρ) is a right regular representation. We take $R \in \operatorname{Ob}\mathcal{F}_{C_0}$ and a natural bijection $\mathscr{A}^r_{(N,\xi)}$ as in (2) of (9.3.4). Put $\varepsilon = \mathscr{A}^r_{(M,\rho)}(id_{(M,\rho)}) : \mathscr{F}_{\boldsymbol{C}}(M,\rho) \to R$. For $f \in \operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F})((N,\xi),(M,\rho))$, the naturality of $\mathscr{A}^r$ implies $\varepsilon\mathscr{F}_{\boldsymbol{C}}(f) = \mathscr{A}^r_{(M,\rho)}(id_{(M,\rho)})\mathscr{F}_{\boldsymbol{C}}(f) = \mathscr{A}^r_{(N,\xi)}(f)$. Hence the composition $\varepsilon_*\mathscr{F}_{\boldsymbol{C}} : \operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F})((N,\xi),(M,\rho)) \to \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N,\xi),R)$ coincides with $\mathscr{A}^r_{(N,\xi)}$. The converse is obvious. □

By the above result and (9.1.18), we have the following.

Corollary 9.3.6 *Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{G} = (G_0, G_1; \sigma', \tau', \varepsilon', \mu')$ be internal categories in $\mathcal{E}$. Consider the fibered category $p_{\boldsymbol{C}} : \mathcal{F}(\boldsymbol{C}) \to \mathcal{E}$ represented by $\boldsymbol{C}$ given in (8.3.18).*

(1) A representation $((G_0,\rho_0),(id_{G_1},\rho_1))$ of $\boldsymbol{G}$ on (G_0,ρ_0) is a left regular representation if and only if there exists a morphism $(id_{G_0},\eta) : (G_0,u) \to (G_0,\rho_0)$ of $\mathcal{F}(\boldsymbol{C})_{G_0}$ such that, for any internal functor $(f_0,f_1) : \boldsymbol{G} \to \boldsymbol{C}$, a map $\boldsymbol{cat}(\mathcal{E})(\boldsymbol{G},\boldsymbol{C})((\rho_0,\rho_1),(f_0,f_1)) \to \Gamma_{\boldsymbol{C}}(G_0)(u,f_0) = \{\varphi \in \mathcal{E}(G_0,C_1)\,|\,\sigma\varphi = u,\ \tau\varphi = f_0\}$ given by $\varphi \mapsto \mu(\eta,\varphi)$ is bijective.

(2) A representation $((G_0,\rho_0),(id_{G_1},\rho_1))$ of $\boldsymbol{G}$ on (G_0,ρ_0) is a right regular representation if and only if there exists a morphism $(id_{G_0},\varepsilon) : (G_0,\rho_0) \to (G_0,v)$ of $\mathcal{F}(\boldsymbol{C})_{G_0}$ such that, for any internal functor $(f_0,f_1) : \boldsymbol{G} \to \boldsymbol{C}$, a map $\boldsymbol{cat}(\mathcal{E})(\boldsymbol{G},\boldsymbol{C})((f_0,f_1),(\rho_0,\rho_1)) \to \Gamma_{\boldsymbol{C}}(G_0)(f_0,v) = \{\varphi \in \mathcal{E}(G_0,C_1)\,|\,\sigma\varphi = f_0,\ \tau\varphi = v\}$ given by $\varphi \mapsto \mu(\varphi,\varepsilon)$ is bijective.

Proof. (1) It follows from (1) of (9.3.5) and (9.1.18) that (G_0,ρ_0) is a left regular representation if and only if there exists a morphism $(id_{G_0},\eta) : (G_0,u) \to (G_0,\rho_0)$ of $\mathcal{F}(\boldsymbol{C})_{G_0}$ such that, for any internal functor $(f_0,f_1) : \boldsymbol{G} \to \boldsymbol{C}$, the following composition is bijective.

$$\boldsymbol{cat}(\mathcal{E})(\boldsymbol{G},\boldsymbol{C})((\rho_0,\rho_1),(f_0,f_1)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}F} \mathcal{F}(\boldsymbol{C})_{G_0}((G_0,\rho_0),(G_0,f_0)) \xrightarrow{(id_{G_0},\eta)^*} \mathcal{F}(\boldsymbol{C})_{G_0}((G_0,u),(G_0,f_0))$$

The above composition maps $\varphi \in \boldsymbol{cat}(\mathcal{E})(\boldsymbol{G},\boldsymbol{C})((\rho_0,\rho_1),(f_0,f_1))$ to a composition $G_0 \xrightarrow{(\eta,\varphi)} C_1 \times_{C_0} C_1 \xrightarrow{\mu} C_1$.

(2) It follows from (2) of (9.3.5) and (9.1.18) that (G_0,ρ_0) is a right regular representation if and only if there exists a morphism $(id_{G_0},\varepsilon) : (G_0,\rho_0) \to (G_0,v)$ of $\mathcal{F}(\boldsymbol{C})_{G_0}$ such that, for any internal functor $(f_0,f_1) : \boldsymbol{G} \to \boldsymbol{C}$, the following composition is bijective.

$$\boldsymbol{cat}(\mathcal{E})(\boldsymbol{G},\boldsymbol{C})((f_0,f_1),(\rho_0,\rho_1)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}F} \mathcal{F}(\boldsymbol{C})_{G_0}((G_0,f_0),(G_0,\rho_0)) \xrightarrow{(id_{G_0},\varepsilon)_*} \mathcal{F}(\boldsymbol{C})_{G_0}((G_0,f_0),(G_0,v))$$

The above composition maps $\varphi : (f_0,f_1) \to (\rho_0,\rho_1)$ to a composition $G_0 \xrightarrow{(\varphi,\varepsilon)} C_1 \times_{C_0} C_1 \xrightarrow{\mu} C_1$. □

Proposition 9.3.7 *The following assertions hold.*

(1) The forgetful functor $\mathscr{F}_{\boldsymbol{C}} : \operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ has a left adjoint if and only if, for every $L \in \operatorname{Ob}\mathcal{F}_{C_0}$, there exist a representation (M_L,ρ_L) of $\boldsymbol{C}$ and a morphism $\eta_L : L \to \mathscr{F}_{\boldsymbol{C}}(M_L,\rho_L)$ of $\mathcal{F}_{C_0}$ such that, for any $(N,\xi) \in \operatorname{Ob}\operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F})$, the following composition is bijective.

$$\operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F})((M_L,\rho_L),(N,\xi)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(M_L,\rho_L),\mathscr{F}_{\boldsymbol{C}}(N,\xi)) \xrightarrow{\eta_L^*} \mathcal{F}_{C_0}(L,\mathscr{F}_{\boldsymbol{C}}(N,\xi))$$

(2) The forgetful functor $\mathscr{F}_{\boldsymbol{C}} : \operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ has a right adjoint if and only if, for every $R \in \operatorname{Ob}\mathcal{F}_{C_0}$, there exist a representation (M_R,ρ_R) of $\boldsymbol{C}$ and a morphism $\varepsilon_R : \mathscr{F}_{\boldsymbol{C}}(M_R,\rho_R) \to R$ of $\mathcal{F}_{C_0}$ such that, for any $(N,\xi) \in \operatorname{Ob}\operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F})$, the following composition is bijective.

$$\operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F})((N,\xi),(M_R,\rho_R)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N,\xi),\mathscr{F}_{\boldsymbol{C}}(M_R,\rho_R)) \xrightarrow{\varepsilon_{R*}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N,\xi),R)$$

Proof. (1) Suppose that $\mathscr{F}_{\boldsymbol{C}}$ has a left adjoint $\mathscr{L}_{\boldsymbol{C}} : \mathcal{F}_{C_0} \to \operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F})$. Let $\eta : id_{\mathcal{F}_{C_0}} \to \mathscr{F}_{\boldsymbol{C}}\mathscr{L}_{\boldsymbol{C}}$ be the unit of this adjunction. For $L \in \operatorname{Ob}\mathcal{F}_{C_0}$, a representation $\mathscr{L}_{\boldsymbol{C}}(L)$ and a morphism $\eta_L : L \to \mathscr{F}_{\boldsymbol{C}}\mathscr{L}_{\boldsymbol{C}}(L)$ satisfies the condition. In fact, for $(N,\xi) \in \operatorname{Ob}\operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F})$, the composition

$$\operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F})(\mathscr{L}_{\boldsymbol{C}}(L),(N,\xi)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}\mathscr{L}_{\boldsymbol{C}}(L),\mathscr{F}_{\boldsymbol{C}}(N,\xi)) \xrightarrow{\eta_L^*} \mathcal{F}_{C_0}(L,\mathscr{F}_{\boldsymbol{C}}(N,\xi))$$

is the adjoint bijection. We show the converse. Define a functor $\mathscr{L}_{\boldsymbol{C}} : \mathcal{F}_{C_0} \to \operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F})$ as follows. For an object L of $\mathcal{F}_{C_0}$, put $\mathscr{L}_{\boldsymbol{C}}(L) = (M_L,\rho_L)$. For a morphism $\varphi : L \to K$ of $\mathcal{F}_{C_0}$, let $\mathscr{L}_{\boldsymbol{C}}(\varphi) : (M_L,\rho_L) \to (M_K,\rho_K)$ be the morphism of $\operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F})$ which maps to $\eta_K\varphi$ by the composition

$$\operatorname{Rep}(\boldsymbol{C}\,;\mathcal{F})((M_L,\rho_L),(M_K,\rho_K)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(M_L,\rho_L),\mathscr{F}_{\boldsymbol{C}}(M_K,\rho_K)) \xrightarrow{\eta_L^*} \mathcal{F}_{C_0}(L,\mathscr{F}_{\boldsymbol{C}}(M_K,\rho_K)).$$

It is easy to verify that $\mathscr{L}_{\boldsymbol{C}}$ is a functor and that it is a left adjoint of $\mathscr{F}_{\boldsymbol{C}}$.

(2) Suppose that $\mathscr{F}_{\boldsymbol{C}}$ has right adjoint $\mathscr{R}_{\boldsymbol{C}} : \mathcal{F}_{C_0} \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$. Let $\varepsilon : \mathscr{F}_{\boldsymbol{C}}\mathscr{R}_{\boldsymbol{C}} \to id_{\mathcal{F}_{C_0}}$ be the counit of this adjunction. For $R \in \mathrm{Ob}\,\mathcal{F}_{C_0}$, a representation $\mathscr{R}_{\boldsymbol{C}}(R)$ and a morphism $\varepsilon_R : \mathscr{F}_{\boldsymbol{C}}\mathscr{R}_{\boldsymbol{C}}(R) \to R$ satisfies the condition. In fact, for $(N,\xi) \in \mathrm{Ob}\,\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$, the composition

$$\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((N,\xi),\mathscr{R}_{\boldsymbol{C}}(R)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N,\xi),\mathscr{F}_{\boldsymbol{C}}\mathscr{R}_{\boldsymbol{C}}(R)) \xrightarrow{\varepsilon_{R*}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(N,\xi),R)$$

is the adjoint bijection. We show the converse. Define a functor $\mathscr{R}_{\boldsymbol{C}} : \mathcal{F}_{C_0} \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ as follows. For an object R of $\mathcal{F}_{C_0}$, put $\mathscr{R}_{\boldsymbol{C}}(R) = (M_R,\rho_R)$. For a morphism $\varphi : Q \to R$ of $\mathcal{F}_{C_0}$, let $\mathscr{R}_{\boldsymbol{C}}(\varphi) : (M_Q,\rho_Q) \to (M_R,\rho_R)$ be the morphism of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ which maps to $\varphi\varepsilon_Q$ by the composition

$$\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M_Q,\rho_Q),(M_R,\rho_R)) \xrightarrow{\mathscr{F}_{\boldsymbol{C}}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(M_Q,\rho_Q),\mathscr{F}_{\boldsymbol{C}}(M_R,\rho_R)) \xrightarrow{\varepsilon_{R*}} \mathcal{F}_{C_0}(\mathscr{F}_{\boldsymbol{C}}(M_Q,\rho_Q),R).$$

It is easy to verify that $\mathscr{R}_{\boldsymbol{C}}$ is a functor and that it is a right adjoint of $\mathscr{F}_{\boldsymbol{C}}$. □

Proposition 9.3.8 *The following assertions hold.*

(1) *Suppose that* $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ *has a left adjoint* $\mathscr{L}_{\boldsymbol{C}}$*. Let us denote by* η *and* ε *the unit and the counit of this adjunction. Put* $T = \mathscr{F}_{\boldsymbol{C}}\mathscr{L}_{\boldsymbol{C}}$ *and consider the monad* $\boldsymbol{T} = (T,\eta,\mathscr{F}_{\boldsymbol{C}}(\epsilon_{\mathscr{L}_{\boldsymbol{C}}}))$ *associated with this adjunction. Then, the comparision functor* $K : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}^{\boldsymbol{T}}$ *given by* $K(M,\xi) = \langle M,\mathscr{F}_{\boldsymbol{C}}(\varepsilon_{(M,\xi)})\rangle$ *is an isomorphism of categories.*

(2) *Suppose that* $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$ *has a right adjoint* $\mathscr{R}_{\boldsymbol{C}}$*. Let us denote by* η *and* ε *the unit and the counit of this adjunction. Put* $T = \mathscr{F}_{\boldsymbol{C}}\mathscr{R}_{\boldsymbol{C}}$ *and consider the comonad* $\boldsymbol{T} = (T,\varepsilon,\mathscr{F}_{\boldsymbol{C}}(\epsilon_L))$ *associated with this adjunction. Then, the comparision functor* $K : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}^{\boldsymbol{T}}$ *given by* $K(M,\xi) = \langle M,\mathscr{F}_{\boldsymbol{C}}(\eta_{(M,\xi)})\rangle$ *is an isomorphism of categories.*

Proof. (1) Let $(M,\xi) \underset{\psi}{\overset{\varphi}{\rightrightarrows}} (N,\zeta)$ be parallel arrows in $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ such that $\mathscr{F}_{\boldsymbol{C}}(M,\xi) \underset{\mathscr{F}_{\boldsymbol{C}}(\psi)}{\overset{\mathscr{F}_{\boldsymbol{C}}(\varphi)}{\rightrightarrows}} \mathscr{F}_{\boldsymbol{C}}(N,\zeta)$ has a split coequalizer in $\mathcal{F}_{C_0}$. Since σ^* preserves split coequalizers and μ^* preserves split epimorphism, $\mathscr{F}_{\boldsymbol{C}}$ creates the coequalizer of $\mathscr{F}_{\boldsymbol{C}}(M,\xi) \underset{\mathscr{F}_{\boldsymbol{C}}(\psi)}{\overset{\mathscr{F}_{\boldsymbol{C}}(\varphi)}{\rightrightarrows}} \mathscr{F}_{\boldsymbol{C}}(N,\zeta)$ by (2) of (9.1.5). Hence, by the theorem of Beck ([12], p.151) the assertion follows.

(2) Let $(M,\xi) \underset{\psi}{\overset{\varphi}{\rightrightarrows}} (N,\zeta)$ be parallel arrows in $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ such that $\mathscr{F}_{\boldsymbol{C}}(M,\xi) \underset{\mathscr{F}_{\boldsymbol{C}}(\psi)}{\overset{\mathscr{F}_{\boldsymbol{C}}(\varphi)}{\rightrightarrows}} \mathscr{F}_{\boldsymbol{C}}(N,\zeta)$ has a split equalizer in $\mathcal{F}_{C_0}$. Since τ^* preserves split equalizers and μ^* preserves split epimorphism, $\mathscr{F}_{\boldsymbol{C}}$ creates the equalizer of $\mathscr{F}_{\boldsymbol{C}}(M,\xi) \underset{\mathscr{F}_{\boldsymbol{C}}(\psi)}{\overset{\mathscr{F}_{\boldsymbol{C}}(\varphi)}{\rightrightarrows}} \mathscr{F}_{\boldsymbol{C}}(N,\zeta)$ by (1) of (9.1.5). Hence, by the theorem of Beck ([12], p.151) the assertion follows. □

9.4 Representations in fibered categories with products

Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category with products and $\boldsymbol{C} = (C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ an internal category in $\mathcal{E}$.

Proposition 9.4.1 *For* $M \in \mathrm{Ob}\,\mathcal{F}_{C_0}$ *and* $\xi \in \mathcal{F}_{C_1}(\sigma^*(M),\tau^*(M))$*, we put* $\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi) : M_{[\sigma,\tau]} \to M$*. Then,* (M,ξ) *is a representation of* $\boldsymbol{C}$ *on* M *if and only if the following diagram commutes and a composition* $M = M_{[\sigma\varepsilon,\tau\varepsilon]} \xrightarrow{M_\varepsilon} M_{[\sigma,\tau]} \xrightarrow{\hat{\xi}} M$ *coincides with the identity morphism of* M*.*

$$\begin{tikzcd}
M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, "M_\mu"] \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(M)"] & M_{[\sigma,\tau]} \arrow[r, "\hat{\xi}"] & M \\
(M_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\hat{\xi}_{[\sigma,\tau]}"] & M_{[\sigma,\tau]} \arrow[ur, "\hat{\xi}"'] &
\end{tikzcd}$$

Proof. We have $P_{\sigma\mu,\tau\mu}(M)_M(\xi_\mu) = \hat{\xi}M_\mu$ and $P_{\sigma\mathrm{pr}_i,\tau\mathrm{pr}_i}(M)_M(\xi_{\mathrm{pr}_i}) = \hat{\xi}M_{\mathrm{pr}_i}$ for $i = 1, 2$ by (1) of (8.4.6). Hence (8.4.3), (8.4.6), (8.4.8), (8.4.15) imply

$$\begin{aligned}
P_{\sigma\mu,\tau\mu}(M)_M(\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}) &= P_{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)_M(\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}) = \hat{\xi}M_{\mathrm{pr}_2}(\hat{\xi}M_{\mathrm{pr}_1})_{[\sigma\mathrm{pr}_2,\tau\mathrm{pr}_2]}\delta_{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1,\tau\mathrm{pr}_2,M}\\
&= \hat{\xi}\hat{\xi}_{[\sigma,\tau]}(M_{[\sigma,\tau]})_{\mathrm{pr}_2}(M_{\mathrm{pr}_1})_{[\sigma\mathrm{pr}_2,\tau\mathrm{pr}_2]}\delta_{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1,\tau\mathrm{pr}_2,M} = \hat{\xi}\hat{\xi}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M)\\
\xi_\varepsilon &= P_{id_{C_0},id_{C_0}}(M)_M(\xi_\varepsilon) = P_{\sigma\varepsilon,\tau\varepsilon}(M)_M(\xi_\varepsilon) = \hat{\xi}M_\varepsilon
\end{aligned}$$

Thus $\xi_\mu = \xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}$ and $\xi_\varepsilon = id_M$ are equivalent to $\hat{\xi}\hat{\xi}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) = \hat{\xi}M_\mu$ and $\hat{\xi}M_\varepsilon = id_M$, respectively. □

Remark 9.4.2 *If we denote $M_{[\sigma,\tau]}$ by $M\times\boldsymbol{C}$ and $M = M^{[id_{C_0},id_{C_0}]}$ by $M\times 1$, $\hat{\xi} : M\times\boldsymbol{C}\to M$ can be regarded as a right action of $\boldsymbol{C}$ on M and $M_\varepsilon : M\times 1\to M\times\boldsymbol{C}$ which is denoted by $M\times\varepsilon$ can be regarded as the unital morphism. Then the equality $\hat{\xi}(M\times\varepsilon) = id_M$ means that the right action $\hat{\xi}$ is untary. Moreover, if we denote $M\times\mu : M\times(\boldsymbol{C}\times\boldsymbol{C})\to M\times\boldsymbol{C}$ instead of $M_\mu : M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}\to M_{[\sigma,\tau]}$ and denote $\hat{\xi}\times id_{\boldsymbol{C}} : (M\times\boldsymbol{C})\times\boldsymbol{C}\to M\times\boldsymbol{C}$ instead of $\hat{\xi}_{[\sigma,\tau]} : (M_{[\sigma,\tau]})_{[\sigma,\tau]}\to M_{[\sigma,\tau]}$, the fact that the following diagram commutes means that the right action $\hat{\xi} : M\times\boldsymbol{C}\to M$ of $\boldsymbol{C}$ is associative.*

$$\begin{tikzcd}
M\times(\boldsymbol{C}\times\boldsymbol{C}) \arrow[r, "M\times\mu"] \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(M)"] & M\times\boldsymbol{C} \arrow[r, "\hat{\xi}"] & M\\
(M\times\boldsymbol{C})\times\boldsymbol{C} \arrow[r, "\hat{\xi}\times id_{\boldsymbol{C}}"] & M\times\boldsymbol{C} \arrow[ur, "\hat{\xi}"'] &
\end{tikzcd}$$

For morphisms $f : X\to Y$, $g : X\to Z$ of $\mathcal{E}$, we define a functor $D_{f,g} : \mathcal{Q}\to\mathcal{E}$ by $D_{f,g}(0) = X$, $D_{f,g}(1) = Y$, $D_{f,g}(2) = Z$, $D_{f,g}(\tau_{01}) = f$, $D_{f,g}(\tau_{02}) = g$. If $h : Y\to V$, $i : Z\to W$ are morphisms of $\mathcal{E}$, we define a natural transformation $\omega(f,g\,;h,i) : D_{f,g}\to D_{hf,ig}$ by $\omega(f,g\,;h,i)_0 = id_X$, $\omega(f,g\,;h,i)_1 = h$, $\omega(f,g\,;h,i)_2 = i$.

Proposition 9.4.3 *Let $(s(C_0), s_{\boldsymbol{C}})$ be the trivial representation associated with a cartesian section $s : \mathcal{E}\to\mathcal{F}$. Put $T = s(1)$. The image of $s_{\boldsymbol{C}}\in\mathcal{F}_{C_1}(\sigma^*s(C_0),\tau^*s(C_0))$ by $P_{\sigma,\tau}(s(C_0))_{s(C_0)} : \mathcal{F}_{C_1}(\sigma^*s(C_0),\tau^*s(C_0))\to\mathcal{F}_{C_0}(s(C_0)_{[\sigma,\tau]}, s(C_0))$ is $o_{C_0}^*(P_{o_{C_1},o_{C_1}}(T)_T(id_{s(C_1)}))\omega(\sigma,\tau\,;o_{C_0},o_{C_0})_T$.*

Proof. It follows from (8.1.26) and the definition of $s_{\boldsymbol{C}}$ that we have $s_{\boldsymbol{C}} = c_{o_{C_0},\tau}(T)^{-1}c_{o_{C_0},\sigma}(T)$. We note that $o_{C_0}\sigma = o_{C_0}\tau = o_{C_1}$ and $s(C_i) = o_{C_i}^*(T)$ for $i = 0, 1$. The following diagram is commutative by (8.4.27).

$$\begin{tikzcd}
\mathcal{F}_{C_1}(s(C_1), s(C_1)) \arrow[r, "c_{o_{C_0},\tau}(T)_*^{-1}"] \arrow[d, "P_{o_{C_1},o_{C_1}}(T)_T"] & \mathcal{F}_{C_1}(s(C_1),\tau^*(s(C_0))) \arrow[r, "c_{o_{C_0},\sigma}(T)^*"] & \mathcal{F}_{C_1}(\sigma^*(s(C_0)),\tau^*(s(C_0))) \arrow[d, "P_{\sigma,\tau}(s(C_0))_{s(C_0)}"]\\
\mathcal{F}_1(T_{[o_{C_1},o_{C_1}]}, T) \arrow[r, "o_{C_0}^*"] & \mathcal{F}_{C_0}(o_{C_0}^*(T_{[o_{C_1},o_{C_1}]}), s(C_0)) \arrow[r, "\omega(\sigma,\tau\,;o_{C_0},o_{C_0})_T^*"] & \mathcal{F}_{C_0}(s(C_0)_{[\sigma,\tau]}, s(C_0))
\end{tikzcd}$$

Hence we have $P_{\sigma,\tau}(s(C_0))_{s(C_0)}(s_{\boldsymbol{C}}) = o_{C_0}^*(P_{o_{C_1},o_{C_1}}(T)_T(id_{s(C_1)}))\omega(\sigma,\tau\,;o_{C_0},o_{C_0})_T$. □

Proposition 9.4.4 *Let $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{D}\to\boldsymbol{C}$ be an internal functor and (M,ξ) a representation of $\boldsymbol{C}$. We denote by $\sigma',\tau' : D_1\to D_0$ the source and target of $\boldsymbol{D}$, respectively. Then,*

$$P_{\sigma',\tau'}(f_0^*(M))_{f_0^*(M)}(\xi_{\boldsymbol{f}}) = f_0^*(\hat{\xi}M_{f_1})\,\omega(\sigma',\tau'; f_0, f_0)_M.$$

Proof. The upper rectangle of the following diagram is commutative by (1) of (8.4.6) and the lower one is commutative (8.4.27).

$$\begin{tikzcd}
\mathcal{F}_{C_1}(\sigma^*(M),\tau^*(M)) \arrow[r, "P_{\sigma,\tau}(M)_M"] \arrow[d, "f_1^\sharp"] & \mathcal{F}_{C_0}(M_{[\sigma,\tau]}, M) \arrow[d, "M_{f_1}^*"]\\
\mathcal{F}_{D_1}((f_0\sigma')^*(M),(f_0\tau')^*(M)) \arrow[r, "P_{f_0\sigma',f_0\tau'}(M)_M"] \arrow[d, "c_{f_0,\tau'}(M)_*^{-1}"] & c\mathcal{F}_{C_0}(M_{[f_0\sigma',f_0\tau']}, M) \arrow[d, "f_0^*"]\\
\mathcal{F}_{D_1}((f_0\sigma')^*(M),\tau'(f_0^*(M))) \arrow[d, "c_{f_0,\sigma'}(M)^*"] & \mathcal{F}_{D_0}(f_0^*(M_{[f_0\sigma',f_0\tau']}), f_0^*(M)) \arrow[d, "\omega(\sigma',\tau';f_0,f_0)_M^*"]\\
\mathcal{F}_{D_1}(\sigma'^*(f_0^*(M)),\tau'^*(f_0^*(M))) \arrow[r, "P_{\sigma',\tau'}(f_0^*(M))_{f_0^*(M)}"] & \mathcal{F}_{D_0}(f_0^*(M)_{[\sigma',\tau']}, f_0^*(M))
\end{tikzcd}$$

The assertion follows from the above diagram and the definition of $\xi_{\boldsymbol{f}}$. $\square$

The following fact is a direct consequence of (8.4.5).

Proposition 9.4.5 *Let (M,ξ) and (N,ζ) be representations of $\boldsymbol{C}$ and $\varphi : M \to N$ a morphism of $\mathcal{F}_{C_0}$. We put $\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi)$ and $\hat{\zeta} = P_{\sigma,\tau}(N)_N(\zeta)$. Then, φ is a morphism of representations if and only if the following diagram is commutative.*

$$\begin{tikzcd} M_{[\sigma,\tau]} \arrow[r, "\hat{\xi}"] \arrow[d, "\varphi_{[\sigma,\tau]}"] & M \arrow[d, "\varphi"] \\ N_{[\sigma,\tau]} \arrow[r, "\hat{\zeta}"] & N \end{tikzcd}$$

Example 9.4.6 *Let $\boldsymbol{C}$ be an internal category in $\mathcal{E}$. For an object $M = (\pi : X \to C_0)$ of $\mathcal{E}^{(2)}_{C_0}$, consider a limit $X \xleftarrow{\sigma_\pi} X \times^{\sigma}_{C_0} C_1 \xrightarrow{\pi_\sigma} C_1$ of a diagram $X \xrightarrow{\pi} C_0 \xleftarrow{\sigma} C_1$. We also consider a limit $X \xleftarrow{\tau_\pi} X \times^{\tau}_{C_0} C_1 \xrightarrow{\pi_\tau} C_1$ of a diagram $X \xrightarrow{\pi} C_0 \xleftarrow{\tau} C_1$. Then, we have $\sigma^*(M) = (\pi_\sigma : X \times^{\sigma}_{C_0} C_1 \to C_1)$, $\tau^*(M) = (\pi_\tau : X \times^{\tau}_{C_0} C_1 \to C_1)$ and $M_{[\sigma,\tau]} = (\tau\pi_\sigma : X \times^{\sigma}_{C_0} C_1 \to C_0)$. We note that, for $\xi \in \mathcal{E}(X \times^{\sigma}_{C_0} C_1, X \times^{\tau}_{C_0} C_1)$, $(\xi, id_{C_1}) \in \mathcal{E}^{(2)}_{C_1}(\sigma^*(M), \tau^*(M))$ if and only if $\pi_\tau\xi = \pi_\sigma$ and that a map $G : \mathcal{E}^{(2)}_{C_0}(M_{[\sigma,\tau]}, M) \to \mathcal{E}(X \times^{\sigma}_{C_0} C_1, X)$ defined by $G(\xi, id_{C_0}) = \tau_\pi\xi$ maps $\mathcal{E}^{(2)}_{C_0}(M_{[\sigma,\tau]}, M)$ injectively onto $\{\alpha \in \mathcal{E}(X \times^{\sigma}_{C_0} C_1, X) \,|\, \pi\alpha = \tau\pi_\sigma\}$. Since*

$$P_{\sigma,\tau}(M)_M : \mathcal{E}^{(2)}_{C_1}(\sigma^*(M), \tau^*(M)) \to \mathcal{E}^{(2)}_{C_0}(M_{[\sigma,\tau]}, M)$$

maps (ξ, id_{C_1}) to $(\tau_\pi\xi, id_{C_0})$, it follows from (9.4.1) that $(M, (\xi, id_{C_1}))$ is a representation of $\boldsymbol{C}$ if and only if $(\pi : X \to C_0,\ \tau_\pi\xi : X \times^{\sigma}_{C_0} C_1 \to X)$ is an internal diagram on $\boldsymbol{C}$. Conversely, for an internal diagram $(\pi : X \to C_0,\ \alpha : X \times^{\sigma}_{C_0} C_1 \to X)$, since $\pi\alpha = \tau\pi_\sigma$, there exists a unique morphism $\xi : X \times^{\sigma}_{C_0} C_1 \to X \times^{\tau}_{C_0} C_1$ of $\mathcal{E}$ that satisfies $\tau_\pi\xi = \alpha$ and $\pi_\tau\xi = \pi_\sigma$. Hence $(M, (\xi, id_{C_1}))$ is a representation of $\boldsymbol{C}$. It can be verified from (9.4.5) that a morphism $(\varphi, id_{C_0}) : M = (\pi : X \to C_0) \to (\rho : Y \to C_0) = N$ of $\mathcal{E}^{(2)}_{C_0}$ defines a morphism of representations $(M, (\xi, id_{C_1})) \to (N, (\zeta, id_{C_1}))$ if and only if $\varphi : X \to Y$ defines a morphism of internal diagrams from $(\pi : X \to C_0,\ \alpha : X \times^{\sigma}_{C_0} C_1 \to X)$ to $(\pi : Y \to C_0,\ \beta : X \times^{\sigma}_{C_0} C_1 \to Y)$. Therefore $\mathrm{Rep}(\boldsymbol{C}; \mathcal{E}^{(2)})$ is isomorphic to the category of internal diagrams $\mathcal{E}^{\boldsymbol{C}}$ on $\boldsymbol{C}$.

We use the same symbols as in (9.4.6). Let $(\pi : X \to C_0,\ \alpha : X \times^{\sigma}_{C_0} C_1 \to X)$ be an internal diagram on $\boldsymbol{C}$. Let $X \times^{\sigma}_{C_0} C_1 \xleftarrow{\tilde{\mathrm{pr}}_{12}} X \times^{\sigma}_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\tilde{\mathrm{pr}}_{23}} C_1 \times_{C_0} C_1$ be a limit of $X \times^{\sigma}_{C_0} C_1 \xrightarrow{\pi_\sigma} C_1 \xleftarrow{\mathrm{pr}_1} C_1 \times_{C_0} C_1$. Then, $X \xleftarrow{\sigma_\pi\tilde{\mathrm{pr}}_{12}} X \times^{\sigma}_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\tilde{\mathrm{pr}}_{23}} C_1 \times_{C_0} C_1$ is a limit of $X \xrightarrow{\pi} C_0 \xleftarrow{\sigma\mathrm{pr}_1} C_1 \times_{C_0} C_1$. We also note that $X \times^{\sigma}_{C_0} C_1 \xleftarrow{\tilde{\mathrm{pr}}_{12}} X \times^{\sigma}_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}} C_1$ is a limit of $X \times^{\sigma}_{C_0} C_1 \xrightarrow{\tau\pi_\sigma} C_0 \xleftarrow{\sigma} C_1$.

$$\begin{tikzcd} X \times^{\sigma}_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "\tilde{\mathrm{pr}}_{23}"] \arrow[d, "\tilde{\mathrm{pr}}_{12}"] & C_1 \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & C_1 \arrow[d, "\sigma"] \\ X \times^{\sigma}_{C_0} C_1 \arrow[r, "\pi_\sigma"] \arrow[d, "\sigma_\pi"] & C_1 \arrow[r, "\tau"] \arrow[d, "\sigma"] & C_0 \\ X \arrow[r, "\pi"] & C_0 & \end{tikzcd}$$

Define a functor $D_\alpha : \mathcal{P} \to \mathcal{E}$ by $D_\alpha(0) = X \times^{\sigma}_{C_0} C_1$, $D_\alpha(1) = C_1$, $D_\alpha(2) = X$, $D_\alpha(3) = D_\alpha(4) = D_\alpha(5) = C_0$ and $D_\alpha(\tau_{01}) = \pi_\sigma$, $D_\alpha(\tau_{02}) = \alpha$, $D_\alpha(\tau_{13}) = \sigma$, $D_\alpha(\tau_{14}) = \tau$, $D_\alpha(\tau_{24}) = D_\alpha(\tau_{25}) = \pi$. For a representation (M,ξ) of $\boldsymbol{C}$, we put $\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi)$. Assume that $\theta_{\pi,\pi,\sigma,\tau}(M) : M_{[\pi\sigma_\pi,\,\tau\pi_\sigma]} \to (M_{[\pi,\pi]})_{[\sigma,\tau]}$ is an isomorphism and define a morphism $\hat{\xi}_\alpha : (M_{[\pi,\pi]})_{[\sigma,\tau]} \to M_{[\pi,\pi]}$ to be the following composition.

$$(M_{[\pi,\pi]})_{[\sigma,\tau]} \xrightarrow{\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}} M_{[\pi\sigma_\pi,\,\tau\pi_\sigma]} = M_{[\sigma\pi_\sigma,\,\pi\alpha]} \xrightarrow{\theta_{D_\alpha}(M)} (M_{[\sigma,\tau]})_{[\pi,\pi]} \xrightarrow{\hat{\xi}_{[\pi,\pi]}} M_{[\pi,\pi]}$$

Proposition 9.4.7 *Assume that $\theta_{\pi,\pi,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M) : M_{[\pi\sigma_\pi\tilde{\mathrm{pr}}_{12},\,\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \to (M_{[\pi,\pi]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an epimorphism. Put $P_{\sigma,\tau}(M_{[\pi,\pi]})^{-1}_{M_{[\pi,\pi]}}(\hat{\xi}_\alpha) = \xi_\alpha$. Then, $(M_{[\pi,\pi]}, \xi_\alpha)$ is a representation of $\boldsymbol{C}$ and $M_\pi : (M_{[\pi,\pi]}, \xi_\alpha) \to (M,\xi)$ is a morphism of representations.*

Proof. The left rectangle of the following diagram is commutative by (8.4.22) and the right rectangle is commutative by (8.4.19).

$$\begin{tikzcd}
(M_{[\pi\sigma_\pi,\,\tau\pi_\sigma]})_{[\sigma,\tau]} \arrow[d, "\theta_{\pi,\pi,\sigma,\tau}(M)_{[\sigma,\tau]}"] & M_{[\pi\sigma_\pi\tilde{\mathrm{pr}}_{12},\,\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[l, "\theta_{\pi\sigma_\pi,\,\tau\pi_\sigma,\sigma,\tau}(M)"'] \arrow[r, "M_{id_X\times_{C_0}\mu}"] \arrow[d, "\theta_{\pi,\pi,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)"] & M_{[\pi\sigma_\pi,\,\tau\pi_\sigma]} \arrow[d, "\theta_{\pi,\pi,\sigma,\tau}(M)"] \\
((M_{[\pi,\pi]})_{[\sigma,\tau]})_{[\sigma,\tau]} & (M_{[\pi,\pi]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[l, "\theta_{\sigma,\tau,\sigma,\tau}(M_{[\pi,\pi]})"'] \arrow[r, "(M_{[\pi,\pi]})_\mu"] & (M_{[\pi,\pi]})_{[\sigma,\tau]}
\end{tikzcd}$$

Since $\pi\alpha=\tau\pi_\sigma$, $\pi_\sigma(\alpha\times_{C_0} id_{C_1})=\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}$ and $\alpha(\alpha\times_{C_0} id_{C_1})=\alpha(id_X\times_{C_0}\mu)$, we can define functors $E,F:\mathcal{P}\to\mathcal{E}$ and a natural transformation $\lambda:E\to D_\alpha$ by $E(0)=F(0)=X\times^{\sigma}_{C_0}C_1\times_{C_0}C_1$, $E(1)=C_1\times_{C_0}C_1$, $F(1)=C_1$, $E(2)=X$, $F(2)=X\times^{\sigma}_{C_0}C_1$, $E(i)=F(i)=C_0$ for $i=3,4,5$, $E(\tau_{01})=\tilde{\mathrm{pr}}_{23}$, $F(\tau_{01})=\mathrm{pr}_1\tilde{\mathrm{pr}}_{23}$, $E(\tau_{02})=\alpha(\alpha\times_{C_0} id_{C_1})$, $F(\tau_{02})=\alpha\times_{C_0} id_{C_1}$, $E(\tau_{13})=\sigma\mathrm{pr}_1$, $F(\tau_{13})=\sigma$, $E(\tau_{14})=\tau\mathrm{pr}_2$, $F(\tau_{14})=\tau$, $E(\tau_{24})=\pi$, $F(\tau_{24})=\sigma\pi_\sigma$, $E(\tau_{25})=\pi$, $F(\tau_{25})=\pi\alpha$ and $\lambda_0=id_X\times_{C_0}\mu$, $\lambda_1=\mu$, $\lambda_2=id_X$, $\lambda_3=\lambda_4=\lambda_5=id_{C_0}$. We also note that $\mathrm{pr}_1\tilde{\mathrm{pr}}_{23}=\pi_\sigma\tilde{\mathrm{pr}}_{12}$. Then, the following diagram commutes by (8.4.21)

$$\begin{tikzcd}
(M_{[\sigma\pi_\sigma,\,\pi\alpha]})_{[\sigma,\tau]} \arrow[d, "\theta_{D_\alpha}(M)_{[\sigma,\tau]}"] & M_{[\sigma\pi_\sigma\tilde{\mathrm{pr}}_{12},\,\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[l, "\theta_{\sigma\pi_\sigma,\pi\alpha,\sigma,\tau}(M)"'] \arrow[r, "\theta_E(M)"] \arrow[d, "\theta_F(M)"] & (M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]})_{[\pi,\pi]} \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(M)_{[\pi,\pi]}"] \\
((M_{[\sigma,\tau]})_{[\pi,\pi]})_{[\sigma,\tau]} & (M_{[\sigma,\tau]})_{[\pi\sigma_\pi,\,\tau\pi_\sigma)} \arrow[l, "\theta_{\pi,\pi,\sigma,\tau}(M_{[\sigma,\tau]})"'] \arrow[r, "\theta_{D_\alpha}(M_{[\sigma,\tau]})"] & ((M_{[\sigma,\tau]})_{[\sigma,\tau]})_{[\pi,\pi]}
\end{tikzcd}$$

and the following diagram commutes by (8.4.18).

$$\begin{tikzcd}
M_{[\sigma\pi_\sigma\tilde{\mathrm{pr}}_{12},\,\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "M_{id_X\times_{C_0}\mu}"] \arrow[d, "\theta_E(M)"] & M_{[\sigma\pi_\sigma,\,\pi\alpha]} \arrow[d, "\theta_{D_\alpha}(M)"] \\
(M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]})_{[\pi,\pi]} \arrow[r, "(M_\mu)_{[\pi,\,\pi]}"] & (M_{[\sigma,\,\tau]})_{[\pi,\,\pi]}
\end{tikzcd}$$

It follows from the above facts and (8.4.17), (8.4.19), (9.4.1) that the following diagram is commutative

$$\begin{tikzcd}
((M_{[\pi,\pi]})_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[d, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}_{[\sigma,\tau]}"] & & (M_{[\pi,\pi]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[ll, "\theta_{\sigma,\tau,\sigma,\tau}(M_{[\pi,\pi]})"'] \arrow[rr, "(M_{[\pi,\pi]})_\mu"] & & (M_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[d, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}"'] \\
(M_{[\pi\sigma_\pi,\,\tau\pi_\sigma]})_{[\sigma,\tau]} \arrow[d, equal] & & M_{[\pi\sigma_\pi\tilde{\mathrm{pr}}_{12},\,\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[u, "\theta_{\pi,\pi,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)"'] \arrow[ll, "\theta_{\pi\sigma_\pi,\,\tau\pi_\sigma,\sigma,\tau}(M)"'] \arrow[rr, "M_{id_X\times_{C_0}\mu}"] \arrow[d, equal] & & M_{[\pi\sigma_\pi,\,\tau\pi_\sigma]} \arrow[d, equal] \\
(M_{[\sigma\pi_\sigma,\,\pi\alpha]})_{[\sigma,\tau]} \arrow[d, "\theta_{D_\alpha}(M)_{[\sigma,\tau]}"] & & M_{[\sigma\pi_\sigma\tilde{\mathrm{pr}}_{12},\,\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[ll, "\theta_{\sigma\pi_\sigma,\pi\alpha,\sigma,\tau}(M)"'] \arrow[rr, "M_{id_X\times_{C_0}\mu}"] \arrow[d, "\theta_E(M)"] \arrow[dddl, "\theta_F(M)"'] & & M_{[\sigma\pi_\sigma,\,\pi\alpha]} \arrow[d, "\theta_{D_\alpha}(M)"] \\
((M_{[\sigma,\tau]})_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[dd, "(\hat{\xi}_{[\pi,\pi]})_{[\sigma,\tau]}"] & & (M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]})_{[\pi,\pi]} \arrow[rr, "(M_\mu)_{[\pi,\pi]}"] \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(M)_{[\pi,\pi]}"] & & (M_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[d, "\hat{\xi}_{[\pi,\pi]}"] \\
& & ((M_{[\sigma,\tau]})_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[dr, "(\hat{\xi}_{[\sigma,\tau]})_{[\pi,\pi]}"] & & M_{[\pi,\pi]} \\
(M_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[dr, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}"'] & (M_{[\sigma,\tau]})_{[\sigma\pi_\sigma,\,\pi\alpha]} \arrow[uul, "\theta_{\pi,\pi,\sigma,\tau}(M_{[\sigma,\tau]})"] \arrow[ur, "\theta_{D_\alpha}(M_{[\sigma,\tau]})"'] \arrow[dr, "\hat{\xi}_{[\sigma\pi_\sigma,\,\pi\alpha]}"] & & (M_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[ur, "\hat{\xi}_{[\pi,\pi]}"] & \\
& M_{[\pi\sigma_\pi,\,\tau\pi_\sigma]} \arrow[r, equal] & M_{[\sigma\pi_\sigma,\,\pi\alpha]} \arrow[ur, "\theta_{D_\alpha}(M)"] & &
\end{tikzcd}$$

Hence $\hat{\xi}_\alpha$ make the diagram of (9.4.1) commute.

Since functors $D_{\pi,\pi,id_{C_0},id_{C_0}}, D_{id_{C_0},id_{C_0},\pi,\pi} : \mathcal{P} \to \mathcal{E}$ are given by

$$\begin{aligned}
D_{\pi,\pi,id_{C_0},id_{C_0}}(i) = D_{id_{C_0},id_{C_0},\pi,\pi}(j) &= X \quad (i = 0,1,\ j = 0,2),\\
D_{\pi,\pi,id_{C_0},id_{C_0}}(i) = D_{id_{C_0},id_{C_0},\pi,\pi}(j) &= C_0 \quad (i = 2,3,4,5,\ j = 1,3,4,5),\\
D_{\pi,\pi,id_{C_0},id_{C_0}}(\tau_{01}) = D_{id_{C_0},id_{C_0},\pi,\pi}(\tau_{02}) &= id_X,\\
D_{\pi,\pi,id_{C_0},id_{C_0}}(\tau_{ij}) = D_{id_{C_0},id_{C_0},\pi,\pi}(\tau_{kl}) &= \pi \quad ((i,j) = (0,2),(1,3),(1,4),(k,l) = (0,1),(1,3),(1,4)),\\
D_{\pi,\pi,id_{C_0},id_{C_0}}(\tau_{2j}) = D_{id_{C_0},id_{C_0},\pi,\pi}(\tau_{2j}) &= id_{C_0} \quad (j = 3,4,5),
\end{aligned}$$

we define natural transformations $\nu : D_{\pi,\pi,id_{C_0},id_{C_0}} \to D_{\pi,\pi,\sigma,\tau}$ and $\kappa : D_{id_{C_0},id_{C_0},\pi,\pi} \to D_\alpha$ by $\nu_0 = \kappa_0 = (id_X, \varepsilon\pi) : X \to X \times^{\sigma}_{C_0} C_1$, $\nu_1 = \kappa_2 = id_X$, $\nu_2 = \kappa_1 = \varepsilon$, $\nu_i = \kappa_i = id_{C_0}$ $(i = 3,4,5)$. Then, the following diagram is commutative by (8.4.17), (8.4.19).

$$\begin{tikzcd}
(M_{[\pi,\pi]})_{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, "\theta_{\pi,\pi,id_{C_0},id_{C_0}}(M)^{-1}"] \arrow[d, "(M_{[\pi,\pi]})_\varepsilon"] & M_{[\pi id_X,\, \tau\varepsilon\pi]} \arrow[r, equal] \arrow[d, "M_{(id_X,\varepsilon\pi)}"] & M_{[id_{C_0}\pi,\, \pi id_X]} \arrow[r, "\theta_{id_{C_0},id_{C_0},\pi,\pi}(M)"] & (M_{[id_{C_0},id_{C_0}]})_{[\pi,\,\pi]} \arrow[d, "(M_\varepsilon)_{[\pi,\,\pi]}"] \\
(M_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[r, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}"] & M_{[\pi\sigma_\pi,\, \tau\pi_\sigma]} \arrow[r, equal] & M_{[\sigma\pi_\sigma,\, \pi\alpha]} \arrow[r, "\theta_{D_\alpha}(M)"] & (M_{[\sigma,\tau]})_{[\pi,\pi]}
\end{tikzcd}$$

The upper row of the above diagram is identified with the identity morphism of $M_{[\pi,\pi]}$. Since $\hat{\xi}M_\varepsilon$ is the identity morphism of M by (9.4.1), $\hat{\xi}_{[\pi,\pi]}(M_\varepsilon)_{[\pi,\,\pi]}$ is the identity morphism of $M_{[\pi,\,\pi]}$. It follows from the above facts and the definition of $\hat{\xi}_\alpha$ that $M_{[\pi,\pi]} = (M_{[\pi,\pi]})_{[\sigma\varepsilon,\tau\varepsilon]} \xrightarrow{(M_{[\pi,\pi]})_\varepsilon} (M_{[\pi,\pi]})_{[\sigma,\tau]} \xrightarrow{\hat{\xi}_\alpha} M_{[\pi,\pi]}$ coincides with the identity morphism of $M_{[\pi,\pi]}$.

By (8.4.8) and (8.4.17), (8.4.19), the following diagram is commutative.

$$\begin{tikzcd}
(M_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[r, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}"] \arrow[d, "(M_\pi)_{[\sigma,\tau]}"] & M_{[\pi\sigma_\pi,\tau\pi_\sigma]} = M_{[\sigma\pi_\sigma,\pi\alpha]} \arrow[r, "\theta_{D_\alpha}(M)"] \arrow[d, "M_{\pi_\sigma}"] & (M_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[r, "\hat{\xi}_{[\pi,\pi]}"] \arrow[d, "(M_{[\sigma,\tau]})_\pi"] & M_{[\pi,\pi]} \arrow[d, "M_\pi"] \\
(M_{[id_{C_0},id_{C_0}]})_{[\sigma,\tau]} \arrow[r, "\theta_{id_{C_0},id_{C_0},\sigma,\tau}(M)^{-1}"] & M_{[id_{C_0}\sigma,\tau id_{C_1}]} = M_{[\sigma id_{C_1},id_{C_0}\tau]} \arrow[r, "\theta_{\sigma,\tau,id_{C_0},id_{C_0}}(M)"] & (M_{[\sigma,\tau]})_{[id_{C_0},id_{C_0}]} \arrow[r, "\hat{\xi}"] & M
\end{tikzcd}$$

Therefore $M_\pi : (M_{[\pi,\pi]}, \xi_\alpha) \to (M, \xi)$ is a morphism of representations by (9.4.5). □

Proposition 9.4.8 *Let $\varphi : (M, \xi) \to (N, \zeta)$ be a morphism of representations of $\boldsymbol{C}$. Assume that the following left morphism is an isomorphism for $L = M, N$ and that the right morphism is an epimorphisms for $L = M, N$*

$$\theta_{\pi,\pi,\sigma,\tau}(L) : L_{[\pi\sigma_\pi,\, \tau\pi_\sigma]} \to (L_{[\pi,\pi]})_{[\sigma,\tau]}, \quad \theta_{\pi,\pi,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(L) : L_{[\pi\sigma_\pi\tilde{\mathrm{pr}}_{12},\, \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \to (L_{[\pi,\pi]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$$

Then, $\varphi_{[\pi,\pi]} : M_{[\pi,\pi]} \to N_{[\pi,\pi]}$ gives a morphism of representations from $(M_{[\pi,\pi]}, \xi_\alpha)$ to $(N_{[\pi,\pi]}, \zeta_\alpha)$.

Proof. The following diagram is commutative by (8.4.3) and (8.4.17).

$$\begin{tikzcd}
(M_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[r, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}"] \arrow[d, "(\varphi_{[\pi,\pi]})_{[\sigma,\tau]}"] & M_{[\pi\sigma_\pi,\, \tau\pi_\sigma]} = M_{[\sigma\pi_\sigma,\, \pi\alpha]} \arrow[r, "\theta_{D_\alpha}(M)"] \arrow[d, "\varphi_{[\pi\sigma_\pi,\, \tau\pi_\sigma]}"] & (M_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[r, "\hat{\xi}_{[\pi,\pi]}"] \arrow[d, "(\varphi_{[\sigma,\tau]})_{[\pi,\pi]}"] & M_{[\pi,\pi]} \arrow[d, "\varphi_{[\pi,\pi]}"] \\
(N_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[r, "\theta_{\pi,\pi,\sigma,\tau}(N)^{-1}"] & N_{[\pi\sigma_\pi,\, \tau\pi_\sigma]} = N_{[\sigma\pi_\sigma,\, \pi\alpha]} \arrow[r, "\theta_{D_\alpha}(N)"] & (N_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[r, "\hat{\zeta}_{[\pi,\pi]}"] & N_{[\pi,\pi]}
\end{tikzcd}$$

Hence the assertion follows. □

Proposition 9.4.9 *Let $(\pi : X \to C_0,\ \alpha : X \times^{\sigma}_{C_0} C_1 \to X)$ and $(\rho : Y \to C_0,\ \beta : Y \times^{\sigma}_{C_0} C_1 \to Y)$ be internal diagrams on $\boldsymbol{C}$ and (M, ξ) a representation of $\boldsymbol{C}$. Assume that the following left morphism is an isomorphism for $\chi = \pi, \rho$ and that the right morphism is an epimorphism for $\chi = \pi, \rho$.*

$$\theta_{\chi,\chi,\sigma,\tau}(M) : M_{[\chi\sigma_\chi,\, \tau\chi_\sigma]} \to (M_{[\chi,\chi]})_{[\sigma,\tau]}, \quad \theta_{\chi,\chi,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M) : M_{[\chi\sigma_\chi\tilde{\mathrm{pr}}_{12},\, \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \to (M_{[\chi,\chi]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$$

If a morphism $f : X \to Y$ of $\mathcal{E}$ defines a morphism of internal diagrams from $(\pi : X \to C_0, \alpha)$ to $(\rho : Y \to C_0, \beta)$, $M_f : M_{[\pi,\pi]} \to M_{[\rho,\rho]}$ is a morphism of representations from $(M_{[\pi,\pi]}, \xi_\alpha)$ to $(M_{[\rho,\rho]}, \xi_\beta)$.

Proof. Define a natural transformation $\lambda : D_\alpha \to D_\beta$ by $\lambda_0 = f \times_{C_0} id_{C_1}$, $\lambda_1 = id_{C_1}$, $\lambda_2 = f$, $\lambda_i = id_{C_0}$ $(i = 3, 4, 5)$. The following diagram is commutative by (8.4.6) and (8.4.18).

$$\begin{tikzcd}
(M_{[\pi,\pi]})_{[\sigma,\tau]} \arrow[r, "\theta_{\pi,\pi,\sigma,\tau}(M)^{-1}"] \arrow[d, "(M_f)_{[\sigma,\tau]}"] & M_{[\pi\sigma_\pi,\,\tau\pi_\sigma]} = M_{[\sigma\pi_\sigma,\,\pi\alpha]} \arrow[r, "\theta_{D_\alpha}(M)"] \arrow[d, "M_{f\times_{C_0} id_{C_1}}"] & (M_{[\sigma,\tau]})_{[\pi,\pi]} \arrow[r, "\hat{\xi}_{[\pi,\pi]}"] \arrow[d, "(M_{[\sigma,\tau]})_f"] & M_{[\pi,\pi]} \arrow[d, "M_f"] \\
(M_{[\rho,\rho]})_{[\sigma,\tau]} \arrow[r, "\theta_{\rho,\rho,\sigma,\tau}(M)^{-1}"] & M_{[\rho\sigma_\rho,\,\tau\rho_\sigma]} = M_{[\sigma\rho_\sigma,\,\rho\beta]} \arrow[r, "\theta_{D_\beta}(M)"] & (M_{[\sigma,\tau]})_{[\rho,\rho]} \arrow[r, "\hat{\zeta}_{[\rho,\rho]}"] & M_{[\rho,\rho]}
\end{tikzcd}$$

Hence the assertion follows. ◻

For an object M of $\mathcal{F}_{C_0}$, we define a morphism $\hat{\mu}_M : (M_{[\sigma,\tau]})_{[\sigma,\tau]} \to M_{[\sigma,\tau]}$ to be the following composition assuming that $\theta_{\sigma,\tau,\sigma,\tau}(M) : M_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \to (M_{[\sigma,\tau]})_{[\sigma,\tau]}$ is an isomorphism.

$$(M_{[\sigma,\tau]})_{[\sigma,\tau]} \xrightarrow{\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}} M_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} = M_{[\sigma\mu,\,\tau\mu]} \xrightarrow{M_\mu} M_{[\sigma,\tau]}$$

Let $C_1 \times_{C_0} C_1 \xleftarrow{\mathrm{pr}_{12}} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_{23}} C_1 \times_{C_0} C_1$ be a limit of a diagram $C_1 \times_{C_0} C_1 \xrightarrow{\mathrm{pr}_2} C_1 \xleftarrow{\mathrm{pr}_1} C_1 \times_{C_0} C_1$.

Proposition 9.4.10 *We assume that $\theta_{\sigma,\tau,\sigma,\tau}(M) : M_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \to (M_{[\sigma,\tau]})_{[\sigma,\tau]}$ is an isomorphism and that $\theta_{\sigma,\tau,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M) : M_{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\,\tau\mathrm{pr}_2\mathrm{pr}_{23}]} \to (M_{[\sigma,\tau]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an epimorphism. Let us denote by μ^l_M a morphism $P_{\sigma,\tau}(M_{[\sigma,\tau]})^{-1}_{M_{[\sigma,\tau]}}(\hat{\mu}_M)$ of $\mathcal{F}_{C_1}$. Then $(M_{[\sigma,\tau]}, \mu^l_M)$ is a representation of $\boldsymbol{C}$. Moreover, if $\xi : \sigma^*(M) \to \tau^*(M)$ is a morphism of $\mathcal{F}_{C_1}$ such that (M, ξ) is a representation of $\boldsymbol{C}$, then $\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi) : M_{[\sigma,\tau]} \to M$ defines a morphism of representations from $(M_{[\sigma,\tau]}, \mu^l_M)$ to (M, ξ).*

Proof. The following diagram is commutative by (8.4.19) and (8.4.22).

$$\begin{tikzcd}
((M_{[\sigma,\tau]})_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}_{[\sigma,\tau]}"] & (M_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]})_{[\sigma,\tau]} = (M_{[\sigma\mu,\,\tau\mu]})_{[\sigma,\tau]} \arrow[r, "(M_\mu)_{[\sigma,\tau]}"] & (M_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}"'] \\
(M_{[\sigma,\tau]})_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[u, "\theta_{\sigma,\tau,\sigma,\tau}(M_{[\sigma,\tau]})"] \arrow[d, equal] & M_{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\,\tau\mathrm{pr}_2\mathrm{pr}_{23}]} \arrow[l, "\theta_{\sigma,\tau,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)"'] \arrow[r, "M_{\mu\times_{C_0} id_{C_1}}"] \arrow[u, "\theta_{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma,\tau}(M)"] \arrow[d, equal] & M_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[d, equal] \\
(M_{[\sigma,\tau]})_{[\sigma\mu,\,\tau\mu]} \arrow[d, "(M_{[\sigma,\tau]})_\mu"] & M_{[\sigma\mu\mathrm{pr}_{12},\,\tau\mu\mathrm{pr}_{23}]} \arrow[l, "\theta_{\sigma,\tau,\sigma\mu,\tau\mu}(M)"'] \arrow[r, "M_{\mu\times_{C_0} id_{C_1}}"] \arrow[d, "M_{id_{C_1}\times_{C_0}\mu}"] & M_{[\sigma\mu,\,\tau\mu]} \arrow[d, "M_\mu"'] \\
(M_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}"] & M_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} = M_{[\sigma\mu,\,\tau\mu]} \arrow[r, "M_\mu"] & M_{[\sigma,\,\tau]}
\end{tikzcd}$$

Since the functor $D_{\sigma,\tau,id_{C_0},id_{C_0}} : \mathcal{P} \to \mathcal{E}$ are given by

$$\begin{array}{ll}
D_{\sigma,\tau,id_{C_0},id_{C_0}}(i) = C_1 \quad (i = 0, 1), & D_{\sigma,\tau,id_{C_0},id_{C_0}}(i) = C_0 \quad (i = 2, 3, 4, 5), \\
D_{\sigma,\tau,id_{C_0},id_{C_0}}(\tau_{01}) = id_{C_1}, & D_{\sigma,\tau,id_{C_0},id_{C_0}}(\tau_{13}) = \sigma, \\
D_{\sigma,\tau,id_{C_0},id_{C_0}}(\tau_{02}) = D_{\sigma,\tau,id_{C_0},id_{C_0}}(\tau_{14}) = \tau, & D_{\sigma,\tau,id_{C_0},id_{C_0}}(\tau_{23}) = D_{\sigma,\tau,id_{C_0},id_{C_0}}(\tau_{24}) = id_{C_0},
\end{array}$$

we define a natural transformations $\nu : D_{\sigma,\tau,id_{C_0},id_{C_0}} \to D_{\sigma,\tau,\sigma,\tau}$ by $\nu_0 = (id_{C_1}, \varepsilon\tau) : C_1 \to C_1 \times_{C_0} C_1$, $\nu_1 = id_{C_1}$, $\nu_2 = \varepsilon$, $\nu_i = \kappa_i = id_{C_0}$ $(i = 3, 4, 5)$. Then, the following diagram is commutative by (8.4.17), (8.4.6).

$$\begin{tikzcd}
(M_{[\sigma,\tau]})_{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, "\theta_{\sigma,\tau,id_{C_0},id_{C_0}}(M)^{-1}"] \arrow[d, "(M_{[\sigma,\tau]})_\varepsilon"] & M_{[\sigma id_{C_1},\,id_{C_0}\tau]} \arrow[r, equal] \arrow[d, "M_{[id_{C_1},\varepsilon\tau]}"] & M_{[\sigma id_{C_1},\,\tau id_{C_1}]} \arrow[r, "M_{id_{C_1}}"] & M_{[\sigma,\tau]} \arrow[d, "id_{M_{[\sigma,\tau]}}"] \\
(M_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}"] & M_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[r, equal] & M_{[\sigma\mu,\tau\mu]} \arrow[r, "M_\mu"] & M_{[\sigma,\tau]}
\end{tikzcd}$$

The upper row of the above diagram is identified with the identity morphism of $M_{[\sigma,\tau]}$ which implies that $\hat{\mu}_M(M_{[\sigma,\tau]})_\varepsilon$ is the identity morphism of $M_{[\sigma,\tau]}$. Thus $(M_{[\sigma,\tau]}, \mu^l_M)$ is a representation of $\boldsymbol{C}$ by (9.4.1).

If (M, ξ) is a representation of $\boldsymbol{C}$, then, $\hat{\xi}\hat{\xi}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) = \hat{\xi}M_\mu$ by (9.4.1). Hence $\hat{\xi}\hat{\xi}_{[\sigma,\tau]} = \hat{\xi}\hat{\mu}_M$ by the definition of $\hat{\mu}_M$ and it follows from (9.4.5) that $\hat{\xi}$ defines a morphism of representations from $(M_{[\sigma,\tau]}, \mu^l_M)$ to (M, ξ). ◻

Proposition 9.4.11 *Assume that $\theta_{\sigma,\tau,\sigma,\tau}(L) : L_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \to (L_{[\sigma,\tau]})_{[\sigma,\tau]}$ is an isomorphism for $L = M, N$ and that $\theta_{\sigma,\tau,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(L) : L_{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\,\tau\mathrm{pr}_2\mathrm{pr}_{23}]} \to (L_{[\sigma,\tau]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an epimorphisms for $L = M, N$. For a morphism $\varphi : M \to N$, $\varphi_{[\sigma,\tau]} : M_{[\sigma,\tau]} \to N_{[\sigma,\tau]}$ defines a morphism of representations from $(M_{[\sigma,\tau]}, \mu^l_M)$ to $(N_{[\sigma,\tau]}, \mu^l_N)$.*

Proof. The following diagram is commtative by (8.4.8) and (8.4.19).

$$\begin{tikzcd}
(M_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}"] \arrow[d, "(\varphi_{[\sigma,\tau]})_{[\sigma,\tau]}"] & M_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[r, equal] \arrow[d, "\varphi_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]}"] & M_{[\sigma\mu,\tau\mu]} \arrow[r, "M_\mu"] & M_{[\sigma,\tau]} \arrow[d, "\varphi_{[\sigma,\tau]}"] \\
(N_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(N)^{-1}"] & N_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[r, equal] & N_{[\sigma\mu,\tau\mu]} \arrow[r, "N_\mu"] & N_{[\sigma,\tau]}
\end{tikzcd}$$

Hence the assertion follows from (9.4.5). □

Remark 9.4.12 *If $\varphi : (M,\xi) \to (N,\zeta)$ is a morphism of representations of $\boldsymbol{C}$, we have the following commutative diagram in* $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$*.*

$$\begin{tikzcd}
(M_{[\sigma,\tau]}, \mu^l_M) \arrow[r, "\hat{\xi}"] \arrow[d, "\varphi_{[\sigma,\tau]}"] & (M,\xi) \arrow[d, "\varphi"] \\
(N_{[\sigma,\tau]}, \mu^l_N) \arrow[r, "\hat{\zeta}"] & (N,\zeta)
\end{tikzcd}$$

Theorem 9.4.13 *Let M be an object of $\mathcal{F}_{C_0}$ and (N,ζ) a representation of $\boldsymbol{C}$. Assume that $\theta_{\sigma,\tau,\sigma,\tau}(L) : L_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \to (L_{[\sigma,\tau]})_{[\sigma,\tau]}$ is an isomorphism for $L = M, N$ and that $\theta_{\sigma,\tau,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(L) : L_{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\,\tau\mathrm{pr}_2\mathrm{pr}_{23}]} \to (L_{[\sigma,\tau]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an epimorphism for $L = M, N$. Then, a map*

$$\Phi : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M_{[\sigma,\tau]}, \mu^l_M), (N,\zeta)) \to \mathcal{F}_{C_0}(M,N)$$

defined by $\Phi(\varphi) = \varphi M_\varepsilon$ is bijective. Hence, if $\theta_{\sigma,\tau,\sigma,\tau}(L)$ is an isomorphism and $\theta_{\sigma,\tau,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(L)$ is an epimorphisms for all $L \in \mathrm{Ob}\,\mathcal{F}_{C_0}$, a functor $\mathscr{L}_{\boldsymbol{C}} : \mathcal{F}_{C_0} \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ defined by $\mathscr{L}_{\boldsymbol{C}}(M) = (M_{[\sigma,\tau]}, \mu^l_M)$ for $M \in \mathrm{Ob}\,\mathcal{F}_{C_0}$ and $\mathscr{L}_{\boldsymbol{C}}(\varphi) = \varphi_{[\sigma,\tau]}$ for $\varphi \in \mathrm{Mor}\mathcal{F}_{C_0}$ is a left adjoint of the forgetful functor $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$.

Proof. We put $\hat{\zeta} = P_{\sigma,\tau}(N)_N(\zeta) : N_{[\sigma,\tau]} \to N$. For $\psi \in \mathcal{F}_{C_0}(M,N)$, it follows from (9.4.11) that we have a morphism $\psi_{[\sigma,\tau]} : (M_{[\sigma,\tau]}, \mu^l_M) \to (N_{[\sigma,\tau]}, \mu^l_N)$ of representations. Since $\hat{\zeta} : (N_{[\sigma,\tau]}, \mu^l_N) \to (N,\zeta)$ is a morphism of representations by (9.4.10), $\hat{\zeta}\psi_{[\sigma,\tau]} : (M_{[\sigma,\tau]}, \mu^l_M) \to (N,\zeta)$ is a morphism of representations. It follows from (8.4.8) and (9.4.1) that we have $\Phi(\hat{\zeta}\psi_{[\sigma,\tau]}) = \hat{\zeta}\psi_{[\sigma,\tau]}M_\varepsilon = \hat{\zeta}N_\varepsilon\psi = \psi$. On the other hand, for $\varphi \in \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M_{[\sigma,\tau]}, \mu^l_M)), (N,\zeta))$, since $\hat{\zeta}\varphi_{[\sigma,\tau]} = \varphi\hat{\mu}_M = \varphi M_\mu \theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}$ by (9.4.5) and the following diagram commutes by (8.4.6) and (8.4.19),

$$\begin{tikzcd}
(M_{[id_{C_0},\,id_{C_0}]})_{[\sigma,\tau]} \arrow[d, "(M_\varepsilon)_{[\sigma,\tau]}"] & M_{[id_{C_0}\sigma,\,\tau id_{C_1}]} \arrow[l, "\theta_{id_{C_0},id_{C_0},\sigma,\tau}(M)"'] \arrow[r, "id_{M_{[\sigma,\tau]}}"] \arrow[d, "M_{(\varepsilon\sigma,id_{C_1})}"] & M_{[\sigma,\tau]} \\
(M_{[\sigma,\tau]})_{[\sigma,\tau]} & M_{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[l, "\theta_{\sigma,\tau,\sigma,\tau}(M)"'] \arrow[r, equal] & M_{[\sigma\mu,\,\tau\mu]} \arrow[u, "M_\mu"']
\end{tikzcd}$$

we have $\hat{\zeta}(\varphi M_\varepsilon)_{[\sigma,\tau]} = \hat{\zeta}\varphi_{[\sigma,\tau]}(M_\varepsilon)_{[\sigma,\tau]} = \varphi M_\mu\theta_{\sigma,\tau,\sigma,\tau}(M)^{-1}(M_\varepsilon)_{[\sigma,\tau]} = \varphi$ by (8.4.3) and (8.4.23). Therefore a correspondence $\psi \mapsto \hat{\zeta}\psi_{[\sigma,\tau]}$ gives the inverse map of Φ. □

For morphisms $f : X \to Y$ and $g : X \to Z$ of $\mathcal{E}$, we denote by $[f,g]_* : \mathcal{F}_Y \to \mathcal{F}_Z$ the functor defined by $[f,g]_*(M) = M_{[f,g]}$ for $M \in \mathrm{Ob}\,\mathcal{F}_Y$ and $[f,g]_*(\varphi) = \varphi_{[f,g]}$ for $\varphi \in \mathrm{Mor}\,\mathcal{F}_Y$.

Proposition 9.4.14 *Let (M,ξ) and (M,ζ) be representations of $\boldsymbol{C}$ on $M \in \mathrm{Ob}\,\mathcal{F}_{C_0}$. We put $\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi)$ and $\hat{\zeta} = P_{\sigma,\tau}(M)_M(\zeta)$. Assume that $[\sigma,\tau]_* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ preserves coequalizers (the presheaf $F^{\sigma,\tau}_K$ on $\mathcal{F}_{C_0}$ is representable for any $K \in \mathrm{Ob}\,\mathcal{F}_{C_0}$, for example. See (8.6.2).) and that $\theta_{\sigma,\tau,\sigma,\tau}(M)$ is an epimorphism. Let $\pi_{\xi,\zeta} : M \to M_{(\xi:\zeta)}$ be a coequalizer of $\hat{\xi}, \hat{\zeta} : M_{[\sigma,\tau]} \to M$.*

(1) There exists unique morphism $\hat{\lambda} : (M_{(\xi:\zeta)})_{[\sigma,\tau]} \to M_{(\xi:\zeta)}$ that makes the following diagram commute.

$$\begin{tikzcd}
M_{[\sigma,\tau]} \arrow[r, "(\pi_{\xi,\zeta})_{[\sigma,\tau]}"] \arrow[d, "\hat{\xi}"] & (M_{(\xi:\zeta)})_{[\sigma,\tau]} \arrow[d, "\hat{\lambda}"] & M_{[\sigma,\tau]} \arrow[l, "(\pi_{\xi,\zeta})_{[\sigma,\tau]}"'] \arrow[d, "\hat{\zeta}"] \\
M \arrow[r, "\pi_{\xi,\zeta}"] & M_{(\xi:\zeta)} & M \arrow[l, "\pi_{\xi,\zeta}"']
\end{tikzcd}$$

(2) Moreover, we assume that $[\sigma\mathrm{pr}_1, \tau\mathrm{pr}_2]_* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ *maps coequalizers to epimorphisms (the presheaf* $F_K^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}$ *on* $\mathcal{F}_{C_0}$ *is representable for any* $K \in \mathrm{Ob}\,\mathcal{F}_{C_0}$*, for example. See (8.6.2).). Put* $\lambda = P_{\sigma,\tau}(M_{(\xi:\zeta)})^{-1}_{M_{(\xi:\zeta)}}(\hat{\lambda})$*. Then,* $(M_{(\xi:\zeta)}, \lambda)$ *is a representation of* $\boldsymbol{C}$ *and* $\pi_{\xi,\zeta}$ *defines morphisms of representations* $(M,\xi) \to (M_{(\xi:\zeta)}, \lambda)$ *and* $(M,\zeta) \to (M_{(\xi:\zeta)}, \lambda)$*.*

(3) Let (N,ν) *be a representation of* $\boldsymbol{C}$*. Suppose that a morphism* $\varphi : M \to N$ *of* $\mathcal{F}_{C_0}$ *gives morphisms* $(M,\xi) \to (N,\nu)$ *and* $(M,\zeta) \to (N,\nu)$ *of* $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$*. Then, there exists unique morphism* $\tilde{\varphi} : (M_{(\xi:\zeta)}, \lambda) \to (N,\nu)$ *of* $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ *that satisfies* $\tilde{\varphi}\pi_{\xi,\zeta} = \varphi$*.*

Proof. (1) Put $\chi = \pi_{\xi,\zeta}\hat{\xi} = \pi_{\xi,\zeta}\hat{\zeta} : M_{[\sigma,\tau]} \to M_{(\xi:\zeta)}$. Then, it follows from (9.4.1) that

$$\chi\hat{\xi}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) = \pi_{\xi,\zeta}\hat{\xi}\hat{\xi}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) = \pi_{\xi,\zeta}\hat{\xi}M_\mu = \pi_{\xi,\zeta}\hat{\zeta}M_\mu = \pi_{\xi,\zeta}\hat{\zeta}\hat{\zeta}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) = \chi\hat{\zeta}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M),$$

which implies $\chi\hat{\xi}_{[\sigma,\tau]} = \chi\hat{\zeta}_{[\sigma,\tau]}$ since $\theta_{\sigma,\tau,\sigma,\tau}(M)$ is an epimorphism. Since $(\pi_{\xi,\zeta})_{[\sigma,\tau]} : M_{[\sigma,\tau]} \to (M_{(\xi:\zeta)})_{[\sigma,\tau]}$ is a coequalizer of $\hat{\xi}_{[\sigma,\tau]}, \hat{\zeta}_{[\sigma,\tau]} : (M_{[\sigma,\tau]})_{[\sigma,\tau]} \to M_{[\sigma,\tau]}$ by the assumption, there exists unique morphism $\hat{\lambda} : (M_{(\xi:\zeta)})_{[\sigma,\tau]} \to M_{(\xi:\zeta)}$ that satisfies $\hat{\lambda}(\pi_{\xi,\zeta})_{[\sigma,\tau]} = \chi$.

(2) By (8.4.3), (8.4.6), (8.4.19) and (9.4.1), the following diagrams are commutative.

$$\begin{tikzcd}
M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M)"] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}"] & (M_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\hat{\xi}_{[\sigma,\tau]}"] \arrow[d, "((\pi_{\xi,\zeta})_{[\sigma,\tau]})_{[\sigma,\tau]}"] & M_{[\sigma,\tau]} \arrow[r, "\hat{\xi}"] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma,\tau]}"] & M \arrow[d, "\pi_{\xi,\zeta}"] \\
(M_{(\xi:\zeta)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, "\theta_{\sigma,\tau,\sigma,\tau}(M_{(\xi:\zeta)})"] & ((M_{(\xi:\zeta)})_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\hat{\lambda}_{[\sigma,\tau]}"] & (M_{(\xi:\zeta)})_{[\sigma,\tau]} \arrow[r, "\hat{\lambda}"] & M_{(\xi:\zeta)}
\end{tikzcd}$$

$$\begin{tikzcd}
M_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, equal] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}"] & M_{[\sigma\mu,\tau\mu]} \arrow[r, "M_\mu"] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma\mu,\tau\mu]}"] & M_{[\sigma,\tau]} \arrow[r, "\hat{\xi}"] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma,\tau]}"] & M \arrow[d, "\pi_{\xi,\zeta}"] \\
(M_{(\xi:\zeta)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, equal] & (M_{(\xi:\zeta)})_{[\sigma\mu,\tau\mu]} \arrow[r, "(M_{(\xi:\zeta)})_\mu"] & (M_{(\xi:\zeta)})_{[\sigma,\tau]} \arrow[r, "\hat{\lambda}"] & M_{(\xi:\zeta)}
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, equal] \arrow[d, "\pi_{\xi,\zeta}"] & M_{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, "M_\varepsilon"] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma\varepsilon,\tau\varepsilon]}"] & M_{[\sigma,\tau]} \arrow[r, "\hat{\xi}"] \arrow[d, "(\pi_{\xi,\zeta})_{[\sigma,\tau]}"] & M \arrow[d, "\pi_{\xi,\zeta}"] \\
M_{(\xi:\zeta)} \arrow[r, equal] & (M_{(\xi:\zeta)})_{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, "(M_{(\xi:\zeta)})_\varepsilon"] & (M_{(\xi:\zeta)})_{[\sigma,\tau]} \arrow[r, "\hat{\lambda}"] & M_{(\xi:\zeta)}
\end{tikzcd}$$

It follows from (9.4.1) that we have

$$\hat{\lambda}\hat{\lambda}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M_{(\xi:\zeta)})(\pi_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} = \pi_{\xi,\zeta}\hat{\xi}\hat{\xi}_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M) = \pi_{\xi,\zeta}\hat{\xi}M_\mu = \hat{\lambda}(M_{(\xi:\zeta)})_\mu(\pi_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$$
$$\hat{\lambda}(M_{(\xi:\zeta)})_\varepsilon\pi_{\xi,\zeta} = \pi_{\xi,\zeta}\hat{\xi}M_\varepsilon = \pi_{\xi,\zeta}$$

Since $\pi_{\xi,\zeta}$ and $(\pi_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ are epimorphisms, it follows that $\hat{\lambda}(\hat{\lambda}_{[\sigma,\tau]})\theta_{\sigma,\tau,\sigma,\tau}(M_{(\xi:\zeta)}) = \hat{\lambda}(M_{(\xi:\zeta)})_\mu$ and $\hat{\lambda}(M_{(\xi:\zeta)})_\varepsilon = id_{M_{(\xi:\zeta)}}$. Therefore λ is a representation of $\boldsymbol{C}$ on $M_{(\xi:\zeta)}$ by (9.4.1). $\pi_{\xi,\zeta} : (M,\xi) \to (M_{(\xi:\zeta)}, \lambda)$ and $\pi_{\xi,\zeta} : (M,\zeta) \to (M_{(\xi:\zeta)}, \lambda)$ are morphisms of representations by the first assertion and (8.4.5).

(3) Put $\hat{\nu} = P_{\sigma,\tau}(N)_N(\nu)$. Since $\varphi\hat{\xi} = \hat{\nu}\varphi_{[\sigma,\tau]} = \varphi\hat{\zeta}$ by (9.4.5), there exists unique morphism $\tilde{\varphi} : M_{(\xi:\zeta)} \to N$ that satisfies $\tilde{\varphi}\pi_{\xi,\zeta} = \varphi$. Then, we have $\tilde{\varphi}\hat{\lambda}(\pi_{\xi,\zeta})_{[\sigma,\tau]} = \tilde{\varphi}\pi_{\xi,\zeta}\hat{\xi} = \varphi\hat{\xi} = \hat{\nu}\varphi_{[\sigma,\tau]} = \hat{\nu}\tilde{\varphi}_{[\sigma,\tau]}(\pi_{\xi,\zeta})_{[\sigma,\tau]}$. Since $(\pi_{\xi,\zeta})_{[\sigma,\tau]}$ is an epimorphism, it follows $\tilde{\varphi}\hat{\lambda} = \hat{\nu}\tilde{\varphi}_{[\sigma,\tau]}$, which implies that $\tilde{\varphi}$ gives a morphism $(M_{(\xi:\zeta)}, \lambda) \to (N,\nu)$ of representations of $\boldsymbol{C}$. □

Remark 9.4.15 *Assume that one of the following conditions.*

(i) $[\sigma,\tau]_* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ *preserves epimorphisms.*
(ii) $\sigma^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_1}$ *preserves epimorphisms.*
(iii) The presheaf $F_N^{\sigma,\tau}$ *on* $\mathcal{F}_{C_0}$ *is representable for* $N \in \mathrm{Ob}\,\mathcal{F}_{C_0}$*.*

For representations (M,ξ), (N,ζ) and (N,ζ') of $\boldsymbol{C}$, suppose that there exists an epimorphism $\varphi : M \to N$ of $\mathcal{F}_{C_0}$ such that $\varphi : (M,\xi) \to (N,\zeta)$ and $\varphi : (M,\xi) \to (N,\zeta')$ are morphisms of $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$. Then, $\sigma^*(\varphi)^* : \mathcal{F}_{C_1}(\sigma^*(N),\tau^*(N)) \to \mathcal{F}_{C_1}(\sigma^*(M),\tau^*(N))$ is injective by the assumption. Hence $\zeta\sigma^*(\varphi) = \tau^*(\varphi)\xi = \zeta'\sigma^*(\varphi)$ implies $\zeta = \zeta'$.

Proposition 9.4.16 *Let (M,ξ), (N,ξ'), (M,ζ) and (N,ζ') be objects of $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$. Put $\hat{\xi} = P_{\sigma,\tau}(M)_M(\xi)$, $\hat{\xi}' = P_{\sigma,\tau}(N)_N(\xi')$, $\hat{\zeta} = P_{\sigma,\tau}(M)_M(\zeta)$ and $\hat{\zeta}' = P_{\sigma,\tau}(N)_N(\zeta')$. Assume that $[\sigma,\tau]_* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ preserves coequalizers and that $[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]_* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ maps coequalizers to epimorphisms (e.g., the presheaves $F_K^{\sigma,\tau}$ and $F_K^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}$ on $\mathcal{F}_{C_0}$ is representable for any $K \in \mathrm{Ob}\,\mathcal{F}_{C_0}$. See (8.6.2)). Suppose that $\pi_{\xi,\zeta} : M \to M_{(\xi:\zeta)}$ is a coequalizer of $\hat{\xi},\hat{\zeta} : M_{[\sigma,\tau]} \to M$ and that $\pi_{\xi',\zeta'} : N \to N_{(\xi':\zeta')}$ is a coequalizer of $\hat{\xi}',\hat{\zeta}' : N_{[\sigma,\tau]} \to N$. We denote by $(M_{(\xi:\zeta)},\lambda)$ and $(N_{(\xi':\zeta')},\lambda')$ the representations of $\boldsymbol{C}$ given in (9.4.14). If a morphism $\varphi : M \to N$ defines morphisms of representations $(M,\xi) \to (N,\xi')$ and $(M,\zeta) \to (N,\zeta')$, then there exists unique morphism $\tilde{\varphi} : (M_{(\xi:\zeta)},\lambda) \to (N_{(\xi':\zeta')},\lambda')$ of representations of $\boldsymbol{C}$ that satisfies $\tilde{\varphi}\pi_{\xi,\zeta} = \pi_{\xi',\zeta'}\varphi$.*

Proof. Since $\pi_{\xi',\zeta'} : N \to N_{(\xi':\zeta')}$ defines morphisms $(N,\xi') \to (N_{(\xi':\zeta')},\lambda')$, $(N,\zeta') \to (N_{(\xi':\zeta')},\lambda')$ of representations of $\boldsymbol{C}$, $\pi_{\xi',\zeta'}\varphi : M \to N_{(\xi':\zeta')}$ defines morphisms $(M,\xi) \to (N_{(\xi':\zeta')},\lambda')$, $(M,\zeta) \to (N_{(\xi':\zeta')},\lambda')$ of representations of $\boldsymbol{C}$. Hence it follows from (3) of (9.4.16) that there exists unique morphism $\tilde{\varphi} : M_{(\xi:\zeta)} \to N_{(\xi':\zeta')}$ that satisfies $\tilde{\varphi}\pi_{\xi,\zeta} = \pi_{\xi',\zeta'}\varphi$ and gives a morphism $(M_{(\xi:\zeta)},\lambda) \to (N_{(\xi':\zeta')},\lambda')$ of representations of $\boldsymbol{C}$. □

9.5 Representations in fibered categories with exponents

Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category with exponents and $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ an internal category in $\mathcal{E}$.

Proposition 9.5.1 *For $M \in \mathrm{Ob}\,\mathcal{F}_{C_0}$ and $\xi \in \mathcal{F}_{C_1}(\sigma^*(M),\tau^*(M))$, we put $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi) : M \to M^{[\sigma,\tau]}$. Then, (M,ξ) is a representation of $\boldsymbol{C}$ on M if and only if the following diagram commutes and a composition $M \xrightarrow{\check{\xi}} M^{[\sigma,\tau]} \xrightarrow{M^{\varepsilon}} M^{[\sigma\varepsilon,\tau\varepsilon]} = M$ coincides with the identity morphism of M.*

$$\begin{tikzcd}
M \arrow[r, "\check{\xi}"] \arrow[dr, "\check{\xi}"'] & M^{[\sigma,\tau]} \arrow[r, "\check{\xi}^{[\sigma,\tau]}"] & (M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(M)"] \\
& M^{[\sigma,\tau]} \arrow[r, "M^{\mu}"] & M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}
\end{tikzcd}$$

and $M^{\varepsilon}\check{\xi} = id_M$.

Proof. We have $E_{\sigma\mu,\tau\mu}(M)_M(\xi_\mu) = M^{\mu}\check{\xi}$ and $E_{\sigma\mu,\tau\mu}(M)_M(\xi_{\mathrm{pr}_i}) = M^{\mathrm{pr}_i}\check{\xi}$ for $i = 1, 2$ by (8.5.6). Hence (8.5.3), (8.5.6), (8.5.8), (8.5.15) imply

$$\begin{aligned}
E_{\sigma\mu,\tau\mu}(M)_M(\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}) &= E_{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)_M(\xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}) = \epsilon_M^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1,\tau\mathrm{pr}_2}(M^{\mathrm{pr}_2}\check{\xi})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1]}M^{\mathrm{pr}_1}\check{\xi} \\
&= \epsilon_M^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1,\tau\mathrm{pr}_2}(M^{\mathrm{pr}_2})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_1]}(M^{[\sigma,\tau]})^{\mathrm{pr}_1}\check{\xi}^{[\sigma,\tau]}\check{\xi} = \theta^{\sigma,\tau,\sigma,\tau}(M)\check{\xi}^{[\sigma,\tau]}\check{\xi}
\end{aligned}$$

Thus $\xi_\mu = \xi_{\mathrm{pr}_2}\xi_{\mathrm{pr}_1}$ and $\xi_\varepsilon = id_M$ are equivalent to $\theta^{\sigma,\tau,\sigma,\tau}(M)\check{\xi}^{[\sigma,\tau]}\check{\xi} = M^{\mu}\check{\xi}$ and $M^{\varepsilon}\check{\xi} = id_M$, respectively. □

Proposition 9.5.2 *Let $(s(C_0), s_{\boldsymbol{C}})$ be the trivial representation associated with a cartesian section $s : \mathcal{E} \to \mathcal{F}$. Put $T = s(1)$. The image of $s_{\boldsymbol{C}} \in \mathcal{F}_{C_1}(\sigma^*s(C_0),\tau^*s(C_0))$ by $E_{\sigma,\tau}(s(C_0))_{s(C_0)} : \mathcal{F}_{C_1}(\sigma^*s(C_0),\tau^*s(C_0)) \to \mathcal{F}_{C_0}(s(C_0), s(C_0)^{[\sigma,\tau]})$ is $\omega(\sigma,\tau;o_{C_0},o_{C_0})^T o_{C_0}^*(E_{o_{C_1},o_{C_1}}(T)_T(id_{s(C_1)}))$.*

Proof. It follows from (8.1.26) and the definition of $s_{\boldsymbol{C}}$ that we have $s_{\boldsymbol{C}} = c_{o_{C_0},\tau}(T)^{-1}c_{o_{C_0},\sigma}(T)$. We note that $o_{C_0}\sigma = o_{C_0}\tau = o_{C_1}$ and $s(C_i) = o_{C_i}^*(T)$ for $i = 0, 1$. The following diagram is commutative by (8.5.27).

$$\begin{tikzcd}
\mathcal{F}_{C_1}(s(C_1), s(C_1)) \arrow[r, "c_{o_{C_0},\tau}(T)_*^{-1}"] \arrow[d, "E_{o_{C_1},o_{C_1}}(T)_T"] & \mathcal{F}_{C_1}(s(C_1), \tau^*(s(C_0))) \arrow[r, "c_{o_{C_0},\sigma}(T)^*"] & \mathcal{F}_{C_1}(\sigma^*(s(C_0)), \tau^*(s(C_0))) \arrow[d, "E_{\sigma,\tau}(s(C_0))_{s(C_0)}"] \\
\mathcal{F}_1(T, T^{[o_{C_1},o_{C_1}]}) \arrow[r, "o_{C_0}^*"] & \mathcal{F}_{C_0}(s(C_0), o_{C_0}^*(T^{[o_{C_1},o_{C_1}]})) \arrow[r, "\omega(\sigma,\tau;o_{C_0},o_{C_0})_*^T"] & \mathcal{F}_{C_0}(s(C_0), s(C_0)^{[\sigma,\tau]})
\end{tikzcd}$$

Hence we have $E_{\sigma,\tau}(s(C_0))_{s(C_0)}(s_{\boldsymbol{C}}) = \omega(\sigma,\tau;o_{C_0},o_{C_0})^T o_{C_0}^*(E_{o_{C_1},o_{C_1}}(T)_T(id_{s(C_1)}))$. □

Proposition 9.5.3 *Let $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{D} \to \boldsymbol{C}$ be an internal functor and (M, ξ) a representation of $\boldsymbol{C}$. Then,*

$$E_{\sigma',\tau'}(f_0^*(M))_{f_0^*(M)}(\xi_{\boldsymbol{f}}) = \omega(\sigma', \tau'; f_0, f_0)^M f_0^*(M^{f_1}\check{\xi}).$$

Proof. The upper rectangle of the following diagram is commutative by (1) of (8.5.6) and the lower one is commutative (8.5.27).

$$\begin{tikzcd}
\mathcal{F}_{C_1}(\sigma^*(M), \tau^*(M)) \arrow[r, "E_{\sigma,\tau}(M)_M"] \arrow[d, "f_1^\sharp"] & \mathcal{F}_{C_0}(M, M^{[\sigma,\tau]}) \arrow[d, "M_*^{f_1}"] \\
\mathcal{F}_{D_1}((f_0\sigma')^*(M), (f_0\tau')^*(M)) \arrow[r, "E_{f_0\sigma', f_0\tau'}(M)_M"] \arrow[d, "c_{f_0,\tau'}(M)_*^{-1}"] & \mathcal{F}_{C_0}(M, M^{[f_0\sigma', f_0\tau']}) \arrow[d, "f_0^*"] \\
\mathcal{F}_{D_1}((f_0\sigma')^*(M), \tau'(f_0^*(M))) \arrow[d, "c_{f_0,\sigma'}(M)^*"] & \mathcal{F}_{D_0}(f_0^*(M), f_0^*(M^{[f_0\sigma', f_0\tau']})) \arrow[d, "\omega(\sigma',\tau';f_0,f_0)_*^M"] \\
\mathcal{F}_{D_1}(\sigma'^*(f_0^*(M)), \tau'^*(f_0^*(M))) \arrow[r, "E_{\sigma',\tau'}(f_0^*(M))_{f_0^*(M)}"] & \mathcal{F}_{D_0}(f_0^*(M), f_0^*(M)^{[\sigma',\tau']})
\end{tikzcd}$$

The assertion follows from the above diagram and the definition of $\xi_{\boldsymbol{f}}$. □

The following fact is a direct consequence of (8.5.5).

Proposition 9.5.4 *Let (M, ξ) and (N, ζ) be representations of $\boldsymbol{C}$ and $\varphi : M \to N$ a morphism of $\mathcal{F}_{C_0}$. We put $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi)$ and $\check{\zeta} = E_{\sigma,\tau}(N)_N(\zeta)$. Then, φ is a morphism of representations if and only if the following diagram is commutative.*

$$\begin{tikzcd}
M \arrow[r, "\check{\xi}"] \arrow[d, "\varphi"] & M^{[\sigma,\tau]} \arrow[d, "\varphi^{[\sigma,\tau]}"] \\
N \arrow[r, "\check{\zeta}"] & N^{[\sigma,\tau]}
\end{tikzcd}$$

For a morphism $\pi : X \to C_0$ of $\mathcal{E}$, we consider a limit $C_1 \xleftarrow{\pi_\tau} C_1 \times^{\tau}_{C_0} X \xrightarrow{\tau_\pi} X$ of a diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{\pi} X$. Let $(\pi : X \to C_0,\ \alpha : C_1 \times^{\tau}_{C_0} X \to X)$ be an internal presheaf on $\boldsymbol{C}$. That is, the following diagrams are commutative.

$$\begin{tikzcd}
C_1 \times^{\tau}_{C_0} X \arrow[r, "\alpha"] \arrow[d, "\pi_\tau"] & X \arrow[d, "\pi"] \\
C_1 \arrow[r, "\sigma"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times_{C_0} C_1 \times^{\tau}_{C_0} X \arrow[r, "id_{C_1}\times\alpha"] \arrow[d, "\mu\times id_X"] & C_1 \times^{\tau}_{C_0} X \arrow[d, "\alpha"] \\
C_1 \times^{\tau}_{C_0} X \arrow[r, "\alpha"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times^{\tau}_{C_0} X \arrow[r, "\alpha"] & X \\
C_0 \times^{\tau}_{C_0} X \arrow[u, "\varepsilon\times id_X"] \arrow[ur, "\mathrm{pr}_2"'] &
\end{tikzcd}$$

Let $C_1 \times^{\tau}_{C_0} X \xleftarrow{\bar{\mathrm{pr}}_{23}} C_1 \times_{C_0} C_1 \times^{\tau}_{C_0} X \xrightarrow{\bar{\mathrm{pr}}_{12}} C_1 \times_{C_0} C_1$ be a limit of $C_1 \times^{\tau}_{C_0} X \xrightarrow{\pi_\tau} C_1 \xleftarrow{\mathrm{pr}_2} C_1 \times_{C_0} C_1$. Then, $X \xleftarrow{\tau_\pi \bar{\mathrm{pr}}_{23}} C_1 \times_{C_0} C_1 \times^{\tau}_{C_0} X \xrightarrow{\bar{\mathrm{pr}}_{12}} C_1 \times_{C_0} C_1$ is a limit of $X \xrightarrow{\pi} C_0 \xleftarrow{\tau\mathrm{pr}_2} C_1 \times_{C_0} C_1$. We also note that $C_1 \times^{\tau}_{C_0} X \xleftarrow{\bar{\mathrm{pr}}_{23}} C_1 \times_{C_0} C_1 \times^{\tau}_{C_0} X \xrightarrow{\mathrm{pr}_1\bar{\mathrm{pr}}_{12}} C_1$ is a limit of $C_1 \times^{\tau}_{C_0} X \xrightarrow{\sigma\pi_\tau} C_0 \xleftarrow{\tau} C_1$.

$$\begin{tikzcd}
C_1 \times_{C_0} C_1 \times^{\tau}_{C_0} X \arrow[r, "\bar{\mathrm{pr}}_{23}"] \arrow[d, "\bar{\mathrm{pr}}_{12}"] & C_1 \times^{\tau}_{C_0} X \arrow[r, "\tau_\pi"] \arrow[d, "\pi_\tau"] & X \arrow[d, "\pi"] \\
C_1 \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & C_1 \arrow[r, "\tau"] \arrow[d, "\sigma"] & C_0 \\
C_1 \arrow[r, "\tau"] & C_0 &
\end{tikzcd}$$

Define a functor $D^\alpha : \mathcal{P} \to \mathcal{E}$ by $D^\alpha(0) = C_1 \times^{\tau}_{C_0} X$, $D^\alpha(1) = X$, $D^\alpha(2) = C_1$, $D^\alpha(3) = D^\alpha(4) = D^\alpha(5) = C_0$ and $D^\alpha(\tau_{01}) = \alpha$, $D^\alpha(\tau_{02}) = \pi_\tau$, $D^\alpha(\tau_{13}) = D^\alpha(\tau_{14}) = \pi$, $D^\alpha(\tau_{24}) = \sigma$, $D^\alpha(\tau_{25}) = \tau$. For a representation (M, ξ) of $\boldsymbol{C}$, we put $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi)$. Assume that $\theta^{\sigma,\tau,\pi,\pi}(M) : (M^{[\pi,\pi]})^{[\sigma,\tau]} \to M^{[\sigma\pi_\tau,\,\pi\tau_\pi]}$ is an isomorphism and define a morphism $\check{\xi}^\alpha : M^{[\pi,\pi]} \to (M^{[\pi,\pi]})^{[\sigma,\tau]}$ to be the following composition.

$$M^{[\pi,\pi]} \xrightarrow{\check{\xi}^{[\pi,\pi]}} (M^{[\sigma,\tau]})^{[\pi,\pi]} \xrightarrow{\theta^{D^\alpha}(M)} M^{[\pi\alpha,\,\tau\pi_\tau]} = M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \xrightarrow{\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}} (M^{[\pi,\pi]})^{[\sigma,\tau]}$$

Proposition 9.5.5 *Assume that $\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\pi,\pi}(M):(M^{[\pi,\pi]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}\to M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\pi\tau_\pi\bar{\mathrm{pr}}_{23}]}$ is a monomorphism. Put $E_{\sigma,\tau}(M^{[\pi,\pi]})^{-1}_{M^{[\pi,\pi]}}(\check{\xi}^\alpha)=\xi^\alpha$. Then, $(M^{[\pi,\pi]},\xi^\alpha)$ is a representation of $\boldsymbol{C}$ and $M^\pi:(M,\xi)\to(M^{[\pi,\pi]},\xi^\alpha)$ is a morphism of representations.*

Proof. The left rectangle of the following diagram is commutative by (8.5.19) and the right rectangle is commutative by (8.5.22).

$$\begin{tikzcd}
(M^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[r, "(M^{[\pi,\pi]})^{\mu}"] \arrow[d, "\theta^{\sigma,\tau,\pi,\pi}(M)"] & (M^{[\pi,\pi]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[d, "\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\pi,\pi}(M)"] & ((M^{[\pi,\pi]})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[l, "\theta^{\sigma,\tau,\sigma,\tau}(M^{[\pi,\pi]})"'] \arrow[d, "(\theta^{\sigma,\tau,\pi,\pi}(M)^{[\sigma,\tau]})"] \\
M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[r, "M^{\mu\times_{C_0}id_X}"] & M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\pi\tau_\pi\bar{\mathrm{pr}}_{23}]} & (M^{[\sigma\pi_\tau,\,\pi\tau_\pi]})^{[\sigma,\tau]} \arrow[l, "\theta^{\sigma,\tau,\sigma\pi_\tau,\,\pi\tau_\pi}(M)"']
\end{tikzcd}$$

Since $\pi\alpha=\sigma\pi_\tau$, $\pi_\tau(id_{C_1}\times_{C_0}\alpha)=\mathrm{pr}_1\bar{\mathrm{pr}}_{12}$ and $\alpha(id_{C_1}\times_{C_0}\alpha)=\alpha(\mu\times_{C_0}id_X)$, we can define functors $E,F:\mathcal{P}\to\mathcal{E}$ and a natural transformation $\lambda:E\to D^\alpha$ by $E(0)=F(0)=C_1\times_{C_0}C_1\times^{\tau}_{C_0}X$, $E(1)=X$, $F(1)=C_1\times^{\tau}_{C_0}X$, $E(2)=C_1\times_{C_0}C_1$, $F(2)=C_1$, $E(i)=F(i)=C_0$ for $i=3,4,5$, $E(\tau_{01})=\alpha(id_{C_1}\times_{C_0}\alpha)$, $F(\tau_{01})=id_{C_1}\times_{C_0}\alpha$, $E(\tau_{02})=\bar{\mathrm{pr}}_{12}$, $F(\tau_{02})=\pi_\tau\bar{\mathrm{pr}}_{23}$, $E(\tau_{13})=\pi$, $F(\tau_{13})=\sigma\pi_\tau$, $E(\tau_{14})=\pi$, $F(\tau_{14})=\pi\tau_\pi$, $E(\tau_{24})=\sigma\mathrm{pr}_1$, $F(\tau_{24})=\sigma\pi_\tau$, $E(\tau_{25})=\tau\mathrm{pr}_2$, $F(\tau_{25})=\pi\alpha$ and $\lambda_0=\mu\times_{C_0}id_X$, $\lambda_1=id_X$, $\lambda_2=\mu$, $\lambda_3=\lambda_4=\lambda_5=id_{C_0}$. We also note that $\mathrm{pr}_2\bar{\mathrm{pr}}_{12}=\pi_\tau\bar{\mathrm{pr}}_{23}$. Then, the following diagram commutes by (8.5.21)

$$\begin{tikzcd}
((M^{[\sigma,\tau]})^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\pi,\pi}(M^{[\sigma,\tau]})"] \arrow[d, "\theta^{D^\alpha}(M)^{[\sigma,\tau]}"] & (M^{[\sigma,\tau]})^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[d, "\theta^{F}(M)"] & ((M^{[\sigma,\tau]})^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[l, "\theta^{D^\alpha}(M^{[\sigma,\tau]})"'] \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(M)^{[\pi,\pi]}"] \\
(M^{[\pi\alpha,\,\tau\pi_\tau]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\pi\alpha,\,\tau\pi_\tau}(M)"] & M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\tau\pi_\tau\bar{\mathrm{pr}}_{23}]} & (M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]})^{[\pi,\pi]} \arrow[l, "\theta^{E}(M)"']
\end{tikzcd}$$

and the following diagram commutes by (8.5.18).

$$\begin{tikzcd}
(M^{[\sigma,\,\tau]})^{[\pi,\,\pi]} \arrow[r, "(M^{\mu})^{[\pi,\pi]}"] \arrow[d, "\theta^{D^\alpha}(M)"] & (M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]})^{[\pi,\pi]} \arrow[d, "\theta^{E}(M)"] \\
M^{[\pi\alpha,\,\tau\pi_\tau]} \arrow[r, "M^{\mu\times_{C_0}id_X}"] & M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\tau\pi_\tau\bar{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

It follows from the above facts and (8.5.17), (8.5.19), (9.5.1) that the following diagram is commutative

$$\begin{tikzcd}
 & & M^{[\pi\alpha,\,\tau\pi_\tau]} \arrow[r, equal] \arrow[dr, "\check{\xi}^{[\pi\alpha,\,\tau\pi_\tau]}"] & M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[dr, "\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}"] & \\
 & (M^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[ur, "\theta^{D^\alpha}(M)"] \arrow[dr, "(\check{\xi}^{[\sigma,\tau]})^{[\pi,\pi]}"] & & (M^{[\sigma,\tau]})^{[\pi\alpha,\,\tau\pi_\tau]} \arrow[dddl, "\theta^{F}(M)"] & (M^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[dd, "(\check{\xi}^{[\pi,\pi]})^{[\sigma,\tau]}"] \\
M^{[\pi,\pi]} \arrow[ur, "\check{\xi}^{[\pi,\pi]}"] \arrow[d, "\check{\xi}^{[\pi,\pi]}"] & & ((M^{[\sigma,\tau]})^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[ur, "\theta^{D^\alpha}(M^{[\sigma,\tau]})"] \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(M)^{[\pi,\pi]}"] & & \\
(M^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[rr, "(M^{\mu})^{[\pi,\pi]}"] \arrow[d, "\theta^{D^\alpha}(M)"] & & (M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]})^{[\pi,\pi]} \arrow[d, "\theta^{E}(M)"] & & ((M^{[\sigma,\tau]})^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[uul, "\theta^{\sigma,\tau,\pi,\pi}(M^{[\sigma,\tau]})"] \arrow[d, "\theta^{D^\alpha}(M)^{[\sigma,\tau]}"] \\
M^{[\pi\alpha,\,\tau\pi_\tau]} \arrow[rr, "M^{\mu\times_{C_0}id_X}"] \arrow[d, equal] & & M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\tau\pi_\tau\bar{\mathrm{pr}}_{23}]} \arrow[d, equal] & & (M^{[\pi\alpha,\,\tau\pi_\tau]})^{[\sigma,\tau]} \arrow[ll, "\theta^{\sigma,\tau,\pi\alpha,\,\tau\pi_\tau}(M)"'] \arrow[d, equal] \\
M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[rr, "M^{\mu\times_{C_0}id_X}"] \arrow[d, "\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}"] & & M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\pi\tau_\pi\bar{\mathrm{pr}}_{23}]} & & (M^{[\sigma\pi_\tau,\,\pi\tau_\pi]})^{[\sigma,\tau]} \arrow[ll, "\theta^{\sigma,\tau,\sigma\pi_\tau,\,\pi\tau_\pi}(M)"'] \arrow[d, "(\theta^{\sigma,\tau,\pi,\pi}(M)^{[\sigma,\tau]})^{-1}"] \\
(M^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[rr, "(M^{[\pi,\pi]})^{\mu}"] & & (M^{[\pi,\pi]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[u, "\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\pi,\pi}(M)"] & & ((M^{[\pi,\pi]})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[ll, "\theta^{\sigma,\tau,\sigma,\tau}(M^{[\pi,\pi]})"']
\end{tikzcd}$$

Hence $\check{\xi}^\alpha$ make the diagram of (9.5.1) commute.

Since functors $D_{\pi,\pi,id_{C_0},id_{C_0}}, D_{id_{C_0},id_{C_0},\pi,\pi} : \mathcal{P} \to \mathcal{E}$ are given by

$$\begin{aligned}
D_{\pi,\pi,id_{C_0},id_{C_0}}(i) &= D_{id_{C_0},id_{C_0},\pi,\pi}(j) = X \quad (i = 0, 1,\ j = 0, 2),\\
D_{\pi,\pi,id_{C_0},id_{C_0}}(i) &= D_{id_{C_0},id_{C_0},\pi,\pi}(j) = C_0 \quad (i = 2, 3, 4, 5,\ j = 1, 3, 4, 5),\\
D_{\pi,\pi,id_{C_0},id_{C_0}}(\tau_{01}) &= D_{id_{C_0},id_{C_0},\pi,\pi}(\tau_{02}) = id_X,\\
D_{\pi,\pi,id_{C_0},id_{C_0}}(\tau_{ij}) &= D_{id_{C_0},id_{C_0},\pi,\pi}(\tau_{kl}) = \pi \quad ((i,j) = (0,2), (1,3), (1,4), (k,l) = (0,1), (1,3), (1,4)),\\
D_{\pi,\pi,id_{C_0},id_{C_0}}(\tau_{2j}) &= D_{id_{C_0},id_{C_0},\pi,\pi}(\tau_{2j}) = id_{C_0} \quad (j = 3, 4, 5),
\end{aligned}$$

we define natural transformations $\nu : D_{id_{C_0},id_{C_0},\pi,\pi} \to D_{\sigma,\tau,\pi,\pi}$ and $\kappa : D_{\pi,\pi,id_{C_0},id_{C_0}} \to D^{\alpha}$ by $\nu_0 = \kappa_0 = (\varepsilon\pi, id_X) : X \to C_1 \times^{\tau}_{C_0} X$, $\nu_1 = \kappa_2 = \varepsilon$, $\nu_2 = \kappa_1 = id_X$, $\nu_i = \kappa_i = id_{C_0}$ $(i = 3, 4, 5)$. Then, the following diagram is commutative by (8.5.17), (8.5.19).

$$\begin{tikzcd}
(M^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[r, "\theta^{D^{\alpha}}(M)"] \arrow[d, "(M^{\varepsilon})^{[\pi,\,\pi]}"] & M^{[\pi\alpha,\,\tau\pi_\tau]} = M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[r, "\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}"] \arrow[d, "M^{(id_X,\,\varepsilon\pi)}"] & (M^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[d, "(M^{[\pi,\pi]})^{\varepsilon}"] \\
(M^{[id_{C_0},id_{C_0}]})^{[\pi,\,\pi]} \arrow[r, "\theta^{\pi,\pi,id_{C_0},id_{C_0}}(M)"] & M^{[id_{C_0}\pi,\,\pi id_X]} = M^{[\pi id_X,\,\tau\varepsilon\pi]} \arrow[r, "\theta^{id_{C_0},id_{C_0},\pi,\pi}(M)^{-1}"] & (M^{[\pi,\pi]})^{[\sigma\varepsilon,\tau\varepsilon]}
\end{tikzcd}$$

The lower row of the above diagram is identified with the identity morphism of $M^{[\pi,\pi]}$. Since $\check{\xi}M^{\varepsilon}$ is the identity morphism of M by (9.5.1), $\check{\xi}^{[\pi,\pi]}(M^{\varepsilon})^{[\pi,\,\pi]}$ is the identity morphism of $M^{[\pi,\,\pi]}$. It follows from the above facts and the definition of $\check{\xi}^{\alpha}$ that $M^{[\pi,\pi]} = (M^{[\pi,\pi]})^{[\sigma\varepsilon,\tau\varepsilon]} \xrightarrow{(M^{[\pi,\pi]})^{\varepsilon}} (M^{[\pi,\pi]})^{[\sigma,\tau]} \xrightarrow{\check{\xi}^{\alpha}} M^{[\pi,\pi]}$ coincides with the identity morphism of $M^{[\pi,\pi]}$.

By (8.5.8) and (8.5.17), (8.5.19), the following diagram is commutative.

$$\begin{tikzcd}
M \arrow[r, "\check{\xi}"] \arrow[d, "M^{\pi}"] & (M^{[\sigma,\tau]})^{[id_{C_0},id_{C_0}]} \arrow[r, "\theta^{id_{C_0},id_{C_0},\sigma,\tau}(M)"] \arrow[d, "(M^{[\sigma,\tau]})^{\pi}"] & M^{[id_{C_0}\sigma,\tau id_{C_1}]} = M^{[\sigma id_{C_1},id_{C_0}\tau]} \arrow[r, "\theta^{id_{C_0},id_{C_0},\sigma,\tau}(M)^{-1}"] \arrow[d, "M^{\pi\tau}"] & (M^{[id_{C_0},id_{C_0}]})^{[\sigma,\tau]} \arrow[d, "(M^{\pi})^{[\sigma,\tau]}"] \\
M^{[\pi,\pi]} \arrow[r, "\check{\xi}^{[\pi,\pi]}"] & (M^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[r, "\theta^{D^{\alpha}}(M)"] & M^{[\pi\alpha,\,\tau\pi_\tau]} = M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[r, "\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}"] & (M^{[\pi,\pi]})^{[\sigma,\tau]}
\end{tikzcd}$$

Therefore $M^{\pi} : (M, \xi) \to (M^{[\pi,\pi]}, \xi^{\alpha})$ is a morphism of representations by (9.5.4). □

Proposition 9.5.6 *Let $\varphi : (M, \xi) \to (N, \zeta)$ be a morphism of representations of $\boldsymbol{C}$. Assume that the following left morphism is an isomorphism for $L = M, N$ and that the right morphism is a monoomorphism for $L = M, N$.*

$$\theta^{\sigma,\tau,\pi,\pi}(L) : (L^{[\pi,\pi]})^{[\sigma,\tau]} \to L^{[\sigma\pi_\tau,\,\pi\tau_\pi]}, \quad \theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\pi,\pi}(L) : (L^{[\pi,\pi]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to L^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\pi\tau_\pi\bar{\mathrm{pr}}_{23}]}$$

Then, $\varphi^{[\pi,\pi]} : M^{[\pi,\pi]} \to N^{[\pi,\pi]}$ gives a morphism of representations from $(M^{[\pi,\pi]}, \xi^{\alpha})$ to $(N^{[\pi,\pi]}, \zeta^{\alpha})$.

Proof. The following diagram is commutative by (8.5.3) and (8.5.17).

$$\begin{tikzcd}
M^{[\pi,\pi]} \arrow[r, "\check{\xi}^{[\pi,\pi]}"] \arrow[d, "\varphi^{[\pi,\pi]}"] & (M^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[r, "\theta^{D^{\alpha}}(M)"] \arrow[d, "(\varphi^{[\sigma,\tau]})^{[\pi,\pi]}"] & M^{[\pi\alpha,\,\tau\pi_\tau]} = M^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[r, "\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}"] \arrow[d, "\varphi^{[\pi\sigma_\pi,\,\tau\pi_\sigma]}"] & (M^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[d, "(\varphi^{[\pi,\pi]})^{[\sigma,\tau]}"] \\
N^{[\pi,\pi]} \arrow[r, "\check{\xi}^{[\pi,\pi]}"] & (N^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[r, "\theta^{D^{\alpha}}(N)"] & N^{[\pi\alpha,\,\tau\pi_\tau]} = N^{[\sigma\pi_\tau,\,\pi\tau_\pi]} \arrow[r, "\theta^{\sigma,\tau,\pi,\pi}(N)^{-1}"] & (N^{[\pi,\pi]})^{[\sigma,\tau]}
\end{tikzcd}$$

Hence the assertion follows. □

Proposition 9.5.7 *Let $(\pi : X \to C_0,\ \alpha : C_1 \times^{\tau}_{C_0} X \to X)$ and $(\rho : Y \to C_0,\ \beta : C_1 \times^{\tau}_{C_0} Y \to Y)$ be internal presheaves on $\boldsymbol{C}$ and (M, ξ) a representation of $\boldsymbol{C}$. Assume that the following left morphism is an isomorphism for $\chi = \pi, \rho$ and that the right morphism is a monomorphism for $\chi = \pi, \rho$.*

$$\theta^{\sigma,\tau,\chi,\chi}(M) : (M^{[\chi,\chi]})^{[\sigma,\tau]} \to M^{[\sigma\chi_\tau,\,\chi\tau_\chi]}, \quad \theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\chi,\chi}(M) : (M^{[\chi,\chi]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to M^{[\sigma\mathrm{pr}_1\bar{\mathrm{pr}}_{12},\,\chi\tau_\chi\bar{\mathrm{pr}}_{23}]}$$

If a morphism $f : X \to Y$ of $\mathcal{E}$ defines a morphism of internal presheaves from $(\pi : X \to C_0, \alpha)$ to $(\rho : Y \to C_0, \beta)$, $M^f : M^{[\rho,\rho]} \to M^{[\pi,\pi]}$ is a morphism of representations from $(M^{[\rho,\rho]}, \xi^{\beta})$ to $(M^{[\pi,\pi]}, \xi^{\alpha})$.

Proof. Define a natural transformation $\lambda : D^{\alpha} \to D^{\beta}$ by $\lambda_0 = id_{C_1} \times_{C_0} f$, $\lambda_1 = f$, $\lambda_2 = id_{C_1}$, $\lambda_i = id_{C_0}$ $(i = 3, 4, 5)$. The following diagram is commutative by (8.5.6) and (8.5.18).

$$\begin{tikzcd}
(M^{[\rho,\rho]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\rho,\rho}(M)^{-1}"] \arrow[d, "(M^f)^{[\sigma,\tau]}"] & M^{[\rho\sigma_\rho,\,\tau\rho_\sigma]} = M^{[\sigma\rho_\sigma,\,\rho\beta]} \arrow[r, "\theta^{D^\beta}(M)"] \arrow[d, "M^{id_{C_1}\times_{C_0}f}"] & (M^{[\sigma,\tau]})^{[\rho,\rho]} \arrow[r, "\hat{\xi}^{[\rho,\rho]}"] \arrow[d, "(M^{[\sigma,\tau]})^f"] & M^{[\rho,\rho]} \arrow[d, "M^f"] \\
(M^{[\pi,\pi]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\pi,\pi}(M)^{-1}"] & M^{[\pi\sigma_\pi,\,\tau\pi_\sigma]} = M^{[\sigma\pi_\sigma,\,\pi\alpha]} \arrow[r, "\theta^{D^\alpha}(M)"] & (M^{[\sigma,\tau]})^{[\pi,\pi]} \arrow[r, "\hat{\xi}^{[\pi,\pi]}"] & M^{[\pi,\pi]}
\end{tikzcd}$$

Hence the assertion follows. ☐

For an object M of $\mathcal{F}_{C_0}$, we define a morphism $\check{\mu}_M : M^{[\sigma,\tau]} \to (M^{[\sigma,\tau]})^{[\sigma,\tau]}$ to be the following composition assuming that $\theta^{\sigma,\tau,\sigma,\tau}(M) : (M^{[\sigma,\tau]})^{[\sigma,\tau]} \to M^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]}$ is an isomorphism.

$$M^{[\sigma,\tau]} \xrightarrow{M^\mu} M^{[\sigma\mu,\,\tau\mu]} = M^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \xrightarrow{\theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}} (M^{[\sigma,\tau]})^{[\sigma,\tau]}$$

Let $C_1\times_{C_0}C_1 \xleftarrow{\mathrm{pr}_{12}} C_1\times_{C_0}C_1\times_{C_0}C_1 \xrightarrow{\mathrm{pr}_{23}} C_1\times_{C_0}C_1$ be a limit of a diagram $C_1\times_{C_0}C_1 \xrightarrow{\mathrm{pr}_2} C_1 \xleftarrow{\mathrm{pr}_1} C_1\times_{C_0}C_1$.

Proposition 9.5.8 *We assume that $\theta^{\sigma,\tau,\sigma,\tau}(M) : (M^{[\sigma,\tau]})^{[\sigma,\tau]} \to M^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]}$ is an isomorphism and that $\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma,\tau}(M) : (M^{[\sigma,\tau]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to M^{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\,\tau\mathrm{pr}_2\mathrm{pr}_{23}]}$ is a monomorphism. Let us denote by μ^r_M a morphism $E_{\sigma,\tau}(M^{[\sigma,\tau]})^{-1}_{M^{[\sigma,\tau]}}(\check{\mu}_M)$ of $\mathcal{F}_{C_1}$. Then, $(M^{[\sigma,\tau]},\mu^r_M)$ is a representation of $\boldsymbol{C}$. Moreover, if $\xi : \sigma^*(M) \to \tau^*(M)$ is a morphism of $\mathcal{F}_{C_1}$ such that (M,ξ) is a representation of $\boldsymbol{C}$, then $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi) : M \to M^{[\sigma,\tau]}$ defines a morphism of representations from (M,ξ) to $(M^{[\sigma,\tau]},\mu^r_M)$.*

Proof. The following diagram is commutative by (8.5.19) and (8.5.22).

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[r, "M^\mu"] \arrow[d, "M^\mu"'] & M^{[\sigma\mu,\,\tau\mu]} = M^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[r, "\theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}"] \arrow[d, "M^{\mu\times_{C_0}id_{C_1}}"] & (M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "(M^{[\sigma,\tau]})^\mu"] \\
M^{[\sigma\mu,\,\tau\mu]} \arrow[r, "M^{id_{C_1}\times_{C_0}\mu}"] \arrow[d, equal] & M^{[\sigma\mu\mathrm{pr}_{12},\,\tau\mu\mathrm{pr}_{23}]} \arrow[r, "\theta^{\sigma\mu,\tau\mu,\sigma,\tau}(M)^{-1}"] \arrow[d, equal] & (M^{[\sigma,\tau]})^{[\sigma\mu,\,\tau\mu]} \arrow[d, equal] \\
M^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[r, "M^{id_{C_1}\times_{C_0}\mu}"] \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}"] & M^{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\,\tau\mathrm{pr}_2\mathrm{pr}_{23}]} \arrow[r, "\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma,\tau}(M)^{-1}"] & (M^{[\sigma,\tau]})^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \\
(M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "(M^\mu)^{[\sigma,\tau]}"] & (M^{[\sigma\mu,\,\tau\mu]})^{[\sigma,\tau]} = (M^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]})^{[\sigma,\tau]} \arrow[r, "(\theta^{\sigma,\tau,\sigma,\tau}(M)^{[\sigma,\tau]})^{-1}"] \arrow[u, "\theta^{\sigma,\tau,\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)"] & ((M^{[\sigma,\tau]})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[u, "\theta^{\sigma,\tau,\sigma,\tau}(M^{[\sigma,\tau]})"]
\end{tikzcd}$$

Since the functor $D_{id_{C_0},id_{C_0},\sigma,\tau} : \mathcal{P} \to \mathcal{E}$ are given by

$$\begin{array}{ll}
D_{id_{C_0},id_{C_0},\sigma,\tau}(i) = C_1 \quad (i=0,2), & D_{id_{C_0},id_{C_0},\sigma,\tau}(i) = C_0 \quad (i=1,3,4,5), \\
D_{id_{C_0},id_{C_0},\sigma,\tau}(\tau_{01}) = D_{id_{C_0},id_{C_0},\sigma,\tau}(\tau_{24}) = \sigma, & D_{id_{C_0},id_{C_0},\sigma,\tau}(\tau_{02}) = id_{C_1} \\
D_{id_{C_0},id_{C_0},\sigma,\tau}(\tau_{13}) = D_{id_{C_0},id_{C_0},\sigma,\tau}(\tau_{14}) = id_{C_0}, & D_{id_{C_0},id_{C_0},\sigma,\tau}(\tau_{25}) = \tau,
\end{array}$$

we define a natural transformations $\nu : D_{id_{C_0},id_{C_0},\sigma,\tau} \to D_{\sigma,\tau,\sigma,\tau}$ by $\nu_0 = (\varepsilon\sigma, id_{C_1}) : C_1 \to C_1\times_{C_0}C_1$, $\nu_1 = \varepsilon$, $\nu_2 = id_{C_1}$, $\nu_i = \kappa_i = id_{C_0}$ $(i=3,4,5)$. Then, the following diagram is commutative by (8.5.17), (8.5.6).

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[r, "M^\mu"] \arrow[d, "id_{M^{[\sigma,\tau]}}"] & M^{[\sigma\mu,\tau\mu]} \arrow[r, equal] & M^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[r, "\theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}"] \arrow[d, "M^{[\varepsilon\sigma,\,id_{C_1}]}"] & (M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "(M^{[\sigma,\tau]})^\varepsilon"] \\
M^{[\sigma,\tau]} \arrow[r, "M^{id_{C_1}}"] & M^{[\sigma id_{C_1},\,\tau id_{C_1}]} \arrow[r, equal] & M^{[id_{C_0}\sigma,\,\tau id_{C_1}]} \arrow[r, "\theta^{id_{C_0},id_{C_0},\sigma,\tau}(M)^{-1}"] & (M^{[\sigma,\tau]})^{[\sigma\varepsilon,\tau\varepsilon]}
\end{tikzcd}$$

The lower row of the above diagram is identified with the identity morphism of $M^{[\sigma,\tau]}$ which implies that $\check{\mu}_M(M^{[\sigma,\tau]})^\varepsilon$ is the identity morphism of $M^{[\sigma,\tau]}$. Thus $(M^{[\sigma,\tau]},\mu^r_M)$ is a representation of $\boldsymbol{C}$ by (9.5.1).

If (M,ξ) is a representation of $\boldsymbol{C}$, then, $\theta^{\sigma,\tau,\sigma,\tau}(M)\check{\xi}^{[\sigma,\tau]}\check{\xi} = M^\mu\check{\xi}$ by (9.5.1). Hence $\check{\xi}^{[\sigma,\tau]}\check{\xi} = \check{\mu}_M\check{\xi}$ by the definition of $\check{\mu}_M$ and it follows from (9.5.4) that $\check{\xi}$ defines a morphism of representations from (M,ξ) to $(M^{[\sigma,\tau]},\mu^r_M)$. ☐

Proposition 9.5.9 *Assume that $\theta^{\sigma,\tau,\sigma,\tau}(L) : (L^{[\sigma,\tau]})^{[\sigma,\tau]} \to L^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]}$ is an isomorphism for $L = M,N$ and that $\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma,\tau}(L) : (L^{[\sigma,\tau]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to L^{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\,\tau\mathrm{pr}_2\mathrm{pr}_{23}]}$ is a monomorphism for $L = M,N$. For a morphism $\varphi : M \to N$, $\varphi^{[\sigma,\tau]} : M^{[\sigma,\tau]} \to N^{[\sigma,\tau]}$ defines a morphism of representations from $(M^{[\sigma,\tau]},\mu^r_M)$ to $(N^{[\sigma,\tau]},\mu^r_N)$.*

Proof. The following diagram is commtative by (8.5.8) and (8.5.19).

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[r, "M^{\mu}"] \arrow[d, "\varphi^{[\sigma,\tau]}"] & M^{[\sigma\mu,\tau\mu]} \arrow[r, equal] & M^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[r, "\theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}"] \arrow[d, "\varphi^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]}"] & (M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "(\varphi^{[\sigma,\tau]})^{[\sigma,\tau]}"] \\
N^{[\sigma,\tau]} \arrow[r, "N^{\mu}"] & N^{[\sigma\mu,\tau\mu]} \arrow[r, equal] & N^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[r, "\theta^{\sigma,\tau,\sigma,\tau}(N)^{-1}"] & (N^{[\sigma,\tau]})^{[\sigma,\tau]}
\end{tikzcd}$$

Hence the assertion follows from (9.5.4). □

Remark 9.5.10 *If $\varphi : (M,\xi) \to (N,\zeta)$ is a morphism of representations of $\boldsymbol{C}$, we have the following commutative diagram in* $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$.

$$\begin{tikzcd}
(M,\xi) \arrow[r, "\hat{\xi}"] \arrow[d, "\varphi"] & (M^{[\sigma,\tau]},\mu^r_M) \arrow[d, "\varphi^{[\sigma,\tau]}"] \\
(N,\zeta) \arrow[r, "\hat{\zeta}"] & (N^{[\sigma,\tau]},\mu^r_N)
\end{tikzcd}$$

Theorem 9.5.11 *Let M be an object of $\mathcal{F}_{C_0}$ and (N,ζ) a representation of $\boldsymbol{C}$. Assume that $\theta^{\sigma,\tau,\sigma,\tau}(L) : (L^{[\sigma,\tau]})^{[\sigma,\tau]} \to L^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]}$ is an isomorphism for $L = M,N$ and that $\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma,\tau}(L) : (L^{[\sigma,\tau]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to L^{[\sigma\mathrm{pr}_1\mathrm{pr}_{12},\,\tau\mathrm{pr}_2\mathrm{pr}_{23}]}$ is a monomorphism for $L = M,N$. Then, a map*

$$\Phi : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),(N^{[\sigma,\tau]},\mu^r_N)) \to \mathcal{F}_{C_0}(M,N)$$

defined by $\Phi(\varphi) = N^{\varepsilon}\varphi$ is bijective. Hence, if $\theta^{\sigma,\tau,\sigma,\tau}(L)$ an isomorphism and $\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma,\tau}(L)$ is a monomorphism for all $L \in \mathrm{Ob}\,\mathcal{F}_{C_0}$, a functor $\mathscr{R}_{\boldsymbol{C}} : \mathcal{F}_{C_0} \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ defined by $\mathscr{R}_{\boldsymbol{C}}(N) = (N^{[\sigma,\tau]},\mu^r_N)$ for $N \in \mathrm{Ob}\,\mathcal{F}_{C_0}$ and $\mathscr{R}_{\boldsymbol{C}}(\varphi) = \varphi^{[\sigma,\tau]}$ for $\varphi \in \mathrm{Mor}\mathcal{F}_{C_0}$ is a right adjoint of the forgetful functor $\mathscr{F}_{\boldsymbol{C}} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F}) \to \mathcal{F}_{C_0}$.

Proof. We put $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi) : M \to M^{[\sigma,\tau]}$. For $\psi \in \mathcal{F}_{C_0}(M,N)$, it follows from (9.5.9) that we have a morphism $\psi^{[\sigma,\tau]} : (M^{[\sigma,\tau]},\mu^r_M) \to (N^{[\sigma,\tau]},\mu^r_N)$ of representations. Since $\check{\xi} : (M,\xi) \to (M^{[\sigma,\tau]},\mu^r_M)$ is a morphism of representations by (9.5.8), $\psi^{[\sigma,\tau]}\check{\xi} : (M,\xi) \to (N^{[\sigma,\tau]},\mu^r_N)$ is a morphism of representations. It follows from (8.5.8) and (9.5.1) that we have $\Phi(\psi^{[\sigma,\tau]}\check{\xi}) = N^{\varepsilon}\psi^{[\sigma,\tau]}\check{\xi} = \psi M^{\varepsilon}\check{\xi} = \psi$. On the other hand, for $\varphi \in \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),(N^{[\sigma,\tau]},\mu^r_N))$, since $\varphi^{[\sigma,\tau]}\check{\xi} = \check{\mu}_N\varphi = N^{\mu}\theta^{\sigma,\tau,\sigma,\tau}(N)^{-1}\varphi$ by (9.5.4) and the following diagram commutes by (8.5.6) and (8.5.19),

$$\begin{tikzcd}
(N^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma,\tau}(N)"] \arrow[d, "(N^{\varepsilon})^{[\sigma,\tau]}"] & N^{[\sigma\mathrm{pr}_1,\,\tau\mathrm{pr}_2]} \arrow[r, equal] \arrow[d, "N^{(id_{C_1},\,\varepsilon\tau)}"] & N^{[\sigma\mu,\,\tau\mu]} \\
(N^{[id_{C_0},\,id_{C_0}]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,id_{C_0},id_{C_0}}(N)"] & N^{[id_{C_0}\sigma,\,\tau id_{C_1}]} & N^{[\sigma,\tau]} \arrow[l, "id_{N^{[\sigma,\tau]}}"'] \arrow[u, "N^{\mu}"']
\end{tikzcd}$$

we have $(N^{\varepsilon}\varphi)^{[\sigma,\tau]}\check{\xi} = (N^{\varepsilon})^{[\sigma,\tau]}\varphi^{[\sigma,\tau]}\check{\xi} = (N^{\varepsilon})^{[\sigma,\tau]}\theta^{\sigma,\tau,\sigma,\tau}(N)^{-1}N^{\mu}\varphi = \varphi$ by (8.5.3) and (8.5.23). Therefore a correspondence $\psi \mapsto \psi^{[\sigma,\tau]}\check{\xi}$ gives the inverse map of Φ. □

For morphisms $f : X \to Y$ and $g : X \to Z$ of $\mathcal{E}$, we denote by $[f,g]^* : \mathcal{F}_Z \to \mathcal{F}_Y$ the functor defined by $[f,g]^*(N) = N^{[f,g]}$ for $N \in \mathrm{Ob}\,\mathcal{F}_Z$ and $[f,g]^*(\varphi) = \varphi^{[f,g]}$ for $\varphi \in \mathrm{Mor}\,\mathcal{F}_Z$.

Proposition 9.5.12 *Let (N,ξ) and (N,ζ) be representations of $\boldsymbol{C}$ on $N \in \mathrm{Ob}\,\mathcal{F}_{C_0}$. We put $\check{\xi} = E_{\sigma,\tau}(N)_N(\xi)$ and $\check{\zeta} = E_{\sigma,\tau}(N)_N(\zeta)$. Assume that $[\sigma,\tau]^* : \mathcal{F}_{C_0} \to \mathcal{F}_{C_0}$ preserves equalizers (the presheaf $F_{\sigma,\tau,K}$ on $\mathcal{F}^{op}_{C_0}$ is representable for any $K \in \mathrm{Ob}\,\mathcal{F}_{C_0}$, for example. See (8.6.2).) and that $\theta^{\sigma,\tau,\sigma,\tau}(N)$ is a monomorphism. Let $\iota_{\xi,\zeta} : N^{(\xi:\zeta)} \to N$ be an equalizer of $\check{\xi},\check{\zeta} : N \to N^{[\sigma,\tau]}$.*

(1) There exists unique morphism $\check{\lambda} : (N^{(\xi:\zeta)})^{[\sigma,\tau]} \to N^{(\xi:\zeta)}$ that makes the following diagram commute.

$$\begin{tikzcd}
N \arrow[d, "\check{\xi}"] & N^{(\xi:\zeta)} \arrow[l, "\iota_{\xi,\zeta}"'] \arrow[r, "\iota_{\xi,\zeta}"] \arrow[d, "\check{\lambda}"] & N \arrow[d, "\check{\zeta}"] \\
N^{[\sigma,\tau]} & (N^{(\xi:\zeta)})^{[\sigma,\tau]} \arrow[l, "(\iota_{\xi,\zeta})^{[\sigma,\tau]}"'] \arrow[r, "(\iota_{\xi,\zeta})^{[\sigma,\tau]}"] & N^{[\sigma,\tau]}
\end{tikzcd}$$

(2) Moreover, we assume that $[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]^*:\mathcal{F}_{C_0}\to\mathcal{F}_{C_0}$ maps equalizers to monomorphisms (the presheaf $F_{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,K}$ on $\mathcal{F}_{C_0}^{op}$ is representable for any $K\in\mathrm{Ob}\,\mathcal{F}_{C_0}$,for example. See (8.6.2).). Put $\lambda=E_{\sigma,\tau}(N^{(\xi:\zeta)})^{-1}_{N^{(\xi:\zeta)}}(\check{\lambda})$. Then, $(N^{(\xi:\zeta)},\lambda)$ is a representation of $\boldsymbol{C}$ and $\iota_{\xi,\zeta}$ defines morphisms of representations $(N^{(\xi:\zeta)},\lambda)\to(N,\xi)$ and $(N^{(\xi:\zeta)},\lambda)\to(N,\zeta)$. Hence $(N^{(\xi:\zeta)},\lambda)$ is a subrepresentation of both (N,ξ) and (N,ζ).

(3) Let (M,ν) be a representation of $\boldsymbol{C}$. Suppose that a morphism $\varphi:M\to N$ of $\mathcal{F}_{C_0}$ gives morphisms $(M,\nu)\to(N,\xi)$ and $(M,\nu)\to(N,\zeta)$ of $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$. Then, there exists unique morphism $\tilde{\varphi}:(M,\nu)\to(N_{(\xi:\zeta)},\lambda)$ of $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$ that satisfies $\iota_{\xi,\zeta}\tilde{\varphi}=\varphi$.

Proof. (1) Put $\chi=\check{\xi}\iota_{\xi,\zeta}=\check{\zeta}\iota_{\xi,\zeta}:N^{(\xi:\zeta)}\to N^{[\sigma,\tau]}$. Then, it follows from (9.5.1) that

$$\theta^{\sigma,\tau,\sigma,\tau}(N)\check{\xi}^{[\sigma,\tau]}\chi=\theta^{\sigma,\tau,\sigma,\tau}(N)\check{\xi}^{[\sigma,\tau]}\check{\xi}\iota_{\xi,\zeta}=N^{\mu}\check{\xi}\iota_{\xi,\zeta}=N^{\mu}\check{\zeta}\iota_{\xi,\zeta}=\theta^{\sigma,\tau,\sigma,\tau}(N)\check{\xi}^{[\sigma,\tau]}\check{\zeta}\iota_{\xi,\zeta}=\theta^{\sigma,\tau,\sigma,\tau}(N)\check{\xi}^{[\sigma,\tau]}\chi,$$

which implies $\check{\xi}^{[\sigma,\tau]}\chi=\check{\zeta}^{[\sigma,\tau]}\chi$ since $\theta^{\sigma,\tau,\sigma,\tau}(N)$ is a monomorphism. Since $(\iota_{\xi,\zeta})^{[\sigma,\tau]}:(N^{(\xi:\zeta)})^{[\sigma,\tau]}\to N^{[\sigma,\tau]}$ is an equalizer of $\check{\xi}^{[\sigma,\tau]},\check{\zeta}^{[\sigma,\tau]}:N^{[\sigma,\tau]}\to(N^{[\sigma,\tau]})^{[\sigma,\tau]}$ by the assumption, there exists unique morphism $\check{\lambda}:N^{(\xi:\zeta)}\to(N^{(\xi:\zeta)})^{[\sigma,\tau]}$ that satisfies $(\iota_{\xi,\zeta})^{[\sigma,\tau]}\check{\lambda}=\chi$.

(2) By (8.5.3), (8.5.6), (8.5.19) and (9.5.1), the following diagrams are commutative.

$$\begin{tikzcd}
N^{(\xi:\zeta)} \arrow[r, "\check{\lambda}"] \arrow[d, "\iota_{\xi,\zeta}"] & (N^{(\xi:\zeta)})^{[\sigma,\tau]} \arrow[r, "\check{\lambda}^{[\sigma,\tau]}"] \arrow[d, "(\iota_{\xi,\zeta})^{[\sigma,\tau]}"] & ((N^{(\xi:\zeta)})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma,\tau}(N^{(\xi:\zeta)})"] \arrow[d, "((\iota_{\xi,\zeta})^{[\sigma,\tau]})^{[\sigma,\tau]}"] & (N^{(\xi:\zeta)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[d, "(\iota_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}"] \\
N \arrow[r, "\check{\xi}"] & N^{[\sigma,\tau]} \arrow[r, "\check{\xi}^{[\sigma,\tau]}"] & (N^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma,\tau}(N)"] & N_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}
\end{tikzcd}$$

$$\begin{tikzcd}
N^{(\xi:\zeta)} \arrow[r, "\check{\lambda}"] \arrow[d, "\iota_{\xi,\zeta}"] & (N^{(\xi:\zeta)})^{[\sigma,\tau]} \arrow[r, "(N^{(\xi:\zeta)})^{\mu}"] \arrow[d, "(\iota_{\xi,\zeta})^{[\sigma,\tau]}"] & (N^{(\xi:\zeta)})^{[\sigma\mu,\tau\mu]} \arrow[r, equal] \arrow[d, "(\iota_{\xi,\zeta})^{[\sigma\mu,\tau\mu]}"] & (N^{(\xi:\zeta)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[d, "(\iota_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}"] \\
N \arrow[r, "\check{\xi}"] & N^{[\sigma,\tau]} \arrow[r, "N^{\mu}"] & N^{[\sigma\mu,\tau\mu]} \arrow[r, equal] & N_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}
\end{tikzcd}$$

$$\begin{tikzcd}
N^{(\xi:\zeta)} \arrow[r, "\check{\lambda}"] \arrow[d, "\iota_{\xi,\zeta}"] & (N^{(\xi:\zeta)})^{[\sigma,\tau]} \arrow[r, "(N^{(\xi:\zeta)})^{\varepsilon}"] \arrow[d, "(\iota_{\xi,\zeta})^{[\sigma,\tau]}"] & (N^{(\xi:\zeta)})^{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, equal] \arrow[d, "(\iota_{\xi,\zeta})^{[\sigma\varepsilon,\tau\varepsilon]}"] & N^{(\xi:\zeta)} \arrow[d, "\iota_{\xi,\zeta}"] \\
N \arrow[r, "\check{\xi}"] & N^{[\sigma,\tau]} \arrow[r, "N^{\varepsilon}"] & N^{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, equal] & N
\end{tikzcd}$$

It follows from (9.5.1) that we have

$$(\iota_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}\theta^{\sigma,\tau,\sigma,\tau}(N^{(\xi:\zeta)})\check{\lambda}^{[\sigma,\tau]}\check{\lambda}=\theta^{\sigma,\tau,\sigma,\tau}(N)\check{\xi}^{[\sigma,\tau]}\check{\xi}\iota_{\xi,\zeta}=N^{\mu}\check{\xi}\iota_{\xi,\zeta}=(\iota_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}(N^{(\xi:\zeta)})^{\mu}\check{\lambda}$$
$$\iota_{\xi,\zeta}(N^{(\xi:\zeta)})^{\varepsilon}\check{\lambda}=N^{\varepsilon}\check{\xi}\iota_{\xi,\zeta}=\iota_{\xi,\zeta}$$

Since $\iota_{\xi,\zeta}$ and $(\iota_{\xi,\zeta})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ are monomorphisms, it follows that $\theta^{\sigma,\tau,\sigma,\tau}(N^{(\xi:\zeta)})\check{\lambda}^{[\sigma,\tau]}\check{\lambda}=(N^{(\xi:\zeta)})^{\mu}\check{\lambda}$ and $N^{\varepsilon}\check{\xi}\iota_{\xi,\zeta}=id_{N^{(\xi:\zeta)}}$. Therefore λ is a representation of $\boldsymbol{C}$ on $N^{(\xi:\zeta)}$ by (9.5.1). $\iota_{\xi,\zeta}:(N^{(\xi:\zeta)},\lambda)\to(N,\xi)$ and $\iota_{\xi,\zeta}:(N^{(\xi:\zeta)},\lambda)\to(N,\zeta)$ are morphisms of representations by the first assertion and (8.5.5).

(3) Put $\check{\nu}=E_{\sigma,\tau}(N)_N(\nu)$. Since $\varphi\check{\xi}=\check{\nu}\varphi^{[\sigma,\tau]}=\varphi\check{\zeta}$ by (9.5.4), there exists unique morphism $\tilde{\varphi}:M\to N^{(\xi:\zeta)}$ that satisfies $\iota_{\xi,\zeta}\tilde{\varphi}=\varphi$. Then, we have $(\iota_{\xi,\zeta})^{[\sigma,\tau]}\check{\lambda}\tilde{\varphi}=\check{\xi}\iota_{\xi,\zeta}\tilde{\varphi}=\check{\xi}\varphi=\varphi^{[\sigma,\tau]}\check{\nu}=(\iota_{\xi,\zeta})^{[\sigma,\tau]}\tilde{\varphi}^{[\sigma,\tau]}\check{\nu}$. Since $(\iota_{\xi,\zeta})^{[\sigma,\tau]}$ is a monomorphism, it follows $\check{\lambda}\tilde{\varphi}=\tilde{\varphi}^{[\sigma,\tau]}\check{\nu}$, which implies that $\tilde{\varphi}$ gives a morphism $(M,\nu)\to(N^{(\xi:\zeta)},\lambda)$ of representations of $\boldsymbol{C}$. □

Remark 9.5.13 *Assume that one of the following conditions.*

(i) $[\sigma,\tau]^*:\mathcal{F}_{C_0}\to\mathcal{F}_{C_0}$ preserves monomorphisms.
(ii) $\sigma^*:\mathcal{F}_{C_0}\to\mathcal{F}_{C_1}$ preserves monomorphisms.
(iii) The presheaf $F_{\sigma,\tau,M}$ on $\mathcal{F}_{C_0}^{op}$ is representable for $M\in\mathrm{Ob}\,\mathcal{F}_{C_0}$.

For representations (M,ξ), (M,ξ') and (N,ζ) of $\boldsymbol{C}$, suppose that there exists a monomorphism $\varphi:M\to N$ of $\mathcal{F}_{C_0}$ such that $\varphi:(M,\xi)\to(N,\zeta)$ and $\varphi:(M,\xi')\to(N,\zeta)$ are morphisms of $\mathrm{Rep}(\boldsymbol{C};\mathcal{F})$. Then, $\tau^*(\varphi)_*:\mathcal{F}_{C_1}(\sigma^*(M),\tau^*(M))\to\mathcal{F}_{C_1}(\sigma^*(M),\tau^*(N))$ is injective by the assumption. Hence $\tau_*(\varphi)\xi=\zeta\sigma^*(\varphi)=\tau^*(\varphi)\xi'$ implies $\xi=\xi'$.

Proposition 9.5.14 *Let (M,ξ), (N,ξ'), (M,ζ) and (N,ζ') be objects of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$. Put $\check{\xi}=E_{\sigma,\tau}(M)_M(\xi)$, $\check{\xi}'=E_{\sigma,\tau}(N)_N(\xi')$, $\check{\zeta}=E_{\sigma,\tau}(M)_M(\zeta)$ and $\check{\zeta}'=E_{\sigma,\tau}(N)_N(\zeta')$. Assume that $[\sigma,\tau]^*:\mathcal{F}_{C_0}\to\mathcal{F}_{C_0}$ preserves equalizers and that $[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]^*:\mathcal{F}_{C_0}\to\mathcal{F}_{C_0}$ map equalizers to monomorphisms (e.g., the presheaves $F_{\sigma,\tau,K}$ and $F_{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,K}$ on $\mathcal{F}_{C_0}^{op}$ is representable for any $K\in\mathrm{Ob}\,\mathcal{F}_{C_0}$. See (8.6.2)). Suppose that $\iota_{\xi,\zeta}:M^{(\xi:\zeta)}\to M$ is an equalizer of $\check{\xi},\check{\zeta}:M\to M^{[\sigma,\tau]}$ and that $\iota_{\xi',\zeta'}:N^{(\xi':\zeta')}\to N$ is an equalizer of $\check{\xi}',\check{\zeta}':N\to N^{[\sigma,\tau]}$. We denote by $(M^{(\xi:\zeta)},\lambda)$ and $(N^{(\xi':\zeta')},\lambda')$ the representations of $\boldsymbol{C}$ given in (9.5.12). If a morphism $\varphi:M\to N$ defines morphisms of representations $(M,\xi)\to(N,\xi')$ and $(M,\zeta)\to(N,\zeta')$, then there exists unique morphism $\tilde{\varphi}:(M^{(\xi:\zeta)},\lambda)\to(N^{(\xi':\zeta')},\lambda')$ of representations that satisfies $\iota_{\xi',\zeta'}\tilde{\varphi}=\varphi\iota_{\xi,\zeta}$.*

Proof. Since $\iota_{\xi,\zeta}:M^{(\xi:\zeta)}\to M$ defines morphisms $(M^{(\xi:\zeta)},\lambda)\to(M,\xi)$, $(M^{(\xi:\zeta)},\lambda)\to(M,\zeta)$ of representations of $\boldsymbol{C}$, $\varphi\iota_{\xi,\zeta}:M^{(\xi:\zeta)}\to N$ defines morphisms $(M^{(\xi:\zeta)},\lambda)\to(N,\xi')$, $(M^{(\xi:\zeta)},\lambda)\to(N,\zeta')$ of representations of $\boldsymbol{C}$. Hence it follows from (3) of (9.5.14) that there exists unique morphism $\tilde{\varphi}:M^{(\xi:\zeta)}\to N^{(\xi':\zeta')}$ that satisfies $\iota_{\xi',\zeta'}\tilde{\varphi}=\varphi\iota_{\xi,\zeta}$ and gives a morphism $(M^{(\xi:\zeta)},\lambda)\to(N^{(\xi':\zeta')},\lambda')$ of representations of $\boldsymbol{C}$. □

9.6 Left induced representations

Let $p:\mathcal{F}\to\mathcal{E}$ be a normalized cloven fibered category. For morphisms $f:X\to Y$, $g:X\to Z$ of $\mathcal{E}$ and an object M of $\mathcal{F}_Y$, we assume that the presheaf $F_{f,g,M}$ on $\mathcal{F}_Z^{op}$ is representable if necessary.

Let $\boldsymbol{C}=(C_0,C_1;\sigma,\tau,\varepsilon,\mu)$ and $\boldsymbol{D}=(D_0,D_1;\sigma',\tau',\varepsilon',\mu')$ be internal categories in $\mathcal{E}$. For an internal functor $\boldsymbol{f}=(f_0,f_1):\boldsymbol{D}\to\boldsymbol{C}$ in $\mathcal{E}$, let $D_0\xleftarrow{\sigma_{f_0}}D_0\times_{C_0}C_1\xrightarrow{f_{0\sigma}}C_1$ be a limit of a diagram $D_0\xrightarrow{f_0}C_0\xleftarrow{\sigma}C_1$. We consider the following diagram whose rectangles are all cartesian.

$$\begin{tikzcd}
D_0\times_{C_0}C_1\times_{C_0}C_1\times_{C_0}C_1 \arrow[r, "\tilde{\mathrm{pr}}_{123}"] \arrow[d, "\tilde{\mathrm{pr}}_{234}"] & D_0\times_{C_0}C_1\times_{C_0}C_1 \arrow[r, "\tilde{\mathrm{pr}}_{12}"] \arrow[d, "\tilde{\mathrm{pr}}_{23}"] & D_0\times_{C_0}C_1 \arrow[r, "\sigma_{f_0}"] \arrow[d, "f_{0\sigma}"] & D_0 \arrow[d, "f_0"] \\
C_1\times_{C_0}C_1\times_{C_0}C_1 \arrow[r, "\mathrm{pr}_{12}"] & C_1\times_{C_0}C_1 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & C_1 \arrow[r, "\sigma"] \arrow[d, "\tau"] & C_0 \\
 & C_1 \arrow[r, "\sigma"] & C_0 &
\end{tikzcd}$$

Let M be an object of $\mathcal{F}_{D_0}$. If $\theta_{\sigma_{f_0},\tau f_{0\sigma},\sigma,\tau}(M):M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}\to(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]}$ is an isomorphism, we define a morphism $\hat{\mu}_{\boldsymbol{f}}(M):(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]}\to M_{[\sigma_{f_0},\tau f_{0\sigma}]}$ to be the following composition.

$$(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]}\xrightarrow{\theta_{\sigma_{f_0},\tau f_{0\sigma},\sigma,\tau}(M)^{-1}}M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}=M_{[\sigma_{f_0}(id_{D_0}\times_{C_0}\mu),\tau f_{0\sigma}(id_{D_0}\times_{C_0}\mu)]}\xrightarrow{M_{id_{D_0}\times_{C_0}\mu}}M_{[\sigma_{f_0},\tau f_{0\sigma}]}$$

We consider the following commutative diagram below.

$$\begin{tikzcd}[column sep=small]
 & & & & D_0\times_{C_0}C_1\times_{C_0}C_1\times_{C_0}C_1 \arrow[dll, "\tilde{\mathrm{pr}}_{123}"'] \arrow[drr, "\tilde{\mathrm{pr}}_{234}"] & & & & & & \\
 & & D_0\times_{C_0}C_1\times_{C_0}C_1 \arrow[dl, "\tilde{\mathrm{pr}}_{12}"'] \arrow[drr, "\tilde{\mathrm{pr}}_{23}"] & & & & C_1\times_{C_0}C_1\times_{C_0}C_1 \arrow[dll, "\mathrm{pr}_{12}"'] \arrow[drr, "\mathrm{pr}_{23}"] & & & & \\
 & D_0\times_{C_0}C_1 \arrow[dl, "\sigma_{f_0}"'] \arrow[dr, "f_{0\sigma}"] & & & C_1\times_{C_0}C_1 \arrow[dll, "\mathrm{pr}_1"'] \arrow[drr, "\mathrm{pr}_2"] & & & & C_1\times_{C_0}C_1 \arrow[dll, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & & \\
D_0 & & C_1 \arrow[drr, "\tau"] & & & & C_1 \arrow[dll, "\sigma"'] \arrow[drr, "\tau"] & & & C_1 \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\
 & & & & C_0 & & & & C_0 & & C_0
\end{tikzcd}$$

Proposition 9.6.1 *Assume that that $\theta_{\sigma_{f_0},\tau f_{0\sigma},\sigma,\tau}(M):M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}\to(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]}$ is an isomorphism and that $\theta_{\sigma_{f_0},\tau f_{0\sigma},\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M):M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234}]}\to(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an epimorphism. We put*

$$\mu^l_{\boldsymbol{f}}(M)=P_{\sigma,\tau}(M_{[\sigma_{f_0},\tau f_{0\sigma}]})^{-1}_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}(\hat{\mu}_{\boldsymbol{f}}(M)):\sigma^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})\to\tau^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]}).$$

Then, $(M_{[\sigma_{f_0},\tau f_{0\sigma}]},\mu^l_{\boldsymbol{f}}(M))$ is a representation of $\boldsymbol{C}$.

Proof. It follows from (8.4.19) that the following diagram is commutative.

$$\begin{tikzcd}
(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, "\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma\varepsilon,\tau\varepsilon}(M)^{-1}"] \arrow[d, "(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{\varepsilon}"] & M_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[d, "M_{id_{D_0}\times_{C_0} C_1 \times_{C_0}\varepsilon}"] \arrow[dr, "id_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}"] & \\
(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(M)^{-1}"] & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "M_{id_{D_0}\times_{C_0}\mu}"] & M_{[\sigma_{f_0},\tau f_{0\sigma}]}
\end{tikzcd}$$

Hence a composition $M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} = (M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma\varepsilon,\tau\varepsilon]} \xrightarrow{(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{\varepsilon}} (M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \xrightarrow{\hat{\mu}_{\boldsymbol{f}}(M)} M_{[\sigma_{f_0},\,\tau f_{0\sigma}]}$ coincides with the identity morphism of $M_{[\sigma_{f_0},\,\tau f_{0\sigma}]}$.
Note that we have the following equalities.

$$\begin{aligned}
\sigma_{f_0}\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123} &= \sigma_{f_0}\tilde{\mathrm{pr}}_{12}(id_{D_0}\times_{C_0} id_{C_0}\times_{C_0}\mu) = \sigma_{f_0}\tilde{\mathrm{pr}}_{12}(id_{D_0}\times_{C_0}\mu\times_{C_0} id_{C_0})\\
\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234} &= \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(id_{D_0}\times_{C_0} id_{C_0}\times_{C_0}\mu) = \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(id_{D_0}\times_{C_0}\mu\times_{C_0} id_{C_0})\\
\sigma_{f_0}\tilde{\mathrm{pr}}_{12} &= \sigma_{f_0}(id_{D_0}\times_{C_0}\mu)\\
\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23} &= \tau f_{0\sigma}(id_{D_0}\times_{C_0}\mu)
\end{aligned}$$

It follows from (2) of (8.4.6), (8.4.19) and (8.4.22) that the following diagram commutes.

$$\begin{tikzcd}
(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[l, "\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(M)"'] \arrow[r, "M_{id_{D_0}\times_{C_0}\mu}"] & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \\
(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[u, "(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{\mu}"] \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})"] & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234}]} \arrow[l, "\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M)"'] \arrow[u, "M_{id_{D_0}\times_{C_0} id_{C_0}\times_{C_0}\mu}"] \arrow[r, "M_{id_{D_0}\times_{C_0}\mu\times_{C_0} id_{C_0}}"] \arrow[d, "\theta_{\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23},\sigma,\tau}(M)"] & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[u, "M_{id_{D_0}\times_{C_0}\mu}"'] \arrow[d, "\theta_{\sigma_{f_0},\tau f_{0\sigma},\sigma,\tau}(M)"'] \\
((M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]})_{[\sigma,\tau]} & (M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]})_{[\sigma,\tau]} \arrow[l, "\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(M)_{[\sigma,\tau]}"'] \arrow[r, "(M_{id_{D_0}\times_{C_0}\mu})_{[\sigma,\tau]}"] & (M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]}
\end{tikzcd}$$

Thus the following diagram commutes.

$$\begin{tikzcd}
(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, "(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{\mu}"] \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})"] & (M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[r, "\hat{\mu}_{\boldsymbol{f}}(M)"] & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \\
((M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "(\hat{\mu}_{\boldsymbol{f}}(M))_{[\sigma,\tau]}"] & (M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[ur, "\hat{\mu}_{\boldsymbol{f}}(M)"'] &
\end{tikzcd}$$

and $\hat{\mu}_{\boldsymbol{f}}(M)$ satisfies the conditions of (9.4.1). □

Proposition 9.6.2 *Let $\varphi : M \to N$ be a morphisms of $\mathcal{F}_{D_0}$. Assume that that the following upper morphism is an isomorphism and that the lower morphism is an epimorphism for $L = M, N$.*

$$\begin{gathered}
\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(L) : L_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \longrightarrow (L_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]}\\
\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(L) : L_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234}]} \longrightarrow (L_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}
\end{gathered}$$

Then, $\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]} : (M_{[\sigma_{f_0},\,\tau f_{0\sigma}]}, \mu^l_{\boldsymbol{f}}(M)) \to (N_{[\sigma_{f_0},\,\tau f_{0\sigma}]}, \mu^l_{\boldsymbol{f}}(N))$ is a morphism of representations of $\boldsymbol{C}$.

Proof. The following diagram is commutative by (8.4.8) and (8.4.19).

$$\begin{tikzcd}
(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(M)^{-1}"] \arrow[d, "(\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]}"] & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "M_{id_{D_0}\times_{C_0}\mu}"] \arrow[d, "\varphi_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}"] & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[d, "\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"] \\
(N_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[r, "\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(N)^{-1}"] & N_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "N_{id_{D_0}\times_{C_0}\mu}"] & N_{[\sigma_{f_0},\,\tau f_{0\sigma}]}
\end{tikzcd}$$

Hence the assertion follows from (9.4.7). □

Let $D_1 \xleftarrow{\tilde{\mathrm{pr}}_1} D_1\times_{C_0} C_1 \xrightarrow{\tilde{\mathrm{pr}}_2} C_1$ be a limit of a diagram $D_1 \xrightarrow{f_0\tau'} C_0 \xleftarrow{\sigma} C_1$. Then, there exists unique morphism $\tau'\times_{C_0} id_{C_1} : D_1\times_{C_0} C_1 \to D_0\times_{C_0} C_1$ that satisfies $\sigma_{f_0}(\tau'\times_{C_0} id_{C_1}) = \tau'\tilde{\mathrm{pr}}_1$ and $f_{0\sigma}(\tau'\times_{C_0} id_{C_1}) = \tilde{\mathrm{pr}}_2$.

$$\begin{tikzcd}
D_1 \times_{C_0} C_1 \arrow[r, "\tilde{\mathrm{pr}}_2"] \arrow[d, "\tilde{\mathrm{pr}}_1"] & C_1 \arrow[d, "\sigma"] \\
D_1 \arrow[r, "f_0\tau'"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
D_1 \times_{C_0} C_1 \arrow[rrd, bend left, "\tilde{\mathrm{pr}}_2"] \arrow[rd, dotted, "\tau' \times_{C_0} id_{C_1}"] \arrow[dd, "\tilde{\mathrm{pr}}_1"] & & \\
& D_0 \times_{C_0} C_1 \arrow[r, "f_{0\sigma}"] \arrow[d, "\sigma_{f_0}"] & C_1 \arrow[d, "\sigma"] \\
D_1 \arrow[r, "\tau'"] & D_0 \arrow[r, "f_0"] & C_0
\end{tikzcd}$$

We note that $D_1 \xleftarrow{\tilde{\mathrm{pr}}_1} D_1 \times_{C_0} C_1 \xrightarrow{\tau' \times_{C_0} id_{C_1}} D_0 \times_{C_0} C_1$ is a limit of a diagram $D_1 \xrightarrow{\tau'} D_0 \xleftarrow{\sigma_{f_0}} D_0 \times_{C_0} C_1$. Since (f_0, f_1) is an internal functor, we also have unique morphism $f_1 \times_{C_0} id_{C_1} : D_1 \times_{C_0} C_1 \to C_1 \times_{C_0} C_1$ that satisfies $\mathrm{pr}_1(f_1 \times_{C_0} id_{C_1}) = f_1\tilde{\mathrm{pr}}_1$ and $\mathrm{pr}_2(f_1 \times_{C_0} id_{C_1}) = \tilde{\mathrm{pr}}_2$. Then, we have

$$\sigma\mu(f_1 \times_{C_0} id_{C_1}) = \sigma\mathrm{pr}_1(f_1 \times_{C_0} id_{C_1}) = \sigma f_1\tilde{\mathrm{pr}}_1 = f_0\sigma'\tilde{\mathrm{pr}}_1$$

which implies that there exists unique morphism $(\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1})) : D_1 \times_{C_0} C_1 \to D_0 \times_{C_0} C_1$ that satisfies $\sigma_{f_0}(\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1})) = \sigma'\tilde{\mathrm{pr}}_1$ and $f_{0\sigma}(\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1})) = \mu(f_1 \times_{C_0} id_{C_1})$. Hence we have

$$\tau f_{0\sigma}(\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1})) = \tau\mu(f_1 \times_{C_0} id_{C_1}) = \tau\mathrm{pr}_2(f_1 \times_{C_0} id_{C_1}) = \tau\tilde{\mathrm{pr}}_2 = \tau f_{0\sigma}(\tau' \times_{C_0} id_{C_1}).$$

Let $D_1 \times_{C_0} C_1 \xleftarrow{\tilde{\mathrm{pr}}_{12}} D_1 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{\tilde{\mathrm{pr}}_3} C_1$ be a limit of a diagram $D_1 \times_{C_0} C_1 \xrightarrow{\sigma\tilde{\mathrm{pr}}_1} D_1 \times_{C_0} C_1 \times_{C_0} C_1 \xleftarrow{\sigma} C_1$.

Assumption 9.6.3 *For a representation (M, ξ) of $\boldsymbol{D}$, we put $\hat{\xi} = P_{\sigma',\tau'}(M)_M : M_{[\sigma',\tau']} \to M$. We assume the following.*

(i) A coequalizer of the following morphisms of $\mathcal{F}_{C_0}$ exists.

$$M_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau f_{0\sigma}(\tau' \times_{C_0} id_{C_1})]} \xrightarrow{\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)} (M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]} \xrightarrow{\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]}} M_{[\sigma_{f_0},\, \tau f_{0\sigma}]}$$

$$M_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau f_{0\sigma}(\tau' \times_{C_0} id_{C_1})]} = M_{[\sigma_{f_0}(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1 \times_{C_0} id_{C_1})),\, \tau f_{0\sigma}(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1 \times_{C_0} id_{C_1}))]} \xrightarrow{M_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1 \times_{C_0} id_{C_1}))}} M_{[\sigma_{f_0},\, \tau f_{0\sigma}]}$$

(ii) Let us denote by $P^{\boldsymbol{f}}_{(M,\xi)} : M_{[\sigma_{f_0},\, \tau f_{0\sigma}]} \to (M,\xi)_{\boldsymbol{f}}$ a coequalizer of the above morphisms. Then $\big(P^{\boldsymbol{f}}_{(M,\xi)}\big)_{[\sigma,\tau]} : (M_{[\sigma_{f_0},\, \tau f_{0\sigma}]})_{[\sigma,\tau]} \to ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]}$ is a coequalizer of the following morphisms.

$$(M_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau f_{0\sigma}(\tau' \times_{C_0} id_{C_1})]})_{[\sigma,\tau]} \xrightarrow{\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)_{[\sigma,\tau]}} ((M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]} \xrightarrow{(\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]}} (M_{[\sigma_{f_0},\, \tau f_{0\sigma}]})_{[\sigma,\tau]}$$

$$(M_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau f_{0\sigma}(\tau' \times_{C_0} id_{C_1})]})_{[\sigma,\tau]} \xrightarrow{(M_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1 \times_{C_0} id_{C_1}))})_{[\sigma,\tau]}} (M_{[\sigma_{f_0},\, \tau f_{0\sigma}]})_{[\sigma,\tau]}$$

(iii) $\theta_{\sigma_{f_0},\tau f_{0\sigma},\sigma,\tau}(M) : M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \to (M_{[\sigma_{f_0},\, \tau f_{0\sigma}]})_{[\sigma,\tau]}$ is an isomorphism.
(iv) The following morphisms are epimorphisms.

$$\theta_{\sigma_{f_0},\tau f_{0\sigma},\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2}(M) : M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234}]} \longrightarrow (M_{[\sigma_{f_0},\, \tau f_{0\sigma}]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$$
$$\theta_{\sigma'\tilde{\mathrm{pr}}_1,\, \tau f_{0\sigma}(\tau' \times_{C_0} id_{C_1}),\sigma,\tau}(M) : M_{[\sigma'\tilde{\mathrm{pr}}_1\bar{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(\tau' \times_{C_0} id_{C_1} \times_{C_0} id_{C_1})]} \longrightarrow (M_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau f_{0\sigma}(\tau' \times_{C_0} id_{C_1})]})_{[\sigma,\tau]}$$
$$(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} : (M_{[\sigma_{f_0},\, \tau f_{0\sigma}]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \longrightarrow ((M,\xi)_{\boldsymbol{f}})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$$

The following diagram commutes.

$$\begin{tikzcd}
D_0 \times_{C_0} C_1 \arrow[d, "\sigma_{f_0}"] & D_0 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[l, "\tilde{\mathrm{pr}}_{12}"'] \arrow[r, "\tilde{\mathrm{pr}}_{23}"] \arrow[d, "id_{D_0} \times_{C_0} \mu"] & C_1 \times_{C_0} C_1 \arrow[d, "\mu"] \\
D_0 & D_0 \times_{C_0} C_1 \arrow[l, "\sigma_{f_0}"'] \arrow[r, "f_{0\sigma}"] & C_1
\end{tikzcd}$$

Hence we have $\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23} = \tau\mu\tilde{\mathrm{pr}}_{23} = \tau f_{0\sigma}(id_{D_0} \times_{C_0} \mu)$ and $\sigma_{f_0}\tilde{\mathrm{pr}}_{12} = \sigma_{f_0}(id_{D_0} \times_{C_0} \mu)$.
Consider the following diagram whose rhombuses are all cartesian.

$$\begin{tikzcd}[column sep=small]
 & & & & D_1\times_{C_0}C_1\times_{C_0}C_1 \arrow[dll, "\bar{\mathrm{pr}}_{12}"'] \arrow[drr, "\tau'\times_{C_0}id_{C_1}\times_{C_0}id_{C_1}"] & & & & & \\
 & & D_1\times_{C_0}C_1 \arrow[dl, "\tilde{\mathrm{pr}}_1"'] \arrow[drr, "\tau'\times_{C_0}id_{C_1}"] & & & & D_0\times_{C_0}C_1\times_{C_0}C_1 \arrow[dll, "\tilde{\mathrm{pr}}_{12}"'] \arrow[drr, "\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}"] & & & \\
 & D_1 \arrow[dl, "\sigma'"'] \arrow[dr, "\tau'"] & & & D_0\times_{C_0}C_1 \arrow[dll, "\sigma_{f_0}"'] \arrow[drr, "\tau f_{0\sigma}"] & & & & C_1 \arrow[dll, "\sigma"'] \arrow[dr, "\tau"] & \\
D_0 & & D_0 & & & & C_0 & & & C_0
\end{tikzcd}$$

It follows from (8.4.22) that

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1\bar{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(\tau'\times_{C_0}id_{C_1}\times_{C_0}id_{C_1})]} \arrow[r, "\theta_{\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0}id_{C_1}),\sigma,\tau}(M)"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}}(M)"] & \big(M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0}id_{C_1})]}\big)_{[\sigma,\tau]} \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)_{[\sigma,\tau]}"] \\
\big(M_{[\sigma',\tau']}\big)_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(M_{[\sigma',\tau']})"] & \big((M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]}\big)_{[\sigma,\tau]}
\end{tikzcd}$$

is commutative. The following diagrams are commutative by (8.4.19), (8.4.17), (8.4.8), respectively.

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1\bar{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(\tau'\times_{C_0}id_{C_1}\times_{C_0}id_{C_1})]} \arrow[r, "M_{id_{D_1}\times_{C_0}\mu}"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}}(M)"] & M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0}id_{C_1})]} \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)"] \\
\big(M_{[\sigma',\tau']}\big)_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "(M_{[\sigma',\tau']})_{id_{D_0}\times_{C_0}\mu}"] & \big(M_{[\sigma',\tau']}\big)_{[\sigma_{f_0},\tau f_{0\sigma}]}
\end{tikzcd}$$

$$\begin{tikzcd}
\big(M_{[\sigma',\tau']}\big)_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(M_{[\sigma',\tau']})"] \arrow[d, "\hat{\xi}_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}"] & \big((M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]}\big)_{[\sigma,\tau]} \arrow[d, "(\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]}"] \\
M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(M)"] & \big(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]}\big)_{[\sigma,\tau]}
\end{tikzcd}$$

$$\begin{tikzcd}
\big(M_{[\sigma',\tau']}\big)_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "(M_{[\sigma',\tau']})_{id_{D_0}\times_{C_0}\mu}"] \arrow[d, "\hat{\xi}_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}"] & \big(M_{[\sigma',\tau']}\big)_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[d, "\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]}"] \\
M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "M_{id_{D_0}\times_{C_0}\mu}"] & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]}
\end{tikzcd}$$

The associativity of μ implies that a diagram

$$\begin{tikzcd}
D_1\times_{C_0}C_1\times_{C_0}C_1 \arrow[r, "id_{D_1}\times_{C_0}\mu"] \arrow[d, "(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))\times_{C_0}id_{C_1}"] & D_1\times_{C_0}C_1 \arrow[d, "(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))"] \\
D_0\times_{C_0}C_1\times_{C_0}C_1 \arrow[r, "id_{D_0}\times_{C_0}\mu"] & D_0\times_{C_0}C_1
\end{tikzcd}$$

is commutative. Hence the following diagram is commutative by (8.4.6).

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1\bar{\mathrm{pr}}_{12},\,\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(\tau'\times_{C_0}id_{C_1}\times_{C_0}id_{C_1})]} \arrow[r, "M_{id_{D_1}\times_{C_0}\mu}"] \arrow[d, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))\times_{C_0}id_{C_1}}"] & M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0}id_{C_1})]} \arrow[d, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}"] \\
M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "M_{id_{D_0}\times_{C_0}\mu}"] & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]}
\end{tikzcd}$$

Moreover, it follows from (8.4.19) that the following diagram commutes.

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1\bar{\mathrm{pr}}_{12},\,\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}(\tau'\times_{C_0}id_{C_1}\times_{C_0}id_{C_1})]} \arrow[r, "\theta_{\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0}id_{C_1}),\sigma,\tau}(M)"] \arrow[d, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))\times_{C_0}id_{C_1}}"] & (M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0}id_{C_1})]})_{[\sigma,\tau]} \arrow[d, "(M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))})_{[\sigma,\tau]}"] \\
M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(M)"] & (M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]}
\end{tikzcd}$$

Since $P^{\boldsymbol{f}}_{(M,\xi)}$ is a coequalizer of $\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]}\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)$ and $M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}$, we have

$$\begin{aligned}
&P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)(\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]}\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M))_{[\sigma,\tau]}\theta_{\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0}id_{C_1}),\sigma,\tau}(M)\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}M_{id_{D_0}\times_{C_0}\mu}\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(M)^{-1}(\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]}\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(M_{[\sigma',\tau']})\theta_{\sigma',\tau',\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}}(M)\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}M_{id_{D_0}\times_{C_0}\mu}\hat{\xi}_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}\theta_{\sigma',\tau',\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}}(M)\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]}(M_{[\sigma',\tau']})_{id_{D_0}\times_{C_0}\mu}\theta_{\sigma',\tau',\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}}(M)\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]}\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)M_{id_{D_1}\times_{C_0}\mu} = P^{\boldsymbol{f}}_{(M,\xi)}M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}M_{id_{D_1}\times_{C_0}\mu}\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}M_{id_{D_0}\times_{C_0}\mu}M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))\times_{C_0}id_{C_1}}\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(M)M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))\times_{C_0}id_{C_1}}\\
&\quad= P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)(M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))})_{[\sigma,\tau]}\theta_{\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0}id_{C_1}),\sigma,\tau}(M).
\end{aligned}$$

Therefore, it follows from the assumption (iv) of (9.6.3) that we have

$$P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)(\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]}\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M))_{[\sigma,\tau]} = P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)(M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))})_{[\sigma,\tau]}.$$

Hence (ii) of (9.6.3) implies that there exists unique morphism $\hat{\xi}_{\boldsymbol{f}} : ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]} \to (M,\xi)_{\boldsymbol{f}}$ that satisfies $\hat{\xi}_{\boldsymbol{f}}(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]} = P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)$. We put $\xi^l_{\boldsymbol{f}} = P_{\sigma,\tau}((M,\xi)_{\boldsymbol{f}})^{-1}_{(M,\xi)_{\boldsymbol{f}}}(\hat{\xi}_{\boldsymbol{f}}) : \sigma^*((M,\xi)_{\boldsymbol{f}}) \to \tau^*((M,\xi)_{\boldsymbol{f}})$.

Proposition 9.6.4 $((M,\xi)_{\boldsymbol{f}}, \xi^l_{\boldsymbol{f}})$ *is a representation of* $\boldsymbol{C}$ *and* $P^{\boldsymbol{f}}_{(M,\xi)} : (M_{[\sigma_{f_0},\,\tau f_{0\sigma}]}, \mu^l_{\boldsymbol{f}}(M)) \to ((M,\xi)_{\boldsymbol{f}}, \xi^l_{\boldsymbol{f}})$ *is a morphism of representations of* $\boldsymbol{C}$.

Proof. It follows from (8.4.8), (9.6.1), (8.4.19) and the definition of $\hat{\xi}_{\boldsymbol{f}}$ that we have

$$\begin{aligned}
\hat{\xi}_{\boldsymbol{f}}((M,\xi)_{\boldsymbol{f}})_{\mu}(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} &= \hat{\xi}_{\boldsymbol{f}}(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{\mu} = P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{\mu}\\
&= P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)\hat{\mu}_{\boldsymbol{f}}(M)_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})\\
&= \hat{\xi}_{\boldsymbol{f}}(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}\hat{\mu}_{\boldsymbol{f}}(M)_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})\\
&= \hat{\xi}_{\boldsymbol{f}}(\hat{\xi}_{\boldsymbol{f}})_{[\sigma,\tau]}((P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]})_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})\\
&= \hat{\xi}_{\boldsymbol{f}}(\hat{\xi}_{\boldsymbol{f}})_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}((M,\xi)_{\boldsymbol{f}}))(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}.
\end{aligned}$$

Since we assume that $(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an epimorphism in (9.6.3), $\hat{\xi}_{\boldsymbol{f}}((M,\xi)_{\boldsymbol{f}})_{\mu} = \hat{\xi}_{\boldsymbol{f}}(\hat{\xi}_{\boldsymbol{f}})_{[\sigma,\tau]}\theta_{\sigma,\tau,\sigma,\tau}((M,\xi)_{\boldsymbol{f}})$ holds. (See the diagram below.)

$$\begin{tikzcd}[column sep=small]
((M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[ddd, "\hat{\mu}_{\boldsymbol{f}}(M)_{[\sigma,\tau]}"'] \arrow[ddr, "((P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]})_{[\sigma,\tau]}"] & (M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[l, "\theta_{\sigma,\tau,\sigma,\tau}(M_{[\sigma_{f_0},\tau f_{0\sigma}]})"'] \arrow[r, "(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{\mu}"] \arrow[d, "(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}"] & (M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[r, "\hat{\mu}_{\boldsymbol{f}}(M)"] \arrow[d, "(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}"] & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[d, "P^{\boldsymbol{f}}_{(M,\xi)}"] \\
 & ((M,\xi)_{\boldsymbol{f}})_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[r, "((M,\xi)_{\boldsymbol{f}})_{\mu}"] \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}((M,\xi)_{\boldsymbol{f}})"] & ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]} \arrow[r, "\hat{\xi}_{\boldsymbol{f}}"] & (M,\xi)_{\boldsymbol{f}} \\
 & (((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "(\hat{\xi}_{\boldsymbol{f}})_{[\sigma,\tau]}"] & ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]} \arrow[ur, "\hat{\xi}_{\boldsymbol{f}}"'] & \\
(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[urr, "(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}"] \arrow[rrr, "\hat{\mu}_{\boldsymbol{f}}(M)"] & & & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[uu, "P^{\boldsymbol{f}}_{(M,\xi)}"']
\end{tikzcd}$$

The following diagram is commutative by (8.4.8) and the definition of $\hat{\xi}_{\boldsymbol{f}}$.

$$\begin{tikzcd}
M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} = (M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, "(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{\varepsilon}"] \arrow[d, "P^{\boldsymbol{f}}_{(M,\xi)}"] & (M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[r, "\hat{\mu}_{\boldsymbol{f}}(M)"] \arrow[d, "(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}"] & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[d, "P^{\boldsymbol{f}}_{(M,\xi)}"] \\
(M,\xi)_{\boldsymbol{f}} = ((M,\xi)_{\boldsymbol{f}})_{[\sigma\varepsilon,\tau\varepsilon]} \arrow[r, "((M,\xi)_{\boldsymbol{f}})_{\varepsilon}"] & ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]} \arrow[r, "\hat{\xi}_{\boldsymbol{f}}"] & (M,\xi)_{\boldsymbol{f}}
\end{tikzcd}$$

Since $\hat{\mu}_{\boldsymbol{f}}(M)(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{\varepsilon}$ is the identity morphism of $M_{[\sigma_{f_0},\,\tau f_{0\sigma}]}$, we have $\hat{\xi}_{\boldsymbol{f}}((M,\xi)_{\boldsymbol{f}})_{\varepsilon}P^{\boldsymbol{f}}_{(M,\xi)} = P^{\boldsymbol{f}}_{(M,\xi)}$ which implies that $\hat{\xi}_{\boldsymbol{f}}((M,\xi)_{\boldsymbol{f}})_{\varepsilon}P^{\boldsymbol{f}}_{(M,\xi)}$ is the identity morphism of $(M,\xi)_{\boldsymbol{f}}$, since $P^{\boldsymbol{f}}_{(M,\xi)}$ is an epimorphism. Hence $((M,\xi)_{\boldsymbol{f}}, \xi^l_{\boldsymbol{f}})$ is a representation of $\boldsymbol{C}$ by (9.4.1). It follows from (9.4.5) and the definition of $\hat{\xi}_{\boldsymbol{f}}$ that $P^{\boldsymbol{f}}_{(M,\xi)}$ is a morphism of representations. $\square$

We assume (9.6.3) also for a representation (N,ζ) of $\boldsymbol{D}$. Let $\varphi : (M,\xi) \to (N,\zeta)$ be a morphism of representations of $\boldsymbol{D}$. The following diagrams are commutative by (8.4.19), (8.4.3) and (8.4.8).

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]} \arrow[r, "\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)"] \arrow[d, "\varphi_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]}"] & (M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[r, "\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]}"] \arrow[d, "(\varphi_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]}"] & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[d, "\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"] \\
N_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]} \arrow[r, "\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(N)"] & (N_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[r, "\hat{\zeta}_{[\sigma_{f_0},\tau f_{0\sigma}]}"] & N_{[\sigma_{f_0},\,\tau f_{0\sigma}]}
\end{tikzcd}$$

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]} \arrow[r, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))}"] \arrow[d, "\varphi_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]}"] & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[d, "\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"] \\
N_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]} \arrow[r, "N_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))}"] & N_{[\sigma_{f_0},\,\tau f_{0\sigma}]}
\end{tikzcd}$$

Hence there exists unique morphism $\varphi_{\boldsymbol{f}} : (M,\xi)_{\boldsymbol{f}} \to (N,\zeta)_{\boldsymbol{f}}$ that satisfies $\varphi_{\boldsymbol{f}}P^{\boldsymbol{f}}_{(M,\xi)} = P^{\boldsymbol{f}}_{(N,\zeta)}\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]}$.

Proposition 9.6.5 *$\varphi_{\boldsymbol{f}} : ((M,\xi)_{\boldsymbol{f}}, \xi^l_{\boldsymbol{f}}) \to ((N,\zeta)_{\boldsymbol{f}}, \zeta^l_{\boldsymbol{f}})$ is a morphism of representations of $\boldsymbol{C}$.*

Proof. It follows from (9.6.2) that the outer rectangle of the following diagram is commutative.

$$\begin{tikzcd}
(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[rrr, "\hat{\mu}_{\boldsymbol{f}}(M)"] \arrow[dr, "(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}"] \arrow[ddd, "(\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]}"] & & & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[dl, "P^{\boldsymbol{f}}_{(M,\xi)}"'] \arrow[ddd, "\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"] \\
 & ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]} \arrow[r, "\hat{\xi}_{\boldsymbol{f}}"] \arrow[d, "(\varphi_{\boldsymbol{f}})_{[\sigma,\tau]}"] & (M,\xi)_{\boldsymbol{f}} \arrow[d, "\varphi_{\boldsymbol{f}}"] & \\
 & ((N,\zeta)_{\boldsymbol{f}})_{[\sigma,\tau]} \arrow[r, "\hat{\zeta}_{\boldsymbol{f}}"] & (N,\zeta)_{\boldsymbol{f}} & \\
(N_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[ur, "(P^{\boldsymbol{f}}_{(N,\zeta)})_{[\sigma,\tau]}"'] \arrow[rrr, "\hat{\mu}_{\boldsymbol{f}}(N)"] & & & N_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[ul, "P^{\boldsymbol{f}}_{(N,\zeta)}"]
\end{tikzcd}$$

Then, by the definitions of $\hat{\xi}_{\boldsymbol{f}}$, $\hat{\zeta}_{\boldsymbol{f}}$ and $\varphi_{\boldsymbol{f}}$, we have

$$\begin{aligned}\varphi_{\boldsymbol{f}}\hat{\xi}_{\boldsymbol{f}}(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]} &= \varphi_{\boldsymbol{f}}P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M) = P^{\boldsymbol{f}}_{(N,\zeta)}\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]}\hat{\mu}_{\boldsymbol{f}}(M) = P^{\boldsymbol{f}}_{(N,\zeta)}\hat{\mu}_{\boldsymbol{f}}(N)(\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]}\\ &= \hat{\zeta}_{\boldsymbol{f}}(P^{\boldsymbol{f}}_{(N,\zeta)})_{[\sigma,\tau]}(\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} = \hat{\zeta}_{\boldsymbol{f}}(\varphi_{\boldsymbol{f}})_{[\sigma,\tau]}(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}.\end{aligned}$$

Since $(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}$ is an epimorphism by (ii) of (9.6.3), the above equality implies $\varphi_{\boldsymbol{f}}\hat{\xi}_{\boldsymbol{f}} = \hat{\zeta}_{\boldsymbol{f}}(\varphi_{\boldsymbol{f}})_{[\sigma,\tau]}$. Therefore $\varphi_{\boldsymbol{f}}$ is a morphism of representations of $\boldsymbol{D}$ by (9.4.5). $\square$

Define functors $S, T, U : \mathcal{P} \to \mathcal{E}$ and natural transformations $\alpha : S \to T$, $\beta : T \to U$ as follows.

$$\begin{array}{llllll}
S(0)=D_1 & S(1)=D_1 & S(2)=D_0 & S(3)=D_0 & S(4)=D_0 & S(5)=D_0\\
S(\tau_{01})=id_{D_1} & S(\tau_{02})=\tau' & S(\tau_{13})=\sigma' & S(\tau_{14})=\tau' & S(\tau_{24})=id_{D_0} & S(\tau_{25})=id_{D_0}\\
T(0)=D_1\times_{C_0}C_1 & T(1)=D_1 & T(2)=D_0\times_{C_0}C_1 & T(3)=D_0 & T(4)=D_0 & T(5)=C_0\\
T(\tau_{01})=\tilde{\mathrm{pr}}_1 & T(\tau_{02})=\tau'\times_{C_0}id_{C_1} & T(\tau_{13})=\sigma' & T(\tau_{14})=\tau' & T(\tau_{24})=\sigma_{f_0} & T(\tau_{25})=\tau f_{0\sigma}\\
U(0)=D_0\times_{C_0}C_1\times_{C_0}C_1 & U(1)=D_0\times_{C_0}C_1 & U(2)=C_1 & U(3)=D_0 & U(4)=C_0 & U(5)=C_0\\
U(\tau_{01})=\tilde{\mathrm{pr}}_{12} & U(\tau_{02})=\mathrm{pr}_2\tilde{\mathrm{pr}}_{23} & U(\tau_{13})=\sigma_{f_0} & U(\tau_{14})=\tau f_{0\sigma} & U(\tau_{24})=\sigma & U(\tau_{25})=\tau\\
\alpha_0=(id_{D_1},f_1\varepsilon'\tau') & \alpha_1=id_{D_1} & \alpha_2=(id_{D_0},f_1\varepsilon') & \alpha_3=id_{D_0} & \alpha_4=id_{D_0} & \alpha_5=f_0\\
\beta_0=(\sigma'\tilde{\mathrm{pr}}_1,f_1\tilde{\mathrm{pr}}_1,\tilde{\mathrm{pr}}_2) & \beta_1=(\sigma',f_1) & \beta_2=f_{0\sigma} & \beta_3=id_{D_0} & \beta_4=f_0 & \beta_5=id_{C_0}
\end{array}$$

Hence if we define functors $S_i, T_i, U_i : \mathcal{Q} \to \mathcal{E}$ for $i = 0, 1, 2$ by

$$\begin{array}{lllll}
S_0(0)=S(0) & S_0(1)=S(3) & S_0(2)=S(5) & S_0(\tau_{01})=S(\tau_{13}\tau_{01}) & S_0(\tau_{02})=S(\tau_{25}\tau_{02})\\
T_0(0)=T(0) & T_0(1)=T(3) & T_0(2)=T(5) & T_0(\tau_{01})=T(\tau_{13}\tau_{01}) & T_0(\tau_{02})=T(\tau_{25}\tau_{02})\\
U_0(0)=U(0) & U_0(1)=U(3) & U_0(2)=U(5) & U_0(\tau_{01})=U(\tau_{13}\tau_{01}) & U_0(\tau_{02})=U(\tau_{25}\tau_{02})\\
S_1(0)=S(1) & S_1(1)=S(3) & S_1(2)=S(4) & S_1(\tau_{01})=S(\tau_{13}) & S_1(\tau_{02})=S(\tau_{14})\\
T_1(0)=T(1) & T_1(1)=T(3) & T_1(2)=T(4) & T_1(\tau_{01})=T(\tau_{13}) & T_1(\tau_{02})=T(\tau_{14})\\
U_1(0)=U(1) & U_1(1)=U(3) & U_1(2)=U(4) & U_1(\tau_{01})=U(\tau_{13}) & U_1(\tau_{02})=U(\tau_{14})\\
S_2(0)=S(2) & S_2(1)=S(4) & S_2(2)=S(5) & S_2(\tau_{01})=S(\tau_{24}) & S_2(\tau_{02})=S(\tau_{25})\\
T_2(0)=T(2) & T_2(1)=T(4) & T_2(2)=T(5) & T_2(\tau_{01})=T(\tau_{24}) & T_2(\tau_{02})=T(\tau_{25})\\
U_2(0)=U(2) & U_2(1)=U(4) & U_2(2)=U(5) & U_2(\tau_{01})=U(\tau_{24}) & U_2(\tau_{02})=U(\tau_{25})
\end{array}$$

and natural transformations $\alpha^i : S_i \to T_i$, $\beta^i : T_i \to U_i$ for $i = 0, 1, 2$ by

$$\begin{array}{lllllllll}
\alpha^0_0=\alpha_0 & \alpha^0_1=\alpha_3 & \alpha^0_2=\alpha_5 & \alpha^1_0=\alpha_1 & \alpha^1_1=\alpha_3 & \alpha^1_2=\alpha_4 & \alpha^2_0=\alpha_2 & \alpha^2_1=\alpha_4 & \alpha^2_2=\alpha_5,\\
\beta^0_0=\beta_0 & \beta^0_1=\beta_3 & \beta^0_2=\beta_5 & \beta^1_0=\beta_1 & \beta^1_1=\beta_3 & \beta^1_2=\beta_4 & \beta^2_0=\beta_2 & \beta^2_1=\beta_4 & \beta^2_2=\beta_5,
\end{array}$$

then we have $S_0 = S_1 = T_1$, $U_1 = T_2$.

For morphisms $f : X \to Y$, $g : X \to Z$ and $k : W \to X$ of $\mathcal{E}$, we denote by $\omega(k\,;f,g) : D_{fk,gk} \to D_{f,g}$ a natural transformation given by $\omega(k\,;f,g)_0 = k$, $\omega(k\,;f,g)_1 = id_Y$, $\omega(k\,;f,g)_2 = id_Z$. We note that $\omega(k\,;f,g)_M = M_k : M_{[fk,gk]} \to M_{[f,g]}$ for $M \in \mathrm{Ob}\,\mathcal{F}_Y$ by (8.4.26).

Lemma 9.6.6 *For a representation (M,ξ) of $\boldsymbol{D}$, the following diagram is commutative.*

$$\begin{tikzcd}
M \arrow[d, equal] & M_{[\sigma',\tau']} \arrow[l, "\hat{\xi}"'] \arrow[r, "\beta^1_M"] & f_0^*(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]}) \arrow[d, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] \\
M_{[id_{D_0},\,id_{D_0}]} \arrow[r, "\alpha^2_M"] & f_0^*(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]}) \arrow[r, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] & f_0^*((M,\xi)_{\boldsymbol{f}})
\end{tikzcd}$$

Proof. The following diagram is commutative by the definition of $P^{\boldsymbol{f}}_{(M,\xi)}$.

$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0}id_{C_1})]} \arrow[r, "\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)"] \arrow[dr, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}"'] & (M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[r, "\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]}"] & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[d, "P^{\boldsymbol{f}}_{(M,\xi)}"] \\
 & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[r, "P^{\boldsymbol{f}}_{(M,\xi)}"] & (M,\xi)_{\boldsymbol{f}}
\end{tikzcd}$$

It follows from (8.4.31) that the following diagram is commutative.

$$
\begin{tikzcd}
M_{[\sigma' id_{D_1}, id_{D_0}\tau']} \arrow[rr, "\alpha^0_M"] \arrow[d, "\theta_{\sigma',\tau',id_{D_0},id_{D_0}}(M)"] & & f_0^*(M_{[\sigma'\tilde{\mathrm{pr}}_1, \tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]}) \arrow[d, "f_0^*(\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M))"] \\
(M_{[\sigma',\tau']})_{[id_{D_0},id_{D_0}]} \arrow[r, "(\alpha^1_M)_{[id_{D_0},id_{D_0}]}"] & (M_{[\sigma',\tau']})_{[id_{D_0},id_{D_0}]} \arrow[r, "\alpha^2_{M_{[\sigma',\tau']}}"] & f_0^*((M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]})
\end{tikzcd}
$$

We note that $\theta_{\sigma',\tau',id_{D_0},id_{D_0}}(M)$ and $(\alpha^1_M)_{[id_{D_0},id_{D_0}]}$ are the identity morphism of $M_{[\sigma',\tau']}$ by (8.4.23) and the definition of α^1_M. Therefore the following diagram commutes by the commutativity of the above diagrams and (8.4.28).

$$
\begin{tikzcd}
M_{[\sigma',\tau']} \arrow[r, "\theta_{\sigma',\tau',id_{D_0},id_{D_0}}(M) = id_{M_{[\sigma',\tau']}}"] \arrow[d, "\alpha^0_M"] & M_{[\sigma',\tau']} \arrow[r, "\hat{\xi}"] \arrow[d, "\alpha^2_{M_{[\sigma',\tau']}}"] & M \arrow[d, "\alpha^2_M"] \\
f_0^*(M_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]}) \arrow[r, "f_0^*(\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M))"] \arrow[dr, "f_0^*(M_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1\times_{C_0} id_{C_1}))})"'] & f_0^*((M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]}) \arrow[r, "f_0^*(\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]})"] & f_0^*(M_{[\sigma_{f_0},\, \tau f_{0\sigma}]}) \arrow[d, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] \\
 & f_0^*(M_{[\sigma_{f_0},\, \tau f_{0\sigma}]}) \arrow[r, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] & f_0^*((M,\xi)_{\boldsymbol{f}})
\end{tikzcd}
$$

We put $\bar{\beta} = \omega((\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1})\,;\sigma_{f_0}, \tau f_{0\sigma}) : T_0 \to T_2$. Then, $\beta^1 = \bar{\beta}\alpha^0$ holds. It follows from (8.4.30) that the following diagram is commutative.

$$
\begin{tikzcd}
M_{[\sigma',\tau']} \arrow[r, "\alpha^0_M"] \arrow[d, "c_{id_{D_0},id_{D_0}}(M)_{[\sigma',\tau']} = id_{M_{[\sigma',\tau']}}"] & f_0^*(M_{[\sigma'\tilde{\mathrm{pr}}_1,\, \tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]}) \arrow[r, "f_0^*(\bar{\beta}_M)"] & f_0^*(M_{[\sigma_{f_0},\, \tau f_{0\sigma}]}) \arrow[d, "c_{id_{C_0},f_0}(M_{[\sigma_{f_0},\, \tau f_{0\sigma}]}) = id_{M_{[\sigma_{f_0},\, \tau f_{0\sigma}]}}"] \\
M_{[\sigma',\tau']} \arrow[rr, "\beta^1_M = (\bar{\beta}\alpha^0)_M"] & & f_0^*(M_{[\sigma_{f_0},\, \tau f_{0\sigma}]})
\end{tikzcd}
$$

Since $\bar{\beta}_M = \omega((\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1})\,;\sigma_{f_0}, \tau f_{0\sigma})_M = M_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1\times_{C_0} id_{C_1}))}$ by (8.4.26), we have

$$f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\alpha^2_M\hat{\xi} = f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})f_0^*(M_{(\sigma'\tilde{\mathrm{pr}}_1,\, \mu(f_1\times_{C_0} id_{C_1}))})\alpha^0_M = f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})f_0^*(\bar{\beta}_M)\alpha^0_M = f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\beta^1_M$$

$\square$

Proposition 9.6.7 *A composition*

$$M = M_{[id_{D_0},\, id_{D_0}]} \xrightarrow{\alpha^2_M} f_0^*(M_{[\sigma_{f_0},\, \tau f_{0\sigma}]}) \xrightarrow{f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})} f_0^*((M,\xi)_{\boldsymbol{f}})$$

defines a morphism $(M,\xi) \to (f_0^*((M,\xi)_{\boldsymbol{f}}), (\xi^l_{\boldsymbol{f}})_{\boldsymbol{f}})$ *of representations of* $\boldsymbol{D}$.

Proof. By applying (8.4.31) to $\beta : \mathcal{P} \to \mathcal{E}$, we see that the following diagram (i) is commutative.

$$
\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]} \arrow[rr, "\beta^0_M = M_{(\sigma'\tilde{\mathrm{pr}}_1, f_1\tilde{\mathrm{pr}}_1, \tilde{\mathrm{pr}}_2)}"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)"] & & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[d, "\theta_{\sigma_{f_0},\, \tau f_{0\sigma},\sigma,\tau}(M)"] \\
(M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[r, "(\beta^1_M)_{[\sigma_{f_0},\tau f_{0\sigma}]}"] & (f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]}))_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[r, "\beta^2_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}"] & (M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]}
\end{tikzcd}
$$

diagram (i)

Let $D_0 \xleftarrow{\hat{\mathrm{pr}}_1} D_0 \times_{C_0} D_1 \xrightarrow{\hat{\mathrm{pr}}_2} D_1$ be a limit of a diagram $D_0 \xrightarrow{f_0} C_0 \xleftarrow{\sigma f_1} D_1$. Define a natural transformation $\bar{\beta}^2 : D_{\hat{\mathrm{pr}}_1, \tau f_1\hat{\mathrm{pr}}_2} \to D_{\sigma f_1, \tau f_1}$ by $\bar{\beta}^2_0 = \hat{\mathrm{pr}}_2$, $\bar{\beta}^2_1 = f_0$, $\bar{\beta}^2_2 = id_{C_0}$. We also consider natural transformations $\omega(id_{D_0} \times_{C_0} f_1\,;\sigma_{f_0}, \tau f_{0\sigma}) : D_{\hat{\mathrm{pr}}_1, \tau f_1\hat{\mathrm{pr}}_2} \to D_{\sigma_{f_0},\tau f_{0\sigma}} = T_2$ and $\omega(f_1\,;\sigma,\tau) : D_{\sigma f_1, \tau f_1} \to D_{\sigma,\tau} = U_2$. Then, we have $\omega(f_1\,;\sigma,\tau)\bar{\beta}^2 = \beta^2\omega(id_{D_0} \times_{C_0} f_1\,;\sigma_{f_0}, \tau f_{0\sigma})$ and it follows from (8.4.30) that the following diagram (ii) is commutative.

diagram (ii)
$$\begin{tikzcd}
f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2]} \arrow[rr, "\bar{\beta}^2_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}"] \arrow[d, "f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{id_{D_0}\times_{C_0} f_1}"] \arrow[rrd, "(\omega(f_1\,;\sigma,\tau)\bar{\beta}^2)_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}"] & & (M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[f_0\sigma',f_0\tau']} \arrow[d, "(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{f_1}"] \\
f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[rr, "\beta^2_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}"'] & & (M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]}
\end{tikzcd}$$

The following diagram is commutative by (8.4.8).

diagram (iii)
$$\begin{tikzcd}
(M_{[\sigma',\tau']})_{[\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2]} \arrow[r, "(\beta^1_M)_{[\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2]}"] \arrow[d, "(M_{[\sigma',\tau']})_{id_{D_0}\times_{C_0} f_1}"] & f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2]} \arrow[d, "f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{id_{D_0}\times_{C_0} f_1}"] \\
(M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[r, "(\beta^1_M)_{[\sigma_{f_0},\tau f_{0\sigma}]}"'] & f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma_{f_0},\tau f_{0\sigma}]}
\end{tikzcd}$$

Define a functor $\gamma : S_0 \to D_{\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2}$ by $\gamma_0 = (\sigma', id_{D_1})$, $\gamma_1 = id_{D_0}$, $\gamma_2 = f_0$, then $\bar{\beta}^2\gamma = \omega(\sigma',\tau'; f_0, f_0)$ holds. It follows from (8.4.30) that

diagram (iv)
$$\begin{tikzcd}
f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma',\tau']} \arrow[d, "\gamma_{f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})}"] \arrow[dr, "\omega(\sigma',\tau';f_0,f_0)_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}"] & \\
f_0^*(f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2]}) \arrow[r, "f_0^*(\bar{\beta}^2_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}})"] & f_0^*((M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[f_0\sigma',f_0\tau']})
\end{tikzcd}$$

is commutative. Moreover, (8.4.28) implies that the following diagram is commutative.

diagram (v)
$$\begin{tikzcd}
(M_{[\sigma',\tau']})_{[\sigma',\tau']} \arrow[r, "(\beta^1_M)_{[\sigma',\tau']}"] \arrow[d, "\gamma_{M_{[\sigma',\tau']}}"] & f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma',\tau']} \arrow[d, "\gamma_{f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})}"] \\
f_0^*((M_{[\sigma',\tau']})_{[\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2]}) \arrow[r, "f_0^*((\beta^1_M)_{[\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2]})"] & f_0^*(f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2]})
\end{tikzcd}$$

The following diagram is commutative by the definition of $\hat{\xi}_{\boldsymbol{f}}$ and (8.4.8), (8.4.19).

diagram (vi)
$$\begin{tikzcd}
f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma',\tau']} \arrow[r, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma',\tau']}"] \arrow[d, "\omega(\sigma',\tau',f_0,f_0)_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}"] & f_0^*((M,\xi)_{\boldsymbol{f}})_{[\sigma',\tau']} \arrow[d, "\omega(\sigma',\tau',f_0,f_0)_{(M,\xi)_{\boldsymbol{f}}}"] \\
f_0^*((M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[f_0\sigma',f_0\tau']}) \arrow[r, "f_0^*((P^{\boldsymbol{f}}_{(M,\xi)})_{[f_0\sigma',f_0\tau']})"'] \arrow[d, "f_0^*((M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{f_1})"] & f_0^*(((M,\xi)_{\boldsymbol{f}})_{[f_0\sigma',f_0\tau']}) \arrow[d, "f_0^*(((M,\xi)_{\boldsymbol{f}})_{f_1})"] \\
f_0^*((M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]}) \arrow[r, "f_0^*((P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]})"] \arrow[d, "f_0^*(\theta_{\sigma_{f_0},\tau f_{0\sigma},\sigma,\tau}(M))^{-1}"] & f_0^*(((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]}) \arrow[dd, "f_0^*(\hat{\xi}_{\boldsymbol{f}})"] \\
f_0^*(M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]}) \arrow[d, "f_0^*(M_{id_{D_0}\times_{C_0}\mu})"] & \\
f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]}) \arrow[r, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] & f_0^*(((M,\xi)_{\boldsymbol{f}}))
\end{tikzcd}$$

Consider natural transformations $\omega(\varepsilon'\,;\sigma',\tau') : S_2 \to S_0$ and $\omega(id_{D_0}\times_{C_0} f_1\,;\sigma_{f_0},\tau f_{0\sigma}) : D_{\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2} \to T_2$. Then, we have $\alpha^2 = \beta^1\omega(\varepsilon'\,;\sigma',\tau')$ and $\omega(id_{D_0}\times_{C_0} f_1\,;\sigma_{f_0},\tau f_{0\sigma})\gamma = \beta^1 = \omega((\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1\times_{C_0} id_{C_1})\,;\sigma_{f_0},\tau f_{0\sigma})\alpha^0$ hold and it follows from (8.4.30) that the following diagrams are commutative.

diagram (vii)
$$\begin{tikzcd}
M = M_{[id_{D_0},id_{D_0}]} \arrow[r, "M_{\varepsilon'}"] \arrow[dr, "\alpha^2_M"'] & M_{[\sigma',\tau']} \arrow[d, "\beta^1_M"] \\
& f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})
\end{tikzcd}$$

$$\begin{tikzcd}
(M_{[\sigma',\tau']})_{[\sigma',\tau']} \arrow[r, "\gamma_{M_{[\sigma',\tau']}}"] \arrow[d, "\alpha^0_{M_{[\sigma',\tau']}}"] \arrow[dr, "\beta^1_{M_{[\sigma',\tau']}}"] & f_0^*((M_{[\sigma',\tau']})_{[\hat{\mathrm{pr}}_1,\tau f_1\hat{\mathrm{pr}}_2]}) \arrow[d, "f_0^*((M_{[\sigma',\tau']})_{id_{D_0}\times_{C_0} f_1})"] \\
f_0^*((M_{[\sigma',\tau']})_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]}) \arrow[r, "f_0^*((M_{[\sigma',\tau']})_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))})"] & f_0^*((M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]})
\end{tikzcd}$$
diagram $(viii)$

We also have the following commutative diagrams by (8.4.28) and (8.4.8).

diagram (ix)
$$\begin{tikzcd}
M_{[\sigma',\tau']} \arrow[r, "(M_{\varepsilon'})_{[\sigma',\tau']}"] \arrow[d, "\alpha^0_M"] & (M_{[\sigma',\tau']})_{[\sigma',\tau']} \arrow[d, "\alpha^0_{M_{[\sigma',\tau']}}"] \\
f_0^*(M_{[\sigma'\tilde{\mathrm{pr}}_1,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]}) \arrow[r, "f_0^*((M_{\varepsilon'})_{[\sigma'\tilde{\mathrm{pr}}_1,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]})"] & f_0^*((M_{[\sigma',\tau']})_{[\sigma'\tilde{\mathrm{pr}}_1,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]})
\end{tikzcd}$$

diagram (x)
$$\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]} \arrow[r, "(M_{\varepsilon'})_{[\sigma'\tilde{\mathrm{pr}}_1,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]}"] \arrow[d, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))}"] & (M_{[\sigma',\tau']})_{[\sigma'\tilde{\mathrm{pr}}_1,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]} \arrow[d, "(M_{[\sigma',\tau']})_{(\sigma'\tilde{\mathrm{pr}}_1,\mu(f_1\times_{C_0} id_{C_1}))}"] \\
M_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[r, "(M_{\varepsilon'})_{[\sigma_{f_0},\tau f_{0\sigma}]}"] & (M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]}
\end{tikzcd}$$

We put $\tilde{\xi}_{\boldsymbol{f}} = P_{\sigma',\tau'}(f_0^*((M,\xi)_{\boldsymbol{f}}))_{f_0^*((M,\xi)_{\boldsymbol{f}})}((\xi^l_{\boldsymbol{f}})_{\boldsymbol{f}})$. Then, $\tilde{\xi}_{\boldsymbol{f}}$ is the following composition by (9.4.4).

$$f_0^*((M,\xi)_{\boldsymbol{f}})_{[\sigma',\tau']} \xrightarrow{\omega(\sigma',\tau';f_0,f_0)_{(M,\xi)_{\boldsymbol{f}}}} f_0^*(((M,\xi)_{\boldsymbol{f}})_{[f_0\sigma',f_0\tau']}) \xrightarrow{f_0^*(((M,\xi)_{\boldsymbol{f}})_{f_1})} f_0^*(((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]}) \xrightarrow{f_0^*(\hat{\xi}_{\boldsymbol{f}})} f_0^*((M,\xi)_{\boldsymbol{f}})$$

We note that $(id_{D_0}\times_{C_0}\mu)(\sigma'\tilde{\mathrm{pr}}_1, f_1\tilde{\mathrm{pr}}_1, \tilde{\mathrm{pr}}_2) = (\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1\times_{C_0} id_{C_1}))$ holds and recall that $P^{\boldsymbol{f}}_{(M,\xi)}$ is a coequalizer of $M_{(\sigma'\tilde{\mathrm{pr}}_1,\mu(f_1\times_{C_0} id_{C_1}))}$ and $\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]}\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)$. We also have $f_0^*(M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))})\alpha^0_M = \beta^1_M$ by (8.4.30). Therefore by the commutativity of diagrams $(i)\sim(ix)$ and (9.6.6), we have

$$\begin{aligned}
\tilde{\xi}_{\boldsymbol{f}}(f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\alpha^2_M)_{[\sigma',\tau']} &= f_0^*(\hat{\xi}_{\boldsymbol{f}})f_0^*(((M,\xi)_{\boldsymbol{f}})_{f_1})\omega(\sigma',\tau';f_0,f_0)_{(M,\xi)_{\boldsymbol{f}}}f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma',\tau']}(\beta^1_M)_{[\sigma',\tau']}(M_{\varepsilon'})_{[\sigma',\tau']}\\
&= f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})f_0^*(M_{id_{D_0}\times_{C_0}\mu})f_0^*(M_{(\sigma'\tilde{\mathrm{pr}}_1,f_1\tilde{\mathrm{pr}}_1,\tilde{\mathrm{pr}}_2)})f_0^*(\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)^{-1})\\
&\quad\ f_0^*((M_{[\sigma',\tau']})_{id_{D_0}\times_{C_0} f_1})\gamma_{M_{[\sigma',\tau']}}(M_{\varepsilon'})_{[\sigma',\tau']}\\
&= f_0^*(P^{\boldsymbol{f}}_{(M,\xi)}M_{(\sigma'\tilde{\mathrm{pr}}_1,\mu(f_1\times_{C_0} id_{C_1}))}\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)^{-1})\\
&\quad\ f_0^*((M_{[\sigma',\tau']})_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))})\alpha^0_{M_{[\sigma',\tau']}}(M_{\varepsilon'})_{[\sigma',\tau']}\\
&= f_0^*(P^{\boldsymbol{f}}_{(M,\xi)}\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]})f_0^*((M_{[\sigma',\tau']})_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))}(M_{\varepsilon'})_{(\sigma'\tilde{\mathrm{pr}}_1,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1}))})\alpha^0_M\\
&= f_0^*(P^{\boldsymbol{f}}_{(M,\xi)}\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]})f_0^*((M_{\varepsilon'})_{[\sigma_{f_0},\tau f_{0\sigma}]}M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))})\alpha^0_M\\
&= f_0^*(P^{\boldsymbol{f}}_{(M,\xi)}(\hat{\xi}M_{\varepsilon'})_{[\sigma_{f_0},\tau f_{0\sigma}]})f_0^*(M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))})\alpha^0_M\\
&= f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\beta^1_M = f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\alpha^2_M\hat{\xi}.
\end{aligned}$$

This shows that $f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\alpha^2_M : M \to f_0^*((M,\xi)_{\boldsymbol{f}})$ defines a morphism $(M,\xi)\to(f_0^*((M,\xi)_{\boldsymbol{f}}),(\xi^l_{\boldsymbol{f}})_{\boldsymbol{f}})$ of representations of $\boldsymbol{D}$. □

We put $(\eta_{\boldsymbol{f}})_{(M,\xi)} = f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})\alpha^2_M : M \to f_0^*((M,\xi)_{\boldsymbol{f}})$.

Remark 9.6.8 *If $\varphi : (M,\xi)\to(N,\zeta)$ is a morphism of representations of $\boldsymbol{D}$, the following diagram is commutative by (8.4.28) and the definition of $\varphi_{\boldsymbol{f}}$.*

$$\begin{tikzcd}
M \arrow[r, "\alpha^2_M"] \arrow[d, "\varphi"] \arrow[rr, bend left, "(\eta_{\boldsymbol{f}})_{(M,\xi)}"] & f_0^*(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]}) \arrow[r, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] \arrow[d, "f_0^*(\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]})"] & f_0^*((M,\xi)_{\boldsymbol{f}}) \arrow[d, "f_0^*(\varphi_{\boldsymbol{f}})"] \\
N \arrow[r, "\alpha^2_N"] \arrow[rr, bend right, "(\eta_{\boldsymbol{f}})_{(N,\zeta)}"'] & f_0^*(N_{[\sigma_{f_0},\,\tau f_{0\sigma}]}) \arrow[r, "f_0^*(P^{\boldsymbol{f}}_{(N,\zeta)})"] & f_0^*((N,\zeta)_{\boldsymbol{f}})
\end{tikzcd}$$

Define a functor $R : \mathcal{P}\to\mathcal{E}$ and a natural transformation $\kappa : U \to R$ by $R(0) = C_1\times_{C_0} C_1$, $R(1) = C_1$, $R(2) = C_1$, $R(i) = C_0$ $(i = 3,4,5)$, $R(\tau_{01}) = \mathrm{pr}_1$, $R(\tau_{02}) = \mathrm{pr}_2$, $R(\tau_{13}) = R(\tau_{24}) = \sigma$, $R(\tau_{14}) = R(\tau_{25}) = \tau$ and $\kappa_0 = \tilde{\mathrm{pr}}_{23}$, $\kappa_1 = f_{0\sigma}$, $\kappa_2 = id_{C_1}$, $\kappa_3 = f_0$, $\kappa_4 = \kappa_5 = id_{C_0}$. We also define functors $R_i : \mathcal{Q}\to\mathcal{E}$ and natural transformations $\kappa^i : U_i \to R_i$ for $i = 0,1,2$ by

$$
\begin{array}{lllll}
R_0(0) = R(0) & R_0(1) = R(3) & R_0(2) = R(5) & R_0(\tau_{01}) = R(\tau_{13}\tau_{01}) & R_0(\tau_{02}) = R(\tau_{25}\tau_{02}) \\
R_1(0) = R(1) & R_1(1) = R(3) & R_1(2) = R(4) & R_1(\tau_{01}) = R(\tau_{13}) & R_1(\tau_{02}) = R(\tau_{14}) \\
R_2(0) = R(2) & R_2(1) = R(4) & R_2(2) = R(5) & R_2(\tau_{01}) = R(\tau_{24}) & R_2(\tau_{02}) = R(\tau_{25})
\end{array}
$$

$$\kappa^0_0 = \kappa_0 \quad \kappa^0_1 = \kappa_3 \quad \kappa^0_2 = \kappa_5 \quad \kappa^1_0 = \kappa_1 \quad \kappa^1_1 = \kappa_3 \quad \kappa^1_2 = \kappa_4 \quad \kappa^2_0 = \kappa_2 \quad \kappa^2_1 = \kappa_4 \quad \kappa^2_2 = \kappa_5.$$

Proposition 9.6.9 *For an object N of $\mathcal{F}_{C_0}$, $\beta^2_N : f_0^*(N)_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \to N_{[\sigma,\tau]}$ defines a morphism of representations $(f_0^*(N)_{[\sigma_{f_0},\,\tau f_{0\sigma}]}, \mu^l_{\boldsymbol{f}}(f_0^*(N))) \to (N_{[\sigma,\tau]}, \mu^l_N)$ under the assumption of (9.6.1) for $M = f_0^*(N)$ and the assumption of (9.4.10) for $M = N$.*

Proof. Since κ^2 is the identity natural transformation and $\kappa^1 = \beta^2$, we have a commutative diagram below by applying (8.4.31) to $\kappa : U \to R$.

$$
\begin{tikzcd}
f_0^*(N)_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\kappa^0_N"] \arrow[d, "\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(f_0^*(N))"] & N_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(N)"] \\
(f_0^*(N)_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[r, "(\beta^2_N)_{[\sigma,\tau]}"] & (N_{[\sigma,\tau]})_{[\sigma,\tau]}
\end{tikzcd}
$$

We consider functors $\omega(\mu\,;\sigma,\tau) : R_0 \to U_2$ and $\omega(id_{D_0}\times_{C_0}\mu\,;\sigma_{f_0},\tau f_{0\sigma}) : U_0 \to T_2$. Then we have $\omega(\mu\,;\sigma,\tau)\kappa^0 = \beta^2\omega(id_{D_0}\times_{C_0}\mu\,;\sigma_{f_0},\tau f_{0\sigma})$. Hence it follows from (8.4.30) that the following diagram is commutative.

$$
\begin{tikzcd}
f_0^*(N)_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\kappa^0_N"] \arrow[d, "f_0^*(N)_{id_{D_0}\times_{C_0}\mu}"] \arrow[dr, "(\omega(\mu\,;\sigma,\tau)\kappa^0)_N = (\beta^2\omega(id_{D_0}\times_{C_0}\mu\,;\sigma_{f_0},\tau f_{0\sigma}))_N" description] & N_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[d, "N_\mu"] \\
f_0^*(N)_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[r, "\beta^2_N"] & N_{[\sigma,\tau]}
\end{tikzcd}
$$

Since $\hat{\mu}_{\boldsymbol{f}}(f_0^*(N)) = f_0^*(N)_{id_{D_0}\times_{C_0}\mu}\theta_{\sigma_{f_0},\,\tau f_{0\sigma},\sigma,\tau}(f_0^*(N))^{-1}$ and $\hat{\mu}_N = N_\mu\theta_{\sigma,\tau,\sigma,\tau}(N)^{-1}$, the commutativity of the above diagrams implies that the following diagram is commutative.

$$
\begin{tikzcd}
(f_0^*(N)_{[\sigma_{f_0},\,\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[r, "\hat{\mu}_{\boldsymbol{f}}(f_0^*(N))"] \arrow[d, "(\beta^2_N)_{[\sigma,\tau]}"] & f_0^*(N)_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[d, "\beta^2_N"] \\
(N_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\hat{\mu}_N"] & N_{[\sigma,\tau]}
\end{tikzcd}
$$

Hence the assertion follows from (9.4.5). ∎

Lemma 9.6.10 *Let (M,ξ) and (N,ζ) be representations of $\boldsymbol{D}$ and $\boldsymbol{C}$, respectively. We put $\hat{\xi} = P_{\sigma'\tau'}(M)_M(\xi)$ and $\hat{\zeta} = P_{\sigma,\tau}(N)_N(\zeta)$. For a morphism $\varphi : (M,\xi) \to \boldsymbol{f}^{\bullet}(N,\zeta)$ of representations of $\boldsymbol{D}$, the following diagram is commutative if $\theta_{\sigma,\tau,\sigma,\tau}(N) : N_{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to (N_{[\sigma,\tau]})_{[\sigma,\tau]}$ is an isomorphism.*

$$
\begin{tikzcd}
M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0}id_{C_1})]} \arrow[rr, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0}id_{C_1}))}"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)"] & & M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[r, "\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"] & f_0^*(N)_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[d, "\beta^2_N"] \\
(M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[d, "\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]}"] & & & N_{[\sigma,\tau]} \arrow[d, "\hat{\zeta}"] \\
M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[r, "\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"] & f_0^*(N)_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[r, "\beta^2_N"] & N_{[\sigma,\tau]} \arrow[r, "\hat{\zeta}"] & N
\end{tikzcd}
$$

Proof. Since $P_{\sigma',\tau'}(f_0^*(N))_{f_0^*(N)}(\zeta_{\boldsymbol{f}})$ is a composition

$$f_0^*(N)_{[\sigma',\tau']} \xrightarrow{\omega(\sigma',\tau';f_0,f_0)_N} f_0^*(N_{[f_0\sigma',f_0\tau']}) \xrightarrow{f_0^*(N_{f_1})} f_0^*(N_{[\sigma,\tau]}) \xrightarrow{f_0^*(\hat{\zeta})} f_0^*(N)$$

by (9.4.4), the following diagram is commutative by (9.4.5).

$$
\begin{tikzcd}
M_{[\sigma',\tau']} \arrow[rrr, "\hat{\xi}"] \arrow[d, "\varphi_{[\sigma',\tau']}"] & & & M \arrow[d, "\varphi"] \\
f_0^*(N)_{[\sigma',\tau']} \arrow[r, "\omega(\sigma',\tau'\,;f_0,f_0)_N"] & f_0^*(N_{[f_0\sigma',f_0\tau']}) \arrow[r, "f_0^*(N_{f_1})"] & f_0^*(N_{[\sigma,\tau]}) \arrow[r, "f_0^*(\hat{\zeta})"] & f_0^*(N)
\end{tikzcd}
$$

It follows from (8.4.28) that the following diagram is commutative.

$$
\begin{tikzcd}
f_0^*(N_{[\sigma,\tau]})_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[r, "\beta^2_{N_{[\sigma,\tau]}}"] \arrow[d, "f_0^*(\hat{\zeta})_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"] & (N_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[d, "\hat{\zeta}_{[\sigma,\tau]}"] \\
f_0^*(N)_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[r, "\beta^2_N"] & N_{[\sigma,\tau]}
\end{tikzcd}
$$

Hence the following diagram (i) is commutative by (8.4.3), (8.4.8) and (8.4.19).

diagram (i)

$$
\begin{tikzcd}
M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[r, "\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"] & f_0^*(N)_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \\
M_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]} \arrow[u, "M_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))}"] \arrow[r, "\varphi_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]}"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)"] & f_0^*(N)_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]} \arrow[u, "f_0^*(N)_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))}"'] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(f_0^*(N))"] \\
(M_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[r, "(\varphi_{[\sigma',\tau']})_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"] \arrow[d, "\hat{\xi}_{[\sigma_{f_0},\tau f_{0\sigma}]}"] & (f_0^*(N)_{[\sigma',\tau']})_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[d, "(\omega(\sigma',\tau'\,;f_0,f_0)_N)_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"] \\
M_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[d, "\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"] & f_0^*(N_{[f_0\sigma',f_0\tau']})_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[d, "f_0^*(N_{f_1})_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"] \\
f_0^*(N)_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[ddr, "\beta^2_N"] & f_0^*(N_{[\sigma,\tau]})_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[l, "f_0^*(\hat{\zeta})_{[\sigma_{f_0},\,\tau f_{0\sigma}]}"'] \arrow[d, "\beta^2_{N_{[\sigma,\tau]}}"] \\
 & (N_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[d, "\hat{\zeta}_{[\sigma,\tau]}"] \\
 & N_{[\sigma,\tau]}
\end{tikzcd}
$$

Define a functor $V:\mathcal{P}\to\mathcal{E}$ and a natural transformation $\lambda: T\to V$ by $V(0)=D_1\times_{C_0}C_1$, $V(1)=D_1$, $V(2)=C_1$, $V(i)=C_0$ $(i=3,4,5)$, $V(\tau_{01})=\tilde{\mathrm{pr}}_1$, $V(\tau_{02})=\tilde{\mathrm{pr}}_2$, $V(\tau_{13})=f_0\sigma'$, $V(\tau_{14})=f_0\tau'$, $V(\tau_{24})=\sigma$, $V(\tau_{25})=\tau$ and $\lambda_0=id_{D_1\times_{C_0}C_1}$, $\lambda_1=id_{D_1}$, $\lambda_2=f_{0\sigma}$, $\lambda_3=\lambda_4=f_0$, $\lambda_5=id_{C_0}$. We also define functors $V_i:\mathcal{Q}\to\mathcal{E}$ and natural transformations $\lambda^i: V_i\to T_i$ for $i=0,1,2$ by

$$
\begin{array}{lllll}
V_0(0)=V(0) & V_0(1)=V(3) & V_0(2)=V(5) & V_0(\tau_{01})=V(\tau_{13}\tau_{01}) & V_0(\tau_{02})=V(\tau_{25}\tau_{02}) \\
V_1(0)=V(1) & V_1(1)=V(3) & V_1(2)=V(4) & V_1(\tau_{01})=V(\tau_{13}) & V_1(\tau_{02})=V(\tau_{14}) \\
V_2(0)=V(2) & V_2(1)=V(4) & V_2(2)=V(5) & V_2(\tau_{01})=V(\tau_{24}) & V_2(\tau_{02})=V(\tau_{25})
\end{array}
$$

$$
\lambda^0_0=\lambda_0 \quad \lambda^0_1=\lambda_3 \quad \lambda^0_2=\lambda_5 \quad \lambda^1_0=\lambda_1 \quad \lambda^1_1=\lambda_3 \quad \lambda^1_2=\lambda_4 \quad \lambda^2_0=\lambda_2 \quad \lambda^2_1=\lambda_4 \quad \lambda^2_2=\lambda_5.
$$

Then, $V_2=U_2$, $\lambda^1=\omega(\sigma',\tau'\,;f_0,f_0)$ and $\lambda^2=\beta^2$ and it follows from (8.4.31) that the following diagram is commutative.

$$
\begin{tikzcd}
f_0^*(N)_{[\sigma'\tilde{\mathrm{pr}}_1,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]} \arrow[rr, "\lambda^0_N"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(f_0^*(N))"] & & N_{[f_0\sigma'\tilde{\mathrm{pr}}_1,\tau\tilde{\mathrm{pr}}_2]} \arrow[d, "\theta_{f_0\sigma',f_0\tau',\sigma,\tau}(N)"'] \\
(f_0^*(N)_{[\sigma',\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[r, "(\omega(\sigma',\tau'\,;f_0,f_0)_N)_{[\sigma_{f_0},\tau f_{0\sigma}]}"] & f_0^*(N_{[f_0\sigma',f_0\tau']})_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[r, "\beta^2_{N_{[f_0\sigma',f_0\tau']}}"] & (N_{[f_0\sigma',f_0\tau']})_{[\sigma,\tau]}
\end{tikzcd}
$$

Consider natural transformations $\omega(\mu(f_1\times_{C_0} id_{C_1})\,;\sigma,\tau): V_0\to U_2$ and $\omega((\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))\,;\,\sigma_{f_0},\tau f_{0\sigma}): T_0\to T_2$. Then, $\omega(\mu(f_1\times_{C_0} id_{C_1})\,;\sigma,\tau)\lambda^0=\beta^2\omega((\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))\,;\,\sigma_{f_0},\tau f_{0\sigma})$ holds and the following diagram is commutative by (8.4.30).

$$
\begin{tikzcd}
f_0^*(N)_{[\sigma'\tilde{\mathrm{pr}}_1,\,\tau f_{0\sigma}(\tau'\times_{C_0} id_{C_1})]} \arrow[r, "\lambda^0_N"] \arrow[d, "f_0^*(N)_{(\sigma'\tilde{\mathrm{pr}}_1,\,\mu(f_1\times_{C_0} id_{C_1}))}"] \arrow[dr, "(\omega(\mu(f_1\times_{C_0} id_{C_1})\,;\,\sigma,\tau)\lambda^0)_N"] & N_{[f_0\sigma'\tilde{\mathrm{pr}}_1,\tau\tilde{\mathrm{pr}}_2]} \arrow[d, "N_{\mu(f_1\times_{C_0} id_{C_1})}"] \\
f_0^*(N)_{[\sigma_{f_0},\,\tau f_{0\sigma}]} \arrow[r, "\beta^2_N"] & N_{[\sigma,\tau]}
\end{tikzcd}
$$

Moreover, the following diagrams are commutative by (9.4.1) and (8.4.28), respectively.

$$
\begin{tikzcd}
N_{[\sigma \mathrm{pr}_1, \tau \mathrm{pr}_2]} \arrow[r, "N_\mu"] \arrow[d, "\theta_{\sigma,\tau,\sigma,\tau}(N)"] & N_{[\sigma,\tau]} \arrow[r, "\hat{\zeta}"] & N \\
(N_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[r, "\hat{\zeta}_{[\sigma,\tau]}"] & N_{[\sigma,\tau]} \arrow[ur, "\hat{\zeta}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
f_0^*(N_{[f_0\sigma', f_0\tau']})_{[\sigma_{f_0}, \tau f_{0\sigma}]} \arrow[r, "\beta^2_{N_{[f_0\sigma', f_0\tau']}}"] \arrow[d, "f_0^*(N_{f_1})_{[\sigma_{f_0}, \tau f_{0\sigma}]}"] & (N_{[f_0\sigma', f_0\tau']})_{[\sigma,\tau]} \arrow[d, "(N_{f_1})_{[\sigma,\tau]}"] \\
f_0^*(N_{[\sigma,\tau]})_{[\sigma_{f_0}, \tau f_{0\sigma}]} \arrow[r, "\beta^2_{N_{[\sigma,\tau]}}"] & (N_{[\sigma,\tau]})_{[\sigma,\tau]}
\end{tikzcd}
$$

Therefore the following diagram (ii) is commutative

diagram (ii)

$$
\begin{tikzcd}
f_0^*(N)_{[\sigma_{f_0}, \tau f_{0\sigma}]} \arrow[rrr, "\beta_N^2"] & & & N_{[\sigma,\tau]} \arrow[dddddd, "\hat{\zeta}"] \\
f_0^*(N)_{[\sigma'\tilde{\mathrm{pr}}_1, \tau f_{0\sigma}(\tau' \times_{C_0} id_{C_1})]} \arrow[u, "f_0^*(N)_{(\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1}))}"'] \arrow[r, "\lambda_N^0"] \arrow[d, "\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(f_0^*(N))"] & N_{[f_0\sigma'\tilde{\mathrm{pr}}_1, \tau\tilde{\mathrm{pr}}_2]} \arrow[dd, "\theta_{f_0\sigma', f_0\tau', \sigma, \tau}(N)"'] \arrow[ddddr, "N_{f_1 \times_{C_0} id_{C_1}}"] & & \\
(f_0^*(N)_{[\sigma',\tau']})_{[\sigma_{f_0}, \tau f_{0\sigma}]} \arrow[d, "(\omega(\sigma',\tau',f_0,f_0)_N)_{[\sigma_{f_0}, \tau f_{0\sigma}]}"] & & & \\
f_0^*(N_{[f_0\sigma', f_0\tau']})_{[\sigma_{f_0}, \tau f_{0\sigma}]} \arrow[r, "\beta^2_{N_{[f_0\sigma', f_0\tau']}}"'] \arrow[d, "f_0^*(N_{f_1})_{[\sigma_{f_0}, \tau f_{0\sigma}]}"] & (N_{[f_0\sigma', f_0\tau']})_{[\sigma,\tau]} \arrow[ddl, "(N_{f_1})_{[\sigma,\tau]}"] & & \\
f_0^*(N_{[\sigma,\tau]})_{[\sigma_{f_0}, \tau f_{0\sigma}]} \arrow[d, "\beta^2_{N_{[\sigma,\tau]}}"] & & & \\
(N_{[\sigma,\tau]})_{[\sigma,\tau]} \arrow[rr, "\theta_{\sigma,\tau,\sigma,\tau}(N)^{-1}"] \arrow[d, "\hat{\zeta}_{[\sigma,\tau]}"] & & N_{[\sigma \mathrm{pr}_1, \tau \mathrm{pr}_2]} \arrow[uuuuur, "N_\mu"] & \\
N_{[\sigma,\tau]} \arrow[rrr, "\hat{\zeta}"] & & & N
\end{tikzcd}
$$

By glueing the right edge of diagram (i) and the left edge of diagram (ii), the assertion follows. □

Recall that $P^{\boldsymbol{f}}_{(M,\xi)} : M_{[\sigma_{f_0}, \tau f_{0\sigma}]} \to (M,\xi)_{\boldsymbol{f}}$ is a coequalizer of the following morphisms.

$$
M_{[\sigma'\tilde{\mathrm{pr}}_1, \tau f_{0\sigma}(\tau' \times_{C_0} id_{C_1})]} \xrightarrow{\theta_{\sigma',\tau',\sigma_{f_0},\tau f_{0\sigma}}(M)} (M_{[\sigma',\tau']})_{[\sigma_{f_0}, \tau f_{0\sigma}]} \xrightarrow{\hat{\xi}_{[\sigma_{f_0}, \tau f_{0\sigma}]}} M_{[\sigma_{f_0}, \tau f_{0\sigma}]}
$$

$$
M_{[\sigma'\tilde{\mathrm{pr}}_1, \tau f_{0\sigma}(\tau' \times_{C_0} id_{C_1})]} \xrightarrow{M_{(\sigma'\tilde{\mathrm{pr}}_1, \mu(f_1 \times_{C_0} id_{C_1}))}} M_{[\sigma_{f_0}, \tau f_{0\sigma}]}
$$

Hence there exists unique morphism ${}^t\varphi : (M,\xi)_{\boldsymbol{f}} \to N$ that satisfies ${}^t\varphi P^{\boldsymbol{f}}_{(M,\xi)} = \hat{\zeta}\beta_N^2 \varphi_{[\sigma_{f_0}, \tau f_{0\sigma}]}$.

Proposition 9.6.11 *Under the assumptions of (9.6.3) for M and the assumptions of (iii) and the first one of (iv) of (9.6.3) for $f_0^*(N)$, ${}^t\varphi : (M,\xi)_{\boldsymbol{f}} \to N$ gives a morphism $((M,\xi)_{\boldsymbol{f}}, \xi^l_{\boldsymbol{f}}) \to (N,\zeta)$ of representations of $\boldsymbol{C}$.*

Proof. It follows from (9.4.10), (9.6.9) and (9.4.11) that $\hat{\zeta}\beta_N^2\varphi_{[\sigma_{f_0}, \tau f_{0\sigma}]} : M_{[\sigma_{f_0}, \tau f_{0\sigma}]} \to N$ gives a morphism $(M_{[\sigma_{f_0}, \tau f_{0\sigma}]}, \mu^l_{\boldsymbol{f}}(M)) \to (N,\zeta)$ of representations of $\boldsymbol{C}$. Hence the outer rectangle of the following diagram is commutative by (9.4.5).

$$
\begin{tikzcd}
(M_{[\sigma_{f_0}, \tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[r, "(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}"] \arrow[d, "\hat{\mu}_{\boldsymbol{f}}(M)"] & ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]} \arrow[r, "{}^t\varphi_{[\sigma,\tau]}"] \arrow[d, "\hat{\xi}_{\boldsymbol{f}}"] & N_{[\sigma,\tau]} \arrow[d, "\hat{\zeta}"] \\
M_{[\sigma_{f_0}, \tau f_{0\sigma}]} \arrow[r, "P^{\boldsymbol{f}}_{(M,\xi)}"] & (M,\xi)_{\boldsymbol{f}} \arrow[r, "{}^t\varphi"] & N
\end{tikzcd}
$$

Since $(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]} : (M_{[\sigma_{f_0}, \tau f_{0\sigma}]})_{[\sigma,\tau]} \to ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]}$ is an epimorphism and the left rectangle of the above diagram is commutative by the definition of $\hat{\xi}_{\boldsymbol{f}}$, the right rectangle of the above diagram is also commutative. Thus the assertion follows from (9.4.5). □

For a morphism $f : X \to Y$ of $\mathcal{E}$, we define a natural transformation $\omega(f) : D_{id_X,id_X} \to D_{id_Y,id_Y}$ by $\omega(f)_0 = \omega(f)_1 = \omega(f)_2 = f$. Since $\iota_{id_Y,id_Y}(M) \in \mathcal{F}_Y(id_Y^*(M), id_Y^*(M_{[id_Y,id_Y]})) = \mathcal{F}_Y(M,M)$ is the identity morphism of $M \in \mathcal{F}_Y$, the following assertion is straightforward from the definition of $\omega(f)_M$.

Proposition 9.6.12 *For an object M of $\mathcal{F}_Y$, $\omega(f)_M : f^*(M) = f^*(M)_{[id_X,id_X]} \to f^*(M_{[id_Y,id_Y]}) = f^*(M)$ is the identity morphism of $f^*(M)$.*

Proposition 9.6.13 *For a morphism $\varphi : (M,\xi) \to \boldsymbol{f}^{\bullet}(N,\zeta)$ of representations of $\boldsymbol{D}$, the following composition coincides with φ.*

$$M \xrightarrow{(\eta_{\boldsymbol{f}})_{(M,\xi)}} f_0^*((M,\xi)_{\boldsymbol{f}}) \xrightarrow{f_0^*({}^t\varphi)} f_0^*(N)$$

Proof. We note that compositions $S_2 \xrightarrow{\alpha^2} T_2 \xrightarrow{\beta^2} U_2$ and $S_2 = D_{id_{D_0},id_{D_0}} \xrightarrow{\omega(f_0)} D_{id_{C_0},id_{C_0}} \xrightarrow{\omega(\varepsilon\,;\sigma,\tau)} U_2$ coincide. Hence the following diagram is commutative by (reffcwp21) and (8.4.30).

$$\begin{tikzcd}
M \arrow[r, "\alpha^2_M"] \arrow[d, "\varphi"] & f_0^*(M_{[\sigma_{f_0},\,\tau f_{0\sigma}]}) \arrow[r, "f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})"] \arrow[d, "f_0^*(\varphi_{[\sigma_{f_0},\,\tau f_{0\sigma}]})"] & f_0^*((M,\xi)_{\boldsymbol{f}}) \arrow[dd, "f_0^*({}^t\varphi)"] \\
f_0^*(N) \arrow[r, "\alpha^2_{f_0^*(N)}"] \arrow[d, "\omega(f_0)_N"] \arrow[dr, "(\beta^2\alpha^2)_N=(\omega(\varepsilon\,;\sigma,\tau)\omega(f_0))_N" description] & f_0^*(f_0^*(N)_{[\sigma_{f_0},\,\tau f_{0\sigma}]}) \arrow[d, "f_0^*(\beta^2_N)"] & \\
f_0^*(N) \arrow[r, "f_0^*(N_\varepsilon)"] & f_0^*(N_{[\sigma,\tau]}) \arrow[r, "f_0^*(\hat{\zeta})"] & f_0^*(N)
\end{tikzcd}$$

Since $\omega(f_0)_N$ is the identity morphism of $f^*(N)$ by (9.6.12) and $\hat{\zeta}N_\varepsilon$ is the identity morphism of N by (9.4.1), the assertion follows. $\square$

Lemma 9.6.14 *For an object M of $\mathcal{F}_{D_0}$, a composition*

$$M_{[\sigma_{f_0},\tau f_{0\sigma}]} \xrightarrow{(\alpha^2_M)_{[\sigma_{f_0},\tau f_{0\sigma}]}} f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma_{f_0},\tau f_{0\sigma}]} \xrightarrow{\beta^2_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}} (M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]} \xrightarrow{\hat{\mu}_{\boldsymbol{f}}(M)} M_{[\sigma_{f_0},\tau f_{0\sigma}]}$$

coincides with the identity morphism of $M_{[\sigma_{f_0},\tau f_{0\sigma}]}$.

Proof. Define a functor $W : \mathcal{P} \to \mathcal{E}$ and a natural transformation $\nu : W \to U$ by $W(0) = W(2) = D_0 \times_{C_0} C_1$, $W(i) = D_0$ $(i = 1,3,4)$, $W(5) = C_0$, $W(\tau_{01}) = \sigma_{f_0}$, $W(\tau_{02}) = id_{D_0 \times_{C_0} C_1}$, $W(\tau_{13}) = W(\tau_{14}) = id_{D_0}$, $W(\tau_{24}) = \sigma_{f_0}$, $W(\tau_{25}) = \tau f_{0\sigma}$ and $\nu_0 = (\sigma_{f_0}, \varepsilon\sigma f_{0\sigma}, f_{0\sigma})$, $\nu_1 = (id_{D_0}, \varepsilon f_0)$, $\nu_2 = f_{0\sigma}$, $\nu_3 = id_{D_0}$, $\nu_4 = f_0$, $\nu_5 = id_{C_0}$. We also define functors $W_i : \mathcal{Q} \to \mathcal{E}$ and natural transformations $\nu^i : W_i \to T_i$ for $i = 0,1,2$ by

$$\begin{array}{lllll}
W_0(0) = W(0) & W_0(1) = W(3) & W_0(2) = W(5) & W_0(\tau_{01}) = W(\tau_{13}\tau_{01}) & W_0(\tau_{02}) = W(\tau_{25}\tau_{02}) \\
W_1(0) = W(1) & W_1(1) = W(3) & W_1(2) = W(4) & W_1(\tau_{01}) = W(\tau_{13}) & W_1(\tau_{02}) = W(\tau_{14}) \\
W_2(0) = W(2) & W_2(1) = W(4) & W_2(2) = W(5) & W_2(\tau_{01}) = W(\tau_{24}) & W_2(\tau_{02}) = W(\tau_{25})
\end{array}$$

$$\nu^0_0 = \nu_0 \quad \nu^0_1 = \nu_3 \quad \nu^0_2 = \nu_5 \quad \nu^1_0 = \nu_1 \quad \nu^1_1 = \nu_3 \quad \nu^1_2 = \nu_4 \quad \nu^2_0 = \nu_2 \quad \nu^2_1 = \nu_4 \quad \nu^2_2 = \nu_5.$$

Then, we have $W_1 = S_2$, $W_2 = T_2$, $\nu^1 = \alpha^2$, $\nu^2 = \beta^2$ and $\nu^0 = \omega((\sigma_{f_0}, \varepsilon\sigma f_{0\sigma}, f_{0\sigma})\,; \sigma_{f_0}\tilde{\mathrm{pr}}_{12}, \tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23})$. It follows from (8.4.31) and the definition of $\hat{\mu}_{\boldsymbol{f}}(M)$ that the following diagram is commutative.

$$\begin{tikzcd}
M_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[rrr, "M_{(\sigma_{f_0},\varepsilon\sigma f_{0\sigma},f_{0\sigma})}"] \arrow[d, "\theta_{id_{D_0},id_{D_0},\sigma_{f_0},f_{0\sigma}}(M)=id_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}"] & & & M_{[\sigma_{f_0}\tilde{\mathrm{pr}}_{12},\tau\mathrm{pr}_2\tilde{\mathrm{pr}}_{23}]} \arrow[d, "\theta_{\sigma_{f_0},f_{0\sigma},\sigma,\tau}(M)"] \arrow[dr, "M_{id_{D_0}\times_{C_0}\mu}"] & \\
M_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[r, "(\alpha^2_M)_{[\sigma_{f_0},\tau f_{0\sigma}]}"] & (f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]}))_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[rr, "\beta^2_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}"] & & (M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[r, "\hat{\mu}_{\boldsymbol{f}}(M)"] & M_{[\sigma_{f_0},\tau f_{0\sigma}]}
\end{tikzcd}$$

Since a composition $D_0 \times_{C_0} C_1 \xrightarrow{(\sigma_{f_0},\,\varepsilon\sigma f_{0\sigma},\,f_{0\sigma})} D_0 \times_{C_0} C_1 \times_{C_0} C_1 \xrightarrow{id_{D_0}\times_{C_0}\mu} D_0 \times_{C_0} C_1$ is the identity morphism of $D_0 \times_{C_0} C_1$, the assertion follows from the commutativity of the above diagram and (8.4.6). $\square$

Under the assumptions of (9.6.3) for M and the assumptions of (iii) and the first one of (iv) of (9.6.3) for $f_0^*(N)$, we define a map

$$\mathrm{ad}^{(M,\xi)}_{(N,\zeta)} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(N,\zeta)) \to \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})((M,\xi),\boldsymbol{f}^\bullet(N,\zeta))$$

by $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}(\psi) = f_0^*(\psi)(\eta_{\boldsymbol{f}})_{(M,\xi)}$.

Proposition 9.6.15 $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}$ *is bijective.*

Proof. We show that a map $\Phi : \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})((M,\xi),\boldsymbol{f}^\bullet(N,\zeta)) \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(N,\zeta))$ defined by $\Phi(\varphi) = {}^t\varphi$ is the inverse of $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}$. $\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}\Phi$ is the identity map of $\mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})((M,\xi),\boldsymbol{f}^\bullet(N,\zeta))$ by (9.6.13). For $\psi \in \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(N,\zeta))$, we put $\varphi = \mathrm{ad}^{(M,\xi)}_{(N,\zeta)}(\psi)$. The following diagram is commutative by (8.4.3), (8.4.28), (9.4.5) and the definition of $\hat{\xi}_{\boldsymbol{f}}$.

$$\begin{tikzcd}
f_0^*(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[rr, "f_0^*(\psi P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma_{f_0},\tau f_{0\sigma}]}"] \arrow[d, "\beta^2_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}"] & & f_0^*(N)_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[d, "\beta^2_N"] \\
(M_{[\sigma_{f_0},\tau f_{0\sigma}]})_{[\sigma,\tau]} \arrow[rr, "(\psi P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}"] \arrow[dr, "(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma,\tau]}"] \arrow[d, "\hat{\mu}_{\boldsymbol{f}}(M)"] & & N_{[\sigma,\tau]} \arrow[d, "\hat{\zeta}"] \\
M_{[\sigma_{f_0},\tau f_{0\sigma}]} \arrow[dr, "P^{\boldsymbol{f}}_{(M,\xi)}"] & ((M,\xi)_{\boldsymbol{f}})_{[\sigma,\tau]} \arrow[ur, "\psi_{[\sigma,\tau]}"] \arrow[d, "\hat{\xi}_{\boldsymbol{f}}"] & N \\
& (M,\xi)_{\boldsymbol{f}} \arrow[ur, "\psi"] &
\end{tikzcd}$$

Hence we have the following equalities by the commutativity of the above diagram and (9.6.14).

$$\begin{aligned}
\hat{\zeta}\beta^2_N\varphi_{[\sigma_{f_0},\tau f_{0\sigma}]} &= \hat{\zeta}\beta^2_N f_0^*(\psi)_{[\sigma_{f_0},\tau f_{0\sigma}]}((\eta_{\boldsymbol{f}})_{(M,\xi)})_{[\sigma_{f_0},\tau f_{0\sigma}]} \\
&= \hat{\zeta}\beta^2_N f_0^*(\psi)_{[\sigma_{f_0},\tau f_{0\sigma}]} f_0^*(P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma_{f_0},\tau f_{0\sigma}]}(\alpha^2_M)_{[\sigma_{f_0},\tau f_{0\sigma}]} \\
&= \hat{\zeta}\beta^2_N f_0^*(\psi P^{\boldsymbol{f}}_{(M,\xi)})_{[\sigma_{f_0},\tau f_{0\sigma}]}(\alpha^2_M)_{[\sigma_{f_0},\tau f_{0\sigma}]} \\
&= \psi P^{\boldsymbol{f}}_{(M,\xi)}\hat{\mu}_{\boldsymbol{f}}(M)\beta^2_{M_{[\sigma_{f_0},\tau f_{0\sigma}]}}(\alpha^2_M)_{[\sigma_{f_0},\tau f_{0\sigma}]} = \psi P^{\boldsymbol{f}}_{(M,\xi)}
\end{aligned}$$

Since we also have $\hat{\zeta}\beta^2_N\varphi_{[\sigma_{f_0},\tau f_{0\sigma}]} = {}^t\varphi P^{\boldsymbol{f}}_{(M,\xi)}$ by the definition of ${}^t\varphi$, it follows that $\Phi(\varphi) = {}^t\varphi = \psi$ which implies that $\Phi\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}$ is the identity map of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(N,\zeta))$. $\square$

Definition 9.6.16 *For a representation* (M,ξ) *of* $\boldsymbol{D}$*, we call* $((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}})$ *the left induced representation of* (M,ξ) *by* $\boldsymbol{f} : \boldsymbol{D} \to \boldsymbol{C}$*.*

The following fact is straightforward from (9.6.8).

Proposition 9.6.17 *The following diagrams are commutative for a morphism* $\varphi : (L,\chi) \to (M,\xi)$ *of* $\mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})$ *and a morphism* $\psi : (N,\zeta) \to (P,\rho)$ *of* $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$*.*

$$\begin{tikzcd}
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(N,\zeta)) \arrow[r, "\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}"] \arrow[d, "\varphi^*_{\boldsymbol{f}}"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})((M,\xi),\boldsymbol{f}^\bullet(N,\zeta)) \arrow[d, "\varphi^*"] \\
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((L,\chi)_{\boldsymbol{f}},\chi^l_{\boldsymbol{f}}),(N,\zeta)) \arrow[r, "\mathrm{ad}^{(L,\chi)}_{(N,\zeta)}"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})((L,\chi),\boldsymbol{f}^\bullet(N,\zeta))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(N,\zeta)) \arrow[r, "\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}"] \arrow[d, "\psi_*"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})((M,\xi),\boldsymbol{f}^\bullet(N,\zeta)) \arrow[d, "\boldsymbol{f}^\bullet(\psi)_*"] \\
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})(((M,\xi)_{\boldsymbol{f}},\xi^l_{\boldsymbol{f}}),(P,\rho)) \arrow[r, "\mathrm{ad}^{(M,\xi)}_{(P,\rho)}"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})((M,\xi),\boldsymbol{f}^\bullet(P,\rho))
\end{tikzcd}$$

9.7 Right induced representations

Let $p : \mathcal{F} \to \mathcal{E}$ be a normalized cloven fibered category. For morphisms $f : X \to Y$, $g : X \to Z$ of $\mathcal{E}$ and an object N of $\mathcal{F}_Z$, we assume that the presheaf $F_N^{f,g}$ on $\mathcal{F}_Y$ is representable if necessary.

Let $\boldsymbol{C} = (C_0, C_1; \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{D} = (D_0, D_1; \sigma', \tau', \varepsilon', \mu')$ be internal categories in $\mathcal{E}$. For an internal functor $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{D} \to \boldsymbol{C}$ in $\mathcal{E}$, let $C_1 \xleftarrow{f_{0\tau}} C_1 \times_{C_0} D_0 \xrightarrow{\tau_{f_0}} D_0$ be a limit of a diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{f_0} D_0$. We consider the following diagram whose rectangles are all cartesian.

$$\begin{tikzcd}
C_1 \times_{C_0} C_1 \times_{C_0} C_1 \times_{C_0} D_0 \arrow[r, "\widetilde{\mathrm{pr}}_{234}"] \arrow[d, "\widetilde{\mathrm{pr}}_{123}"] & C_1 \times_{C_0} C_1 \times_{C_0} D_0 \arrow[r, "\widetilde{\mathrm{pr}}_{23}"] \arrow[d, "\widetilde{\mathrm{pr}}_{12}"] & C_1 \times_{C_0} D_0 \arrow[r, "\tau_{f_0}"] \arrow[d, "f_{0\tau}"] & D_0 \arrow[d, "f_0"] \\
C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_{23}"] & C_1 \times_{C_0} C_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & C_1 \arrow[r, "\tau"] \arrow[d, "\sigma"] & C_0 \\
 & C_1 \arrow[r, "\tau"] & C_0 &
\end{tikzcd}$$

Let N be an object of $\mathcal{F}_{D_0}$. If $\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N) : (N^{[\sigma f_{0\tau},\, \tau_{f_0}]})^{[\sigma,\tau]} \to N^{[\sigma \mathrm{pr}_1 \widetilde{\mathrm{pr}}_{12},\, \tau_{f_0} \widetilde{\mathrm{pr}}_{23}]}$ is an isomorphism, we define a morphism $\check{\mu}_{\boldsymbol{f}}(N) : N^{[\sigma f_{0\tau},\, \tau_{f_0}]} \to (N^{[\sigma f_{0\tau},\, \tau_{f_0}]})^{[\sigma,\tau]}$ to be the following composition.

$$N^{[\sigma f_{0\tau},\, \tau_{f_0}]} \xrightarrow{N^{\mu \times_{C_0} id_{D_0}}} N^{[\sigma f_{0\tau}(\mu \times_{C_0} id_{D_0}),\, \tau_{f_0}(\mu \times_{C_0} id_{D_0})]} = N^{[\sigma \mathrm{pr}_1 \widetilde{\mathrm{pr}}_{12}, \tau_{f_0} \widetilde{\mathrm{pr}}_{23}]} \xrightarrow{\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N)^{-1}} (N^{[\sigma f_{0\tau},\, \tau_{f_0}]})^{[\sigma,\tau]}$$

We consider the following commutative diagram below.

$$\begin{tikzcd}[column sep=small]
 & & & & C_1 \times_{C_0} C_1 \times_{C_0} C_1 \times_{C_0} D_0 \arrow[dl, "\widetilde{\mathrm{pr}}_{123}"'] \arrow[dr, "\widetilde{\mathrm{pr}}_{234}"] & & & \\
 & & & C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[dr, "\mathrm{pr}_{23}"] & & C_1 \times_{C_0} C_1 \times_{C_0} D_0 \arrow[dl, "\widetilde{\mathrm{pr}}_{12}"'] \arrow[dr, "\widetilde{\mathrm{pr}}_{23}"] & & \\
 & & C_1 \times_{C_0} C_1 \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & & C_1 \times_{C_0} C_1 \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & & C_1 \times_{C_0} D_0 \arrow[dl, "f_{0\tau}"'] \arrow[dr, "\tau_{f_0}"] & \\
 & C_1 \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & & C_1 \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & & C_1 \arrow[dl, "\sigma"'] & & D_0 \\
C_0 & & C_0 & & C_0 & & &
\end{tikzcd}$$

Proposition 9.7.1 *Assume that that $\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N) : (N^{[\sigma f_{0\tau},\, \tau_{f_0}]})^{[\sigma,\tau]} \to N^{[\sigma \mathrm{pr}_1 \widetilde{\mathrm{pr}}_{12}, \tau_{f_0} \widetilde{\mathrm{pr}}_{23}]}$ is an isomorphism and that $\theta^{\sigma \mathrm{pr}_1, \tau \mathrm{pr}_2, \sigma f_{0\tau},\, \tau_{f_0}}(N) : (N^{[\sigma f_{0\tau},\, \tau_{f_0}]})^{[\sigma \mathrm{pr}_1, \tau \mathrm{pr}_2]} \to N^{[\sigma \mathrm{pr}_1 \widetilde{\mathrm{pr}}_{12} \widetilde{\mathrm{pr}}_{123}, \tau_{f_0} \widetilde{\mathrm{pr}}_{23} \widetilde{\mathrm{pr}}_{234}]}$ is a monomorphism. We put $\mu^r_{\boldsymbol{f}}(N) = E_{\sigma,\tau}(N^{[\sigma f_{0\tau},\, \tau_{f_0}]})^{-1}_{N^{[\sigma f_{0\tau},\, \tau_{f_0}]}}(\check{\mu}_{\boldsymbol{f}}(N)) : \sigma^*(N^{[\sigma f_{0\tau},\, \tau_{f_0}]}) \to \tau^*(N^{[\sigma f_{0\tau},\, \tau_{f_0}]})$. Then, $(N^{[\sigma f_{0\tau},\, \tau_{f_0}]}, \mu^r_{\boldsymbol{f}}(N))$ is a representation of $\boldsymbol{C}$.*

Proof. It follows from (8.5.19) that the following diagram is commutative.

$$\begin{tikzcd}
N^{[\sigma f_{0\tau},\, \tau_{f_0}]} \arrow[r, "N^{\mu \times_{C_0} id_{D_0}}"] \arrow[dr, "id_{N^{[\sigma f_{0\tau},\, \tau_{f_0}]}}"'] & N^{[\sigma \mathrm{pr}_1 \widetilde{\mathrm{pr}}_{12},\, \tau_{f_0} \widetilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N)^{-1}"] \arrow[d, "N^{\varepsilon \times_{C_0} id_{C_1 \times_{C_0} D_0}}"] & (N^{[\sigma f_{0\tau},\, \tau_{f_0}]})^{[\sigma,\tau]} \arrow[d, "(N^{[\sigma f_{0\tau},\, \tau_{f_0}]})^{\varepsilon}"] \\
 & N^{[\sigma f_{0\tau}, \tau_{f_0}]} \arrow[r, "\theta_{\sigma\varepsilon,\tau\varepsilon,\sigma f_{0\tau},\, \tau_{f_0}}(N)^{-1}"] & (N^{[\sigma f_{0\tau},\, \tau_{f_0}]})^{[\sigma\varepsilon,\tau\varepsilon]}
\end{tikzcd}$$

Hence a composition $N^{[\sigma f_{0\tau},\, \tau_{f_0}]} \xrightarrow{\check{\mu}_{\boldsymbol{f}}(N)} (N^{[\sigma f_{0\tau},\, \tau_{f_0}]})^{[\sigma,\tau]} \xrightarrow{(N^{[\sigma f_{0\tau},\, \tau_{f_0}]})^{\varepsilon}} (N^{[\sigma f_{0\tau},\, \tau_{f_0}]})^{[\sigma\varepsilon,\tau\varepsilon]} = N^{[\sigma f_{0\tau},\, \tau_{f_0}]}$ coincides with the identity morphism of $N^{[\sigma f_{0\tau},\, \tau_{f_0}]}$.
Note that we have the following equalities.

$$\begin{aligned}
\sigma \mathrm{pr}_1 \widetilde{\mathrm{pr}}_{12} \widetilde{\mathrm{pr}}_{123} &= \sigma \mathrm{pr}_1 \widetilde{\mathrm{pr}}_{12}(\mu \times_{C_0} id_{C_0} \times_{C_0} id_{D_0}) = \sigma \mathrm{pr}_1 \widetilde{\mathrm{pr}}_{12}(id_{C_0} \times_{C_0} \mu \times_{C_0} id_{D_0}) \\
\tau_{f_0} \widetilde{\mathrm{pr}}_{23} \widetilde{\mathrm{pr}}_{234} &= \tau_{f_0} \widetilde{\mathrm{pr}}_{23}(\mu \times_{C_0} id_{C_0} \times_{C_0} id_{D_0}) = \tau_{f_0} \widetilde{\mathrm{pr}}_{23}(id_{C_0} \times_{C_0} \mu \times_{C_0} id_{D_0}) \\
\sigma \mathrm{pr}_1 \widetilde{\mathrm{pr}}_{12} &= \sigma f_{0\tau}(\mu \times_{C_0} id_{D_0}) \\
\tau_{f_0} \widetilde{\mathrm{pr}}_{23} &= \tau_{f_0}(\mu \times_{C_0} id_{D_0})
\end{aligned}$$

It follows from (2) of (8.5.6), (8.5.19) and (8.5.22) that the following diagram commutes.

$$
\begin{tikzcd}
N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "N^{\mu\times_{C_0} id_{D_0}}"] \arrow[d, "N^{\mu\times_{C_0} id_{D_0}}"] & N^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "N^{\mu\times_{C_0} id_{C_0}\times_{C_0} id_{D_0}}"] & (N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \arrow[l, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N)"'] \arrow[d, "(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{\mu}"'] \\
N^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[r, "N^{id_{C_0}\times_{C_0}\mu\times_{C_0} id_{D_0}}"] & N^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123},\tau_{f_0}\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234}]} & (N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma \mathrm{pr}_1,\tau \mathrm{pr}_2]} \arrow[l, "\theta^{\sigma \mathrm{pr}_1,\tau \mathrm{pr}_2,\sigma f_{0\tau},\tau_{f_0}}(N)"'] \\
(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \arrow[u, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N)"] \arrow[r, "(N^{\mu\times_{C_0} id_{D_0}})^{[\sigma,\tau]}"] & (N^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]})^{[\sigma,\tau]} \arrow[u, "\theta^{\sigma,\tau,\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}}(N)"] & ((N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[l, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N)^{[\sigma,\tau]}"'] \arrow[u, "\theta^{\sigma,\tau,\sigma,\tau}(N^{[\sigma f_{0\tau},\tau_{f_0}]})"']
\end{tikzcd}
$$

Thus the following diagram commutes.

$$
\begin{tikzcd}
N^{[\sigma f_{0\tau},\,\tau_{f_0}]} \arrow[r, "\check{\mu}_{\boldsymbol{f}}(N)"] \arrow[dr, "\check{\mu}_{\boldsymbol{f}}(N)"'] & (N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\check{\mu}_{\boldsymbol{f}}(N)^{[\sigma,\tau]}"] & ((N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(N^{[\sigma f_{0\tau},\,\tau_{f_0}]})"] \\
& (N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "(N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{\mu}"] & (N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma \mathrm{pr}_1,\tau \mathrm{pr}_2]}
\end{tikzcd}
$$

and $\check{\mu}_{\boldsymbol{f}}(N)$ satisfies the conditions of (9.5.1). □

Proposition 9.7.2 *Let $\varphi : M \to N$ be a morphisms of $\mathcal{F}_{D_0}$. Assume that that the following upper morphism is an isomorphism and that the lower morphism is a monomorphism for $L = M, N$.*

$$\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(L) : (L^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma,\tau]} \longrightarrow L^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}$$
$$\theta^{\sigma \mathrm{pr}_1,\tau \mathrm{pr}_2,\sigma f_{0\tau},\,\tau_{f_0}}(L) : (L^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma \mathrm{pr}_1,\tau \mathrm{pr}_2]} \longrightarrow L^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123},\tau_{f_0}\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234}]}$$

Then, $\varphi^{[\sigma f_{0\tau},\,\tau_{f_0}]} : (M^{[\sigma f_{0\tau},\,\tau_{f_0}]}, \mu^r_{\boldsymbol{f}}(M)) \to (N^{[\sigma f_{0\tau},\,\tau_{f_0}]}, \mu^r_{\boldsymbol{f}}(N))$ is a morphism of representations of $\boldsymbol{C}$.

Proof. The following diagram is commutative by (8.5.8) and (8.5.19).

$$
\begin{tikzcd}
M^{[\sigma f_{0\tau},\,\tau_{f_0}]} \arrow[r, "M^{\mu\times_{C_0} id_{D_0}}"] \arrow[d, "\varphi^{[\sigma f_{0\tau},\,\tau_{f_0}]}"] & M^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(M)^{-1}"] \arrow[d, "\varphi^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}"] & (M^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[d, "(\varphi^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma,\tau]}"] \\
N^{[\sigma f_{0\tau},\,\tau_{f_0}]} \arrow[r, "N^{\mu\times_{C_0} id_{D_0}}"] & N^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[r, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N)^{-1}"] & (N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma,\tau]}
\end{tikzcd}
$$

Hence the assertion follows from (9.5.5). □

Let $C_1 \xleftarrow{\tilde{\mathrm{pr}}_1} C_1 \times_{C_0} D_1 \xrightarrow{\tilde{\mathrm{pr}}_2} D_1$ be a limit of a diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{f_0\sigma'} D_1$. Then, there exists unique morphism $id_{C_1} \times_{C_0} \sigma' : C_1 \times_{C_0} D_1 \to C_1 \times_{C_0} D_0$ that satisfies $\tau_{f_0}(id_{C_1} \times_{C_0} \sigma') = \sigma'\tilde{\mathrm{pr}}_2$ and $f_{0\tau}(id_{C_1} \times_{C_0} \sigma') = \tilde{\mathrm{pr}}_1$.

$$
\begin{tikzcd}
C_1 \times_{C_0} D_1 \arrow[r, "\tilde{\mathrm{pr}}_2"] \arrow[d, "\tilde{\mathrm{pr}}_1"] & D_1 \arrow[d, "f_0\sigma'"] \\
C_1 \arrow[r, "\tau"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \times_{C_0} D_1 \arrow[dr, dotted, "id_{C_1}\times_{C_0}\sigma'"] \arrow[drr, bend left, "\tilde{\mathrm{pr}}_1"] \arrow[dd, "\tilde{\mathrm{pr}}_2"] & & \\
& C_1 \times_{C_0} D_0 \arrow[r, "f_{0\tau}"] \arrow[d, "\tau_{f_0}"] & C_1 \arrow[d, "\tau"] \\
D_1 \arrow[r, "\sigma'"] & D_0 \arrow[r, "f_0"] & C_0
\end{tikzcd}
$$

We note that $C_1 \times_{C_0} D_0 \xleftarrow{id_{C_1}\times_{C_0}\sigma'} C_1 \times_{C_0} D_1 \xrightarrow{\tilde{\mathrm{pr}}_2} D_1$ is a limit of a diagram $C_1 \times_{C_0} D_0 \xrightarrow{\tau_{f_0}} D_0 \xleftarrow{\sigma'} D_1$. Since (f_0, f_1) is an internal functor, we also have unique morphism $id_{C_1} \times_{C_0} f_1 : C_1 \times_{C_0} D_1 \to C_1 \times_{C_0} C_1$ that satisfies $\mathrm{pr}_1(id_{C_1} \times_{C_0} f_1) = \tilde{\mathrm{pr}}_1$ and $\mathrm{pr}_2(id_{C_1} \times_{C_0} f_1) = f_1\tilde{\mathrm{pr}}_2$. Then, we have

$$\tau\mu(id_{C_1} \times_{C_0} f_1) = \tau \mathrm{pr}_2(id_{C_1} \times_{C_0} f_1) = \tau f_1\tilde{\mathrm{pr}}_2 = f_0\tau'\tilde{\mathrm{pr}}_2$$

which implies that there exists unique morphism $(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2) : C_1 \times_{C_0} D_1 \to C_1 \times_{C_0} D_0$ that satisfies $\tau_{f_0}(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2) = \tau'\tilde{\mathrm{pr}}_2$ and $f_{0\tau}(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2) = \mu(id_{C_1} \times_{C_0} f_1)$. Hence we have

$$\sigma f_{0\tau}(\mu(id_{C_1} \times_{C_0} f_1), \tau'\tilde{\mathrm{pr}}_2) = \sigma\mu(id_{C_1} \times_{C_0} f_1) = \sigma \mathrm{pr}_1(id_{C_1} \times_{C_0} f_1) = \sigma\tilde{\mathrm{pr}}_1 = \sigma f_{0\tau}(id_{C_1} \times_{C_0} \sigma').$$

Let $C_1 \xleftarrow{\tilde{\mathrm{pr}}_1} C_1 \times_{C_0} C_1 \times_{C_0} D_1 \xrightarrow{\tilde{\mathrm{pr}}_{23}} C_1 \times_{C_0} D_1$ be a limit of a diagram $C_1 \xrightarrow{\tau} C_0 \xleftarrow{\sigma\tilde{\mathrm{pr}}_1} C_1 \times_{C_0} D_1$.

Assumption 9.7.3 *For a representation (N,ζ) of $\boldsymbol{D}$, we put $\check{\zeta} = E_{\sigma',\tau'}(N)_N : N \to N^{[\sigma',\tau']}$. We assume the following.*

(i) An equalizer of the following morphisms of $\mathcal{F}_{C_0}$ exists.

$$N^{[\sigma f_{0\tau},\tau_{f0}]} \xrightarrow{\check{\zeta}^{[\sigma f_{0\tau},\tau_{f0}]}} (N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f0}]} \xrightarrow{\theta^{\sigma f_{0\tau},\tau_{f0},\sigma',\tau'}(N)} N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}$$

$$N^{[\sigma f_{0\tau},\tau_{f0}]} \xrightarrow{N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}} N^{[\sigma f_{0\tau}(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2),\,\tau_{f0}(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)]} = N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}$$

(ii) Let us denote by $E^{\boldsymbol{f}}_{(N,\zeta)} : (N,\zeta)^{\boldsymbol{f}} \to N^{[\sigma f_{0\tau},\tau_{f0}]}$ an equalizer of the above morphisms. Then $\left(E^{\boldsymbol{f}}_{(N,\zeta)}\right)^{[\sigma,\tau]} : ((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]} \to (N^{[\sigma f_{0\tau},\tau_{f0}]})^{[\sigma,\tau]}$ is an equalizer of the following morphisms.

$$(N^{[\sigma f_{0\tau},\tau_{f0}]})^{[\sigma,\tau]} \xrightarrow{(\check{\zeta}^{[\sigma f_{0\tau},\tau_{f0}]})^{[\sigma,\tau]}} ((N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f0}]})^{[\sigma,\tau]} \xrightarrow{\theta^{\sigma f_{0\tau},\tau_{f0},\sigma',\tau'}(N)^{[\sigma,\tau]}} (N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]})^{[\sigma,\tau]}$$

$$(N^{[\sigma f_{0\tau},\tau_{f0}]})^{[\sigma,\tau]} \xrightarrow{(N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)})^{[\sigma,\tau]}} (N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]})^{[\sigma,\tau]}$$

(iii) $\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f0}}(N) : (N^{[\sigma f_{0\tau},\,\tau_{f0}]})^{[\sigma,\tau]} \to N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f0}\tilde{\mathrm{pr}}_{23}]}$ is an isomorphism.

(iv) The following morphisms are monomorphisms.

$$\theta^{\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2,\sigma f_{0\tau},\,\tau_{f0}}(N) : (N^{[\sigma f_{0\tau},\,\tau_{f0}]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \to N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}\tilde{\mathrm{pr}}_{123},\tau_{f0}\tilde{\mathrm{pr}}_{23}\tilde{\mathrm{pr}}_{234}]}$$
$$\theta^{\sigma,\tau,\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2}(N) : (N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]})^{[\sigma,\tau]} \longrightarrow N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}(id_{C_1}\times_{C_0}id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2\tilde{\mathrm{pr}}_{23}]}$$
$$(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} : ((N,\zeta)^{\boldsymbol{f}})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \longrightarrow (N^{[\sigma f_{0\tau},\tau_{f0}]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$$

The following diagram commutes.

$$\begin{tikzcd}
C_1\times_{C_0}C_1 \arrow[d, "\mu"] & C_1\times_{C_0}C_1\times_{C_0}D_0 \arrow[l, "\tilde{\mathrm{pr}}_{12}"'] \arrow[r, "\tilde{\mathrm{pr}}_{23}"] \arrow[d, "\mu\times_{C_0}id_{D_0}"] & C_1\times_{C_0}D_0 \arrow[d, "\tau_{f0}"] \\
C_1 & C_1\times_{C_0}D_0 \arrow[l, "f_{0\tau}"'] \arrow[r, "\tau_{f0}"] & D_0
\end{tikzcd}$$

Hence we have $\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12} = \sigma\mu\tilde{\mathrm{pr}}_{12} = \sigma f_{0\tau}(\mu\times_{C_0}id_{D_0})$ and $\tau_{f0}\tilde{\mathrm{pr}}_{23} = \tau_{f0}(\mu\times_{C_0}id_{D_0})$.

Consider the following diagram whose rhombuses are all cartesian.

$$\begin{tikzcd}[column sep=small]
& & & & C_1\times_{C_0}C_1\times_{C_0}D_1 \arrow[dll, "id_{C_1}\times_{C_0}id_{C_1}\times_{C_0}\sigma'"'] \arrow[drr, "\tilde{\mathrm{pr}}_{23}"] & & & & \\
& & C_1\times_{C_0}C_1\times_{C_0}D_0 \arrow[dl, "\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}"'] \arrow[drr, "\tilde{\mathrm{pr}}_{23}"] & & & & C_1\times_{C_0}D_1 \arrow[dll, "id_{C_1}\times_{C_0}\sigma'"'] \arrow[dr, "\tilde{\mathrm{pr}}_2"] & & \\
& C_1 \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & & & C_1\times_{C_0}D_0 \arrow[dll, "\sigma f_{0\tau}"'] \arrow[drr, "\tau_{f0}"] & & & D_1 \arrow[dl, "\sigma'"'] \arrow[dr, "\tau'"] & \\
C_0 & & C_0 & & & & D_0 & & D_0
\end{tikzcd}$$

It follows from (8.5.22) that

$$\begin{tikzcd}
((N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f0}]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f0}}(N^{[\sigma',\tau']})"] \arrow[d, "\theta^{\sigma f_{0\tau},\tau_{f0},\sigma',\tau'}(N)^{[\sigma,\tau]}"] & (N^{[\sigma',\tau']})^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "\theta^{\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f0}\tilde{\mathrm{pr}}_{23},\sigma',\tau'}(N)"] \\
(N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2}(N)"] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}(id_{C_1}\times_{C_0}id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

is commutative. The following diagrams are commutative by (8.5.19), (8.5.17), (8.5.8), respectively.

$$\begin{tikzcd}
(N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f0}]} \arrow[r, "(N^{[\sigma',\tau']})^{\mu\times_{C_0}id_{D_0}}"] \arrow[d, "\theta^{\sigma f_{0\tau},\tau_{f0},\sigma',\tau'}(N)"] & (N^{[\sigma',\tau']})^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "\theta^{\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f0}\tilde{\mathrm{pr}}_{23},\sigma',\tau'}(N)"] \\
N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \arrow[r, "N^{\mu\times_{C_0}id_{D_1}}"] & N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}(id_{C_1}\times_{C_0}id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

$$\begin{tikzcd}
(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N)"] \arrow[d, "(\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]}"] & N^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "\check{\zeta}^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}"] \\
((N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N^{[\sigma',\tau']})"] & (N^{[\sigma',\tau']})^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

$$\begin{tikzcd}
N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "N^{\mu\times_{C_0} id_{D_0}}"] \arrow[d, "\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]}"] & N^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "\check{\zeta}^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}"] \\
(N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "(N^{[\sigma',\tau']})^{\mu\times_{C_0} id_{D_0}}"] & (N^{[\sigma',\tau']})^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

The associativity of μ implies that a diagram

$$\begin{tikzcd}
C_1\times_{C_0} C_1\times_{C_0} D_1 \arrow[r, "\mu\times_{C_0} id_{D_1}"] \arrow[d, "id_{C_1}\times_{C_0}(\mu(id_{C_1}\times_{C_0} f_1),\,\tau'\tilde{\mathrm{pr}}_2)"] & C_1\times_{C_0} D_1 \arrow[d, "(\mu(id_{C_1}\times_{C_0} f_1),\,\tau'\tilde{\mathrm{pr}}_2)"] \\
C_1\times_{C_0} C_1\times_{C_0} D_0 \arrow[r, "\mu\times_{C_0} id_{D_0}"] & C_1\times_{C_0} D_0
\end{tikzcd}$$

is commutative. Hence the following diagram is commutative by (8.5.6).

$$\begin{tikzcd}
N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "N^{\mu\times_{C_0} id_{D_0}}"] \arrow[d, "N^{(\mu(id_{C_1}\times_{C_0} f_1),\,\tau'\tilde{\mathrm{pr}}_2)}"] & N^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "N^{id_{C_1}\times_{C_0}(\mu(id_{C_1}\times_{C_0} f_1),\,\tau'\tilde{\mathrm{pr}}_2)}"] \\
N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\,\tau'\tilde{\mathrm{pr}}_2]} \arrow[r, "N^{\mu\times_{C_0} id_{D_1}}"] & N^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12}(id_{C_1}\times_{C_0} id_{C_1}\times_{C_0}\sigma'),\,\tau'\tilde{\mathrm{pr}}_2\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

Moreover, it follows from (8.5.19) that the following diagram commutes.

$$\begin{tikzcd}
(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N)"] \arrow[d, "(N^{(\mu(id_{C_1}\times_{C_0} f_1),\,\tau'\tilde{\mathrm{pr}}_2)})^{[\sigma,\tau]}"] & N^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[d, "N^{id_{C_1}\times_{C_0}(\mu(id_{C_1}\times_{C_0} f_1),\,\tau'\tilde{\mathrm{pr}}_2)}"] \\
(N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\,\tau'\tilde{\mathrm{pr}}_2]})^{[\sigma,\tau]} \arrow[r, "\theta^{\sigma,\tau,\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2}(N)"] & N^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12}(id_{C_1}\times_{C_0} id_{C_1}\times_{C_0}\sigma'),\,\tau'\tilde{\mathrm{pr}}_2\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

Since $E^{\boldsymbol{f}}_{(N,\zeta)}$ is an equalizer of $\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]}$ and $N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}$, we have

$$\begin{aligned}
&\theta^{\sigma,\tau,\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2}(N)(\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]}\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= \theta^{\sigma,\tau,\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2}(N)\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)^{[\sigma,\tau]}(\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]}\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N)^{-1}N^{\mu\times_{C_0} id_{D_0}}E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= \theta^{\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23},\sigma',\tau'}(N)\check{\zeta}^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}N^{\mu\times_{C_0} id_{D_0}}E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= \theta^{\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23},\sigma',\tau'}(N)(N^{[\sigma',\tau']})^{\mu\times_{C_0} id_{D_0}}\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]}E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= N^{\mu\times_{C_0} id_{D_1}}\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]}E^{\boldsymbol{f}}_{(N,\zeta)} = N^{\mu\times_{C_0} id_{D_1}}N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= N^{id_{C_1}\times_{C_0}(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}N^{\mu\times_{C_0} id_{D_0}}E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= N^{id_{C_1}\times_{C_0}(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N)\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)} \\
&\quad= \theta^{\sigma,\tau,\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2}(N)(N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)})^{[\sigma,\tau]}\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)}.
\end{aligned}$$

Therefore, it follows from the assumption (iv) of (9.7.3) that we have

$$(\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]}\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)} = (N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)})^{[\sigma,\tau]}\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)}.$$

Hence (ii) of (9.7.3) implies that there exists unique morphism $\check{\zeta}_{\boldsymbol{f}} : (N,\zeta)^{\boldsymbol{f}} \to ((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]}$ that satisfies $(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}\check{\zeta}_{\boldsymbol{f}} = \check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)}$. We put $\zeta^r_{\boldsymbol{f}} = E_{\sigma,\tau}((N,\zeta)^{\boldsymbol{f}})^{-1}_{(N,\zeta)^{\boldsymbol{f}}}(\check{\zeta}_{\boldsymbol{f}}) : \sigma^*((N,\zeta)^{\boldsymbol{f}}) \to \tau^*((N,\zeta)^{\boldsymbol{f}})$.

Proposition 9.7.4 *$((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}})$ is a representation of $\boldsymbol{C}$ and $E^{\boldsymbol{f}}_{(N,\zeta)} : ((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}}) \to (N^{[\sigma f_{0\tau},\,\tau_{f_0}]},\mu^r_{\boldsymbol{f}}(N))$ is a morphism of representations of $\boldsymbol{C}$.*

Proof. It follows from (8.5.8), (9.7.1), (8.5.19) and the definition of $\check{\zeta}_{\boldsymbol{f}}$ that we have

$$\begin{aligned}
(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}((N,\zeta)^{\boldsymbol{f}})^{\mu}\check{\zeta}_{\boldsymbol{f}} &= (N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{\mu}(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}\check{\zeta}_{\boldsymbol{f}} = (N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{\mu}\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)} \\
&= \theta^{\sigma,\tau,\sigma,\tau}(N^{[\sigma f_{0\tau},\,\tau_{f_0}]})\check{\mu}_{\boldsymbol{f}}(N)^{[\sigma,\tau]}\check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)} \\
&= \theta^{\sigma,\tau,\sigma,\tau}(N^{[\sigma f_{0\tau},\,\tau_{f_0}]})\check{\mu}_{\boldsymbol{f}}(N)^{[\sigma,\tau]}(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}\check{\zeta}_{\boldsymbol{f}} \\
&= \theta^{\sigma,\tau,\sigma,\tau}(N^{[\sigma f_{0\tau},\,\tau_{f_0}]})((E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]})^{[\sigma,\tau]}(\check{\zeta}_{\boldsymbol{f}})^{[\sigma,\tau]}\check{\zeta}_{\boldsymbol{f}} \\
&= (E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}\theta^{\sigma,\tau,\sigma,\tau}((N,\zeta)^{\boldsymbol{f}}))(\check{\zeta}_{\boldsymbol{f}})^{[\sigma,\tau]}\check{\zeta}_{\boldsymbol{f}}.
\end{aligned}$$

Since we assume that $(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is a monomorphism in (9.7.3), $((N,\zeta)^{\boldsymbol{f}})^{\mu}\check{\zeta}_{\boldsymbol{f}} = \theta^{\sigma,\tau,\sigma,\tau}((N,\zeta)^{\boldsymbol{f}})(\check{\zeta}_{\boldsymbol{f}})^{[\sigma,\tau]}\check{\zeta}_{\boldsymbol{f}}$ holds. (See the diagram below.)

$$\begin{tikzcd}
N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[rrrr, "\check{\mu}_{\boldsymbol{f}}(N)"] & & & & (N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \arrow[ddd, "\check{\mu}_{\boldsymbol{f}}(N)^{[\sigma,\tau]}"] \\
& ((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]} \arrow[urrr, "(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}"] \arrow[r, "\check{\zeta}^{[\sigma,\tau]}"'] & (((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}((N,\zeta)^{\boldsymbol{f}})"'] \arrow[ddrr, "((E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]})^{[\sigma,\tau]}"] & & \\
(N,\zeta)^{\boldsymbol{f}} \arrow[uu, "E^{\boldsymbol{f}}_{(N,\zeta)}"] \arrow[ur, "\check{\zeta}_{\boldsymbol{f}}"] \arrow[r, "\check{\zeta}_{\boldsymbol{f}}"] \arrow[d, "E^{\boldsymbol{f}}_{(N,\zeta)}"] & ((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]} \arrow[r, "((N,\zeta)^{\boldsymbol{f}})^{\mu}"] \arrow[d, "(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}"] & ((N,\zeta)^{\boldsymbol{f}})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[d, "(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}"] & & \\
N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "\check{\mu}_{\boldsymbol{f}}(N)"] & (N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{\mu}"] & (N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} & & ((N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[ll, "\theta^{\sigma,\tau,\sigma,\tau}(N^{[\sigma f_{0\tau},\tau_{f_0}]})"']
\end{tikzcd}$$

The following diagram is commutative by (8.5.8) and the definition of $\check{\zeta}_{\boldsymbol{f}}$.

$$\begin{tikzcd}
(N,\zeta)^{\boldsymbol{f}} \arrow[r, "\check{\zeta}_{\boldsymbol{f}}"] \arrow[d, "E^{\boldsymbol{f}}_{(N,\zeta)}"] & ((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]} \arrow[r, "((N,\zeta)^{\boldsymbol{f}})^{\varepsilon}"] \arrow[d, "(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}"] & ((N,\zeta)^{\boldsymbol{f}})^{[\sigma\varepsilon,\tau\varepsilon]} = (N,\zeta)^{\boldsymbol{f}} \arrow[d, "E^{\boldsymbol{f}}_{(N,\zeta)}"] \\
N^{[\sigma f_{0\tau},\,\tau_{f_0}]} \arrow[r, "\check{\mu}_{\boldsymbol{f}}(N)"] & (N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "(N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{\varepsilon}"] & (N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma\varepsilon,\tau\varepsilon]} = N^{[\sigma f_{0\tau},\,\tau_{f_0}]}
\end{tikzcd}$$

Since $(N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{\varepsilon}\check{\mu}_{\boldsymbol{f}}(N)$ is the identity morphism of $N^{[\sigma f_{0\tau},\,\tau_{f_0}]}$, we have $((N,\zeta)^{\boldsymbol{f}})^{\varepsilon}\check{\zeta}_{\boldsymbol{f}}E^{\boldsymbol{f}}_{(N,\zeta)} = E^{\boldsymbol{f}}_{(N,\zeta)}$ which implies that $E^{\boldsymbol{f}}_{(N,\zeta)}\check{\zeta}_{\boldsymbol{f}}((N,\zeta)^{\boldsymbol{f}})^{\varepsilon}$ is the identity morphism of $(N,\zeta)^{\boldsymbol{f}}$, since $E^{\boldsymbol{f}}_{(N,\zeta)}$ is a monomorphism. Hence $((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}})$ is a representation of $\boldsymbol{C}$ by (9.5.1). It follows from (9.5.4) and the definition of $\check{\zeta}_{\boldsymbol{f}}$ that $E^{\boldsymbol{f}}_{(N,\zeta)}$ is a morphism of representations. □

We assume (9.7.3) also for a representation (M,ξ) of $\boldsymbol{D}$. Let $\varphi : (M,\xi) \to (N,\zeta)$ be a morphism of representations of $\boldsymbol{D}$. The following diagrams are commutative by (8.5.19), (8.5.3) and (8.5.8).

$$\begin{tikzcd}
M^{[\sigma f_{0\tau},\,\tau_{f_0}]} \arrow[r, "\check{\xi}^{[\sigma f_{0\tau},\tau_{f_0}]}"] \arrow[d, "\varphi^{[\sigma f_{0\tau},\,\tau_{f_0}]}"] & (M^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(M)"] \arrow[d, "(\varphi^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]}"] & M^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \arrow[d, "\varphi^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}"] \\
N^{[\sigma f_{0\tau},\,\tau_{f_0}]} \arrow[r, "\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]}"] & (N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)"] & N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

$$\begin{tikzcd}
M^{[\sigma f_{0\tau},\,\tau_{f_0}]} \arrow[r, "M^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}"] \arrow[d, "\varphi^{[\sigma f_{0\tau},\,\tau_{f_0}]}"] & M^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \arrow[d, "\varphi^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}"] \\
N^{[\sigma f_{0\tau},\,\tau_{f_0}]} \arrow[r, "N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}"] & N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

Hence there exists unique morphism $\varphi^{\boldsymbol{f}} : (M,\xi)^{\boldsymbol{f}} \to (N,\zeta)^{\boldsymbol{f}}$ that satisfies $E^{\boldsymbol{f}}_{(N,\zeta)}\varphi^{\boldsymbol{f}} = \varphi^{[\sigma f_{0\tau},\,\tau_{f0}]}E^{\boldsymbol{f}}_{(M,\xi)}$.

Proposition 9.7.5 *$\varphi^{\boldsymbol{f}} : ((M,\xi)^{\boldsymbol{f}}, \xi^r_{\boldsymbol{f}}) \to ((N,\zeta)^{\boldsymbol{f}}, \zeta^r_{\boldsymbol{f}})$ is a morphism of representations of $\boldsymbol{C}$.*

Proof. It follows from (9.7.2) that the inner rectangle of the following diagram is commutative.

$$\begin{tikzcd}
(M,\xi)^{\boldsymbol{f}} \arrow[rrr, "\check{\xi}_{\boldsymbol{f}}"] \arrow[dr, "E^{\boldsymbol{f}}_{(M,\xi)}"] \arrow[ddd, "\varphi^{\boldsymbol{f}}"'] & & & ((M,\xi)^{\boldsymbol{f}})^{[\sigma,\tau]} \arrow[dl, "(E^{\boldsymbol{f}}_{(M,\xi)})^{[\sigma,\tau]}"'] \arrow[ddd, "(\varphi^{\boldsymbol{f}})^{[\sigma,\tau]}"] \\
& M^{[\sigma f_{0\tau},\,\tau_{f0}]} \arrow[r, "\check{\mu}_{\boldsymbol{f}}(M)"] \arrow[d, "\varphi^{[\sigma f_{0\tau},\,\tau_{f0}]}"] & (M^{[\sigma f_{0\tau},\,\tau_{f0}]})^{[\sigma,\tau]} \arrow[d, "(\varphi^{[\sigma f_{0\tau},\,\tau_{f0}]})^{[\sigma,\tau]}"] & \\
& N^{[\sigma f_{0\tau},\,\tau_{f0}]} \arrow[r, "\check{\mu}_{\boldsymbol{f}}(N)"] & (N^{[\sigma f_{0\tau},\,\tau_{f0}]})^{[\sigma,\tau]} & \\
(N,\zeta)^{\boldsymbol{f}} \arrow[rrr, "\check{\zeta}_{\boldsymbol{f}}"'] \arrow[ur, "E^{\boldsymbol{f}}_{(N,\zeta)}"'] & & & ((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]} \arrow[ul, "(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}"]
\end{tikzcd}$$

Then, by the definitions of $\check{\xi}_{\boldsymbol{f}}$, $\check{\zeta}_{\boldsymbol{f}}$ and $\varphi^{\boldsymbol{f}}$, we have

$$\begin{aligned}
(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}\check{\zeta}_{\boldsymbol{f}}\varphi^{\boldsymbol{f}} &= \check{\mu}_{\boldsymbol{f}}(N)E^{\boldsymbol{f}}_{(N,\zeta)}\varphi^{\boldsymbol{f}} = \check{\mu}_{\boldsymbol{f}}(N)\varphi^{[\sigma f_{0\tau},\,\tau_{f0}]}E^{\boldsymbol{f}}_{(M,\xi)} = (\varphi^{[\sigma f_{0\tau},\,\tau_{f0}]})^{[\sigma,\tau]}\check{\mu}_{\boldsymbol{f}}(M)E^{\boldsymbol{f}}_{(M,\xi)} \\
&= (\varphi^{[\sigma f_{0\tau},\,\tau_{f0}]})^{[\sigma,\tau]}(E^{\boldsymbol{f}}_{(M,\xi)})^{[\sigma,\tau]}\check{\xi}_{\boldsymbol{f}} = (E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}\check{\xi}_{\boldsymbol{f}}(\varphi^{\boldsymbol{f}})^{[\sigma f_{0\tau},\,\tau_{f0}]}.
\end{aligned}$$

Since $(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}$ is an epimorphism by (*ii*) of (9.7.3), the above equality implies $\check{\zeta}_{\boldsymbol{f}}\varphi^{\boldsymbol{f}} = (\varphi^{\boldsymbol{f}})^{[\sigma,\tau]}\check{\xi}_{\boldsymbol{f}}$. Therefore $\varphi^{\boldsymbol{f}}$ is a morphism of representations of $\boldsymbol{D}$ by (9.5.4). $\square$

Define functors $S, T, U : \mathcal{P} \to \mathcal{E}$ and natural transformations $\alpha : S \to T$, $\beta : T \to U$ as follows.

$S(0)=D_1$	$S(1)=D_0$	$S(2)=D_1$	$S(3)=D_0$	$S(4)=D_0$	$S(5)=D_0$
$S(\tau_{01})=\sigma'$	$S(\tau_{02})=id_{D_1}$	$S(\tau_{13})=id_{D_0}$	$S(\tau_{14})=id_{D_0}$	$S(\tau_{24})=\sigma'$	$S(\tau_{25})=\tau'$
$T(0)=C_1\times_{C_0}D_1$	$T(1)=C_1\times_{C_0}D_0$	$T(2)=D_1$	$T(3)=C_0$	$T(4)=D_0$	$T(5)=D_0$
$T(\tau_{01})=id_{C_1}\times_{C_0}\sigma'$	$T(\tau_{02})=\tilde{\mathrm{pr}}_2$	$T(\tau_{13})=\sigma f_{0\tau}$	$T(\tau_{14})=\tau_{f_0}$	$T(\tau_{24})=\sigma'$	$T(\tau_{25})=\tau'$
$U(0)=C_1\times_{C_0}C_1\times_{C_0}D_0$	$U(1)=C_1$	$U(2)=C_1\times_{C_0}D_0$	$U(3)=C_0$	$U(4)=C_0$	$U(5)=D_0$
$U(\tau_{01})=\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}$	$U(\tau_{02})=\tilde{\mathrm{pr}}_{23}$	$U(\tau_{13})=\sigma$	$U(\tau_{14})=\tau$	$U(\tau_{24})=\sigma f_{0\tau}$	$U(\tau_{25})=\tau_{f_0}$
$\alpha_0=(f_1\varepsilon'\sigma', id_{D_1})$	$\alpha_1=(f_1\varepsilon', id_{D_0})$	$\alpha_2=id_{D_1}$	$\alpha_3=f_0$	$\alpha_4=id_{D_0}$	$\alpha_5=id_{D_0}$
$\beta_0=(\tilde{\mathrm{pr}}_1, f_1\tilde{\mathrm{pr}}_2, \tau'\tilde{\mathrm{pr}}_2)$	$\beta_1=f_{0\tau}$	$\beta_2=(f_1,\tau')$	$\beta_3=id_{C_0}$	$\beta_4=f_0$	$\beta_5=id_{D_0}$

Hence if we define functors $S_i, T_i, U_i : \mathcal{Q} \to \mathcal{E}$ for $i = 0, 1, 2$ by

$$\begin{array}{lllll}
S_0(0)=S(0) & S_0(1)=S(3) & S_0(2)=S(5) & S_0(\tau_{01})=S(\tau_{13}\tau_{01}) & S_0(\tau_{02})=S(\tau_{25}\tau_{02}) \\
T_0(0)=T(0) & T_0(1)=T(3) & T_0(2)=T(5) & T_0(\tau_{01})=T(\tau_{13}\tau_{01}) & T_0(\tau_{02})=T(\tau_{25}\tau_{02}) \\
U_0(0)=U(0) & U_0(1)=U(3) & U_0(2)=U(5) & U_0(\tau_{01})=U(\tau_{13}\tau_{01}) & U_0(\tau_{02})=U(\tau_{25}\tau_{02}) \\
S_1(0)=S(1) & S_1(1)=S(3) & S_1(2)=S(4) & S_1(\tau_{01})=S(\tau_{13}) & S_1(\tau_{02})=S(\tau_{14}) \\
T_1(0)=T(1) & T_1(1)=T(3) & T_1(2)=T(4) & T_1(\tau_{01})=T(\tau_{13}) & T_1(\tau_{02})=T(\tau_{14}) \\
U_1(0)=U(1) & U_1(1)=U(3) & U_1(2)=U(4) & U_1(\tau_{01})=U(\tau_{13}) & U_1(\tau_{02})=U(\tau_{14}) \\
S_2(0)=S(2) & S_2(1)=S(4) & S_2(2)=S(5) & S_2(\tau_{01})=S(\tau_{24}) & S_2(\tau_{02})=S(\tau_{25}) \\
T_2(0)=T(2) & T_2(1)=T(4) & T_2(2)=T(5) & T_2(\tau_{01})=T(\tau_{24}) & T_2(\tau_{02})=T(\tau_{25}) \\
U_2(0)=U(2) & U_2(1)=U(4) & U_2(2)=U(5) & U_2(\tau_{01})=U(\tau_{24}) & U_2(\tau_{02})=U(\tau_{25})
\end{array}$$

and natural transformations $\alpha^i : S_i \to T_i$, $\beta^i : T_i \to U_i$ for $i = 0, 1, 2$ by

$$\begin{array}{lllllllll}
\alpha^0_0=\alpha_0 & \alpha^0_1=\alpha_3 & \alpha^0_2=\alpha_5 & \alpha^1_0=\alpha_1 & \alpha^1_1=\alpha_3 & \alpha^1_2=\alpha_4 & \alpha^2_0=\alpha_2 & \alpha^2_1=\alpha_4 & \alpha^2_2=\alpha_5, \\
\beta^0_0=\beta_0 & \beta^0_1=\beta_3 & \beta^0_2=\beta_5 & \beta^1_0=\beta_1 & \beta^1_1=\beta_3 & \beta^1_2=\beta_4 & \beta^2_0=\beta_2 & \beta^2_1=\beta_4 & \beta^2_2=\beta_5,
\end{array}$$

then we have $S_0 = S_2 = T_2$, $U_2 = T_1$.

We note that $\omega(k\,; f, g)^N = N^k : N^{[f,g]} \to N^{[fk,gk]}$ for morphisms $f : X \to Y$, $g : X \to Z$ and $k : W \to X$ of $\mathcal{E}$ and $N \in \mathrm{Ob}\,\mathcal{F}_Z$ by (8.5.26).

Lemma 9.7.6 *For a representation (N,ζ) of $\boldsymbol{D}$, the following diagram is commutative.*

$$\begin{tikzcd}
f_0^*((N,\zeta)^{\boldsymbol{f}}) \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] \arrow[d, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] & f_0^*(N^{[\sigma f_{0\tau},\,\tau_{f_0}]}) \arrow[r, "\alpha^{1N}"] & N^{[id_{D_0},\,id_{D_0}]} \arrow[d, equal] \\
f_0^*(N^{[\sigma f_{0\tau},\,\tau_{f_0}]}) \arrow[r, "\beta^{2N}"] & N^{[\sigma',\tau']} & N \arrow[l, "\check{\zeta}"']
\end{tikzcd}$$

Proof. The following diagram is commutative by the definition of $E^{\boldsymbol{f}}_{(N,\zeta)}$.

$$\begin{tikzcd}
(N,\zeta)^{\boldsymbol{f}} \arrow[r, "E^{\boldsymbol{f}}_{(N,\zeta)}"] \arrow[d, "E^{\boldsymbol{f}}_{(N,\zeta)}"] & N^{[\sigma f_{0\tau},\,\tau_{f_0}]} \arrow[dr, "N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}"] & \\
N^{[\sigma f_{0\tau},\,\tau_{f_0}]} \arrow[r, "\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]}"] & (N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)"] & N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

It follows from (8.5.30) that the following diagram is commutative.

$$\begin{tikzcd}
f_0^*((N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]}) \arrow[r, "\alpha^{1N^{[\sigma',\tau']}}"] \arrow[d, "f_0^*(\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N))"] & (N^{[\sigma',\tau']})^{[id_{D_0},id_{D_0}]} \arrow[r, "(\alpha^{2N})^{[id_{D_0},id_{D_0}]}"] & (N^{[\sigma',\tau']})^{[id_{D_0},id_{D_0}]} \arrow[d, "\theta^{\sigma',\tau',id_{D_0},id_{D_0}}(N)"] \\
f_0^*(N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}) \arrow[rr, "\alpha^{0N}"] & & N^{[\sigma' id_{D_1},id_{D_0}\tau']}
\end{tikzcd}$$

We note that $\theta^{\sigma',\tau',id_{D_0},id_{D_0}}(N)$ and $(\alpha^{2N})^{[id_{D_0},id_{D_0}]}$ are the identity morphism of $N^{[\sigma',\tau']}$ by (8.5.23) and the definition of α^{2N}. Therefore the following diagram commutes by the commutativity of the above diagrams and (8.5.28).

$$\begin{tikzcd}
f_0^*((N,\zeta)^{\boldsymbol{f}}) \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] \arrow[d, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] & f_0^*(N^{[\sigma f_{0\tau},\,\tau_{f_0}]}) \arrow[dr, "f_0^*(N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)})"] & \\
f_0^*(N^{[\sigma f_{0\tau},\,\tau_{f_0}]}) \arrow[r, "f_0^*(\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]})"] \arrow[d, "\alpha^{1N}"] & f_0^*((N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]}) \arrow[r, "f_0^*(\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N))"] \arrow[d, "\alpha^{1N^{[\sigma',\tau']}}"] & f_0^*(N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}) \arrow[d, "\alpha^{0N}"] \\
N \arrow[r, "\check{\zeta}"] & N^{[\sigma',\tau']} \arrow[r, "\theta^{\sigma',\tau',id_{D_0},id_{D_0}}(N)=id_{N^{[\sigma',\tau']}}"] & N^{[\sigma',\tau']}
\end{tikzcd}$$

We put $\bar{\beta} = \omega((\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)\,;\sigma f_{0\tau},\tau_{f_0}) : T_0 \to T_1$. Then, $\beta^2 = \bar{\beta}\alpha^0$ holds. It follows from (8.5.29) that the following diagram is commutative.

$$\begin{tikzcd}
f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]}) \arrow[r, "f_0^*(\bar{\beta}^N)"] \arrow[d, "c_{id_{C_0},f_0}(N^{[\sigma f_{0\tau},\tau_{f_0}]})=id_{N^{[\sigma f_{0\tau},\tau_{f_0}]}}"] & f_0^*(N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}) \arrow[r, "\alpha^{0N}"] & N^{[\sigma',\tau']} \arrow[d, "c_{id_{D_0},id_{D_0}}(N)^{[\sigma',\tau']}=id_{N^{[\sigma',\tau']}}"] \\
f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]}) \arrow[rr, "\beta^{2N}=(\bar{\beta}\alpha^0)^N"] & & N^{[\sigma',\tau']}
\end{tikzcd}$$

Since $\bar{\beta}^N = \omega((\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)\,;\sigma f_{0\tau},\tau_{f_0})_N = N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}$ by (8.5.26), we have

$$\check{\zeta}\alpha^{1N} f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) = \alpha^{0N} f_0^*(N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}) f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) = \alpha^{0N} f_0^*(\bar{\beta}^N) f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) = \beta^{2N} f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}).$$

□

Proposition 9.7.7 *A composition*

$$f_0^*((N,\zeta)^{\boldsymbol{f}}) \xrightarrow{f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})} f_0^*(N^{[\sigma f_{0\tau},\,\tau_{f_0}]}) \xrightarrow{\alpha^{1N}} N^{[id_{D_0},\,id_{D_0}]} = N$$

defines a morphism $(f_0^*((N,\zeta)^{\boldsymbol{f}}),(\zeta^r_{\boldsymbol{f}})_{\boldsymbol{f}}) \to (N,\zeta)$ *of representations of* $\boldsymbol{D}$.

Proof. By applying (8.5.30) to $\beta : \mathcal{P} \to \mathcal{E}$, we see that the following diagram (i) is commutative.

$$\begin{tikzcd}
(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\beta^{1N^{[\sigma f_{0\tau},\tau_{f_0}]}}"] \arrow[d, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N)"] & f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "(\beta^{2N})^{[\sigma f_{0\tau},\tau_{f_0}]}"] & (N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)"] \\
N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[rr, "\beta^{0N}=N^{(\tilde{\mathrm{pr}}_1,f_1\tilde{\mathrm{pr}}_2,\tau'\tilde{\mathrm{pr}}_2)}"] & & N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

diagram (i)

Let $D_1 \xleftarrow{\hat{\mathrm{pr}}_1} D_1 \times_{C_0} D_0 \xrightarrow{\hat{\mathrm{pr}}_2} D_0$ be a limit of a diagram $D_1 \xrightarrow{\tau f_1} C_0 \xleftarrow{f_0} D_0$. Define a natural transformation $\bar{\beta}^1 : D_{\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2} \to D_{\sigma f_1,\tau f_1}$ by $\bar{\beta}^1_0 = \hat{\mathrm{pr}}_1$, $\bar{\beta}^1_1 = id_{C_0}$, $\bar{\beta}^1_2 = f_0$. We also consider natural transformations $\omega(f_1 \times_{C_0} id_{D_0}\,; \sigma f_{0\tau}, \tau_{f_0}) : D_{\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2} \to D_{\sigma f_{0\tau},\tau_{f_0}} = T_1$ and $\omega(f_1\,;\sigma,\tau) : D_{\sigma f_1,\tau f_1} \to D_{\sigma,\tau} = U_1$. Then, we have $\omega(f_1\,;\sigma,\tau)\bar{\beta}^1 = \beta^1\omega(f_1 \times_{C_0} id_{D_0}\,;\sigma f_{0\tau},\tau_{f_0})$ and it follows from (8.5.29) that the following diagram (ii) is commutative.

diagram (ii)
$$\begin{tikzcd}
(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\beta^{1N^{[\sigma f_{0\tau},\tau_{f_0}]}}"] \arrow[d, "(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{f_1}"] \arrow[dr, "(\omega(f_1\,;\sigma,\tau)\bar{\beta}^1)^{N^{[\sigma f_{0\tau},\tau_{f_0}]}}"] & f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{f_1\times_{C_0} id_{D_0}}"] \\
(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma f_1,\tau f_1]} \arrow[r, "\bar{\beta}^{1N^{[\sigma f_{0\tau},\tau_{f_0}]}}"] & f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}
\end{tikzcd}$$

The following diagram is commutative by (8.5.8).

diagram (iii)
$$\begin{tikzcd}
f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "(\beta^{2N})^{[\sigma f_{0\tau},\tau_{f_0}]}"] \arrow[d, "f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{f_1\times_{C_0} id_{D_0}}"] & (N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "(N^{[\sigma',\tau']})^{f_1\times_{C_0} id_{D_0}}"] \\
f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]} \arrow[r, "(\beta^{2N})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}"] & (N^{[\sigma',\tau']})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}
\end{tikzcd}$$

Define a functor $\gamma : S_0 \to D_{\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2}$ by $\gamma_0 = (id_{D_1},\tau')$, $\gamma_1 = f_0$, $\gamma_2 = id_{D_0}$, then $\bar{\beta}^1\gamma = \omega(\sigma',\tau'; f_0, f_0)$ holds. It follows from (8.5.29) that

diagram (iv)
$$\begin{tikzcd}
f_0^*((N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma f_1,\tau f_1]}) \arrow[r, "f_0^*(\bar{\beta}^{1N^{[\sigma f_{0\tau},\tau_{f_0}]}})"] \arrow[dr, "\omega(\sigma',\tau';f_0,f_0)^{N^{[\sigma f_{0\tau},\tau_{f_0}]}}"'] & f_0^*(f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}) \arrow[d, "\gamma^{f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})}"] \\
 & f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma',\tau']}
\end{tikzcd}$$

is commutative. Moreover, (8.5.28) implies that the following diagram is commutative.

diagram (v)
$$\begin{tikzcd}
f_0^*(f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}) \arrow[r, "f_0^*((\beta^{2N})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]})"] \arrow[d, "\gamma^{f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})}"] & f_0^*((N^{[\sigma',\tau']})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}) \arrow[d, "\gamma^{N^{[\sigma',\tau']}}"] \\
f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma',\tau']} \arrow[r, "(\beta^{2N})^{[\sigma',\tau']}"] & (N^{[\sigma',\tau']})^{[\sigma',\tau']}
\end{tikzcd}$$

The following diagram is commutative by the definition of $\check{\zeta}_{\boldsymbol{f}}$ and (8.5.8), (8.5.19).

diagram (vi)
$$\begin{tikzcd}
f_0^*(((N,\zeta)^{\boldsymbol{f}})) \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] \arrow[dd, "f_0^*(\check{\zeta}_{\boldsymbol{f}})"] & f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]}) \arrow[d, "f_0^*(N^{\mu\times_{C_0} id_{D_0}})"] \\
 & f_0^*(N^{[\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}) \arrow[d, "f_0^*(\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N))^{-1}"] \\
f_0^*(((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]}) \arrow[r, "f_0^*((E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]})"] \arrow[d, "f_0^*(((N,\zeta)_{\boldsymbol{f}})^{f_1})"] & f_0^*((N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]}) \arrow[d, "f_0^*((N^{[\sigma f_{0\tau},\tau_{f_0}]})^{f_1})"] \\
f_0^*(((N,\zeta)^{\boldsymbol{f}})^{[f_0\sigma',f_0\tau']}) \arrow[r, "f_0^*((E^{\boldsymbol{f}}_{(N,\zeta)})^{[f_0\sigma',f_0\tau']})"] \arrow[d, "\omega(\sigma',\tau',f_0,f_0)^{(N,\zeta)^{\boldsymbol{f}}}"] & f_0^*((N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[f_0\sigma',f_0\tau']}) \arrow[d, "\omega(\sigma',\tau',f_0,f_0)^{N^{[\sigma f_{0\tau},\tau_{f_0}]}}"] \\
f_0^*((N,\zeta)^{\boldsymbol{f}})^{[\sigma',\tau']} \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma',\tau']}"] & f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma',\tau']}
\end{tikzcd}$$

Consider natural transformations $\omega(\varepsilon'\,;\sigma',\tau') : S_1 \to S_2$ and $\omega(f_1 \times_{C_0} id_{D_0}\,;\sigma f_{0\tau},\tau_{f_0}) : D_{\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2} \to T_2$. Then, we have $\alpha^1 = \beta^2\omega(\varepsilon'\,;\sigma',\tau')$ and $\omega(f_1\times_{C_0} id_{D_0}\,;\sigma f_{0\tau},\tau_{f_0})\gamma = \beta^2 = \omega((\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)\,;\,\sigma f_{0\tau},\tau_{f_0})\alpha^0$ hold and it follows from (8.5.29) that the following diagrams are commutative.

diagram (vii)
$$\begin{tikzcd}
f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]}) \arrow[r, "\beta^{2N}"] \arrow[dr, "\alpha^{1N}"'] & N^{[\sigma',\tau']} \arrow[d, "N^{\varepsilon'}"] \\
& N^{[id_{D_0},id_{D_0}]} = N
\end{tikzcd}$$

$$\begin{tikzcd}
f_0^*((N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]}) \arrow[r, "f_0^*((N^{[\sigma',\tau']})^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)})"] \arrow[d, "f_0^*((N^{[\sigma',\tau']})^{f_1\times_{C_0}id_{D_0}})"] \arrow[dr, "\beta^{2N^{[\sigma',\tau']}}"] & f_0^*((N^{[\sigma',\tau']})^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}) \arrow[d, "\alpha^{0N^{[\sigma',\tau']}}"] \\
f_0^*((N^{[\sigma',\tau']})^{[\sigma f_1\hat{\mathrm{pr}}_1,\hat{\mathrm{pr}}_2]}) \arrow[r, "\gamma_{N^{[\sigma',\tau']}}"] & (N^{[\sigma',\tau']})^{[\sigma',\tau']}
\end{tikzcd}$$
diagram $(viii)$

We also have the following commutative diagrams by (8.5.28) and (8.5.8).

diagram (ix)
$$\begin{tikzcd}
f_0^*((N^{[\sigma',\tau']})^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}) \arrow[r, "f_0^*((N^{\varepsilon'})^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]})"] \arrow[d, "\alpha^{0N^{[\sigma',\tau']}}"] & f_0^*(N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}) \arrow[d, "\alpha^{0N}"] \\
(N^{[\sigma',\tau']})^{[\sigma',\tau']} \arrow[r, "(N^{\varepsilon'})^{[\sigma',\tau']}"] & N^{[\sigma',\tau']}
\end{tikzcd}$$

diagram (x)
$$\begin{tikzcd}
(N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "(N^{\varepsilon'})^{[\sigma f_{0\tau},\tau_{f_0}]}"] \arrow[d, "(N^{[\sigma',\tau']})^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}"] & N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}"] \\
(N^{[\sigma',\tau']})^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2)]} \arrow[r, "(N^{\varepsilon'})^{(\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2))}"] & N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2)]}
\end{tikzcd}$$

We put $\tilde{\zeta}_{\boldsymbol{f}} = E_{\sigma',\tau'}(f_0^*((N,\zeta)^{\boldsymbol{f}}))_{f_0^*((N,\zeta)^{\boldsymbol{f}})}((\zeta^r_{\boldsymbol{f}})_{\boldsymbol{f}})$. Then, $\tilde{\zeta}_{\boldsymbol{f}}$ is the following composition by (9.5.3).

$$f_0^*((N,\zeta)^{\boldsymbol{f}}) \xrightarrow{f_0^*(\check{\zeta}_{\boldsymbol{f}})} f_0^*(((N,\zeta)^{\boldsymbol{f}})^{[\sigma,\tau]}) \xrightarrow{f_0^*(((N,\zeta)^{\boldsymbol{f}})^{f_1})} f_0^*(((N,\zeta)^{\boldsymbol{f}})^{[f_0\sigma',f_0\tau']}) \xrightarrow{\omega(\sigma',\tau';f_0,f_0)^{(N,\zeta)^{\boldsymbol{f}}}} f_0^*((N,\zeta)^{\boldsymbol{f}})^{[\sigma',\tau']}$$

We note that $(\mu\times_{C_0} id_{D_0})(\tilde{\mathrm{pr}}_1, f_1\tilde{\mathrm{pr}}_2,\tau'\tilde{\mathrm{pr}}_2) = (\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)$ holds and recall that $E^{\boldsymbol{f}}_{(N,\zeta)}$ is an equalizer of $N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}$ and $\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]}$. We also have $\alpha^{0N}f_0^*(N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}) = \beta^{2N}$ by (8.5.29). Therefore by the commutativity of diagrams $(i)\sim(ix)$ and (9.7.6), we have

$$\begin{aligned}
(\alpha^{1N}f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}))^{[\sigma',\tau']}\tilde{\zeta}_{\boldsymbol{f}} &= (N^{\varepsilon'})^{[\sigma',\tau']}(\beta^{2N})^{[\sigma',\tau']}f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma',\tau']}\omega(\sigma',\tau';f_0,f_0)^{(N,\zeta)^{\boldsymbol{f}}}f_0^*(((N,\zeta)^{\boldsymbol{f}})^{f_1})f_0^*(\check{\zeta}^{\boldsymbol{f}}) \\
&= (N^{\varepsilon'})^{[\sigma',\tau']}\gamma_{N^{[\sigma',\tau']}}f_0^*((N^{[\sigma',\tau']})^{f_1\times_{C_0}id_{D_0}})f_0^*(\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)^{-1}) \\
&\quad\ f_0^*(N^{(\tilde{\mathrm{pr}}_1,f_1\tilde{\mathrm{pr}}_2,\tau'\tilde{\mathrm{pr}}_2)})f_0^*(N^{\mu\times_{C_0}id_{D_0}})f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) \\
&= (N^{\varepsilon'})^{[\sigma',\tau']}\alpha^{0N^{[\sigma',\tau']}}f_0^*((N^{[\sigma',\tau']})^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}) \\
&\quad\ f_0^*(\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)^{-1}N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}E^{\boldsymbol{f}}_{(N,\zeta)}) \\
&= \alpha^{0N}f_0^*((N^{\varepsilon'})^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}(N^{[\sigma',\tau']})^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]}E^{\boldsymbol{f}}_{(N,\zeta)}) \\
&= \alpha^{0N}f_0^*(N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}(N^{\varepsilon'})^{[\sigma f_{0\tau},\tau_{f_0}]}\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]}E^{\boldsymbol{f}}_{(N,\zeta)}) \\
&= \alpha^{0N}f_0^*(N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)})f_0^*((N^{\varepsilon'}\check{\zeta})^{[\sigma f_{0\tau},\tau_{f_0}]}E^{\boldsymbol{f}}_{(N,\zeta)}) \\
&= \beta^{2N}f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) = \check{\zeta}\alpha^{1N}f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}).
\end{aligned}$$

This shows that $\alpha^{1N}f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) : f_0^*((N,\zeta)^{\boldsymbol{f}}) \to N$ defines a morphism $(f_0^*((N,\zeta)^{\boldsymbol{f}}),(\zeta^r_{\boldsymbol{f}})_{\boldsymbol{f}}) \to (N,\zeta)$ of representations of $\boldsymbol{D}$. □

We put $\varepsilon^{\boldsymbol{f}}_{(N,\zeta)} = \alpha^{1N}f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)}) : f_0^*((N,\zeta)^{\boldsymbol{f}}) \to N$.

Remark 9.7.8 *If $\varphi : (M,\xi) \to (N,\zeta)$ is a morphism of representations of $\boldsymbol{D}$, the following diagram is commutative by (8.5.28) and the definition of $\varphi^{\boldsymbol{f}}$.*

$$\begin{tikzcd}
f_0^*((M,\xi)^{\boldsymbol{f}}) \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(M,\xi)})"] \arrow[d, "\varphi"] \arrow[rr, bend left, "\varepsilon^{\boldsymbol{f}}_{(M,\xi)}"] & f_0^*(M^{[\sigma f_{0\tau},\,\tau_{f_0}]}) \arrow[r, "\alpha^{1M}"] \arrow[d, "f_0^*(\varphi^{[\sigma f_{0\tau},\,\tau_{f_0}]})"] & M \arrow[d, "f_0^*(\varphi^{\boldsymbol{f}})"] \\
f_0^*((N,\zeta)^{\boldsymbol{f}}) \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] \arrow[rr, bend right, "\varepsilon^{\boldsymbol{f}}_{(N,\zeta)}"'] & f_0^*(N^{[\sigma f_{0\tau},\,\tau_{f_0}]}) \arrow[r, "\alpha^{1N}"] & N
\end{tikzcd}$$

Define a functor $R : \mathcal{P} \to \mathcal{E}$ and a natural transformation $\kappa : U \to R$ by $R(0) = C_1 \times_{C_0} C_1$, $R(1) = C_1$, $R(2) = C_1$, $R(i) = C_0$ $(i = 3,4,5)$, $R(\tau_{01}) = \mathrm{pr}_1$, $R(\tau_{02}) = \mathrm{pr}_2$, $R(\tau_{13}) = R(\tau_{24}) = \sigma$, $R(\tau_{14}) = R(\tau_{25}) = \tau$ and $\kappa_0 = \tilde{\mathrm{pr}}_{12}$, $\kappa_1 = id_{C_1}$, $\kappa_2 = f_{0\tau}$, $\kappa_3 = \kappa_4 = id_{C_0}$, $\kappa_5 = f_0$. We also define functors $R_i : \mathcal{Q} \to \mathcal{E}$ and natural transformations $\kappa^i : U_i \to R_i$ for $i = 0,1,2$ by

$$\begin{array}{lllll}
R_0(0) = R(0) & R_0(1) = R(3) & R_0(2) = R(5) & R_0(\tau_{01}) = R(\tau_{13}\tau_{01}) & R_0(\tau_{02}) = R(\tau_{25}\tau_{02}) \\
R_1(0) = R(1) & R_1(1) = R(3) & R_1(2) = R(4) & R_1(\tau_{01}) = R(\tau_{13}) & R_1(\tau_{02}) = R(\tau_{14}) \\
R_2(0) = R(2) & R_2(1) = R(4) & R_2(2) = R(5) & R_2(\tau_{01}) = R(\tau_{24}) & R_2(\tau_{02}) = R(\tau_{25})
\end{array}$$
$$\kappa^0_0 = \kappa_0 \quad \kappa^0_1 = \kappa_3 \quad \kappa^0_2 = \kappa_5 \quad \kappa^1_0 = \kappa_1 \quad \kappa^1_1 = \kappa_3 \quad \kappa^1_2 = \kappa_4 \quad \kappa^2_0 = \kappa_2 \quad \kappa^2_1 = \kappa_4 \quad \kappa^2_2 = \kappa_5.$$

Proposition 9.7.9 *For an object M of $\mathcal{F}_{C_0}$, $\beta^{1M} : M^{[\sigma,\tau]} \to f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]}$ defines a morphism of representations $(M^{[\sigma,\tau]}, \mu^r_M) \to (f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]}, \mu^r_{\boldsymbol{f}}(f_0^*(M)))$ under the assumption of (9.7.1) for $N = f_0^*(M)$ and the assumption of (9.5.8).*

Proof. Since κ^1 is the identity natural transformation and $\kappa^2 = \beta^1$, we have a commutative diagram below by applying (8.5.30) to $\kappa : U \to R$.

$$\begin{tikzcd}
(M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "(\beta^{1M})^{[\sigma,\tau]}"] \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(M)"] & (f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \arrow[d, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(f_0^*(M))"] \\
M^{[\sigma \mathrm{pr}_1,\tau \mathrm{pr}_2]} \arrow[r, "\kappa^{0M}"] & f_0^*(M)^{[\sigma \mathrm{pr}_1 \tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

We consider functors $\omega(\mu\,;\sigma,\tau) : R_0 \to U_1$ and $\omega(\mu \times_{C_0} id_{D_0}\,;\sigma f_{0\tau},\tau_{f_0}) : U_0 \to T_1$. Then we have $\omega(\mu\,;\sigma,\tau)\kappa^0 = \beta^1\omega(\mu \times_{C_0} id_{D_0}\,;\sigma f_{0\tau},\tau_{f_0})$. Hence it follows from (8.5.29) that the following diagram is commutative.

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[r, "\beta^{1M}"] \arrow[d, "M^\mu"] \arrow[dr, "(\omega(\mu\,;\sigma,\tau)\kappa^0)^M = (\beta^1\omega(\mu\times_{C_0} id_{D_0}\,;\sigma f_{0\tau},\tau_{f_0}))^M"] & f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "f_0^*(M)^{\mu\times_{C_0} id_{D_0}}"] \\
M^{[\sigma \mathrm{pr}_1,\tau \mathrm{pr}_2]} \arrow[r, "\kappa^{0M}"] & f_0^*(M)^{[\sigma \mathrm{pr}_1 \tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]}
\end{tikzcd}$$

Since $\check{\mu}_{\boldsymbol{f}}(f_0^*(M)) = \theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(f_0^*(M))^{-1} f_0^*(M)^{\mu\times_{C_0} id_{D_0}}$ and $\check{\mu}_M = \theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}M^\mu$, the commutativity of the above diagrams implies that the following diagram is commutative.

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[r, "\beta^{1M}"] \arrow[d, "\check{\mu}_M"] & f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "\check{\mu}_{\boldsymbol{f}}(f_0^*(M))"] \\
(M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "(\beta^{1M})^{[\sigma,\tau]}"] & (f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]}
\end{tikzcd}$$

Hence the assertion follows from (9.5.4). □

Lemma 9.7.10 *Let (M,ξ) and (N,ζ) be representations of $\boldsymbol{C}$ and $\boldsymbol{D}$, respectively. We put $\check{\xi} = E_{\sigma,\tau}(M)_M(\xi)$ and $\check{\zeta} = E_{\sigma',\tau'}(N)_N(\zeta)$. For a morphism $\varphi : \boldsymbol{f}^\bullet(M,\xi) \to (N,\zeta)$ of representations of $\boldsymbol{D}$, the following diagram is commutative if $\theta^{\sigma,\tau,\sigma,\tau}(M) : (M^{[\sigma,\tau]})^{[\sigma,\tau]} \to M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}$ is an isomorphism.*

$$\begin{tikzcd}
M \arrow[r, "\check{\xi}"] \arrow[d, "\check{\xi}"] & M^{[\sigma,\tau]} \arrow[r, "\beta^{1M}"] & f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "\varphi^{[\sigma f_{0\tau},\tau_{f_0}]}"] & N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]}"] \\
M^{[\sigma,\tau]} \arrow[d, "\beta^{1M}"] & & & (N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)"] \\
f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[rr, "\varphi^{[\sigma f_{0\tau},\tau_{f_0}]}"] & & N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}"] & N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

Proof. Since $E_{\sigma',\tau'}(f_0^*(M))^{f_0^*(M)}(\xi_{\boldsymbol{f}})$ is a composition

$$f_0^*(M) \xrightarrow{f_0^*(\check{\xi})} f_0^*(M^{[\sigma,\tau]}) \xrightarrow{f_0^*(M^{f_1})} f_0^*(M^{[f_0\sigma',f_0\tau']}) \xrightarrow{\omega(\sigma',\tau';f_0,f_0)^M} f_0^*(M)^{[\sigma',\tau']}$$

by (9.5.3), the following diagram is commutative by (9.5.4).

$$\begin{tikzcd}
f_0^*(M) \arrow[r, "f_0^*(\check{\xi})"] \arrow[d, "\varphi"] & f_0^*(M^{[\sigma,\tau]}) \arrow[r, "f_0^*(M^{f_1})"] & f_0^*(M^{[f_0\sigma',f_0\tau']}) \arrow[r, "\omega(\sigma',\tau';f_0,f_0)^M"] & f_0^*(M)^{[\sigma',\tau']} \arrow[d, "\varphi^{[\sigma',\tau']}"] \\
N \arrow[rrr, "\check{\zeta}"] & & & N^{[\sigma',\tau']}
\end{tikzcd}$$

It follows from (8.5.28) that the following diagram is commutative.

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[r, "\beta^{1M}"] \arrow[d, "\check{\xi}^{[\sigma,\tau]}"] & f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "f_0^*(\check{\xi})^{[\sigma f_{0\tau},\tau_{f_0}]}"] \\
(M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "\beta^{1M^{[\sigma,\tau]}}"] & f_0^*(M^{[\sigma,\tau]})^{[\sigma f_{0\tau},\tau_{f_0}]}
\end{tikzcd}$$

Hence the following diagram (i) is commutative by (8.5.3), (8.5.8) and (8.5.19).

diagram (i)

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[d, "\check{\xi}^{[\sigma,\tau]}"] \arrow[ddr, "\beta^{1M}"] & \\
(M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "\beta^{1M^{[\sigma,\tau]}}"] & \\
f_0^*(M^{[\sigma,\tau]})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "f_0^*(M^{f_1})^{[\sigma f_{0\tau},\tau_{f_0}]}"] & f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[l, "f_0^*(\check{\xi})^{[\sigma f_{0\tau},\tau_{f_0}]}"'] \arrow[d, "\varphi^{[\sigma f_{0\tau},\tau_{f_0}]}"] \\
f_0^*(M^{[f_0\sigma',f_0\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "(\omega(\sigma',\tau';f_0,f_0)^M)^{[\sigma f_{0\tau},\tau_{f_0}]}"] & N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]}"] \\
(f_0^*(M)^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "(\varphi^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]}"] \arrow[d, "\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(f_0^*(M))"] & (N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)"] \\
f_0^*(M)^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \arrow[r, "\varphi^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}"] & N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \\
f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[u, "f_0^*(M)^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}"] \arrow[r, "\varphi^{[\sigma f_{0\tau},\tau_{f_0}]}"] & N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[u, "N^{(\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)}"']
\end{tikzcd}$$

Define a functor $V : \mathcal{P} \to \mathcal{E}$ and a natural transformation $\lambda : T \to V$ by $V(0) = C_1 \times_{C_0} D_1$, $V(1) = C_1$, $V(2) = D_1$, $V(i) = C_0$ $(i = 3, 4, 5)$, $V(\tau_{01}) = \tilde{\mathrm{pr}}_1$, $V(\tau_{02}) = \tilde{\mathrm{pr}}_2$, $V(\tau_{13}) = \sigma$, $V(\tau_{14}) = \tau$, $V(\tau_{24}) = f_0\sigma'$, $V(\tau_{25}) = f_0\tau'$ and $\lambda_0 = id_{C_1\times_{C_0}D_1}$, $\lambda_1 = f_{0\tau}$, $\lambda_2 = id_{D_1}$, $\lambda_3 = id_{C_0}$, $\lambda_4 = \lambda_5 = f_0$. We also define functors $V_i : \mathcal{Q} \to \mathcal{E}$ and natural transformations $\lambda^i : V_i \to T_i$ for $i = 0, 1, 2$ by

$$\begin{array}{lllll}
V_0(0) = V(0) & V_0(1) = V(3) & V_0(2) = V(5) & V_0(\tau_{01}) = V(\tau_{13}\tau_{01}) & V_0(\tau_{02}) = V(\tau_{25}\tau_{02}) \\
V_1(0) = V(1) & V_1(1) = V(3) & V_1(2) = V(4) & V_1(\tau_{01}) = V(\tau_{13}) & V_1(\tau_{02}) = V(\tau_{14}) \\
V_2(0) = V(2) & V_2(1) = V(4) & V_2(2) = V(5) & V_2(\tau_{01}) = V(\tau_{24}) & V_2(\tau_{02}) = V(\tau_{25})
\end{array}$$

$$\lambda_0^0=\lambda_0 \quad \lambda_1^0=\lambda_3 \quad \lambda_2^0=\lambda_5 \quad \lambda_0^1=\lambda_1 \quad \lambda_1^1=\lambda_3 \quad \lambda_2^1=\lambda_4 \quad \lambda_0^2=\lambda_2 \quad \lambda_1^2=\lambda_4 \quad \lambda_2^2=\lambda_5.$$

Then, $V_1=U_1$, $\lambda^2=\omega(\sigma',\tau'\,;f_0,f_0)$ and $\lambda^1=\beta^1$ and it follows from (8.5.30) that the following diagram is commutative.

$$\begin{tikzcd}
(M^{[f_0\sigma',f_0\tau']})^{[\sigma,\tau]} \arrow[r, "\beta^{1M^{[f_0\sigma',f_0\tau']}}"] \arrow[d, "\theta^{\sigma,\tau,f_0\sigma',f_0\tau'}(M)"] & f_0^*(M^{[f_0\sigma',f_0\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "(\omega(\sigma',\tau'\,;f_0,f_0)^M)^{[\sigma f_{0\tau},\tau_{f_0}]}"] & (f_0^*(M)^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(f_0^*(M))"'] \\
M^{[\sigma\tilde{\mathrm{pr}}_1,f_0\tau'\mathrm{pr}_2]} \arrow[rr, "\lambda^{0M}"] & & f_0^*(M)^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

Consider natural transformations $\omega(\mu(id_{C_1}\times_{C_0} f_1)\,;\sigma,\tau):V_0\to U_1$ and $\omega((\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)\,;\sigma f_{0\tau},\tau_{f_0}):T_0\to T_1$. Then, $\omega(\mu(id_{C_1}\times_{C_0} f_1)\,;\sigma,\tau)\lambda^0=\beta^1\omega((\mu(id_{C_1}\times_{C_0} f_1),\tau'\tilde{\mathrm{pr}}_2)\,;\sigma f_{0\tau},\tau_{f_0})$ holds and the following diagram is commutative by (8.5.29).

$$\begin{tikzcd}
M^{[\sigma,\tau]} \arrow[r, "\beta^{1M}"] \arrow[d, "M^{\mu(id_{C_1}\times_{C_0}f_1)}"] \arrow[dr, "(\omega(\mu(id_{C_1}\times_{C_0}f_1)\,;\sigma,\tau)\lambda^0)^M"] & f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "f_0^*(M)^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}"] \\
M^{[\sigma'\tilde{\mathrm{pr}}_1,f_0\tau'\tilde{\mathrm{pr}}_2]}) \arrow[r, "\lambda^{0M}"] & f_0^*(M)^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}
\end{tikzcd}$$

Moreover, the following diagrams are commutative by (9.5.1) and (8.5.28), respectively.

$$\begin{tikzcd}
M \arrow[r, "\check{\xi}"] \arrow[dr, "\check{\xi}"'] & M^{[\sigma,\tau]} \arrow[r, "\check{\xi}^{[\sigma,\tau]}"] & (M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "\theta^{\sigma,\tau,\sigma,\tau}(M)"] \\
& M^{[\sigma,\tau]} \arrow[r, "M^{\mu}"] & M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]}
\end{tikzcd}
\qquad
\begin{tikzcd}
(M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[r, "\beta^{1M^{[\sigma,\tau]}}"] \arrow[d, "(M^{f_1})^{[\sigma,\tau]}"] & f_0^*(M^{[\sigma,\tau]})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "f_0^*(M^{f_1})^{[\sigma f_{0\tau},\tau_{f_0}]}"] \\
(M^{[f_0\sigma',f_0\tau']})^{[\sigma,\tau]} \arrow[r, "\beta^{1M^{[f_0\sigma',f_0\tau']}}"] & f_0^*(M^{[f_0\sigma',f_0\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]}
\end{tikzcd}$$

Therefore the following diagram (ii) is commutative

diagram (ii)

$$\begin{tikzcd}
M \arrow[rrrr, "\check{\xi}"] \arrow[dddddd, "\check{\xi}"] & & & & M^{[\sigma,\tau]} \arrow[d, "\check{\xi}^{[\sigma,\tau]}"] \\
& M^{[\sigma\mathrm{pr}_1,\tau\mathrm{pr}_2]} \arrow[rrr, "\theta^{\sigma,\tau,\sigma,\tau}(M)^{-1}"] \arrow[ddddrr, "M^{id_{C_1}\times_{C_0}f_1}"] & & & (M^{[\sigma,\tau]})^{[\sigma,\tau]} \arrow[d, "\beta^{1M^{[\sigma,\tau]}}"] \arrow[ddl, "(M^{f_1})^{[\sigma,\tau]}"'] \\
& & & & f_0^*(M^{[\sigma,\tau]})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "f_0^*(M^{f_1})^{[\sigma f_{0\tau},\tau_{f_0}]}"'] \\
& & & (M^{[f_0\sigma',f_0\tau']})^{[\sigma,\tau]} \arrow[r, "\beta^{1M^{[f_0\sigma',f_0\tau']}}"] \arrow[dd, "\theta^{\sigma,\tau,f_0\sigma',f_0\tau'}(M)"] & f_0^*(M^{[f_0\sigma',f_0\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "(\omega(\sigma',\tau'\,;f_0,f_0)^M)^{[\sigma f_{0\tau},\tau_{f_0}]}"'] \\
& & & & (f_0^*(M)^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(f_0^*(M))"'] \\
& & & M^{[\sigma\tilde{\mathrm{pr}}_1,f_0\tau'\mathrm{pr}_2]} \arrow[r, "\lambda^{0M}"] & f_0^*(M)^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]} \\
M^{[\sigma,\tau]} \arrow[uuuuur, "M^{\mu}"] \arrow[rrrr, "\beta^{1M}"] & & & & f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[u, "f_0^*(M)^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}"']
\end{tikzcd}$$

By glueing the left edge of diagram (i) and the right edge of diagram (ii), the assertion follows. □

Recall that $E^{\boldsymbol{f}}_{(N,\zeta)}:(N,\zeta)^{\boldsymbol{f}}\to N^{[\sigma f_{0\tau},\tau_{f_0}]}$ is an equalizer of the following morphisms.

$$N^{[\sigma f_{0\tau},\tau_{f_0}]}\xrightarrow{\check{\zeta}^{[\sigma f_{0\tau},\tau_{f_0}]}}(N^{[\sigma',\tau']})^{[\sigma f_{0\tau},\tau_{f_0}]}\xrightarrow{\theta^{\sigma f_{0\tau},\tau_{f_0},\sigma',\tau'}(N)}N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}$$

$$N^{[\sigma f_{0\tau},\tau_{f_0}]}\xrightarrow{N^{(\mu(id_{C_1}\times_{C_0}f_1),\tau'\tilde{\mathrm{pr}}_2)}}N^{[\sigma f_{0\tau}(id_{C_1}\times_{C_0}\sigma'),\tau'\tilde{\mathrm{pr}}_2]}$$

Hence there exists unique morphism ${}^t\varphi:M\to(N,\zeta)^{\boldsymbol{f}}$ that satisfies $E^{\boldsymbol{f}}_{(N,\zeta)}\,{}^t\varphi=\varphi^{[\sigma f_{0\tau},\tau_{f_0}]}\beta^{1M}\check{\xi}$.

Proposition 9.7.11 *Under the assumptions of (9.7.3) for N and the assumptions of (iii) and the first one of (iv) of (9.7.3) for $f_0^*(M)$, ${}^t\varphi : M \to (N,\zeta)^{\boldsymbol{f}}$ gives a morphism $(M,\xi) \to ((N,\zeta)^{\boldsymbol{f}}, \zeta^r_{\boldsymbol{f}})$ of representations of $\boldsymbol{C}$.*

Proof. It follows from (9.5.8), (9.7.9) and (9.5.9) that $\varphi^{[\sigma f_{0\tau},\tau_{f_0}]}\beta^{1N}\check{\xi} : M \to N^{[\sigma f_{0\tau},\tau_{f_0}]}$ gives a morphism $(M,\xi) \to (N^{[\sigma f_{0\tau},\tau_{f_0}]}, \mu^r_{\boldsymbol{f}}(N))$ of representations of $\boldsymbol{C}$. Hence the outer rectangle of the following diagram is commutative by (9.5.4).

$$\begin{tikzcd}
M \arrow[r, "{}^t\varphi"] \arrow[d, "\check{\xi}"] & (N,\xi)_{\boldsymbol{f}} \arrow[r, "E^{\boldsymbol{f}}_{(N,\zeta)}"] \arrow[d, "\check{\xi}_{\boldsymbol{f}}"] & N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "\check{\mu}_{\boldsymbol{f}}(N)"] \\
M^{[\sigma,\tau]} \arrow[r, "{}^t\varphi^{[\sigma,\tau]}"] & ((N,\xi)_{\boldsymbol{f}})^{[\sigma,\tau]} \arrow[r, "(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]}"] & (N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]}
\end{tikzcd}$$

Since $(E^{\boldsymbol{f}}_{(N,\zeta)})^{[\sigma,\tau]} : (M^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \to ((M,\xi)_{\boldsymbol{f}})^{[\sigma,\tau]}$ is a monomorphism and the right rectangle of the above diagram is commutative by the definition of $\check{\xi}_{\boldsymbol{f}}$, the left rectangle of the above diagram is also commutative. Thus the assertion follows from (9.5.4). □

Proposition 9.7.12 *For a morphism $\varphi : \boldsymbol{f}^{\bullet}(M,\xi) \to (N,\zeta)$ of representations of $\boldsymbol{D}$, the following composition coincides with φ.*

$$f_0^*(M) \xrightarrow{f_0^*({}^t\varphi)} f_0^*((N,\zeta)^{\boldsymbol{f}}) \xrightarrow{\varepsilon^{\boldsymbol{f}}_{(M,\xi)}} N$$

Proof. We note that compositions $S_1 \xrightarrow{\alpha^1} T_1 \xrightarrow{\beta^1} U_1$ and $S_1 = D_{id_{D_0},id_{D_0}} \xrightarrow{\omega(f_0)} D_{id_{C_0},id_{C_0}} \xrightarrow{\omega(\varepsilon\,;\sigma,\tau)} U_1$ coincide. Hence the following diagram is commutative by (8.5.28) and (8.5.29).

$$\begin{tikzcd}
f_0^*(M) \arrow[r, "f_0^*(\check{\xi})"] \arrow[dd, "f_0^*({}^t\varphi)"] & f_0^*(M^{[\sigma,\tau]}) \arrow[r, "f_0^*(M^{\varepsilon})"] \arrow[d, "f_0^*(\beta^{1M})"] \arrow[dr, "(\beta^1\alpha^1)^M = (\omega(\varepsilon\,;\sigma,\tau)\omega(f_0))^M"] & f_0^*(M) \arrow[d, "\omega(f_0)^M"] \\
 & f_0^*(f_0^*(M)^{[\sigma f_{0\tau},\tau_{f_0}]}) \arrow[r, "\alpha^{1f_0^*(M)}"] \arrow[d, "f_0^*(\varphi^{[\sigma f_{0\tau},\tau_{f_0}]})"] & f_0^*(M) \arrow[d, "\varphi"] \\
f_0^*((N,\zeta)^{\boldsymbol{f}}) \arrow[r, "f_0^*(E^{\boldsymbol{f}}_{(N,\zeta)})"] & f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]}) \arrow[r, "\alpha^{1N}"] & N
\end{tikzcd}$$

Since $\omega(f_0)^N$ is the identity morphism of $f^*(N)$ by (9.6.12) and $M^{\varepsilon}\check{\zeta}$ is the identity morphism of N by (9.5.1), the assertion follows. □

Lemma 9.7.13 *For an object N of $\mathcal{F}_{D_0}$, a composition*

$$N^{[\sigma f_{0\tau},\tau_{f_0}]} \xrightarrow{\check{\mu}_{\boldsymbol{f}}(N)} (N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \xrightarrow{\beta^{1N^{[\sigma f_{0\tau},\tau_{f_0}]}}} f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma f_{0\tau},\tau_{f_0}]} \xrightarrow{(\alpha^{1N})^{[\sigma f_{0\tau},\tau_{f_0}]}} N^{[\sigma f_{0\tau},\tau_{f_0}]}$$

coincides with the identity morphism of $N^{[\sigma f_{0\tau},\tau_{f_0}]}$.

Proof. Define a functor $W : \mathcal{P} \to \mathcal{E}$ and a natural transformation $\nu : W \to U$ by $W(0) = W(1) = C_1 \times_{C_0} D_0$, $W(i) = D_0$ $(i = 2,4,5)$, $W(3) = C_0$, $W(\tau_{01}) = id_{C_1 \times_{C_0} D_0}$, $W(\tau_{02}) = \tau_{f_0}$, $W(\tau_{13}) = \sigma f_{0\tau}$, $W(\tau_{14}) = \tau_{f_0}$, $W(\tau_{24}) = W(\tau_{25}) = id_{D_0}$ and $\nu_0 = (f_{0\tau}, \varepsilon\tau f_{0\tau}, \tau_{f_0})$, $\nu_1 = f_{0\tau}$, $\nu_2 = (\varepsilon f_0, id_{D_0})$, $\nu_3 = id_{C_0}$, $\nu_4 = f_0$, $\nu_5 = id_{D_0}$. We also define functors $W_i : \mathcal{Q} \to \mathcal{E}$ and natural transformations $\nu^i : W_i \to T_i$ for $i = 0,1,2$ by

$$\begin{array}{lllll}
W_0(0) = W(0) & W_0(1) = W(3) & W_0(2) = W(5) & W_0(\tau_{01}) = W(\tau_{13}\tau_{01}) & W_0(\tau_{02}) = W(\tau_{25}\tau_{02}) \\
W_1(0) = W(1) & W_1(1) = W(3) & W_1(2) = W(4) & W_1(\tau_{01}) = W(\tau_{13}) & W_1(\tau_{02}) = W(\tau_{14}) \\
W_2(0) = W(2) & W_2(1) = W(4) & W_2(2) = W(5) & W_2(\tau_{01}) = W(\tau_{24}) & W_2(\tau_{02}) = W(\tau_{25})
\end{array}$$

$$\nu^0_0 = \nu_0 \quad \nu^0_1 = \nu_3 \quad \nu^0_2 = \nu_5 \quad \nu^1_0 = \nu_1 \quad \nu^1_1 = \nu_3 \quad \nu^1_2 = \nu_4 \quad \nu^2_0 = \nu_2 \quad \nu^2_1 = \nu_4 \quad \nu^2_2 = \nu_5.$$

Then, we have $W_1 = T_1$, $W_2 = S_1$, $\nu^1 = \beta^1$, $\nu^2 = \alpha^1$ and $\nu^0 = \omega((f_{0\tau}, \varepsilon\tau f_{0\tau}, \tau_{f_0})\,;\sigma\mathrm{pr}_1\tilde{\mathrm{pr}}_{12}, \tau_{f_0}\tilde{\mathrm{pr}}_{23})$. It follows from (8.5.30) and the definition of $\check{\mu}_{\boldsymbol{f}}(N)$ that the following diagram is commutative.

$$\begin{tikzcd}
N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "\check{\mu}_{\boldsymbol{f}}(N)"] \arrow[dr, "N^{\mu}\times_{C_0} id_{D_0}"'] & (N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma,\tau]} \arrow[r, "\beta^{1N^{[\sigma f_{0\tau},\tau_{f_0}]}}"] \arrow[d, "\theta^{\sigma,\tau,\sigma f_{0\tau},\tau_{f_0}}(N)"] & f_0^*(N^{[\sigma f_{0\tau},\tau_{f_0}]})^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[r, "(\alpha^{1N})^{[\sigma f_{0\tau},\tau_{f_0}]}"] & N^{[\sigma f_{0\tau},\tau_{f_0}]} \arrow[d, "\theta^{\sigma f_{0\tau},\tau_{f_0},id_{D_0},id_{D_0}}(id_{D_0}^*(N))=id_{N^{[\sigma f_{0\tau},\tau_{f_0}]}}"'] \\
 & N^{[\sigma \mathrm{pr}_1\tilde{\mathrm{pr}}_{12},\tau_{f_0}\tilde{\mathrm{pr}}_{23}]} \arrow[rr, "N^{(f_{0\tau},\varepsilon\tau f_{0\tau},\tau_{f_0})}"] & & N^{[\sigma f_{0\tau},\tau_{f_0}]}
\end{tikzcd}$$

Since a composition $C_1\times_{C_0} D_0 \xrightarrow{(f_{0\tau},\varepsilon\tau f_{0\tau},\tau_{f_0})} C_1\times_{C_0} C_1\times_{C_0} D_0 \xrightarrow{\mu\times_{C_0} id_{D_0}} C_1\times_{C_0} D_0$ is the identity morphism of $C_1\times_{C_0} D_0$, the assertion follows from the commutativity of the above diagram and (8.5.6). $\square$

Under the assumptions of (9.7.3) for N and the assumptions of (iii) and the first one of (iv) of (9.7.3) for $f_0^*(M)$, we define a map

$$\mathrm{ad}_{(N,\zeta)}^{(M,\xi)} : \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((N,\zeta)^{\boldsymbol{f}},\zeta_{\boldsymbol{f}}^r)) \to \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^{\bullet}(M,\xi),(N,\zeta))$$

by $\mathrm{ad}_{(N,\zeta)}^{(M,\xi)}(\psi) = \varepsilon_{(M,\xi)}^{\boldsymbol{f}} f_0^*(\psi)$.

Proposition 9.7.14 $\mathrm{ad}_{(N,\zeta)}^{(M,\xi)}$ *is bijective.*

Proof. We show that a map $\Phi : \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^{\bullet}(M,\xi),(N,\zeta)) \to \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((N,\zeta)^{\boldsymbol{f}},\zeta_{\boldsymbol{f}}^r))$ defined by $\Phi(\varphi) = {}^t\varphi$ is the inverse of $\mathrm{ad}_{(N,\zeta)}^{(M,\xi)}$. $\mathrm{ad}_{(N,\zeta)}^{(M,\xi)}\Phi$ is the identity map of $\mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^{\bullet}(M,\xi),(N,\zeta))$ by (9.7.12). For $\psi \in \mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((N,\zeta)^{\boldsymbol{f}},\zeta_{\boldsymbol{f}}^r))$, we put $\varphi = \mathrm{ad}_{(N,\zeta)}^{(M,\xi)}(\psi)$. The following diagram is commutative by (8.5.3), (8.5.28), (9.5.4) and the definition of $\check{\zeta}_{\boldsymbol{f}}$.

$$\begin{tikzcd}
 & (N,\zeta)_{\boldsymbol{f}} \arrow[d, "\check{\zeta}_{\boldsymbol{f}}"] \arrow[dr, "E_{(N,\zeta)}^{\boldsymbol{f}}"] & \\
M \arrow[ur, "\psi"] \arrow[d, "\check{\xi}"] & ((N,\zeta)_{\boldsymbol{f}})^{[\sigma,\tau]} \arrow[dr, "(E_{(N,\zeta)}^{\boldsymbol{f}})^{[\sigma,\tau]}"] & N^{[\sigma f_{0\tau},\,\tau_{f_0}]} \arrow[d, "\check{\mu}_{\boldsymbol{f}}(N)"] \\
M^{[\sigma,\tau]} \arrow[ur, "\psi^{[\sigma,\tau]}"] \arrow[rr, "(E_{(N,\zeta)}^{\boldsymbol{f}}\psi)^{[\sigma,\tau]}"] \arrow[d, "\beta^{1M}"] & & (N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma,\tau]} \arrow[d, "\beta^{1N^{[\sigma f_{0\tau},\,\tau_{f_0}]}}"] \\
f_0^*(M)^{[\sigma f_{0\tau},\,\tau_{f_0}]} \arrow[rr, "f_0^*(E_{(N,\zeta)}^{\boldsymbol{f}}\psi)^{[\sigma f_{0\tau},\,\tau_{f_0}]}"] & & f_0^*(N^{[\sigma f_{0\tau},\,\tau_{f_0}]})^{[\sigma f_{0\tau},\,\tau_{f_0}]}
\end{tikzcd}$$

Hence we have the following equalities by the commutativity of the above diagram and (9.7.13).

$$\begin{aligned}
\varphi^{[\sigma f_{0\tau},\,\tau_{f_0}]}\beta^{1M}\check{\xi} &= (\varepsilon_{(M,\xi)}^{\boldsymbol{f}})^{[\sigma f_{0\tau},\,\tau_{f_0}]} f_0^*(\psi)^{[\sigma f_{0\tau},\,\tau_{f_0}]}\beta^{1M}\check{\xi} \\
&= (\alpha^{1N})^{[\sigma f_{0\tau},\,\tau_{f_0}]} f_0^*(E_{(N,\zeta)}^{\boldsymbol{f}})^{[\sigma f_{0\tau},\,\tau_{f_0}]} f_0^*(\psi)^{[\sigma f_{0\tau},\,\tau_{f_0}]}\beta^{1M}\check{\xi} \\
&= (\alpha^{1N})^{[\sigma f_{0\tau},\,\tau_{f_0}]}\beta^{1N^{[\sigma f_{0\tau},\,\tau_{f_0}]}}\check{\mu}_{\boldsymbol{f}}(N) E_{(N,\zeta)}^{\boldsymbol{f}}\psi = E_{(N,\zeta)}^{\boldsymbol{f}}\psi
\end{aligned}$$

Since we also have $\varphi^{[\sigma f_{0\tau},\,\tau_{f_0}]}\beta^{1M}\check{\xi} = E_{(M,\xi)}^{\boldsymbol{f}}\,{}^t\varphi$ by the definition of ${}^t\varphi$, it follows that $\Phi(\varphi) = {}^t\varphi = \psi$ which implies that $\Phi\mathrm{ad}_{(N,\zeta)}^{(M,\xi)}$ is the identity map of $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((N,\zeta)^{\boldsymbol{f}},\zeta_{\boldsymbol{f}}^r))$. $\square$

Definition 9.7.15 *For a representation* (N,ζ) *of* $\boldsymbol{D}$*, we call* $((N,\zeta)^{\boldsymbol{f}},\xi_{\boldsymbol{f}}^r)$ *the left induced representation of* (N,ζ) *by* $\boldsymbol{f} : \boldsymbol{D} \to \boldsymbol{C}$.

The following fact is straightforward from (9.7.8).

Proposition 9.7.16 *The following diagrams are commutative for a morphism* $\varphi : (L,\chi) \to (M,\xi)$ *of* $\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})$ *and a morphism* $\psi : (N,\zeta) \to (P,\rho)$ *of* $\mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})$.

$$\begin{tikzcd}
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}})) \arrow[r, "\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}"] \arrow[d, "\varphi^*"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^{\bullet}(M,\xi),(N,\zeta)) \arrow[d, "\boldsymbol{f}^{\bullet}(\varphi)^*"] \\
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((L,\chi),((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}})) \arrow[r, "\mathrm{ad}^{(L,\chi)}_{(N,\zeta)}"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^{\bullet}(L,\chi),(N,\zeta))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((N,\zeta)^{\boldsymbol{f}},\zeta^r_{\boldsymbol{f}})) \arrow[r, "\mathrm{ad}^{(M,\xi)}_{(N,\zeta)}"] \arrow[d, "\psi^{\boldsymbol{f}}_*"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^{\bullet}(M,\xi),(N,\zeta)) \arrow[d, "\psi_*"] \\
\mathrm{Rep}(\boldsymbol{C}\,;\mathcal{F})((M,\xi),((P,\rho)^{\boldsymbol{f}},\rho^r_{\boldsymbol{f}})) \arrow[r, "\mathrm{ad}^{(M,\xi)}_{(P,\rho)}"] & \mathrm{Rep}(\boldsymbol{D}\,;\mathcal{F})(\boldsymbol{f}^{\bullet}(M,\xi),(P,\rho))
\end{tikzcd}$$

9.8 Representations in fibered category of modules

We call an internal category in $\mathcal{A}lg^{op}_{K_*}$ a Hopf algebroid. Namely, a Hopf algebroid $\boldsymbol{\Gamma}$ consists of two objects A_*, Γ_* of $\mathcal{A}lg_{K_*}$ and four morphisms $\sigma,\tau : A_* \to \Gamma_*$, $\varepsilon : \Gamma_* \to A_*$, $\mu : \Gamma_* \to \Gamma_* \otimes_{A_*} \Gamma_*$ of $\mathcal{A}lg_{K_*}$ which satisfy $\varepsilon\sigma = \varepsilon\tau = id_{A_*}$ and make the following diagrams commute. We regard Γ_* as a left A_*-module by σ and a right A_*-module by τ.

$$\begin{tikzcd}
A_* \arrow[r, "\sigma"] \arrow[d, "\sigma"] & \Gamma_* \arrow[d, "\mu"] & A_* \arrow[l, "\tau"'] \arrow[d, "\tau"] \\
\Gamma \arrow[r, "i_1"] & \Gamma_* \otimes_{A_*} \Gamma_* & \Gamma \arrow[l, "i_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma_* \arrow[r, "\mu"] \arrow[d, "\mu"] & \Gamma_* \otimes_{A_*} \Gamma_* \arrow[d, "id_{\Gamma_*}\otimes_{A_*}\mu"] \\
\Gamma_* \otimes_{A_*} \Gamma_* \arrow[r, "\mu\otimes_{A_*} id_{\Gamma_*}"] & \Gamma_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_*
\end{tikzcd}$$

$$\begin{tikzcd}
& \Gamma_* \arrow[dl, "j_1"'] \arrow[d, "\mu"] \arrow[dr, "j_2"] & \\
\Gamma_* \otimes_{A_*} A_* & \Gamma_* \otimes_{A_*} \Gamma_* \arrow[l, "id_{\Gamma_*}\otimes_{A_*}\varepsilon"'] \arrow[r, "\varepsilon\otimes_{A_*} id_{\Gamma_*}"] & A_* \otimes_{A_*} \Gamma_*
\end{tikzcd}$$

Here, $i_1, i_2 : \Gamma_* \to \Gamma_* \otimes_{A_*} \Gamma_*$ and $j_1 : A_* \to A_* \otimes_{A_*} \Gamma_*$, $j_2 : A_* \to \Gamma_* \otimes_{A_*} A_*$ are maps defined by $i_1(x) = x \otimes 1$, $i_2(x) = 1 \otimes x$ and $j_1(a) = a \otimes 1$, $j_2(a) = 1 \otimes a$.

Let $\boldsymbol{\Gamma} = (A_*, \Gamma_*, \sigma, \tau, \varepsilon, \mu)$ be a Hopf algebroid in $\mathcal{C}$ and $\boldsymbol{M} = (A_*, M_*, \alpha)$ an object of $\mathcal{M}od(\mathcal{C},\mathcal{M})_{A_*}$. For a morphism $\boldsymbol{\xi} : \sigma^*(\boldsymbol{M}) \to \tau^*(\boldsymbol{M})$ of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{A_*}$, we put $\hat{\boldsymbol{\xi}} = P_{\sigma,\tau}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) \in \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{A_*}(\boldsymbol{M}_{[\sigma,\tau]}, \boldsymbol{M})$. For a morphism $f : A_* \to B_*$ of $\mathcal{A}lg_{K_*}$, we denote by ${}_fB_*$ a left A_*-module defined by ${}_fB_* = B_*$ as a K_*-module, with left A_*-module structure map $A_*\otimes_{K_*}{}_fB_* \to {}_fB_*$ given by $a\otimes b \to f(a)b$. Then, if we put $\boldsymbol{\xi} = (id_{\Gamma_*}, \xi)$, ξ is a right Γ_*-module homomorphism from $M_*\otimes_{A_*\,\tau}\Gamma_*$ to $M_*\otimes_{A_*\,\sigma}\Gamma_*$. Since $\boldsymbol{M}_{[\sigma,\tau]} = (A_*, M_*\otimes_{A_*}\Gamma_*, \alpha_\sigma(id_{M_*\otimes_{A_*}\Gamma_*}\otimes_{K_*}\tau))$ and $\hat{\boldsymbol{\xi}} = (id_{A_*}, \hat{\xi})$ for a homomorphism $\hat{\xi} = \xi i_\tau(\boldsymbol{M}) : M_* \to M_*\otimes_{A_*\,\sigma}\Gamma_*$ of right A_*-modules by (3) of (8.7.7), the following result follows from (9.4.1) and (8.7.7).

Proposition 9.8.1 *$\boldsymbol{\xi}$ defines a representation of $\boldsymbol{\Gamma}$ on $\boldsymbol{M}$ if and only if a composition*

$$M_* \xrightarrow{\hat{\xi}} M_*\otimes_{A_*}\Gamma_* \xrightarrow{id_{M_*}\otimes_{A_*}\varepsilon} M_*\otimes_{A_*} A_* \xrightarrow{\bar{\alpha}} M_*$$

is the identity morphism of M_* and the following diagram commute.

$$\begin{tikzcd}
M_* \arrow[r, "\hat{\xi}"] \arrow[dr, "\hat{\xi}"'] & M_*\otimes_{A_*}\Gamma_* \arrow[r, "\hat{\xi}\otimes_{A_*} id_{\Gamma_*}"] & (M_*\otimes_{A_*}\Gamma_*)\otimes_{A_*}\Gamma_* \arrow[d, "\tilde{\theta}_{\sigma,\tau,\sigma,\tau}(\boldsymbol{M})"] \\
& M_*\otimes_{A_*}\Gamma_* \arrow[r, "id_{M_*}\otimes_{A_*}\mu"] & M_*\otimes_{A_*}(\Gamma_*\otimes_{A_*}\Gamma_*)
\end{tikzcd}$$

Here, $\bar{\alpha} : M_*\otimes_{A_*} A_* \to M_*$ is the isomorphism induced by α and $\Gamma_*\otimes_{A_*}\Gamma_*$ is regarded as a left A_*-module by $i_1\sigma$, a right A_*-module by $i_2\tau$.

The following result follows from (9.4.5) and (8.7.7).

Proposition 9.8.2 *Let $(\boldsymbol{M}, \boldsymbol{\xi})$ and $(\boldsymbol{N}, \boldsymbol{\zeta})$ be representations of $\boldsymbol{\Gamma}$ and $\boldsymbol{\varphi} : \boldsymbol{M} \to \boldsymbol{N}$ a morphism of $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{A_*}$. Suppose that $\boldsymbol{M} = (A_*, M_*, \alpha)$, $\boldsymbol{N} = (A_*, N_*, \beta)$ and $\boldsymbol{\varphi} = (id_{A_*}, \varphi)$ for objects M_*, N_* and a morphism $\varphi : N_* \to M_*$ of $\mathcal{M}$. We put $P_{\sigma,\tau}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{A_*}, \hat{\xi}) \in \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{A_*}(\boldsymbol{M}_{[\sigma,\tau]}, \boldsymbol{M})$ and $P_{\sigma,\tau}(\boldsymbol{N})_{\boldsymbol{N}}(\boldsymbol{\zeta}) = (id_{A_*}, \hat{\zeta}) \in \mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{A_*}(\boldsymbol{N}_{[\sigma,\tau]}, \boldsymbol{N})$. Then, $\boldsymbol{\varphi}$ gives a morphism $(\boldsymbol{M}, \boldsymbol{\xi}) \to (\boldsymbol{N}, \boldsymbol{\zeta})$ of representations if and only if the following diagram in $\mathcal{M}$ is commutative.*

$$\begin{tikzcd} N_* \arrow[r, "\hat{\zeta}"] \arrow[d, "\varphi"] & N_* \otimes_{A_*} \Gamma_* \arrow[d, "\varphi\otimes_{A_*} id_{\Gamma_*}"] \\ M_* \arrow[r, "\hat{\xi}"] & M_* \otimes_{A_*} \Gamma_* \end{tikzcd}$$

If a morphism $\hat{\xi} : M_* \to M_* \otimes_{A_*} \Gamma_*$ of right A_*-modules satisfies the conditions of (9.8.1), a pair $(M_*, \hat{\xi})$ is usually called a right Γ-comodule. It follows from the above fact that, the category of representations of $\boldsymbol{\Gamma}$ is isomorphic to the opposite category of the category of right Γ-comodules.

Proposition 9.8.3 *Suppose that K_* is an object of $\mathcal{C}$ and let $\boldsymbol{M} = (K_*, M_*, \alpha)$ be an object of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K_*}$.*
(1) The trivial representation $(\eta^*_{A_*}(\boldsymbol{M}), \boldsymbol{\phi_M})$ associated with $\boldsymbol{M}$ is described as follows. Define a map

$$\phi_{\boldsymbol{M}} : (M_* \otimes_{K_*} A_*) \otimes_{A_*\,\tau} \Gamma \to (M_* \otimes_{K_*} A_*) \otimes_{A_*\,\sigma} \Gamma$$

by $\phi_{\boldsymbol{M}}((x \otimes a) \otimes b) = (x \otimes 1) \otimes \tau(a)b$, then the morphism $\boldsymbol{\phi_M} : \sigma^*\eta^*_{A_*}(\boldsymbol{M}) \to \tau^*\eta^*_{A_*}(\boldsymbol{M})$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{\Gamma_*}$ is $(id_{A_*}, \phi_{\boldsymbol{M}})$.
(2) Define a map $\hat{\phi}_{\boldsymbol{M}} : M_* \otimes_{K_*} A_* \to (M_* \otimes_{K_*} A_*) \otimes_{A_*\,\sigma} \Gamma$ by $\hat{\phi}_{\boldsymbol{M}}(x \otimes a) = (x \otimes 1) \otimes \tau(a)$. If we put $\hat{\boldsymbol{\phi}}_{\boldsymbol{M}} = P_{\sigma,\tau}(\eta^*_{A_*}(\boldsymbol{M}))_{\eta^*_{A_*}(\boldsymbol{M})}(\boldsymbol{\phi_M}) : \eta^*_{A_*}(\boldsymbol{M})_{[\sigma,\tau]} \to \eta^*_{A_*}(\boldsymbol{M})$, then we have $\hat{\boldsymbol{\phi}}_{\boldsymbol{M}} = (id_{A_*}, \hat{\phi}_{\boldsymbol{M}})$.

Proof. (1) Since $\boldsymbol{\phi_M} = \boldsymbol{c}_{o_{A_*},\tau}(\boldsymbol{M})^{-1}\boldsymbol{c}_{o_{A_*},\sigma}(\boldsymbol{M})$, the assertion follows from (8.7.6).
(2) This is a direct consequence of (3) of (8.7.7). □

Definition 9.8.4 *Suppose that K_* is an object of $\mathcal{C}$. We denote by $\boldsymbol{K}$ an object (K_*, K_*, μ_{K_*}) of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{K_*}$. For a representation $(\boldsymbol{M}, \boldsymbol{\xi})$ of $\boldsymbol{\Gamma}$, we call a morphism $(\boldsymbol{M}, \boldsymbol{\xi}) \to (\eta^*_{A_*}(\boldsymbol{K}), \boldsymbol{\phi_K})$ a primitive element of $(\boldsymbol{M}, \boldsymbol{\xi})$.*

Proposition 9.8.5 *Let $(\boldsymbol{M}, \boldsymbol{\xi})$ be a representation of $\boldsymbol{\Gamma}$ and put $\boldsymbol{M} = (A_*.M_*, \alpha)$. For a morphism $\varphi : K_* \to M_*$ of $\mathcal{M}$, $(id_*, \varphi) : (\boldsymbol{M}, \boldsymbol{\xi}) \to (\eta^*_{A_*}(\boldsymbol{K}), \boldsymbol{\phi_K})$ is a primitive element of $(\boldsymbol{M}, \boldsymbol{\xi})$ if and only if $\hat{\xi}(\varphi(1)) = \varphi(1) \otimes 1$. Hence if we define a K_*-submodule $P(\boldsymbol{M}, \boldsymbol{\xi})$ of M_* by $P(\boldsymbol{M}, \boldsymbol{\xi}) = \{x \in M_* \,|\, \hat{\xi}(x) = x \otimes 1\}$, a correspondence $(id_*, \varphi) \mapsto \varphi(1)$ gives a bijection from the set of primitive elements of $(\boldsymbol{M}, \boldsymbol{\xi})$ to $P(\boldsymbol{M}, \boldsymbol{\xi})$.*

Proof. We identify $\eta^*_{A_*}(\boldsymbol{K})$ with (A_*, A_*, μ_{A_*}). It follws from (9.8.5) that the Γ-comodule structure $\hat{\phi}_{\boldsymbol{K}} : A_* \to A_* \otimes_{A_*} \Gamma_*$ is a homomorphism of right A_*-modules which is given by $\hat{\phi}_{\boldsymbol{K}}(a) = 1 \otimes \tau(a)$. Hence a morphism $(id_{A_*}, \varphi) : \boldsymbol{M} \to \eta^*_{A_*}(\boldsymbol{K})$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{A_*}$ gives a morphism $(\boldsymbol{M}, \boldsymbol{\xi}) \to (\eta^*_{A_*}(\boldsymbol{K}), \boldsymbol{\phi_K})$ of representations of $\boldsymbol{\Gamma}$ if and only if $\varphi : A_* \to M_*$ is a homomorphism of right A_*-modules and $\hat{\xi}(\varphi(1)) = \varphi(1) \otimes 1$ □

Proposition 9.8.6 *Let $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{\Gamma} \to \boldsymbol{\Delta}$ be a morphism of Hopf algebroids. We put $\boldsymbol{\Gamma} = (A_*, \Gamma, \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{\Delta} = (B_*, \Delta_*, \sigma', \tau', \varepsilon', \mu')$. For an object $\boldsymbol{M} = (A_*, M_*, \alpha)$ of $\mathcal{M}od(\mathcal{C}, \mathcal{M})_{A_*}$ and a representation $(\boldsymbol{M}, \boldsymbol{\xi})$ on $\boldsymbol{M}$, we put $P_{\sigma,\tau}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{A_*}, \hat{\xi})$ and $P_{\sigma',\tau'}(f_0^*(\boldsymbol{M}))_{f_0^*(\boldsymbol{M})}(\boldsymbol{\xi_f}) = (id_{B_*}, \hat{\xi}_{\boldsymbol{f}})$. Then, $\hat{\xi}_{\boldsymbol{f}}$ is the following composition.*

$$M_* \otimes_{A_*} B_* \xrightarrow{(id_{M_*}\otimes_{A_*} f_1)\hat{\xi}\otimes_{A_*} id_{B_*}} (M_* \otimes_{A_*} \Delta_*) \otimes_{A_*} B_* \xrightarrow{\tilde{\omega}(\sigma',\tau';f_0,f_0)_{\boldsymbol{M}}} (M_* \otimes_{A_*} B_*) \otimes_{B_*} \Delta_*$$

Here, $\tilde{\omega}(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}$ is a map given by $\tilde{\omega}(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}(x \otimes r \otimes s) = x \otimes 1 \otimes r\tau'(s)$.

Proof. It follows from (9.4.4) and (5) of (8.7.7) that we have the following equalities in $\mathcal{M}od(\mathcal{C}, \mathcal{M})^{op}_{B_*}$.

$$\begin{aligned} P_{\sigma',\tau'}(f_0^*(\boldsymbol{M}))_{f_0^*(\boldsymbol{M})}(\boldsymbol{\xi_f}) &= \omega(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}\, f_0^*(\boldsymbol{M}_{f_1}\hat{\boldsymbol{\xi}}) \\ &= (id_{B_*}, \tilde{\omega}(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}) f_0^*((id_{A_*}, id_{M_*} \otimes_{A_*} f_1)(id_{A_*}, \hat{\xi})) \\ &= (id_{B_*}, \tilde{\omega}(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}) f_0^*(id_{A_*}, (id_{M_*} \otimes_{A_*} f_1)\hat{\xi}) \\ &= (id_{B_*}, \tilde{\omega}(\sigma', \tau'; f_0, f_0)_{\boldsymbol{M}}((id_{M_*} \otimes_{A_*} f_1)\hat{\xi} \otimes_{A_*} id_{B_*})) \end{aligned}$$

Hence the assertion follows from (8.7.11). □

For a Hopf algebroid $\boldsymbol{\Gamma}$, we call an internal diagram on $\boldsymbol{\Gamma}$ in $\mathcal{A}lg^{op}_{K_*}$ a $\boldsymbol{\Gamma}$-comodule algebra. Namely, if $\boldsymbol{\Gamma} = (A_*, \Gamma_*, \sigma, \tau, \varepsilon, \mu)$, a $\boldsymbol{\Gamma}$-commdule algebra consists of a pair $(\pi : A_* \to B_*, \gamma : B_* \to B_* \otimes_{A_*} \Gamma_*)$ of morphisms of $\mathcal{A}lg_{K_*}$ which make the following diagrams commute.

$$\begin{tikzcd} A_* \arrow[r, "\tau"] \arrow[d, "\pi"] & \Gamma_* \arrow[d, "j_2"] \\ B_* \arrow[r, "\gamma"] & B_* \otimes_{A_*} \Gamma_* \end{tikzcd} \qquad \begin{tikzcd} B_* \arrow[r, "\gamma"] \arrow[d, "\gamma"] & B_* \otimes_{A_*} \Gamma_* \arrow[d, "id_{B_*} \otimes_{A_*} \mu"] \\ B_* \otimes_{A_*} \Gamma_* \arrow[r, "\gamma \otimes_{A_*} id_{\Gamma_*}"] & B_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_* \end{tikzcd} \qquad \begin{tikzcd} B_* \arrow[r, "\gamma"] \arrow[dr, "\tilde{j}_1"'] & B_* \otimes_{A_*} \Gamma_* \arrow[d, "id_{B_*} \otimes_{A_*} \varepsilon"] \\ & B_* \otimes_{A_*} A_* \end{tikzcd}$$

Here, $\tilde{j}_1 : B_* \to B_* \otimes_{A_*} A_*$ and $j_2 : \Gamma_* \to B_* \otimes_{A_*} \Gamma_*$ are maps defined by $\tilde{j}_1(b) = b \otimes 1$, $j_2(x) = 1 \otimes x$. We define a functor $D_\gamma : \mathcal{P} \to \mathcal{A}lg^{op}_{K_*}$ by $D_\gamma(0) = B_* \otimes_{A_*} \Gamma_*$, $D_\gamma(1) = \Gamma_*$, $D_\gamma(2) = B_*$, $D_\gamma(3) = D_\gamma(4) = D_\gamma(5) = A_*$, $D_\gamma(\tau_{01}) = j_2$, $D_\gamma(\tau_{02}) = \gamma$, $D_\gamma(\tau_{13}) = \sigma$, $D_\gamma(\tau_{14}) = \tau$, $D_\gamma(\tau_{24}) = D_\gamma(\tau_{25}) = \pi$. We also define a map $j_1 : B_* \to B_* \otimes_{A_*} \Gamma_*$ by $j_1(b) = b \otimes 1$. For a representation $(\boldsymbol{M}, \boldsymbol{\xi})$ of $\boldsymbol{C}$, we put $\hat{\boldsymbol{\xi}} = P_{\sigma,\tau}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi})$. We define a morphism $\hat{\boldsymbol{\xi}}_\gamma : \boldsymbol{M}_{[\pi,\pi]} \to (\boldsymbol{M}_{[\pi,\pi]})_{[\sigma,\tau]}$ of $\mathcal{M}od(\mathcal{A}lg_{K_*}, \mathcal{M}od_{K_*})_{B_*}$ to be the following composition.

$$\boldsymbol{M}_{[\pi,\pi]} \xrightarrow{\hat{\boldsymbol{\xi}}_{[\pi,\pi]}} (\boldsymbol{M}_{[\sigma,\tau]})_{[\pi,\pi]} \xrightarrow{\theta_{D_\gamma}(\boldsymbol{M})} \boldsymbol{M}_{[j_2\sigma,\, \gamma\pi]} = \boldsymbol{M}_{[j_1\pi,\, j_2\tau]} \xrightarrow{\theta_{\pi,\pi,\sigma,\tau}(\boldsymbol{M})^{-1}} (\boldsymbol{M}_{[\pi,\pi]})_{[\sigma,\tau]}$$

Proposition 9.8.7 *If $\boldsymbol{M} = (A_*, M_*, \alpha)$ and $\hat{\boldsymbol{\xi}} = (id_{A*}, \hat{\xi})$ for a map $\hat{\xi} : M_* \to M_* \otimes_{A_*} \Gamma_*$, we define a map $\hat{\xi}_\gamma : M_* \otimes_{A_*} B_* \to (M_* \otimes_{A_*} B_*) \otimes_{A_*} \Gamma_*$ to be a composition of $\hat{\xi} \otimes_{A_*} id_{B_*} : M_* \otimes_{A_*} B_* \to (M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} B_*$ and a map $(M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} B_* \to (M_* \otimes_{A_*} B_*) \otimes_{A_*} \Gamma_*$ given by $x \otimes g \otimes b \mapsto x \otimes (1 \otimes g)\gamma(b)$ Then, we have $\hat{\boldsymbol{\xi}}_\gamma = (id_{A*}, \hat{\xi}_\gamma)$.*

Proof. It follows from the definition of $\hat{\boldsymbol{\xi}}_\gamma$ that $\hat{\xi}_\gamma$ is the following composition.

$$M_* \otimes_{A_*} B_* \xrightarrow{\hat{\xi} \otimes_{A_*} id_{B_*}} (M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} B_* \xrightarrow{\tilde{\theta}_{D_\gamma}(\boldsymbol{M})} M_* \otimes_{A_*} (B_* \otimes_{A_*} \Gamma_*) \xrightarrow{\tilde{\theta}_{\pi,\pi,\sigma,\tau}(\boldsymbol{M})^{-1}} (M_* \otimes_{A_*} B_*) \otimes_{A_*} \Gamma_*$$

Hence the assertion follows from (8.7.9). □

We define a morphism $\hat{\boldsymbol{\mu}}_{\boldsymbol{M}} : \boldsymbol{M}_{[\sigma,\tau]} \to (\boldsymbol{M}_{[\sigma,\tau]})_{[\sigma,\tau]}$ to be the following composition.

$$\boldsymbol{M}_{[\sigma,\tau]} \xrightarrow{\boldsymbol{M}_\mu} \boldsymbol{M}_{[\mu\sigma, \mu\tau]} = \boldsymbol{M}_{[i_1\sigma, i_2\tau]} \xrightarrow{\theta_{\sigma,\tau,\sigma,\tau}(\boldsymbol{M})^{-1}} (\boldsymbol{M}_{[\sigma,\tau]})_{[\sigma,\tau]}$$

Proposition 9.8.8 *If $\boldsymbol{M} = (A_*, M_*, \alpha)$, we define a map $\hat{\mu}_{\boldsymbol{M}} : M_* \otimes_{A_*} \Gamma_* \to (M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} \Gamma_*$ to be the following composition.*

$$M_* \otimes_{A_*} \Gamma_* \xrightarrow{id_{M_*} \otimes_{A_*} \mu} M_* \otimes_{A_*} (\Gamma_* \otimes_{A_*} \Gamma_*) \xrightarrow{\tilde{\theta}_{\sigma,\tau,\sigma,\tau}(\boldsymbol{M})^{-1}} (M_* \otimes_{A_*} \Gamma_*) \otimes_{A_*} \Gamma_*$$

Then, we have $\hat{\boldsymbol{\mu}}_{\boldsymbol{M}} = (id_{A_*}, \hat{\mu}_{\boldsymbol{M}})$.

Proof. The assertion is a direct consequence of (8.7.7) and (8.7.11). □

The following assertion is a direct consequence of (8.7.7).

Proposition 9.8.9 *For morphisms $\lambda : R_* \to S_*$ and $\nu : T_* \to S_*$ of $\mathcal{A}lg_{K_*}$, $[\lambda, \nu]_* : \mathcal{M}od(\mathcal{A}lg_{K_*}, \mathcal{M}od_{K_*})_{R_*} \to \mathcal{M}od(\mathcal{A}lg_{K_*}, \mathcal{M}od_{K_*})_{T_*}$ preserves coequalizers. It preserves equalizers λ is flat.*

(9.4.14) implies the following result.

Proposition 9.8.10 *Let $(\boldsymbol{M}, \boldsymbol{\xi})$ and $(\boldsymbol{M}, \boldsymbol{\zeta})$ be representations of $\boldsymbol{\Gamma}$ on $\boldsymbol{M} = (A_*, M_*, \alpha) \in$ Ob $\mathcal{M}od(\mathcal{C}, \mathcal{M})$. We put $P_{\sigma,\tau}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\xi}) = (id_{A_*}, \hat{\xi})$ and $P_{\sigma,\tau}(\boldsymbol{M})_{\boldsymbol{M}}(\boldsymbol{\zeta}) = (id_{A_*}, \hat{\zeta})$. Assume that $\sigma : A_* \to \Gamma_*$ is flat.*

(1) Let $\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}} : M_{(\boldsymbol{\xi}:\boldsymbol{\zeta})*} \to M_*$ be the kernel of $\hat{\xi} - \hat{\zeta} : M_* \to M_* \otimes_{A_*} \Gamma_*$. There exists unique homomorphism $\hat{\lambda} : M_{(\boldsymbol{\xi}:\boldsymbol{\zeta})*} \to M_{(\boldsymbol{\xi}:\boldsymbol{\zeta})*} \otimes_{A_*} \Gamma_*$ of right A_*-modules that makes the following diagram commute. Here we put $\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} = (A_*, M_{(\boldsymbol{\xi}:\boldsymbol{\zeta})*}, \bar{\alpha})$ where $\bar{\alpha} : M_{(\boldsymbol{\xi}:\boldsymbol{\zeta})*} \otimes_{K_*} A_* \to M_{(\boldsymbol{\xi}:\boldsymbol{\zeta})*}$ is the map induced by $\alpha : M_* \otimes_{K_*} A_* \to M_*$.

$$\begin{tikzcd} \boldsymbol{M} \arrow[d, "(id_{A_*}, \hat{\xi})"] & \boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} \arrow[l, "\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}}"'] \arrow[r, "\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}}"] \arrow[d, "(id_{A_*}, \hat{\lambda})"] & \boldsymbol{M} \arrow[d, "(id_{A_*}, \hat{\zeta})"] \\ \boldsymbol{M}_{[\sigma,\tau]} & (\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})})_{[\sigma,\tau]} \arrow[l, "(\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}})_{[\sigma,\tau]}"'] \arrow[r, "(\kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}})_{[\sigma,\tau]}"] & \boldsymbol{M}_{[\sigma,\tau]} \end{tikzcd}$$

(2) Put $\hat{\boldsymbol{\lambda}} = (id_{A_*}, \hat{\lambda}) : \boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} \to (\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})})_{[\sigma,\tau]}$ *and* $\boldsymbol{\lambda} = P_{\sigma,\tau}(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})})^{-1}_{\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}}(\hat{\boldsymbol{\lambda}}) : \sigma^*(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}) \to \tau^*(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})})$*. Then,* $(\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \boldsymbol{\lambda})$ *is a representation of* $\boldsymbol{\Gamma}$ *and a morphism* $\boldsymbol{\kappa}_{\boldsymbol{\xi},\boldsymbol{\zeta}} = (id_{A_*}, \kappa_{\boldsymbol{\xi},\boldsymbol{\zeta}}) : \boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})} \to \boldsymbol{M}$ *of* $\mathcal{M}od(\mathcal{C},\mathcal{M})$ *defines morphisms of representations* $(\boldsymbol{M},\boldsymbol{\xi}) \to (\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \boldsymbol{\lambda})$ *and* $(\boldsymbol{M},\boldsymbol{\zeta}) \to (\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \boldsymbol{\lambda})$*.*

(3) Let $(\boldsymbol{N},\boldsymbol{\nu})$ *be a representation of* $\boldsymbol{\Gamma}$*. Suppose that a morphism* $\boldsymbol{\varphi} : \boldsymbol{M} \to \boldsymbol{N}$ *of* $\mathcal{M}od(\mathcal{C},\mathcal{M})^{op}_{A_*}$ *gives morphisms* $(\boldsymbol{M},\boldsymbol{\xi}) \to (\boldsymbol{N},\boldsymbol{\nu})$ *and* $(\boldsymbol{M},\boldsymbol{\zeta}) \to (\boldsymbol{N},\boldsymbol{\nu})$ *of representations of* $\boldsymbol{\Gamma}$*. Then, there exists unique morphism* $\tilde{\boldsymbol{\varphi}} : (\boldsymbol{M}_{(\boldsymbol{\xi}:\boldsymbol{\zeta})}, \lambda) \to (\boldsymbol{N},\boldsymbol{\nu})$ *of representations of* $\boldsymbol{\Gamma}$ *that satisfies* $\tilde{\boldsymbol{\varphi}}\pi_{\boldsymbol{\xi},\boldsymbol{\zeta}} = \boldsymbol{\varphi}$*.*

Let $\boldsymbol{\Gamma} = (A_*, \Gamma, \sigma, \tau, \varepsilon, \mu)$ and $\boldsymbol{\Delta} = (B_*, \Delta_*, \sigma', \tau', \varepsilon', \mu')$ be Hopf algebroids. We regard $\boldsymbol{\Gamma}$ as a left A_*-module by σ and a right A_*-module by τ. Similarly, we regard $\boldsymbol{\Delta}$ as a left A_*-module by σ' and a right A_*-module by τ'. Let $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{\Gamma} \to \boldsymbol{\Delta}$ be a morphism of Hopf algebroids. Regard B_* as an A_*-algebra by f_0 and define maps $f_{0\sigma} : \Gamma_* \to B_* \otimes_{A_*} \Gamma_*$ and $\sigma_{f_0} : B_* \to B_* \otimes_{A_*} \Gamma_*$ by $f_{0\sigma}(x) = 1 \otimes x$ and $\sigma_{f_0}(b) = b \otimes 1$, respectively. Let us consider the following diagram in $\mathcal{C}$ whose rectangles are all cocartesian.

$$\begin{tikzcd}
 & A_* \arrow[r, "\sigma"] \arrow[d, "\tau"] & \Gamma_* \arrow[d, "i_2"] & \\
A_* \arrow[r, "\sigma"] \arrow[d, "f_0"] & \Gamma_* \arrow[r, "i_1"] \arrow[d, "f_{0\sigma}"] & \Gamma_* \otimes_{A_*} \Gamma_* \arrow[r, "id_{\Gamma_*} \otimes_{A_*} i_1"] \arrow[d, "f_{0\sigma} \otimes_{A_*} id_{\Gamma_*}"] & \Gamma_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_* \arrow[d, "f_{0\sigma} \otimes_{A_*} id_{\Gamma_* \otimes_{A_*} \Gamma_*}"] \\
B_* \arrow[r, "\sigma_{f_0}"] & B_* \otimes_{A_*} \Gamma_* \arrow[r, "id_{B_*} \otimes_{A_*} i_1"] & B_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_* \arrow[r, "id_{B_* \otimes_{A_*} \Gamma_*} \otimes_{A_*} i_1"] & B_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_* \otimes_{A_*} \Gamma_*
\end{tikzcd}$$

To be continued.

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Appendix A

Categories for mathematicians reading SGA

A.1 Preliminaries

Definition A.1.1 *A universe $\mathcal{U}$ is a non-empty set satisfying the following properties.*

U1) If $x \in \mathcal{U}$ and $y \in x$, then $y \in \mathcal{U}$.
U2) If $x, y \in \mathcal{U}$, then $\{x, y\} \in \mathcal{U}$.
U3) If $x \in \mathcal{U}$, then $P(x) = \{y | \, y \subset x\} \in \mathcal{U}$.
U4) If $(x_i)_{i \in I}$ is a family of elements of $\mathcal{U}$ and $I \in \mathcal{U}$, then $\bigcup_{i \in I} x_i \in \mathcal{U}$.

The above axioms imply the following facts.

1) If $x \in \mathcal{U}$, $\{x\} \in \mathcal{U}$.
2) If x is a subset of $y \in \mathcal{U}$, $x \in \mathcal{U}$.
3) If $x, y \in \mathcal{U}$, the ordered pair $(x, y) = \{\{x\}, \{x, y\}\}$ belongs to $\mathcal{U}$.
4) If $x, y \in \mathcal{U}$, the union $x \cup y$ and the product $x \times y$ belong to $\mathcal{U}$.
5) If $(x_i)_{i \in I}$ is a family of elements of $\mathcal{U}$ and $I \in \mathcal{U}$, then $\prod_{i \in I} x_i \in \mathcal{U}$.
6) If $x \in \mathcal{U}$, then $\mathrm{card}(x) < \mathrm{card}(\mathcal{U})$. In particular, $\mathcal{U} \notin \mathcal{U}$.

From now on, we assume that a universe contains an infinite set unless otherwise stated. We also assume the following axiom.

U5) For any set x, there exists a universe $\mathcal{U}$ such that $x \in \mathcal{U}$.

Definition A.1.2 *Let $\mathcal{U}$ be a universe.*

1) We say that a set is $\mathcal{U}$-small (or small for short) if it is isomorphic to an element of $\mathcal{U}$.

2) A category $\mathcal{C}$ is called a $\mathcal{U}$-category if the set of morphisms $\mathcal{C}(X, Y)$ is $\mathcal{U}$-small for any objects X, Y of $\mathcal{C}$.

3) Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is an element of $\mathcal{U}$ (resp. $\mathcal{U}$-small) if $\mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Mor}\,\mathcal{C}$ are elements of $\mathcal{U}$ (resp. $\mathcal{U}$-small).

We denote by $\mathcal{U}$-**Ens** the category of sets belonging to $\mathcal{U}$.

Proposition A.1.3 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ denotes the category of functors from $\mathcal{C}$ to $\mathcal{D}$.*

1) If $\mathcal{C}$ and $\mathcal{D}$ are elements of $\mathcal{U}$ (resp. $\mathcal{U}$-small), $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ is an element of $\mathcal{U}$ (resp. $\mathcal{U}$-small).

2) If $\mathcal{C}$ is $\mathcal{U}$-small and $\mathcal{D}$ is a $\mathcal{U}$-category, $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ is a $\mathcal{U}$-category.

Let $F : \mathcal{C} \to \mathcal{A}$ be a functor.

Definition A.1.4 *1) F is called faithful (resp. full, fully faithful) if $F : \mathcal{C}(X, Y) \to \mathcal{A}(F(X), F(Y))$ is injective (resp. surjective, bijective) for any object X and Y.*

2) F is called an equivalence if there is a functor $G : \mathcal{A} \to \mathcal{C}$ and natural equivalences $GF \to 1_{\mathcal{C}}$ and $FG \to 1_{\mathcal{A}}$.

3) F preserves (co)limits for a functor $D : \mathcal{D} \to \mathcal{C}$ if, for every (co)limiting cone $(f_i : L \to D(i))_{i \in \mathrm{Ob}\mathcal{D}}$ (resp. $(f_i : D(i) \to L)_{i \in \mathrm{Ob}\mathcal{D}}$) in $\mathcal{C}$, $(F(f_i) : F(L) \to FD(i))_{i \in \mathrm{Ob}\mathcal{D}}$ (resp. $(F(f_i) : FD(i) \to F(L))_{i \in \mathrm{Ob}\mathcal{D}}$ is a (co)limiting cone in $\mathcal{A}$.

4) F reflects (co)limits for a functor $D : \mathcal{D} \to \mathcal{C}$ if each cone $(f_i : L \to D(i))_{i\in \mathrm{Ob}\mathcal{D}}$ (resp. $(f_i : D(i) \to L)_{i\in \mathrm{Ob}\mathcal{D}}$) in $\mathcal{C}$ such that $(F(f_i) : F(L) \to FD(i))_{i\in \mathrm{Ob}\mathcal{D}}$ (resp. $(F(f_i) : FD(i) \to F(L))_{i\in \mathrm{Ob}\mathcal{D}}$ is a (co)limiting cone in $\mathcal{A}$ is a (co)limiting cone.

5) F creates (co)limits for a functor $D : \mathcal{D} \to \mathcal{C}$ if, for every (co)limiting cone $(g_i : M \to FD(i))_{i\in \mathrm{Ob}\mathcal{D}}$ (resp. $(g_i : FD(i) \to M)_{i\in \mathrm{Ob}\mathcal{D}}$) in $\mathcal{A}$, there exists a unique pair of an object L of $\mathcal{C}$ with $F(L) = M$ and a cone $(f_i : L \to D(i))_{i\in \mathrm{Ob}\mathcal{D}}$ (resp. $(f_i : D(i) \to L)_{i\in \mathrm{Ob}\mathcal{D}}$) with $F(f_i) = g_i$ and this cone is a (co)limiting cone in $\mathcal{C}$.

6) F is said to be left exact if F preserves finite limits.

Definition A.1.5 *Let $\mathcal{C}$ be a $\mathcal{U}$-category and X an object of $\mathcal{C}$. We define a functor $h_X : \mathcal{C}^{op} \to \mathcal{U}$-**Ens** as follows.*
1) If $\mathcal{C}(Y,X) \in \mathcal{U}$, we set $h_X(Y) = \mathcal{C}(Y,X)$ and $\varphi_{Y,X} = id_{\mathcal{C}(Y,X)} : \mathcal{C}(Y,X) \to h_X(Y)$.
2) If $\mathcal{C}(Y,X) \notin \mathcal{U}$, we choose a set $M(Y,X) \in \mathcal{U}$ and a bijection $\varphi_{Y,X} : \mathcal{C}(Y,X) \to M(Y,X)$ and we set $h_X(Y) = M(Y,X)$.
For a morphism $f : Y \to Z$, $h_X(f) : h_X(Z) \to h_X(Y)$ is defined to be $\varphi_{Y,X} f^* \varphi^{-1}_{Z,X}$. We call h_X the functor represented by X.

A morphism $f : X \to Y$ of $\mathcal{C}$ defines a natural transformation $h_f : h_X \to h_Y$ by $(h_f)_Z = \varphi_{Z,Y} f_* \varphi^{-1}_{Z,X}$.

We denote by the $\widehat{\mathcal{C}}_{\mathcal{U}}$ the category of functors from $\mathcal{C}^{op}$ to $\mathcal{U}$-**Ens** and call an object of $\widehat{\mathcal{C}}_{\mathcal{U}}$ a $\mathcal{U}$-presheaf on $\mathcal{C}$.

Proposition A.1.6 *(Yoneda's lemma) Let F be a $\mathcal{U}$-presheaf on $\mathcal{C}$ and X an object of $\mathcal{C}$. There is a natural bijection $\theta_F : F(X) \to \widehat{\mathcal{C}}_{\mathcal{U}}(h_X, F)$ given by $(\theta_F(x))_Y(\varphi) = F(\varphi)(x)$.*

Proof. We define $\theta_F^{-1} : \widehat{\mathcal{C}}_{\mathcal{U}}(h_X, F) \to F(X)$ by $\theta_F^{-1}(\psi) = \psi_X(id_X)$. Then, it is easy to verify that $\theta_F^{-1}\theta_F = id_{F(X)}$ and $\theta_F\theta_F^{-1} = id_{\widehat{\mathcal{C}}_{\mathcal{U}}(h_X,F)}$. $\square$

Corollary A.1.7 *There is a fully faithful functor $h : \mathcal{C} \to \widehat{\mathcal{C}}$ defined by $X \mapsto h_X$ on objects and $(f : X \to Y) \mapsto (h_f : h_X \to h_Y)$ on morphisms.*

Proof. By (A.1.6), $\theta^{-1}_{h_Y} : \widehat{\mathcal{C}}_{\mathcal{U}}(h_X, h_Y) \to h_Y(X) \cong \mathcal{C}(X,Y)$ gives the inverse of $h : \mathcal{C}(X,Y) \to \widehat{\mathcal{C}}_{\mathcal{U}}(h_X, h_Y)$. $\square$

Definition A.1.8 *Let $\mathcal{C}$ be a category.*

1) A morphism $f : X \to Y$ is called a monomorphism (resp. epimorphism) if $f_* : \mathcal{C}(Z,X) \to \mathcal{C}(Z,Y)$ (resp. $f^* : \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$) is injective for any object Z. We often use an arrow $\rightarrowtail$ for a monomorphism.

2) Let us denote by $\widetilde{P}(X)$ the set of monomorphisms whose codomains are X. We define a relation $\prec$ in $\widetilde{P}(X)$ as follows. For $\sigma_1 : Y_1 \rightarrowtail X$ and $\sigma_2 : Y_2 \rightarrowtail X$ monomorphisms, we write $\sigma_2 \prec \sigma_1$ if there exists a morphism $f : Y_1 \to Y_2$ such that $\sigma_2 f = \sigma_1$. Thus $(\widetilde{P}(X), \prec)$ is a partially ordered set. We say that σ_1 and σ_2 are equivalent if both $\sigma_2 \prec \sigma_1$ and $\sigma_1 \prec \sigma_2$ hold. This is an equivalence relation in $\widetilde{P}(X)$ and $\mathrm{Sub}(X)$ denotes the quotient set of $\widetilde{P}(X)$ modulo this relation. We call an element of $\mathrm{Sub}(X)$ a subobject of X. The relation $\prec$ in $\widetilde{P}(X)$ defines a relation $\subset$ in $\mathrm{Sub}(X)$ uniquely so that the quotient map preserves the relations. Then, $(\mathrm{Sub}(X), \subset)$ is an ordered set.

Definition A.1.9 *Let $\mathcal{C}$ be a category.*

1) A pair of morphisms $R \overset{f}{\underset{g}{\rightrightarrows}} X$ of $\mathcal{C}$ is called an equivalence relation if $(f_*, g_*) : \mathcal{C}(Y,R) \to \mathcal{C}(Y,X) \times \mathcal{C}(Y,X)$ is injective and its image is an equivalence relation on $\mathcal{C}(Y,X)$ for any object Y.

2) A pair of morphisms $Z \overset{f}{\underset{g}{\rightrightarrows}} X$ is called a kernel pair of a morphism $p : X \to Y$ if it is an equivalence relation such that $\varphi, \psi \in \mathcal{C}(W,X)$ are equivalent if and only if $p\varphi = p\psi$.

3) An equivalence relation is said to be effective if it is a kernel pair of a certain morphism.

4) A pair of morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ of $\mathcal{C}$ is called a reflexive pair if there exists a morphism $s : Y \to X$ such that $fs = gs = 1_Y$.

Definition A.1.10 *1) If $(X_i \xrightarrow{\iota_i} X)_{i\in I}$ is a colimiting cone of a diagram $(X_i \xrightarrow{\alpha} X_j)$ of $\mathcal{C}$, we say X is a universal colimit provided that for each morphism $f : Y \to X$, the pull-back $\tilde{\iota}_i : Y \times_X X_i \to Y$ of ι_i*

along f exists for any $i \in I$ and the cone $(Y \times_X X_i \xrightarrow{\tilde{\iota}_i} Y)_{i\in I}$ is a colimit for the "pulled-back" diagram $(Y \times_X X_i \xrightarrow{id_Y \times \alpha} Y \times_X X_j)$.

2) If X is a coproduct of a family of objects $(X_i)_{i\in I}$ of $\mathcal{C}$, we say that it is disjoint provided that each canonical inclusion $\nu_i : X_i \to X$ is a monomorphism and that for each pair of distinct indices (i, j),

$$\begin{array}{ccc} 0_{\mathcal{C}} & \longrightarrow & X_j \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X \end{array}$$

is a pull-back, where $0_{\mathcal{C}}$ is an initial object of $\mathcal{C}$. Moreover, X is said to be universally disjoint if for each morphism $f : Y \to X$, the pull-back $\tilde{\nu}_i : Y \times_X X_i \to Y$ of ν_i along f exists for any $i \in I$ and Y is a disjoint coproduct of $(Y \times_X X_i)_{i\in I}$.

Definition A.1.11 *Let $\mathcal{C}$ be a category.*

1) A family of morphisms $(f_i : X_i \to X)_{i\in I}$ of $\mathcal{C}$ is called an epimorphic family if

$$e : \mathcal{C}(X, Z) \to \prod_{i\in I} \mathcal{C}(X_i, Z)$$

defined by $e(\varphi) = (\varphi f_i)_{i\in I}$ is injective for any object Z.

2) It is called a strict epimorphic family if the image of e consists of family $(g_i)_{i\in I}$ such that, for any $i, j \in I$, any object W of $\mathcal{C}$ and any morphisms $u : W \to X_i$, $v : W \to X_j$ satisfying $f_i u = f_j v$, $g_i u = g_j v$ holds.

3) It is called effective if a pull-back

$$\begin{array}{ccc} X_i \times_X X_j & \xrightarrow{q_{ij}} & X_j \\ \downarrow{\scriptstyle p_{ij}} & & \downarrow{\scriptstyle f_j} \\ X_i & \xrightarrow{f_i} & X \end{array}$$

exists for any $i, j \in I$ and

$$\mathcal{C}(X, Z) \xrightarrow{e} \prod_{i\in I} \mathcal{C}(X_i, Z) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{i,j\in I} \mathcal{C}(X_i \times_X X_j, Z)$$

is an equalizer for any object Z, where α and β are given by $\mathrm{pr}_{ij}\alpha((\varphi_i)_{i\in I}) = \varphi_i p_{ij}$ and $\mathrm{pr}_{ij}\beta((\varphi_i)_{i\in I}) = \varphi_j q_{ij}$, respectively.

4) An epimorphic family $(f_i : X_i \to X)_{i\in I}$ is said to be universal (resp. universal effective) if for any morphism $g : Y \to X$, a pull-back f_i' of f_i along g exists for each $i \in I$ and $(f_i' : X_i \times_X Y \to Y)_{i\in I}$ is an epimorphic (resp. effective epimorphic) family.

5) An epimorphism $p : X \to Z$ in $\mathcal{C}$ is said to be regular if it is a coequalizer of a certain pair of morphisms.

If a family of morphisms $(f_i : X_i \to X)_{i\in I}$ is a colimiting cone of a certain diagram with vertices indexed by I, it is a strict epimorphic family. We also remark that if $\mathcal{C}$ is a category with finite limits, the notion of a strict (resp. universal strict) epimorphic family coincides with that of an effective (resp. universal effective) epimorphic family.

Definition A.1.12 *Let $\mathcal{C}$ be a category.*

1) A family of morphisms $(f_i : X \to X_i)_{i\in I}$ of $\mathcal{C}$ is called an monomorphic family if

$$e : \mathcal{C}(Z, X) \to \prod_{i\in I} \mathcal{C}(Z, X_i)$$

defined by $e(\varphi) = (f_i\varphi)_{i\in I}$ is injective for any object Z.

2) It is called a strict monomorphic family the image of e consists of family $(g_i)_{i\in I}$ such that, for any $i, j \in I$, any object W of $\mathcal{C}$ and any morphisms $u : X_i \to W$, $v : X_j \to W$ satisfying $uf_i = vf_j$, $ug_i = vg_j$ holds.

3) A monomorphism $i : A \to X$ in $\mathcal{C}$ is said to be regular if it is an equalizer of a certain pair of morphisms.

Definition A.1.13 *Let $(F_i : \mathcal{C} \to \mathcal{D}_i)_{i \in I}$ be a family of functors.*

1) $(F_i : \mathcal{C} \to \mathcal{D}_i)_{i \in I}$ is said to be faithful if for any $X, Y \in \mathrm{Ob}\,\mathcal{C}$, the map $\mathcal{C}(X,Y) \to \prod_{i \in I} \mathcal{D}_i(F_i(X), F_i(Y))$ defined by $f \mapsto (F_i(f))_{i \in I}$ is injective.

2) $(F_i : \mathcal{C} \to \mathcal{D}_i)_{i \in I}$ is said to be conservative (resp. conservative for monomorphisms, resp. conservative for strict monomorphisms) if a morphism (resp. mono-morphism, resp. strict monomorphism) $f : X \to Y$ of $\mathcal{C}$ such that $F_i(f)$ is an isomorphism for all $i \in I$ is an isomorphism.

Definition A.1.14 *Let $\mathcal{C}$ be a category and $\mathcal{G}$ a full subcategory of $\mathcal{C}$.*

1) $\mathcal{G}$ is called a generating subcategory of $\mathcal{C}$ by epimorphisms (resp. strict epimorphisms) if for any object X of $\mathcal{C}$, $\bigcup_{Y \in \mathrm{Ob}\,\mathcal{G}} \mathcal{C}(Y,X)$ is an epimorphic (resp. strict epimorphic) family.

2) $\mathcal{G}$ is called a generating subcategory of $\mathcal{C}$ (resp. a generating subcategory of $\mathcal{C}$ for monomorphisms, resp. a generating subcategory of $\mathcal{C}$ for strict monomorphisms) if a morphism (resp. monomorphism, resp. strict monomorphism) $u : X \to Y$ such that $u_* : \mathcal{C}(Z,X) \to \mathcal{C}(Z,Y)$ is bijective for any object Z of $\mathcal{G}$ is an isomorphism.

3) Let G be a set of objects and $\mathcal{G}$ the full subcategory of $\mathcal{C}$ such that $\mathrm{Ob}\,\mathcal{G} = G$. G is called a generator of $\mathcal{C}$ by strict epimorphisms (resp. epimorphisms) if $\mathcal{G}$ is a generating subcategory of $\mathcal{C}$ by strict epimorphisms (resp. epimorphisms). G is called a generator of $\mathcal{C}$ (resp. a generator of $\mathcal{C}$ for monomorphisms, resp. a generator for strict monomorphisms) if $\mathcal{G}$ is a generating subcategory of $\mathcal{C}$ (resp. a generating subcategory of $\mathcal{C}$ for monomorphisms, resp. a generating subcategory of $\mathcal{C}$ for strict monomorphisms).

Definition A.1.15 *Let $T : \mathcal{A} \to \mathcal{C}$ and $S : \mathcal{B} \to \mathcal{C}$ be functors. We define the "comma category" $(T{\downarrow}S)$ as follows. Objects of $(T{\downarrow}S)$ are triples $\langle X, f, Y\rangle$ with $X \in \mathrm{Ob}\,\mathcal{A}$, $Y \in \mathrm{Ob}\,\mathcal{B}$ and $f \in \mathcal{C}(T(X), S(Y))$. Morphisms $\langle X, f, Y\rangle \to \langle Z, g, W\rangle$ are pairs $\langle \varphi, \psi\rangle$ of morphisms $\varphi : X \to Z$ in $\mathcal{A}$ and $\psi : Y \to W$ in $\mathcal{B}$ such that $gT(\varphi) = S(\psi)f$. The composite of $\langle \varphi, \psi\rangle : \langle X, f, Y\rangle \to \langle Z, g, W\rangle$ and $\langle \lambda, \mu\rangle : \langle Z, g, W\rangle \to \langle U, h, V\rangle$ is defined by $\langle \lambda\varphi, \mu\psi\rangle$.*

If $\mathcal{A}$ is a category consisting of a single object 1 and a single morphism id_1 and T is the functor given by $T(1) = X$, we denote $(T{\downarrow}S)$ by $(X{\downarrow}S)$. In this case, we denote by $\langle f, Y\rangle$ an object $\langle X, f, Y\rangle$ and by ψ a morphism $\langle id_X, \psi\rangle$ in $(X{\downarrow}S)$. Similarly, if $\mathcal{B}$ is a category consisting of a single object 1 and a single morphism id_1 and S is the functor given by $S(1) = Y$, we denote $(T{\downarrow}S)$ by $(T{\downarrow}Y)$. In this case, we denote by $\langle X, f\rangle$ an object $\langle X, f, Y\rangle$ and by φ a morphism $\langle \varphi, id_Y\rangle$ of $(T{\downarrow}Y)$. Moreover, if $\mathcal{A} = \mathcal{C}$ and T is the identity functor of $\mathcal{C}$, $(id_{\mathcal{C}}{\downarrow}Y)$ is usually denoted by $\mathcal{C}/Y$.

We have functors $P : (T{\downarrow}S) \to \mathcal{A}$, $Q : (T{\downarrow}S) \to \mathcal{B}$ and $R : (T{\downarrow}S) \to \mathcal{C}^{(2)}$ given by $P\langle X, f, Y\rangle = X$, $P\langle \varphi, \psi\rangle = \varphi$, $Q\langle X, f, Y\rangle = Y$, $Q\langle \varphi, \psi\rangle = \psi$ and $R\langle X, f, Y\rangle = f$, $R\langle \varphi, \psi\rangle = (T(\varphi), S(\psi))$, where $\mathcal{C}^{(2)}$ is the category of morphisms of $\mathcal{C}$ given by $\mathrm{Ob}\,\mathcal{C}^{(2)} = \mathrm{Mor}\,\mathcal{C}$ and $\mathcal{C}^{(2)}(f,g) = \{(\varphi,\psi) |\, \varphi \in \mathcal{C}(\mathrm{dom}(f), \mathrm{dom}(g)),\ \psi \in \mathcal{C}(\mathrm{codom}(f), \mathrm{codom}(g))\ \psi f = g\varphi\}$.

Let $T, T' : \mathcal{A} \to \mathcal{C}$ and $S, S' : \mathcal{B} \to \mathcal{C}$ be functors and $\alpha : T' \to T$, $\beta : S \to S'$ natural transformations. Define a functor $(\alpha{\downarrow}\beta) : (T{\downarrow}S) \to (T'{\downarrow}S')$ by $\alpha\langle X, f, Y\rangle = \langle X, \beta_Y f \alpha_X, Y\rangle$, $(\alpha{\downarrow}\beta)\langle \varphi, \psi\rangle = \langle \varphi, \psi\rangle$. In particular, if $\alpha : X \to X'$ and $\beta : Y \to Y'$ are morphisms in $\mathcal{C}$, we have functors $(\alpha{\downarrow}id_S) : (X'{\downarrow}S) \to (X{\downarrow}S)$ and $(id_T{\downarrow}\beta) : (T{\downarrow}Y) \to (T{\downarrow}Y')$ which are given by $(\alpha{\downarrow}id_S)\langle f, Y\rangle = \langle f\alpha, Y\rangle$ and $(id_T{\downarrow}\beta)\langle X, f\rangle = \langle X, \beta f\rangle$, respectively.

Remark A.1.16 *For a functor $F : \mathcal{C} \to \mathcal{D}$, we denote by $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ the functor given by $F^{op}(X) = F(X)$ and $F^{op}(f) = F(f)$ for $X \in \mathrm{Ob}\,\mathcal{C}^{op} = \mathrm{Ob}\,\mathcal{C}$ and $X \in \mathrm{Mor}\,\mathcal{C}^{op} = \mathrm{Mor}\,\mathcal{C}$. The opposite category $(T{\downarrow}S)^{op}$ of $(T{\downarrow}S)$ is $(S^{op}{\downarrow}T^{op})$.*

A.2 Adjoints

For a functor $F : \mathcal{C} \to \mathcal{D}$ and an object Y of $\mathcal{D}$, we define a $\mathcal{U}$-presheaf h_Y^F on $\mathcal{C}$ by $h_Y^F(X) = \mathcal{D}(F(X), Y)$. Suppose that h_Y^F is representable for any object Y of $\mathcal{D}$, namely, there exists an object $G(Y)$ of $\mathcal{C}$ and natural equivalence $\varphi(Y) : h_Y^F \to h_{G(Y)}$ for any $Y \in \mathrm{Ob}\,\mathcal{D}$. For $Y \in \mathrm{Ob}\,\mathcal{D}$, we denote by $\varepsilon_Y : F(G(Y)) \to Y$ the image of $id_{G(Y)} \in \mathcal{C}(G(Y), G(Y))$ by $\varphi(Y)^{-1}_{G(Y)} : \mathcal{C}(G(Y), G(Y)) \to \mathcal{D}(F(G(Y)), Y)$. For $X \in \mathrm{Ob}\,\mathcal{C}$, we denote by $\eta_X : X \to G(F(X))$ the image of $id_{F(X)} \in \mathcal{D}(F(X), F(X))$ by $\varphi(F(X))_X : \mathcal{D}(F(X), F(X)) \to \mathcal{C}(X, G(F(X)))$.

Proposition A.2.1 *A composition $F(X) \xrightarrow{F(\eta_X)} F(G(F(X))) \xrightarrow{\varepsilon_{F(X)}} F(X)$ is the identity morphism of $F(X)$.*

Proof. In fact, since

$$\begin{tikzcd}
\mathcal{D}(F(G(F(X))), F(X)) \arrow[r, "\varphi(F(X))_{G(F(X))}"] \arrow[d, "F(\eta_X)^*"] & \mathcal{C}(G(F(X)), G(F(X))) \arrow[d, "\eta_X^*"] \\
\mathcal{D}(F(X), F(X)) \arrow[r, "\varphi(F(X))_X"] & \mathcal{C}(X, G(F(X)))
\end{tikzcd}$$

is commutative by the naturality of $\varphi(F(X))$, we have

$$\varphi(F(X))_X(\varepsilon_{F(X)}F(\eta_X)) = \varphi(F(X))_{G(F(X))}(\varepsilon_{F(X)})\eta_X = \eta_X = \varphi(F(X))_X(id_{F(X)})$$

and this implies $\varepsilon_{F(X)}F(\eta_X) = id_{F(X)}$. □

A morphism $f : Y \to Z$ of $\mathcal{D}$ defines a morphism $h_f^F : h_Y^F \to h_Z^F$ of $\mathcal{U}$-presheaves by $(h_f^F)_X(\alpha) = f\alpha$ for $\alpha \in h_Y^F(X) = \mathcal{D}(F(X), Y)$. We define a natural transformation $\theta : h_{G(Y)} \to h_{G(Z)}$ between representable functors to be a composition $h_{G(Y)} \xrightarrow{\varphi(Y)^{-1}} h_Y^F \xrightarrow{h_f^F} h_Z^F \xrightarrow{\varphi(Z)} h_{G(Z)}$. It follows from Yoneda's lemma that θ is induced by a morphism $G(f) : G(Y) \to G(Z)$ of $\mathcal{C}$ which is the image of $f\varepsilon_Y \in \mathcal{D}(F(G(Y)), Z)$ by $\varphi(Z)_{G(Y)} : \mathcal{D}(F(G(Y)), Z) \to \mathcal{C}(G(Y), G(Z))$. Thus we have the following fact.

Proposition A.2.2 *The following diagram is commutative for any $X \in \operatorname{Ob}\mathcal{C}$.*

$$\begin{tikzcd}
\mathcal{D}(F(X), Y) \arrow[r, "\varphi(Y)_X"] \arrow[d, "f_*"] & \mathcal{C}(X, G(Y)) \arrow[d, "G(f)_*"] \\
\mathcal{D}(F(X), Z) \arrow[r, "\varphi(Z)_X"] & \mathcal{C}(X, G(Z))
\end{tikzcd}$$

For a morphism $g : Z \to W$ of $\mathcal{D}$, it follows from $G(gf) = \varphi(W)_{G(Y)}(gf\varepsilon_Y)$ and the commutativity of the following diagram that $G(gf) = G(g)G(f)$.

$$\begin{tikzcd}
\mathcal{D}(F(G(Y)), Y) \arrow[r, "f_*"] \arrow[d, "\varphi(Y)_{G(Y)}"] & \mathcal{D}(F(G(Y)), Z) \arrow[r, "g_*"] \arrow[d, "\varphi(Z)_{G(Y)}"] & \mathcal{D}(F(G(Y)), W) \arrow[d, "\varphi(W)_{G(Y)}"] \\
\mathcal{C}(G(Y), G(Y)) \arrow[r, "G(f)_*"] & \mathcal{C}(G(Y), G(Z)) \arrow[r, "G(g)_*"] & \mathcal{C}(G(Y), G(W))
\end{tikzcd}$$

Since $G(id_Y)$ is the identity morphism of $G(Y)$ by the definition of $G(id_Y)$, we have a functor $G : \mathcal{D} \to \mathcal{C}$ which is called a right adjoint of F.

Proposition A.2.3 *A composition $G(Y) \xrightarrow{\eta_{G(Y)}} G(F(G(Y))) \xrightarrow{G(\varepsilon_Y)} G(Y)$ is the identity morphism of $G(Y)$.*

Proof. By the commutativity of

$$\begin{tikzcd}
\mathcal{D}(F(G(Y)), F(G(Y))) \arrow[r, "\varphi(F(G(Y)))_{G(Y)}"] \arrow[d, "\varepsilon_{Y*}"] & \mathcal{C}(G(Y), G(F(G(Y)))) \arrow[d, "G(\varepsilon_Y)_*"] \\
\mathcal{D}(F(G(Y)), Y) \arrow[r, "\varphi(Y)_{G(Y)}"] & \mathcal{C}(G(Y), G(Y))
\end{tikzcd},$$

we have $G(\varepsilon_Y)\eta_{G(Y)} = G(\varepsilon_Y)\varphi(F(G(Y)))_{G(Y)}(id_{F(G(Y))}) = \varphi(Y)_{G(Y)}(\varepsilon_Y) = id_{G(Y)}$. □

Proposition A.2.4 *The following diagrams commute for morphisms $f : X \to Y$ in $\mathcal{C}$ and $g : W \to Z$ in $\mathcal{D}$.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\eta_X"] & Y \arrow[d, "\eta_Y"] \\
G(F(X)) \arrow[r, "G(F(f))"] & G(F(Y))
\end{tikzcd}
\qquad
\begin{tikzcd}
F(G(Z)) \arrow[r, "F(G(g))"] \arrow[d, "\varepsilon_Z"] & F(G(W)) \arrow[d, "\varepsilon_W"] \\
Z \arrow[r, "g"] & W
\end{tikzcd}$$

Proof. The right rectangle of the following diagram commutes by the naturality of $\varphi(F(Y))$ and it follows from (A.2.2) that the left rectangle is commutative.

$$\begin{tikzcd}
\mathcal{D}(F(X),F(X)) \arrow[r, "F(f)_*"] \arrow[d, "\varphi(F(X))_X"] & \mathcal{D}(F(X),F(Y)) \arrow[d, "\varphi(F(Y))_X"] & \mathcal{D}(F(Y),F(Y)) \arrow[l, "F(f)^*"'] \arrow[d, "\varphi(F(Y))_Y"] \\
\mathcal{C}(X,G(F(X))) \arrow[r, "G(F(f))_*"] & \mathcal{C}(X,G(F(Y))) & \mathcal{D}(Y,G(F(Y))) \arrow[l, "f^*"']
\end{tikzcd}$$

Hence we have

$$\begin{aligned}G(F(f))\eta_X &= G(F(f))_*(\varphi(F(X))_X(id_{F(X)})) = \varphi(F(Y))_X(F(f)_*(id_{F(X)})) = \varphi(F(Y))_X(F(f)) \\ &= \varphi(F(Y))_X(F(f)^*(id_{F(Y)})) = f^*(\varphi(F(Y))_Y(id_{F(Y)})) = \eta_Y f.\end{aligned}$$

The the left rectangle of the following diagram is commutative by the naturality of $\varphi(W)$ and the right rectangle is commutative by (A.2.2).

$$\begin{tikzcd}
\mathcal{D}(F(G(W)),W) \arrow[r, "F(G(g))^*"] \arrow[d, "\varphi(W)_{G(W)}"] & \mathcal{D}(F(G(Z)),W) \arrow[d, "\varphi(W)_{G(Z)}"] & \mathcal{D}(F(G(Z)),Z) \arrow[l, "g_*"'] \arrow[d, "\varphi(Z)_{G(Z)}"] \\
\mathcal{C}(G(W),G(W)) \arrow[r, "G(g)^*"] & \mathcal{C}(G(Z),G(W)) & \mathcal{C}(G(Z),G(Z)) \arrow[l, "G(g)_*"']
\end{tikzcd}$$

Therefore we have

$$\begin{aligned}\varphi(W)_{G(Z)}(\varepsilon_W F(G(g))) &= \varphi(W)_{G(Z)}(F(G(g))^*(\varepsilon_W)) = G(g)^*(\varphi(W)_{G(W)}(\varepsilon_W)) = G(g)^*(id_{G(W)}) = G(g) \\ &= G(g)_*(id_{G(Z)}) = G(g)_*(\varphi(Z)_{G(Z)}(\varepsilon_Z)) = \varphi(W)_{G(Z)}(g_*(\varepsilon_Z)) = \varphi(W)_{G(Z)}(g\varepsilon_Z).\end{aligned}$$

Since $\varphi(W)_{G(Z)} : \mathcal{D}(F(G(Z)),W) \to \mathcal{C}(G(Z),G(W))$ is bijective, the assertion follows. □

Thus we have a natural transformations $\eta : id_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to id_{\mathcal{D}}$. η is called the unit and ε is called the counit of the adjunction.

Proposition A.2.5 *The natural bijection $\varphi(Y)_X : \mathcal{D}(F(X),Y) \to \mathcal{C}(X,G(Y))$ and its inverse is given by $\varphi(Y)_X(\alpha) = G(\alpha)\eta_X$ and $\varphi(Y)_X^{-1}(\beta) = \varepsilon_Y F(\beta)$ for $\alpha \in \mathcal{D}(F(X),Y)$ and $\beta \in \mathcal{C}(X,G(Y))$, respectively.*

Proof. The commutativity of the following left diagram implies $\varphi(Y)_X(\alpha) = G(\alpha)\eta_X$ and the commutativity of the following right diagram implies $\varphi(Y)_X^{-1}(\beta) = \varepsilon_Y F(\beta)$.

$$\begin{tikzcd}
\mathcal{D}(F(X),F(X)) \arrow[r, "\varphi(F(X))_X"] \arrow[d, "\alpha_*"] & \mathcal{C}(X,G(F(X))) \arrow[d, "G(\alpha)_*"] \\
\mathcal{D}(F(X),Y) \arrow[r, "\varphi(Y)_X"] & \mathcal{C}(X,G(Y))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D}(F(G(Y)),Y) \arrow[r, "\varphi(Y)_{G(Y)}"] \arrow[d, "F(\beta)^*"] & \mathcal{C}(G(Y),G(Y)) \arrow[d, "\beta^*"] \\
\mathcal{D}(F(X),Y) \arrow[r, "\varphi(Y)_X"] & \mathcal{C}(X,G(Y))
\end{tikzcd}$$

□

A.3 Miscellaneous results

Let $\mathcal{C}$ be a category.

Proposition A.3.1 *Suppose that the following diagram on the left is a cartesian square in $\mathcal{C}$. Then the diagram in the middle is a cartesian square if and only if the right diagram is so.*

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "q"] & Z \arrow[d, "r"] \\
V \arrow[r, "k"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[d, "q"] \\
U \arrow[r, "h"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "gf"] \arrow[d, "p"] & Z \arrow[d, "r"] \\
U \arrow[r, "kh"] & W
\end{tikzcd}$$

Proof. An easy diagram chasing. □

Proposition A.3.2 *1) If $E \xrightarrow{e} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is an equalizer and $e' : V \to W$ is a pull-back of e along a morphism $h : Z \to X$, then e' is an equalizer of fh and gh.*

2) If $R \overset{f}{\underset{g}{\rightrightarrows}} X$ is an effective equivalence relation and its coequalizer exists, then it is a kernel pair of its coequalizer.

3) A pair of morphisms $Z \overset{f}{\underset{g}{\rightrightarrows}} X$ is a kernel pair of a morphism $p : X \to Y$ if and only if

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & X \arrow[d, "p"] \\ X \arrow[r, "p"] & Y \end{tikzcd}$$

is a pull-back diagram. Hence if $\mathcal{C}$ is a category with pull-backs, a kernel pair of a morphism always exists.

4) The following conditions on a morphism $f : X \to Y$ are equivalent.

(1) f is a monomorphism.

(2) f has a kernel pair of the form $Z \overset{g}{\underset{g}{\rightrightarrows}} X$.

(3) $X \overset{id_X}{\underset{id_X}{\rightrightarrows}} X$ is a kernel pair of f.

(4) If $Z \overset{g}{\underset{h}{\rightrightarrows}} X$ is a kernel pair of f and $\Delta : X \to Z$ is a unique morphism satisfying $g\Delta = h\Delta = id_X$, then Δ is an epimorphism.

Hence a monomorphism is preserved by a left exact functor.

Proof. 1) An easy diagram chasing.

2) Suppose that $R \overset{f}{\underset{g}{\rightrightarrows}} X$ is a kernel pair of $f : X \to Z$ and let $p : X \to Y$ be the coequalizer. Then, there is a unique morphism $g : Y \to Z$ such that $gp = f$. The result follows from (A.3.6) below.

3), 4) Straightforward form the definition. □

Proposition A.3.3 *1) A faithful functor reflects monomorphic families and epimorphic families.*

2) A fully faithful functor F reflects limits and colimits, hence in particular, isomorphisms.

3) If $F : \mathcal{C} \to \mathcal{A}$ is an equivalence, F preserves and reflects limits and colimits.

Proof. Direct consequences from the definitions. □

Proposition A.3.4 *Let $\sigma_1 : Y_1 \rightarrowtail X$ and $\sigma_2 : Y_2 \rightarrowtail X$ be monomorphisms. There exists a morphism $\iota : Y_1 \to Y_2$ satisfying $\sigma_2\iota = \sigma_1$ if and only if the image of $h_{\sigma_1} : h_{Y_1} \to h_X$ is contained in that of $h_{\sigma_2} : h_{Y_2} \to h_X$. Hence σ_1 and σ_2 are equivalent monomorphisms if and only if these images coincide.*

Proof. If there exists such ι, $h_{\sigma_1} = h_{\sigma_2}h_\iota$ implies the assertion. Conversely, suppose that the image of $h_{\sigma_1} : h_{Y_1} \to h_X$ is contained in that of $h_{\sigma_2} : h_{Y_2} \to h_X$. Then, $\sigma_1 = h_{\sigma_1}(id_{Y_1})$ is contained in $h_{\sigma_2}(h_{Y_2}(Y_1))$. Hence there exists $\iota : Y_1 \to Y_2$ satisfying $\sigma_2\iota = \sigma_1$. □

Proposition A.3.5 *If $g : Z \to W$ is a monomorphism and the following diagram on the left is commutative. Then, the diagram on the right is a cartesian square.*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "h"] & X \arrow[d, "k"] \\ Z \arrow[r, "g"] & W \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "{(h,id_Y)}"] & X \arrow[d, "{(k,id_X)}"] \\ Z \times Y \arrow[r, "g\times f"] & W \times X \end{tikzcd}$$

Proof. Let $a : U \to X$ and $b : U \to Z \times Y$ be morphisms satisfying $(k, id_X)a = (g \times id_Y)b$. Then, we have $ka = g\mathrm{pr}_1 b$ and $a = f\mathrm{pr}_2 b$ and it follows that $g\mathrm{pr}_1 b = ka = kf\mathrm{pr}_2 b = gh\mathrm{pr}_2 b$. Since g is a monomorphism, $\mathrm{pr}_1 b = h\mathrm{pr}_2 b$. Hence $(h, id_Y)\mathrm{pr}_2 b = (h\mathrm{pr}_2 b, \mathrm{pr}_2 b) = (\mathrm{pr}_1 b, \mathrm{pr}_2 b) = b$. If $c : U \to Y$ satisfies $(h, id_Y)c = b$, $\mathrm{pr}_2 b = \mathrm{pr}_2(h, id_Y)c = c$. Therefore $\mathrm{pr}_2 b : U \to Y$ is the unique morphism satisfying $f\mathrm{pr}_2 b = a$ and $(h, id_Y)\mathrm{pr}_2 b = b$. □

Proposition A.3.6 *Let $i : W \to V$ be a morphism and consider the following commutative squares.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "k"] \\ Z \arrow[r, "g"] & W \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "ik"] \\ Z \arrow[r, "ig"] & V \end{tikzcd}$$

1) If the above diagram on the right is a cartesian square, so is the diagram on the left.

2) If $i : W \to V$ is a monomorphism and the left diagram is a cartesian square, so is the diagram on the right.

3) Suppose that g and k are universal epimorphisms whose pull-backs of a morphism always exist. If the above diagram on the right is a cartesian square, i is a monomorphism.

Proof. 1) and 2) follow from an easy diagram chasing. For 3), let $\alpha, \beta : U \to W$ be morphisms such that $i\alpha = i\beta$. By the assumption, there are following pull-backs.

$$
\begin{tikzcd}
A \arrow[r, "\bar{\alpha}"] \arrow[d, "\bar{k}"] & Y \arrow[d, "k"] \\
U \arrow[r, "\alpha"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\bar{g}"] \arrow[d, "\bar{\beta}"] & U \arrow[d, "\beta"] \\
Z \arrow[r, "g"] & W
\end{tikzcd}
$$

Since a pull-back of k along $\alpha\bar{g}$ exist, pull-back of $\bar{k}$ along $\bar{g}$ also exists by (A.3.1). Hence we have the following commutative square whose edges are all epimorphisms.

$$
\begin{tikzcd}
C \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{k}"] & A \arrow[d, "\bar{k}"] \\
A \arrow[r, "\bar{g}"] & U
\end{tikzcd}
$$

Then, $ik\bar{\alpha}\tilde{g} = i\alpha\bar{k}\tilde{g} = i\beta\bar{g}\tilde{k} = ig\bar{\beta}\tilde{k}$. Hence there exists a unique morphism $\gamma : C \to X$ satisfying $f\gamma = \bar{\alpha}\tilde{g}$ and $h\gamma = \bar{\beta}\tilde{k}$. Thus we have $\alpha\bar{k}\tilde{g} = k\bar{\alpha}\tilde{g} = kf\gamma = gh\gamma = g\bar{\beta}\tilde{k} = \beta\bar{g}\tilde{k} = \beta\bar{k}\tilde{g}$. Since both $\bar{k}$ and $\tilde{g}$ are epimorphisms, we have $\alpha = \beta$. □

Proposition A.3.7 *Let $\boldsymbol{G} = (G, \eta, \mu)$ be a monad on $\mathcal{C}$, then the forgetful functor $U : \mathcal{C}^{\boldsymbol{G}} \to \mathcal{C}$ creates limits.*

Proof. Let $D : \mathcal{D} \to \mathcal{C}^{\boldsymbol{G}}$ be a functor and $(g_i : M \to UD(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$ be a limiting cone in $\mathcal{C}$. Put $D(i) = \langle D_i, \nu_i \rangle$, then $(\nu_i G(g_i) : G(M) \to UD(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a cone in $\mathcal{C}$. Hence there exists a unique morphism $\nu : G(M) \to M$ satisfying $\nu_i G(g_i) = g_i \nu$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. Then, $g_i \nu G(\nu) = \nu_i G(g_i) G(\nu) = \nu_i G(\nu_i G(g_i)) = \nu_i \mu_{D_i} G^2(g_i) = \nu_i G(g_i) \mu_M = g_i G(\nu) \mu_M$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. Thus $\nu G(\nu) = G(\nu)\mu_M$ and $\langle M, \nu \rangle$ is a $\boldsymbol{G}$-algebra. Since $\nu_i G(g_i) = g_i \nu$, g_i gives a morphism of $\boldsymbol{G}$-algebras. □

Proposition A.3.8 *Let $\mathcal{C}$ be a category and X, Y objects of $\mathcal{C}$. Suppose that there is an isomorphism $\varphi_Z : \mathcal{C}(Z, X) \to \mathcal{C}(Z, Y)$ for each object Z which is natural in Z. Then $\varphi_X(id_X) : X \to Y$ is an isomorphism with inverse $\varphi_Y^{-1}(id_Y)$.*

Proof. By the naturality, the following squares commute.

$$
\begin{tikzcd}
\mathcal{C}(X,X) \arrow[r, "\varphi_X"] \arrow[d, "\varphi_Y^{-1}(id_Y)^*"] & \mathcal{C}(X,Y) \arrow[d, "\varphi_Y^{-1}(id_Y)^*"] \\
\mathcal{C}(Y,X) \arrow[r, "\varphi_Y"] & \mathcal{C}(Y,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(Y,Y) \arrow[r, "\varphi_Y^{-1}"] \arrow[d, "\varphi_X(id_X)^*"] & \mathcal{C}(Y,X) \arrow[d, "\varphi_X(id_X)^*"] \\
\mathcal{C}(X,Y) \arrow[r, "\varphi_X^{-1}"] & \mathcal{C}(X,X)
\end{tikzcd}
$$

The commutativity of the left square implies $\varphi_X(id_X)\varphi_Y^{-1}(id_Y) = id_Y$ and the right one implies $\varphi_Y^{-1}(id_Y)\varphi_X(id_X) = id_X$. □

Proposition A.3.9 *1) Let $\mathcal{C}$ be a category and $f : X \to Y$ a morphism of $\mathcal{C}$ such that, for any morphism $g : Z \to Y$, a pull-back $Z \times_Y X \to X$ of g along f exists. This gives a pull-back functor $f^* : \mathcal{C}/Y \to \mathcal{C}/X$ sending $(Z \xrightarrow{g} Y)$ to $(Z \times_Y X \xrightarrow{\mathrm{pr}_2} X)$. Then, $f^* : \mathcal{C}/Y \to \mathcal{C}/X$ has a left adjoint $\Sigma_f : \mathcal{C}/X \to \mathcal{C}/Y$.*

2) Let $\mathcal{C}$ be a category and X be an object of $\mathcal{C}$ such that, for any object Y, a product $Y \times X$ exists. This gives a functor $X^* : \mathcal{C} \to \mathcal{C}/X$ sending Y to $(Y \times X \xrightarrow{\mathrm{pr}_2} X)$. Then, $X^* : \mathcal{C} \to \mathcal{C}/X$ has a left adjoint $\Sigma_X : \mathcal{C}/X \to \mathcal{C}$.

Proof. Define Σ_f by $\Sigma_f(Z \xrightarrow{g} X) = (Z \xrightarrow{g} X \xrightarrow{f} Y)$ and $\Sigma_f(\varphi : g \to h) = (\varphi : fg \to fh)$ for $Z \xrightarrow{g} X$ and $W \xrightarrow{h} X$. Define a map $\mathcal{C}/Y(\Sigma_f(Z \xrightarrow{g} X), (W \xrightarrow{h} Y)) \to \mathcal{C}/X((Z \xrightarrow{g} X), f^*(W \xrightarrow{h} Y))$ by $(\varphi : Z \to W) \mapsto ((\varphi, g) : Z \to W \times_Y X)$. The inverse of this map is given by $(\psi : Z \to W \times_Y X) \mapsto (\mathrm{pr}_1\psi : Z \to W)$.

The proof of 2) is similar. □

If $\mathcal{C}$ has a terminal object 1, $\mathcal{C}/1$ is identified with $\mathcal{C}$. If f is the unique morphism $X \to 1$, f^* and Σ_f are identified with X^* and Σ_X, respectively.

The unit $\eta_f : 1_{\mathcal{C}/X} \to f^*\Sigma_f$ and the counit $\varepsilon_f : \Sigma_f f^* \to 1_{\mathcal{C}/Y}$ of this adjunction is given as follows. For $Z \xrightarrow{g} X$, $(id_Z, g) : Z \to Z \times_Y X$ defines a morphism $(\eta_f)_g : (Z \xrightarrow{g} X) \to f^*\Sigma_f(Z \xrightarrow{g} X) = (Z \times_Y X \xrightarrow{g \times id_X} X \times_Y X \xrightarrow{\mathrm{pr}_2} X)$. For $W \xrightarrow{h} Y$, $\mathrm{pr}_1 : W \times_Y X \to W$ defines a morphism $(\varepsilon_f)_h : (W \times_Y X \xrightarrow{\mathrm{pr}_2} X \xrightarrow{f} Y) \to (W \xrightarrow{h} Y)$.

$$\begin{tikzcd} Z \times_Y X \arrow[r, "g \times id_X"] \arrow[d, "\mathrm{pr}_1"] & X \times_Y X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} W \times_Y X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & X \arrow[d, "f"] \\ W \arrow[r, "h"] & Y \end{tikzcd}$$

Definition A.3.10 *A category is said to be connected if, for any pair (X, Y) of objects, there exist a finite number of objects $X = X_0, X_1, X_2, \ldots, X_n = Y$ such that $\mathcal{C}(X_{2i-2}, X_{2i-1}) \neq \emptyset$ and $\mathcal{C}(X_{2i}, X_{2i-1}) \neq \emptyset$ for $0 \leq i \leq [\frac{n}{2}]$.*

Proposition A.3.11 *1) $\Sigma_f : \mathcal{C}/X \to \mathcal{C}/Y$ (resp. $\Sigma_X : \mathcal{C}/X \to \mathcal{C}$) creates limits of functors from connected categories and arbitrary colimits.*

2) $\Sigma_f : \mathcal{C}/X \to \mathcal{C}/Y$ (resp. $\Sigma_X : \mathcal{C}/X \to \mathcal{C}$) preserves monomorphic families.

3) $\Sigma_X : \mathcal{C}/X \to \mathcal{C}$ reflects regular epimorphisms.

Proof. 1) Let $\mathcal{D}$ be a connected category and $D : \mathcal{D} \to \mathcal{C}/X$ a functor such that $D(i) = (X_i \xrightarrow{p_i} X)$. Suppose that $((L \xrightarrow{p} Y) \xrightarrow{\pi_i} (X_i \xrightarrow{fp_i} Y))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of $\Sigma_f D : \mathcal{D} \to \mathcal{C}/Y$. For a morphism $\theta : i \to j$ in $\mathcal{D}$, we have $p_j\pi_j = p_j D(\theta)\pi_i = p_i\pi_i$. Since $\mathcal{D}$ is connected, it follows that $p_j\pi_j = p_i\pi_i$ for any pair of objects (i, j) of $\mathcal{D}$. We set $q = p_i\pi_i$, then it is easy to verify that $((L \xrightarrow{q} X) \xrightarrow{\pi_i} (X_i \xrightarrow{p_i} X))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of $D : \mathcal{D} \to \mathcal{C}/X$.

Let $D : \mathcal{D} \to \mathcal{C}/X$ be a functor such that $D(i) = (X_i \xrightarrow{p_i} X)$. Suppose that $(\Sigma_f D(i) \xrightarrow{\lambda_i} (L \xrightarrow{p} Y))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $\Sigma_f D : \mathcal{D} \to \mathcal{C}/Y$. Since $(p_i : (X_i \xrightarrow{fp_i} Y) \to (X \xrightarrow{f} Y))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a cone in $\mathcal{C}/Y$, there is a unique morphism $q : L \to X$ such that $fq = p$ and $q\lambda_i = p_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. We show that $(D(i) \xrightarrow{\lambda_i} (L \xrightarrow{q} X))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of D. Suppose that $(D(i) \xrightarrow{\mu_i} (Z \xrightarrow{r} X))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a cone of D. Then $(\Sigma_f D(i) \xrightarrow{\mu_i} (Z \xrightarrow{fr} Y))_{i \in \mathrm{Ob}\,\mathcal{D}}$ is a cone of $\Sigma_f D$ and there is a unique morphism $s : (L \xrightarrow{p} Y) \to (Z \xrightarrow{fr} Y)$ in $\mathcal{C}/Y$ such that $\mu_i = s\lambda_i$. Since $r\mu_i = p_i$ and $frs = p$, the uniqueness of q implies that $rs = q$. Hence $s : (L \xrightarrow{q} X) \to (Z \xrightarrow{r} Z)$ is a morphism in $\mathcal{C}/X$.

2) Let $(s_i : (Z \xrightarrow{t} X) \to (X_i \xrightarrow{f_i} X))_{i \in I}$ be a monomorphic family in $\mathcal{C}/X$. Suppose that $g, h : (W \xrightarrow{u} Y) \to (Z \xrightarrow{ft} Y)$ are morphisms in $\mathcal{C}/Y$ such that $s_i g = s_i h$ for any $i \in I$ in $\mathcal{C}/Y$. Set $v = tg : W \to X$. Since $tg = f_i s_i g = f_i s_i h = th$, $g, h : (W \xrightarrow{v} X) \to (Z \xrightarrow{t} X)$ are morphisms in $\mathcal{C}/X$ such that $s_i g = s_i h$ for any $i \in I$ in $\mathcal{C}/X$. Therefore we have $g = h$ by the assumption.

3) Let $g : (Y \xrightarrow{p} X) \to (Z \xrightarrow{q} X)$ be a morphism in $\mathcal{C}/X$ such that $\Sigma_X(g) : Y \to Z$ is a regular epimorphism. Suppose that $\Sigma_X(g)$ is a coequalizer of $W \overset{a}{\underset{b}{\rightrightarrows}} Y$. Then, $pa = q\Sigma_X(g)a = q\Sigma_X(g)b = pb$ and we set $r = pa : W \to X$. Hence we have a pair of morphisms $a, b : (W \xrightarrow{r} X) \to (Y \xrightarrow{p} X)$ in $\mathcal{C}/X$. We claim that g is a coequalizer of them. Let $h : (Y \xrightarrow{p} X) \to (U \xrightarrow{s} X)$ be a morphism satisfying $ha = hb$. There is a unique morphism $t : Z \to U$ in $\mathcal{C}$ such that $t\Sigma_X(g) = \Sigma_X(h)$. Then we have $st\Sigma_X(g) = s\Sigma_X(h) = p = q\Sigma_X(g)$. Since $\Sigma_X(g)$ is an epimorphism, $st = q$ and we have a morphism $t : (Z \xrightarrow{q} X) \to (U \xrightarrow{s} X)$ such that $tg = h$. □

We remark that, since Σ_f and Σ_X do not preserve terminal objects, they do not preserve products.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor. For an object X of $\mathcal{C}$, we denote by $F/X : \mathcal{C}/X \to \mathcal{C}'/F(X)$ the functor given by $(Z \xrightarrow{p} X) \mapsto (F(Z) \xrightarrow{F(p)} F(X))$.

Proposition A.3.12 *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor and $G : \mathcal{C}' \to \mathcal{C}$ a left adjoint of F. We denote by $\eta : id_{\mathcal{C}'} \to FG$, $\varepsilon : GF \to id_{\mathcal{C}}$ the unit, counit of the adjunction.*

1) For an object X of $\mathcal{C}$, $\Sigma_{\varepsilon_X}(G/F(X)) : \mathcal{C}'/F(X) \to \mathcal{C}/X$ is a left adjoint of F/X.

2) Suppose that a morphism $\eta_Y : Y \to FG(Y)$ in $\mathcal{C}'$ has a pull-back along an arbitrary morphism. Then, $\eta_Y^*(F/G(Y)) : \mathcal{C}/G(Y) \to \mathcal{C}'/Y$ is a right adjoint of $G/Y : \mathcal{C}'/Y \to \mathcal{C}/G(Y)$.

3) Under the assumption of 2), if, for any $(W \xrightarrow{q} G(Y)) \in \mathrm{Ob}\,\mathcal{C}/G(Y)$, G preserves a pull-back of $F(q) : F(W) \to FG(Y)$ along η_Y and the following square is a pull-back (for example, ε is a natural equivalence), $(G/Y)\eta_Y^*(F/G(Y))$ is naturally equivalent to the identity functor of $\mathcal{C}/G(Y)$.

$$\begin{tikzcd} GF(W) \arrow[r, "\varepsilon_W"] \arrow[d, "GF(q)"] & W \arrow[d, "q"] \\ GFG(Y) \arrow[r, "\varepsilon_{G(Y)}"] & G(Y) \end{tikzcd}$$

Proof. 1) Let $(Z \xrightarrow{p} F(X))$ be an object of $\mathcal{C}'/F(X)$ and $(W \xrightarrow{q} X)$ an object of $\mathcal{C}/X$. By the adjointness of G and F, the natural bijection $\mathcal{C}(G(Z), W) \to \mathcal{C}'(Z, F(W))$ induces a bijection $\mathcal{C}/X(\Sigma_{\varepsilon_X}(G/F(X))(Z \xrightarrow{p} F(X)), (W \xrightarrow{q} X)) \to \mathcal{C}'/F(X)((Z \xrightarrow{p} F(X)), (F/X)(W \xrightarrow{q} X))$.

2) Let $(Z \xrightarrow{p} Y)$ be an object of $\mathcal{C}'/Y$ and $(W \xrightarrow{q} G(Y))$ an object of $\mathcal{C}/G(Y)$. η_Y^* has a left adjoint Σ_{η_Y} (A.3.9), there is a natural bijection $(\mathcal{C}'/Y)((Z \xrightarrow{p} Y), \eta_Y^*(F/G(Y))(W \xrightarrow{q} G(Y))) \to (\mathcal{C}'/FG(Y))(\Sigma_{\eta_Y}(Z \xrightarrow{p} Y), (F/G(Y))(W \xrightarrow{q} G(Y)))$. Moreover, $F/G(Y) : \mathcal{C}/G(Y) \to \mathcal{C}'/FG(Y)$ has a left adjoint $\Sigma_{\varepsilon_{G(Y)}}(G/FG(Y))$. Since $\varepsilon_{G(Y)}G(\eta_Y) = id_{G(Y)}$, we have $G/Y = \Sigma_{\varepsilon_{G(Y)}}(G/FG(Y))\Sigma_{\eta_Y}$.

3) For $(W \xrightarrow{q} G(Y)) \in \operatorname{Ob}\mathcal{C}/G(Y)$, set $(\bar{W} \xrightarrow{\bar{q}} Y) = \eta_Y^*(F(W) \xrightarrow{F(q)} FG(Y))$. By the assumption, both squares of the following diagram is a pull-back.

$$\begin{tikzcd}
G(\bar{W}) \arrow[r] \arrow[d, "G(\bar{q})"] & GF(W) \arrow[r, "\varepsilon_W"] \arrow[d, "GF(q)"] & W \arrow[d, "q"] \\
G(Y) \arrow[r, "G(\eta_Y)"] & GFG(Y) \arrow[r, "\varepsilon_{G(Y)}"] & G(Y)
\end{tikzcd}$$

Note that the composition of the lower row of the above diagram is the identity morphism of $G(Y)$. Since the outer rectangle is a pull-back by (A.3.1), the composition of the upper row is an isomorphism. Hence the assertion follows. □

Proposition A.3.13 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.*

1) If F has a left adjoint, F preserves limits and in particular, terminal objects. Moreover, F preserves monomorphic families and strict monomorphic families.

2) If F has a right adjoint, F preserves colimits and in particular, initial objects. Moreover, F preserves epimorphic families and strict epimorphic families.

Proof. 1) We only show that F preserves strict monomorphic families. Other statements are straightforward. Let $(f_i : X \to X_i)_{i\in I}$ be a strict epimorphic family in $\mathcal{C}$. For $Y \in \operatorname{Ob}\mathcal{D}$, suppose that $(g_i)_{i\in I} \in \prod_{i\in I} \mathcal{D}(Y, F(X_i))$ satisfies "$ug_i = vg_j$ *holds for any* $i, j \in I$ *and morphisms* $u : F(X_i) \to W$, $v : F(X_j) \to W$ *in* $\mathcal{D}$ *such that* $uF(f_i) = vF(X_j)$". Let us denote by $g_i' : L(Y) \to X_i$ the adjoint of g_i. Then, for any $i, j \in I$ and morphisms $p : X_i \to U$, $q : X_j \to U$ in $\mathcal{C}$ such that $pf_i = qf_j$, we have $F(p)g_i = F(q)g_j$, namely, $pg_i' = qg_j'$. Hence there exists a morphism $g' : L(Y) \to X$ such that $g_i' = g'f_i$ for any $i \in I$. The adjoint $g : Y \to F(X)$ of g' satisfies $g_i = gF(f_i)$ for any $i \in I$. Therefore $(F(f_i) : F(X) \to F(X_i))_{i\in I}$ is a strict epimorphic family in $\mathcal{C}$.

2) The dual of 1). □

Proposition A.3.14 *If $(\mathcal{C}, \otimes)$ is a closed monoidal category, the functor $(-) \otimes X : \mathcal{C} \to \mathcal{C}$ preserves colimits for any object X of $\mathcal{C}$.*

Proof. A right adjoint of $(-) \otimes X$ exists by definition. □

Proposition A.3.15 *If 0 is an initial object of a category and $i : X \to 0$ is a monomorphism, then i is an isomorphism.*

Proof. Let $r : 0 \to X$ be the unique morphism, then $ir = id_0$. Hence we have $iri = i$. Since i is a monomorphism, it follows $ri = id_X$. □

Proposition A.3.16 *Let $\mathcal{C}$ be a category.*

1) Let X be a universally disjoint coproduct of a family of objects $(X_i)_{i\in I}$ of $\mathcal{C}$. Then, if $i \neq j$, $X_i \times_X X_j$ is a strict initial object of $\mathcal{C}$.

2) Suppose that $\mathcal{C}$ has pull-backs and universal coproducts. Then, every coproduct in $\mathcal{C}$ is disjoint if and only if, for each pair of objects X and Y, the following is a pull-back diagram.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Y \arrow[d, "\nu_2"] \\
X \arrow[r, "\nu_1"] & X \coprod Y
\end{tikzcd}$$

Proof. 1) Suppose that there is a morphism $f : Y \to X_i \times_X X_j$. Consider the following diagram, where each square is a pull-back.

$$\begin{array}{ccccccc} Y\times_X X_k & \xrightarrow{f\times 1_{X_k}} & X_i\times_X X_j\times_X X_k & \longrightarrow & X_i\times_X X_k & \xrightarrow{\mathrm{pr}_2} & X_k \\ \downarrow{\scriptstyle \mathrm{pr}_{Yk}} & & \downarrow{\scriptstyle \mathrm{pr}_{12}} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \nu_k} \\ Y & \xrightarrow{f} & X_i\times_X X_j & \longrightarrow & X_i & \xrightarrow{\nu_i} & X \end{array}$$

Since ν_k is a monomorphism, so is the each vertical morphisms. By (A.3.15), pr_{12} in the above diagram is an isomorphism, hence so is pr_{Yk}. But, since Y is a disjoint coproduct of $Y\times_X X_k$'s, we have the following pull-back diagram, where all arrows are isomorphisms.

$$\begin{array}{ccc} 0 & \longrightarrow & Y\times_X X_j \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{Yj}} \\ Y\times_X X_i & \xrightarrow{\mathrm{pr}_{Yi}} & Y \end{array}$$

2) The "only if" part is obvious from the definition. First, we show that the canonical morphisms $\nu_1 : X \to X\coprod Y$, $\nu_2 : Y \to X\coprod Y$ are monomorphisms. Let $Z \underset{t}{\overset{s}{\rightrightarrows}} X$ be the kernel pair of ν_1. Then, there is a unique morphism $e : X \to Z$ such that $se = te = id_X$. Since the unique morphism $c : 0 \to X$ is a pull-back of ν_2 along ν_1 by the assumption and s is a pull-back of ν_1 along ν_1, it follows from the universality of coproducts that X is a coproduct of 0 and Z. Thus $id_Z : Z \to Z$ and $ec : 0 \to Z$ induce a unique morphism $f : X \to Z$ satisfying $fs = id_Z$ and $fc = ec$. Hence $f = fid_X = fse = id_Ze = e$ and it follows that s is an isomorphism. By (A.3.2), ν_1 is a monomorphism. Similarly, ν_2 is also a monomorphism.

Let $(\nu_i : X_i \to \coprod_{i\in I} X_i)_{i\in I}$ be a coproduct in $\mathcal{C}$. Since $\coprod_{k\in I} X_k \cong X_i \coprod(\coprod_{k\neq i} X_k)$, each s_i is a monomorphism by the previous result. If i and j are distinct indices in I, it follows from $\coprod_{k\in I} X_k \cong (X_i \coprod X_j)\coprod(\coprod_{k\neq i,j} X_k)$ and the previous result that $X_i \coprod X_j$ is a subobject of $\coprod_{k\in I} X_k$. Therefore (A.3.6) implies that the pull-back of ν_j along ν_i is the unique morphism $0 \to X_i$. □

Proposition A.3.17 *Let $(s_i : X \to X_i)_{i\in I}$ be a family of morphisms in a category $\mathcal{C}$ such that a kernel pair $Z_i \underset{b_i}{\overset{a_i}{\rightrightarrows}} X$ of s_i exists for each $i \in I$. Set $I_* = I\cup\{0,\infty\}$ and $Z_0 = Z_\infty = X$, $a_0 = b_0 = a_\infty = b_\infty = id_X$ and we denote by $\Delta_i : X \to Z_i$ the unique morphism satisfying $a_i\Delta_i = b_i\Delta_i = id_X$ for $i \in I_*$. Then, $(s_i : X \to X_i)_{i\in I}$ is a monomorphic family if and only if $(X \xrightarrow{\Delta_i} Z_i)_{i\in I_*}$ is a limiting cone of a diagram $(Z_i \xrightarrow{a_i} Z_0,\ Z_i \xrightarrow{b_i} Z_\infty)_{i\in I}$.*

Proof. Suppose that $(s_i : X \to X_i)_{i\in I}$ is a monomorphic family. Let $(Y \xrightarrow{f_i} Z_i)_{i\in I_*}$ be a cone of the diagram $(Z_i \xrightarrow{a_i} Z_0,\ Z_i \xrightarrow{b_i} Z_\infty)_{i\in I}$. Then, $a_if_i = f_0$ and $b_if_i = f_\infty$, hence $s_if_0 = s_ia_if_i = s_ib_if_i = s_if_\infty$ for any $i \in I$. Thus we have $f_0 = f_\infty$ by the assumption. Since $a_i\Delta_if_0 = f_0 = a_if_i$, $b_i\Delta_if_0 = f_0 = f_\infty = b_if_i$ and $Z_i \underset{b_i}{\overset{a_i}{\rightrightarrows}} Z_0$ is a monomorphic pair, it follows $f_i = \Delta_if_0$ for any $i \in I_*$. Conversely, let $f_0, f_\infty : Y \to X$ be morphisms satisfying $s_if_0 = s_if_\infty$ for any $i \in I$. Then we have a family of morphisms $(f_i : Y \to Z_i)_{i\in I_*}$ such that $a_if_i = f_0$, $b_if_i = f_\infty$, hence there is a morphism $g : Y \to X$ such that $f_i = \Delta_ig$ for any $i \in I_*$. Thus we have $f_0 = \Delta_0g = \Delta_\infty g = f_\infty$. □

Corollary A.3.18 *Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$ such that a kernel pair of each morphism with domain X exists. If $F : \mathcal{C} \to \mathcal{D}$ is a functor preserving (finite) limits, F preserves (finite) monomorphic families with domain X.*

Proposition A.3.19 *Let $\mathcal{C}$ be a category which has pull-back of reflexive pairs (A.1.9). Then, a pair of morphisms $R \underset{g}{\overset{f}{\rightrightarrows}} X$ of $\mathcal{C}$ is an equivalence relation if and only if it satisfies the following conditions.*

(1) $f, g : R \to X$ is a monomorphic pair.

(2) $R \underset{g}{\overset{f}{\rightrightarrows}} X$ is a reflexive pair.

(3) There exists a morphism $\tau : R \to R$ such that $g\tau = f$ and $f\tau = g$.

(4) If $\begin{array}{ccc} T & \xrightarrow{q} & R \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} \\ R & \xrightarrow{g} & X \end{array}$ is a pull-back, then there is a morphism $t : T \to R$ satisfying $ft = fp$ and $gt = gq$.

Proof. Let $R \overset{f}{\underset{g}{\rightrightarrows}} X$ be an equivalence relation in $\mathcal{C}$. The condition (1) is obvious. Since the image of $(f_*, g_*) : \mathcal{C}(X, R) \to \mathcal{C}(X, X) \times \mathcal{C}(X, X)$ is an equivalence relation on $\mathcal{C}(X, X)$, it contains (id_X, id_X). Hence there exists $r : X \to R$ such that $fr = gr = id_X$.

Since the image of $(f_*, g_*) : \mathcal{C}(R, R) \to \mathcal{C}(R, X) \times \mathcal{C}(R, X)$ is an equivalence relation on $\mathcal{C}(R, X)$ and $(f, g) = (f_*, g_*)(id_R)$ belongs to the image, (g, f) is also contained in the image. Hence there exists $\tau : R \to R$ such that $f\tau = g$ and $g\tau = f$.

Since the image of $(f_*, g_*) : \mathcal{C}(T, R) \to \mathcal{C}(T, X) \times \mathcal{C}(T, X)$ is an equivalence relation on $\mathcal{C}(T, X)$ and both $(fp, fq) = (fp, gp) = (f_*, g_*)(p)$ and $(fq, gq) = (f_*, g_*)(q)$ belong to the image, (fp, gq) is also contained in the image. Hence there exists a morphism $t : T \to R$ satisfying $ft = fp$ and $gt = gq$.

Conversely, suppose that a pair of morphisms $R \overset{f}{\underset{g}{\rightrightarrows}} X$ satisfies the conditions (1)$\sim$(4). For any object Y of $\mathcal{C}$ and $\alpha \in \mathcal{C}(Y, X)$, we have (α, α)=$((fr)_*(\alpha), (gr)_*(\alpha))$=$(f_*, g_*)(r\alpha)$. Thus the image of (f_*, g_*) contains the diagonal subset.

If $(\alpha, \beta) \in \mathcal{C}(Y, X) \times \mathcal{C}(Y, X)$ belongs to the image of (f_*, g_*), then $(\alpha, \beta) = (f_*, g_*)(\lambda)$ for some $\lambda \in \mathcal{C}(Y, R)$ and we have $(\beta, \alpha) = (g\lambda, f\lambda) = (f\tau\lambda, g\tau\lambda) = (f_*, g_*)(\tau\lambda)$. Hence the image of (f_*, g_*) is symmetric.

Suppose that $(\alpha, \beta), (\beta, \gamma) \in \mathcal{C}(Y, X) \times \mathcal{C}(Y, X)$ belong to the image of (f_*, g_*), then $(\alpha, \beta) = (f_*, g_*)(\lambda)$, $(\beta, \gamma) = (f_*, g_*)(\mu)$ for some $\lambda, \mu \in \mathcal{C}(Y, R)$. Since $g\lambda = f\mu = \beta$, there exists a unique morphism $\nu : Y \to T$ such that $p\nu = \lambda$ and $q\nu = \mu$. Then, $(\alpha, \gamma) = (f\lambda, g\mu) = (fp\nu, gq\nu) = (ft\nu, gt\nu) = (f_*, g_*)(t\nu)$. Thus the image of (f_*, g_*) is transitive. □

It follows from (3) that $f\tau^2 = f$ and $g\tau^2 = g$. Hence $\tau^2 = id_R$ by (1) and τ is an isomorphism.

Corollary A.3.20 *Suppose that $\mathcal{C}$ is a category which has pull-backs of reflexive pairs. If a functor $F : \mathcal{C} \to \mathcal{D}$ preserves pull-backs of reflexive pairs and monomorphic pairs of morphisms, F preserves equivalence relations. In particular, if $\mathcal{C}$ is a category with finite limits and F is left exact, F preserves equivalence relations.*

Proposition A.3.21 *Let $\mathcal{C}$ be a category with finite products and X, Y objects of $\mathcal{C}$. Set*

$$\Gamma = \{G \in \mathrm{Sub}(X \times Y) \,|\, G \overset{i_G}{\rightarrowtail} X \times Y \xrightarrow{p_1} X \textit{ is an isomorphism.}\}$$

and define maps $\Phi : \Gamma \to \mathcal{C}(X, Y)$, $\Psi : \mathcal{C}(X, Y) \to \Gamma$ as follows. For $G \in \Gamma$, $\Phi(G) : X \to Y$ is a composite $X \xrightarrow{(p_1 i_G)^{-1}} G \overset{i_G}{\rightarrowtail} X \times Y \xrightarrow{p_2} Y$. For $f \in \mathcal{C}(X, Y)$, $\Psi(f)$ is the subobject of $X \times Y$ represented by $(id_X, f) : X \to X \times Y$. Then, Ψ is the inverse of Φ.

Proof. For $G \in \Gamma$, the isomorphism $p_1 i_G : G \to X$ gives $G = \Psi\Phi(G)$ in $\mathrm{Sub}(X \times Y)$. $\Phi\Psi(f) = f$ is obvious for $f \in \mathcal{C}(X, Y)$. □

Definition A.3.22 *Let $\mathcal{C}$ be a category.*

(1) Two morphisms $p : X \to Y$ and $i : Z \to W$ are said to be orthogonal if the following left diagram is commutative, there exits a unique morphism $s : Y \to Z$ that makes the following right diagram commute.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "p"'] & Z \arrow[d, "i"] \\ Y \arrow[r, "v"'] & W \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "u"] \arrow[d, "p"'] & Z \arrow[d, "i"] \\ Y \arrow[r, "v"'] \arrow[ur, dotted, "s"] & W \end{tikzcd}$$

If p and i are orthgonal, we denote this by $p \perp i$.

(2) For a class C of morphisms in $\mathcal{C}$, we put

$$C^{\perp} = \{i \in \mathrm{Mor}\,\mathcal{C} \,|\, p \perp i \textit{ if } p \in C\}, \qquad {}^{\perp}C = \{p \in \mathrm{Mor}\,\mathcal{C} \,|\, p \perp i \textit{ if } i \in C\}.$$

(3) Let E be the class of all epimorphisms in $\mathcal{C}$. A monomorphism $i : Z \to W$ in $\mathcal{C}$ is called a strong monomorphism if i belongs to $E^{\perp}$.

(4) Let M be the class of all monomorphisms in $\mathcal{C}$. An epimorphism $p : X \to Y$ in $\mathcal{C}$ is called a strong epimorphism if p belongs to ${}^{\perp}M$.

Proposition A.3.23 *Let C be a class of morphisms in $\mathcal{C}$.*

(1) If D is a class of morphisms in $\mathcal{C}$ which contains C, then $C^{\perp} \supset D^{\perp}$ and ${}^{\perp}C \supset {}^{\perp}D$.

(2) $C \subset {}^{\perp}(C^{\perp})$ and $C \subset ({}^{\perp}C)^{\perp}$ hold.

(3) $({}^{\perp}(C^{\perp}))^{\perp} = C^{\perp}$ and ${}^{\perp}(({}^{\perp}C)^{\perp}) = {}^{\perp}C$ hold.

Proof. (1) Since $f \in C$ implies $f \in D$, the assertion is straightforward from the definition (A.3.22).

(2) For $p \in C$, we have $p \perp j$ for any $j \in C^{\perp}$, which shows $p \in {}^{\perp}(C^{\perp})$. Thus we have $C \subset {}^{\perp}(C^{\perp})$. For $i \in C$, we have $p \perp i$ for any $p \in {}^{\perp}C$, which shows $i \in ({}^{\perp}C)^{\perp}$. Thus we have $C \subset ({}^{\perp}C)^{\perp}$.

(3) It follows from (1) and (2) that we have $({}^{\perp}(C^{\perp}))^{\perp} \subset C^{\perp}$ and ${}^{\perp}(({}^{\perp}C)^{\perp}) \subset {}^{\perp}C$. Suppose that $i \in C^{\perp}$ and $p \in {}^{\perp}(C^{\perp})$. Then, $p \perp j$ for any $j \in C^{\perp}$ in particular, we have $p \perp i$. Hence $p \perp i$ holds for any $p \in {}^{\perp}(C^{\perp})$, which implies $i \in ({}^{\perp}(C^{\perp}))^{\perp}$. Thus we have $C^{\perp} \subset ({}^{\perp}(C^{\perp}))^{\perp}$. Suppose that $i \in {}^{\perp}C$ and $p \in ({}^{\perp}C)^{\perp}$. Then, $p \perp j$ for any $j \in {}^{\perp}C$ in particular, we have $p \perp i$. Hence $p \perp i$ holds for any $p \in ({}^{\perp}C)^{\perp}$, which implies $i \in (({}^{\perp}C)^{\perp})^{\perp}$. Thus we have ${}^{\perp}C \subset (({}^{\perp}C)^{\perp})^{\perp}$. □

Proposition A.3.24 *(1) A regular monomorphism (A.1.12) is a strong monomorphism.*

(2) A regular epimorphism (A.1.11) is a strong epimorphism.

Proof. (1) Suppose that i is an equalizer of $f, g : W \to V$ and the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{v} & W \end{array}$$

Then, we have $fvp = fiu = giu = gvp$. Hence if p is an epimorphism, it follows that $fv = gv$. Since i is an equalizer of $f, g : W \to V$, there exists a unique $s : Y \to Z$ that satisfies $v = is$. Then, $isp = vp = iu$ which implies $sp = u$ since i is a monomorphism.

(2) Suppose that p is a coequalizer of $f, g : U \to X$ and the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{v} & W \end{array}$$

Then, we have $iuf = vpf = vpg = iug$. Hence if i is a monomorphism, it follows that $uf = ug$. Since p is a coequalizer of $f, g : U \to X$, there exists a unique $s : Y \to Z$ that satisfies $u = sp$. Then, $isp = iu = vp$ which implies $is = v$ since p is an epimorphism. □

A.4 Limits, colimits and generators

Let $\mathcal{C}$ and $\mathcal{A}$ be categories and X an object of $\mathcal{C}$. Define an "evaluation functor" $E_X : \mathrm{Funct}(\mathcal{C}, \mathcal{A}) \to \mathcal{A}$ by $F \mapsto F(X)$, $(\varphi : F \to G) \mapsto (\varphi_X : F(X) \to G(X))$.

Proposition A.4.1 *Let $D : \mathcal{D} \to \mathrm{Funct}(\mathcal{C}, \mathcal{A})$ be a functor. If, for each $X \in \mathrm{Ob}\,\mathcal{C}$, there exists a colimiting (resp. limiting) cone $(E_X D(i) \xrightarrow{\iota_{iX}} L_X)_{i \in \mathrm{Ob}\,\mathcal{D}}$ (resp. $(L_X \xrightarrow{\pi_{iX}} E_X D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$) in $\mathcal{A}$, then there exists a unique cone $(D(i) \xrightarrow{\iota_i} L)_{i \in \mathrm{Ob}\,\mathcal{D}}$ (resp. $(L \xrightarrow{\pi_i} D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$) in $\mathrm{Funct}(\mathcal{C}, \mathcal{A})$ such that $L(X) = L_X$. Moreover, $(D(i) \xrightarrow{\iota_i} L)_{i \in \mathrm{Ob}\,\mathcal{D}}$ (resp. $(L \xrightarrow{\pi_i} D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$) is a colimiting (resp. limiting) cone of a functor $D : \mathcal{D} \to \mathrm{Funct}(\mathcal{C}, \mathcal{A})$. Conversely, if $(D(i) \xrightarrow{\iota_i} L)_{i \in \mathrm{Ob}\,\mathcal{D}}$ (resp. $(L \xrightarrow{\pi_i} D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$) is a colimiting (resp. limiting) cone of a functor $D : \mathcal{D} \to \mathrm{Funct}(\mathcal{C}, \mathcal{A})$ and $\mathcal{A}$ is cocomplete (resp. complete), so is $(E_X D(i) \xrightarrow{\iota_{iX}} L_X)_{i \in \mathrm{Ob}\,\mathcal{D}}$ (resp. $(L_X \xrightarrow{\pi_{iX}} E_X D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$) in $\mathcal{A}$.*

Proof. For a morphism $\varphi : X \to Y$ in $\mathcal{C}$, since

$$\left(E_X D(i) \xrightarrow{\iota_{iY} D(i)(\varphi)} L_Y\right)_{i \in \mathrm{Ob}\,\mathcal{D}} \qquad \left(\text{resp. } \left(L_X \xrightarrow{D(i)(\varphi)\pi_{iX}} E_Y D(i)\right)_{i \in \mathrm{Ob}\,\mathcal{D}}\right)$$

is a cone, there is a unique morphism $L_\varphi : L_X \to L_Y$ such that $L_\varphi \iota_{iX} = \iota_{iY} D(i)(\varphi)$ (resp. $\pi_{iY} L_\varphi = D(i)(\varphi)\pi_{iX}$) for any $i \in \mathrm{Ob}\,\mathcal{D}$. Define a functor $L : \mathcal{C} \to \mathcal{A}$ by $L(X) = L_X$ and $L(\varphi) = L_\varphi$. It is a routine to verify that L is a functor and $\iota_{iX} : D(i)(X) \to L(X)$ (resp. $\pi_{iX} : L(X) \to D(i)(X)$) is natural in X. Thus we have a cone $(D(i) \xrightarrow{\iota_i} L)_{i \in \mathrm{Ob}\,\mathcal{D}}$ (resp. $(L \xrightarrow{\pi_i} D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$) and the uniqueness of L is obvious. Let $(D(i) \xrightarrow{\sigma_i} M)_{i \in \mathrm{Ob}\,\mathcal{D}}$ (resp. $(M \xrightarrow{\lambda_i} D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$) be a cone in $\mathrm{Funct}(\mathcal{C}, \mathcal{A})$. For each $X \in \mathrm{Ob}\,\mathcal{C}$, there is a unique morphism $\rho_X : L(X) = L_X \to M(X)$ (resp. $\rho_X : M(X) \to L_X = L(X)$) such that $\sigma_{iX} = \rho_X \iota_{iX}$ (resp. $\lambda_{iX} = \pi_{iX}\rho_X$) for any $i \in \mathrm{Ob}\,\mathcal{D}$. For a morphism $\varphi : X \to Y$ in $\mathcal{C}$, since $(E_X D(i) \xrightarrow{M(\varphi)\sigma_{iX}} M(Y))_{i \in \mathrm{Ob}\,\mathcal{D}}$ (resp. $(M(X) \xrightarrow{\lambda_{iY} M(\varphi)}$

$E_Y D(i))_{i\in\mathrm{Ob}\,\mathcal{D}})$ is a cone and $M(\varphi)\rho_X\iota_{iX} = M(\varphi)\sigma_{iX} = \sigma_{iY}D(i)(\varphi) = \rho_Y\iota_{iY}D(i)(\varphi) = \rho_Y L(\varphi)\iota_{iX}$ (resp. $\pi_{iY}\rho_Y M(\varphi) = \lambda_{iY}M(\varphi) = D(i)(\varphi)\lambda_{iX} = \pi_{iX}\rho_X D(i)(\varphi) = \pi_{iY}L(\varphi)\rho_X)$ for any $i \in \mathrm{Ob}\,\mathcal{D}$ by the naturality of ι_i (resp. π_i), we have $M(\varphi)\rho_X = \rho_Y L(\varphi)$ (resp. $\rho_Y M(\varphi) = L(\varphi)\rho_X$), namely, ρ is natural. The converse statement is obvious. $\square$

In particular, since $\mathcal{U}$-**Ens** is both $\mathcal{U}$-complete and $\mathcal{U}$-cocomplete, so is the category $\widehat{\mathcal{C}}_{\mathcal{U}}$ of presheaves of $\mathcal{U}$-sets.

Yoneda's lemma implies the following result.

Proposition A.4.2 *Let F be a presheaf on $\mathcal{C}$ and $h : \mathcal{C} \to \widehat{\mathcal{C}}$ the Yoneda embedding ((A.1.7)). Consider a comma category $(h{\downarrow}F)$ and a functor $hP : (h{\downarrow}F) \to \widehat{\mathcal{C}}$. Then, $(hP\langle Y, f\rangle \xrightarrow{f} F)_{\langle Y,f\rangle\in\mathrm{Ob}\,(h\downarrow F)}$ is a colimiting cone of hP.*

Proof. Let $\mathcal{C}_F$ denotes a category with objects (Y, y) for $Y \in \mathrm{Ob}\,\mathcal{C}$, $y \in F(Y)$ and morphisms $\alpha : (Y, y) \to (Z, z)$ such that $F(\alpha)(z) = y$. Define functors $\Psi : \mathcal{C}_F \to \widehat{\mathcal{C}}$ and $\Theta : \mathcal{C}_F \to (h{\downarrow}F)$ by $\Psi(Y, y) = h_Y$, $\Psi(\alpha) = h_\alpha$ and $\Theta(Y, y) = \langle Y, \theta_F(y)\rangle$, $\Theta(\alpha) = h_\alpha$. Then $hP\Theta = \Psi$ and it follows from (A.1.6) that Θ is an isomorphism of categories. Hence it suffices to show that $(\Psi(Y, y) \xrightarrow{\theta_F(y)} F)_{(Y,y)\in\mathrm{Ob}\,\mathcal{C}_F}$ is a colimiting cone of Ψ.

Fix an object X of $\mathcal{C}$, we claim that $(\Psi(Y, y)(X) \xrightarrow{\theta_F(y)_X} F(X))_{(Y,y)\in\mathrm{Ob}\,\mathcal{C}_F}$ is a colimiting cone of $E_X\Psi : \mathcal{C}_F \to \mathcal{U}$-**Ens**. Then, the result follows from (A.4.1). For a cone $(\Psi(Y, y)(X) \xrightarrow{\alpha_{(Y,y)}} C)_{(Y,y)\in\mathrm{Ob}\,\mathcal{C}_F}$, define $\rho : F(X) \to C$ by $\rho(x) = \alpha_{(X,x)}(id_X)$ for $x \in F(X)$. Then, for any $(Y, y) \in \mathrm{Ob}\,\mathcal{C}_F$ and $\varphi \in \Psi(Y, y)(X) = h_Y(X)$, since $\varphi : (X, F(\varphi)(y)) \to (Y, y)$ is a morphism in $\mathcal{C}_F$, $\alpha_{(X,F(\varphi)(y))} = \alpha_{(Y,y)}h_\varphi$. It follows that $\rho\theta_F(y)_X(\varphi) = \rho(F(\varphi)(y)) = \alpha_{(X,F(\varphi)(y))}(id_X) = \alpha_{(X,F(\varphi)(y))}(id_X) = \alpha_{(Y,y)}h_\varphi(id_X) = \alpha_{(Y,y)}(\varphi)$. Thus we have $\rho\theta_F(y)_X = \alpha_{(Y,y)}$ for any $(Y, y) \in \mathrm{Ob}\,\mathcal{C}_F$. Since $x \in F(X)$ is the image of $id_X \in h_X(X) = \Psi(X, x)$ by $\theta_F(x)_X$, ρ is the unique map satisfying $\rho\theta_F(y)_X = \alpha_{(Y,y)}$. $\square$

Proposition A.4.3 *Colimits indexed by $\mathcal{U}$-set in $\mathcal{U}$-**Ens** are universal.*

Proof. Let $D : \mathcal{D} \to \mathcal{U}$-**Ens** be a functor and suppose that a cone $(D(j) \xrightarrow{f_j} S)_{j\in\mathrm{Ob}\,\mathcal{D}}$ and a map $f : T \to S$ are given. Define a functor $D_F : \mathcal{D} \to \mathcal{U}$-**Ens** by $D_S(j) = D(j) \times_S T$ and we claim that $(D(j) \times_S T \xrightarrow{\iota_j\times id_T} (\varinjlim D) \times_S T)_{j\in\mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $D_F : \mathcal{D} \to \mathcal{U}$-**Ens**. Suppose that $(D(j) \times_S T \xrightarrow{g_j} U)_{j\in\mathrm{Ob}\,\mathcal{D}}$ is a cone of D_F. Recall that $\varinjlim D$ is given by the coequalizer $\coprod_{f\in\mathrm{Mor}\,\mathcal{D}} D(\mathrm{dom}(f)) \overset{s}{\underset{t}{\rightrightarrows}} \coprod_{j\in\mathrm{Ob}\,\mathcal{D}} D(i) \xrightarrow{q} \varinjlim D$, where s and t are given by $s(x) = x \in D(\mathrm{dom}(f))$ and $t(x) = D(f)(x) \in D(\mathrm{codom}(f))$ on $D(\mathrm{dom}(f))$. For any $(x, y) \in (\varinjlim D) \times_S T$, choose $z \in D(j)$ such that $q(z) = x$. Then, $(z, y) \in D(j) \times_S T$ and define $h : (\varinjlim D) \times_S T \to U$ by $h(x, y) = g_j(z, y)$. It is easy to check that this definition does not depend upon the choice of z and the uniqueness of h is clear. $\square$

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories and $D : \mathcal{C}\times\mathcal{D} \to \mathcal{E}$ a functor. For $(i, j) \in \mathrm{Ob}\,(\mathcal{C}\times\mathcal{D})$, we denote by $D_j : \mathcal{C} \to \mathcal{E}$ and $D^i : \mathcal{D} \to \mathcal{E}$ the functors defined by $D_j(k) = D(k, j)$, $D_j(f : k \to l) = D(f, id_j)$ and $D^i(m) = D(i, m)$, $D^i(g : m \to n) = D(id_i, g)$, respectively. Suppose that $\varinjlim_k D_j$ and $\varprojlim_m D^i$ exist for each $(i, j) \in \mathrm{Ob}\,(\mathcal{C}\times\mathcal{D})$ and let $(D(k, j) \xrightarrow{\iota_{k,j}} \varinjlim_k D_j)_{k\in\mathrm{Ob}\,\mathcal{C}}$, $(\varprojlim_m D^i \xrightarrow{\rho_{i,m}} D(i, m))_{m\in\mathrm{Ob}\,\mathcal{C}}$ be colimiting, limiting cones respectively. Then, we have functors $\mathcal{D} \to \mathcal{E}$ and $\mathcal{C} \to \mathcal{E}$ given by $j \mapsto \varinjlim_k D_j$ and $i \mapsto \varprojlim_m D^i$. Moreover, if these functors have a limit and a colimit respectively, we have a "canonical" morphism $\kappa : \varinjlim_i \varprojlim_m D^i \to \varprojlim_j \varinjlim_k D_j$ defined as follows. Let $(\varprojlim_j \varinjlim_k D_j \xrightarrow{\rho_j} \varinjlim_k D_j)_{j\in\mathrm{Ob}\,\mathcal{D}}$ and $(\varprojlim_m D^i \xrightarrow{\iota_i} \varinjlim_i \varprojlim_m D^i)_{i\in\mathrm{Ob}\,\mathcal{C}}$ be limiting, colimiting cones respectively. For each $i \in \mathcal{C}$, there is a unique morphism $\alpha_i : \varprojlim_m D^i \to \varprojlim_j \varinjlim_k D_j$ satisfying $\rho_j\alpha_i = \iota_{i,j}\rho_{i,j}$ for any $j \in \mathrm{Ob}\,\mathcal{D}$. Then, $(\varprojlim_m D^i \xrightarrow{\alpha_i} \varprojlim_j \varinjlim_k D_j)_{i\in\mathrm{Ob}\,\mathcal{C}}$ is a cone and we have a unique morphism κ satisfying $\kappa\iota_i = \alpha_i$ for any $i \in \mathrm{Ob}\,\mathcal{C}$.

Proposition A.4.4 *If $\mathcal{C}$ is a $\mathcal{U}$-small filtered category, $\mathcal{D}$ is a finite category and $\mathcal{E}$ is the category of $\mathcal{U}$-set, κ is bijective.*

Proof. The filtered colimit $\varinjlim_k D_j$ is the quotient set of $\coprod_{k\in\mathrm{Ob}\,\mathcal{C}} D(k, j)$ by the equivalence relation "$x \sim y \Leftrightarrow D(f, id_j)(x) = D(g, id_j)(y)$ *for some* $f : k \to l$, $g : k' \to l$", where $x \in D(k, j)$ and $y \in D(k', j)$. We write (x, k)

for the class of $x \in D(k,j)$. For $((x_j,k_j))_{j\in \mathrm{Ob}\,\mathcal{D}} \in \varprojlim_j \varinjlim_k D_j \subset \prod_{j\in \mathrm{Ob}\,\mathcal{D}} \varinjlim_k D_j$, since $\mathrm{Ob}\,\mathcal{D}$ is finite and $\mathcal{C}$ is filtered, there exist $k \in \mathrm{Ob}\,\mathcal{D}$ and $y_j \in D(k,j)$ for $j \in \mathrm{Ob}\,\mathcal{D}$ such that $(x_j,k_j) = (y_j,k)$. If $u : j \to m$ is a morphism in $\mathcal{D}$, $(\varinjlim_k D_j(u))(y_j,k) = (y_m,k)$ implies $(D(id_k,u)(y_j),m) = (y_m,k)$ hence there exists a morphism $h_u : k \to l_u$ such that $D(h_u,id_m)D(id_k,u)(y_j) = D(h_u,id_m)(y_m)$. Since $\mathrm{Mor}\,\mathcal{D}$ is finite and $\mathcal{C}$ is filtered, there exist morphisms $p_u : l_u \to n$ in $\mathcal{C}$ such that $p_u h_u = p_v h_v$ for any $u,v \in \mathrm{Mor}\,\mathcal{D}$. Thus we have a morphism $q : k \to n$ satisfying $D(q,id_m)D(id_k,u)(y_j) = D(q,id_m)(y_m)$ for any $(u : j \to m) \in \mathrm{Mor}\,\mathcal{D}$. Therefore we have $D(id_n,u)D(q,id_j)(y_j) = D(q,id_m)(y_m)$ and this implies that $(D(q,id_j)(y_j))_{j\in\mathrm{Ob}\,\mathcal{D}} \in \varprojlim_j D^n$. We define $\kappa^{-1} : \varprojlim_j \varinjlim_k D_j \to \varinjlim_i \varprojlim_m D^i$ by $\kappa^{-1}(((y_j,k))_{j\in\mathrm{Ob}\,\mathcal{D}}) = ((D(q,id_j)(y_j))_{j\in\mathrm{Ob}\,\mathcal{D}},n)$. It is easy to verify that κ^{-1} is well-defined and this is the inverse of κ. □

Proposition A.4.5 *Let $\mathcal{C}$ be a category with pull-backs.*

1) Let $f : Y \to X$ be a morphism in $\mathcal{C}$. Suppose that $D : \mathcal{D} \to \mathcal{C}/X$ is a functor such that $\Sigma_X D : \mathcal{D} \to \mathcal{C}$ has a universal colimit. Then, the pull-back functor $f^* : \mathcal{C}/X \to \mathcal{C}/Y$ preserves the colimit of D. Hence if every (finite) colimit (resp. coproduct) in $\mathcal{C}$ is universal, $f^* : \mathcal{C}/X \to \mathcal{C}/Y$ preserves (finite) colimits (resp. coproducts).

2) Let $(f_i : X_i \to X)_{i\in I}$ and $(g_j : Y_j \to X)_{j\in J}$ be family of morphisms of $\mathcal{C}$ such that $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$ have universal coproducts. We denote by $f : \coprod_{i\in I} X_i \to X$ and $g : \coprod_{j\in J} Y_j \to X$ the morphisms induced by $(f_i : X_i \to X)_{j\in J}$ and $(g_j : Y_j \to X)_{j\in J}$, respectively. For each $(i,j) \in I \times J$, form a pull-back

$$\begin{tikzcd} X_i \times_X Y_j \arrow[r, "f_{ij}"] \arrow[d, "g_{ji}"] & Y_j \arrow[d, "g_j"] \\ X_i \arrow[r, "f_i"] & X \end{tikzcd} .$$

Then, the coproduct of $(X_i \times_X Y_j)_{(i,j)\in I\times J}$ exists and the following square is a pull-back, where $\bar{f}$ and $\bar{g}$ denote the morphisms induced by f_{ij}'s and g_{ji}'s.

$$\begin{tikzcd} \coprod_{(i,j)\in I\times J} (X_i \times_X Y_j) \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"] & \coprod_{j\in J} Y_j \arrow[d, "g"] \\ \coprod_{i\in I} X_i \arrow[r, "f"] & X \end{tikzcd}$$

Proof. 1) We put $D(i) = (E_i \xrightarrow{p_i} X)$ and let $(E_i \xrightarrow{\iota_i} E)_{i\in\mathrm{Ob}\,\mathcal{D}}$ be a colimiting cone of $\Sigma_X D$. There is a unique morphism $p : E \to X$ such that $p\iota_i = p_i$. Since Σ_X creates colimits (A.3.11), $L = (E \xrightarrow{p} X)$ is a colimit of D. Set $f^*(L) = (E' \xrightarrow{p'} Y)$ and $\bar{f} : E' \to E$ denotes the morphism that covers f. Let $\iota'_i : E'_i \to E'$ be the pull-back of ι_i along $\bar{f}$. Then, the outer rectangle of the following diagram is a pull-back.

$$\begin{tikzcd} E'_i \arrow[r, "\iota'_i"] \arrow[d] & E' \arrow[r, "p'"] \arrow[d, "\bar{f}"] & Y \arrow[d, "f"] \\ E_i \arrow[r, "\iota_i"] & E \arrow[r, "p"] & X \end{tikzcd}$$

It follows from the assumption that $(E'_i \xrightarrow{\iota'_i} E')_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $\Sigma_Y f^*D$. Since $\Sigma_Y : \mathcal{C}/Y \to \mathcal{C}$ creates colimits, $((E'_i \xrightarrow{p'\iota'_i} Y) \xrightarrow{\iota'_i} (E' \xrightarrow{p'} Y))_{i\in\mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of f^*D. Hence the assertion follows from the fact that $p'\iota'_i$ is a pull-back of $p\iota_i$ along f.

2) Let $\bar{f}_j : \coprod_{i\in I}(X_i \times_X Y_j) \to Y_j$ be the morphism induced by $(f_{ij})_{i\in I}$ and $\tilde{g}_j : \coprod_{i\in I}(X_i \times_X Y_j) \to \coprod_{i\in I} X_i$ the morphism induced by $(g_{ij})_{i\in I}$. By the above result, the following square on the left is a pull-back. Hence, again by the above result, the right square is also a pull-back.

$$\begin{tikzcd} \coprod_{i\in I}(X_i \times_X Y_j) \arrow[r, "\tilde{g}_j"] \arrow[d, "\bar{f}_j"] & \coprod_{i\in I} X_i \arrow[d, "f"] \\ Y_j \arrow[r, "g_j"] & X \end{tikzcd} \qquad \begin{tikzcd} \coprod_{j\in J}\coprod_{i\in I}(X_i \times_X Y_j) \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"] & \coprod_{j\in J} Y_j \arrow[d, "g"] \\ \coprod_{i\in I} X_i \arrow[r, "f"] & X \end{tikzcd}$$

□

Proposition A.4.6 *If $F : \mathcal{C} \to \mathcal{D}$ is a fully faithful functor which has the following property $(*)$, then F preserves limits.*

$(*)$ *For each object Z of $\mathcal{D}$, $(FP\langle X, f\rangle \xrightarrow{f} Z)_{\langle X,f\rangle\in \mathrm{Ob}\,(F\downarrow Z)}$ is a colimiting cone of $FP : (F\downarrow Z) \to \mathcal{D}$.*

Proof. Let $(L \xrightarrow{p_i} X_i)_{i\in I}$ be a limiting cone of a diagram $(X_i \xrightarrow{f_{ij}} X_j)_{i,j\in I}$ in $\mathcal{C}$. Suppose that $(Z \xrightarrow{q_i} F(X_i))_{i\in I}$ is a cone of a diagram $(F(X_i) \xrightarrow{F(f_{ij})} F(X_j))_{i,j\in I}$. For each $\langle X, f\rangle \in \mathrm{Ob}\,(F\downarrow Z)$ and $i \in I$, there exists a unique morphism $s_i^f : X \to X_i$ in $\mathcal{C}$ such that $F(s_i^f) = q_i f$. Hence $(X \xrightarrow{s_i^f} X_i)_{i\in I}$ is a cone of $(X_i \xrightarrow{f_{ij}} X_j)_{i,j\in I}$ and there exists a unique morphism $\varphi^f : X \to L$ such that $p_i\varphi^f = s_i^f$ for any $i \in I$.

For any morphism $\alpha : \langle X, f\rangle \to \langle Y, g\rangle$ in $(F\downarrow Z)$ and $i \in I$, $F(p_i\varphi^g\alpha) = F(s_i^g\alpha) = q_i gF(\alpha) = q_i f = F(s_i^f) = F(p_i\varphi^f)$, hence we have $p_i\varphi^g\alpha = p_i\varphi^f$. Thus $\varphi^g\alpha = \varphi^f$ and $(FP\langle X, f\rangle \xrightarrow{F(\varphi^f)} F(L))_{\langle X,f\rangle\in\mathrm{Ob}\,(F\downarrow Z)}$ is a cone of FP. There exists a unique morphism $h : Z \to F(L)$ such that $hf = F(\varphi^f)$ for any $\langle X, f\rangle \in \mathrm{Ob}\,(F\downarrow Z)$. Then, for any $\langle X, f\rangle \in \mathrm{Ob}\,(F\downarrow Z)$ and $i \in I$, $F(p_i)hf = F(p_i)F(\varphi^f) = F(s_i^f) = q_i f$, hence $F(p_i)h = q_i$.

Suppose that a morphism $k : Z \to F(L)$ also satisfies $F(p_i)k = q_i$ for any $i \in I$. For any $\langle X, f\rangle \in \mathrm{Ob}\,(F\downarrow Z)$, there exists a unique $\psi^f : X \to L$ such that $F(\psi^f) = kf$ and we have $F(p_i\psi^f) = F(p_i)kf = q_i f = F(p_i)hf = F(p_i\varphi^f)$. Since F is faithful, $p_i\psi^f = p_i\varphi^f$ for any $i \in I$. Then $\psi^f = \varphi^f$ which implies $kf = hf$. This shows the uniqueness of h. We conclude that $(F(L) \xrightarrow{F(p_i)} F(X_i))_{i\in I}$ is a limiting cone of a diagram $(F(X_i) \xrightarrow{F(f_{ij})} F(X_j))_{i,j\in I}$ in $\mathcal{D}$. □

Proposition A.4.7 *Let $(\mathcal{C}_i)_{i\in I}$ be a family of categories and $D : \mathcal{D} \to \prod_{i\in I}\mathcal{C}_i$ a functor. A cone $((X_i)_{i\in I} \xrightarrow{(p_i)_{i\in I}} D(k))_{k\in\mathrm{Ob}\,\mathcal{D}}$ (resp. $(D(k) \xrightarrow{(s_i)_{i\in I}} (X_i)_{i\in I})_{k\in\mathrm{Ob}\,\mathcal{D}}$) is a limiting (resp. colimiting) cone of D if and only if $(X_i \xrightarrow{p_i} P_iD(k))_{k\in\mathrm{Ob}\,\mathcal{D}}$ (resp. $(P_iD(k) \xrightarrow{s_i} X_i)_{k\in\mathrm{Ob}\,\mathcal{D}}$) is a limiting (resp. colimiting) cone of P_iD for each $i \in I$, where $P_i : \prod_{i\in I}\mathcal{C}_i \to \mathcal{C}_i$ is the projection functor.*

Proof. Suppose that $((X_i)_{i\in I} \xrightarrow{(p_{ki})_{i\in I}} D(k))_{k\in\mathrm{Ob}\,\mathcal{D}}$ (resp. $(D(k) \xrightarrow{(s_{ki})_{i\in I}} (X_i)_{i\in I})_{k\in\mathrm{Ob}\,\mathcal{D}}$) is a limiting (resp. colimiting) cone of D. Let $(Y \xrightarrow{f_k} P_jD(k))_{k\in\mathrm{Ob}\,\mathcal{D}}$ (resp. $(P_jD(k) \xrightarrow{f_k} Y)_{k\in\mathrm{Ob}\,\mathcal{D}}$) be a cone of P_jD and set $Y_i = X_i$, $q_{ik} = p_{ik}$ (resp. $t_{ik} = s_{ik}$) if $i \neq j$, $Y_j = Y$, $q_{jk} = f_k$ (resp. $t_{jk} = f_k$). Then, $((Y_i)_{i\in I} \xrightarrow{(q_{ik})_{i\in I}} D(k))_{k\in\mathrm{Ob}\,\mathcal{D}}$ (resp. $(D(k) \xrightarrow{(t_{ik})_{i\in I}} (Y_i)_{i\in I})_{k\in\mathrm{Ob}\,\mathcal{D}}$) is a cone of D. There exists a unique morphism $(g_i)_{i\in I} : (Y_i)_{i\in I} \to (X_i)_{i\in I}$ (resp. $(g_i)_{i\in I} : (X_i)_{i\in I} \to (Y_i)_{i\in I}$) such that $q_{ik} = p_{ik}g_i$ (resp. $t_{ik} = g_is_{ik}$) for each $i \in I$ and $k \in \mathrm{Ob}\,\mathcal{D}$. In particular, $q_{jk} = f_kg_j$ (resp. $t_{jk} = g_jf_k$) for any $k \in \mathrm{Ob}\,\mathcal{D}$. Suppose that $g : Y_j \to X_j$ (resp. $g : X_j \to Y_j$) satisfies $q_{jk} = f_kg$ (resp. $t_{jk} = gf_k$) for any $k \in \mathrm{Ob}\,\mathcal{D}$. Set $h_i = id_{X_i}$ if $i \neq j$, $h_j = g$, then $(h_i)_{i\in I} : (Y_i)_{i\in I} \to (X_i)_{i\in I}$ (resp. $(h_i)_{i\in I} : (X_i)_{i\in I} \to (Y_i)_{i\in I}$) satisfies $q_{ik} = p_{ik}h_i$ (resp. $t_{ik} = h_is_{ik}$) for each $i \in I$ and $k \in \mathrm{Ob}\,\mathcal{D}$. The uniqueness of $(g_i)_{i\in I}$ implies $g = g_j$.

The converse is clear. □

Proposition A.4.8 *Let $\mathcal{A}$ be a $\mathcal{U}$-small category. For a category $\mathcal{C}$ and $i \in \mathrm{Ob}\,\mathcal{A}$, we denote by $\mathcal{C}_i$ a copy of $\mathcal{C}$. Define a functor $E : \mathrm{Funct}(\mathcal{A}, \mathcal{C}) \to \prod_{i\in\mathrm{Ob}\,\mathcal{A}}\mathcal{C}_i$ by $E(F) = (F(i))_{i\in\mathrm{Ob}\,\mathcal{A}}$ and $E(f) = (f_i)_{i\in\mathrm{Ob}\,\mathcal{A}}$. Then, E creates limits and colimits.*

Proof. Let $D : \mathcal{D} \to \mathrm{Funct}(\mathcal{A}, \mathcal{C})$ be a functor and $\big((L_i)_{i\in\mathrm{Ob}\,\mathcal{A}} \xrightarrow{(p_{d,i})_{i\in\mathrm{Ob}\mathcal{A}, d\in\mathrm{Ob}\mathcal{D}}} (D(d)(i))_{i\in\mathrm{Ob}\,\mathcal{A}}\big)_{d\in\mathrm{Ob}\,\mathcal{D}}$ $\big($*resp.* $\big((D(d)(i))_{i\in\mathrm{Ob}\,\mathcal{A}} \xrightarrow{(\iota_{d,i})_{i\in\mathrm{Ob}\mathcal{A}, d\in\mathrm{Ob}\mathcal{D}}} (L_i)_{i\in\mathrm{Ob}\,\mathcal{A}}\big)_{d\in\mathrm{Ob}\,\mathcal{D}}\big)$ a limiting cone (resp. colimiting cone) of $ED : \mathcal{D} \to \prod_{i\in\mathrm{Ob}\,\mathcal{A}}\mathcal{C}_i$. Define a functor $F : \mathcal{A} \to \mathcal{C}$ as follows. Put $F(i) = L_i$ and, for a morphism $\alpha : i \to j$

in $\mathcal{A}$, $F(\alpha) : L_i \to L_j$ is the unique morphism induced by $(D(d)(i) \xrightarrow{D(d)(\alpha)} D(d)(j))_{d\in \mathrm{Ob}\,\mathcal{D}}$. For each $d \in \mathrm{Ob}\,\mathcal{D}$, morphisms $p_{d,i} : F(i) \to D(d)(i)$ (resp. $\iota_{d,i} : D(d)(i) \to F(i)$) ($i \in \mathrm{Ob}\,\mathcal{A}$) in $\mathcal{C}$ induce a morphism $p_d : F \to D(d)$ (resp. $\iota : D(d) \to F$) in $\mathrm{Funct}(\mathcal{A},\mathcal{C})$. It is easy to verify that $(p_d : F \to D(d))_{d\in \mathrm{Ob}\,\mathcal{D}}$ (resp. $(\iota : D(d) \to F)_{d\in \mathrm{Ob}\,\mathcal{D}}$) is a limiting cone (resp. colimiting cone) of D. Hence E creates limits and colimits. □

Let $\mathcal{C}$ be a $\mathcal{U}$-category and $\mathcal{G}$ a subcategory of $\mathcal{C}$. Define a functor $\varPhi : \mathcal{C} \to \widehat{\mathcal{G}} = \mathrm{Funct}(\mathcal{G},\mathcal{U}\text{-}\mathbf{Ens})$ to be the composition $\mathcal{C} \xrightarrow{h} \widehat{\mathcal{C}} \xrightarrow{\iota^*} \widehat{\mathcal{G}}$, where $\iota^* : \widehat{\mathcal{C}} \to \widehat{\mathcal{G}}$ is given by restricting the domain of a presheaf to $\mathcal{G}$. Then, the following result is easily verified from the definition.

Proposition A.4.9 *$\mathcal{G}$ is a generating subcategory by epimorphisms if and only if $\varPhi$ is faithful.*

Proposition A.4.10 *Let $\mathcal{C}$ be a $\mathcal{U}$-category and $\mathcal{G}$ a full subcategory of $\mathcal{C}$. We denote by $i : \mathcal{G} \to \mathcal{C}$ the inclusion functor.*

1) The following properties are equivalent.

i) $\mathcal{G}$ is a generating subcategory by strict epimorphisms.

ii) For any $X \in \mathrm{Ob}\,\mathcal{C}$, $(iP\langle Y,p\rangle \xrightarrow{p} X)_{\langle Y,p\rangle\in \mathrm{Ob}\,(i\downarrow X)}$ is a colimiting cone for the functor $iP : (i\downarrow X) \to \mathcal{C}$.

iii) The functor $\varPhi : \mathcal{C} \to \widehat{\mathcal{G}}$ in (A.4.9) is fully faithful.

2) Consider the following conditions.

i) $\mathcal{G}$ is a generating subcategory by strict epimorphisms.
ii) $\mathcal{G}$ is a generating subcategory by epimorphisms.
iii) $\mathcal{G}$ is a generating subcategory.
iv) $\mathcal{G}$ is a generating subcategory for monomorphisms.
v) $\mathcal{G}$ is a generating subcategory for strict monomorphisms.

Then, i) ⇒ ii), i) ⇒ iii) ⇒ iv) ⇒ v).

3) There are the following implications between the conditions of 2).

a) If an epimorphic family in $\mathcal{C}$ is a strict epimorphic family, ii) implies i). If a monomorphism in $\mathcal{C}$ is strict, v) implies iv).

b) If $\mathcal{C}$ has equalizers (resp. a kernel pair of each morphism), iii) implies ii) (resp. iv) implies iii)).

c) If, for any family $(f_i : X_i \to X)_{i\in I}$ of morphisms in $\mathcal{C}$, there exist an epimorphic family $(p_i : X_i \to Y)_{i\in I}$ and a monomorphism $j : Y \to X$ such that $f_i = jp_i$ ($i \in I$) and $(p_i : X_i \to Y)_{i\in I}$ is strict (resp. j is strict), then iv) implies i) (resp. v) implies ii)).

Proof. 1) *i) ⇒ ii)*: Let $(iP\langle Y,p\rangle \xrightarrow{g_p} Z)_{\langle Y,p\rangle\in \mathrm{Ob}\,(i\downarrow X)}$ be a cone in $\mathcal{C}$. Suppose that $u : W \to Y$ and $v : W \to Y'$ are morphisms in $\mathcal{C}$ with $Y, Y' \in \mathcal{G}$ satisfying $up = vq$ for $p : Y \to X$, $q : Y' \to X$. Then, $g_pu = g_{up} = g_{vq} = g_qv$ and this implies that there exists a unique morphism $h : X \to Z$ such that $hp = g_p$ for any $\langle Y,p\rangle \in \mathrm{Ob}\,(i\downarrow X)$. Therefore $(iP\langle Y,p\rangle \xrightarrow{p} X)_{\langle Y,p\rangle\in \mathrm{Ob}\,(i\downarrow X)}$ is a colimiting cone of the functor $iP : (i\downarrow X) \to \mathcal{C}$.

ii) ⇒ iii): Obvious. *iii) ⇒ iv)*: Let $f, g : X \to Z$ be morphisms in $\mathcal{C}$ such that $\varPhi(f) = \varPhi(g)$. Since $(iP\langle Y,p\rangle \xrightarrow{p} X)_{\langle Y,p\rangle\in \mathrm{Ob}\,(i\downarrow X)}$ is a colimiting cone for the functor $iP : (i\downarrow X) \to \mathcal{C}$ and $fp = \varPhi(f)_Y(p) = \varPhi(g)_Y(p) = gp$ for any $\langle Y,p\rangle \in \mathrm{Ob}\,(i\downarrow X)$, we have $f = g$ and $\varPhi$ is faithful.

Let $\alpha : \varPhi(X) \to \varPhi(Z)$ be a morphism in $\widehat{\mathcal{G}}$. Then $(iP\langle Y,p\rangle \xrightarrow{\alpha_Y(p)} Z)_{\langle Y,p\rangle\in \mathrm{Ob}\,(i\downarrow X)}$ is a cone for the functor $iP : (i\downarrow X) \to \mathcal{C}$. Since $(iP\langle Y,p\rangle \xrightarrow{p} X)_{\langle Y,p\rangle\in \mathrm{Ob}\,(i\downarrow X)}$ is a colimiting cone for the functor $iP : (i\downarrow X) \to \mathcal{C}$, there is a unique morphism $f : X \to Z$ such that $fp = \alpha_Y(p)$ for any $\langle Y,p\rangle \in \mathrm{Ob}\,(i\downarrow X)$. Hence we have $\varPhi(f) = \alpha$ and $\varPhi$ is full.

iii) ⇒ i): By (A.4.9), $\bigcup_{Y\in \mathrm{Ob}\,\mathcal{G}} \mathcal{C}(Y,X)$ is an epimorphic family. Assume that $(g_p : iP\langle Y,p\rangle \to Z)_{\langle Y,p\rangle\in \mathrm{Ob}(i\downarrow X)}$ satisfies for any $p : Y \to X$, $q : Y' \to X$ with $Y, Y' \in \mathrm{Ob}\,\mathcal{G}$, "$pu = qv$ for $u : W \to Y$, $v : W \to Y' \Rightarrow g_pu = g_qv$". Then, it is easy to verify that a map $\alpha_Y : \varPhi(X)(Y) \to \varPhi(Z)(Y)$ defined by $\alpha_Y(p) = g_p$ gives a morphism $\alpha : \varPhi(X) \to \varPhi(Z)$ in $\widehat{\mathcal{G}}$. There exists a morphism $f : X \to Z$ such that $\varPhi(f) = \alpha$, that is, $fp = g_p$ for any $\langle Y,p\rangle \in \mathrm{Ob}\,(i\downarrow X)$. This shows that $(g_p : iP\langle Y,p\rangle \to Z)_{\langle Y,p\rangle\in \mathrm{Ob}\,(i\downarrow X)}$ is strict.

2) Implications *i) ⇒ ii)*, *iii) ⇒ iv) ⇒ v)* are trivial. We show *i) ⇒ iii)*. Let $f : X \to Y$ be a morphism in $\mathcal{C}$ such that $f_* : \mathcal{C}(Z,X) \to \mathcal{C}(Z,Y)$ is bijective for any $Z \in \mathrm{Ob}\,\mathcal{G}$, in other words, $\varPhi(f) : \varPhi(X) \to \varPhi(Y)$ is an isomorphism in $\widehat{\mathcal{G}}$. Since $\varPhi$ is fully faithful by 1), $\varPhi$ reflects isomorphisms. Hence f is an isomorphism.

3) *a)* is trivial.

b) iii) ⇒ ii): Suppose that $f, g \in \mathcal{C}(U,V)$ satisfy $fh = gh$ for any $X \in G$ and $h \in \mathcal{C}(X,U)$. Let $e : E \to U$ be an equalizer of f and g. Then, there exists $h' \in \mathcal{C}(X,E)$ such that $eh' = h$. Hence $e_* : \mathcal{C}(X,E) \to \mathcal{C}(X,U)$ is surjective. Since e is a monomorphism, e_* is injective. Thus e is an isomorphism and we have $f = g$.

iv) $\Rightarrow$ *iii)*: Let $f : X \to Y$ be a morphism in $\mathcal{C}$ such that $f_* : \mathcal{C}(Z, X) \to \mathcal{C}(Z, Y)$ is bijective for any $Z \in \mathrm{Ob}\,\mathcal{G}$. It suffices to show that f is a monomorphism. Let $K \overset{g}{\underset{h}{\rightrightarrows}} X$ be a kernel pair of f. Then, $f_*(g) = fg = fh = f_*(h)$ implies $g = h$ and f is a monomorphism.

c) iv) $\Rightarrow$ *i)* (resp. *v)* $\Rightarrow$ *ii)*): For any object X of $\mathcal{C}$, there exist a monomorphism (resp. strict monomorphism) $j : X' \to X$ and $g_p : Y \to X'$ for each $\langle Y, p\rangle \in \mathrm{Ob}\,(i{\downarrow}X)$ such that $p = jg_p$ and $(g_p : Y \to X')_{p \in \mathrm{Ob}(i\downarrow X)}$ is a strict epimorphic family (resp. epimorphic family). Then, $j_* : \mathcal{C}(Y, X') \to \mathcal{C}(Y, X)$ is bijective for any $Y \in \mathrm{Ob}\,\mathcal{G}$. In fact, if $p \in \mathcal{C}(Y, X)$, we have $j_*(g_p) = jg_p = p$. Hence j is an isomorphism by assumption and $\bigcup_{Y \in \mathrm{Ob}\,\mathcal{G}} \mathcal{C}(Y, X)$ is a strict epimorphic family (resp. epimorphic family). $\square$

Proposition A.4.11 *1) Let $\mathcal{C}$ be a category and $\mathcal{G}$ a generating subcategory of $\mathcal{C}$ for monomorphisms. Suppose that a pull-back of a monomorphism along a monomorphism always exists in $\mathcal{C}$. If $\sigma : Y \to X$ and $\tau : Z \to X$ are monomorphisms such that $\mathrm{Im}(\sigma_* : \mathcal{C}(U, Y) \to \mathcal{C}(U, X)) \subset \mathrm{Im}(\tau_* : \mathcal{C}(U, Z) \to \mathcal{C}(U, X))$ for any $U \in \mathrm{Ob}\,\mathcal{G}$, then there exists a unique monomorphism $\iota : Y \to Z$ satisfying $\tau\iota = \sigma$. Hence if $\mathrm{Im}(\sigma_* : \mathcal{C}(U, Y) \to \mathcal{C}(U, X)) = \mathrm{Im}(\tau_* : \mathcal{C}(U, Z) \to \mathcal{C}(U, X))$ for any $U \in \mathrm{Ob}\,\mathcal{G}$, σ and τ represents the same subobject of X.*

2) Let $\mathcal{C}$ be a category and $\mathcal{G}$ a generating subcategory of $\mathcal{C}$ by epimorphisms. If $\sigma : Y \to X$ and $\tau : Z \to X$ are morphisms such that τ is a strict monomorphism and $\mathrm{Im}(\sigma_* : \mathcal{C}(U, Y) \to \mathcal{C}(U, X)) \subset \mathrm{Im}(\tau_* : \mathcal{C}(U, Z) \to \mathcal{C}(U, X))$ for any $U \in \mathrm{Ob}\,\mathcal{G}$, then there exists a unique morphism $\iota : Y \to Z$ satisfying $\tau\iota = \sigma$. Hence if both σ and τ are strict monomorphisms and $\mathrm{Im}(\sigma_* : \mathcal{C}(U, Y) \to \mathcal{C}(U, X)) = \mathrm{Im}(\tau_* : \mathcal{C}(U, Z) \to \mathcal{C}(U, X))$ for any $U \in \mathrm{Ob}\,\mathcal{G}$, σ and τ represents the same subobject of X.

Proof. 1) Let $\bar{\tau} : V \to Y$ be a pul-back of τ along σ. For any $U \in \mathrm{Ob}\,\mathcal{G}$, $\bar{\sigma}_* : \mathcal{C}(U, V) \to \mathcal{C}(U, Y)$ is bijective. In fact, for any $f \in \mathcal{C}(U, Y)$, there exists $g \in \mathcal{C}(U, Z)$ such that $\sigma f = \tau g$ by assumption. Thus $\bar{\tau}$ is an isomorphism and ι is defined by $\bar{\sigma}\bar{\tau}^{-1}$, where $\bar{\sigma} : V \to Z$ is a pul-back of σ along τ.

2) Suppose $u\tau = v\tau$ for $u, v : X \to W$. Then, for any $U \in \mathrm{Ob}\,\mathcal{G}$ and $f : U \to Y$, there exists $g \in \mathcal{C}(U, Z)$ such that $\sigma f = \tau g$ by assumption. Hence $u\sigma f = u\tau g = v\tau g = v\sigma f$ and this implies $u\sigma = v\sigma$. Since τ is a strict monomorphism there exists a morphism $\iota : Y \to Z$ satisfying $\tau\iota = \sigma$. $\square$

Corollary A.4.12 *Under the assumptions of (A.4.11) 1) (resp.(A.4.11) 2)), the cardinal number of the set of subobjects (resp. strict subobjects) of X is smaller than or equal to $\prod\limits_{U \in \mathrm{Ob}\,\mathcal{G}} 2^{\mathrm{card}(\mathcal{C}(U,X))}$.*

Proposition A.4.13 *Let $\mathcal{C}$ be a category and $\mathcal{G}$ a generating subcategory of $\mathcal{C}$ by epimorphisms. If $p : X \to Y$ and $q : X \to Z$ are morphisms such that p is a strict epimorphism and $\{(u, v) \in \mathcal{C}(U, X) \times \mathcal{C}(U, X) |\, pu = pv\} \subset \{(u, v) \in \mathcal{C}(U, X) \times \mathcal{C}(U, X) |\, qu = qv\}$ for any $U \in \mathrm{Ob}\,\mathcal{G}$, then there exists a unique morphism $r : Y \to Z$ satisfying $pr = q$. Hence if both p and q are strict epimorphisms and $\{(u, v) \in \mathcal{C}(U, X) \times \mathcal{C}(U, X) |\, pu = pv\} = \{(u, v) \in \mathcal{C}(U, X) \times \mathcal{C}(U, X) |\, qu = qv\}$ for any $U \in \mathrm{Ob}\,\mathcal{G}$, p and q represents the same quotient object of X.*

Proof. Suppose that $pu = pv$ holds for $u, v : W \to X$, then for any $U \in \mathrm{Ob}\,\mathcal{G}$ and $f : U \to W$, we have $quf = qvf$. Since $\mathcal{G}$ a generating subcategory of $\mathcal{C}$ by epimorphisms, it follows $qu = qv$, hence there exists a unique morphism $r : Y \to Z$ satisfying $pr = q$. $\square$

Corollary A.4.14 *Under the assumptions of (A.4.13), the cardinal number of the set of strict quotient objects of X is smaller than or equal to $\prod\limits_{U \in \mathrm{Ob}\,\mathcal{G}} 2^{\mathrm{card}(\mathcal{C}(U,X))^2}$.*

Proposition A.4.15 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor with a left adjoint $L : \mathcal{D} \to \mathcal{C}$ and G a set of objects of $\mathcal{D}$. We set $G_L = \{L(X) |\, X \in \mathrm{Ob}\,\mathcal{D}\}$.*

1) If G is a generator of $\mathcal{D}$ by epimorphisms and F is faithful, then G_L is a generator of $\mathcal{C}$ by epimorphisms.

2) If G is a generator of $\mathcal{D}$ and F reflects isomorphisms, then G_L is a generator of $\mathcal{C}$.

3) If G is a generator of $\mathcal{D}$ for monomorphisms and F has the following property, then G_L is a generator of $\mathcal{C}$ for monomorphisms.

($*$) *If u is a monomorphism in $\mathcal{C}$ such that $F(u)$ is an isomorphism, u is an isomorphism.*

4) If G is a generator of $\mathcal{D}$ for strict monomorphisms and F has the following property, then G_L is a generator of $\mathcal{C}$ for strict monomorphisms.

($*$) *If u is a strict monomorphism in $\mathcal{C}$ such that $F(u)$ is an isomorphism, u is an isomorphism.*

Proof. 1) Let $f, g : Y \to Z$ be morphisms in $\mathcal{C}$ such that $fh = gh$ for any $X \in G$ and morphism $h : L(X) \to Y$ in $\mathcal{C}$. Then, for any $X \in G$ and morphism $h' : X \to F(Y)$ in $\mathcal{D}$, $F(f)h' = F(g)h'$. Hence $F(f) = F(g)$ and this implies $f = g$.

2),3),4) Let $f : Y \to Z$ be a morphism (resp. monomorphism, strict monomorphism) in $\mathcal{C}$ such that $f_* : \mathcal{C}(L(X), Y) \to \mathcal{C}(L(X), Z)$ is bijective for any $X \in G$. Note that since F has a left adjoint, it preserves monomorphisms and strict monomorphisms (A.3.13). Then, $F(f)_* : \mathcal{D}(X, F(Y)) \to \mathcal{D}(X, F(Z))$ is bijective for any $X \in G$ and it follows that $F(f)$ is an isomorphism. Thus f is an isomorphism. $\square$

Proposition A.4.16 *Let $(\mathcal{C}_i)_{i\in I}$ be a family of categories and G_i be a set of objects of $\mathcal{C}_i$. Put $G = \{(X_i)_{i\in I} \in \mathrm{Ob}\prod\limits_{i\in I}\mathcal{C}_i |\, X_i \in G_i\}$.*

1) If each G_i is a generator of $\mathcal{C}_i$ by epimorphisms, then G is a generator of $\prod\limits_{i\in I}\mathcal{C}_i$ by epimorphisms.

2) If each G_i is a generator of $\mathcal{C}_i$ by strict epimorphisms, then G is a generator of $\prod\limits_{i\in I}\mathcal{C}_i$ by strict epimorphisms.

3) If each G_i is a generator of $\mathcal{C}_i$, then G is a generator of $\prod\limits_{i\in I}\mathcal{C}_i$.

4) If each G_i is a generator of $\mathcal{C}_i$ for monomorphisms, then G is a generator of $\prod\limits_{i\in I}\mathcal{C}_i$ for monomorphisms.

5) If each G_i is a generator of $\mathcal{C}_i$ for strict monomorphisms, then G is a generator of $\prod\limits_{i\in I}\mathcal{C}_i$ for strict monomorphisms.

Proof. Assertions 1), 3), 4) and 5) is straightforward. We show 2). For $(Y_i)_{i\in I} \in \mathrm{Ob}\prod\limits_{i\in I}\mathcal{C}_i$, set $M = \bigcup_{(X_i)_{i\in I}\in G}(\prod\limits_{i\in I}\mathcal{C}_i)((X_i)_{i\in I}, (Y_i)_{i\in I})$. We denote by $(X_{\mu i})_{i\in I}$ the domain of $\mu = (\mu_i)_{i\in I} \in M$. Let $((f_{\mu i})_{i\in I} : (X_{\mu i})_{i\in I} \to (Z_i)_{i\in I})_{\mu\in M}$ be a family of morphisms in $\prod\limits_{i\in I}\mathcal{C}_i$ such that $(f_{\mu i})_{i\in I}(g_i)_{i\in I} = (f_{\nu i})_{i\in I}(h_i)_{i\in I}$ holds if $(\mu_i)_{i\in I}(g_i)_{i\in I} = (\nu_i)_{i\in I}(h_i)_{i\in I}$ for $\mu = (\mu_i)_{i\in I}, \nu = (\nu_i)_{i\in I} \in M$ and $(g_i)_{i\in I} : (W_i)_{i\in I} \to (X_{\mu i})_{i\in I}$, $(h_i)_{i\in I} : (W_i)_{i\in I} \to (X_{\nu i})_{i\in I}$. We note that $M = \{(\mu_i)_{i\in I} |\, \mu_i \in M_i\}$ where $M_i = \bigcup_{X_i\in G_i}\mathcal{C}_i(X_i, Y_i)$. For each $i \in I$, $f_{\mu i}g_i = f_{\nu i}h_i$ holds if $\mu_i g_i = \nu_i h_i$ for $\mu_i, \nu_i \in M_i$ and $g_i : W_i \to X_{\mu i}$, $h_i : W_i \to X_{\nu i}$. Since each M_i is a strict epimorphic family, there exists a unique morphism $k_i : Y_i \to Z_i$ such that $f_{\mu i} = k_i\mu_i$. m$\prod\limits_{i\in I}\mathcal{C}_i$. $\square$

A.5 Cofinal functors and exact functors

Definition A.5.1 *1) A functor $F : \mathcal{D} \to \mathcal{C}$ is called cofinal if $(i{\downarrow}F)$ is non-empty and connected for any $i \in \mathrm{Ob}\,\mathcal{C}$.*

2) A functor $F : \mathcal{D} \to \mathcal{C}$ is called co-cofinal if $F^{op} : \mathcal{D}^{op} \to \mathcal{C}^{op}$ is cofinal, that is, $(F{\downarrow}i)$ is non-empty and connected for any $i \in \mathrm{Ob}\,\mathcal{C}$.

Proposition A.5.2 *Let $F : \mathcal{D} \to \mathcal{C}$ be a cofinal functor.*

1) Suppose that $G : \mathcal{C} \to \mathcal{E}$ is a functor such that a colimit of G exists. If $\left(G(i) \xrightarrow{\iota_i} C\right)_{i\in\mathrm{Ob}\,\mathcal{C}}$ is a colimiting cone of G, then $\left(GF(j) \xrightarrow{\iota_{F(j)}} C\right)_{j\in\mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of GF.

2) Suppose that $G : \mathcal{C}^{op} \to \mathcal{E}$ is a functor such that a limit of G exists. If $\left(L \xrightarrow{\pi_i} G(i)\right)_{i\in\mathrm{Ob}\,\mathcal{C}}$ is a limiting cone of G, then $\left(L \xrightarrow{\pi_{F(j)}} GF(j)\right)_{j\in\mathrm{Ob}\,\mathcal{D}}$ is a limiting cone of GF^{op}.

Proof. 1) Let $\left(GF(j) \xrightarrow{\theta_j} D\right)_{j\in\mathrm{Ob}\,\mathcal{D}}$ be a cone of GF. For any $i \in \mathrm{Ob}\,\mathcal{C}$ and $\langle f, j\rangle, \langle g, k\rangle \in \mathrm{Ob}\,(i{\downarrow}F)$, we claim that $\theta_j G(f) = \theta_k G(g)$. Assume that there exists a morphism $\tau : \langle f, j\rangle \to \langle g, k\rangle$ in $(i{\downarrow}F)$. Then $F(\tau)f = g$ and $\theta_k GF(\tau) = \theta_j$ since we have a morphism $\tau : j \to k$ in $\mathcal{D}$ and $\left(GF(j) \xrightarrow{\theta_j} D\right)_{j\in\mathrm{Ob}\,\mathcal{D}}$ is a cone of GF. Hence $\theta_j G(f) = \theta_k GF(\tau)G(f) = \theta_k G(F(\tau)f) = \theta_k G(g)$. Since $(i{\downarrow}F)$ is connected, $\theta_j G(f) = \theta_k G(g)$ hold for any pair of objects $\langle f, j\rangle, \langle g, k\rangle$ of $(i{\downarrow}F)$.

We define $g_i : G(i) \to C$ as follows. Choose $\langle f, j\rangle \in \mathrm{Ob}\,(i{\downarrow}F)$ and put $g_i = \theta_j G(f)$. By the above argument, this definition of g_i does not depend on the choice of $\langle f, j\rangle \in \mathrm{Ob}\,(i{\downarrow}F)$. Hence, for $j \in \mathrm{Ob}\,\mathcal{D}$, taking $\langle id_{F(j)}, j\rangle \in \mathrm{Ob}\,(F(j)\downarrow F)$ we have $g_{F(j)} = \theta_j G(id_{F(j)}) = \theta_j$. Let $\sigma : h \to i$ be a morphism in $\mathcal{C}$. We choose $\langle f\sigma, j\rangle \in \mathrm{Ob}\,(h\downarrow F)$ and define $g_h : G(h) \to D$ by $g_h = \theta_j G(f\sigma)$. Then, we have $g_i G(\sigma) = g_h$ and $\left(G(i) \xrightarrow{g_i} D\right)_{i\in \mathrm{Ob}\,\mathcal{C}}$ is a cone of G. Hence there exists a unique morphism $\lambda : C \to D$ satisfying $\lambda\iota_i = g_i$ for any $i \in \mathrm{Ob}\,\mathcal{C}$. In particular, we have $\lambda\iota_{F(j)} = g_{F(j)} = \theta_j$ for any $j \in \mathrm{Ob}\,\mathcal{D}$. Suppose that $\mu : C \to D$ also satisfies $\mu\iota_{F(j)} = \theta_j$ for any $j \in \mathrm{Ob}\,\mathcal{D}$. For any $i \in \mathrm{Ob}\,\mathcal{C}$, choose $\langle f, j\rangle \in \mathrm{Ob}\,(i\downarrow F)$. Since $\iota_i = \iota_{F(j)}G(f)$, we have $\mu\iota_i = \mu\iota_{F(j)}G(f) = \theta_j G(f) = g_i = \lambda\iota_i$. Therefore $\mu = \lambda$ by the uniqueness of λ.

2) Applying the result of 1) to $G^{op} : \mathcal{C} \to \mathcal{E}^{op}$, $\left(G^{op}F(j) \xrightarrow{\pi_{F(j)}} L\right)_{j\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of $G^{op}F$. Hence the assertion follows. □

Let $\mathcal{C}$ be a $\mathcal{U}$-category. For $X \in \mathrm{Ob}\,\mathcal{C}$, we denote by $h^X : \mathcal{C} \to \mathcal{U}$-**Ens** a functor defined by $h^X(Y) = \mathcal{C}(X, Y)$ for $X \in \mathrm{Ob}\,\mathcal{C}$ and $h^X(f : Y \to Z) = (f_* : \mathcal{C}(X, Y) \to \mathcal{C}(X, Z))$.

Lemma A.5.3 *Let $\mathcal{C}$ be a $\mathcal{U}$-small category. For $i \in \mathrm{Ob}\,\mathcal{C}$, $\left(h^i(j) \to \{1\}\right)_{j\in\mathrm{Ob}\mathcal{C}}$ is a colimiting cone of h^i.*

Proof. Let $\left(h^i(j) \xrightarrow{\iota_j} X\right)_{j\in\mathrm{Ob}\mathcal{C}}$ be a cone of h^i. Define a map $\eta : \{1\} \to X$ by $\eta(1) = \iota_i(id_i)$. Then, for $j \in \mathrm{Ob}\,\mathcal{C}$ and $f \in h^i(j)$, since $f = h^i(f)(id_i)$, we have $\iota_j(f) = \iota_j(h^i(f)(id_i)) = \iota_i(id_i) = \eta(1) = \eta c_j(f)$, where $c_j : h^i(j) \to \{1\}$ is the unique map. Suppose that there exists a map $\eta' : \{1\} \to X$ satisfying $\eta' c_j = \iota_j$ for each $j \in \mathrm{Ob}\,\mathcal{C}$. Then, $\eta'(1) = \eta' c_i(id_i) = \iota_i(id_i) = \eta(1)$ which implies $\eta' = \eta$. Hence the assertion follows. □

Let $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{E}$ be functors. If a colimit of GF exist, let $\left(GF(j) \xrightarrow{\eta_j} L\right)_{j\in \mathrm{Ob}\,\mathcal{D}}$ be a colimiting cone of FG. If $\left(G(i) \xrightarrow{\iota_i} C\right)_{i\in \mathrm{Ob}\,\mathcal{C}}$ is a cone of G, then $\left(GF(j) \xrightarrow{\iota_{F(j)}} C\right)_{j\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of GF. Hence there is a unique morphism $\gamma^C_{F,G} : L \to C$ satisfying $\gamma^C_{F,G}\eta_j = \iota_{F(j)}$ for any $j \in \mathrm{Ob}\,\mathcal{D}$. If $\left(G(i) \xrightarrow{\iota_i} C\right)_{i\in \mathrm{Ob}\,\mathcal{C}}$ is a colimiting cone of G, we denote $\gamma^C_{F,G}$ by $\gamma_{F,G}$.

If a limit of GF exist, let $\left(L \xrightarrow{\rho_j} GF(j)\right)_{j\in \mathrm{Ob}\,\mathcal{D}}$ be a limiting cone of GF. If $\left(M \xrightarrow{\pi_i} G(i)\right)_{i\in \mathrm{Ob}\,\mathcal{C}}$ is a cone of G, then $\left(M \xrightarrow{\pi_{F(j)}} GF(j)\right)_{j\in \mathrm{Ob}\,\mathcal{D}}$ is a cone of GF. Hence there is a unique morphism $\gamma^{F,G}_M : M \to L$ satisfying $\rho_j\gamma^{F,G}_M = \pi_{G(j)}$ for any $j \in \mathrm{Ob}\,\mathcal{E}$. If $\left(M \xrightarrow{\pi_i} G(i)\right)_{i\in \mathrm{Ob}\,\mathcal{C}}$ is a limiting cone of F, we denote $\gamma^{F,G}_M$ by $\gamma^{F,G}$.

Proposition A.5.4 *For a functor $F : \mathcal{D} \to \mathcal{C}$ between $\mathcal{U}$-small categories, the following conditions are equivalent.*

(i) F is cofinal.
*(ii) For any functor $G : \mathcal{C} \to \mathcal{U}$-**Ens**, the natural map $\gamma_{F,G} : \varinjlim GF \to \varinjlim G$ is bijective.*
(iii) $(h^iF(j) \to \{1\})_{j\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of h^iF for any $i \in \mathrm{Ob}\,\mathcal{C}$.

Proof. It follows from 1) of (A.5.2) that (i) implies (ii). Since $\left(h^i(j) \to \{1\}\right)_{j\in\mathrm{Ob}\mathcal{C}}$ is a colimiting cone of h^i for any $i \in \mathrm{Ob}\,\mathcal{C}$ by (A.5.3), (i) also implies (iv) by 1) of (A.5.2). It is clear that (ii) implies (iv). Assume (iii). For each $i \in \mathrm{Ob}\,\mathcal{C}$, there exists $j_0 \in \mathrm{Ob}\,\mathcal{D}$ such that $h^iF(j_0)$ is not empty by the assumption. Suppose $f_0 \in h^iF(j_0)$, then $\langle f_0, j_0\rangle \in \mathrm{Ob}\,(i\downarrow F)$ which implies that $(i\downarrow F)$ is not empty. Define maps $s, t : \coprod_{\tau\in\mathrm{Mor}\mathcal{D}} \mathcal{C}(i, F(\mathrm{dom}(\tau))) \to \coprod_{j\in \mathrm{Ob}\,\mathcal{D}} \mathcal{C}(i, F(j))$ by $s(f) = f \in \mathcal{C}(i, F(\mathrm{dom}(\tau)))$ and $t(f) = F(\tau)f \in \mathcal{C}(i, F(\mathrm{codom}(\tau)))$. Then,

$$\coprod_{\tau\in\mathrm{Mor}\mathcal{D}} \mathcal{C}(i, F(\mathrm{dom}(\tau))) \underset{t}{\overset{s}{\rightrightarrows}} \coprod_{j\in \mathrm{Ob}\,\mathcal{D}} \mathcal{C}(i, F(j)) \to \{1\}$$

is a coequalizer by the assumption. Suppose $\langle g, k\rangle, \langle h, l\rangle \in \mathrm{Ob}\,(i\downarrow F)$. Then, $g \in h^iF(k) = \mathcal{C}(i, F(k))$ and $h \in h^iF(l) = \mathcal{C}(i, F(l))$ are both mapped to 1 by $\coprod_{j\in \mathrm{Ob}\,\mathcal{D}} \mathcal{C}(i, F(j)) \to \{1\}$. Hence g and h are equivalent by the equivalence relation $\sim$ on $\coprod_{j\in \mathrm{Ob}\,\mathcal{D}} \mathcal{C}(i, F(j))$ generated by $s(f) \sim t(f)$. This implies that $(i\downarrow F)$ is connected. Therefore F is cofinal. □

Lemma A.5.5 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and Y an object of $\mathcal{D}$. If $\mathcal{C}$ is a category with finite colimits (resp. finite limits) and F preserves finite colimits (resp. finite limits), then $(F\downarrow Y)$ (resp. $(Y\downarrow F)^{op}$) is a filtered category.*

Proof. Let $\langle X, f\rangle$ and $\langle Z, g\rangle$ be objects of $(F\downarrow Y)$. We denote by $\nu_1 : X \to X\coprod Z$ and $\nu_2 : Z \to X\coprod Z$ the canonical morphisms. Then, there is an isomorphism $k : F(X)\coprod F(Z) \to F(X\coprod Z)$ such that $k\nu_i' = F(\nu_i)$, where $\nu_1' : F(X) \to F(X)\coprod F(Z)$ and $\nu_2' : F(Z) \to F(X)\coprod F(Z)$ are the canonical morphisms. We also have a morphism $h : F(X)\coprod F(Z) \to Y$ such that $h\nu_1' = f$ and $h\nu_2' = g$. Hence we have morphisms $\nu_1 : \langle X, f\rangle \to \langle X\coprod Z, hk^{-1}\rangle$ and $\nu_2 : \langle Z, g\rangle \to \langle X\coprod Z, hk^{-1}\rangle$ in $(F\downarrow Y)$.

Let $\varphi, \psi : \langle X, f\rangle \to \langle Z, g\rangle$ be morphisms in $(F\downarrow Y)$ and $\rho : Z \to W$ a coequalizer of $X \underset{\psi}{\overset{\varphi}{\rightrightarrows}} Z$ in $\mathcal{C}$. Then, $F(\rho) : F(Z) \to F(W)$ is a coequalizer of $F(X) \underset{F(\psi)}{\overset{F(\varphi)}{\rightrightarrows}} F(Z)$ in $\mathcal{D}$. Since $gF(\varphi) = gF(\psi) = f$, there is a unique morphism $h : F(W) \to Y$ such that $hF(\rho) = g$. Hence we have a morphism $\rho : \langle Z, g\rangle \to \langle W, h\rangle$ in $(F\downarrow Y)$ such that $\rho\varphi = \rho\psi$. Proof of the dual statement is similar. □

A.6 Kan extensions

Definition A.6.1 *Let $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{C}'$ a functor. We denote by $F^* : \mathrm{Funct}(\mathcal{C}', \mathcal{D}) \to \mathrm{Funct}(\mathcal{C}, \mathcal{D})$ defined by $F^*(T) = TF$ and $F^*(\varphi) = \varphi_F$.*

Proposition A.6.2 *Let $\mathcal{U}$ be a fixed universe. If $\mathcal{D}$ is $\mathcal{U}$-complete (resp. $\mathcal{U}$-cocomplete), F^* preserves $\mathcal{U}$-limits (resp. $\mathcal{U}$-colimits).*

Proof. Let $D : \mathcal{A} \to \mathrm{Funct}(\mathcal{C}', \mathcal{D})$ be a functor and $(L \xrightarrow{\pi_i} D(i))_{i\in \mathrm{Ob}\,\mathcal{A}}$ (resp. $(D(i) \xrightarrow{\iota_i} L)_{i\in \mathrm{Ob}\,\mathcal{A}}$) a limiting (resp. colimiting) cone of D. It follows from (A.4.1) that, for $X \in \mathrm{Ob}\,\mathcal{C}$, $(LF(X) \xrightarrow{\pi_{iF(X)}} D(i)F(X))_{i\in \mathrm{Ob}\,\mathcal{A}}$ (resp. $(D(i)F(X) \xrightarrow{\iota_{iF(X)}} LF(X))_{i\in \mathrm{Ob}\,\mathcal{A}}$) is a limiting (resp. colimiting) cone of $E_{F(X)}D$. Hence $(LF \xrightarrow{\pi_{iF}} D(i)F)_{i\in \mathrm{Ob}\,\mathcal{A}}$ (resp. $(D(i)F \xrightarrow{\iota_{iF}} LF)_{i\in \mathrm{Ob}\,\mathcal{A}}$) is a limiting (resp. colimiting) cone of DF by (A.4.1). □

Definition A.6.3 *For functors $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C} \to \mathcal{A}$, a left Kan extension of G along F is a pair (L, η) of a functor $L : \mathcal{C}' \to \mathcal{A}$ and a natural transformation $\eta : G \to LF$ such that for any functor $H : \mathcal{C}' \to \mathcal{A}$, the assignment $\sigma \mapsto \sigma_F\eta$ gives a bijection $\mathrm{Funct}(\mathcal{C}', \mathcal{A})(L, H) \xrightarrow{\cong} \mathrm{Funct}(\mathcal{C}, \mathcal{A})(G, F^*(H))$. We denote L by $F_!(G)$.*

Definition A.6.4 *For functors $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C} \to \mathcal{A}$, a right Kan extension of G along F is a pair (R, ε) of a functor $R : \mathcal{C}' \to \mathcal{A}$ and a natural transformation $\varepsilon : RF \to G)$ such that for any functor $H : \mathcal{C}' \to \mathcal{A}$, the assignment $\tau \mapsto \varepsilon\tau_F$ is a bijection $\mathrm{Funct}(\mathcal{C}', \mathcal{A})(H, R) \xrightarrow{\cong} \mathrm{Funct}(\mathcal{C}, \mathcal{A})(F^*(H), G)$. We denote R by $F_*(G)$.*

Proposition A.6.5 *Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C} \to \mathcal{A}$ be functors. Assume that, for each object Y of $\mathcal{C}'$, the composite $(F\downarrow Y) \xrightarrow{P} \mathcal{C} \xrightarrow{G} \mathcal{A}$ has a colimit with a colimiting cone $(GP\langle X, f\rangle \xrightarrow{\lambda^Y_{\langle X,f\rangle}} L(Y))_{\langle X,f\rangle\in \mathrm{Ob}\,(F\downarrow Y)}$. Each morphism $g : Y \to Z$ in $\mathcal{C}'$ induces a unique morphism $L(g) : L(Y) \to L(Z)$ commuting with the colimiting cones. This defines a functor $L : \mathcal{C}' \to \mathcal{A}$. For each $X \in \mathrm{Ob}\,\mathcal{C}$, set $\eta_X = \lambda^{F(X)}_{\langle X, id_{F(X)}\rangle} : G(X) \to LF(X)$. Then, we have a natural transformation $\eta : G \to LF$ and (L, η) is a left Kan extension of G along F.*

Proof. Let $g : Y \to Z$ be a morphism in $\mathcal{C}'$. Consider the functor $(id_F\downarrow g) : (F\downarrow Y) \to (F\downarrow Z)$ defined in (A.1.15). Then, we have a cone

$$\left(GP(id_F\downarrow g)\langle X, f\rangle \xrightarrow{\lambda^Z_{(id_F\downarrow g)\langle X,f\rangle}} L(Z)\right)_{\langle X,f\rangle\in \mathrm{Ob}\,(F\downarrow Y)}.$$

Since $GP(id_F\downarrow g)\langle X, f\rangle = G(X)$ for any $\langle X, f\rangle \in \mathrm{Ob}, (F\downarrow Y)$, there exists a unique morphism $L(g) : L(Y) \to L(Z)$ such that $L(g)\lambda^Y_{\langle X,f\rangle} = \lambda^Z_{(id_F\downarrow g)\langle X,f\rangle}$ for any $\langle X, f\rangle \in \mathrm{Ob}\,(F\downarrow Y)$. It is easy to verify that this choice of $L(g)$ makes L a functor.

Let $h : V \to W$ be a morphism in $\mathcal{C}$. It follows from the definition of $LF(h) : LF(V) \to LF(W)$ that $LF(h)\eta_V = LF(h)\lambda^{F(V)}_{\langle V, id_{F(V)}\rangle} = \lambda^{F(W)}_{(id_F\downarrow F(h))\langle V, id_{F(V)}\rangle} = \lambda^{F(W)}_{\langle V, F(h)\rangle} = \lambda^{F(W)}_{\langle W, id_{F(W)}\rangle}GP(h) = \eta_W G(h)$. Therefore $\eta : G \to LF$ is natural.

Let $H : \mathcal{C}' \to \mathcal{A}$ be a functor and $\alpha : G \to HF$ be a natural transformation. We construct a natural transformation $\sigma : L \to H$ as follows. For $Y \in \mathrm{Ob}\,\mathcal{C}'$, $(GP\langle X, f\rangle = G(X) \xrightarrow{H(f)\alpha_X} H(Y))_{\langle X,f\rangle\in \mathrm{Ob}\,(F\downarrow Y)}$ is a cone. In fact, if $\varphi : \langle X, f\rangle \to \langle W, k\rangle$ is a morphism in $(F\downarrow Y)$, $H(k)\alpha_W GP(\varphi) = H(k)\alpha_W G(\varphi) = H(k)HF(\varphi)\alpha_X = H(kF(\varphi))\alpha_X = H(f)\alpha_X$. Thus we have a unique morphism $\sigma_Y : L(Y) \to H(Y)$ such that $\sigma_Y\lambda^Y_{\langle X,f\rangle} = H(f)\alpha_X$ for any $\langle X, f\rangle \in \mathrm{Ob}\,(F\downarrow Y)$.

To show the naturality of σ, take a morphism $g : Y \to Z$ in $\mathcal{C}'$. For each $\langle X, f\rangle \in \mathrm{Ob}\,(F{\downarrow}Y)$, since $H(g)\sigma_Y\lambda^Y_{\langle X,f\rangle} = H(g)H(f)\alpha_X = H(gf)\alpha_X = \sigma_Z\lambda^Z_{\langle X,gf\rangle} = \sigma_Z\lambda^Z_{(id_F\downarrow g)\langle X,f\rangle} = \sigma_Z L(g)\lambda^Y_{\langle X,f\rangle}$, we have $H(g)\sigma_Y = \sigma_Z L(g)$.

Finally, we show that the correspondence $\alpha \mapsto \sigma$ gives the inverse correspondence of the assignment $\sigma \mapsto \sigma_F\eta$. For given $\alpha \in \mathrm{Funct}(\mathcal{C}, \mathcal{A})(G, HF)$, construct $\sigma \in \mathrm{Funct}(\mathcal{C}', \mathcal{A})(L, H)$ as above, then for any $X \in \mathrm{Ob}\,\mathcal{C}$, $\sigma_{F(X)}\eta_X = \sigma_{F(X)}\lambda^{F(X)}_{\langle X,id_{F(X)}\rangle} = \alpha_X$. Conversely, for given $\sigma \in \mathrm{Funct}(\mathcal{C}', \mathcal{A})(L, H)$, apply the above construction to $\sigma_F\eta$ to have a natural transformation $\sigma' : L \to H$. Since $\sigma'_Y\lambda^Y_{\langle X,f\rangle} = H(f)\sigma_{F(X)}\eta_X = \sigma_Y L(f)\lambda^{F(X)}_{\langle X,id_{F(X)}\rangle} = \sigma_Y\lambda^Y_{(id_F\downarrow f)\langle X,id_Y\rangle} = \sigma_Y\lambda^Y_{\langle X,f\rangle}$ for any $\langle X, f\rangle \in \mathrm{Ob}\,(F{\downarrow}Y)$, we have $\sigma'_Y = \sigma_Y$. □

Proposition A.6.6 *Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C} \to \mathcal{A}$ be functors. Assume that, for each object Y of $\mathcal{C}'$, the composite $(Y{\downarrow}F) \xrightarrow{Q} \mathcal{C} \xrightarrow{G} \mathcal{A})$ has a limit with a limiting cone $(R(Y) \xrightarrow{\lambda^Y_{\langle f,X\rangle}} GQ\langle f, X\rangle)_{\langle f,X\rangle\in\mathrm{Ob}\,(Y\downarrow F)}$. Each morphism $g : Y \to Z$ in $\mathcal{C}'$ induces a unique morphism $R(g) : R(Y) \to R(Z)$ commuting with the limiting cones. This defines a functor $R : \mathcal{C}' \to \mathcal{A}$. For each $X \in \mathrm{Ob}\,\mathcal{C}$, set $\varepsilon_X = \lambda^{F(X)}_{\langle id_{F(X)},X\rangle} : RF(X) \to G(X)$. Then, we have a natural transformation $\varepsilon : RF \to G$ and $(R, \varepsilon))$ is a right Kan extension of G along F.*

Proof. Let $g : Y \to Z$ be a morphism in $\mathcal{C}'$. Consider the functor $(g{\downarrow}id_F) : (Z{\downarrow}F) \to (Y{\downarrow}F)$ defined in (A.1.15). Then, we have a cone

$$\left(R(Y) \xrightarrow{\lambda^Y_{(g\downarrow id_F)\langle f,X\rangle}} GQ(g{\downarrow}id_F)\langle f, X\rangle\right)_{\langle f,X\rangle\in\mathrm{Ob}\,(Z\downarrow F)}.$$

Since $GQ(g{\downarrow}id_F)\langle f, X\rangle = G(X)$ for any $\langle f, X\rangle \in (Z{\downarrow}F)$, there exists a unique morphism $R(g) : R(Y) \to R(Z)$ such that $\lambda^Z_{\langle f,X\rangle}R(g) = \lambda^Y_{(g\downarrow id_F)\langle f,X\rangle}$ for any $\langle f, X\rangle \in \mathrm{Ob}\,(Z{\downarrow}F)$. It is easy to verify that this choice of $R(g)$ makes R a functor.

Let $h : V \to W$ be a morphism in $\mathcal{C}$. It follows from the definition of $RF(h) : RF(V) \to RF(W)$ that $\varepsilon_W RF(h) = \lambda^{F(W)}_{\langle id_{F(W)},W\rangle}RF(h) = \lambda^{F(V)}_{(F(h)\downarrow id_F)\langle id_{F(W)},W\rangle} = \lambda^{F(V)}_{\langle F(h),W\rangle} = GQ(h)\lambda^{F(V)}_{\langle id_{F(V)},V\rangle} = G(h)\varepsilon_V$. Therefore $\eta : G \to LF$ (resp. $\varepsilon : RF \to G$) is natural.

Let $H : \mathcal{C}' \to \mathcal{A}$ be a functor and $\beta : HF \to G$ be a natural transformation. We construct a natural transformation $\tau : H \to R$ as follows. For $Y \in \mathrm{Ob}\,\mathcal{C}'$, $(H(Y) \xrightarrow{\beta_X H(f)} G(X) = GQ\langle f, X\rangle)_{\langle f,X\rangle\in\mathrm{Ob}\,(Y\downarrow F)}$ is a cone. In fact, if $\varphi : \langle f, X\rangle \to \langle k, W\rangle$ is a morphism in $(Y{\downarrow}F)$, $GQ(\varphi)\beta_X H(f) = G(\varphi)\beta_X H(f) = \beta_W HF(\varphi)H(f) = \beta_W H(F(\varphi)f) = \beta_W H(k)$. Thus we have a unique morphism $\tau_Y : H(Y) \to R(Y)$ such that $\lambda^Y_{\langle f,X\rangle}\tau_Y = \beta_X H(f)$ for any $\langle f, X\rangle \in \mathrm{Ob}\,(Y{\downarrow}F)$.

To show the naturality of τ, take a morphism $g : Y \to Z$ in $\mathcal{C}'$. For each $\langle f, X\rangle \in \mathrm{Ob}\,(Z{\downarrow}F)$, since $\lambda^Z_{\langle f,X\rangle}\tau_Z H(g) = \beta_X H(f)\ H(g) = \beta_X H(fg) = \lambda^Y_{(g\downarrow id_F)\langle f,X\rangle}\tau_Y = \lambda^Z_{\langle f,X\rangle}R(g)\tau_Y$, we have $\tau_Z H(g) = R(g)\tau_Y$).

Finally, we show that the correspondence $\beta \mapsto \tau$ gives the inverse correspondence of the assignment $\tau \mapsto \varepsilon\tau_F$. For given $\beta \in \mathrm{Funct}(\mathcal{C}, \mathcal{A})(HF, G)$, construct $\tau \in \mathrm{Funct}(\mathcal{C}', \mathcal{A})(H, R))$ as above, then for any $X \in \mathrm{Ob}\,\mathcal{C}$, $\varepsilon_X\tau_{F(X)} = \lambda^{F(X)}_{\langle id_{F(X)},FX\rangle}\tau_{F(X)} = \beta_X$. Conversely, for given $\tau \in \mathrm{Funct}(\mathcal{C}', \mathcal{A})(H, R)$, apply the above construction to $\varepsilon\tau_F$ to have a natural transformation $\tau' : H \to R$. Since $\tau'_Y\lambda^Y_{\langle X,f\rangle} = H(f)\tau_{F(X)}\eta_X = \tau_Y R(f)\lambda^{F(X)}_{\langle X,id_{F(X)}\rangle} = \tau_Y\lambda^Y_{(F\downarrow f)\langle X,id_Y\rangle} = \tau_Y\lambda^Y_{\langle X,f\rangle}$ for any $\lambda^Y_{\langle f,X\rangle}\tau'_Y = \varepsilon_X\tau_{F(X)}H(f) = \lambda^{F(X)}_{\langle id_{F(X)},X\rangle}R(f)\tau_Y = \lambda^Y_{(f\downarrow F)\langle id_Y,X\rangle}\tau_Y = \lambda^Y_{\langle f,X\rangle}\tau_Y$ for any $\langle f, X\rangle \in \mathrm{Ob}\,(Y{\downarrow}F)$, we have $\tau'_Y = \tau_Y$. □

Corollary A.6.7 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $\mathcal{U}$ a universe. If $\mathcal{C}$ is $\mathcal{U}$-small and $\mathcal{A}$ is $\mathcal{U}$-cocomplete (resp. $\mathcal{U}$-complete), then for any functor $G : \mathcal{C} \to \mathcal{A}$, the left (resp. right) Kan extension of along F exists and a left (resp. right) adjoint of $F^* : \mathrm{Funct}(\mathcal{D}, \mathcal{A}) \to \mathrm{Funct}(\mathcal{C}, \mathcal{A})$ exists.*

Proposition A.6.8 *1) Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor and $\mathcal{A}$ a category. Suppose that, for any functor $G : \mathcal{C} \to \mathcal{A}$, a left (resp. right) Kan extension $F_!(G)$ (resp. $F_*(G)$) along F exists. Then, $F_!$ (resp. F_*) gives a functor $\mathrm{Funct}(\mathcal{C}, \mathcal{A}) \to \mathrm{Funct}(\mathcal{C}', \mathcal{A})$ which is a left (resp. right) adjoint of $F^* : \mathrm{Funct}(\mathcal{C}', \mathcal{A}) \to \mathrm{Funct}(\mathcal{C}, \mathcal{A})$ with unit η (resp. counit ε).*

2) Let $F_1 : \mathcal{C}_1 \to \mathcal{C}_2$, $F_2 : \mathcal{C}_2 \to \mathcal{C}_3$ and $G : \mathcal{C}_1 \to \mathcal{A}$ be functors. If left (resp. right) Kan extensions $(F_{1!}(G), \eta_1)$ and $(F_{2!}F_{1!}(G), \eta_2)$ (resp. $(F_{1*}(G), \varepsilon_1)$ and $(F_{2*}F_{1*}(G), \varepsilon_2)$) exist, then $(F_{2!}F_{1!}(G), \eta_{2F_1}\eta_1)$ (resp. $(F_{2*}F_{1*}(G), \varepsilon_1\varepsilon_{2F_1})$) is a left (resp. right) Kan extension of G along F_2F_1.

Proof. 1) Let $\varphi : G \to H$ be a morphism in $\mathrm{Funct}(\mathcal{C}, \mathcal{A})$. We define $F_!(\varphi) : F_!(G) \to F_!(H)$ (resp. $F_*(\varphi) : F_*(G) \to F_*(H)$) to be the map that corresponds to a composite $G \xrightarrow{\varphi} H \xrightarrow{\eta} F^*F_!(H)$ (resp. $F^*F_*(G) \xrightarrow{\varepsilon}$

$G \xrightarrow{\varphi} H$) by the $\mathrm{Funct}(\mathcal{C}',\mathcal{A})(F_!(G),F_!(H)) \xrightarrow{\cong} \mathrm{Funct}(\mathcal{C},\mathcal{A})(G,F^*F_!(H))$ (resp. $\mathrm{Funct}(\mathcal{C}',\mathcal{A})(F_*(G),F_*(H)) \xrightarrow{\cong} \mathrm{Funct}(\mathcal{C}',\mathcal{A})(F^*F_*(G),H)$) given by $\sigma \mapsto \sigma_F\eta$ (resp. $\tau \mapsto \varepsilon\tau_F$). Then, it is easy to verify that $F_!$ (resp. F_*) is a functor and the correspondence $\sigma \mapsto \sigma_F\eta$ (resp. $\tau \mapsto \varepsilon\tau_F$) is a bijection $\mathrm{Funct}(\mathcal{C}',\mathcal{A})(F_!(G),H) \xrightarrow{\cong} \mathrm{Funct}(\mathcal{C},\mathcal{A})(G,F^*(H))$ (resp. $\mathrm{Funct}(\mathcal{C}',\mathcal{A})(H,F_*(G)) \xrightarrow{\cong} \mathrm{Funct}(\mathcal{C}',\mathcal{A})(F^*(H),G)$) which is natural in both G and H.

2) It is easy to verify that $\sigma \mapsto \sigma_{F_2F_1}\eta_{2F_1}\eta_1$ and $\tau \mapsto \varepsilon_1\varepsilon_{2F_1}\tau_{F_2F_1}$ give bijections $\mathrm{Funct}(\mathcal{C}_3,\mathcal{A})(F_{2!}F_{1!}(G),H) \to \mathrm{Funct}(\mathcal{C}_1,\mathcal{A})(G,(F_2F_1)^*(H))$ and $\mathrm{Funct}(\mathcal{C}_1,\mathcal{A})(G,F_{2*}F_{1*}(H)) \to \mathrm{Funct}(\mathcal{C}_3,\mathcal{A})((F_2F_1)^*(G),H)$ for a functor $H : \mathcal{C}_3 \to \mathcal{A}$. □

Proposition A.6.9 *Let $\mathcal{C}$ be a finitely cocomplete $\mathcal{U}$-small category and $\mathcal{A}$ a $\mathcal{U}$-cocomplete and finitely complete category. Suppose that $\mathcal{U}$-colimits commute with finite limits in $\mathcal{A}$. If $F : \mathcal{C} \to \mathcal{C}'$ is a functor preserving finite colimits, $F_! : \mathrm{Funct}(\mathcal{C},\mathcal{A}) \to \mathrm{Funct}(\mathcal{C}',\mathcal{A})$ is left exact.*

Proof. Let $G : \mathcal{C} \to \mathcal{A}$ be a functor and Y an object of $\mathcal{C}'$. Since $(F{\downarrow}Y)$ is a filtered category by (A.5.5), $F_!(G)(Y) = \varinjlim_{(F\downarrow Y)} GP$ is a filtered colimit in $\mathcal{A}$. If $D : \mathcal{D} \to \mathrm{Funct}(\mathcal{C},\mathcal{A})$ is a functor with $\mathcal{D}$ a finite category, $F_!(\varprojlim_{\mathcal{D}} D)(Y) = \varinjlim_{(F\downarrow Y)}(\varprojlim_{\mathcal{D}} D)P = \varinjlim_{(F\downarrow Y)}(\varprojlim_{\mathcal{D}} D(i)P) \cong \varprojlim_{\mathcal{D}}(\varinjlim_{(F\downarrow Y)} D(i)P) = \varprojlim_{\mathcal{D}} F_!(D(i))$. □

Proposition A.6.10 *1) Let $F : \mathcal{C} \to \mathcal{C}'$ be a fully faithful functor and $G : \mathcal{C} \to \mathcal{A}$ a functor. Suppose that the condition in (A.6.5) (resp.(A.6.6)) holds. Then, $\eta : G \to F^*F_!(G)$ (resp. $\varepsilon : F^*F_*(G) \to G$) is a natural equivalence. Hence if the condition in (A.6.5) (resp.(A.6.6)) holds for any object of $\mathrm{Funct}(\mathcal{C},\mathcal{A})$, then $\mathrm{Funct}(\mathcal{C},\mathcal{A})$ can be regarded as a coreflexive (resp. reflexive) category of $\mathrm{Funct}(\mathcal{C}',\mathcal{A})$ with inclusion functor $F_!$ (resp. F_*) and a reflection F^*.*

2) Suppose that both left and right Kan extensions along $F : \mathcal{C} \to \mathcal{C}'$ of every object of $\mathrm{Funct}(\mathcal{C},\mathcal{A})$ exists, then the unit $\eta : id_{\mathrm{Funct}(\mathcal{C},\mathcal{A})} \to F^*F_!$ of the adjunction is an isomorphism if and only if so is the counit $\varepsilon : F^*F_* \to id_{\mathrm{Funct}(\mathcal{C},\mathcal{A})}$ of the adjunction.

Proof. 1) Let Z be an object of $\mathcal{C}$. Since F is fully faithful, there exists a unique morphism $\tilde{f} : X \to Z$ (resp. $\tilde{f} : Z \to X$) in $\mathcal{C}$ such that $F(\tilde{f}) = f$ for each object $\langle X, f\rangle$ (resp. $\langle f, X\rangle$) of $(F{\downarrow}F(Z))$ (resp. $(F(Z){\downarrow}F)$). Hence $\tilde{f}$ gives a unique morphism $\tilde{f} : \langle X, f\rangle \to \langle Z, id_{F(Z)}\rangle$ (resp. $\tilde{f} : \langle id_{F(Z)}, Z\rangle \to \langle f, X\rangle$) in $(F{\downarrow}F(Z))$ (resp. $(F(Z){\downarrow}F)$). Therefore $\langle Z, id_{F(Z)}\rangle$ (resp. $\langle id_{F(Z)}, Z\rangle$) is a terminal (resp. initial) object of $(F{\downarrow}F(Z))$ (resp. $(F(Z){\downarrow}F)$). It follows that for each object Z of $\mathcal{C}$, $(GP\langle X, f\rangle \xrightarrow{G(\tilde{f})} G(Z))_{\langle X,f\rangle \in \mathrm{Ob}\,(F\downarrow F(Z))}$ (resp. $(G(Z) \xrightarrow{G(\tilde{f})} GQ\langle f, X\rangle)_{\langle f,X\rangle \in \mathrm{Ob}\,(F(Z)\downarrow F)}$) is a colimiting (resp. limiting) cone for a functor $GP : (F{\downarrow}F(Z)) \to \mathcal{A}$ (resp. $GQ : (F(Z){\downarrow}F) \to \mathcal{A}$). Thus $\eta_Z = \lambda^{F(Z)}_{\langle Z,id\rangle}$ (resp. $\varepsilon_Z = \lambda^{F(Z)}_{\langle id,Z\rangle}$) is an isomorphism.

2) Generally, suppose that a functor $F : \mathcal{C} \to \mathcal{D}$ has both left and right adjoints $L, R : \mathcal{D} \to \mathcal{C}$. Let $\eta : id_{\mathcal{D}} \to FL$ and $\varepsilon : FR \to id_{\mathcal{D}}$ be the unit and the counit of the adjunctions. For any $X, Y \in \mathrm{Ob}\,\mathcal{D}$, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}(L(X),R(Y)) \arrow[r, "\Phi"] \arrow[d, "\Psi"] & \mathcal{D}(X,FR(Y)) \arrow[d, "\varepsilon_{Y*}"] \\
\mathcal{D}(FL(X),Y) \arrow[r, "\eta_X^*"] & \mathcal{D}(X,Y)
\end{tikzcd}$$

Here Φ and Ψ denote the bijections $f \mapsto F(f)\eta_X$ and $f \mapsto \varepsilon_Y F(f)$. Hence η is an isomorphism if and only if ε is so. □

Proposition A.6.11 *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor with a left adjoint G. We denote by $\eta : id_{\mathcal{C}'} \to FG$ and $\varepsilon : GF \to id_{\mathcal{C}}$ the unit and counit of the adjunction. Then, for any category $\mathcal{A}$ and functors $H : \mathcal{C} \to \mathcal{A}$, $K : \mathcal{C}' \to \mathcal{A}$, there is a natural bijection $\mathrm{Funct}(\mathcal{C}',\mathcal{A})(K,G^*(H)) \to \mathrm{Funct}(\mathcal{C},\mathcal{A})(F^*(K),H)$ given by $\sigma \mapsto H(\varepsilon)\sigma_F$. The inverse of this map is given by $\tau \mapsto \tau_G K(\eta)$. This shows that a right Kan extension of H along F is $G^*(H)$ and a left Kan extension of K along G is $F^*(K)$, namely, $F_* = G^*$ and $G_! = F^*$.*

Proof. The assertion follows from equalities $F(\varepsilon)\eta_F = id_F$ and $\varepsilon_G G(\eta) = id_G$.

Consider the case $\mathcal{A} = \mathcal{U}$-**Ens**. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor and regard this as a functor $\mathcal{C}^{op} \to \mathcal{C}'^{op}$. If $\mathcal{C}$ is $\mathcal{U}$-small, then for any presheaf G of $\mathcal{U}$-sets, left and right Kan extensions $F_!(G)$ and $F_*(G)$ of G along F exist. Hence $F^* : \widehat{\mathcal{C}'} \to \widehat{\mathcal{C}}$ has both left and right adjoints $F_!, F_* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$. □

Proposition A.6.12 *Let $\mathcal{C}$, $\mathcal{C}'$ be $\mathcal{U}$-categories and $F : \mathcal{C} \to \mathcal{C}'$ a functor. We denote by $h : \mathcal{C} \to \widehat{\mathcal{C}}$ and $h' : \mathcal{C}' \to \widehat{\mathcal{C}'}$ the Yoneda embeddings.*

1) If G is a presheaf on $\mathcal{C}$ such that the right Kan extension $F_*(G)$ of G along F exists, then for any $Y \in \mathrm{Ob}\,\mathcal{C}'$, there is an isomorphism $F_*(G)(Y) \cong \widehat{\mathcal{C}}(F^*(h'_Y), G)$ which is natural in Y. Moreover, if $F_*(G)$ exists for any $G \in \mathrm{Ob}\,\widehat{\mathcal{C}}$, the above isomorphism is also natural in G.

2) For any object Z of $\mathcal{C}$, the left Kan extension $F_!(h_Z)$ is given by $h'_{F(Z)}$ and the unit $\eta : h_Z \to F_!(h_Z)F = h'_{F(Z)}F$ given by $f \mapsto F(f)$. Hence if $F^* : \widehat{\mathcal{C}'} \to \widehat{\mathcal{C}}$ has a left adjoint $F_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$, we can choose $F_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ such that the following square commutes.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "h"] & \mathcal{C}' \arrow[d, "h'"] \\ \widehat{\mathcal{C}} \arrow[r, "F_!"] & \widehat{\mathcal{C}'} \end{tikzcd}$$

Moreover, there is a colimiting cone $(h'FP\langle X, f\rangle \xrightarrow{\lambda_{\langle X,f\rangle}} F_!(G))_{\langle X,f\rangle \in \mathrm{Ob}\,(h\downarrow id_G)}$.

3) Suppose that $F^* : \widehat{\mathcal{C}'} \to \widehat{\mathcal{C}}$ has a left adjoint $F_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$. Let H be a presheaf on $\mathcal{C}'$. For $\langle X, f\rangle \in \mathrm{Ob}\,(h\downarrow id_{HF})$, $\mu_{\langle X,f\rangle} : h'FP\langle X, f\rangle \to H$ denotes the unique morphism satisfying $(\mu_{\langle X,f\rangle})_{F(X)}(id_{F(X)}) = f_X(id_X)$. Then, a family of morphisms $(h'FP\langle X, f\rangle \xrightarrow{\mu_{\langle X,f\rangle}} H)_{\langle X,f\rangle \in \mathrm{Ob}\,(h\downarrow id_{HF})}$ is a cone and the counit $\varepsilon_H : F_!F^*(H) \to H$ is the unique morphism such that $\varepsilon_H \lambda_{\langle X,f\rangle} = \mu_{\langle X,f\rangle}$, where $\lambda_{\langle X,f\rangle}$ is the morphism given in 2) for $G = HF$.

4) If $\mathcal{C}$ is a $\mathcal{U}$-small category with finite limits and F is left exact, $F_!$ is left exact.

5) If F has a left adjoint $G : \mathcal{C}' \to \mathcal{C}$, $G^* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ is a left adjoint of F^*. Hence $G^* = F_!$, similarly, $F^* = G_*$.

Proof. 1) By the definition of the right Kan extension and the Yoneda's lemma, there is a natural equivalence $\widehat{\mathcal{C}}(F^*(h'_Y), G) \cong \widehat{\mathcal{C}'}(h'_Y, F_*(G)) \cong F_*(G)(Y)$.

2) For each $Z \in \mathrm{Ob}\,\mathcal{C}$, $Y \in \mathrm{Ob}\,\mathcal{C}'$ and $\langle X, f\rangle \in \mathrm{Ob}\,(F\downarrow Y)$, let $\alpha^Y_{\langle X,f\rangle} : h_Z P\langle X, f\rangle \to h'_{F(Z)}$ be a composite $h_Z(X) = \mathcal{C}^{op}(Z, X) \xrightarrow{F} \mathcal{C}'^{op}(F(Z), F(X)) \xrightarrow{f_*} \mathcal{C}'^{op}(F(Z), Y) = h'_{F(Z)}(Y)$. We claim that $(h_Z P\langle X, f\rangle \xrightarrow{\alpha^Y_{\langle X,f\rangle}} h'_{F(Z)}(Y))_{\langle X,f\rangle \in \mathrm{Ob}\,(F\downarrow Y)}$ is a colimiting cone for a functor $h_Z P : (F\downarrow Y) \to \mathcal{U}$-**Ens**. In fact, since $\alpha^Y_{\langle Z,g\rangle}(id_Z) = g$ for each $g \in h'_{F(Z)}(Y) = \mathcal{C}'^{op}(F(Z), Y)$, $(\alpha^Y_{\langle X,f\rangle} : h_Z P\langle X, f\rangle \to h'_{F(Z)}(Y))_{\langle X,f\rangle \in \mathrm{Ob}\,(F\downarrow Y)}$ is an epimorphic family. Let $(h_Z P\langle X, f\rangle \xrightarrow{\iota_{\langle X,f\rangle}} M)_{\langle X,f\rangle \in \mathrm{Ob}\,(F\downarrow Y)}$ be a cone. Define $\rho : h'_{F(Z)}(Y) \to M$ by $\rho(g) = \iota_{\langle Z,g\rangle}(id_Z)$. Since each $k \in h_Z(X) = h_Z P\langle X, f\rangle$ defines a morphism $k : \langle Z, fF(k)\rangle \to \langle X, f\rangle$ in $(F\downarrow Y)$, we have $\iota_{\langle X,f\rangle}(k) = \iota_{\langle X,f\rangle}(h_Z P(k)(id_Z)) = \iota_{\langle Z,fF(k)\rangle}(id_Z)$
$= \rho(fF(k)) = \rho\alpha^Y_{\langle X,f\rangle}(k)$. Thus we see $\rho\alpha^Y_{\langle X,f\rangle} = \iota_{\langle X,f\rangle}$ and this shows the assertion. $\eta_Z = \alpha^{F(Z)}_{\langle Z,id_Z\rangle} : h_Z \to h'_{F(Z)}F$ maps $f \in h_Z(W)$ to $F(f) \in h'_{F(Z)}(F(W))$ by the definition. Let $g : Y \to W$ be a morphism in $\mathcal{C}'^{op}$, then $h'_Z(g)\alpha^Y_{\langle X,f\rangle} = g_* f_* F = (gf)_* F = \alpha^Y_{\langle X,gf\rangle} = \alpha^W_{(id_F\downarrow g)\langle X,f\rangle}$. Since $(h_Z P\langle X, f\rangle \xrightarrow{\alpha^Y_{\langle X,f\rangle}} h'_{F(Z)}(Y))_{\langle X,f\rangle \in \mathrm{Ob}\,(F\downarrow Y)}$ is a colimiting cone for a functor $h_Z P : (F\downarrow Y) \to \mathcal{U}$-**Ens**, the above facts shows that we can define $F_!(h_Z)$ to be $h'_{F(Z)}$.

By (A.4.2), $(hP\langle X, f\rangle \xrightarrow{f} G)_{\langle X,f\rangle \in \mathrm{Ob}\,(h\downarrow id_G)}$ is a colimiting cone. Since $F_!$ has a right adjoint F^*, $F_!$ preserves colimits. Hence $(h'FP\langle X, f\rangle = F_!hP\langle X, f\rangle \xrightarrow{F_!(f)} F_!(G))_{\langle X,f\rangle \in \mathrm{Ob}\,(h\downarrow id_G)}$ is a colimiting cone.

3) Let $\varphi : \langle X, f\rangle \to \langle Y, g\rangle$ be a morphism in $(h\downarrow id_{HF})$. Then, $gh_\varphi = f : h_X \to HF$ and it follows that $(\mu_{\langle Y,g\rangle} h'FP(\varphi))_{F(X)}(id_{F(X)}) = (\mu_{\langle Y,g\rangle})_{F(X)}(h'_{F(\varphi)})_{F(X)}(id_{F(X)}) = (\mu_{\langle Y,g\rangle})_{F(X)}(F(\varphi)) =$
$(\mu_{\langle Y,g\rangle})_{F(X)} h'_{F(Y)}(F(\varphi))(id_{F(Y)}) = HF(\varphi)(\mu_{\langle Y,g\rangle})_{F(Y)}(id_{F(Y)}) = HF(\varphi)g_Y(id_Y) = g_X h_Y(\varphi)(id_Y) = f_X(id_X)$
$= (\mu_{\langle X,f\rangle})_{F(X)}(id_{F(X)})$. Therefore we have $\mu_{\langle Y,g\rangle} h'FP(\varphi) = \mu_{\langle X,f\rangle}$. This shows the first assertion and there exists a unique morphism $\varepsilon_H : F_!F^*(H) \to H$ such that $\varepsilon_H \lambda_{\langle X,f\rangle} = \mu_{\langle X,f\rangle}$. The adjoint of ε_H is a composite $F^*(H) \xrightarrow{\eta_{F^*(H)}} F^*F_!F^*(H) \xrightarrow{F^*(\varepsilon_H)} F^*(H)$ and we show that it is the identity morphism of $F^*(H)$. For any $\langle X, f\rangle \in \mathrm{Ob}\,(h\downarrow id_{F^*(H)})$, the left square of the following diagram commutes by the naturality of the unit $\eta : id_{\widehat{\mathcal{C}}} \to F^*F_!$ and so does the right one by the definition of ε_H.

$$\begin{tikzcd} hP\langle X, f\rangle \arrow[r, "\eta_{h_X}"] \arrow[d, "f"] & F^*F_!(hP\langle X, f\rangle) \arrow[d, "F^*F_!(f)"] \arrow[dr, "F^*(\mu_{\langle X,f\rangle})"] & \\ F^*(H) \arrow[r, "\eta_{F^*(H)}"] & F^*F_!F^*(H) \arrow[r, "F^*(\varepsilon_H)"] & F^*(H) \end{tikzcd}$$

Since $(\eta_{h_X})_X(id_X) = F(id_X) = id_{F(X)}$ by 2), we have $(F^*(\varepsilon_H)\eta_{F^*(H)}f)_X(id_X) = (F^*(\mu_{\langle X,f\rangle})\eta_{h_X})_X(id_X) = (\mu_{\langle X,f\rangle})_{F(X)}(\eta_{h_X})_X(id_X) = (\mu_{\langle X,f\rangle})_{F(X)}(id_{F(X)}) = f_X(id_X)$. Therefore we have $F^*(\varepsilon_H)\eta_{F^*(H)}f = f$ for any $\langle X, f\rangle \in \mathrm{Ob}\,(h{\downarrow}id_{F^*(H)})$. Since $(hP\langle X,f\rangle \xrightarrow{f} F^*(H))_{\langle X,f\rangle\in \mathrm{Ob}\,(h\downarrow id_{F^*(H)})}$ is an epimorphic family, it follows $F^*(\varepsilon_H)\eta_{F^*(H)} = id_{F^*(H)}$.

4) This follows from (A.6.9).

5) If we regard F, G as functors $F : \mathcal{C}^{op} \to \mathcal{C}'^{op}$, $G : \mathcal{C}'^{op} \to \mathcal{C}^{op}$, then G is a right adjoint of F. Hence the result follows from (A.6.11). □

Proposition A.6.13 *Under the assumptions of (A.6.12), the following conditions are equivalent.*

i) $F : \mathcal{C} \to \mathcal{C}'$ is fully faithful.

ii) $F_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}'$ is fully faithful.

iii) The unit $\eta : id_{\widehat{\mathcal{C}}} \to F^*F_!$ of the adjunction is an isomorphism.

iv) $F_* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}'$ is fully faithful.

v) The counit $\varepsilon : F^*F_* \to id_{\widehat{\mathcal{C}}}$ of the adjunction is an isomorphism.

Proof. The equivalences $ii) \Leftrightarrow iii)$ and $iv) \Leftrightarrow v)$ are general properties of adjoint functors. Implications $i) \Rightarrow iii)$ and $i) \Rightarrow v)$ are shown in (A.6.10). We also showed $iii) \Leftrightarrow v)$ in (A.6.10). It follows from 2) of (A.6.12) that $ii)$ implies $i)$.

Let $\mathcal{C}$ be a $\mathcal{U}$-category and X a presheaf on $\mathcal{C}$. Consider a comma category $(h{\downarrow}X)$ ($h : \mathcal{C} \to \widehat{\mathcal{C}}$ the Yoneda embedding). We denote by $P_X : (h{\downarrow}X) \to \mathcal{C}$ the projection functor. For a morphism $\alpha : X \to Y$ in $\widehat{\mathcal{C}}$, we set $P_\alpha = (id_h{\downarrow}\alpha) : (h{\downarrow}X) \to (h{\downarrow}Y)$. Then, $P_Y P_\alpha = P_X$.

For a presheaf F on $(h{\downarrow}X)$, let F_X be a presheaf on $\mathcal{C}$ defined as follows. Set $F_X(Z) = \coprod_{f\in\widehat{\mathcal{C}}(h_Z,X)} F\langle Z, f\rangle$ for $Z \in \mathrm{Ob}\,\mathcal{C}$. For $f \in \widehat{\mathcal{C}}(h_Z, X)$, a morphism $\sigma : Z \to W$ in $\mathcal{C}^{op}$ defines a morphism $\sigma_f : \langle Z, f\rangle \to \langle W, fh_\sigma\rangle$ in $(h{\downarrow}X)^{op}$. Let $F_X(\sigma) : F_X(Z) \to F_X(W)$ be the map induced by the composite $F\langle Z,f\rangle \xrightarrow{F(\sigma_f)} F\langle W, fh_\sigma\rangle \xrightarrow{\nu} \coprod_{g\in\widehat{\mathcal{C}}(h_W,X)} F\langle W, g\rangle$, where ν is the canonical morphism. Note that there is a morphism $p_X(F) : F_X \to X$ in $\widehat{\mathcal{C}}$ given by $(p_X(F))_Z(F\langle Z, f\rangle) = \{f_Z(id_Z)\}$.

If $\varphi : F \to G$ is a morphism in $\widehat{(h{\downarrow}X)}$, we have a morphism $\varphi_X : F_X \to G_X$ in $\widehat{\mathcal{C}}$ defined by $(\varphi_X)_Z = \coprod_{f\in\widehat{\mathcal{C}}(h_Z,X)} \varphi_{\langle Z,f\rangle}$. We define a functor $e_X : \widehat{(h{\downarrow}X)} \to \widehat{\mathcal{C}}/X$ by $e_X(F) = (p_X(F) : F_X \to X)$ for $F \in \mathrm{Ob}\widehat{(h{\downarrow}X)}$ and $e_X(\varphi) = \varphi_X$ for a morphism φ in $\widehat{(h{\downarrow}X)}$. □

Proposition A.6.14 *1) $e_X P_X^* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}/X$ is naturally equivalent to a functor X^* given in (A.3.9).*

2) For a morphism $\alpha : X \to Y$ in $\widehat{\mathcal{C}}$, a composition $\widehat{(h{\downarrow}Y)} \xrightarrow{P_\alpha^*} \widehat{(h{\downarrow}X)} \xrightarrow{e_X} \widehat{\mathcal{C}}/X$ is naturally equivalent to $\widehat{(h{\downarrow}Y)} \xrightarrow{e_Y} \widehat{\mathcal{C}}/Y \xrightarrow{\alpha^*} \widehat{\mathcal{C}}/X$.

3) Define a functor $H : (h{\downarrow}X) \to \widehat{\mathcal{C}}/X$ by $H\langle Z, g\rangle = (h_Z \xrightarrow{g} X)$ and $H(\varphi) = h_\varphi$. Then, H is naturally equivalent to a composition $(h{\downarrow}X) \xrightarrow{h'} \widehat{(h{\downarrow}X)} \xrightarrow{e_X} \widehat{\mathcal{C}}/X$, where $h' : (h{\downarrow}X) \to \widehat{(h{\downarrow}X)}$ is the Yoneda embedding.

Proof. 1) For $F \in \mathrm{Ob}\,\widehat{\mathcal{C}}$, since $e_X P_X^*(F) = (p_X(FP_X) : (FP_X)_X \to X)$ and $(FP_X)_X(Z) = \coprod_{f\in\widehat{\mathcal{C}}(h_Z,X)} FP_X\langle Z, f\rangle$, a map $FP_X\langle Z, f\rangle = F(Z) \to F(Z) \times X(Z)$ $x \mapsto (x, f_Z(id_Z))$ induces a natural equivalence $\psi_F : (FP_X)_X \to F \times X$ such that $\mathrm{pr}_2\psi_F = p_X(FP_X)$, where $\mathrm{pr}_2 : F \times X \to X$ is the projection. Since ψ_F is natural in F, this gives a natural equivalence $\psi : e_X P_X^* \to X^*$.

2) For $Z \in \mathrm{Ob}\,\mathcal{C}$ and $g \in \widehat{\mathcal{C}}(h_Z, X)$, a map $F\langle Z, \alpha g\rangle \to F\langle Z, \alpha g\rangle \times X(Z)$ $x \mapsto (x, g_Z(id_Z))$ induces a bijection $P_\alpha^*(F)_Y(Z) = \coprod_{g\in\widehat{\mathcal{C}}(h_Z,X)} F\langle Z, \alpha g\rangle \to (F_Y \times_Y X)(Z) = (\coprod_{k\in\widehat{\mathcal{C}}(h_Z,Y)} F\langle Z, k\rangle) \times_{Y(Z)} X(Z)$ which is natural in Z and commutes with projections.

3) Define a natural equivalence $\omega^X : e_X h' \to H$ as follows. For $\langle Z, g\rangle \in \mathrm{Ob}\,(h{\downarrow}X)$, let $\chi^X_{\langle Z,g\rangle} : P_{X!}(h'_{\langle Z,g\rangle}) \to h_Z$ be a natural transformation given by $(\chi^X_{\langle Z,g\rangle})_W(\alpha) = P_X(\alpha)$ $(\alpha \in h'_{\langle Z,g\rangle}\langle W, k\rangle)$. Then, it is easy to see that $g\chi^X_{\langle Z,g\rangle} = p_X(h'_{\langle Z,g\rangle})$. $\omega^X_{\langle Z,g\rangle}$ is define to be the morphism induced by $\chi^X_{\langle Z,g\rangle}$. Define $(\chi^X_{\langle Z,g\rangle})^{-1} : h_Z \to P_{X!}(h'_{\langle Z,g\rangle})$ as follows. For $W \in \mathrm{Ob}\,\mathcal{C}$ and $\alpha \in h_Z(W)$, let $\bar{\alpha} : \langle W, fh_\alpha\rangle \to \langle Z, g\rangle$ be the unique morphism in $(h{\downarrow}X)$ such that $P_X(\bar{\alpha}) = \alpha$. We set $(\chi^X_{\langle Z,g\rangle})^{-1}_W(\alpha) = \nu_{fh_\alpha}(\bar{\alpha})$, where $\nu_g : (h{\downarrow}X)(\langle W, k\rangle, \langle Z, g\rangle) \to$

$\coprod_{l\in\widehat{\mathcal{C}}(h_Z,X)} (h{\downarrow}X)(\langle W,l\rangle,\langle Z,g\rangle)$ denotes the canonical morphism into the k-th summand. It is easy to verify that $(\chi^X_{\langle Z,g\rangle})^{-1}$ is the inverse of $\chi^X_{\langle Z,g\rangle}$. Hence ω^X is an equivalence. $\square$

Let us denote by $\epsilon_X : \Sigma_X X^* \to id_{\widehat{\mathcal{C}}}$ the counit of the adjunction of $X^* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}/X$ and $\Sigma_X : \widehat{\mathcal{C}}/X \to \widehat{\mathcal{C}}$ ((A.3.9)). Then, for $F \in \mathrm{Ob}\,\widehat{\mathcal{C}}$, $(\epsilon_X)_F : \Sigma_X X^*(F) = F \times X \to F$ is the projection onto the first component.

Proposition A.6.15 *1) A composition $\Sigma_X e_X : \widehat{(h{\downarrow}X)} \to \widehat{\mathcal{C}}$ is a left adjoint of $P^*_X : \widehat{\mathcal{C}} \to \widehat{(h{\downarrow}X)}$. In fact, if we set $P_{X!} = \Sigma_X e_X$, the unit $\eta^X : id_{\widehat{(h\downarrow X)}} \to P^*_X P_{X!}$ and the counit $\varepsilon^X : P_{X!}P^*_X \to id_{\widehat{\mathcal{C}}}$ are given as follows. For $F \in \mathrm{Ob}\,\widehat{(h{\downarrow}X)}$ and $\langle Z,f\rangle \in \mathrm{Ob}\,(h{\downarrow}X)$, $(\eta^X_F)_{\langle Z,f\rangle} : F\langle Z,f\rangle \to \coprod_{g\in\widehat{\mathcal{C}}(h_Z,X)} F\langle Z,g\rangle = F_X P_X\langle Z,f\rangle$ is the canonical map into the f-th summand. Consider the natural equivalence $\Sigma_X(\psi) : P_{X!}P^*_X \to \Sigma_X X^*$ induced by the natural equivalence $\psi : e_X P^*_X \to X^*$ given in (A.6.14). Then, $\varepsilon^X = \epsilon_X \Sigma_X(\psi) : P_{X!}P^*_X \to id_{\widehat{\mathcal{C}}}$.*

2) The adjoint $q_X(F) : F \to P^*_X(X)$ of $p_X(F) : P_{X!}(F) = F_X \to X$ is the morphism such that $q_X(F)_{\langle Z,f\rangle}$ is the constant map onto $f_Z(id_Z) \in X(Z)$ for each $\langle Z,f\rangle \in \mathrm{Ob}\,(h{\downarrow}X)$.

Proof. 1) It is easy to verify the equalities $\varepsilon^X_{P_{X!}} P_{X!}(\eta^X) = id_{P_{X!}}$ and $P^*_X(\varepsilon^X)(\eta^X_{P^*_X}) = id_{P^*_X}$.

2) Since $q_X(F) = p_X(F)_{P_X}\eta^X_F$, the assertion follows from 1). $\square$

Proposition A.6.16 *$e_X : \widehat{(h{\downarrow}X)} \to \widehat{\mathcal{C}}/X$ is an equivalence of categories.*

Proof. We define $e_X^{-1} : \widehat{\mathcal{C}}/X \to \widehat{(h{\downarrow}X)}$ as follows. For $(q : H \to X) \in \mathrm{Ob}\,\widehat{\mathcal{C}}/X$, we set $e_X^{-1}(q : H \to X)\langle Z,f\rangle = q_Z^{-1}(f_Z(id_Z))$ ($\langle Z,f\rangle \in \mathrm{Ob}\,(h{\downarrow}X)$). If $\sigma : \langle Z,f\rangle \to \langle W,g\rangle$ is a morphism in $(h{\downarrow}X)^{op}$, it follows from the following commutative diagram and $g = fh_\sigma$ in $\widehat{\mathcal{C}}$ that $H(\sigma) : H(Z) \to H(W)$ maps $q_Z^{-1}(f_Z(id_Z))$ into $q_W^{-1}(g_W(id_W))$.

$$\begin{tikzcd}
H(Z) \arrow[r, "q_Z"] \arrow[d, "H(\sigma)"] & X(Z) \arrow[d, "X(\sigma)"] & h_Z(Z) \arrow[l, "f_Z"'] \arrow[d, "h_Z(\sigma)"] \\
H(W) \arrow[r, "q_W"] & X(W) & h_Z(W) \arrow[l, "f_W"']
\end{tikzcd}$$

Let $e_X^{-1}(q : H \to X)(\sigma) : q_Z^{-1}(f_Z(id_Z)) \to q_W^{-1}(g_W(id_W))$ be the restriction of $H(\sigma)$ to $q_Z^{-1}(f_Z(id_Z))$. For a morphism $\varphi : (q : H \to X) \to (r : K \to X)$ in $\widehat{\mathcal{C}}/X$, we define $e_X^{-1}(\varphi)_{\langle Z,f\rangle} : q_Z^{-1}(f_Z(id_Z)) \to r_Z^{-1}(f_Z(id_Z))$ to be the restriction of $\varphi_Z : H(Z) \to K(Z)$ to $q_Z^{-1}(f_Z(id_Z))$. It is easy to verify that $e_X^{-1}e_X = id_{\widehat{(h\downarrow X)}}$. For $(q : H \to X) \in \mathrm{Ob}\,\widehat{\mathcal{C}}/X$ and $Z \in \mathrm{Ob}\,\mathcal{C}$, let $\lambda_{qZ} : e_X^{-1}(q : H \to X)_X(Z) = \coprod_{f\in\widehat{\mathcal{C}}(h_Z,X)} q_Z^{-1}(f_Z(id_Z)) \to H(Z)$ be the map induced by the inclusion map $q_Z^{-1}(f_Z(id_Z)) \hookrightarrow H(Z)$. λ_{qZ} is bijective and satisfies $q_Z\lambda_{qZ} = p(e_X^{-1}(q : H \to X)_X)_Z : e_X^{-1}(q : H \to X)_X(Z) \to H(Z)$. Moreover, λ_{qZ} is natural in Z and we have a natural equivalence $\lambda_q : e_X^{-1}(q : H \to X)_X \to H$. If $\varphi : (q : H \to X) \to (r : K \to X)$ is a morphism in $\widehat{\mathcal{C}}/X$, we verify $\lambda_r e_X^{-1}(\varphi)_X = \varphi\lambda_q$. Thus we have a natural equivalence $\lambda : e_X e_X^{-1} \to id_{\widehat{\mathcal{C}}/X}$. $\square$

Corollary A.6.17 *$P_{X!} : \widehat{(h{\downarrow}X)} \to \widehat{\mathcal{C}}$ preserves monomorphic families, pull-backs and $\mathcal{U}$-colimits.*

Proof. Since $\widehat{\mathcal{C}}$ is $\mathcal{U}$-complete, $\mathcal{U}$-cocomplete and $\Sigma_X : \widehat{\mathcal{C}}/X \to \widehat{\mathcal{C}}$ creates pull-backs and colimits by (A.3.11), Σ_X preserves pull-backs and $\mathcal{U}$-colimits. Moreover Σ_X preserves monomorphic families. Thus the result follows from the fact that $P_{X!} = \Sigma_X e_X$ and e_X is an equivalence. $\square$

For $(Y \xrightarrow{\alpha} X) \in \mathrm{Ob}\,\widehat{\mathcal{C}}/X$, we set $[\alpha] = e_X^{-1}(Y \xrightarrow{\alpha} X)$. We denote by $P_{[\alpha]} : (h'{\downarrow}[\alpha]) \to (h{\downarrow}X)$ the projection functor. Define $Q_\alpha : (h'{\downarrow}[\alpha]) \to (h{\downarrow}Y)$ as follows. Note that for $\langle\langle Z,f\rangle,\sigma\rangle \in \mathrm{Ob}(h'{\downarrow}[\alpha])$, $\sigma_{\langle Z,f\rangle}(id_{\langle Z,f\rangle}) \in [\alpha]\langle Z,f\rangle = \alpha_Z^{-1}(f_Z(id_Z)) \subset Y(Z)$. Set $Q_\alpha\langle\langle Z,f\rangle,\sigma\rangle = \langle Z, \theta_Y(\sigma_{\langle Z,f\rangle}(id_{\langle Z,f\rangle}))\rangle$, where $\theta_Y : Y(Z) \to \widehat{\mathcal{C}}(h_Z,Y)$ is the bijection given in (A.1.6). If $\xi : \langle\langle Z,f\rangle,\sigma\rangle \to \langle\langle W,g\rangle,\tau\rangle$ is a morphism in $(h'{\downarrow}[\alpha])$, $P_{[\alpha]}(\xi) : \langle Z,f\rangle \to \langle W,g\rangle$ is a morphism in $(h{\downarrow}X)$ hence $\sigma = \tau h'_{P_{[\alpha]}(\xi)}$ and $f = gh_{P_X P_{[\alpha]}(\xi)}$. Note that the following diagram commutes where b_1, b_2 and b_3 are the bijections induced by the inclusion morphisms.

$$\begin{tikzcd}
\coprod_{k\in\widehat{\mathcal{C}}(h_V,X)} h'_{\langle Z,f\rangle}\langle V,k\rangle \arrow[r, "\coprod_{k\in\widehat{\mathcal{C}}(h_V,X)}(h'_{P_{[\alpha]}(\xi)})_{\langle V,k\rangle}"] \arrow[d, "b_1"] & \coprod_{k\in\widehat{\mathcal{C}}(h_V,X)} h'_{\langle W,g\rangle}\langle V,k\rangle \arrow[r, "\coprod_{k\in\widehat{\mathcal{C}}(h_V,X)}\tau_{\langle V,k\rangle}"] \arrow[d, "b_2"] & \coprod_{k\in\widehat{\mathcal{C}}(h_V,X)} \alpha_X^{-1}(k_V(id_V)) \arrow[d, "b_3"] \\
h_Z(V) \arrow[r, "(h_{P_X P_{[\alpha]}(\xi)})_V"] & h_W(V) \arrow[r, "\theta_Y(\tau_{\langle W,g\rangle}(id_{\langle W,g\rangle}))_V"] & Y(V)
\end{tikzcd}$$

Since the composition of the above left vertical maps is $\coprod_{k\in\widehat{\mathcal{C}}(h_V,X)}\sigma_{\langle V,k\rangle}$, it follows from $b_3\coprod_{k\in\widehat{\mathcal{C}}(h_V,X)}\sigma_{\langle V,k\rangle} = \theta_Y(\sigma_{\langle Z,f\rangle}(id_{\langle Z,f\rangle}))_V b_1$ that the composition of the above right vertical maps is $\theta_Y(\sigma_{\langle Z,f\rangle}(id_{\langle Z,f\rangle}))_V$. Thus $P_X P_{[\alpha]}(\xi) : Z \to W$ defines a morphism $Q_\alpha(\xi) : Q_\alpha\langle\langle Z,f\rangle,\sigma\rangle \to Q_\alpha\langle\langle W,g\rangle,\tau\rangle$ in $(h{\downarrow}Y)$.

Proposition A.6.18 *1) $Q_\alpha : (h'{\downarrow}[\alpha]) \to (h{\downarrow}Y)$ is an isomorphism of categories. Moreover, the composition $(h'{\downarrow}[\alpha]) \xrightarrow{Q_\alpha} (h{\downarrow}Y) \xrightarrow{(h\downarrow\alpha)} (h{\downarrow}X)$ coincides with the projection functor $P_{[\alpha]} : (h'{\downarrow}[\alpha]) \to (h{\downarrow}X)$.*

2) Define a functor $\Theta : \widehat{\mathcal{C}}/Y \to (\widehat{\mathcal{C}}/X)/(Y \xrightarrow{\alpha} X)$ by $(Z \xrightarrow{p} Y) \mapsto ((Z \xrightarrow{\alpha p} X) \xrightarrow{p} (Y \xrightarrow{\alpha} X))$. Then, the following diagram commutes up to natural equivalence.

$$\begin{tikzcd}
\widehat{(h{\downarrow}Y)} \arrow[r, "e_Y"] \arrow[d, "Q_\alpha^*"] & \widehat{\mathcal{C}}/Y \arrow[r, "\Theta"] & (\widehat{\mathcal{C}}/X)/(Y \xrightarrow{\alpha} X) \\
\widehat{(h'{\downarrow}[\alpha])} \arrow[r, "e_{[\alpha]}"] & \widehat{(h{\downarrow}X)} \arrow[r, "e_X/[\alpha]"] & (\widehat{\mathcal{C}}/X)/e_X[\alpha] \arrow[u, "\Sigma_{\lambda_\alpha}"]
\end{tikzcd}$$

Proof. 1) The inverse Q_α^{-1} of Q_α is given as follows. For $\langle Z,f\rangle \in \mathrm{Ob}\,(h{\downarrow}Y)$, $Q_\alpha^{-1}\langle Z,f\rangle = \langle\langle Z,\alpha f\rangle, \theta_{[\alpha]}(f_Z(id_Z))\rangle$. If $\varphi : \langle Z,f\rangle \to \langle W,g\rangle$ is a morphism in $(h{\downarrow}Y)$, φ defines a morphism $\varphi' : \langle Z,\alpha f\rangle \to \langle W,\alpha g\rangle$. We can easily verify that $\theta_{[\alpha]}(g_W(id_W))h'_{\varphi'} = \theta_{[\alpha]}(f_Z(id_Z))$. Hence φ' gives a morphism $Q_\alpha^{-1}(\varphi) : Q_\alpha^{-1}\langle Z,f\rangle \to Q_\alpha^{-1}\langle W,g\rangle$.

2) For $F \in \mathrm{Ob}\,\widehat{(h{\downarrow}Y)}$, $\Sigma_{\lambda_\alpha}(e_X/[\alpha])e_{[\alpha]}Q_\alpha^*(F)$ is given by

$$\Big(\Big(((FQ_\alpha)_{[\alpha]})_X \xrightarrow{p_X((FQ_\alpha)_{[\alpha]})} X\Big) \xrightarrow{\lambda_\alpha e_X(p_{[\alpha]}(FQ_\alpha))} (Y \xrightarrow{\alpha} X)\Big).$$

By the definition of Q_α, we have, for $Z \in \mathrm{Ob}\,\mathcal{C}$,

$$((FQ_\alpha)_{[\alpha]})_X(Z) = \coprod_{g\in\widehat{\mathcal{C}}(h_Z,X)}\ \coprod_{k\in\widehat{(h\downarrow X)}(h'_{\langle Z,g\rangle},[\alpha])} F\langle Z, \theta_Y(k_{\langle Z,g\rangle}(id_{\langle Z,g\rangle}))\rangle.$$

On the other hand, we have $\Theta e_Y(F) = ((F_Y \xrightarrow{\alpha p_Y(F)} X) \xrightarrow{p_Y(F)} (Y \xrightarrow{\alpha} X))$. Set $I_Z = \bigcup_{g\in\widehat{\mathcal{C}}(h_Z,X)}\{g\} \times \widehat{(h{\downarrow}X)}(h'_{\langle Z,g\rangle},[\alpha])$ and define a map $\kappa : I_Z \to \widehat{\mathcal{C}}(h_Z,Y)$ by $I_Z(g,k) = \theta_Y(k_{\langle Z,g\rangle}(id_{\langle Z,g\rangle}))$. Then it is easy to check that κ is bijective and it follows that we have a natural equivalence $\xi : ((FQ_\alpha)_{[\alpha]})_X \to F_Y$ such that $\alpha p_Y(F)\xi = p_X((FQ_\alpha)_{[\alpha]})$ and $p_Y(F)\xi = \lambda_\alpha e_X(p_{[\alpha]}(FQ_\alpha))$. $\square$

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between $\mathcal{U}$-small categories and H a presheaf on $\mathcal{C}$. We choose $F_!$ so that $F_!(h_Z) = h_{F(Z)}$ for any $Z \in \mathrm{Ob}\,\mathcal{C}$ (A.6.12). $F_! : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ induces a functor $F/H : (h{\downarrow}H) \to (h{\downarrow}F_!(H))$ by $\langle Z,f\rangle \mapsto \langle F(Z), F_!(f)\rangle$ $\varphi \mapsto F(\varphi)$. Then, the following diagram commutes.

$$\begin{tikzcd}
(h{\downarrow}H) \arrow[r, "F/H"] \arrow[d, "P_H"] & (h{\downarrow}F_!(H)) \arrow[d, "P_{F_!(H)}"] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{C}'
\end{tikzcd}$$

It follows from (A.6.8) that we can define left Kan extensions $(P_{F_!(H)}(F/H))_!, (FP_H)_! : \widehat{(h{\downarrow}H)} \to \widehat{\mathcal{C}'}$ by $(P_{F_!(H)!}(F/H))_! = P_{F_!(H)!}(F/H)_!$ and $(FP_H)_! = F_!P_{H!}$. Since Kan extensions are determined up to natural equivalences, there is a unique natural equivalence $\kappa : P_{F_!(H)!}(F/H)_! \to F_!P_{H!}$ such that $(FP_H)^*(\kappa)P_H^*(\eta)\eta^H = (F/H)^*(\eta^{F_!(H)})\eta'$, where $\eta : id_{\widehat{\mathcal{C}}} \to F^*F_!$ and $\eta' : id_{\widehat{(h\downarrow H)}} \to (F/H)^*(F/H)_!$ are the counits. In other words, the following diagram commutes for any $K \in \mathrm{Ob}\,(h{\downarrow}H)$ and $L \in \mathrm{Ob}\,\widehat{\mathcal{C}'}$, where the vertical maps are bijections given by adjoints.

$$\begin{tikzcd}
\widehat{\mathcal{C}'}(F_!P_{H!}(K),L) \arrow[r, "\kappa_K^*"] \arrow[d] & \widehat{\mathcal{C}}(P_{F_!(H)!}(F/H)_!(K),L) \arrow[d] \\
hcc(P_{H!}(K),F^*(L)) \arrow[d] & (h\widehat{\downarrow F_!(H)})((F/H)_!(K),P^*_{F_!(H)}(L)) \arrow[d] \\
\widehat{(h\downarrow H)}(K,P_H^*F^*(L)) \arrow[r, equal] & \widehat{(h\downarrow H)}(K,(F/H)^*P^*_{F_!(H)}(L))
\end{tikzcd}$$

Proposition A.6.19 *For any $K \in \mathrm{Ob}\,(h\downarrow H)$, $F_!(p_H(K))\kappa_K = p_{F_!(H)}((F/H)_!(K)) : P_{F_!(H)}((F/H)_!(K)) \to F_!(H)$ holds. Hence $\kappa : P_{F_!(H)!}(F/H)_! \to F_!P_{H!}$ defines a natural equivalence of functors $e_{F_!(H)}(F/H)_! \to F_!/He_H$. It follows that $(F/H)_!$ preserves terminal objects.*

Proof. Let $q_H(K) \in \widehat{\mathcal{C}'}(K, HP_H)$ be the adjoint of $p_H(K) \in \widehat{\mathcal{C}}(P_{H!}(K),H)$. It follows from (A.6.15) that $P_H^*(\eta)q_H(K)_{\langle Z,f\rangle} : K\langle Z,f\rangle \to H(Z) \to F_!(H)(F(Z))$ is the constant map onto $(\eta_H)_Z(f_Z(id_Z))$. Note that $(\eta_H)_Z(f_Z(id_Z)) = F^*F_!(f)(\eta_{h_Z})_Z(id_Z) = F^*F_!(f)(id_{F(Z)})$ by the naturality of η and $F_!(h_Z) = h_{F(Z)}$ ((A.6.12)). On the other hand, since the adjoint $q_{F_!(H)}((F/H)_!(K)) \in (h\downarrow F_!(H))((F/H)_!(K), F_!(H)P_{F_!(H)})$ of $p_{F_!(H)}((F/H)_!(K)) \in \widehat{\mathcal{C}}(P_{F_!(H)!}(F/H)_!(K), F_!(H))$ is the morphism such that $q_{F_!(H)}((F/H)_!(K))_{\langle Z,f\rangle}$ is the constant map onto $g_W(id_W) \in F_!(H)(W)$, it follows that $(q_{F_!(H)}((F/H)_!(K))_{F/H})_{\langle Z,f\rangle}\eta'_{\langle Z,f\rangle} = q_{F_!(H)}((F/H)_!(K))_{\langle F(Z),F_!(f)\rangle}\eta'_{\langle Z,f\rangle} : K\langle Z,f\rangle \to F_!(H)(F(Z))$ is the constant map onto $F_!(f)_Z(id_{F(Z)})$. Then, the first assertion follows from the following commutative diagram, where we set $\mathcal{D} = (h\downarrow F_!(H))$.

$$\begin{tikzcd}
\widehat{\mathcal{C}}(P_{H!}(K),H) \arrow[r, "F_!"] \arrow[dr, "\eta_*"] \arrow[dd] & \widehat{\mathcal{C}'}(F_!P_{H!}(K),F_!(H)) \arrow[r, "\kappa_K^*"] \arrow[d] & \widehat{\mathcal{C}}(P_{F_!(H)!}(F/H)_!(K),F_!(H)) \arrow[d] \\
 & \widehat{\mathcal{C}}(P_{H!}(K),F_!(H)F) \arrow[d] & \widehat{\mathcal{D}}((F/H)_!(K),F_!(H)P_{F_!(H)}) \arrow[d] \\
\widehat{(h\downarrow H)}(K,HP_H) \arrow[r, "P_H^*(\eta)"] & \widehat{(h\downarrow H)}(K,F_!(H)FP_H) \arrow[r, equal] & \widehat{(h\downarrow H)}(K,F_!(H)P_{F_!(H)}(F/H))
\end{tikzcd}$$

Since $F_!/H : \widehat{\mathcal{C}}/H \to \widehat{\mathcal{C}'}/F_!(H)$ maps the terminal object id_H of $\widehat{\mathcal{C}}/H$ to the terminal object $id_{F_!(H)}$ of $\widehat{\mathcal{C}'}/F_!(H)$ and e_H, $e_{F_!(H)}$ are equivalences, F/H preserves terminal objects. □

A.7 Localization

Let $\mathcal{C}$ ba a category and $\mathcal{S}$ be a family of morphisms in $\mathcal{C}$.

Definition A.7.1 *A localization of $\mathcal{C}$ by $\mathcal{S}$ is the data of a category $\mathcal{C}_\mathcal{S}$ and a functor $Q : \mathcal{C} \to \mathcal{C}_\mathcal{S}$ satisfing the following conditions.*

(i) $Q(s)$ is an isomorphism if $s \in \mathcal{S}$.
(ii) If $F : \mathcal{C} \to \mathcal{A}$ is a functor such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, then there exist a unique functor $F_\mathcal{S} : \mathcal{C}_\mathcal{S} \to \mathcal{A}$ satisfying $F = F_\mathcal{S}Q$.

Remark A.7.2 *If a localization of $\mathcal{C}$ by $\mathcal{S}$ exists, it is unique up to isomorphism of categories.*

There is a weaker notion of localization.

Definition A.7.3 *A weak localization of $\mathcal{C}$ by $\mathcal{S}$ is the data of a category $\mathcal{C}_\mathcal{S}$ and a functor $Q : \mathcal{C} \to \mathcal{C}_\mathcal{S}$ satisfing the following conditions.*

(i) $Q(s)$ is an isomorphism if $s \in \mathcal{S}$.
(ii) If $F : \mathcal{C} \to \mathcal{A}$ is a functor such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, then there exist a functor $F_\mathcal{S} : \mathcal{C}_\mathcal{S} \to \mathcal{A}$ and an equivalence $F \to F_\mathcal{S}Q$ of functors.
(iii) $Q^* : \mathrm{Funct}(\mathcal{C}_\mathcal{S},\mathcal{A}) \to \mathrm{Funct}(\mathcal{C},\mathcal{A})$ is fully faithful for any category $\mathcal{A}$.

Remark A.7.4 *1) Suppose that $G : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ is also a functor such that GQ is equivalent to F. Let us denote by $\tau : F \to F_{\mathcal{S}}Q$ and $\sigma : F \to GQ$ the equvalences. Then, by the condition (iii) above, there exist unique natural transformations $\alpha : F_{\mathcal{S}} \to G$ and $\beta : G \to F_{\mathcal{S}}$ satisfying $\alpha_Q = \sigma\tau^{-1}$ and $\beta_Q = \tau\sigma^{-1}$. Hence we have $Q^*(\beta\alpha) = \beta_Q\alpha_Q = id_{F_{\mathcal{S}}Q} = Q^*(id_{F_{\mathcal{S}}})$ and $Q^*(\alpha\beta) = \alpha_Q\beta_Q = id_{GQ} = Q^*(id_G)$ which imply that α is a natural equivalence whose inverse is β.*

2) If a data of a category $\mathcal{D}$ and a functor $R : \mathcal{C} \to \mathcal{D}$ is also a localization of $\mathcal{C}$ by $\mathcal{S}$, there exist functors $R_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}} \to \mathcal{D}$, $Q_{\mathcal{S}} : \mathcal{D} \to \mathcal{C}_{\mathcal{S}}$ and equivalences $\tau : R \to R_{\mathcal{S}}Q$, $\sigma : Q \to Q_{\mathcal{S}}R$. Since $Q_{\mathcal{S}}(\tau) : Q_{\mathcal{S}}R \to Q_{\mathcal{S}}R_{\mathcal{S}}Q$ and $R_{\mathcal{S}}(\sigma) : R_{\mathcal{S}}Q \to R_{\mathcal{S}}Q_{\mathcal{S}}R$ are equivalences, we have equivalences $Q_{\mathcal{S}}(\tau)\sigma : Q \to Q_{\mathcal{S}}R_{\mathcal{S}}Q$ and $R_{\mathcal{S}}(\sigma)\tau : R \to R_{\mathcal{S}}Q_{\mathcal{S}}R$. Then, there are unique natural transformations $\alpha : id_{\mathcal{C}_{\mathcal{S}}} \to Q_{\mathcal{S}}R_{\mathcal{S}}Q$ and $\beta : id_{\mathcal{D}} \to R_{\mathcal{S}}Q_{\mathcal{S}}$ satisfying $Q^*(\alpha) = Q_{\mathcal{S}}(\tau)\sigma$ and $R^*(\beta) = R_{\mathcal{S}}(\sigma)\tau$. Since fully faithful functors reflects isomorphisms, α and β are equivalences. Thus $\mathcal{D}$ is equivalent to $\mathcal{C}_{\mathcal{S}}$.

Let $\mathcal{U}$ be a universe and suppose that $\mathcal{C}$ is a $\mathcal{U}$-small category. For a family $\mathcal{S}$ of morphisms in $\mathcal{C}$, we construct a category $\mathcal{C}_{\mathcal{S}}$ as follows. Set $\mathrm{Ob}\,\mathcal{C}_{\mathcal{S}} = \mathrm{Ob}\,\mathcal{C}$ and let us denote by $(\mathrm{Mor}\,\mathcal{C})^n$ the n-fold direct product of $\mathrm{Mor}\,\mathcal{C}$ and by $\sigma, \tau : \mathrm{Mor}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}$ the maps associating to each morphism its source and target, respectively. For $X, Y \in \mathrm{Ob}\,\mathcal{C}_{\mathcal{S}}$ and a non-negative integer n, we denote by $\mathcal{W}_n(X,Y)$ a subset of $(\mathrm{Mor}\,\mathcal{C})^{2n+1}$ consisting of elements $(f_1, f_2, \dots, f_{2n+1})$ satisfying the following conditions.

(*i*) $\sigma(f_1) = X$ and $\tau(f_{2n+1}) = Y$.
(*ii*) $f_{2i} \in \mathcal{S}$ for $i = 1, 2, \dots, n$.
(*iii*) $\sigma(f_{2i}) = \sigma(f_{2i+1})$ and $\tau(f_{2i}) = \tau(f_{2i-1})$ for $i = 1, 2, \dots, n$.

For $X, Y, Z \in \mathrm{Ob}\,\mathcal{C}_{\mathcal{S}}$, define a map $(\tilde{\mu}_{X,Y,Z})_{m,n} : \mathcal{W}_m(X,Y) \times \mathcal{W}_n(Y,Z) \to \mathcal{W}_{m+n}(X,Z)$ by

$$(\tilde{\mu}_{X,Y,Z})_{m,n}((f_1, f_2, \dots, f_{2m+1}), (g_1, g_2, \dots, g_{2n+1})) = (f_1, f_2, \dots, f_{2m}, g_1 f_{2m+1}, g_2, \dots, g_{2n+1}).$$

We set $\mathcal{W}(X,Y) = \coprod_{n\geq 0} \mathcal{W}_n(X,Y)$ and let $\tilde{\mu}_{X,Y,Z} : \mathcal{W}(X,Y) \times \mathcal{W}(Y,Z) \to \mathcal{W}(X,Z)$ be the map induced by $(\tilde{\mu}_{X,Y,Z})_{m,n}$'s. Define an equivalence relation $\equiv$ on $\mathcal{W}(X,Y)$ generated by the following types of relations.

(*i*) $(f_1, f_2, \dots, f_{2i-2}, f_{2i-1}, f_{2i}, f_{2i+1}, \dots, f_{2m+1}) \equiv (f_1, f_2, \dots, f_{2i-2}, f_{2i+1}g, f_{2i+2}, \dots, f_{2m+1})$ if $f_{2i-1} = f_{2i}g$ for some $i = 1, 2, \dots, m$ and $g \in \mathrm{Mor}\,\mathcal{C}$.
(*ii*) $(f_1, f_2, \dots, f_{2i-1}, f_{2i}, f_{2i+1}, f_{2i+2}, \dots, f_{2m+1}) \equiv (f_1, f_2, \dots, f_{2i-2}, gf_{2i-1}, f_{2i+2}, \dots, f_{2m+1})$ if $f_{2i+1} = gf_{2i}$ for some $i = 1, 2, \dots, m$ and $g \in \mathrm{Mor}\,\mathcal{C}$.

Let $\mathcal{C}_{\mathcal{S}}(X,Y)$ be the quotient set of $\mathcal{W}(X,Y)$ by the relation $\equiv$. It is easy to verify that the above $\tilde{\mu}_{X,Y,Z}$ induces a map $\mu_{X,Y,Z} : \mathcal{C}_{\mathcal{S}}(X,Y) \times \mathcal{C}_{\mathcal{S}}(Y,Z) \to \mathcal{C}_{\mathcal{S}}(X,Z)$. We denote by $id_X \in \mathcal{C}_{\mathcal{S}}(X,X)$ the class of the identity morphism of X in $\mathcal{W}_0(X,X) = \mathcal{C}(X,X)$. Thus we have a category $\mathcal{C}_{\mathcal{S}}$.

Define a functor $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ by $Q(X) = X$ and letting $Q : \mathcal{C}(X,Y) \to \mathcal{C}_{\mathcal{S}}(Q(X), Q(Y))$ be the composition of the inclusion map $\mathcal{C}(X,Y) = \mathcal{W}_0(X,Y) \to \mathcal{W}(X,Y)$ and the quotient map $\mathcal{W}(X,Y) \to \mathcal{C}_{\mathcal{S}}(X,Y)$.

Proposition A.7.5 *The functor $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ constructed above is a localization of $\mathcal{C}$ by $\mathcal{S}$.*

Proof. Suppose that $s : X \to Y$ belongs to $\mathcal{S}$. Let $s' \in \mathcal{C}_{\mathcal{S}}(Y,X)$ be the class of $(id_Y, s, id_X) \in \mathcal{W}_1(Y,X)$. Since $(\tilde{\mu}_{X,Y,X})_{0,1}((s), (id_Y, s, id_X)) = (s, s, id_X) \equiv (id_X)$ and $(\tilde{\mu}_{Y,X,Y})_{1,0}((id_Y, s, id_X), (s)) = (id_Y, s, s) \equiv (id_Y)$, we have $Q(s)s' = id_X$ and $s'Q(s) = id_Y$. Thus $Q(s)$ is an isomorphism and $Q(s)^{-1} = s'$ is represented by $(id_Y, s, id_X) \in \mathcal{W}_1(Y,X)$. For $(f_1, f_2, \dots, f_{2n+1}) \in \mathcal{W}_n(X,Y)$, let $\varphi \in \mathcal{C}_{\mathcal{S}}(X,Y)$ be the class of $(f_1, f_2, \dots, f_{2n+1})$. Since $Q(f_i)$ is represented by $(f_i) \in \mathcal{W}_0(\sigma(f_i), \tau(f_i))$ and $Q(f_{2i})^{-1}$ is represented by $\bigl(id_{\tau(f_{2i})}, f_{2i}, id_{\sigma(f_{2i})}\bigr) \in \mathcal{W}_1(\tau(f_{2i}), \sigma(f_{2i}))$, $Q(f_{2n+1})Q(f_{2n})^{-1}Q(f_{2n-1})Q(f_{2n-2})^{-1}\cdots Q(f_3)Q(f_2)^{-1}Q(f_1)$ is also represented by $(f_1, f_2, \dots, f_{2n+1})$, hence $\varphi = Q(f_{2n+1})Q(f_{2n})^{-1}Q(f_{2n-1})Q(f_{2n-2})^{-1}\cdots Q(f_3)Q(f_2)^{-1}Q(f_1)$.

Suppose that $F : \mathcal{C} \to \mathcal{A}$ is a functor such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$. We define a functor $F_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ as follows. Set $F_{\mathcal{S}}(X) = F(X)$ for $X \in \mathrm{Ob}\,\mathcal{C}_{\mathcal{S}} = \mathrm{Ob}\,\mathcal{C}$. For $X, Y \in \mathrm{Ob}\,\mathcal{C}_{\mathcal{S}}$, define a map $\widetilde{F}_{X,Y} : \mathcal{W}(X,Y) \to \mathcal{A}(X,Y)$ by

$$\widetilde{F}_{X,Y}(f_1, f_2, \dots, f_{2n+1}) = F(f_{2n+1})F(f_{2n})^{-1}F(f_{2n-1})F(f_{2n-2})^{-1}\cdots F(f_3)F(f_2)^{-1}F(f_1)$$

if $(f_1, f_2, \dots, f_{2n+1}) \in \mathcal{W}_n(X,Y)$. It is easy to verify that $\widetilde{F}_{X,Y}(f_1, f_2, \dots, f_{2m+1}) = \widetilde{F}_{X,Y}(g_1, g_2, \dots, g_{2n+1})$ if $(f_1, f_2, \dots, f_{2m+1}) = (g_1, g_2, \dots, g_{2n+1})$. Hence $\widetilde{F}_{X,Y}$ induces a map $F_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}}(X,Y) \to \mathcal{A}(X,Y)$. We also have

$$\begin{aligned}\widetilde{F}_{X,Z}\tilde{\mu}_{X,Y,Z}((f_1, f_2, \dots, f_{2m+1}), (g_1, g_2, \dots, g_{2n+1})) &= \widetilde{F}_{X,Z}(f_1, f_2, \dots, f_{2m}, g_1 f_{2m+1}, g_2, \dots, g_{2n+1}) \\ &= \widetilde{F}_{Y,Z}(g_1, g_2, \dots, g_{2n+1})\widetilde{F}_{X,Y}(f_1, f_2, \dots, f_{2m+1})\end{aligned}$$

and $F_{\mathcal{S}}Q(f) = F(f)$ for $f \in \mathcal{C}(X, Y)$. Thus we have a functor $F_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ satisfying $F = F_{\mathcal{S}}Q$.

Let $G : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ be a functor satisfying $F = GQ$. Then, $G(X) = GQ(X) = F(X) = F_{\mathcal{S}}Q(X) = F_{\mathcal{S}}(X)$ for $X \in \mathrm{Ob}\,\mathcal{C}$. For $\varphi \in \mathcal{C}_{\mathcal{S}}(X, Y)$, suppose that $(f_1, f_2, \ldots, f_{2n+1}) \in \mathcal{W}_n(X, Y)$ is a representative of φ. Then,

$$\begin{aligned} G(\varphi) &= G(Q(f_{2n+1})Q(f_{2n})^{-1}Q(f_{2n-1})Q(f_{2n-2})^{-1} \cdots Q(f_3)Q(f_2)^{-1}Q(f_1)) \\ &= GQ(f_{2n+1})GQ(f_{2n})^{-1}GQ(f_{2n-1})GQ(f_{2n-2})^{-1} \cdots GQ(f_3)GQ(f_2)^{-1}GQ(f_1) \\ &= F(f_{2n+1})F(f_{2n})^{-1}F(f_{2n-1})F(f_{2n-2})^{-1} \cdots F(f_3)F(f_2)^{-1}F(f_1) \\ &= \widetilde{F}_{X,Y}(f_1, f_2, \ldots, f_{2n+1}) = F_{\mathcal{S}}(\varphi). \end{aligned}$$

This shows the uniqueness of $F_{\mathcal{S}}$. □

Lemma A.7.6 *Let $Q : \mathcal{C} \to \mathcal{D}$ be a functor. Suppose that $\mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{D}$ and $F(X) = X$ for any $X \in \mathrm{Ob}\,\mathcal{C}$. Then, $Q^* : \mathrm{Funct}(\mathcal{D}, \mathcal{A}) \to \mathrm{Funct}(\mathcal{C}, \mathcal{A})$ is faithful for any category $\mathcal{A}$.*

Proof. Let $\alpha, \beta : F \to G$ be morphisms in $\mathrm{Funct}(\mathcal{D}, \mathcal{A})$ such that $Q^*(\alpha) = Q^*(\beta) : \mathcal{C} \to \mathcal{A}$, namely $\alpha_{Q(X)} = \beta_{Q(X)} : FQ(X) \to GQ(X)$ for any $X \in \mathrm{Ob}\,\mathcal{C}$. Since $Q(X) = X$ for any $X \in \mathrm{Ob}\,\mathcal{C}$, we have $\alpha_X = \beta_X$. Hence $\alpha = \beta$ and Q^* is faithful. □

Proposition A.7.7 *If $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ is a strong localization of $\mathcal{C}$ by $\mathcal{S}$, then $Q^* : \mathrm{Funct}(\mathcal{C}_{\mathcal{S}}, \mathcal{A}) \to \mathrm{Funct}(\mathcal{C}, \mathcal{A})$ is fully faithful for any category $\mathcal{A}$. Hence a localization of $\mathcal{C}$ by $\mathcal{S}$ is a weak localization of $\mathcal{C}$ by $\mathcal{S}$.*

Proof. Since a localization of $\mathcal{C}$ by $\mathcal{S}$ is unique up to isomorphism of categories, we may assume that $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ is the functor constructed above. It follows from (A.7.6) that Q^* is faithful.

Let $\tilde{\alpha} : Q^*(F) \to Q^*(G)$ be a morphism in $\mathrm{Funct}(\mathcal{C}, \mathcal{A})$ for $F, G \in \mathrm{Ob}\,\mathrm{Funct}(\mathcal{C}_{\mathcal{S}}, \mathcal{A})$. For $X \in \mathrm{Ob}\,\mathcal{C}$, let $\alpha_X : F(X) \to G(X)$ be a morphism in $\mathcal{A}$ defined by $\alpha_X = \tilde{\alpha}_X : F(X) = F(Q(X)) \to G(Q(X)) = G(X)$. For a morphism $\varphi : X \to Y$ in $\mathcal{C}_{\mathcal{S}}$, let $(f_1, f_2, \ldots, f_{2n+1}) \in \mathcal{W}_n(X, Y)$ be a representative of φ. By the naturality of $\tilde{\alpha}_X$, the following diagram commutes for $i = 1, 2, \ldots, 2n + 1$.

$$\begin{array}{ccc} FQ(\sigma(f_i)) & \xrightarrow{\tilde{\alpha}_{\sigma(f_i)}} & GQ(\sigma(f_i)) \\ \downarrow{\scriptstyle FQ(f_i)} & & \downarrow{\scriptstyle GQ(f_i)} \\ FQ(\tau(f_i)) & \xrightarrow{\tilde{\alpha}_{\tau(f_i)}} & GQ(\tau(f_i)) \end{array}$$

Since $\varphi = Q(f_{2n+1})Q(f_{2n})^{-1}Q(f_{2n-1})Q(f_{2n-2})^{-1} \cdots Q(f_3)Q(f_2)^{-1}Q(f_1)$ as we have shown in the proof of (A.7.5), we have $F(\varphi) = FQ(f_{2n+1})FQ(f_{2n})^{-1}FQ(f_{2n-1})FQ(f_{2n-2})^{-1} \cdots FQ(f_3)FQ(f_2)^{-1}FQ(f_1)$ and $G(\varphi) = GQ(f_{2n+1})GQ(f_{2n})^{-1}GQ(f_{2n-1})GQ(f_{2n-2})^{-1} \cdots GQ(f_3)GQ(f_2)^{-1}GQ(f_1)$. It follows from the commutativity of the above diagram that the following diagram commutes.

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ \downarrow{\scriptstyle F(\varphi)} & & \downarrow{\scriptstyle G(\varphi)} \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array}$$

Hence $\alpha \in \mathrm{Funct}(\mathcal{C}_{\mathcal{S}}, \mathcal{A})(F, G)$ and $Q^* : \mathrm{Funct}(\mathcal{C}_{\mathcal{S}}, \mathcal{A})(F, G) \to \mathrm{Funct}(\mathcal{C}, \mathcal{A})(FQ, GQ)$ maps α to $\tilde{\alpha}$. It follows that Q^* is full. □

Lemma A.7.8 *Consider functors $Q : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{A}$. $Q^* : \mathrm{Funct}(\mathcal{D}, \mathcal{A})(F, G) \to \mathrm{Funct}(\mathcal{C}, \mathcal{A})(FQ, GQ)$ is bijective for any functor $F : \mathcal{D} \to \mathcal{A}$ if the following conditions are satisfied.*

(i) For any $U, Z \in \mathcal{C}$ and $t \in \mathcal{D}(Q(U), Q(Z))$, there exist morphisms $g : Z \to W$ and $f : U \to W$ in $\mathcal{C}$ such that $Q(g)$ is an isomorphism and $t = Q(g)^{-1}Q(f)$.

(ii) For any $X \in \mathrm{Ob}\,\mathcal{D}$, there exist $Y \in \mathrm{Ob}\,\mathcal{C}$ and a morphism $s : X \to Q(Y)$ in $\mathcal{D}$ which satisfy the following conditions.

(a) $G(s)$ is an isomorphism.

(b) For any $Z \in \mathrm{Ob}\,\mathcal{C}$ and any morphism $t : X \to Q(Z)$ in $\mathcal{D}$, there exist morphisms $s' : Z \to W$ and $t' : Y \to W$ in $\mathcal{C}$ such that $GQ(s')$ is a monomorphism and the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{s} & Q(Y) \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle Q(t')} \\ Q(Z) & \xrightarrow{Q(s')} & Q(W) \end{array}$$

Proof. Suppose that $\varphi, \psi : F \to G$ satisfy $\varphi_{Q(Y)} = \psi_{Q(Y)}$ for any $Y \in \mathrm{Ob}\,\mathcal{C}$. For $X \in \mathrm{Ob}\,\mathcal{D}$, there exist $Y \in \mathrm{Ob}\,\mathcal{C}$ and a morphism $s : X \to Q(Y)$ in $\mathcal{D}$ such that $G(s)$ is an isomorphism by the assumption. Then the following diagram commutes and we have $G(s)\varphi_X = G(s)\psi_X$.

$$\begin{array}{ccccc} F(X) & \xrightarrow{F(s)} & FQ(Y) & \xleftarrow{F(s)} & F(X) \\ \downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \varphi_{Q(Y)}=\psi_{Q(Y)}} & & \downarrow{\scriptstyle \psi_X} \\ G(X) & \xrightarrow{G(s)} & GQ(Y) & \xleftarrow{G(s)} & G(X) \end{array}$$

Since $G(s)$ is an isomorphism, we have $\varphi_X = \psi_X$. Hence $Q^* : \mathrm{Funct}(\mathcal{D}, \mathcal{A})(F, G) \to \mathrm{Funct}(\mathcal{C}, \mathcal{A})(FQ, GQ)$ is injective.

Let $\theta : FQ \to GQ$ be a morphism in $\mathrm{Funct}(\mathcal{C}, \mathcal{A})$. For each $X \in \mathrm{Ob}\,\mathcal{D}$, choose a morphism $s : X \to Q(Y)$ in $\mathcal{D}$ satisfying the conditions (a) and (b). Define $\varphi_X : F(X) \to G(X)$ by $\varphi_X = G(s)^{-1}\theta_Y F(s)$. Let $f : X \to V$ a morphism in $\mathcal{D}$. Choose morphisms $s : X \to Q(Y)$ and $v : V \to Q(Z)$ in $\mathcal{D}$ satisfying the conditions (a) and (b). Applying the condition (b) to $s : X \to Q(Y)$ and $vf : X \to Q(Z)$, there exist morphisms $s' : Z \to W$ and $t' : Y \to W$ in $\mathcal{C}$ such that $GQ(s')$ is a monomorphism and $Q(t')s = Q(s')vf$. By the commutativity of the following diagram, we have $GQ(s')G(v)\varphi_V F(f) = \theta_W FQ(s')F(v)F(f) = \theta_W F(Q(s')vf) = \theta_W F(Q(t')s) = \theta_W FQ(t')F(s) = GQ(t')G(s)\varphi_X = G(Q(t')s)\varphi_X = G(Q(s')vf)\varphi_X = GQ(s')G(v)G(f)\varphi_X$.

$$\begin{array}{ccccccccc} F(X) & \xrightarrow{F(s)} & FQ(Y) & \xrightarrow{FQ(t')} & FQ(W) & \xleftarrow{FQ(s')} & FQ(Z) & \xleftarrow{F(v)} & F(V) \\ \downarrow{\scriptstyle \varphi_X} & & \downarrow{\scriptstyle \theta_Y} & & \downarrow{\scriptstyle \theta_W} & & \downarrow{\scriptstyle \theta_Z} & & \downarrow{\scriptstyle \varphi_V} \\ G(X) & \xrightarrow[\cong]{G(s)} & GQ(Y) & \xrightarrow{GQ(t')} & GQ(W) & \xleftarrow[\cong]{GQ(s')} & GQ(Z) & \xleftarrow[\cong]{G(v)} & G(V) \end{array}$$

Since $GQ(s')$ is a monomorphism and $G(v)$ is an isomorphism, it follows that $\varphi_V F(f) = G(f)\varphi_X$. In particular, taking $f = id_X$, we see that φ_X does not depend on the choice of s. Thus we have a morphism $\varphi : F \to G$. It remains to show $Q^*(\varphi) = \theta$, that is $\varphi_{Q(U)} = \theta_U$ for any $U \in \mathrm{Ob}\,\mathcal{C}$. Clearly, $s = id_{Q(U)}$ satisfies the condition (a). For any $Z \in \mathrm{Ob}\,\mathcal{C}$ and any morphism $t : Q(U) \to Q(Z)$ in $\mathcal{D}$, there exist morphisms $g : Z \to W$ and $f : U \to W$ in $\mathcal{C}$ such that $Q(g)$ is an isomorphism and $t = Q(g)^{-1}Q(f)$. We set $s' = g$ and $t' = f$, then we see that $s = id_{Q(U)}$ also satisfies the condition (b). Since the definition of $\varphi_{Q(U)}$ does not depend on the choice of s, we have $\varphi_{Q(U)} = G(s)^{-1}\theta_U F(s) = \theta_U$. □

Definition A.7.9 *A family $\mathcal{S}$ of morphisms in a category $\mathcal{C}$ is called a right multiplicative system if it satisfies the following conditions.*

(i) Every isomorphism belongs to $\mathcal{S}$.

(ii) $\mathcal{S}$ is closed under the composition of morphisms.

(iii) Let $f : X \to Y$ and $s : X \to Z$ be morphisms in $\mathcal{C}$. If s belongs to $\mathcal{S}$, there exist a morphism $g : Z \to W$ and a morphism $t : Y \to W$ which belongs to $\mathcal{S}$ satisfying $tf = gs$.

(iv) Let $f, g : X \to Y$ be morphisms in $\mathcal{C}$. If there exists a morphism $s : W \to X$ which belongs to $\mathcal{S}$ satisfying $fs = gs$, there exists a morphism $t : Y \to Z$ which belongs to $\mathcal{S}$ satisfying $tf = tg$.

Definition A.7.10 *A family $\mathcal{S}$ of morphisms in a category $\mathcal{C}$ is called a left multiplicative system if it satisfies the following conditions.*

(i) Every isomorphism belongs to $\mathcal{S}$.

(ii) $\mathcal{S}$ is closed under the composition of morphisms.

(iii) Let $f : Y \to X$ and $s : Z \to X$ be morphisms in $\mathcal{C}$. If s belongs to $\mathcal{S}$, there exist a morphism $g : W \to Z$ and a morphism $t : W \to Y$ which belongs to $\mathcal{S}$ satisfying $ft = sg$.

(iv) Let $f, g : Y \to X$ be morphisms in $\mathcal{C}$. If there exists a morphism $s : X \to W$ which belongs to $\mathcal{S}$ satisfying $sf = sg$, there exists a morphism $t : Z \to Y$ which belongs to $\mathcal{S}$ satisfying $ft = gt$.

Remark A.7.11 *Let $\mathcal{S}$ be a family of morphisms in $\mathcal{C}$. Then, $\mathcal{S}$ is a right multiplicative system in $\mathcal{C}$ if and only if $\mathcal{S}$ is a left multiplicative system in $\mathcal{C}^{op}$.*

Suppose that a family $\mathcal{S}$ of morphisms in a category $\mathcal{C}$ satisfies (i) and (ii) of (A.7.9). For $X \in \mathrm{Ob}\,\mathcal{C}$, we define categories $\mathcal{S}^X$, $\mathcal{S}_X$ and functors $\alpha^X : \mathcal{S}^X \to \mathcal{C}$, $\alpha_X : \mathcal{S}_X \to \mathcal{C}$ as follows. Set $\mathrm{Ob}\,\mathcal{S}^X = \{s \in \mathcal{S} \mid \sigma(s) = X\}$, $\mathcal{S}^X(s,t) = \{f \in \mathcal{C}(\tau(s),\tau(t)) \mid fs = t\}$ and $\mathrm{Ob}\,\mathcal{S}^X = \{s \in \mathcal{S} \mid \tau(s) = X\}$, $\mathcal{S}_X(s,t) = \{f \in \mathcal{C}(\sigma(s),\sigma(t)) \mid tf = s\}$. We set $\alpha^X(s) = \tau(s)$, $\alpha^X(f : s \to t) = (f : \tau(s) \to \tau(t))$ and $\alpha_X(s) = \sigma(s)$, $\alpha^X(f : s \to t) = (f : \sigma(s) \to \sigma(t))$.

Proposition A.7.12 *Let $\mathcal{S}$ be a right multiplicative system in $\mathcal{C}$.*

(1) For morphisms $f : X \to Y$ and $s : X \to Z$ in $\mathcal{C}$ such that $s \in \mathcal{S}$, we define a category $Sq(f,s)$ as follows. $\mathrm{Ob}\,Sq(f,s)$ consists of diagrams $Z \xrightarrow{g} W \xleftarrow{t} Y$ such that $tf = gs$ and $t \in \mathcal{S}$. For $(Z \xrightarrow{g} W \xleftarrow{t} Y), (Z \xrightarrow{h} V \xleftarrow{u} Y) \in \mathrm{Ob}\,Sq(f,s)$, we set $Sq(f,s)((Z \xrightarrow{g} W \xleftarrow{t} Y), (Z \xrightarrow{h} V \xleftarrow{u} Y)) = \{\varphi \in \mathcal{C}(W,V) \mid \varphi g = h,\ \varphi t = u\}$. Then, $Sq(f,s)$ is a filtered category.

(2) For morphisms $f, g : X \to Y$, let us denote by $\mathcal{S}^Y(f,g)$ the full subcategory of $\mathcal{S}^Y$ consisting objects $(s : Y \to Z)$ which satisfy $sf = sg$. Then $\mathcal{S}^Y(f,g)$ is a filtered category. In particular, $\mathcal{S}^X = \mathcal{S}^X(id_X, id_X)$ is a filtered category.

Proof. (1) $Sq(f,s)$ is not empty by (iii) of (A.7.9). For $(Z \xrightarrow{g} W \xleftarrow{t} Y), (Z, \xrightarrow{g} W' \xleftarrow{t'} Y) \in \mathrm{Ob}\,Sq(f,s)$, there exist morphisms $v : W \to V$ and $v' : W' \to V$ satisfying $vt = v't'$ and $v \in \mathcal{S}$ by (iii) of (A.7.9). Then, $vgs = vtf = v't'f = v'g's$ and this implies the existence of $(u : V \to U) \in \mathcal{S}$ satisfying $uvg = uv'g'$ by (iv) of (A.7.9). We put $k = uvg = uv'g'$ and $w = uvt = uv't'$. Since $u, v, t \in \mathcal{S}$, $w \in \mathcal{S}$ by (ii) of (A.7.9). Thus we have morphisms $uv : (Z \xrightarrow{g} W \xleftarrow{t} Y) \to (Z \xrightarrow{k} U \xleftarrow{w} Y)$ and $uv' : (Z \xrightarrow{g'} W' \xleftarrow{t'} Y) \to (Z \xrightarrow{k} U \xleftarrow{w} Y)$.

If $\varphi, \psi : (Z \xrightarrow{g} W \xleftarrow{t} Y) \to (Z' \xrightarrow{h} W' \xleftarrow{u} Y)$ are morphisms in $Sq(f,s)$, we have $h = \varphi g = \psi g$ and $u = \varphi t = \psi t$. Hence, by (iv) of (A.7.9), there exists a morphism $\theta : V \to U$ satisfying $\theta\varphi = \theta\psi$ in $\mathcal{C}$. Then, $\theta : (Z \xrightarrow{h} V \xleftarrow{u} Y) \to (Z \xrightarrow{\theta h} U \xleftarrow{\theta u} Y)$ satisfies $\theta\varphi = \theta\psi$ in $Sq(f,s)$.

(2) For $s, t \in \mathrm{Ob}\,\mathcal{S}^Y(f,g)$, there exist a morphism $p : \tau(s) \to W$ and a morphism $q : \tau(t) \to W$ which belongs to $\mathcal{S}$ satisfying $ps = qt$ by (iii) of (A.7.9). We put $h = ps = qt$, then $(h : Y \to W)$ is an object of $\mathcal{S}^Y(f,g)$ and we have morphisms $p : (s : Y \to \tau(s)) \to (h : Y \to W)$ and $q : (t : Y \to \tau(t)) \to (h : Y \to W)$ in $\mathcal{S}^Y(f,g)$. Let $p, q : s \to t$ be morphisms in $\mathcal{S}^Y(f,g)$. Since $ps = qs = t$ holds in $\mathcal{C}$ and $s \in \mathcal{S}$, there exists a morphism $u : \tau(t) \to Z$ which belongs to $\mathcal{S}$ satisfying $up = uq$. Hence we have a morphism $u : t \to ut$ in $\mathcal{S}^Y(f,g)$ satisfying $up = uq$ in $\mathcal{S}^Y(f,g)$. □

Remark A.7.13 *We have shown that, for $(Z \xrightarrow{g} W \xleftarrow{t} Y), (Z, \xrightarrow{g} W' \xleftarrow{t'} Y) \in \mathrm{Ob}\,Sq(f,s)$, there exist morphisms $\varphi : (Z \xrightarrow{g} W \xleftarrow{t} Y) \to (Z \xrightarrow{k} U \xleftarrow{w} Y)$ and $\psi : (Z \xrightarrow{g'} W' \xleftarrow{t'} Y) \to (Z \xrightarrow{k} U \xleftarrow{w} Y)$ such that $\varphi : W \to U$ belongs to $\mathcal{S}$.*

Let $\mathcal{S}$ be a right multiplicative system in a $\mathcal{U}$-category $\mathcal{C}$. For $X, Y \in \mathrm{Ob}\,\mathcal{C}$, we define a functor $D_{X,Y} : \mathcal{S}^Y \to \mathcal{U}$-**Ens** by $D_{X,Y}(s : Y \to V) = \mathcal{C}(X,V)$ and $D_{X,Y}(f : s \to t) = (f_* : \mathcal{C}(X,\tau(s)) \to \mathcal{C}(X,\tau(t)))$. We denote by $\mathcal{C}^r_{\mathcal{S}}(X,Y)$ the colimit of $D_{X,Y}$ and by $\left(D_{X,Y}(s) \xrightarrow{\iota_{X,Y,s}} \mathcal{C}^r_{\mathcal{S}}(X,Y)\right)_{s \in \mathrm{Ob}\,\mathcal{S}^Y}$ the colimiting cone of $D_{X,Y}$.

For $X, Y, Z \in \mathrm{Ob}\,\mathcal{C}^r_{\mathcal{S}} = \mathrm{Ob}\,\mathcal{C}$ and $(s : Y \to U) \in \mathrm{Ob}\,\mathcal{S}^Y$, $(t : Z \to V) \in \mathrm{Ob}\,\mathcal{S}^Z$, we define a map $\bar{\mu}_{X,Y,Z,s,t} : D_{X,Y}(s) \times D_{Y,Z}(t) \to \mathcal{C}^r_{\mathcal{S}}(X,Z)$ as follows. For $\bar{\alpha} \in D_{X,Y}(s) = \mathcal{C}(X,U)$ and $\bar{\beta} \in D_{Y,Z}(t) = \mathcal{C}(Y,V)$, there exist $\bar{\gamma} \in \mathcal{C}(U,W)$ and $u \in \mathcal{C}(V,W) \cap \mathcal{S}$ satisfying $u\bar{\beta} = \bar{\gamma}s$ by (iii) of (A.7.9). We regard $(ut : Z \to W)$ as an object of $\mathcal{S}^Z$ and define $\bar{\mu}_{X,Y,Z,s,t}(\bar{\alpha},\bar{\beta})$ to be the image of $\bar{\gamma}\bar{\alpha} \in \mathcal{C}(X,W) = D_{X,Z}(ut)$ by $\iota_{X,Z,ut} : D_{X,Z}(ut) \to \mathcal{C}^r_{\mathcal{S}}(X,Z)$. Suppose that $\tilde{\gamma} \in \mathcal{C}(U,W')$ and $u' \in \mathcal{C}(V,W') \cap \mathcal{S}$ also satisfy $u'\bar{\beta} = \tilde{\gamma}s$. Since $(U \xrightarrow{\bar{\gamma}} W \xleftarrow{u} V)$ and $(U \xrightarrow{\tilde{\gamma}} W' \xleftarrow{u'} V)$ are objects of $Sq(\bar{\beta},s)$, it follows from (1) of (A.7.12) that there exist morphisms $\varphi : (U \xrightarrow{\bar{\gamma}} W \xleftarrow{u} V) \to (U \xrightarrow{\delta} W'' \xleftarrow{w} V)$ and $\psi : (U \xrightarrow{\tilde{\gamma}} W' \xleftarrow{u'} V) \to (U \xrightarrow{\delta} W'' \xleftarrow{w} V)$ in $Sq(\bar{\beta},s)$. By (A.7.13), we may assume $\varphi \in \mathcal{S}$. Hence $\varphi ut = wt = \psi u't : Z \to W''$ is an object of $\mathcal{S}^Z$ and we can regard $\varphi : ut \to \varphi ut$ and $\psi : u't \to \psi u't$ as morphisms in $\mathcal{S}^Z$. Since $\delta = \varphi\bar{\gamma} = \psi\tilde{\gamma}$, $D_{X,Z}(\varphi) : D_{X,Z}(ut) \to D_{X,Z}(\varphi ut)$ maps $\bar{\gamma}\bar{\alpha}$ to $\delta\bar{\alpha}$ and $D_{X,Z}(\psi) : D_{X,Z}(u't) \to D_{X,Z}(\psi u't)$ maps $\tilde{\gamma}\bar{\alpha}$ to $\delta\tilde{\alpha}$. Therefore $\iota_{X,Z,ut}(\bar{\gamma}\bar{\alpha}) = \iota_{X,Z,\varphi ut}(\delta\bar{\alpha}) = \iota_{X,Z,\psi u't}(\delta\bar{\alpha}) = \iota_{X,Z,u't}(\tilde{\gamma}\bar{\alpha})$ and we see that $\iota_{X,Z,ut}(\bar{\gamma}\bar{\alpha})$ does not depend on the choice of $\bar{\gamma}$ and u.

For $\bar{\alpha} \in D_{X,Y}(s) = \mathcal{C}(X,U)$ and $\bar{\beta} \in D_{Y,Z}(t) = \mathcal{C}(Y,V)$, we take $\bar{\gamma} \in \mathcal{C}(U,W)$ and $u \in \mathcal{C}(V,W) \cap \mathcal{S}$ satisfying $u\bar{\beta} = \bar{\gamma}s$. Let $g : (t : Z \to V) \to (t' : Z \to V')$ be a morphism in $\mathcal{S}^Z$. Again, using (iii) of (A.7.9), there exist $\bar{\delta} \in \mathcal{C}(W,W')$ and $v \in \mathcal{C}(V',W') \cap \mathcal{S}$ satisfying $vg = \bar{\delta}u$. Then, since $v(g\bar{\beta}) = (\bar{\delta}\bar{\gamma})s$ and $\bar{\delta}$ is regarded as a morphism $\bar{\delta} : (ut : Z \to W) \to (vt' : Z \to W')$ in $\mathcal{S}^Z$, $\bar{\mu}_{X,Y,Z,s,t'}(\bar{\alpha}, D_{Y,Z}(g)(\bar{\beta})) = \bar{\mu}_{X,Y,Z,s,t'}(\bar{\alpha}, g\bar{\beta}) = \iota_{X,Z,vt'}(\bar{\delta}\bar{\gamma}\bar{\alpha}) = \iota_{X,Z,vt'}(D_{X,Z}(\bar{\delta})(\bar{\gamma}\bar{\alpha})) = \iota_{X,Z,ut}(\bar{\gamma}\bar{\alpha}) = \bar{\mu}_{X,Y,Z,s,t}(\bar{\alpha},\bar{\beta})$. Let $f : (s : Y \to U) \to (s' : Y \to U')$ be a morphism in $\mathcal{S}^Y$. Since $s' = fs : Y \to U'$ belongs to $\mathcal{S}$, there exist $\bar{\eta} \in \mathcal{C}(U',S)$ and $u' \in \mathcal{C}(V,S) \cap \mathcal{S}$ satisfying $u'\bar{\beta} = \bar{\eta}fs$.

Then, there exist $w \in \mathcal{C}(S,T)$ and $u'' \in \mathcal{C}(W,T) \cap \mathcal{S}$ satisfying $wu' = u''u$. Hence $u''\bar{\gamma}s = u''u\bar{\beta} = wu'\bar{\beta} = w\bar{\eta}fs$ and it follows from (iv) of (A.7.9), there exists $v' \in \mathcal{C}(T,T') \cap \mathcal{S}$ satisfying $v'u''\bar{\gamma} = v'w\bar{\eta}f$. We regard $v'u'' : W \to T'$ as a morphism $v'u'' : (ut : Z \to W) \to (v'u''ut : Z \to T')$ in $\mathcal{S}^Z$. It follows that $\bar{\mu}_{X,Y,Z,s',t}(D_{X,Y}(f)(\bar{\alpha}), \bar{\beta}) = \iota_{X,Z,v'u''ut}(v'w\bar{\eta}f\bar{\alpha}) = \iota_{X,Z,v'u''ut}(v'u''\bar{\gamma}\bar{\alpha}) = \iota_{X,Z,v'u''ut}(D_{X,Z}(v'u'')(\bar{\gamma}f\bar{\alpha})) = \iota_{X,Z,ut}(\bar{\gamma}f\bar{\alpha}) = \bar{\mu}_{X,Y,Z,s,t}(\bar{\alpha}, \bar{\beta})$.

Therefore, for $\bar{\alpha} \in D_{X,Y}(s) = \mathcal{C}(X,U)$, $\bar{\beta} \in D_{Y,Z}(t) = \mathcal{C}(Y,V)$ and $(f : (s : Y \to U) \to (s' : Y \to U')) \in \operatorname{Mor} \mathcal{S}^Y$, $(g : (t : Z \to V) \to (t' : Z \to V')) \in \operatorname{Mor} \mathcal{S}^Z$, we have

$$\bar{\mu}_{X,Y,Z,s',t'}(D_{X,Y}(f)(\bar{\alpha}), D_{Y,Z}(g)(\bar{\beta})) = \bar{\mu}_{X,Y,Z,s,t}(\bar{\alpha}, \bar{\beta}).$$

Define a map $\mu_{X,Y,Z} : \mathcal{C}^r_{\mathcal{S}}(X,Y) \times \mathcal{C}^r_{\mathcal{S}}(Y,Z) \to \mathcal{C}^r_{\mathcal{S}}(X,Z)$ as follows. For $\alpha \in \mathcal{C}^r_{\mathcal{S}}(X,Y)$ and $\beta \in \mathcal{C}^r_{\mathcal{S}}(Y,Z)$, choose $(s : Y \to U) \in \operatorname{Ob} \mathcal{S}^Y$, $(t : Z \to V) \in \operatorname{Ob} \mathcal{S}^Z$ and $\bar{\alpha} \in D_{X,Y}(s) = \mathcal{C}(X,U)$, $\bar{\beta} \in D_{Y,Z}(t) = \mathcal{C}(Y,V)$ satisfying $\iota_{X,Y,s}(\bar{\alpha}) = \alpha$ and $\iota_{Y,Z,s}(\bar{\beta}) = \beta$. $\mu_{X,Y,Z}(\alpha, \beta)$ is defined to be $\bar{\mu}_{X,Y,Z,s,t}(\bar{\alpha}, \bar{\beta})$. By the above argument, $\mu_{X,Y,Z}(\alpha, \beta)$ does not depend on the choices of $(s : Y \to U) \in \operatorname{Ob} \mathcal{S}^Y$, $(t : Z \to V) \in \operatorname{Ob} \mathcal{S}^Z$ and $\bar{\alpha} \in D_{X,Y}(s) = \mathcal{C}(X,U)$, $\bar{\beta} \in D_{Y,Z}(t) = \mathcal{C}(Y,V)$.

For $X \in \operatorname{Ob} \mathcal{C}_{\mathcal{S}}$, since $id_X \in D_{X,X}(id_X : X \to X) = \mathcal{C}(X,X)$, we set $id_X = \iota_{X,X,id_X}(id_X) \in \mathcal{C}^r_{\mathcal{S}}(X,X)$.

Proposition A.7.14 *(1) For $X, Y, Z, W \in \operatorname{Ob} \mathcal{C}_{\mathcal{S}}$, the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{C}^r_{\mathcal{S}}(X,Y) \times \mathcal{C}^r_{\mathcal{S}}(Y,Z) \times \mathcal{C}^r_{\mathcal{S}}(Z,W) \arrow[r, "\mu_{X,Y,Z} \times id_{\mathcal{C}^r_{\mathcal{S}}(Z,W)}"] \arrow[d, "id_{\mathcal{C}^r_{\mathcal{S}}(X,Y)} \times \mu_{Y,Z,W}"] & \mathcal{C}^r_{\mathcal{S}}(X,Z) \times \mathcal{C}^r_{\mathcal{S}}(Z,W) \arrow[d, "\mu_{X,Z,W}"] \\
\mathcal{C}^r_{\mathcal{S}}(X,Y) \times \mathcal{C}^r_{\mathcal{S}}(Y,W) \arrow[r, "\mu_{X,Y,W}"] & \mathcal{C}^r_{\mathcal{S}}(X,W)
\end{tikzcd}$$

(2) For $X, Y \in \operatorname{Ob} \mathcal{C}_{\mathcal{S}}$ and $\alpha \in \mathcal{C}^r_{\mathcal{S}}(X,Y)$, $\mu_{X,X,Y}(id_X, \alpha) = \mu_{X,Y,Y}(\alpha, id_Y) = \alpha$.

Proof. (1) For $\alpha \in \mathcal{C}^r_{\mathcal{S}}(X,Y)$, $\beta \in \mathcal{C}^r_{\mathcal{S}}(Y,Z)$ and $\gamma \in \mathcal{C}^r_{\mathcal{S}}(Z,W)$, we choose $(s : Y \to U) \in \operatorname{Ob} \mathcal{S}^Y$, $(t : Z \to V) \in \operatorname{Ob} \mathcal{S}^Z$, $(u : W \to T) \in \operatorname{Ob} \mathcal{S}^W$ and $\bar{\alpha} \in D_{X,Y}(s) = \mathcal{C}(X,U)$, $\bar{\beta} \in D_{Y,Z}(t) = \mathcal{C}(Y,V)$, $\bar{\gamma} \in D_{Z,W}(u) = \mathcal{C}(Z,T)$. Take $\bar{\delta} \in \mathcal{C}(U,S)$, $u \in \mathcal{C}(V,S) \cap \mathcal{S}$ and $\bar{\eta} \in \mathcal{C}(V,P)$, $w \in \mathcal{C}(T,P) \cap \mathcal{S}$ satisfying $v\bar{\beta} = \bar{\delta}s$ and $w\bar{\gamma} = \bar{\eta}t$. Then, $\mu_{X,Y,Z}(\alpha, \beta) = \bar{\mu}_{X,Y,Z,s,t}(\bar{\alpha}, \bar{\beta}) = \iota_{X,Z,vt}(\bar{\delta}\bar{\alpha})$ and $\mu_{Y,Z,W}(\beta, \gamma) = \bar{\mu}_{Y,Z,W,t,u}(\bar{\beta}, \bar{\gamma}) = \iota_{Y,W,wu}(\bar{\eta}\bar{\beta})$. Moreover, there exist $\bar{\varepsilon} \in \mathcal{C}(S,Q)$, $p \in \mathcal{C}(P,Q) \cap \mathcal{S}$ satisfying $p\bar{\eta} = \bar{\varepsilon}v$. Since $pw : T \to Q$ belongs to $\mathcal{S}$ and $pw\bar{\gamma} = p\bar{\eta}t = \bar{\varepsilon}vt$, we have $\mu_{X,Z,W}(\mu_{X,Y,Z}(\alpha, \beta), \gamma) = \bar{\mu}_{X,Z,W,vt,u}(\bar{\delta}\bar{\alpha}, \bar{\gamma}) = \iota_{X,W,pwu}(\bar{\varepsilon}(\bar{\delta}\bar{\alpha}))$. On the other hand, since $p : P \to Q$ belongs to $\mathcal{S}$ and $p\bar{\eta}\bar{\beta} = \bar{\varepsilon}v\bar{\beta} = \bar{\varepsilon}\bar{\delta}s$, we have $\mu_{X,Y,W}(\alpha, \mu_{Y,Z,W}(\beta, \gamma)) = \bar{\mu}_{X,Y,W,s,wu}(\bar{\alpha}, \bar{\eta}\bar{\beta}) = \iota_{X,W,pwu}((\bar{\varepsilon}\bar{\delta})\bar{\alpha})$.

(2) Choose $(s : Y \to U) \in \operatorname{Ob} \mathcal{S}^Y$ and $\bar{\alpha} \in D_{X,Y}(s) = \mathcal{C}(X,U)$ satisfying $\iota_{X,Y,s}(\bar{\alpha}) = \alpha$. Then, since id_U belongs to $\mathcal{S}$ and $id_U\bar{\alpha} = \bar{\alpha}id_X$, we have $\mu_{X,X,Y}(id_X, \alpha) = \bar{\mu}_{X,X,Y,id_X,s}(id_X, \bar{\alpha}) = \iota_{X,Y,id_U s}(\bar{\alpha}id_X) = \iota_{X,Y,s}(\bar{\alpha}) = \alpha$. Similarly, since s belongs to $\mathcal{S}$ and $sid_Y = id_U s$, we have $\mu_{X,Y,Y}(\alpha, id_Y) = \bar{\mu}_{X,Y,Y,s,id_Y}(\bar{\alpha}, id_Y) = \iota_{X,Y,sid_Y}(id_U\bar{\alpha}) = \iota_{X,Y,s}(\bar{\alpha}) = \alpha$. □

By the above result, we have a category $\mathcal{C}^r_{\mathcal{S}}$. Define a functor $Q^r_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}^r_{\mathcal{S}}$ by $Q^r_{\mathcal{S}}(X) = X$ and $Q^r_{\mathcal{S}} = \iota_{X,Y,id_Y} : \mathcal{C}(X,Y) = D_{X,Y}(id_Y) \to \mathcal{C}^r_{\mathcal{S}}(X,Y)$ for $X, Y \in \operatorname{Ob} \mathcal{C}$.

For a morphism $\varphi : Z \to X$ in $\mathcal{C}$, a family of maps $(\varphi^* : \mathcal{C}(X, \tau(s)) \to \mathcal{C}(Z, \tau(s)))_{s \in \operatorname{Ob} \mathcal{S}^Y}$ defines a natural transformation $D_{\varphi,Y} : D_{X,Y} \to D_{Z,Y}$. Thus we have a map $\varphi^* : \mathcal{C}^r_{\mathcal{S}}(X,Y) \to \mathcal{C}^r_{\mathcal{S}}(Z,Y)$.

Lemma A.7.15 *For $\alpha \in \mathcal{C}^r_{\mathcal{S}}(X,Y)$, we have $\varphi^*(\alpha) = \mu_{Z,X,Y}(Q^r_{\mathcal{S}}(\varphi), \alpha)$.*

Proof. Choose $(s : Y \to U) \in \operatorname{Ob} \mathcal{S}^Y$ and $\bar{\alpha} \in D_{X,Y}(s) = \mathcal{C}(X,U)$ satisfying $\iota_{X,Y,s}(\bar{\alpha}) = \alpha$. Then, since id_U belongs to $\mathcal{S}$ and $id_U\bar{\alpha} = \bar{\alpha}id_X$, we have $\mu_{Z,X,Y}(Q^r_{\mathcal{S}}(\varphi), \alpha) = \bar{\mu}_{Z,X,Y,id_X,s}(\varphi, \bar{\alpha}) = \iota_{Z,Y,id_U s}(\bar{\alpha}\varphi) = \iota_{Z,Y,s}(\bar{\alpha}\varphi) = \iota_{Z,Y,s}((D_{\varphi,Y})_s(\bar{\alpha})) = \varphi^*(\iota_{X,Y,s}(\bar{\alpha})) = \varphi^*(\alpha)$. □

Lemma A.7.16 *If $\mathcal{S}$ is a right multiplicative system in $\mathcal{C}$ and $(\varphi : Z \to X) \in \mathcal{S}$, then $\varphi^* : \mathcal{C}^r_{\mathcal{S}}(X,Y) \to \mathcal{C}^r_{\mathcal{S}}(Z,Y)$ is bijective for any $Y \in \operatorname{Ob} \mathcal{C}$.*

Proof. For $\alpha \in \mathcal{C}^r_{\mathcal{S}}(Z,Y)$, we choose $(s : Y \to V) \in \operatorname{Ob} \mathcal{S}^Y$ and $\bar{\alpha} \in D_{Z,Y}(s) = \mathcal{C}(Z,V)$ such that $\iota_{Z,Y,s}(\bar{\alpha}) = \alpha$. Then, by (iii) of (A.7.9), there exist $g \in \mathcal{C}(X,W)$ and $t \in \mathcal{C}(V,W) \cap \mathcal{S}$ satisfying $t\bar{\alpha} = g\varphi$. We can regard $ts : Y \to W$ as an object of $\mathcal{S}^Y$ and g as an element of $D_{X,Y}(ts) = \mathcal{C}(X,W)$ and t as a morphism $s \to ts$ in $\mathcal{S}^Y$. Hence $\varphi^*(\iota_{X,Y,ts}(g)) = \iota_{Z,Y,ts}\varphi^*(g) = \iota_{Z,Y,ts}(g\varphi) = \iota_{Z,Y,ts}(t\bar{\alpha}) = \iota_{Z,Y,ts}(D_{Z,Y}(t : s \to ts)(\bar{\alpha})) = \iota_{Z,Y,s}(\bar{\alpha}) = \alpha$. It follows that φ^* is surjective.

Suppose that $\varphi^*(\alpha) = \varphi^*(\beta)$ for $\alpha, \beta \in \mathcal{C}^r_{\mathcal{S}}(X,Y)$. Since $\mathcal{S}^Y$ is filtered, there exist $(s : Y \to V) \in \operatorname{Ob} \mathcal{S}$ and $\bar{\alpha}, \bar{\beta} \in D_{X,Y}(s) = \mathcal{C}(X,V)$ satisfying $\iota_{X,Y,s}(\bar{\alpha}) = \alpha$ and $\iota_{X,Y,s}(\bar{\beta}) = \beta$. Since $\iota_{X,Y,s}(\bar{\alpha}\varphi) = \iota_{X,Y,s}(\varphi^*(\bar{\alpha})) =$

$\varphi^*(\iota_{X,Y,s}(\bar{\alpha})) = \varphi^*(\alpha) = \varphi^*(\beta) = \varphi^*(\iota_{X,Y,s}(\bar{\beta})) = \iota_{X,Y,s}(\varphi^*(\bar{\beta})) = \iota_{X,Y,s}(\bar{\beta}\varphi)$ and $\mathcal{S}^Y$ is filtered, there exists a morphism $f : (s : Y \to V) \to (t : Y \to W)$ in $\mathcal{S}^Y$ satisfying $f\bar{\alpha}\varphi = f\bar{\beta}\varphi$. Then, by (iv) of (A.7.9), there exists a morphism $\psi : W \to U$ which belongs to $\mathcal{S}$ satisfying $\psi f\bar{\alpha} = \psi f\bar{\beta}$. We regard $\psi f s : Y \to U$ as an object of $\mathcal{S}^Y$ and $\psi f : V \to U$ as a morphism $s \to \psi f s$. Then, $\alpha = \iota_{X,Y,s}(\bar{\alpha}) = \iota_{X,Y,\psi f s} D_{X,Y}(\psi f)(\bar{\alpha}) = \iota_{X,Y,\psi f s}(\psi f\bar{\alpha}) = \iota_{X,Y,\psi f s}(\psi f\bar{\beta}) = \iota_{X,Y,\psi f s} D_{X,Y}(\psi f)(\bar{\beta}) = \iota_{X,Y,s}(\bar{\beta}) = \beta$. Thus φ^* is injective. $\square$

Proposition A.7.17 *Let $\mathcal{S}$ be a right multiplicative system in $\mathcal{C}$.*

1) If $(s : Z \to X) \in \mathcal{S}$, then $Q^r_{\mathcal{S}}(s) : Z \to X$ is an isomorphism in $\mathcal{C}^r_{\mathcal{S}}$.

2) For any morphism $\alpha : X \to Y$ in $\mathcal{C}^r_{\mathcal{S}}$, there exist morphisms $f : X \to Z$ and $s : Y \to Z$ in $\mathcal{C}$ such that $s \in \mathcal{S}$ and $\alpha = Q^r_{\mathcal{S}}(s)^{-1}Q^r_{\mathcal{S}}(f)$.

3) For $f, g \in \mathcal{C}(X,Y)$, $Q^r_{\mathcal{S}}(f) = Q^r_{\mathcal{S}}(g)$ if and only if $sf = sg$ for some $(s : Y \to Z) \in \mathcal{S}$.

Proof. 1) It follows from (A.7.15) and (A.7.16) that the map $Q^r_{\mathcal{S}}(s)^* : \mathcal{C}^r_{\mathcal{S}}(X,Y) \to \mathcal{C}^r_{\mathcal{S}}(Z,Y)$ given by $\alpha \mapsto \alpha Q^r_{\mathcal{S}}(s)$ is bijective for any $Y \in \operatorname{Ob}\mathcal{C}^r_{\mathcal{S}}$. Hence there exists a unique $\sigma \in \mathcal{C}^r_{\mathcal{S}}(X,Z)$ satisfying $\sigma Q^r_{\mathcal{S}}(s) = id_Z$. Then, both $Q^r_{\mathcal{S}}(s)\sigma$ and id_X are mapped to $Q^r_{\mathcal{S}}(s)$ by $Q^r_{\mathcal{S}}(s)^* : \mathcal{C}^r_{\mathcal{S}}(X,X) \to \mathcal{C}^r_{\mathcal{S}}(Z,X)$ and this implies that $Q^r_{\mathcal{S}}(s)\sigma = id_X$. We also give how to get the inverse morphism of $Q^r_{\mathcal{S}}(s)$ as follows. By (iii) of (A.7.9), there exist $g \in \mathcal{C}(X,W)$ and $t \in \mathcal{C}(Z,W) \cap \mathcal{S}$ satisfying $t = t id_Z = gs$. We can regard $t : Z \to W$ as an object of $\mathcal{S}^Z$ and g as an element of $D_{X,Z}(t) = \mathcal{C}(X,W)$ and t as a morphism $id_Z \to t$ in $\mathcal{S}^Z$. Since $Q^r_{\mathcal{S}}(s) = \iota_{Z,X,id_X}(s)$, we have $\iota_{X,Z,t}(g)Q^r_{\mathcal{S}}(s) = \bar{\mu}_{Z,X,Z,id_X,t}(s,g) = \iota_{Z,Z,t}(gs) = \iota_{Z,Z,t}(t id_Z) = \iota_{Z,Z,t}(D_{Z,Z}(t : id_Z \to t)(id_Z)) = \iota_{Z,Z,id_Z}(id_Z) = id_Z$. Hence $\iota_{X,Z,t}(g) \in \mathcal{C}^r_{\mathcal{S}}(X,Z)$ is the inverse of $Q^r_{\mathcal{S}}(s)$.

2) Choose $(s : Y \to Z) \in \operatorname{Ob}\mathcal{S}^Y$ and $f \in D_{X,Y}(s) = \mathcal{C}(X,Z)$ satisfying $\iota_{X,Y,s}(f) = \alpha$. Since id_Z belongs to $\mathcal{S}$, we have $Q^r_{\mathcal{S}}(s)\alpha = \bar{\mu}_{X,Y,Z,s,id_Z}(f,s) = \iota_{X,Z,id_Z id_Z}(id_Z f) = \iota_{X,Z,id_Z}(f) = Q^r_{\mathcal{S}}(f)$. Since $Q^r_{\mathcal{S}}(s)$ is an isomorphism by (A.7.17), we have $\alpha = Q^r_{\mathcal{S}}(s)^{-1}Q^r_{\mathcal{S}}(f)$.

3) Suppose $Q^r_{\mathcal{S}}(f) = Q^r_{\mathcal{S}}(g)$. Since $\iota_{X,Y,id_Y}(f) = Q^r_{\mathcal{S}}(f) = Q^r_{\mathcal{S}}(g) = \iota_{X,Y,id_Y}(g)$, there exist a morphism $s : (id_Y : Y \to Y) \to (s : Y \to Z)$ satisfying $sf = D_{X,Y}(s)(f) = D_{X,Y}(s)(g) = sg$. $\square$

Theorem A.7.18 *Let $\mathcal{S}$ be a right multiplicative system in $\mathcal{C}$. Then, $Q^r_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}^r_{\mathcal{S}}$ is a localization of $\mathcal{C}$ by $\mathcal{S}$.*

Proof. By 1) of (A.7.17), (i) of (A.7.1) is satisfied. Suppose that $F : \mathcal{C} \to \mathcal{A}$ is a functor such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$. We define a functor $F_{\mathcal{S}} : \mathcal{C}^r_{\mathcal{S}} \to \mathcal{A}$ as follows. Set $F_{\mathcal{S}}(X) = F(X)$ for any $X \in \operatorname{Ob}\mathcal{C}^r_{\mathcal{S}} = \operatorname{Ob}\mathcal{C}$. We define a family of maps $\left(D_{X,Y}(s) \xrightarrow{\lambda_{X,Y,s}} \mathcal{A}(F(X),F(Y))\right)_{s \in \operatorname{Ob}\mathcal{S}^Y}$ by $\lambda_s(f) = F(s)^{-1}F(f)$ for $(s : Y \to Z) \in \operatorname{Ob}\mathcal{S}^Y$ and $f \in D_{X,Y}(s) = \mathcal{C}(X,Z)$. If $\varphi : (s : Y \to Z) \to (t : Y \to W)$ is a morphism in $\mathcal{S}^Y$, then $t = \varphi s$. Hence we have $F(t) = F(\varphi)F(s)$ and, since both $F(t)$ and $F(s)$ are isomorphisms, so is $F(\varphi)$. It follows $\lambda_{X,Y,t}(D_{X,Y}(\varphi)(f)) = F(t)^{-1}F(D_{X,Y}(\varphi)(f)) = F(s)^{-1}F(\varphi)^{-1}F(\varphi f) = F(s)^{-1}F(f) = \lambda_{X,Y,s}(f)$. Thus $\left(D_{X,Y}(s) \xrightarrow{\lambda_{X,Y,s}} \mathcal{A}(F(X),F(Y))\right)_{s \in \operatorname{Ob}\mathcal{S}^Y}$ is a cone of $D_{X,Y}$ and let $F_{\mathcal{S}} : \mathcal{C}^r_{\mathcal{S}}(X,Y) \to \mathcal{A}(X,Y)$ be the unique map induced by this cone. For $X \in \operatorname{Ob}\mathcal{C}$, since $id_X \in \mathcal{C}(X,X) = D_{X,X}(id_X)$ is mapped to $F(id_X)^{-1}F(id_X) = id_{F(X)}$ and $id_X \in \mathcal{C}^r_{\mathcal{S}}(X,X)$ by λ_{X,Y,id_X} and ι_{X,X,id_X}, respectively, we have $F_{\mathcal{S}}(id_X) = id_{F(X)}$. For $\alpha \in \mathcal{C}^r_{\mathcal{S}}(X,Y)$ and $\beta \in \mathcal{C}^r_{\mathcal{S}}(Y,Z)$, we choose $(s : Y \to U) \in \operatorname{Ob}\mathcal{S}^Y$, $(t : Z \to V) \in \operatorname{Ob}\mathcal{S}^Z$ and $f \in D_{X,Y}(s) = \mathcal{C}(X,U)$, $g \in D_{Y,Z}(t) = \mathcal{C}(Y,V)$. Take $h \in \mathcal{C}(U,W)$, $u \in \mathcal{C}(V,W) \cap \mathcal{S}$ satisfying $ug = hs$. Then, $F(u)F(g) = F(h)F(s)$ and it follows $F(u)^{-1}F(h) = F(g)F(s)^{-1}$. Hence we have $F_{\mathcal{S}}(\beta\alpha) = F_{\mathcal{S}}(\bar{\mu}_{X,Y,Z,s,t}(f,g)) = F_{\mathcal{S}}(\iota_{X,Z,ut}(hf)) = \lambda_{X,Z,ut}(hf) = F(ut)^{-1}F(hf) = F(t)^{-1}F(u)^{-1}F(h)F(f) = F(t)^{-1}F(g)F(s)^{-1}F(f) = \lambda_{Y,Z,t}(g)\lambda_{X,Y,s}(f) = F_{\mathcal{S}}(\beta)F_{\mathcal{S}}(\alpha)$. Therefore $F_{\mathcal{S}}$ is a functor. For a morphism $f : X \to Y$ in $\mathcal{C}$, since $Q^r_{\mathcal{S}}(f) = \iota_{X,Y,id_Y}(f)$, we have $F_{\mathcal{S}}Q^r_{\mathcal{S}}(f) = \lambda_{X,Y,id_Y}(f) = F(id_Y)^{-1}F(f) = F(f)$. Hence $F_{\mathcal{S}}Q^r_{\mathcal{S}} = F$. Suppose that a functor $G : \mathcal{C}^r_{\mathcal{S}} \to \mathcal{A}$ also satisfies $GQ^r_{\mathcal{S}} = F$. For any morphism $\alpha : X \to Y$ in $\mathcal{C}^r_{\mathcal{S}}$, there exist morphisms $f : X \to Z$ and $s : Y \to Z$ in $\mathcal{C}$ such that $s \in \mathcal{S}$ and $\alpha = Q^r_{\mathcal{S}}(s)^{-1}Q^r_{\mathcal{S}}(f)$ by 2) of (A.7.17). Then, $G(\alpha) = G\left(Q^r_{\mathcal{S}}(s)^{-1}Q^r_{\mathcal{S}}(f)\right) = GQ^r_{\mathcal{S}}(s)^{-1}GQ^r_{\mathcal{S}}(f) = F(s)^{-1}F(f) = \lambda_{X,Y,s}(f) = F_{\mathcal{S}}(\alpha)$, thus $G = F$. We conclude that (ii) of (A.7.1) is satisfied. $\square$

Suppose that $\mathcal{S}$ is a left multiplicative system in a $\mathcal{U}$-category $\mathcal{C}$, we define a category $\mathcal{C}^l_{\mathcal{S}}$ and a functor $Q^l_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}^l_{\mathcal{S}}$ as follows. For $X, Y \in \operatorname{Ob}\mathcal{C}$, we define a functor $E_{X,Y} : \mathcal{S}^{op}_X \to \mathcal{U}$-**Ens** by $E_{X,Y}(s : V \to X) = \mathcal{C}(V,Y)$ and $E_{X,Y}(f : t \to s) = (f^* : \mathcal{C}(\sigma(s),Y) \to \mathcal{C}(\sigma(t),Y))$. We denote by $\mathcal{C}^l_{\mathcal{S}}(X,Y)$ the colimit of $E_{X,Y}$ and by $\left(E_{X,Y}(s) \xrightarrow{\kappa_{X,Y,s}} \mathcal{C}^l_{\mathcal{S}}(X,Y)\right)_{s \in \operatorname{Ob}\mathcal{S}_X}$ the colimiting cone of $E_{X,Y}$.

For $X, Y, Z \in \operatorname{Ob}\mathcal{C}^l_{\mathcal{S}} = \operatorname{Ob}\mathcal{C}$ and $(s : U \to X) \in \operatorname{Ob}\mathcal{S}_X$, $(t : V \to Y) \in \operatorname{Ob}\mathcal{S}_Y$, we define a map $\bar{\mu}_{X,Y,Z,s,t} : E_{X,Y}(s) \times E_{Y,Z}(t) \to \mathcal{C}^l_{\mathcal{S}}(X,Z)$ as follows. For $\bar{\alpha} \in E_{X,Y}(s) = \mathcal{C}(U,Y)$ and $\bar{\beta} \in E_{Y,Z}(t) = \mathcal{C}(V,Z)$, there exist $\bar{\gamma} \in \mathcal{C}(W,V)$ and $u \in \mathcal{C}(W,U) \cap \mathcal{S}$ satisfying $\bar{\alpha}u = t\bar{\gamma}$ by (iii) of (A.7.10). We regard $(su : W \to X)$ as an object of

$\mathcal{S}_X$ and define $\bar{\mu}_{X,Y,Z,s,t}(\bar{\alpha},\bar{\beta})$ to be the image of $\bar{\beta}\bar{\gamma} \in \mathcal{C}(W,Z) = E_{X,Z}(su)$ by $\kappa_{X,Z,su} : E_{X,Z}(su) \to \mathcal{C}^l_{\mathcal{S}}(X,Z)$. It can be verified that this definition of $\bar{\mu}_{X,Y,Z,s,t}(\bar{\alpha},\bar{\beta})$ does not depend on the choice of $\bar{\gamma}$ and u.

For $\bar{\alpha} \in E_{X,Y}(s) = \mathcal{C}(U,Y)$, $\bar{\beta} \in E_{Y,Z}(t) = \mathcal{C}(Y,V)$ and $(f : (s' : U' \to X) \to (s : U \to X)) \in \operatorname{Mor}\mathcal{S}_X$, $(g : (t' : V' \to Y) \to (t : V \to Y)) \in \operatorname{Mor}\mathcal{S}_Y$, we can also verify the following equality.

$$\bar{\mu}_{X,Y,Z,s',t'}(E_{X,Y}(f)(\bar{\alpha}), E_{Y,Z}(g)(\bar{\beta})) = \bar{\mu}_{X,Y,Z,s,t}(\bar{\alpha},\bar{\beta}).$$

Hence $\bar{\mu}_{X,Y,Z,s,t}$'s ($s \in \operatorname{Ob}\mathcal{S}_X$, $t \in \operatorname{Ob}\mathcal{S}_Y$) induce a map $\mu_{X,Y,Z} : \mathcal{C}^l_{\mathcal{S}}(X,Y) \times \mathcal{C}^l_{\mathcal{S}}(Y,Z) \to \mathcal{C}^l_{\mathcal{S}}(X,Z)$. For $X \in \operatorname{Ob}\mathcal{C}^l_{\mathcal{S}}$, the identity morphism id_X of X is given by $\kappa_{X,X,id_X}(id_X)$, here we regard id_X as an object of $\mathcal{S}_X$ and an element of $E_{X,X}(id_X) = \mathcal{C}(X,X)$. We can verify that $\mathcal{C}^l_{\mathcal{S}}$ is a category. A functor $Q^l_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}^l_{\mathcal{S}}$ is defined by $Q^l_{\mathcal{S}}(X) = X$ for $X \in \operatorname{Ob}\mathcal{C}$ and $Q^l_{\mathcal{S}}(f : X \to Y) = \kappa_{X,Y,id_X}(f)$.

As we have shown (A.7.17) and (A.7.18), we can show the following results.

Proposition A.7.19 *Let $\mathcal{S}$ be a right multiplicative system in $\mathcal{C}$.*

1) If $(s : Z \to X) \in \mathcal{S}$, then $Q^l_{\mathcal{S}}(s) : Z \to X$ is an isomorphism in $\mathcal{C}^l_{\mathcal{S}}$.

2) For any morphism $\alpha : X \to Y$ in $\mathcal{C}^l_{\mathcal{S}}$, there exist morphisms $f : Z \to Y$ and $s : Z \to X$ in $\mathcal{C}$ such that $s \in \mathcal{S}$ and $\alpha = Q^l_{\mathcal{S}}(f)Q^l_{\mathcal{S}}(s)^{-1}$.

3) For $f, g \in \mathcal{C}(X,Y)$, $Q^l_{\mathcal{S}}(f) = Q^l_{\mathcal{S}}(g)$ if and only if $fs = gs$ for some $(s : Z \to X) \in \mathcal{S}$.

Theorem A.7.20 *Let $\mathcal{S}$ be a left multiplicative system in $\mathcal{C}$. Then, $Q^l_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}^l_{\mathcal{S}}$ is a localization of $\mathcal{C}$ by $\mathcal{S}$.*

Since a localization of $\mathcal{C}$ by $\mathcal{S}$ is uniquely determines up to isomorphism of categories, we denote both $\mathcal{C}^r_{\mathcal{S}}$ and $\mathcal{C}^l_{\mathcal{S}}$ by $\mathcal{C}_{\mathcal{S}}$, both $Q^r_{\mathcal{S}}$ and $Q^l_{\mathcal{S}}$ by $Q_{\mathcal{S}}$.

A.8 Regular category

Definition A.8.1 *We say that $\mathcal{C}$ is a regular category if it satisfies the following R1∼R3. If a regular category satisfies R4 below, it is called an exact category.*

R1) Each morphism of $\mathcal{C}$ has a kernel pair.
R2) Every kernel pair has a coequalizer.
R3) Each regular epimorphism has a pull-back along an arbitrary morphism, which is also a regular epimorphism.
R4) Every equivalence relation is effective.

Proposition A.8.2 *Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in a category satisfying R3 and $U \overset{a}{\underset{b}{\rightrightarrows}} X$, $V \overset{c}{\underset{d}{\rightrightarrows}} Y$ kernel pairs of f, g, respectively. If $p : X \to Y$ is a regular epimorphism such that $f = gp$, then the unique morphism $\tilde{p} : U \to V$ satisfying $c\tilde{p} = pa$ and $d\tilde{p} = pb$ is an epimorphism.*

Proof. We can form a pull-back $\begin{array}{ccc} W & \xrightarrow{c'} & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{c} & Y \end{array}$ and q is a regular epimorphism by R3. Then, $\begin{array}{ccc} W & \xrightarrow{c'} & X \\ \downarrow{\scriptstyle dq} & & \downarrow{\scriptstyle gp} \\ Y & \xrightarrow{g} & Z \end{array}$ is a pull-back by (A.3.1) and $\begin{array}{ccc} U & \xrightarrow{a} & X \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle gp} \\ X & \xrightarrow{gp} & Z \end{array}$ is a pull-back by assumption. Hence there is a unique morphism $r : U \to W$ satisfying $dqr = pb$ and $c'r = a$. Again by (A.3.1), $\begin{array}{ccc} U & \xrightarrow{r} & W \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle dq} \\ X & \xrightarrow{p} & Y \end{array}$ is a pull-back. Thus r is a regular epimorphism. Since $cqr = pc'r = pa$ and $dqr = pb$, we have $qr = \tilde{p}$ by the uniqueness of $\tilde{p}$ and it is a composite of epimorphisms. $\square$

Theorem A.8.3 *Every morphism $f : X \to Y$ in a regular category has a factorization $f = ip$ with p a regular epimorphism and i a monomorphism.*

Proof. Consider a kernel pair $U \overset{a}{\underset{b}{\rightrightarrows}} X$ of f and let $p : X \to Z$ be a coequalizer of this pair. Then there exists a unique morphism $i : Z \to Y$ satisfying $f = ip$. Let $V \overset{c}{\underset{d}{\rightrightarrows}} Z$ be a kernel pair of i and $\tilde{p} : U \to V$ the unique morphism satisfying $c\tilde{p} = pa$ and $d\tilde{p} = pb$. Since $\tilde{p}$ is an epimorphism by (A.8.2) and $pa = pb$, we have $c = d$. This implies that i is a monomorphism. □

Proposition A.8.4 *Let*
$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"] & Y \arrow[d, "f"] \\ X \arrow[r, "i"] & W \end{tikzcd}$$
be a commutative diagram in an arbitrary category. If p is a regular epimorphism and i is a monomorphism, there exists a unique morphism $h : Y \to Z$ such that $hp = f$ and $ih = g$.

Proof. Let p be a coequalizer of $U \overset{a}{\underset{b}{\rightrightarrows}} X$. Since $ifa = gpa = gpb = ifb$ and i is a monomorphism, we have $fa = fb$ and there is a unique morphism $h : Y \to Z$ satisfying $hp = f$. Thus $ihp = if = gp$ and p is an epimorphism, we have $ih = g$. □

Corollary A.8.5 *A morphism in an arbitrary category which is both monomorphism and regular epimorphism is an isomorphism.*

Proof. Suppose that $p : X \to Y$ is both monomorphism and regular epimorphism. Apply the above result for $Z = X$, $W = Y$, $i = p$, $f = id_X$ and $g = id_Y$. □

Corollary A.8.6 *1) Let $i : W \to Z$ be a monomorphism and $p : X \to Z$ a regular epimorphism in an arbitrary category. If $p = ik$ for some morphism $k : X \to W$, i is an isomorphism.*

2) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in an arbitrary category such that $g = iq$ for a regular epimorphism $q : Y \to W$ and a monomorphism $i : W \to Z$. If gf is a regular epimorphism, so is g. In particular, a split epimorphism in a regular category is a regular epimorphism.

Proof. 1) Apply (A.8.4) to a commutative square
$$\begin{tikzcd} X \arrow[r, two heads, "p"] \arrow[d, "k"] & Z \arrow[d, "id_Z"] \\ W \arrow[r, tail, "i"] & Z \end{tikzcd}$$
Then, we have a morphism $s : Z \to W$ satisfying $is = id_Z$. Hence $isi = i$ and, since i is a monomorphism, $si = id_W$.

2) Applying the above result for $p = gf$, $k = qf$, we see that i is an isomorphism. Thus g is a regular epimorphism. □

Corollary A.8.7 *The composite of two regular epimorphisms in a regular category is a regular epimorphism.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be regular epimorphisms and $fg = ip$ a factorization of fg with p a regular epimorphism and i a monomorphism. Applying (A.8.4) to
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[d, "g"] \\ W \arrow[r, "i"] & Z, \end{tikzcd}$$
there exists a morphism $h : Y \to W$ satisfying $hf = p$ and $ih = g$. Since g is a regular epimorphism, so is i by (A.8.6). Hence i is an isomorphism by (A.8.5). □

Corollary A.8.8 *Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in a regular category and $U \overset{a}{\underset{b}{\rightrightarrows}} X$, $V \overset{c}{\underset{d}{\rightrightarrows}} Y$ kernel pairs of f, g, respectively. If $p : X \to Y$ is a regular epimorphism such that $f = gp$, then the unique morphism $\tilde{p} : U \to V$ satisfying $c\tilde{p} = pa$ and $d\tilde{p} = pb$ is a regular epimorphism.*

Proof. The result follows from (A.8.7) and the proof of (A.8.2). $\square$

Corollary A.8.9 *1) Let* $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ Z & \xrightarrow{g} & W \end{array}$ *be a commutative square in an arbitrary category. If* $f = ip$ *and* $g = jq$ *are factorizations of* f, g *with* $p : X \to U$, $q : Z \to V$ *regular epimorphisms and* $i : U \to Y$, $j : V \to Y$ *monomorphisms, then there is a unique morphism* $\varphi : U \to V$ *satisfying* $\varphi p = qh$ *and* $j\varphi = ik$.

2) Let $f : X \to Y$ *be a morphism in an arbitrary category. If* $f = ip = jq$ *are factorizations of* f *with* $p : X \to U$, $q : X \to V$ *regular epimorphisms and* $i : U \to Y$, $j : V \to Y$ *monomorphisms, then there is a unique isomorphism* $\varphi : U \to V$ *satisfying* $\varphi p = q$ *and* $j\varphi = i$.

Proof. 1) By (A.8.4), there is a morphism $\varphi : U \to V$ satisfying $\varphi p = qh$ and $j\varphi = ik$.

2) Apply the above result for $h = id_X$, $k = id_Y$. Then, there are morphisms $\varphi : U \to V$ and $\psi : V \to U$ satisfying $\varphi p = q$, $j\varphi = i$ and $\psi q = p$, $i\psi = j$. Then, we have $\psi\varphi p = \psi q = p$ and $\varphi\psi q = \varphi p = q$. Since p and q are epimorphisms, it follows that $\psi\varphi = id_Z$ and $\varphi\psi = id_W$. $\square$

Definition A.8.10 *A category is said to be balanced if a morphism which is both monomorphism and epimorphism is an isomorphism.*

Proposition A.8.11 *If* $\mathcal{C}$ *is a category in which every monomorphism is an equalizer of a certain pair of morphisms,* $\mathcal{C}$ *is balanced.*

Proof. Let $f : X \to Y$ be a morphism which is both monomorphism and epimorphism. There exist morphisms $g, h : Y \to Z$ such that f is an equalizer of g and h. Since f is an epimorphism and $gf = hf$, we have $g = h$. Then id_Y factors through f, namely f has a right inverse f'. Hence $ff'f = f$ and this implies f' is also a left inverse of f. $\square$

Proposition A.8.12 *An epimorphism in a balanced regular category is a regular epimorphism.*

Proof. Let $f : X \to Y$ be an epimorphism and $f = ip$ a factorization of f with p a regular epimorphism and i a monomorphism. Then, i is an epimorphism, hence an isomorphism. Therefore f is a regular epimorphism. $\square$

Definition A.8.13 *Let* $Z \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$ *be a diagram in a category.*

1) If $Z \underset{h}{\overset{g}{\rightrightarrows}} X$ *is a kernel pair of* f*, the above diagram is called left exact.*

2) Let $W \underset{b}{\overset{a}{\rightrightarrows}} X$ *be a kernel pair of* f*. If* f *is a coequalizer of* $Z \underset{h}{\overset{g}{\rightrightarrows}} X$ *and the morphism* $Z \to W$ *induced by* g *and* h *is a regular epimorphism, the above diagram is called right exact.*

3) The above diagram is said to be exact if it is both left and right exact, that is, $Z \underset{h}{\overset{g}{\rightrightarrows}} X$ *is a kernel pair of* f *and* f *is a coequalizer of* $Z \underset{h}{\overset{g}{\rightrightarrows}} X$ *.*

Proposition A.8.14 *1) If* $p : X \to Y$ *is a regular epimorphism in an arbitrary category, it is a coequalizer of its kernel pair* $Z \underset{g}{\overset{f}{\rightrightarrows}} X$ *and* $Z \underset{g}{\overset{f}{\rightrightarrows}} X \xrightarrow{p} Y$ *is exact.*

2) Let $Z \underset{g}{\overset{f}{\rightrightarrows}} X$ *be a kernel pair of a morphism* $h : X \to W$ *and* $p : X \to Y$ *be a coequalizer of* f *and* g*. Then* $Z \underset{g}{\overset{f}{\rightrightarrows}} X$ *is a kernel pair of* p *and* $Z \underset{g}{\overset{f}{\rightrightarrows}} X \xrightarrow{p} Y$ *is exact.*

Proof. 1) Let p be a coequalizer of a pair $W \underset{b}{\overset{a}{\rightrightarrows}} X$ and $h : X \to V$ a morphism satisfying $hf = hg$. Then, there is a unique morphism $k : W \to Z$ such that $fk = a$ and $gk = b$. Thus $ha = hfk = hgk = hb$ and there is a unique morphism $h' : Y \to V$ such that $h'f = h$.

2) There exists a unique morphism $k : Y \to W$ such that $kp = h$. Then the result follows from (A.3.6). $\square$

Definition A.8.15 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ a functor.*

1) F is said to be quasi-exact if it preserves exact sequences.

2) F is said to be exact if it is quasi-exact and left exact.

3) F is said to be reflexively (quasi-)exact if it is (quasi-)exact and reflects isomorphisms.

Proposition A.8.16 *1) A quasi-exact functor from a category which satisfies R1 preserves regular epimorphisms and right exact sequences. Hence a quasi-exact functor preserving monomorphisms preserves factorizations.*

2) Let $(\mathcal{C}_i)_{i\in I}$ be a family of categories and $P_i : \prod_{i\in I} \mathcal{C}_i \to \mathcal{C}_i$ denotes the projection functor. A diagram $Z \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$ in the product category $\prod_{i\in I} \mathcal{C}_i$ is exact if and only if $P_i(Z) \underset{P_i(h)}{\overset{P_i(g)}{\rightrightarrows}} P_i(X) \xrightarrow{P_i(f)} P_i(Y)$ is exact for each $i \in I$.

3) Let $(\mathcal{C}_i)_{i\in I}$ be a family of regular (resp. exact) categories. Then, the product category $\prod_{i\in I} \mathcal{C}_i$ is also regular (resp. exact).

Proof. 1) Let $f : X \to Y$ be a regular epimorphism and $Z \underset{b}{\overset{a}{\rightrightarrows}} X$ a kernel pair of f. Since $Z \underset{b}{\overset{a}{\rightrightarrows}} X \xrightarrow{f} Y$ is exact by (A.8.14), it is preserved by a quasi-exact functor. Thus a regular epimorphism f is preserved by a quasi-exact functor.

Let $Z \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$ be a right exact sequence and $W \underset{b}{\overset{a}{\rightrightarrows}} X$ a kernel pair of f. Then, the morphism $p : Z \to W$ induced by g and h is a regular epimorphism and $W \underset{b}{\overset{a}{\rightrightarrows}} X \xrightarrow{f} Y$ is exact by (A.8.14). If F is a quasi-exact functor, $F(p) : F(Z) \to F(W)$ is a regular epimorphism and $F(W) \underset{F(b)}{\overset{F(a)}{\rightrightarrows}} F(X) \xrightarrow{F(f)} F(Y)$ is exact. We note that $F(f)$ is a coequalizer of $F(Z) \underset{F(h)}{\overset{F(g)}{\rightrightarrows}} F(X)$ by the dual result of (A.3.6). Hence $F(Z) \underset{F(h)}{\overset{F(g)}{\rightrightarrows}} F(X) \xrightarrow{F(f)} F(Y)$ is right exact. 2) The assertion follows from (A.4.7).

3) If $(\mathcal{C}_i)_{i\in I}$ is a family of regular categories, so is $\prod_{i\in I} \mathcal{C}_i$ by (A.4.7). Assume that each $\mathcal{C}_i$ is exact and let $R \underset{h}{\overset{g}{\rightrightarrows}} X$ be an equivalence relation in $\prod_{i\in I} \mathcal{C}_i$. It follows from (A.3.18) and (A.4.7) that P_i preserves monomorphic families. Hence by (A.3.20) and (A.4.7), P_i preserves equivalence relations. Thus

$$P_i(R) \underset{P_i(h)}{\overset{P_i(g)}{\rightrightarrows}} P_i(X)$$

is an equivalence relation in $\mathcal{C}_i$, which is a kernel pair of a certain morphism, say $f_i : P_i(X) \to Y_i$. Again by (A.4.7), $R \underset{h}{\overset{g}{\rightrightarrows}} X$ is a kernel pair of $(f_i)_{i\in I} : X \to (Y_i)_{i\in I}$. $\square$

The following result is a direct consequence of (A.8.14) and (A.8.16).

Corollary A.8.17 *1) A functor is quasi-exact (resp. exact) if it preserves kernel pairs (resp. finite limits) and regular epimorphisms.*

2) An exact functor from a category which satisfies R1 preserves finite limits and regular epimorphisms.

Proposition A.8.18 *Let $Z_i \underset{h_i}{\overset{g_i}{\rightrightarrows}} X_i \xrightarrow{f_i} Y_i$ $(i = 1, 2)$ be exact (resp. left exact, right exact) sequences in a regular category. If products $X_1 \times X_2$, $Y_1 \times Y_2$ and $Z_1 \times Z_2$ exist, $Z_1 \times Z_2 \underset{h_1\times h_2}{\overset{g_1\times g_2}{\rightrightarrows}} X_1 \times X_2 \xrightarrow{f_1\times f_2} Y_1 \times Y_2$ is an exact (resp. left exact, right exact) sequence. Therefore if a finite product of exact (resp. left exact, right exact) sequences exists, it is exact (resp. left exact, right exact).*

Proof. We first show that a product of regular epimorphisms are regular epimorphisms. Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be regular epimorphisms. Consider a pull-back $\tilde{f}_1 : Z \to Y_1 \times Y_2$ of f_1 along the projection $\mathrm{pr}_1 :$

$Y_1 \times Y_2 \to Y_1$, then $\tilde{f}_1$ is a regular epimorphism. Let us denote by $p_1 : Z \to X_1$ the pull-back of pr_1 along f_1. Since $\mathrm{pr}_1(f_1 \times f_2) = f_1\mathrm{pr}_1$, there exists a unique morphism $g : X_1 \times X_2 \to Z$ such that $\tilde{f}_1 g = f_1 \times f_2$ and $p_1 g = \mathrm{pr}_1$. An easy diagram chasing shows that

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "g"] & X_2 \arrow[d, "f_2"] \\ Z \arrow[r, "\mathrm{pr}_2\tilde{f}_1"] & Y_2 \end{tikzcd}$$

is a pull-back, thus g is a regular epimorphism. We note that $\mathrm{pr}_1\tilde{f}_1 g = f_1 p_1 g = f_1\mathrm{pr}_1$ and $\mathrm{pr}_2\tilde{f}_1 g = f_2\mathrm{pr}_2$, thus $\tilde{f}_1 g = f_1 \times f_2$. By (A.8.7), $\tilde{f}_1 g = f_1 \times f_2$ is a regular epimorphism.

Suppose that $Z_i \overset{g_i}{\underset{h_i}{\rightrightarrows}} X_i \xrightarrow{f_i} Y_i$ $(i = 1, 2)$ are right exact sequences. Let $W_i \overset{a_i}{\underset{b_i}{\rightrightarrows}} X_i$ be a kernel pair of f_i and $p_i : Z_i \to W_i$ be the regular epimorphism satisfying $a_i p_i = g_i$ and $b_i p_i = h_i$. Since a product of left exact sequences is left exact, $W_1 \times W_2 \overset{a_1\times a_2}{\underset{b_1\times b_2}{\rightrightarrows}} X_1 \times X_2$ is a kernel pair of $f_1 \times f_2$, which is a regular epimorphism. Hence, by (A.8.14), $W_1 \times W_2 \overset{a_1\times a_2}{\underset{b_1\times b_2}{\rightrightarrows}} X_1 \times X_2 \xrightarrow{f_1\times f_2} Y_1 \times Y_2$ is exact. Since $p_1 \times p_2$ is a regular epimorphism, $Z_1 \times Z_2 \overset{g_1\times g_2}{\underset{h_1\times h_2}{\rightrightarrows}} X_1 \times X_2 \xrightarrow{f_1\times f_2} Y_1 \times Y_2$ is right exact. □

Corollary A.8.19 *Let X be an object of a regular category $\mathcal{C}$ such that a product $Y \times X$ exists for any object Y. Then the product functor $(-) \times X : \mathcal{C} \to \mathcal{C}$ is exact.*

Proof. Since $X \overset{id_X}{\underset{id_X}{\rightrightarrows}} X \xrightarrow{id_X} X$ is exact, $(-)\times X$ preserves exact sequences by the previous result. $(-)\times X$ also preserves finite limits. □

Corollary A.8.20 *Let $\mathcal{C}$ be a regular category with finite powers, then for any positive integer n, the n-th power functor $(-)^n : \mathcal{C} \to \mathcal{C}$, $X \mapsto X^n$ is exact.*

Proposition A.8.21 *If a terminal object exists in a regular category $\mathcal{C}$, each object of $\mathcal{C}$ has finite powers.*

Proof. Let us denote by 1 a terminal object of $\mathcal{C}$. For an object X of $\mathcal{C}$, $X \times X$ is the kernel pair of the unique morphism $X \to 1$. Let $p_i^2 : X \times X \to X$ $(i = 1, 2)$ be the canonical projections. Since the diagonal morphism $\Delta : X \to X \times X$ is a right inverse of p_i^2, Δ is a regular epimorphism by (A.8.6). Suppose that we have an $(n-1)$-th power X^{n-1} with projections $p_i^{n-1} : X^{n-1} \to X$. Then, n-th power X^n is given by the pull-back square

$$\begin{tikzcd} X^n \arrow[r, "q"] \arrow[d, "p"] & X \times X \arrow[d, "p_1^2"] \\ X^{n-1} \arrow[r, "p_{n-1}^{n-1}"] & X \end{tikzcd} .$$

We define $p_i^n : X^n \to X$ by $p_i^n = p_i^{n-1}p$ for $1 \le i \le n-1$, $p_n^n = p_2^2 q$. Suppose that morphisms $f_i : Y \to X$ $(1 \le i \le n)$ are given. Then we have a unique morphism $g : Y \to X^{n-1}$ satisfying $p_i^{n-1}g = f_i$ for $1 \le i \le n-1$ by the inductive hypothesis. We also have a unique morphism $h : Y \to X \times X$ satisfying $p_1^2 h = f_{n-1}$ and $p_2^2 h = f_n$. Hence $p_{n-1}^{n-1}g = f_{n-1} = p_1^2 h$ and there exists a unique morphism $f : Y \to X^n$ such that $pf = g$ and $qf = h$. Therefore $p_i^n f = p_i^{n-1}pf = p_i^{n-1}g = f_i$ for $1 \le i \le n-1$ and $p_n^n f = p_2^2 qf = p_2^2 h = f_n$.

Let $f' : Y \to X^n$ be a morphism satisfying $p_i^n f' = f_i$ for $1 \le i \le n$. Then $pf' : Y \to X^{n-1}$ and $qf' : Y \to X \times X$ satisfies $p_i^{n-1}pf' = p_i^n f' = f_i$ for $1 \le i \le n-1$ and $p_i^2 qf' = f_{n-2+i}$ for $i = 1, 2$, respectively. The uniqueness of g and h implies $pf' = g$ and $qf' = h$. By the uniqueness of f we have $f' = f$. □

We denote by $\widehat{\mathcal{C}}$ the category of presheaves of sets on a category $\mathcal{C}$ and $\underline{1}$ the terminal object of $\widehat{\mathcal{C}}$.

Proposition A.8.22 *Suppose that $\mathcal{C}$ is a regular category. If there exist an object X of $\mathcal{C}$ and a natural transformation $c : \underline{1} \to h_X$, $\mathcal{C}$ has a terminal object, where h_X denotes a presheaf represented by X.*

Proof. For an object Y, let $k_Y : Y \to X$ be the unique element of $c_Y(\underline{1}(Y))$. Factor k_Y as $Y \overset{l_Y}{\twoheadrightarrow} T_Y \overset{i_Y}{\rightarrowtail} X$, where l_Y is a regular epimorphism and i_Y is a monomorphism. By the naturality of c, we have $k_X k_Y = k_Y$ and we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "l_Y"] \arrow[d, "l_X k_Y"'] & T_Y \arrow[d, "i_X"] \\ T_X \arrow[r, "i_X"] & X \end{tikzcd} .$$

It follows from (A.8.4) that there exists a morphism $f : T_Y \to T_X$ such that $f l_Y = l_X k_Y$ and $i_X f = i_Y$. We show that T_X is a terminal object. The naturality of c implies $k_{T_X} l_X = k_X = i_X l_X$. Since l_X is an epimorphism, we have $k_{T_X} = i_X$. Again by the naturality of c, we have $k_{T_X} g = k_Y$ for any morphism $g : Y \to T_X$. Then, $i_X g = k_{T_X} g = k_Y = i_Y l_Y = i_X f l_Y = i_X l_X k_Y$ and since i_X is a monomorphism, we have $g = l_X k_Y$. Thus $\mathcal{C}(Y, T_X)$ consists of a single element $l_X k_Y$ and T_X is a terminal object. □

Proposition A.8.23 *Let $\mathcal{C}$ be a regular category and $f, g : X \to Y$ morphisms in $\mathcal{C}$ such that a product $Y \times Y$ exists. Suppose that the following conditions hold. Then, the image of $(f, g) : X \to Y \times Y$ is an equivalence relation on Y.*

(1) $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ *is a reflexive pair.*

(2) *There exists a morphism $\tau : X \to X$ in $\mathcal{C}$ satisfying $f\tau = g$ and $g\tau = f$.*

(3) *If* $\begin{tikzcd} T \arrow[r, "q"] \arrow[d, "p"'] & X \arrow[d, "f"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$ *is a pull-back, then the image of $(fq, gp) : T \to Y \times Y$ is contained in the image of $(f, g) : X \to Y \times Y$.*

Proof. Let $X \overset{\pi}{\twoheadrightarrow} R \overset{i}{\rightarrowtail} Y \times Y$ be a factorization of $(f, g) : X \to Y \times Y$. We denote by $p_i : Y \times Y \to Y$ $(i = 1, 2)$ the projection onto the i-th component. Put $f' = p_1 i$ and $g' = p_2 i$, then $f', g' : R \to Y$ is a monomorphic pair and we have $f = f'\pi$, $g = g'\pi$. Let $s : Y \to X$ be a morphism satisfying $fs = gs = id_Y$, then $f'\pi s = g'\pi s = id_Y$, hence $R \overset{f'}{\underset{g'}{\rightrightarrows}} Y$ is a reflexive pair.

Let $\tau' = (p_2, p_1) : Y \times Y \to Y \times Y$ be the switching map. Then $p_1 i\pi\tau = f\tau = g = p_2 i\pi = p_1\tau' i\pi$, $p_2 i\pi\tau = g\tau = f = p_1 i\pi = p_2\tau' i\pi$, hence the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "\tau"] \arrow[d, "\pi"'] & X \arrow[r, "\pi"] & R \arrow[d, "i"] \\ R \arrow[r, "i"] & Y \times Y \arrow[r, "\tau'"] & Y \times Y \end{tikzcd}$$

It follows from (A.8.4) that there exists a morphism $\tau'' : R \to R$ such that $\tau''\pi = \pi\tau$ and $i\tau'' = \tau' i$. Thus $f'\tau''\pi = f'\pi\tau = f\tau = g = g'\pi$, $g'\tau''\pi = g'\pi\tau = g\tau = f = f'\pi$ and we have $f'\tau'' = g'$, $g'\tau'' = f'$.

Since f' is a regular epimorphism by (A.8.6), we can form a pull-back of f' along g'. Consider the following diagram, where $\pi_1 : U \to T'$ and $\pi_2 : V \to T'$ are pull-backs of a regular epimorphism $\pi : X \to R$ along $q' : T' \to R$ and $p' : T' \to R$, respectively.

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, "\beta"'] & U \arrow[r, "u"] \arrow[d, "\pi_1"] & X \arrow[d, "\pi"] \\ V \arrow[r, "\pi_2"] \arrow[d, "v"'] & T' \arrow[r, "q'"] \arrow[d, "p'"] & R \arrow[d, "f'"] \\ X \arrow[r, "\pi"] & R \arrow[r, "g'"] & Y \end{tikzcd}$$

Let $\rho : T \to T'$ be the unique morphism satisfying $\rho q' = \pi q$, $\rho p' = \pi p$. Then, there are unique morphisms $\alpha : T \to U$ and $\beta : T \to V$ satisfying $\pi_1\alpha = \rho$, $u\alpha = q$, $\pi_2\beta = \rho$, $v\beta = p$. Since the upper left square of the diagram is a pull-back and both π_1 and π_2 are regular epimorphisms, α and β are regular epimorphisms. Hence ρ is a regular epimorphism by (A.8.7). By the assumption, there is a morphism $t : T \to R$ such that $it = (fp, gq)$. Then, $(f'p', g'q')\rho = (f'p'\pi_2\beta, g'q'\pi_1\alpha) = (f'\pi v\beta, g'\pi u\alpha) = (fp, gq) = it$. Applying (A.8.4) to the following square, we have a morphism $t' : T' \to R$ such that $it' = (f'p', g'p')$.

$$\begin{tikzcd} T \arrow[r, "t"] \arrow[d, "\rho"] & R \arrow[d, "i"] \\ T' \arrow[r, "(f'p',g'p')"] & Y \times Y \end{tikzcd}$$

□

Proposition A.8.24 *Let $\mathcal{C}$ be a with coproducts indexed by sets I and $I \times I$. Suppose that a family of morphisms $(f_i : X_i \to X)_{i \in I}$ is given such that, for $i, j \in I$, the pull-back of f_j along f_i exists. We denote by $f : \coprod_{i \in I} X_i \to X$ the morphism induced by $(f_i : X_i \to X)_{i \in I}$.*

1) If $(f_i : X_i \to X)_{i \in I}$ is a strict epimorphic family in $\mathcal{C}$, then f is a regular epimorphism.

2) If coproducts in $\mathcal{C}$ are universal and f is a regular epimorphism, $(f_i : X_i \to X)_{i \in I}$ is a strict epimorphic family. Moreover, if $\mathcal{C}$ is a regular category, $(f_i : X_i \to X)_{i \in I}$ is a universal strict epimorphic family.

Proof. 1) For $i, j \in I$, consider a cartesian square

$$\begin{tikzcd} X_i \times_X X_j \arrow[r, "q_{ij}"] \arrow[d, "p_{ij}"] & X_j \arrow[d, "f_j"] \\ X_i \arrow[r, "f_i"] & X \end{tikzcd}$$

and let $p, q : \coprod_{i,j \in I} X_i \times_X X_j \to \coprod_{i \in I} X_i$ be morphisms induced by p_{ij}, q_{ij}, respectively. Then, it is clear that f is a coequalizer of p and q.

2) Suppose that coproducts in $\mathcal{C}$ are universal and that $f : \coprod_{i \in I} X_i \to X$ is a regular epimorphism. Then, $\coprod_{i,j \in I} X_i \times_X X_j \overset{p}{\underset{q}{\rightrightarrows}} \coprod_{i \in I} X_i$ is a kernel pair of f. It follows from (A.8.14) that f is a coequalizer of p and q, which implies that $(f_i : X_i \to X)_{i \in I}$ is a strict epimorphic family.

Since a pull-back of f is also a regular epimorphism if $\mathcal{C}$ is regular, it follows that $(f_i : X_i \to X)_{i \in I}$ is a universal strict epimorphic family. □

It follows from the above result that, if $\mathcal{C}$ is a finitely complete regular $\mathcal{U}$-category with universal $\mathcal{U}$-small coproducts, then a strict epimorphic family indexed by a $\mathcal{U}$-small set is universal.

An object X of a category $\mathcal{C}$ is said to be connected if X is not isomorphic to the coproduct of two objects which are not initial.

Proposition A.8.25 *Let $\mathcal{C}$ be a $\mathcal{U}$-category with finite coproducts. For an object X of $\mathcal{C}$, we denote by $h^X : \mathcal{C} \to \mathcal{U}$-**Ens** the functor defined by $h^X(Y) = \mathcal{C}(X, Y)$.*

1) Assume that if $X = X_1 \coprod X_2$, the canonical morphisms $\nu_i : X_i \to X_1 \coprod X_2$ $(i = 1, 2)$ are monomorphisms. If h^X preserves finite coproducts, X is connected and it is not an initial object.

2) Assume that finite coproducts in $\mathcal{C}$ are universal and disjoint. If $X \in \mathrm{Ob}\,\mathcal{C}$ is connected and it is not an initinal object, h^X preserves finite coproducts.

Proof. 1) If X is an initial object, $h^X(Y)$ consists of a single element for any object Y and this contradicts the assumption. Suppose $X = X_1 \coprod X_2$ and $\nu_i : X_i \to X_1 \coprod X_2$ $(i = 1, 2)$ denote the canonical morphisms. Then, the map $h^X(X_1) \coprod h^X(X_2) \to h^X(X)$ induced by $h^X(\nu_i)$ is bijective. Hence there exists a morphism $p : X \to X_k$ such that $\nu_k p = id_X$ for some $k \in \{1, 2\}$. Since ν_k is a monomorphism, it follows (A.4.6) that ν_k is an isomorphism. We claim that X_{3-k} is initial. Otherwise, there is an object Y such that $\mathcal{C}(X_{3-k}, Y)$ has more than one element. Since $\nu_i^* : \mathcal{C}(X, Y) \to \mathcal{C}(X_i, Y)$ induce a bijection $b : \mathcal{C}(X, Y) \to \mathcal{C}(X_1, Y) \times \mathcal{C}(X_2, Y)$ such that $\mathrm{pr}_i b = \nu_i^*$ and pr_k is not injective, ν_k^* is not bijective.

2) Let $f : X \to Y_1 \coprod Y_2$ be a morphism and consider pull-backs of f along the canonical morphisms $\iota_i : Y_i \to Y_1 \coprod Y_2$, which are monomorphisms by the assumption.

$$\begin{tikzcd} X_i \arrow[r, "\bar{\iota}_i"] \arrow[d, "f_i"] & X \arrow[d, "f"] \\ Y_i \arrow[r, "\iota_i"] & Y \end{tikzcd}$$

Then, by the assumption, the morphism $g : X_1 \coprod X_2 \to X$ induced by $\bar{\iota}_i : X_i \to X$ is an isomorphism. By the connectivity of X, either X_1 or X_2 is initial. If X_2 (resp. X_1) is initial, $\bar{\iota}_1$ (resp. $\bar{\iota}_2$) is an isomorphism and f factors through ι_1 (resp. ι_2). Thus the map $\mathcal{C}(X, Y_1) \coprod \mathcal{C}(X, Y_2) \to \mathcal{C}(X, Y_1 \coprod Y_2)$ induced by $\iota_{i*} : \mathcal{C}(X, Y_i) \to \mathcal{C}(X, Y_1 \coprod Y_2)$ is surjective. If a morphism $f : X \to Y_1 \coprod Y_2$ factors through both ι_1 and ι_2, f factors through the strict initial object by (A.3.16). This contradicts the assumption. Hence the above map is injective. □

Let $D : I \to \mathcal{C}$ be a functor and $\mathcal{C}/D$ a category of cones of D defined as follows. The objects of $\mathcal{C}/D$ are cones $(X \xrightarrow{p_i} D(i))_{i \in \mathrm{Ob}\, I}$ of D and

$$\mathcal{C}/D((X \xrightarrow{p_i} D(i))_{i \in \mathrm{Ob}\, I}, (Y \xrightarrow{q_i} D(i))_{i \in \mathrm{Ob}\, I}) = \{f \in \mathcal{C}(X, Y) |\, q_i f = p_i \text{ for any } i \in I\}.$$

Let $U : \mathcal{C}/D \to \mathcal{C}$ be the forgetful functor defined by $U((X \xrightarrow{p_i} D(i))_{i \in \mathrm{Ob}\, I}) = X$.

Proposition A.8.26 *1) U creates colimits.*
2) U creates limits of a functor $E : J \to \mathcal{C}/D$ such that J has a terminal objects.
3) U preserves monomorphic families.

Proof. 1) Put $E(j) = (X_j \xrightarrow{p_{ij}} D(i))_{i \in \mathrm{Ob}\, I}$. For any morphism $\theta : j \to k$ in J, since $E(\theta) : X_j \to X_k$ is a morphism in $\mathcal{C}/D$, $p_{ik} U E(\theta) = p_{ij}$. Hence $(X_j \xrightarrow{p_{ij}} D(i))_{j \in \mathrm{Ob}\, J}$ is a cone of UE.

Let $(X_j \xrightarrow{\iota_j} C)_{j \in \mathrm{Ob}\, J}$ be a colimiting cone of a functor $UE : J \to \mathcal{C}$. There exists a unique morphism $s_i : C \to D(i)$ such that $p_{ij} = s_i \iota_j$ for any $j \in \mathrm{Ob}\, J$. Then for any morphism $\lambda : i \to m$ in I, $D(\lambda) s_i \iota_j = D(\lambda) p_{ij} = p_{mj} = s_m \iota_j$ for any $j \in \mathrm{Ob}\, J$. Hence we have $D(\lambda) s_i = s_m$ and $(C \xrightarrow{s_i} D(i))_{i \in \mathrm{Ob}\, I}$ is an object of $\mathcal{C}/D$. It is easy to verify that $((X_j \xrightarrow{p_{ij}} D(i))_{i \in \mathrm{Ob}\, I} \xrightarrow{\iota_j} (C \xrightarrow{s_i} D(i))_{i \in \mathrm{Ob}\, I})$ is a colimiting cone of E.

2) Suppose that J has a terminal object j_0 and let $(C \xrightarrow{\pi_j} X_j)_{j \in \mathrm{Ob}\, J}$ be a limiting cone of a functor $UE : J \to \mathcal{C}$. We set $\rho_i = p_{ij_0} \pi_{j_0}$, then for any $j \in J$, $p_{ij} \pi_j = p_{ij_0} E(c_j) \pi_j = \rho_i$, where $c_j : j \to j_0$ is the unique morphism. For any morphism $\lambda : i \to m$ in I, $D(\lambda) \rho_i = D(\lambda) p_{ij_0} \pi_{j_0} = p_{mj_0} \pi_{j_0} = \rho_m$, hence $(C \xrightarrow{\rho_i} D(i))_{i \in \mathrm{Ob}\, I})$ is an object of $\mathcal{C}/D$. It is easy to verify that $((C \xrightarrow{\rho_i} D(i))_{i \in \mathrm{Ob}\, I}) \xrightarrow{\pi_j} (X_j \xrightarrow{p_{ij}} D(i))_{i \in \mathrm{Ob}\, I})$ is a limiting cone of E.

3) Let $(s_j : (X \xrightarrow{p_i} D(i))_{i \in \mathrm{Ob}\, I} \to (X_j \xrightarrow{p_{ij}} D(i))_{i \in \mathrm{Ob}\, I})_{j \in J}$ be a monomorphic family in $\mathcal{C}/D$. Suppose that $f, g : Y \to X$ are morphisms in $\mathcal{C}$ satisfying $s_j f = s_j g$ for any $j \in J$. Set $t_i = p_i f$, then $p_i g = p_{ij} s_j g = p_{ij} s_j f = p_i f = t_i$ and $f, g : (Y \xrightarrow{t_i} D(i))_{i \in \mathrm{Ob}\, I} \to (X \xrightarrow{p_i} D(i))_{i \in \mathrm{Ob}\, I}$ are morphisms in $\mathcal{C}/D$. Hence we have $f = g$. □

Proposition A.8.27 *Let $\mathcal{C}$ be a category satisfying R1 in (A.8.1) and $D : I \to \mathcal{C}$ a functor. Then, the forgetful functor $U : \mathcal{C}/D \to \mathcal{C}$ (A.8.26)preserves regular epimorphism.*

Proof. It follows from (A.8.26) that $\mathcal{C}/D$ also satisfies R1. Let $p : (X \xrightarrow{p_i} D(i))_{i \in \mathrm{Ob}\, I} \to (Z \xrightarrow{r_i} D(i))_{i \in \mathrm{Ob}\, I}$ be an regular epimorphism in $\mathcal{C}/D$ and $(Y \xrightarrow{q_i} D(i))_{i \in \mathrm{Ob}\, I} \overset{f}{\underset{g}{\rightrightarrows}} (X \xrightarrow{p_i q} D(i))_{i \in \mathrm{Ob}\, I}$ a kernel pair of p. By (A.8.26), $Y \overset{f}{\underset{g}{\rightrightarrows}} X \xrightarrow{p} Z$ is left exact. Let $q : X \to W$ be a coequalizer of $Y \overset{f}{\underset{g}{\rightrightarrows}} X$ in $\mathcal{C}$, then there exists a cone $(W \xrightarrow{s_i} D(i))_{i \in \mathrm{Ob}\, I}$ of D such that q is a coequalizer of f and g in $\mathcal{C}/D$ by (A.8.26). Hence there is an isomorphism $k : (Z \xrightarrow{r_i} D(i))_{i \in \mathrm{Ob}\, I} \to (W \xrightarrow{s_i} D(i))_{i \in \mathrm{Ob}\, I}$ in $\mathcal{C}/D$ such that $kp = q$. Therefore $p : X \to Z$ is an regular epimorphism in $\mathcal{C}$. □

Theorem A.8.28 *Let $\mathcal{C}$ be a regular (resp. exact) category and $D : I \to \mathcal{C}$ a functor. Then the category $\mathcal{C}/D$ of cones of D is regular (resp. exact).*

Proof. R1 and R2 follow from (A.8.26).

Let $p : (X \xrightarrow{p_i} D(i))_{i \in \mathrm{Ob}\, I} \to (Z \xrightarrow{r_i} D(i))_{i \in \mathrm{Ob}\, I}$ be an regular epimorphism in $\mathcal{C}/D$ and $f : (Y \xrightarrow{q_i} D(i))_{i \in \mathrm{Ob}\, I} \to (Z \xrightarrow{r_i} D(i))_{i \in \mathrm{Ob}\, I}$ a morphism in $\mathcal{C}/D$. Then $p : X \to Z$ is a regular epimorphism by (A.8.26) and the pull-back $p' : W \to Y$ of p along f exists in $\mathcal{C}$, which is a regular epimorphism. Hence (A.8.26) implies that there exists a unique cone $(W \xrightarrow{s_i} D(i))_{i \in \mathrm{Ob}\, I}$ of D such that p' is a morphism of $\mathcal{C}/D$ and p' is a pull-back of p along f in $\mathcal{T}/D$. Form a kernel pair of p' in $\mathcal{C}$ and lift it to $\mathcal{C}/D$ using (A.8.26), then p' is a colimiting of its kernel pair in $\mathcal{C}/D$ by (A.8.26). Therefore p' is a regular epimorphism in $\mathcal{C}/D$.

Suppose that $\mathcal{C}$ is exact and let $(R \xrightarrow{q_i} D(i))_{i\in \mathrm{Ob}\, I} \overset{f}{\underset{g}{\rightrightarrows}} (X \xrightarrow{p_i} D(i))_{i\in \mathrm{Ob}\, I}$ be an equivalence relation in $\mathcal{C}/D$. It follows from (A.3.18) and (A.8.26) that $U : \mathcal{C}/D \to \mathcal{C}$ preserves equivalence relations. Hence $R \overset{f}{\underset{g}{\rightrightarrows}} X$ is a kernel pair of a certain morphism $p : X \to Z$ in $\mathcal{C}$. We may assume that p is a coequalizer of f and g by (A.8.14). By (A.8.26), there exists a unique cone $(Z \xrightarrow{q_i} D(i))_{i\in \mathrm{Ob}\, I}$ such that p is a coequalizer of f and g in $\mathcal{C}/D$. Since $R \overset{f}{\underset{g}{\rightrightarrows}} X$ is a kernel pair of a $p : X \to Z$ in $\mathcal{C}$, $(R \xrightarrow{q_i} D(i))_{i\in \mathrm{Ob}\, I} \overset{f}{\underset{g}{\rightrightarrows}} (X \xrightarrow{p_i} D(i))_{i\in \mathrm{Ob}\, I}$ is a kernel pair of p in $\mathcal{C}/D$ by (A.8.26). □

A.9 Subobjects in a regular category

Let $\mathcal{C}$ be a category. Recall that $(\mathrm{Sub}(X), \subset)$ denotes the ordered set of subobjects of $X \in \mathrm{Ob}\,\mathcal{C}$. We regard $\mathrm{Sub}(X)$ as a category.

If $f : Y \to X$ is a morphism of $\mathcal{C}$ such that each monomorphism $Z \rightarrowtail X$ has a pull-back along f, then we have a map $f^* : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ which maps a subobject represented by a monomorphism σ to a subobject represented by the pull-back of σ along f.

Proposition A.9.1 *Let $\mathcal{C}$ be a category with pull-backs of monomorphisms. For any $Z_1, Z_2 \in \mathrm{Sub}(X)$, $f^*(Z_1 \cap Z_2) = f^*(Z_1) \cap f^*(Z_2)$. Hence f^* preserves the order and it is regarded as a functor.*

Proof. We have the following pull-back squares.

$$(1)\quad \begin{array}{ccc} Z_1\cap Z_2 & \xrightarrow{\bar\sigma_1} & Z_2 \\ \downarrow{\scriptstyle \bar\sigma_2} & & \downarrow{\scriptstyle \sigma_2} \\ Z_1 & \xrightarrow{\sigma_1} & X \end{array} \qquad (2)\quad \begin{array}{ccc} f^*(Z_1\cap Z_2) & \xrightarrow{f_{12}} & Z_1\cap Z_2 \\ \downarrow{\scriptstyle \sigma_{12}} & & \downarrow{\scriptstyle \sigma_1\bar\sigma_2=\sigma_2\bar\sigma_1} \\ Y & \xrightarrow{f} & X \end{array}$$

$$(3)\quad \begin{array}{ccc} f^*(Z_1) & \xrightarrow{f_1} & Z_1 \\ \downarrow{\scriptstyle f^*(\sigma_1)} & & \downarrow{\scriptstyle \sigma_1} \\ Y & \xrightarrow{f} & X \end{array} \qquad (4)\quad \begin{array}{ccc} f^*(Z_2) & \xrightarrow{f_2} & Z_2 \\ \downarrow{\scriptstyle f^*(\sigma_2)} & & \downarrow{\scriptstyle \sigma_2} \\ Y & \xrightarrow{f} & X \end{array}$$

Since $f\sigma_{12} = \sigma_1\bar\sigma_2 f_{12} = \sigma_2\bar\sigma_1 f_{12}$, there exist morphisms $f^*(\bar\sigma_2) : f^*(Z_1 \cap Z_2) \to f^*(Z_1)$ and $f^*(\bar\sigma_1) : f^*(Z_1 \cap Z_2) \to f^*(Z_2)$ such that $\sigma_{12} = f^*(\sigma_1)f^*(\bar\sigma_2) = f^*(\sigma_2)f^*(\bar\sigma_1)$, $\bar\sigma_2 f_{12} = f_1 f^*(\bar\sigma_2)$ and $\bar\sigma_1 f_{12} = f_2 f^*(\bar\sigma_1)$. Applying (A.3.1) to the above diagrams (2) and (3), the following square (5) is a pull-back.

$$(5)\quad \begin{array}{ccc} f^*(Z_1\cap Z_2) & \xrightarrow{f_{12}} & Z_1\cap Z_2 \\ \downarrow{\scriptstyle f^*(\bar\sigma_2)} & & \downarrow{\scriptstyle \bar\sigma_2} \\ f^*(Z_1) & \xrightarrow{f_1} & Z_1 \end{array}$$

Again by applying (A.3.1) to (1) and (5), the following square (6) is a pull-back. Finally, since we have $\bar\sigma_1 f_{12} = f_2 f^*(\bar\sigma_1)$, $\sigma_1 f_1 = f f^*(\sigma_1)$ and the diagram (4) is a pull-back, the diagram (7) is a pull-back by (A.3.1).

$$(6)\quad \begin{array}{ccc} f^*(Z_1\cap Z_2) & \xrightarrow{f^*(\bar\sigma_2)} & f^*(Z_2) \\ \downarrow{\scriptstyle \bar\sigma_1 f_{12}} & & \downarrow{\scriptstyle \sigma_1 f_2} \\ Z_2 & \xrightarrow{\sigma_2} & X \end{array} \qquad (7)\quad \begin{array}{ccc} f^*(Z_1\cap Z_2) & \xrightarrow{f^*(\bar\sigma_2)} & f^*(Z_2) \\ \downarrow{\scriptstyle f^*(\bar\sigma_1)} & & \downarrow{\scriptstyle f^*(\sigma_1)} \\ f^*(Z_2) & \xrightarrow{f^*(\sigma_2)} & Y \end{array}$$

□

We note that if $i : Y \to X$ is a monomorphism, we have an injection $i_! : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ defined by $i_!(Z \overset{\sigma}{\rightarrowtail} Y) = (Z \overset{i\sigma}{\rightarrowtail} X)$. Thus $\mathrm{Sub}(Y)$ is identified with a subset $\{Z \in \mathrm{Sub}(X) | \, Z \subset Y\}$ of $\mathrm{Sub}(X)$.

Proposition A.9.2 *Let $\mathcal{C}$ be a category with pull-backs of monomorphisms and $i : Z \rightarrowtail X$, $j : W \rightarrowtail Y$ monomorphisms. If the following square on the left is a pull-back, the right one commutes.*

$$\begin{array}{ccc} W & \xrightarrow{j} & Y \\ \downarrow g & & \downarrow f \\ Z & \xrightarrow{i} & X \end{array} \qquad \begin{array}{ccc} \mathrm{Sub}(Z) & \xrightarrow{i_!} & \mathrm{Sub}(X) \\ \downarrow g^* & & \downarrow f^* \\ \mathrm{Sub}(W) & \xrightarrow{j_!} & \mathrm{Sub}(Y) \end{array}$$

Proof. For any $U \in \mathrm{Sub}(Z)$ represented by a monomorphism $\sigma : U \rightarrowtail Z$, consider pull-back of σ along g.

$$\begin{array}{ccccc} V & \xrightarrow{\bar\sigma} & W & \xrightarrow{j} & Y \\ \downarrow h & & \downarrow g & & \downarrow f \\ U & \xrightarrow{\sigma} & Z & \xrightarrow{i} & X \end{array}$$

By (A.3.1), the outer rectangle of the above diagram is a pull-back. Hence $f^*i_!(U) \in \mathrm{Sub}(Y)$ is represented by $j\bar\sigma : V \to Y$ which also represents $j_!g^*(U)$.

If $f : X \to Y$ is a morphism in a category such that f has a factorization $f = ip$ with $p : X \to Z$ a regular epimorphism and $i : Z \to Y$ a monomorphism, then (A.8.9) shows that the subobject of Y represented by i does not depend on the choice of the factorization. We call this subobject the image of f.

If $\mathcal{C}$ is a regular category, for a morphism $f : X \to Y$, we can define a functor $f_! : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ by $f_!(Z \overset{\sigma}{\rightarrowtail} X) = (\textit{the image of} Z \overset{\sigma}{\rightarrowtail} X \xrightarrow{f} Y)$ and $f_!(Z \xrightarrow{\iota} W) = (\textit{the morphism induced by } \iota \textit{ and } id_Y)$ (A.8.9). $\square$

Proposition A.9.3 *Let $\mathcal{C}$ be a regular category. Suppose that $f : Y \to X$ is a morphism of $\mathcal{C}$ such that each monomorphism $Z \rightarrowtail X$ has a pull-back along f.*

1) $f_! : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ is a left adjoint of $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$. In fact, the unit $\eta : id_{\mathrm{Sub}(X)} \to f^*f_!$ and the counit $\varepsilon : f_!f^* \to id_{\mathrm{Sub}(Y)}$ are given as follows. For $(Z \overset{\sigma}{\rightarrowtail} X) \in \mathrm{Ob}\,\mathrm{Sub}(X)$, let $\bar\sigma : f^*f_!(Z) \rightarrowtail X$ be the pull-back of the image $f_!(Z) \rightarrowtail Y$ of $Z \overset{\sigma}{\rightarrowtail} X$ along f, then $\eta_Z : Z \to f^*f_!(Z)$ is the unique morphism satisfying $\bar\sigma\eta_Z = \sigma$. For $(W \overset{\tau}{\rightarrowtail} Y) \in \mathrm{Ob}\,\mathrm{Sub}(Y)$, since $f^*(W) \overset{f^*(\tau)}{\rightarrowtail} X \xrightarrow{f} Y$ factors through τ, there is a unique morphism $\varepsilon_W : f_!f^*(W) \to W$ in $\mathrm{Sub}(Y)$.

2) $f^*f_! = id_{\mathrm{Sub}(X)}$ holds, if f is a monomorphism and $f_!f^* = id_{\mathrm{Sub}(Y)}$ holds, if f is a regular epimorphism.

Proof. 1) Let Z be an object of $\mathrm{Sub}(X)$ and W an object of $\mathrm{Sub}(Y)$. Suppose that $f_!(Z) \subset W$ in $\mathrm{Sub}(Y)$. Since

$$\begin{array}{ccc} f^*(W) & \longrightarrow & W \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is a pull-back, $Z \rightarrowtail X$ and $Z \twoheadrightarrow f_!(Z) \rightarrowtail W$ induces a unique morphism $Z \rightarrowtail f^*(W)$ over X. Thus we have $Z \subset f^*(W)$ in $\mathrm{Sub}(X)$. Suppose that $Z \subset f^*(W)$ in $\mathrm{Sub}(X)$. Apply (A.8.4) to

$$\begin{array}{ccc} Z & \longrightarrow & f_!(Z) \\ \downarrow & & \downarrow \\ W & \longrightarrow & Y \end{array},$$

where the left vertical morphism is the composition $Z \rightarrowtail f^*(W) \to W$. Then we have a unique monomorphism $f_!(Z) \to W$ over Y. Therefore $f_!(Z) \subset W$.

2) If f is a monomorphism, then $f_! : \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ is fully faithful. Hence the unit $\eta : id_{\mathrm{Sub}(X)} \to f^*f_!$ is a natural equivalence. Namely, for any subobject Z of X, $f^*f_!(Z)$ and Z are the same. Thus η_Z is an identity morphism. If f is a regular epimorphism, then $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is fully faithful. In fact, since $\mathrm{Sub}(Y)$ is an ordered set, f^* is faithful. Let $Z \overset{\sigma}{\rightarrowtail} Y$ and $W \overset{\tau}{\rightarrowtail} Y$ be subobjects of Y such that $f^*(Z) \overset{\iota}{\rightarrowtail} f^*(W)$ in $\mathrm{Sub}(X)$, then the pull-back $f' : f^*(Z) \to Z$ of f along σ is a regular epimorphism. Let us denote by $f'' : f^*(W) \to W$ the pull-back of f along τ. Applying (A.8.4) to the following square, we have a morphism $\iota' : Z \to W$ satisfying $\sigma = \tau\iota'$.

$$\begin{array}{ccc} f^*(Z) & \xrightarrow{f''\iota} & W \\ \downarrow f' & & \downarrow \tau \\ Z & \xrightarrow{\sigma} & Y \end{array}$$

Therefore f^* is also full. It follows that the counit $\varepsilon : f_! f^* \to id_{\mathrm{Sub}(Y)}$ is a natural equivalence, hence ε_W is an identity morphism for any $W \in \mathrm{Sub}(Y)$. □

Proposition A.9.4 *Let $\mathcal{C}$ is a regular category such that each monomorphism has a pull-back along arbitrary morphism.*

1) If the following square on the left is a pull-back, the right one commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "k"] \\ Z \arrow[r, "g"] & W \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sub}(Y) \arrow[r, "f^*"] \arrow[d, "k_!"] & \mathrm{Sub}(X) \arrow[d, "h_!"] \\ \mathrm{Sub}(W) \arrow[r, "g^*"] & \mathrm{Sub}(Z) \end{tikzcd}$$

2) If the following square on the left is a pull-back and k is a monomorphism, the right one is also a pull-back.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "k"] \\ Z \arrow[r, "g"] & W \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sub}(X) \arrow[r, "f_!"] \arrow[d, "h_!"] & \mathrm{Sub}(Y) \arrow[d, "k_!"] \\ \mathrm{Sub}(Z) \arrow[r, "g_!"] & \mathrm{Sub}(W) \end{tikzcd}$$

Proof. 1) Let $U \overset{\sigma}{\rightarrowtail} Y$ be a subobject of Y. Since each square on the left and middle is a pull-back, there exists a unique morphism $\bar{h} : f^*(U) \to g^* k_!(U)$ such that $\bar{\tau}\bar{h} = h\bar{\sigma}$. It follows from (A.3.1) that the square on the right is a pull-back.

$$\begin{tikzcd} f^*(U) \arrow[r, "\bar{\sigma}"] \arrow[d] & X \arrow[r, "h"] \arrow[d, "f"] & Z \arrow[d, "g"] \\ U \arrow[r, "\sigma"] & Y \arrow[r, "k"] & W \end{tikzcd} \qquad \begin{tikzcd} g^* k_!(U) \arrow[r, "\bar{\tau}"] \arrow[d] & Z \arrow[d, "g"] \\ k_!(U) \arrow[r, "\tau"] & W \end{tikzcd} \qquad \begin{tikzcd} f^*(U) \arrow[r, "\bar{h}"] \arrow[d] & g^* k_!(U) \arrow[d] \\ U \arrow[r, "p"] & k_!(U) \end{tikzcd}$$

Since p is a regular epimorphism, so is $\bar{h}$. On the other hand, since $\bar{\tau}$ is a pull-back of a monomorphism τ, $\bar{\tau}$ is a monomorphism. Hence $f^*(U) \overset{\bar{h}}{\twoheadrightarrow} g^* k_!(U) \overset{\bar{\tau}}{\rightarrowtail} Z$ is a mono-epi factorization of $h\bar{\sigma}$, we have $h_! f^*(U) = g^* k_!(U)$ in $\mathrm{Sub}(Z)$ by (A.8.9).

2) Suppose that $S \overset{\sigma}{\rightarrowtail} Y$ and $T \overset{\tau}{\rightarrowtail} Z$ satisfies $k_!(S) = g_!(T)$. Since k is a monomorphism, $S \overset{k\sigma}{\rightarrowtail} W$ is the image of $g\tau$. Hence there is a regular epimorphism $p : T \twoheadrightarrow S$ satisfying $g\tau = k\sigma p$. Then, we have a unique monomorphism $\tau' : T \rightarrowtail X$ satisfying $h\tau' = \tau$ and $f\tau' = \sigma p$. Since h is a monomorphism, the first equality shows that $h_!(T \overset{\tau'}{\rightarrowtail} X) = (T \overset{\tau}{\rightarrowtail} Z)$. Since p is a regular epimorphism, the second equality shows that $f_!(T \overset{\tau'}{\rightarrowtail} X) = (S \overset{\sigma}{\rightarrowtail} Y)$. The uniqueness of the subobject U of X satisfying $h_!(U) = T$ and $f_!(U) = S$ follows from the fact that h is a monomorphism. □

Proposition A.9.5 *Let $(\sigma_i : X_i \rightarrowtail X)_{i \in I}$ be subobjects of X.*

1) If there exists a limiting cone $(L \overset{\tau_i}{\to} X_i)_{i \in I}$ of a diagram $(\sigma_i : X_i \rightarrowtail X)_{i \in I}$ exists, then $\sigma_i\tau_i : L \rightarrowtail X$ is the lower bound of $\{\sigma_i | \, i \in I\}$ in the ordered set $(\mathrm{Sub}(X), \subset)$.

2) If there exists a coproduct $\coprod_{i \in I} X_i$ and the morphism $\sigma : \coprod_{i \in I} X_i \to X$ induced by σ_i's has a factorization $\sigma = jp$ with $p : \coprod_{i \in I} X_i \to U$ a regular epimorphism and $j : U \to X$ a monomorphism, then j is the upper bound of $\{\sigma_i | \, i \in I\}$ in the ordered set $(\mathrm{Sub}(X), \subset)$.

Proof. 1) We first note that each τ_i is a monomorphism. In fact, let $f, g : Z \to L$ be morphisms such that $\tau_i f = \tau_i g$. Then, for any $j \in I$, $\sigma_j\tau_j f = \sigma_i\tau_i f = \sigma_i\tau_i g = \sigma_j\tau_j g$. Since σ_j is a monomorphism, we have $\tau_j f = \tau_j g$ for any $j \in I$. It follows that $f = g$.

Let $\tau : Y \rightarrowtail X$ be a monomorphism such that there is a family of morphisms $(Y \overset{\alpha_i}{\longrightarrow} X_i)_{i \in I}$ satisfying $\sigma_i\alpha_i = \tau$. There exists a unique morphism $\beta : Y \to L$ such that $\alpha_i = \tau_i\beta$. This shows that $L \rightarrowtail X$ is the lower bound of $\{\sigma_i | \, i \in I\}$.

2) Suppose that p is a coequalizer of $R \overset{f}{\underset{g}{\rightrightarrows}} \coprod_{i \in I} X_i$. Let $\tau : Y \to X$ be a monomorphism such that there exist morphisms $\alpha_i : X_i \to Y$ $(i \in I)$ satisfying $\tau\alpha_i = \sigma_i$. We denote by $\alpha : \coprod_{i \in I} X_i \to Y$ the morphism induced

by α_i's. Then $\tau\alpha = \sigma$, hence $\tau\alpha f = \sigma f = jpf = jpg = \sigma g = \tau\alpha g$. Since τ is a monomorphism, α equalizes f and g and we have a morphism $\iota : U \to Y$ such that $\iota p = \alpha$. Therefore, we have $\tau\iota p = \tau\alpha = \sigma = jp$. Since p is an epimorphism, $\tau\iota = j$.

We denote the above C (resp. U) by $X_1 \cap X_2$ (resp. $X_1 \cup X_2$), and call the intersection (resp. union) of X_1 and X_2. □

Corollary A.9.6 *If $\mathcal{C}$ is a regular category with pull-backs of monomorphisms and finite coproducts, the ordered set $(\mathrm{Sub}(X), \subset)$ is finitely complete and cocomplete. Moreover, $\mathrm{Sub}(X)$ is a lattice with operations $\cap$ and $\cup$.*

Proof. It is obvious that the class of $id_X : X \to X$ is the maximum element of $\mathrm{Sub}(X)$. Let 0 be an initial object (empty coproduct) of $\mathcal{C}$ and $n_X : 0 \to X$ the unique morphism. Consider the mono-epi factorization $0 \xrightarrow{p} 0_X \overset{\nu}{\rightarrowtail} X$ and we claim that 0_X is the minimum element of $\mathrm{Sub}(X)$. In fact, for a monomorphism $i : Y \rightarrowtail X$, we have $n_X = in_Y$. Let $R \underset{g}{\overset{f}{\rightrightarrows}} 0$ be a kernel pair of p. Then, $in_Y f = n_X f = \nu pf = \nu pg = n_X g = in_Y g$ and, since i is a monomorphism, we have $n_Y f = n_Y g$. Hence there exists a unique morphism $m : 0_X \to Y$ such that $mp = n_Y$. It follows that $imp = in_Y = n_X = \nu p$. Since p is an epimorphism, we have $im = \nu$. This shows that 0_X is a subobject of Y.

For $Y, Z, W \in \mathrm{Sub}(X)$, the following equalities are obvious. $(Y \cup Z) \cup W = Y \cup (Z \cup W)$, $(Y \cap Z) \cap W = Y \cap (Z \cap W)$, $Y \cup Z = Z \cup Y$, $Y \cap Z = Z \cap Y$, $Y \cup Y = Y$, $Y \cap Y = Y$. It is easy to verify that $Y \subset Z$ holds if and only if $Y \cap Z = Y$ or $Y \cup Z = Z$ holds. Hence we have equalities $Y \cup 0_X = Y$, $Y \cap X = Y$, $(Y \cap Z) \cup Z = Z$, $Y \cap (Y \cup Z) = Y$. □

Proposition A.9.7 *Let $\mathcal{C}$ be a regular category with finite limits and $(\sigma_i : X_i \rightarrowtail X)_{i\in I}$ a family of monomorphisms. Suppose that a coproduct of $(X_i)_{i\in I}$ exists and it is universal. Then, the union $\bigcup_{i\in I} X_i$ exists by (A.9.5). The family of inclusion morphisms $(X_j \xrightarrow{\tau_j} \bigcup_{i\in I} X_j)_{j\in I}$ is a colimiting cone of the diagram $(X_i \cap X_j \overset{\kappa_{ij}}{\rightarrowtail} X_j)_{i,j\in I}$ of the inclusion morphisms.*

Proof. Let $\rho : \coprod_{i\in I} X_i \to X$ be the morphism induced by σ_i's and $\coprod_{i\in I} X_i \xrightarrow{p} \bigcup_{i\in I} X_i \overset{\iota}{\rightarrowtail} X$ a mono-epi factorization of ρ. We denote by $\nu_i : X_i \to \coprod_{i\in I} X_i$ and $\nu_{ij} : X_i \cap X_j \to \coprod_{(i,j)\in I\times I} (X_i \cap X_j)$ the canonical morphisms. Then, $\tau_i = p\nu_i$. Define $f, g : \coprod_{(i,j)\in I\times I} (X_i \cap X_j) \to \coprod_{i\in I} X_i$ to be the morphisms satisfying $f\nu_{ij} = \nu_i\kappa_{ji}$, $g\nu_{ij} = \nu_j\kappa_{ij}$. Then, $\coprod_{(i,j)\in I\times I} (X_i \cap X_j) \underset{g}{\overset{f}{\rightrightarrows}} \coprod_{i\in I} X_i$ is a kernel pair of ρ by (A.4.5). Suppose that morphisms $u_i : X_i \to Z$ $(i \in I)$ satisfy $u_i\kappa_{ji} = u_j\kappa_{ij}$. Let $\varphi : \coprod_{i\in I} X_i \to Z$ be the morphism induced by u_i's. Then, we have $\varphi f\nu_{ij} = \varphi\nu_i\kappa_{ji} = u_i\kappa_{ji} = u_j\kappa_{ij} = \varphi\nu_j\kappa_{ij} = \varphi g\nu_{ij}$ for any $i, j \in I$. Hence $\varphi f = \varphi g$. Since f, g is also a kernel pair of the regular epimorphism p, there is a unique morphism $\bar{\varphi} : \bigcup_{i\in I} X_i \to Z$ satisfying $\bar{\varphi}p = \varphi$ which is equivalent to $\bar{\varphi}\tau_i = u_i$ $(i \in I)$. □

Proposition A.9.8 *Let $\mathcal{C}$ be a regular category with finite limits, Y a subobject of $X \in \mathrm{Ob}\,\mathcal{C}$ and $(\sigma_i : X_i \rightarrowtail X)_{i\in I}$ a family of monomorphisms. Suppose that coproducts of $(X_i)_{i\in I}$, $(Y \cap X_i)_{i\in I}$ exists and they are universal. Then, $Y \cap (\bigcup_{i\in I} X_i) = \bigcup_{i\in I}(Y \cap X_i)$. Hence if $\mathcal{C}$ has universal finite coproducts, the lattice $(\mathrm{Sub}(X), \cap, \cup)$ is distributive.*

Proof. We use the same notations as in (A.9.7). Set $Y' = Y \cap (\bigcup_{i\in I} X_i)$ and $\sigma : Y' \to \bigcup_{i\in I} X_i$, $\alpha_i : Y \cap X_i = Y' \cap X_i \to Y'$, $\beta_{ij} : Y \cap X_i \cap X_j \to Y \cap X_j$ denote the inclusion morphisms. By the definition of $\bigcup_{i\in I} X_i$, $p : \coprod_{i\in I} X_i \to \bigcup_{i\in I} X_i$ is a regular epimorphism. It follows from (A.4.5) that the morphism $p' : \coprod_{i\in I}(Y \cap X_i) \to Y'$ induced by α_i's is a pull-back of p along σ. Moreover, since $\mathcal{C}$ is a regular category, p' is a regular epimorphism. By (A.4.5), the coproduct $\coprod_{(i,j)\in I\times I} (Y \cap X_i \cap X_j)$ exists and a pair of morphisms $f', g' : \coprod_{(i,j)\in I\times I} (Y \cap X_i \cap X_j) \to \coprod_{j\in I}(Y \cap X_j)$ satisfying $f'\lambda_{ij} = \mu_i\beta_{ji}$, $g'\lambda_{ij} = \mu_j\beta_{ij}$ for any $i, j \in I$ is the kernel pair of p'. Here $\lambda_{ij} : Y \cap X_i \cap X_j \to \coprod_{(i,j)\in I\times I} (Y \cap X_i \cap X_j)$ and $\mu_i : Y \cap X_i \to \coprod_{j\in I}(Y \cap X_j)$ denote the canonical morphisms. It is clear that the morphism $q : \coprod_{j\in I}(Y \cap X_j) \to \bigcup_{i\in I}(Y \cap X_i)$ defining $\bigcup_{i\in I}(Y \cap X_i)$ coequalizes

f' and g'. Hence there is a unique morphism $\eta : Y' \to \bigcup_{i\in I}(Y \cap X_i)$ satisfying $\eta p' = q$. Obviously, η commutes with inclusion morphisms $Y' \to X$ and $\bigcup_{i\in I}(Y \cap X_i) \to X$. Therefore $Y' \subset \bigcup_{i\in I}(Y \cap X_i)$. Since $Y \cap X_i \subset Y'$, $\bigcup_{i\in I}(Y \cap X_i) \subset Y'$ is clear. □

Proposition A.9.9 *Let $\mathcal{C}$ be a regular category with pull-backs of monomorphisms and finite coproducts.*

1) Suppose that finite coproducts in $\mathcal{C}$ is universal. For $X \in \mathrm{Ob}\,\mathcal{C}$, if there exist subobjects $X_1, X_2 \in \mathrm{Sub}(X)$ such that $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = 0$, then the morphism $f : X_1 \coprod X_2 \to X$ induced by the inclusion morphisms $\iota_i : X_i \rightarrowtail X$ $(i = 1, 2)$ is an isomorphism. Hence, if X is connected, then $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = 0$ in $\mathrm{Sub}(X)$ imply $(X_1, X_2) = (X, 0)$ or $(0, X)$.

2) Suppose that finite coproducts in $\mathcal{C}$ is disjoint. For $X \in \mathrm{Ob}\,\mathcal{C}$, if there exists an isomorphism $f : X_1 \coprod X_2 \to X$, then $X = X_1 \cup X_2$ and $X_1 \cap X_2 = 0$, where we regard X_i as a subobject of X by the morphism $X_i \xrightarrow{\nu_i} X_1 \coprod X_2 \xrightarrow[\cong]{f} X$. Hence, if $X_1 \cup X_2 = X$ and $X_1 \cap X_2 = 0$ in $\mathrm{Sub}(X)$ imply $(X_1, X_2) = (X, 0)$ or $(0, X)$, then X is connected.

Proof. 1) Since f is a composition of a regular epimorphism $p : X_1 \coprod X_2 \to X_1 \cup X_2$ and an isomorphism $i : X_1 \cup X_2 \to X$, f is a regular epimorphism. By the assumption, the identity morphisms of $X_1 \coprod X_2$ is the kernel pair of f (See the proof of 4.24). Hence f is also an isomorphism by (A.3.2) and it follows from (A.8.5) that f is an isomorphism.

2) We first note that Since $X_1 \coprod X_2$ is disjoint, the canonical morphism $\nu_i : X_i \to X_1 \coprod X_2$ is a monomorphism and $X_1 \cap X_2 = 0$. Since f is an isomorphism, $X = X_1 \cup X_2$ by the definition of $X_1 \cup X_2$. □

Let X be an object of a category $\mathcal{C}$ and denote by $EQ(X)$ the set of all equivalence relations on X. Two equivalence relations $R \overset{a}{\underset{b}{\rightrightarrows}} X$ and $S \overset{c}{\underset{d}{\rightrightarrows}} X$ are said to be equivalent if there is an isomorphism $\varphi : R \to S$ in $\mathcal{C}$ satisfying $a\varphi = c$ and $b\varphi = d$. We denote by $E(X)$ the quotient set of $EQ(X)$ by this equivalence relation. If a product $X \times X$ exists, $E(X)$ is regarded as a subset of $\mathrm{Sub}(X \times X)$. If R and S are elements of $E(X)$ represented by equivalence relations $R \overset{a}{\underset{b}{\rightrightarrows}} X$ and $S \overset{c}{\underset{d}{\rightrightarrows}} X$, a morphism $\varphi : R \to S$ is a morphism $\varphi : R \to S$ in $\mathcal{C}$ satisfying $a\varphi = c$ and $b\varphi = d$. Note that since (c, d) is a monomorphic pair, φ is necessarily a monomorphism in $\mathcal{C}$. Thus we also denote by $E(X)$ the category of equivalence relations on X with these morphisms. We also note that $E(X)$ is an ordered set.

Two regular epimorphisms $p : X \to Y$ and $q : X \to Z$ are said to be equivalent if there exists an isomorphism $r : Y \to Z$ satisfying $rp = q$. Let $RE(X)$ be the set of all regular epimorphisms with domain X. We denote by $Q(X)$ the quotient set of $RE(X)$ by the equivalence relation. An element of $Q(X)$ is called a quotient object of X. If Y and Z are quotient objects of X represented by regular epimorphisms $p : X \to Y$ and $q : X \to Z$, a morphism $f : Y \to X$ of quotient objects is a morphism $f : Y \to Z$ in $\mathcal{C}$ satisfying $fp = q$. We note that f is a regular epimorphism in $\mathcal{C}$ by (A.8.6). Hence we also regard $Q(X)$ as a category of quotient objects, which is an ordered set.

If every equivalence relation on X has a coequalizer, we can define a functor $C : E(X) \to Q(X)$ as follows. $C(R \overset{a}{\underset{b}{\rightrightarrows}} X) = (\textit{the coequalizer of } a \textit{ and } b)$ and for a morphism $\varphi : (R \overset{a}{\underset{b}{\rightrightarrows}} X) \longrightarrow (S \overset{c}{\underset{d}{\rightrightarrows}} X)$ of equivalence relations with coequalizers $p : X \to Y$ and $q : X \to Z$, $C(\varphi) : Y \to Z$ is the unique morphism satisfying $C(\varphi)p = q$.

If every regular epimorphism with domain X has a kernel pair, we can define a functor $K : Q(X) \to E(X)$ as follows. $K(p : X \to Y) = (\textit{the kernel pair of } p)$ and for a morphism $\psi : (p : X \to Y) \to (q : X \to Z)$ of regular epimorphisms with kernel pairs $R \overset{a}{\underset{b}{\rightrightarrows}} X$ and $S \overset{a}{\underset{d}{\rightrightarrows}} X$, $K(\psi) : R \to S$ is the unique morphism satisfying $cK(\psi) = a$ and $dK(\psi) = b$.

The next result is a direct consequence of (A.8.14).

Proposition A.9.10 *If every equivalence relation on X has a coequalizer and every regular epimorphism with domain X has a kernel pair, $CK : Q(X) \to Q(X)$ is the identity functor. Moreover, if every equivalence relation on X is effective, $KC : E(X) \to E(X)$ is the identity functor. In particular, if $\mathcal{C}$ is an exact category, $E(X)$ is isomorphic to $Q(X)$ for every object X.*

A.10 Additive exact category

Let $\mathcal{A}$ be a regular category which is preadditive, that, is, the following conditions hold for any objects A, B, C of $\mathcal{A}$.

(1) $\mathcal{A}(A, B)$ has a structure of an abelian group.

(2) The composition $\mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ of morphisms is biadditive.

Take an object A of $\mathcal{A}$ and $0 : A \to A$ denotes the zero map, that is, the unit of $\mathcal{A}(A, A)$, then for any morphism $f : B \to A$, $0f = (0 + 0)f = 0f + 0f$ implies that $0f : B \to A$ is the zero map. Consider the kernel pair $P \overset{p_1}{\underset{p_2}{\rightrightarrows}} A$ of 0. Since $0id_A = 0 = 00$, there exists a unique morphism $h : A \to P$ satisfying $p_1h = id_A$ and $p_2h = 0$. Let $\nu : A \to N$ be the coequalizer of $P \overset{p_1}{\underset{p_2}{\rightrightarrows}} A$. For any morphism $f : N \to B$, we have $f\nu = f\nu p_1h = f\nu p_2h = f\nu 0 = 0$. Since ν is an epimorphism, it follows $f = 0$. Therefore $\mathcal{A}(N, B) = \{0\}$ and N is an initial object of $\mathcal{A}$. We also denote N by 0.

0 is also a terminal object of $\mathcal{A}$. In fact, since $id_0 = 0 \in \mathcal{A}(0, 0) = \{0\}$, for any morphism $f : B \to 0$, $f = id_0f = 0f = 0$. Thus we have shown

Proposition A.10.1 *A preadditive regular category $\mathcal{A}$ has a null (= initial and terminal) object 0 and the unique morphism $A \to 0$ is a regular epimorphism.*

The next assertion follows from R3 and the above result.

Proposition A.10.2 *Finite products exist in $\mathcal{A}$. Thus finite biproducts exist in $\mathcal{A}$.*

Proposition A.10.3 *Morphisms in $\mathcal{A}$ have kernels.*

Proof. Let $f : A \to B$ be a morphism. Form a kernel pair $C \overset{g}{\underset{h}{\rightrightarrows}} A$ and factorize $g-h : C \to A$ as $g-h = ip$ with $p : C \to D$ a regular epimorphism and $i : D \to A$ a monomorphism. Then, $fip = f(g - h) = fg - fh = 0$ and this implies $fi = 0$. Let $j : E \to A$ be a morphism satisfying $fj = 0$. There exists a morphism $k : E \to C$ such that $gk = j$ and $hk = 0$. Hence $j = (g - h)k = ipk$ and j factors through i. Since i is a monomorphism, this factorization is unique. Therefore i is a kernel of f. □

Proposition A.10.4 *Let A be an object of preadditive category $\mathcal{A}$ such that the product $A \times A$ exists. A subobject $i = (p_1, p_2) : R \rightarrowtail A \times A$ is an equivalence relation if R contains the diagonal subobject.*

Proof. For any object B, the image of $(p_{1*}, p_{2*}) : \mathcal{A}(B, R) \to \mathcal{A}(B, A) \times \mathcal{A}(B, A)$ is a subgroup of $\mathcal{A}(B, A) \times \mathcal{A}(B, A)$ containing the diagonal subgroup. Hence it suffices to show that for an abelian group G, if E is a subgroup of $G \times G$ containing the diagonal subgroup, then E is an equivalence relation on G. If $(x, y) \in E$, then $(y, x) = (y, y) - (x, y) + (x, x) \in E$. If $(x, y), (y, z) \in E$, then $(x, z) = (x, y) - (y, y) + (y, z) \in E$. □

Proposition A.10.5 *Every monomorphism of a preadditive exact category has a cokernel and it is a kernel of its cokernel.*

Proof. Let $f : B \to A$ be a monomorphism and define morphisms $g : B \oplus A \to A \oplus A$ and $h : A \to B \oplus A$ by $g = (fp_1 + p_2, p_2)$ and $h = (0, id_A)$, where $p_1 : B \oplus A \to B$ and $p_2 : B \oplus A \to A$ are projections. Then, gh is the diagonal morphism and the image of g contains the diagonal subobject. Moreover, g is a monomorphism. In fact, suppose that $k, l : C \to B \oplus A$ are morphisms satisfying $gk = gl$. Then we have $fp_1k + p_2k = fp_1l + p_2l$ and $p_2k = p_2l$. Since f is a monomorphism, it follows that $p_1k = p_1l$.

Hence $B \oplus A \overset{fp_1+p_2}{\underset{p_2}{\rightrightarrows}} A$ is an equivalence relation on A by (A.10.4) and it is a kernel pair of a certain morphism $p : A \to D$. We may assume that p is a coequalizer of $fp_1 + p_2, p_2$ by (A.3.6). Then, since p_1 has a right inverse $(id_B, 0) : B \to B \oplus A$, $p(fp_1 + p_2) = pp_2$ implies that $pf = 0$. Suppose $pk = 0(= p0)$ for $k : C \to A$. Then, there exists a morphism $s : C \to B \oplus A$ such that $fp_1s + p_2s = k$ and $p_2s = 0$. Hence k factors through f uniquely and f is a kernel of p. Suppose $lf = 0$ for $l : A \to C$, then $l(fp_1 + p_2) = lp_2$. Thus we have a unique morphism $q : D \to C$ such that $qp = l$ and p is a cokernel of f. □

Corollary A.10.6 *Morphisms in a preadditive exact category have cokernels.*

Proof. Let $f : A \to B$ be a morphism and $f = ip$ be an factorization of f such that $p : A \to C$ is an regular epimorphism and $i : C \to B$ is a monomorphism. By (A.10.5), the cokernel $q : B \to D$ is exists and it is also a cokernel of f. In fact, for a morphism $g : B \to E$, $gf = 0$ if and only if $gi = 0$. □

Corollary A.10.7 *A preadditive exact category is balanced.*

Proof. Let $f : B \to A$ be a monomorphism. It is a kernel of its cokernel $p : A \to C$, that is, f is an equalizer of $A \overset{p}{\underset{0}{\rightrightarrows}} C$. Then the assertion follows from (A.8.14). □

Proposition A.10.8 *Every epimorphism in a preadditive exact category is a cokernel of a certain morphism.*

Proof. Let $f : A \to B$ be an epimorphism and $R \overset{g}{\underset{h}{\rightrightarrows}} A$ the kernel pair of f. By (A.8.15) and the (A.10.7), f is a regular epimorphism. Hence f is a coequalizer of its kernel pair by (A.8.17). Then, it is obvious that f is a cokernel of $g - h : R \to A$. □

By (A.10.1), (A.10.2), (A.10.3), (A.10.5), (A.10.6) and (A.10.8), we have the following result.

Theorem A.10.9 *A preadditive exact category is an abelian category.*

A.11 Finitary algebraic theory

We denote by $\mathcal{N}$ the full subcategory of the category of finite sets with objects $\{1, 2, \ldots, n\}$ for $n \in \boldsymbol{N}$. Put $\langle n\rangle = \{1, 2, \ldots, n\}$

Definition A.11.1 *Let $\mathcal{T}$ be a category with coproducts and $\omega_s : \mathcal{N} \to \mathcal{T}$ $(s = 1, 2, \ldots, k)$ functors preserving coproducts such that a map $\mathrm{Ob}\,\mathcal{N}^k \to \mathrm{Ob}\,\mathcal{T}$ defined by*
$$(\langle n_1\rangle, \langle n_2\rangle, \ldots, \langle n_k\rangle) \mapsto \omega_1(\langle n_1\rangle) \coprod \omega_2(\langle n_2\rangle) \coprod \cdots \coprod \omega_k(\langle n_k\rangle)$$
is bijective. We call $(\mathcal{T}; \omega_1, \ldots, \omega_k)$ a k-fold finitary algebraic theory.

Set $\omega_s(\langle n\rangle) = [n]_s$ ($[n]_1 = [n]$ if $k = 1$). Then, $[n]_s$ is the n-fold coproduct of $[1]_s$ and $[0]_s$ is the unique initial object in $\mathcal{T}$. We set $[0]_s = 0$. Hence if $\mathcal{C}$ is a category with finite products (finite powers when $k = 1$) and $F : \mathcal{T}^{op} \to \mathcal{C}$ is a product preserving functor, $F\Bigl(\coprod\limits_{s=1}^{k} [n_s]_s\Bigr) = \prod\limits_{s=1}^{k} F([1]_s)^{n_s}$ for $n_1, \ldots, n_s \in \boldsymbol{N}$ and $F([0]_s)$ is a terminal object of $\mathcal{C}$.

Example A.11.2 *1) Define a functor $\bar\omega_s : \mathcal{N} \to \mathcal{N}^k$ by*

$$(\textit{the } i\textit{-th component of } \bar\omega_s(\langle n\rangle)) = \begin{cases} \langle 0\rangle & \textit{if } i \neq s \\ \langle n\rangle & \textit{if } i = s \end{cases} \qquad (\textit{the } i\textit{-th component of } \bar\omega_s(f)) = \begin{cases} id_{\langle 0\rangle} & \textit{if } i \neq s \\ f & \textit{if } i = s \end{cases}$$

for $n \in \boldsymbol{N}$ and a morphism f in $\mathcal{N}$. Then, $(\mathcal{N}^k; \bar\omega_1, \ldots, \bar\omega_k)$ is a k-fold finitary algebraic theory. We call this the trivial k-fold finitary algebraic theory. For a k-fold finitary algebraic theory $(\mathcal{T}; \omega_1, \ldots, \omega_k)$, there is a functor $T_{\mathcal{T}} : \mathcal{N}^k \to \mathcal{T}$ defined by $T_{\mathcal{T}}(\langle n_1\rangle, \ldots, \langle n_k\rangle) = \coprod\limits_{s=1}^{k} \omega_s(\langle n_s\rangle)$ and $T_{\mathcal{T}}(f_1, \ldots, f_k) = \coprod\limits_{s=1}^{k} \omega_s(f_s)$. Then, $T_{\mathcal{T}}$ preserves coproducts and $T_{\mathcal{T}}\bar\omega_s = \omega_s$.

2) Let $(\mathcal{T}_t; \omega_1^t, \ldots, \omega_{k_t}^t)$ $(t = 1, 2, \ldots, m)$ be k_t-fold finitary algebraic theories. Define $\omega_{st} : \mathcal{N} \to \prod\limits_{t=1}^{m} \mathcal{T}_t$ by

$$(\textit{the } i\textit{-th component of } \omega_{st}(\langle n\rangle)) = \begin{cases} \omega_s^i(\langle 0\rangle) & \textit{if } i \neq t \\ \omega_s^t(\langle n\rangle) & \textit{if } i = t \end{cases} \qquad (\textit{the } i\textit{-th component of } \omega_{st}(f)) = \begin{cases} \omega_s^i(id_{\langle 0\rangle}) & \textit{if } i \neq t \\ \omega_s^t(f) & \textit{if } i = t \end{cases}$$

for $n \in \boldsymbol{N}$ and a morphism f in $\mathcal{N}$. Then, $\Bigl(\prod\limits_{t=1}^{m} \mathcal{T}_t; \omega_{st}\ (1 \leq t \leq m,\ 1 \leq s \leq k_t)\Bigr)$ is a $(k_1 + \cdots + k_m)$-fold finitary algebraic theory. We call this the product of $(\mathcal{T}_t; \omega_1^t, \ldots, \omega_{k_t}^t)$ $(t = 1, 2, \ldots, m)$. The projection functor $\rho_l : \prod\limits_{t=1}^{m} \mathcal{T}_t \to \mathcal{T}_l$ preserves coproducts and $\rho_l\omega_{sl} = \omega_s^l$. We also have functors $T_l : \mathcal{T}_l \to \prod\limits_{t=1}^{m} \mathcal{T}_t$ defined by

$$(\textit{the } i\textit{-th component of } T_l(x)) = \begin{cases} 0 & \textit{if } i \neq l \\ x & \textit{if } i = l \end{cases} \qquad (\textit{the } i\textit{-th component of } T_l(f)) = \begin{cases} id_0 & \textit{if } i \neq l \\ f & \textit{if } i = l \end{cases}$$

for an object x of $\mathcal{T}_l$ and a morphism f in $\mathcal{T}_l$.

Definition A.11.3 *Let $(\mathcal{T};\omega_1,\ldots,\omega_k)$ be a k-fold finitary algebraic theory.*

1) Let $\mathcal{C}$ be a category and $(X_1,X_2,\ldots,X_k)$ an object of $\mathcal{C}^k$. A $\mathcal{T}$-structure on $(X_1,X_2,\ldots,X_k)$ is a product preserving functor $F:\mathcal{T}^{op}\to\widehat{\mathcal{C}}$ such that $F([1]_s)=h_{X_s}$ and we also call F a $\mathcal{T}$-model in $\mathcal{C}$.

2) Let F and G be $\mathcal{T}$-models in $\mathcal{C}$ such that $F([1]_s)=h_{X_s}$, $G([1]_s)=h_{Y_s}$ and $(f_1,f_2,\ldots,f_k):(X_1,X_2,\ldots,X_k)\to(Y_1,Y_2,\ldots,Y_k)$ a morphism in $\mathcal{C}^k$. If $\prod\limits_{s=1}^{k}h_{f_s}^{n_s}:F\Big(\coprod\limits_{s=1}^{k}[n_s]_s\Big)\to G\Big(\coprod\limits_{s=1}^{k}[n_s]_s\Big)$ defines a natural transformation $F\to G$, $(f_1,f_2,\ldots,f_k)$ is called a morphism $F\to G$ of $\mathcal{T}$-models. We denote by $\mathcal{T}(\mathcal{C})$ the category of $\mathcal{T}$-models and by $U_{\mathcal{T}}:\mathcal{T}(\mathcal{C})\to\mathcal{C}^k$ the forgetful functor $U_{\mathcal{T}}(F)=(X_1,X_2,\ldots,X_k)$.

3) Let $(\mathcal{T}';\omega'_1,\ldots,\omega'_l)$ be an l-fold finitary algebraic theory. If $T:\mathcal{T}'\to\mathcal{T}$ is a functor such that $T\omega'_s=\omega_{\tau(s)}$ for some map $\tau:\{1,2,\ldots,l\}\to\{1,2,\ldots,k\}$ and T preserves coproducts, we call T a morphism of finitary algebraic theories. We define a functor $T^*:\mathcal{T}(\mathcal{C})\to\mathcal{T}'(\mathcal{C})$ by $T^*(F)=FT$, $T^*(f_1,f_2,\ldots,f_k)=(f_{\tau(1)},f_{\tau(2)},\ldots,f_{\tau(l)})$.

Definition A.11.4 *Let $(\mathcal{T};\omega_1,\ldots,\omega_k)$, $(\mathcal{T}_0;\bar\omega_1,\ldots,\bar\omega_{k_0})$ be finitary algebraic theories and $T_0:\mathcal{T}_0\to\mathcal{T}$ a morphism of finitary algebraic theories such that $T_0\bar\omega_s=\omega_{\sigma(s)}$ for some map $\sigma:\{1,2,\ldots,k_0\}\to\{1,2,\ldots,k\}$.*

1) Let $\mathcal{C}$ be a category and $\mathcal{A}$ a subcategory of $\mathcal{T}_0(\mathcal{C})$. We define a subcategory $\mathcal{T}(\mathcal{C};T_0,\mathcal{A})$ of $\mathcal{C}$ by

$$\mathrm{Ob}\,\mathcal{T}(\mathcal{C};T_0,\mathcal{A})=\{F\in\mathrm{Ob}\,\mathcal{T}(\mathcal{C})|\,T_0^*(F)\in\mathrm{Ob}\,\mathcal{A}\},\quad \mathrm{Mor}\,\mathcal{T}(\mathcal{C};T_0,\mathcal{A})=\{f\in\mathrm{Mor}\,\mathcal{T}(\mathcal{C})|\,T_0^*(f)\in\mathrm{Mor}\,\mathcal{A}\}.$$

In the case $\mathrm{Ob}\,\mathcal{A}=\{F_0\}$, $\mathrm{Mor}\,\mathcal{A}=\{id_{F_0}\}$, we denote $\mathcal{T}(\mathcal{C};T_0,\mathcal{A})$ by $\mathcal{T}(\mathcal{C};T_0,F_0)$.

2) Let $\{\bar\sigma_1,\bar\sigma_2,\ldots,\bar\sigma_{k-m}\}$ ($\bar\sigma_1<\bar\sigma_2<\cdots<\bar\sigma_{k-m}$) be the complement of the image of σ and $P:\mathcal{C}^k\to\mathcal{C}^{k-m}$ the projection functor onto $\bar\sigma_1,\bar\sigma_2,\ldots,\bar\sigma_{k-m}$ components. We define the forgetful functor $\widetilde{U}_{\mathcal{T}}:\mathcal{T}(\mathcal{C};T_0,\mathcal{A})\to\mathcal{C}^{k-m}$ by $\widetilde{U}_{\mathcal{T}}=PU_{\mathcal{T}}$.

3) Let $(\mathcal{T}';\omega'_1,\ldots,\omega'_l)$, $(\mathcal{T}'_0;\bar\omega'_1,\ldots,\bar\omega'_{l_0})$ be finitary algebraic theories and $T'_0:\mathcal{T}'_0\to\mathcal{T}'$, $T:\mathcal{T}'\to\mathcal{T}$, $\overline{T}:\mathcal{T}'_0\to\mathcal{T}_0$ morphisms of finitary algebraic theories such that $T'_0\bar\omega'_s=\omega'_{\sigma'(s)}$, $T\omega'_s=\omega_{\tau(s)}$, $\overline{T}\bar\omega'_s=\bar\omega_{\tau_0(s)}$ for each s and that $T_0\overline{T}=TT'_0$. If $\mathcal{A}$ and $\mathcal{A}'$ are subcategories of $\mathcal{T}_0(\mathcal{C})$ and $\mathcal{T}'_0(\mathcal{C})$ respectively and $\overline{T}^*:\mathcal{T}_0(\mathcal{C})\to\mathcal{T}'_0(\mathcal{C})$ maps $\mathcal{A}$ into $\mathcal{A}'$, $T^*:\mathcal{T}(\mathcal{C})\to\mathcal{T}'(\mathcal{C})$ maps $\mathcal{T}(\mathcal{C};T_0,\mathcal{A})$ into $\mathcal{T}'(\mathcal{C};T'_0,\mathcal{A}')$. We also denote by T^* the functor from $\mathcal{T}(\mathcal{C};T_0,\mathcal{A})$ to $\mathcal{T}'(\mathcal{C};T'_0,\mathcal{A}')$ induced by T^*.

Remark A.11.5 *Suppose that $\mathcal{T}(\mathcal{C};T_0,F_0)$ is not empty. Put $m=\mathrm{card}(\mathrm{Im}\,\sigma)$. If F is an object of $\mathcal{T}(\mathcal{C};T_0,F_0)$, $F([1]_{\sigma(s)})=F_0([1]_s)$ for each $1\le s\le k_0$. Hence $\sigma(s)=\sigma(t)$ implies $F_0([1]_s)=F_0([1]_t)$ and there exist m objects $Z_1,Z_2,\ldots,Z_m$ of $\mathcal{C}$, a surjection $\tilde\sigma:\{1,2,\ldots,k_0\}\to\{1,2,\ldots,m\}$ and an injection $\hat\sigma:\{1,2,\ldots,m\}\to\{1,2,\ldots,k\}$ such that $\sigma=\hat\sigma\tilde\sigma$ and $F_0([1]_s)=h_{Z_{\tilde\sigma(s)}}$. Thus for any $F\in\mathrm{Ob}\,\mathcal{T}(\mathcal{C};T_0,F_0)$, $F([1]_{\hat\sigma(s)})=h_{Z_s}$ ($s=1,2,\ldots,m$). If σ is injective, we can assume that $\tilde\sigma$ is the identity map.*

Definition A.11.6 *Suppose that the functor $T_0:\mathcal{T}_0\to\mathcal{T}$ in (A.11.4) satisfies the following conditions.*

(1) $\sigma:\{1,2,\ldots,k_0\}\to\{1,2,\ldots,k\}$ is injective.

(2) For any $\coprod\limits_{s=1}^{k_0}[n'_s]_s\in\mathrm{Ob}\,\mathcal{T}_0$ and $\coprod\limits_{s=1}^{k}[n_s]_s\in\mathrm{Ob}\,\mathcal{T}$, the following composition is bijective, where $\nu:\coprod\limits_{s=1}^{k_0}[n_{\sigma(s)}]_{\sigma(s)}\to\coprod\limits_{s=1}^{k}[n_s]_s$ is the canonical morphism in $\mathcal{T}$.

$$\mathcal{T}_0\Big(\coprod_{s=1}^{k_0}[n'_s]_s,\coprod_{s=1}^{k_0}[n_{\sigma(s)}]_s\Big)\xrightarrow{T_0}\mathcal{T}\Big(\coprod_{s=1}^{k_0}[n'_s]_{\sigma(s)},\coprod_{s=1}^{k_0}[n_{\sigma(s)}]_{\sigma(s)}\Big)\xrightarrow{\nu_*}\mathcal{T}\Big(\coprod_{s=1}^{k_0}[n'_s]_{\sigma(s)},\coprod_{s=1}^{k}[n_s]_s\Big)$$

For an object F_0 of $\mathcal{T}_0(\mathcal{C})$, we call an object of $\mathcal{T}(\mathcal{C};T_0,F_0)$ an F_0-module and $\mathcal{T}(\mathcal{C};T_0,F_0)$ the category of F_0-modules.

Proposition A.11.7 *Let $\mathcal{C}$ be a category.*

1) Let $(\mathcal{T}_0;\bar\omega_1,\ldots,\bar\omega_{k_0})$ be a finitary algebraic theory and $T_0:\mathcal{T}_0\to\mathcal{N}^k$ a morphism of finitary algebraic theories to a trivial finitary algebraic theory such that $T_0\bar\omega_s=\omega_{\sigma(s)}$ ($1\le s\le k_0$) for an injection $\sigma:\{1,2,\ldots,k_0\}\to\{1,2,\ldots,k\}$. For any object F_0 of $\mathcal{T}_0(\mathcal{C})$, the forgetful functor $\widetilde{U}_{\mathcal{N}^k}:\mathcal{N}^k(\mathcal{C};T_0,F_0)\to\mathcal{C}^{k-k_0}$ is an isomorphism of categories.

2) In the situation of (A.11.4), let $\{\bar\sigma'_1,\bar\sigma'_2,\ldots,\bar\sigma'_{l-d}\}$ ($\bar\sigma'_1<\bar\sigma'_2<\cdots<\bar\sigma'_{l-d}$) be the compliment of the image of σ'. Suppose that $\mathrm{Im}\,\sigma'=\tau^{-1}(\mathrm{Im}\,\sigma)$. Then, there exists a unique $1\le\beta(s)\le k-m$ such that $\tau(\bar\sigma'_s)=\bar\sigma_{\beta(s)}$

for each $1 \le s \le l-d$. Let us denote by $\Pi : \mathcal{C}^{k-m} \to \mathcal{C}^{l-d}$ the functor given by $\Pi(X_1, X_2, \dots, X_{k-m}) = (X_{\beta(1)}, X_{\beta(2)}, \dots, X_{\beta(l-d)})$ and $\Pi(f_1, f_2, \dots, f_{k-m}) = (f_{\beta(1)}, f_{\beta(2)}, \dots, f_{\beta(l-d)})$. Then, we have $\widetilde{U}_{\mathcal{T}'}T^* = \Pi\widetilde{U}_{\mathcal{T}} : \mathcal{T}(\mathcal{C}; T_0, \mathcal{A}) \to \mathcal{C}^{l-d}$.

3) Let F_0 be an object of $\mathcal{T}_0(\mathcal{C})$. Then the inclusion functor $\mathcal{T}(\mathcal{C}; T_0, F_0) \to \mathcal{T}(\mathcal{C})$ reflects limits and colimits. The forgetful functor $\widetilde{U}_{\mathcal{T}} : \mathcal{T}(\mathcal{C}; T_0, F_0) \to \mathcal{C}^{k-m}$ is faithful and reflects isomorphisms.

Proof. 1) Suppose that $F_0([1]_s) = h_{Z_s}$ for $1 \le s \le k_0$. The inverse $\widetilde{U}^{-1}_{\mathcal{N}^k} : \mathcal{C}^{k-k_0} \to \mathcal{N}^k(\mathcal{C}; T_0, F_0)$ is defined as follows. For $(X_1, X_2, \dots, X_{k-k_0}) \in \operatorname{Ob}\mathcal{C}^{k-k_0}$ and $n_s \in \boldsymbol{N}$ $(s = 1, 2, \dots, k)$, we set

$$(\widetilde{U}^{-1}_{\mathcal{N}^k}(X_1, X_2, \dots, X_{k-k_0}))(\langle n_1\rangle, \langle n_2\rangle, \dots, \langle n_k\rangle) = \prod_{s=1}^{k_0} h_{Z_s}^{n_{\sigma(s)}} \times \prod_{s=1}^{k-k_0} h_{X_s}^{n_{\bar\sigma_s}}.$$

If $\varphi_s : \langle m_s\rangle \to \langle n_s\rangle$ $(s = 1, 2, \dots, k)$ are morphisms in $\mathcal{N}$, for $Y \in \operatorname{Ob}\mathcal{C}$ and $(x_{\sigma(s)1}, x_{\sigma(s)2}, \dots, x_{\sigma(s)n_{\sigma(s)}}) \in h_{Z_s}^{n_{\sigma(s)}}(Y)$ $(1 \le s \le k_0)$, $(x_{\bar\sigma_s 1}, x_{\bar\sigma_s 2}, \dots, x_{\bar\sigma_s n_{\sigma(s)}}) \in h_{X_s}^{n_{\bar\sigma_s}}(Y)$ $(1 \le s \le k - k_0)$, set

$$\begin{aligned}(\widetilde{U}^{-1}_{\mathcal{N}^k}&(X_1, X_2, \dots, X_{k-k_0}))(\varphi_1, \varphi_2, \dots, \varphi_k)_Y(((x_{s1}, x_{s2}, \dots, x_{sn_s}))_{s=1,2,\dots,k}) \\ &= ((x_{s\varphi_s(1)}, x_{s\varphi_s(2)}, \dots, x_{s\varphi_s(m_s)}))_{s=1,2,\dots,k}.\end{aligned}$$

For a morphism $(f_1, f_2, \dots, f_{k-k_0}) : (X_1, X_2, \dots, X_{k-k_0}) \to (Y_1, Y_2, \dots, Y_{k-k_0})$ in $\mathcal{C}^{k-k_0}$, the following diagram commutes.

$$\begin{array}{ccc}
\prod_{s=1}^{k_0} h_{Z_s}^{n_{\sigma(s)}} \times \prod_{s=1}^{k-k_0} h_{X_s}^{n_{\bar\sigma_s}} & \xrightarrow{(\widetilde{U}^{-1}_{\mathcal{N}^k}(X_1,\dots,X_{k-k_0}))(\varphi_1,\dots,\varphi_k)} & \prod_{s=1}^{k_0} h_{Z_s}^{m_{\sigma(s)}} \times \prod_{s=1}^{k-k_0} h_{X_s}^{m_{\bar\sigma_s}} \\
\Big\downarrow {\scriptstyle \prod_{s=1}^{k_0} h_{id_{Z_s}}^{n_{\sigma(s)}} \times \prod_{s=1}^{k-k_0} h_{f_s}^{n_{\bar\sigma_s}}} & & \Big\downarrow {\scriptstyle \prod_{s=1}^{k_0} h_{id_{Z_s}}^{m_{\sigma(s)}} \times \prod_{s=1}^{k-k_0} h_{f_s}^{m_{\bar\sigma_s}}} \\
\prod_{s=1}^{k_0} h_{Z_s}^{n_{\sigma(s)}} \times \prod_{s=1}^{k-k_0} h_{X_s}^{n_{\bar\sigma_s}} & \xrightarrow{(\widetilde{U}^{-1}_{\mathcal{N}^k}(Y_1,\dots,Y_{k-k_0}))(\varphi_1,\dots,\varphi_k)} & \prod_{s=1}^{k_0} h_{Z_s}^{m_{\sigma(s)}} \times \prod_{s=1}^{k-k_0} h_{X_s}^{m_{\bar\sigma_s}}
\end{array}$$

Hence $(\tilde f_1, \tilde f_2, \dots, \tilde f_k)$ $(\tilde f_{\sigma(s)} = id_{Z_s}, \tilde f_{\bar\sigma_s} = f_s)$ is a morphism in $\mathcal{N}^k(\mathcal{C}; T_0, F_0)$ and set $\widetilde{U}^{-1}_{\mathcal{N}^k}(f_1, f_2, \dots, f_{k-k_0}) = (\tilde f_1, \tilde f_2, \dots, \tilde f_k)$.

2) The assumption implies that $\tau(\{\bar\sigma'_1, \bar\sigma'_2, \dots, \bar\sigma'_{l-d}\}) \subset \{\bar\sigma_1, \bar\sigma_2, \dots, \bar\sigma_{k-m}\}$. Hence there exists a unique $1 \le \beta(s) \le k-m$ such that $\tau(\bar\sigma'_s) = \bar\sigma_{\beta(s)}$ for each $1 \le s \le l-d$. If F is an object of $\mathcal{T}(\mathcal{C}; T_0, \mathcal{A})$ such that $F([1]_{\bar\sigma_s}) = h_{X_s}$, then $FT([1]_{\bar\sigma'_s}) = FT(\omega'_{\bar\sigma'_s}(\langle 1\rangle)) = F\omega_{\tau(\bar\sigma'_s)}(\langle 1\rangle) = F([1]_{\bar\sigma_{\beta(s)}}) = h_{X_{\beta(s)}}$, thus $\widetilde{U}_{\mathcal{T}'}T^*(F) = \widetilde{U}_{\mathcal{T}'}(FT) = \Pi\widetilde{U}_{\mathcal{T}}(F)$. For a morphism $(f_1, \dots, f_k)$ in $\mathcal{T}(\mathcal{C}; T_0, \mathcal{A})$, we have $\widetilde{U}_{\mathcal{T}'}T^*(f_1, \dots, f_k) = \widetilde{U}_{\mathcal{T}'}(f_{\tau(1)}, \dots, f_{\tau(k)}) = (f_{\bar\sigma_{\beta(1)}}, \dots, f_{\bar\sigma_{\beta(k)}}) = \Pi(f_{\bar\sigma_1}, \dots, f_{\bar\sigma_k}) = \Pi\widetilde{U}_{\mathcal{T}}(f_1, \dots, f_k)$.

3) The assertions are obvious. $\square$

Note that (A.11.2) and (A.11.7) imply that the forgetful functor $\widetilde{U}_{\mathcal{T}}$ is regarded as a special case of a functor T^* in (A.11.4).

Definition A.11.8 *Suppose that a morphism of finitary algebraic theories $T_0 : \mathcal{T}_0 \to \mathcal{T}$ satisfies the conditions of (A.11.6). For an object $n = \coprod_{s=1}^{k} [n_s]_s$ of $\mathcal{T}$, we set $n_\sigma = \coprod_{s=1}^{k_0} [n_{\sigma(s)}]_s \in \operatorname{Ob}\mathcal{T}_0$ and $n_{\bar\sigma} = \coprod_{s=1}^{k-k_0} [n_{\bar\sigma_s}]_{\bar\sigma_s} \in \operatorname{Ob}\mathcal{T}$. Then $n = T_0(n_\sigma) \coprod n_{\bar\sigma}$ in $\mathcal{T}$.*

1) For a morphism $\varphi : G_0 \to F_0$ in $\mathcal{T}_0(\mathcal{C})$, we construct a functor $\varphi^\sharp : \mathcal{T}(\mathcal{C}; T_0, F_0) \to \mathcal{T}(\mathcal{C}; T_0, G_0)$ as follows. Set $\varphi^\sharp(F)(n) = G_0(n_\sigma) \times F(n_{\bar\sigma})$ for $n = \coprod_{s=1}^{k} [n_s]_s \in \operatorname{Ob}\mathcal{T}$. Let $\theta : n' \to n$ be a morphism in $\mathcal{T}$ and $\nu'_1 : n'_\sigma \to n'$, $\nu'_2 : n'_{\bar\sigma} \to n'$, $\nu_1 : T_0(n_\sigma) \to n$ the canonical morphisms. By the assumption, there is a unique morphism $\theta' : n'_\sigma \to n_\sigma$ in $\mathcal{T}_0$ such that $\nu_1 T_0(\theta') = \theta\nu'_1$. We define $\varphi^\sharp(F)(\theta\nu'_1)$ and $\varphi^\sharp(F)(\theta\nu'_2)$ to be the following compositions.

$$G_0(n_\sigma) \times F(n_{\bar\sigma}) \xrightarrow{\mathrm{pr}_1} G_0(n_\sigma) \xrightarrow{G_0(\theta')} G_0(n'_\sigma)$$

$$G_0(n_\sigma) \times F(n_{\bar\sigma}) \xrightarrow{\varphi\times id"} F_0(n_\sigma) \times F(n_{\bar\sigma}) = F(n) \xrightarrow{F(\theta\nu'_2)} F(n'_{\bar\sigma})$$

We set $\varphi^\sharp(F)(\theta) = (\varphi^\sharp(F)(\theta\nu'_1), \varphi^\sharp(F)(\theta\nu'_2))$. It is easy to verify that $\varphi^\sharp(F)$ is an object of $\mathcal{T}(\mathcal{C}; T_0, G_0)$. If $f : F \to G$ is a morphism in $\mathcal{T}(\mathcal{C}; T_0, F_0)$, define $\varphi^\sharp(f) : \varphi^\sharp(F) \to \varphi^\sharp(G)$ by $\varphi^\sharp(f)_n = id_{G_0(n_\sigma)} \times f_{n_{\bar\sigma}}$.

2) We note that $\varphi \times id : G_0(n_\sigma) \times F(n_{\bar{\sigma}}) \to F_0(n_\sigma) \times F(n_{\bar{\sigma}})$ defines a morphism $\tilde{\varphi}_F : \varphi^\sharp(F) \to F$ in $\mathcal{T}(\mathcal{C})$.

3) If G is an object of $\mathcal{T}(\mathcal{C}; T_0, G_0)$ and $f : G \to F$ is a morphism in $\mathcal{T}(\mathcal{C})$ such that $T_0^*(f) = \varphi$, then $id \times f : G(n) = G_0(n_\sigma) \times G(n_{\bar{\sigma}}) \to G_0(n_\sigma) \times F(n_{\bar{\sigma}})$ defines a morphism $f_\sharp : G \to \varphi^\sharp(F)$ in $\mathcal{T}(\mathcal{C}; T_0, G_0)$.

The following result is easily verified from the above construction.

Proposition A.11.9 *We use the same notations as in (A.11.4) and suppose that morphisms of finitary algebraic theories $T_0 : \mathcal{T}_0 \to \mathcal{T}$, $T_0' : \mathcal{T}_0' \to \mathcal{T}'$ satisfy the conditions of (A.11.6).*

1) For a morphism $\varphi : G_0 \to F_0$ in $\mathcal{T}_0(\mathcal{C})$, the following diagram commutes if $\operatorname{Im}\sigma' = \tau^{-1}(\operatorname{Im}\sigma)$.

$$\begin{tikzcd}
\mathcal{T}(\mathcal{C}; T_0, F_0) \arrow[r, "\varphi^\sharp"] \arrow[d, "T^*"] & \mathcal{T}(\mathcal{C}; T_0, G_0) \arrow[d, "T^*"] \\
\mathcal{T}'(\mathcal{C}; T_0', \overline{T}^*(F_0)) \arrow[r, "(\overline{T}^*(\varphi))^\sharp"] & \mathcal{T}'(\mathcal{C}; T_0', \overline{T}^*(G_0))
\end{tikzcd}$$

Moreover, for each object F of $\mathcal{T}(\mathcal{C}; T_0, F_0)$,

$$\widetilde{\overline{T}^*(\varphi)}_{T^*(F)} = T^*(\tilde{\varphi}_F) : (\overline{T}^*(\varphi))^\sharp(T^*(F)) = T^*(\varphi^\sharp(F)) \to T^*(F)$$

2) $T_0^*(\tilde{\varphi}_F) = \varphi$ holds. Moreover, if $\psi : H_0 \to G_0$ is a morphism in $\mathcal{T}_0(\mathcal{C})$, then $(\varphi\psi)^\sharp = \psi^\sharp\varphi^\sharp$ and $(\widetilde{\varphi\psi})_F = \tilde{\varphi}_F \tilde{\psi}_{\varphi^\sharp(F)}$ hold.

3) If $f : G \to F$ is a morphism in $\mathcal{T}(\mathcal{C}; T_0, F_0)$, then the following diagram commutes.

$$\begin{tikzcd}
\varphi^\sharp(G) \arrow[r, "\tilde{\varphi}_G"] \arrow[d, "\varphi^\sharp(f)"] & G \arrow[d, "f"] \\
\varphi^\sharp(F) \arrow[r, "\tilde{\varphi}_F"] & F
\end{tikzcd}$$

4) Let G and H be objects of $\mathcal{T}(\mathcal{C}; T_0, G_0)$ and $\mathcal{T}(\mathcal{C}; T_0, H_0)$, respectively. If $f : G \to F$ and $g : H \to G$ are morphisms in $\mathcal{T}(\mathcal{C})$ such that $T_0^*(f) = \varphi$ and $T_0^*(g) = \psi$, then we have $\tilde{\varphi}_F f_\sharp = f$ and $(fg)_\sharp = \psi^\sharp(f_\sharp)g_\sharp$.

5) Let $f : F \to F'$ and $g : G \to G'$ be morphisms of $\mathcal{T}(\mathcal{C}; T_0, F_0)$ and $\mathcal{T}(\mathcal{C}; T_0, G_0)$, respectively. If $u : G \to F$ and $v : G' \to F'$ are morphisms in $\mathcal{T}(\mathcal{C})$ such that $T_0^*(u) = T_0^*(v) = \varphi$ and $fu = vg$, then the following diagram commutes.

$$\begin{tikzcd}
G \arrow[r, "u_\sharp"] \arrow[d, "g"] & \varphi^\sharp(F) \arrow[d, "\varphi^\sharp(f)"] \\
G' \arrow[r, "v_\sharp"] & \varphi^\sharp(F')
\end{tikzcd}$$

Remark A.11.10 *If $\mathcal{C}$ is a category with finite products (finite powers when $k = 1$), a product preserving functor $F : \mathcal{T}^{op} \to \widehat{\mathcal{C}}$ satisfying $F([1]_s) = h_{X_s}$ for some $X_s \in \operatorname{Ob}\mathcal{C}$ $(s = 1, 2, \ldots, k)$ uniquely factors through the embedding $h : \mathcal{C} \to \widehat{\mathcal{C}}$. In this case, $\mathcal{T}(\mathcal{C})$ is regarded as the category of product preserving functors $\mathcal{T}^{op} \to \mathcal{C}$ and natural transformation between them. In particular, since $\widehat{\mathcal{C}}$ is complete, $\mathcal{T}(\widehat{\mathcal{C}})$ is the category of product preserving functors $\mathcal{T}^{op} \to \widehat{\mathcal{C}}$. Thus the embedding $h : \mathcal{C} \to \widehat{\mathcal{C}}$ defines a fully faithful functor $h_{\mathcal{T}} : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\widehat{\mathcal{C}})$ and $\mathcal{T}(\mathcal{C})$ is regarded as a full subcategory of $\mathcal{T}(\widehat{\mathcal{C}})$. More generally, let $\mathcal{D}$ be a subcategory of $\widehat{\mathcal{C}}$ such that the inclusion functor $i : \mathcal{D} \to \widehat{\mathcal{C}}$ creates finite products. Suppose that $h : \mathcal{C} \to \widehat{\mathcal{C}}$ factors through $\mathcal{D}$ and let $j : \mathcal{C} \to \mathcal{D}$ be the functor such that $h = ij$. Then, $\mathcal{T}(\mathcal{C})$ is regarded as a full subcategory of $\mathcal{T}(\mathcal{D})$. In fact, j induces a fully faithful functor $j_{\mathcal{T}} : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\mathcal{D})$ which maps $\operatorname{Ob}\mathcal{T}(\mathcal{C})$ onto $\{F \in \operatorname{Ob}\mathcal{T}(\mathcal{D}) \mid F([1]_s) = h_{X_s}$ for some $X_s \in \operatorname{Ob}\mathcal{C}$ for $s = 1, 2, \ldots, k\}$. Moreover, the following diagram commutes.*

$$\begin{tikzcd}
\mathcal{T}(\mathcal{C}) \arrow[r, "j_{\mathcal{T}}"] \arrow[d, "T_0^*"] & \mathcal{T}(\mathcal{D}) \arrow[d, "T_0^*"] \\
\mathcal{T}_0(\mathcal{C}) \arrow[r, "j_{\mathcal{T}}"] & \mathcal{T}_0(\mathcal{D})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T}(\mathcal{C}) \arrow[r, "j_{\mathcal{T}}"] \arrow[d, "U_{\mathcal{T}}"] & \mathcal{T}(\mathcal{D}) \arrow[d, "U_{\mathcal{T}}"] \\
\mathcal{C}^k \arrow[r, "j^k"] & \mathcal{D}^k
\end{tikzcd}$$

Hence if F_0 is an object of $\mathcal{T}_0(\mathcal{C})$, $j_{\mathcal{T}}$ restricts a fully faithful functor $j_{\mathcal{T}} : \mathcal{T}(\mathcal{C}; T_0, F_0) \to \mathcal{T}(\mathcal{D}; T_0, j_{\mathcal{T}}(F_0))$ which maps $\mathrm{Ob}\,\mathcal{T}(\mathcal{C}; T_0, \mathcal{C})$ onto $\{F \in \mathrm{Ob}\,\mathcal{T}(\mathcal{D}; T_0, j_{\mathcal{T}}(F_0)) \,|\, F([1]_s) = h_{X_s}$ for some $X_s \in \mathrm{Ob}\,\mathcal{C}$ for $s = 1, 2, \ldots, k\}$.

We also remark that if there is a morphism $[1]_s \to [0]_s$ in $\mathcal{T}$ for some s and $\mathcal{C}$ is a regular category such that $\mathcal{T}(\mathcal{C})$ is a nonempty category, it follows from (A.8.21) and (A.8.22) that $\mathcal{C}$ has a terminal object and finite powers.

Let $\mathcal{C}$ be a category and $\mathcal{D}$ a category with finite products (resp. powers). Then the functor category $\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})$ has finite products (resp. powers).

Proposition A.11.11 *Let $(\mathcal{T}; \omega_1, \ldots, \omega_k)$ be a k-fold finitary algebraic theory, $\mathcal{C}$ a category and $\mathcal{D}$ a category with finite products (resp. powers if $k = 1$). There is an isomorphism of categories $\Phi : \mathcal{T}(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})) \to \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}))$. Moreover, for a morphism $T : \mathcal{T}' \to \mathcal{T}$ of finitary algebraic theories, the following diagrams commute.*

$$\begin{array}{ccc} \mathcal{T}(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})) & \xrightarrow{\Phi} & \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D})) \\ \big\downarrow{\scriptstyle T^*} & & \big\downarrow{\scriptstyle \mathrm{Funct}(id_{\mathcal{C}^{op}}, T^*)} \\ \mathcal{T}'(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})) & \xrightarrow{\Phi} & \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}'(\mathcal{D})) \end{array} \qquad \begin{array}{ccc} \mathcal{T}(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})) & \xrightarrow{\Phi} & \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D})) \\ \big\downarrow{\scriptstyle U_{\mathcal{T}}} & & \big\downarrow{\scriptstyle \mathrm{Funct}(id_{\mathcal{C}^{op}}, U_{\mathcal{T}})} \\ \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})^k & \xleftarrow{\rho} & \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}^k) \end{array}$$

Here ρ denotes the canonical isomorphism. In particular, if $\mathcal{D}$ is the category of $\mathcal{U}$-sets, we have an isomorphism of categories $\mathcal{T}(\widehat{\mathcal{C}}_{\mathcal{U}}) \cong \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}))$.

Proof. For an object X of $\mathcal{C}$, we denote by $E_X : \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}) \to \mathcal{D}$ the evaluation functor at X. Let $F : \mathcal{T}^{op} \to \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})$ be a product preserving functor. We define $\Phi(F) : \mathcal{C}^{op} \to \mathcal{T}(\mathcal{D})$ by $\Phi(F)(X) = E_X F$ $(X \in \mathrm{Ob}\,\mathcal{C})$ and $\Phi(F)(f) = (F([1]_1)(f), F([1]_2)(f), \ldots, F([1]_k)(f))$ $(f \in \mathrm{Mor}\,\mathcal{C})$. For a morphism $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) : F \to G$ in $\mathcal{T}(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}))$, define $\Phi(\alpha) : \Phi(F) \to \Phi(G)$ by $\Phi(\alpha)_X = (\alpha_{1X}, \alpha_{2X}, \ldots, \alpha_{kX})$. The inverse $\Phi^{-1} : \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D})) \to \mathcal{T}(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}))$ is defined as follows. For a functor $K : \mathcal{C}^{op} \to \mathcal{T}(\mathcal{D})$, define $\Phi^{-1}(K) : \mathcal{T}^{op} \to \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})$ by $(\Phi^{-1}(K)(A))(X) = K(X)(A)$ $(A \in \mathrm{Ob}\,\mathcal{T},\ X \in \mathrm{Ob}\,\mathcal{C})$, $(\Phi^{-1}(K)(A))(f) = K(f)_A$ $(f \in \mathrm{Mor}\,\mathcal{C})$ and $\Phi^{-1}(K)(\varphi)_X = K(X)(\varphi)$ $(\varphi \in \mathrm{Mor}\,\mathcal{T})$. For a morphism $\psi : K \to L$ in $\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}))$, set $\psi_X = (\psi_{X1}, \psi_{X2}, \ldots, \psi_{Xk})$ and define $\Phi^{-1}(\psi) = (\psi_1, \psi_2, \ldots, \psi_k) : \Phi^{-1}(K) \to \Phi^{-1}(L)$ by $(\psi_s)_X = \psi_{Xs}$. □

Let $T_0 : \mathcal{T}_0 \to \mathcal{T}$ be a morphism of finitary algebraic theories and F_0 an object of $\mathcal{T}_0(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}))$. Define a subcategory $\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}); T_0, F_0)$ of $\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}))$ by

$$\mathrm{Ob}\,\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}); T_0, F_0) = \{F : \mathcal{C}^{op} \to \mathcal{T}(\mathcal{D}) \,|\, T_0^* F = \Phi_0(F_0)\},$$

where $\Phi_0 : \mathcal{T}_0(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})) \to \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}_0(\mathcal{D}))$ is the isomorphism in (A.11.11), and

$$\mathrm{Mor}\,\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}); T_0, F_0) = \{(\theta : F \to G) \in \mathrm{Mor}\,\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D})) \,|\, T_0^*(\theta) = id_{\Phi_0(F_0)}\}.$$

By the construction of Φ, we can verify the following fact.

Proposition A.11.12 *$\Phi : \mathcal{T}(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})) \to \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}))$ gives an isomorphism $\mathcal{T}(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}); T_0, F_0) \to \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}); T_0, F_0)$. Moreover, for morphisms of finitary algebraic theories $T_0' : \mathcal{T}_0' \to \mathcal{T}'$, $T : \mathcal{T}' \to \mathcal{T}$ and $\overline{T} : \mathcal{T}_0' \to \mathcal{T}_0$ satisfying $T_0 \overline{T} = T T_0'$, the following diagrams commute.*

$$\begin{array}{ccc} \mathcal{T}(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}); T_0, F_0) & \xrightarrow{\Phi} & \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}); T_0, F_0) \\ \big\downarrow{\scriptstyle T^*} & & \big\downarrow{\scriptstyle \mathrm{Funct}(id_{\mathcal{C}^{op}}, T^*)} \\ \mathcal{T}'(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}); T_0', \overline{T}^*(F_0)) & \xrightarrow{\Phi} & \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}'(\mathcal{D}); T_0', \overline{T}^*(F_0)) \end{array}$$

$$\begin{array}{ccc} \mathcal{T}(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}); T_0, F_0) & \xrightarrow{\Phi} & \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}); T_0, F_0) \\ \big\downarrow{\scriptstyle \widetilde{U}_{\mathcal{T}}} & & \big\downarrow{\scriptstyle \mathrm{Funct}(id_{\mathcal{C}^{op}}, \widetilde{U}_{\mathcal{T}})} \\ \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D})^{k-m} & \xleftarrow{\rho} & \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}^{k-m}) \end{array}$$

Here ρ denotes the canonical isomorphism. In particular, if $\mathcal{D}$ is the category of $\mathcal{U}$-sets, we have an isomorphism of categories $\mathcal{T}(\widehat{\mathcal{C}}_{\mathcal{U}}; T_0, F_0) \cong \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}); T_0, F_0)$.

Lemma A.11.13 *With the same notations as above, suppose that both T_0 and T_0' satisfy the conditions of (A.11.6) and that $\mathrm{Im}\,\sigma' = \tau^{-1}(\mathrm{Im}\,\sigma)$ holds. If*

$$T^* : \mathcal{T}(\mathcal{D}; T_0, E_X F_0) \to \mathcal{T}'(\mathcal{D}; T_0', \overline{T}^*(E_X F_0))$$

has a left adjoint for any object X of $\mathcal{C}$,

$$T^* : \mathcal{T}(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}); T_0, F_0) \to \mathcal{T}'(\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{D}); T_0', \overline{T}^*(F_0))$$

also has a left adjoint.

Proof. By (A.11.12), it suffices to show that

$$\mathrm{Funct}(id_{\mathcal{C}^{op}}, T^*) : \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}); T_0, F_0) \to \mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}'(\mathcal{D}); T_0', \overline{T}^*(F_0))$$

has a left adjoint. Let $L_X : \mathcal{T}'(\mathcal{D}; T_0', \overline{T}^*(E_X F_0)) \to \mathcal{T}(\mathcal{D}; T_0, E_X F_0)$ be a left adjoint of $T^* : \mathcal{T}(\mathcal{D}; T_0, E_X F_0) \to \mathcal{T}'(\mathcal{D}; T_0', \overline{T}^*(E_X F_0))$ and $F : \mathcal{C}^{op} \to \mathcal{T}'(\mathcal{D})$ an object of $\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}'(\mathcal{D}); T_0', \overline{T}^*(F_0))$. Since $T_0'^*(F(X)) = \Phi_0(\overline{T}^*(F_0))(X) = E_X \overline{T}^*(F_0) = \overline{T}^*(E_X F_0)$ for each object X of $\mathcal{C}$, $F(X)$ is an object of $\mathcal{T}'(\mathcal{D}; T_0', \overline{T}^*(E_X F_0))$. We set $L(F)(X) = L_X(F(X))$, then $T_0^* L(F)(X) = T_0^* L_X(F(X)) = E_X F_0 = \Phi_0(F_0)(X)$.

For a morphism $f : X \to Y$ in $\mathcal{C}$, we put $\Phi_0(F_0)(f) = E_f : E_Y F_0 \to E_X F_0$, which is a morphism in $\mathcal{T}_0(\mathcal{D})$. Hence $\overline{T}^*(E_f) : \overline{T}^*(E_Y F_0) \to \overline{T}^*(E_X F_0)$ is a morphism in $\mathcal{T}_0'(\mathcal{D})$ and it follows from (A.11.9) that the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{T}(\mathcal{D}; T_0, E_X F_0) & \xrightarrow{(E_f)^\sharp} & \mathcal{T}(\mathcal{D}; T_0, E_Y F_0) \\
\Big\downarrow{\scriptstyle T^*} & & \Big\downarrow{\scriptstyle T^*} \\
\mathcal{T}'(\mathcal{D}; T_0', \overline{T}^*(E_X F_0)) & \xrightarrow{(\overline{T}^*(E_f))^\sharp} & \mathcal{T}'(\mathcal{D}; T_0', \overline{T}^*(E_Y F_0))
\end{array}$$

We note that $F(f) : F(Y) \to F(X)$ is a morphism in $\mathcal{T}'(\mathcal{D})$ such that $T_0'^*(F(f)) = \overline{T}^*(E_f)$. Hence, by (A.11.8), we have a morphism $F(f)_\sharp : F(Y) \to (\overline{T}^*(E_f))^\sharp(F(X))$ in $\mathcal{T}'(\mathcal{D}; T_0', \overline{T}^*(E_Y F_0))$. We denote by $\eta_X : id \to T^* L_X$ and $\varepsilon_X : L_X T^* \to id$ the unit and the counit of the adjunction, respectively. Let us define $L(F)(f) : L_Y(F(Y)) \to L_X(F(X))$ to be the composition

$$L_Y(F(Y)) \longrightarrow (E_f)^\sharp(L_X(F(X))) \xrightarrow{\widetilde{E_f}_{L_X(F(X))}} L_X(F(X)),$$

where the first morphism is the adjoint of the following composition.

$$(\overline{T}^*(E_f))^\sharp(\eta_X) F(f)_\sharp : F(Y) \to (\overline{T}^*(E_f))^\sharp(T^* L_X(F(X))) = T^*(E_f)^\sharp(L_X(F(X)))$$

Namely, $L(F)(f) = \widetilde{E_f}_{L_X(F(X))} \varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(\eta_X)) L_Y(F(f)_\sharp)$. Since $T_0^*(L(F)(f)) =$
$T_0^*(\widetilde{E_f}_{L_X(F(X))}) T_0^*(\varepsilon_Y) T_0^* L_Y((\overline{T}^*(E_f))^\sharp(\eta_X)) T_0^*(L_Y(F(f)_\sharp)) = E_f = \Phi_0(F_0)(f)$, we have $T_0^* L(F) = \Phi_0(F_0)$.

It is obvious that $L(F)(id_X)$ is an identity morphism of $L(F)(X)$. For morphisms $f : X \to Y$ and $g : Y \to Z$, we verify that $L(F)(gf) = L(F)(f)L(F)(g)$ as follows.

$$\begin{aligned}
&L(F)(f)L(F)(g) = \widetilde{E_f}_{L_X(F(X))} \varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(\eta_X)) L_Y(F(f)_\sharp) \widetilde{E_g}_{L_Y(F(Y))} \varepsilon_Z L_Z((\overline{T}^*(E_g))^\sharp(\eta_Y)) L_Z(F(g)_\sharp) \\
&= \widetilde{E_f}_{L_X(F(X))} \varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(\eta_X)) \widetilde{E_g}_{L_Y((\overline{T}^*(E_f))^\sharp(F(Y)))} (E_g)^\sharp(L_Y(F(f)_\sharp)) \varepsilon_Z L_Z((\overline{T}^*(E_g))^\sharp(\eta_Y)) L_Z(F(g)_\sharp) \\
&= \widetilde{E_f}_{L_X(F(X))} \varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(\eta_X)) \widetilde{E_g}_{L_Y((\overline{T}^*(E_f))^\sharp(F(Y)))} \varepsilon_Z L_Z((\overline{T}^*(E_g))^\sharp(T^* L_Y(F(f)_\sharp))) \\
&\qquad\qquad L_Z((\overline{T}^*(E_g))^\sharp(\eta_Y)) L_Z(F(g)_\sharp) \\
&= \widetilde{E_f}_{L_X(F(X))} \varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(\eta_X)) \widetilde{E_g}_{L_Y((\overline{T}^*(E_f))^\sharp(F(Y)))} \varepsilon_Z L_Z((\overline{T}^*(E_g))^\sharp(\eta_Y)) L_Z((\overline{T}^*(E_g))^\sharp(F(f)_\sharp)) L_Z(F(g)_\sharp) \\
&= \widetilde{E_f}_{L_X(F(X))} \varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(\eta_X)) \widetilde{E_g}_{L_Y((\overline{T}^*(E_f))^\sharp(F(Y)))} \varepsilon_Z L_Z((\overline{T}^*(E_g))^\sharp(\eta_Y)) L_Z(F(gf)_\sharp) \\
&= \widetilde{E_f}_{L_X(F(X))} \varepsilon_Y \widetilde{E_g}_{L_Y T^*(E_f)^\sharp(L_X(F(Y)))} (E_g)^\sharp(L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))) \varepsilon_Z L_Z((\overline{T}^*(E_g))^\sharp(\eta_Y)) L_Z(F(gf)_\sharp) \\
&= \widetilde{E_f}_{L_X(F(X))} \widetilde{E_g}_{(E_f)^\sharp(L_X(F(Y)))} (E_g)^\sharp(\varepsilon_Y) (E_g)^\sharp(L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))) \varepsilon_Z L_Z((\overline{T}^*(E_g))^\sharp(\eta_Y)) L_Z(F(gf)_\sharp) \\
&= \widetilde{E_{gf}}_{L_X(F(X))} (E_g)^\sharp(\varepsilon_Y) (E_g)^\sharp(L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))) \varepsilon_Z L_Z((\overline{T}^*(E_g))^\sharp(\eta_Y)) L_Z(F(gf)_\sharp) \\
&= \widetilde{E_{gf}}_{L_X(F(X))} (E_g)^\sharp(\varepsilon_Y) \varepsilon_Z L_Z T^*((E_g)^\sharp(L_Y((\overline{T}^*(E_f))^\sharp(\eta_X)))) L_Z((\overline{T}^*(E_g))^\sharp(\eta_Y)) L_Z(F(gf)_\sharp) \\
&= \widetilde{E_{gf}}_{L_X(F(X))} \varepsilon_Z L_Z T^*((E_g)^\sharp(\varepsilon_Y)) L_Z T^*((E_g)^\sharp L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))) L_Z((\overline{T}^*(E_g))^\sharp(\eta_Y)) L_Z(F(gf)_\sharp) \\
&= \widetilde{E_{gf}}_{L_X(F(X))} \varepsilon_Z L_Z T^*((E_g)^\sharp(\varepsilon_Y)) L_Z((\overline{T}^*(E_g))^\sharp(\eta_Y)) L_Z((\overline{T}^*(E_g))^\sharp(\overline{T}^*(E_f))^\sharp(\eta_X)) L_Z(F(gf)_\sharp) \\
&= \widetilde{E_{gf}}_{L_X(F(X))} \varepsilon_Z L_Z((\overline{T}^*(E_g))^\sharp(T^*(\varepsilon_Y)\eta_Y)) L_Z((\overline{T}^*(E_{gf}))^\sharp(\eta_X)) L_Z(F(gf)_\sharp) \\
&= \widetilde{E_{gf}}_{L_X(F(X))} \varepsilon_Z L_Z((\overline{T}^*(E_{gf}))^\sharp(\eta_X)) L_Z(F(gf)_\sharp) = L(F)(gf)
\end{aligned}$$

Thus we have verified that $L(F)$ is an object of $\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}(\mathcal{D}); T_0, F_0)$.

If $\alpha : F \to G$ is a morphism in $\mathrm{Funct}(\mathcal{C}^{op}, \mathcal{T}'(\mathcal{D}); T_0', \overline{T}^*(F_0))$, define $L(\alpha) : L(F) \to L(G)$ by $L(\alpha)_X = L_X(\alpha_X)$. For a morphism $f : X \to Y$ in $\mathcal{C}$, we show that $L(\alpha)_X L(F)(f) = L(G)(f) L(\alpha)_Y$ as follows.
$L(\alpha)_X L(F)(f) = L_X(\alpha_X) \widetilde{E_f}_{L_X(F(X))} \varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(\eta_X)) L_Y(F(f)_\sharp)$

$= \widetilde{E_f}_{L_X(G(X))} E_f^\sharp(L_X(\alpha_X))\varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))L_Y(F(f)_\sharp)$
$= \widetilde{E_f}_{L_X(G(X))}\varepsilon_Y L_Y T^* E_f^\sharp(L_X(\alpha_X))L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))L_Y(F(f)_\sharp)$
$= \widetilde{E_f}_{L_X(G(X))}\varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(T^*L_X(\alpha_X)))L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))L_Y(F(f)_\sharp)$
$= \widetilde{E_f}_{L_X(G(X))}\varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))L_Y((\overline{T}^*(E_f))^\sharp(\alpha_X))L_Y(F(f)_\sharp)$
$= \widetilde{E_f}_{L_X(G(X))}\varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))L_Y(F(f)_\sharp)L_Y(\alpha_Y) = L(G)(f)L_Y(\alpha_Y)$
Therefore $L(\alpha)$ is a morphism in $\mathrm{Funct}(\mathcal{C}^{op},\mathcal{T}(\mathcal{D});T_0,F_0)$. Thus we have a functor

$$L : \mathrm{Funct}(\mathcal{C}^{op},\mathcal{T}'(\mathcal{D});T_0',\overline{T}^*(F_0)) \to \mathrm{Funct}(\mathcal{C}^{op},\mathcal{T}(\mathcal{D});T_0,F_0).$$

Next, we define the following natural transformations which are the unit and the counit of the adjunction.

$$\eta : id \to \mathrm{Funct}(id_{\mathcal{C}^{op}},T^*)L \qquad \varepsilon : L\mathrm{Funct}(id_{\mathcal{C}^{op}},T^*) \to id$$

For $F \in \mathrm{Ob}\,\mathrm{Funct}(\mathcal{C}^{op},\mathcal{T}'(\mathcal{D});T_0',\overline{T}^*(F_0))$ and $G \in \mathrm{Ob}\,\mathrm{Funct}(\mathcal{C}^{op},\mathcal{T}(\mathcal{D});T_0,F_0)$, $\eta_F : F \to T^*L(F)$ and $\varepsilon_G : L(T^*G) \to G$ are defined by $(\eta_F)_X = \eta_X : F(X) \to T^*(L_X(F(X)))$ and $(\varepsilon_G)_X = \varepsilon_X : L_X(T^*G(X)) \to G(X)$. We claim that η_F and ε_G are natural. In fact, let $f : X \to Y$ be a morphism in $\mathcal{C}$, then
$T^*(L(F)(f))(\eta_F)_Y = T^*(\widetilde{E_f}_{L_X(F(X))})T^*(\varepsilon_Y)T^*L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))T^*L_Y(F(f)_\sharp)\eta_Y$
$= T^*(\widetilde{E_f}_{L_X(F(X))})T^*(\varepsilon_Y)T^*L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))\eta_Y F(f)_\sharp$
$= T^*(\widetilde{E_f}_{L_X(F(X))})T^*(\varepsilon_Y)\eta_Y(\overline{T}^*(E_f))^\sharp(\eta_X)F(f)_\sharp = T^*(\widetilde{E_f}_{L_X(F(X))})(\overline{T}^*(E_f))^\sharp(\eta_X)F(f)_\sharp$
$= \widetilde{\overline{T}^*(E_f)}_{T^*L_X(F(X))}(\overline{T}^*(E_f))^\sharp(\eta_X)F(f)_\sharp = \eta_X\widetilde{\overline{T}^*(E_f)}_{F(X)}F(f)_\sharp = \eta_X F(f)(\varepsilon_G)_X L(T^*G)(f)$
$= \varepsilon_X\widetilde{E_f}_{L_X(T^*G(X))}\varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))L_Y(T^*G(f)_\sharp) = \widetilde{E_f}_{G(X)}E_f^\sharp(\varepsilon_X)\varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))L_Y(T^*G(f)_\sharp)$
$= \widetilde{E_f}_{G(X)}\varepsilon_Y L_Y T^*(E_f^\sharp(\varepsilon_X))L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))L_Y(T^*G(f)_\sharp)$
$= \widetilde{E_f}_{G(X)}\varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(T^*(\varepsilon_X)))L_Y((\overline{T}^*(E_f))^\sharp(\eta_X))L_Y(T^*G(f)_\sharp)$
$= \widetilde{E_f}_{G(X)}\varepsilon_Y L_Y((\overline{T}^*(E_f))^\sharp(T^*(\varepsilon_X)\eta_X))L_Y(T^*G(f)_\sharp) = \widetilde{E_f}_{G(X)}\varepsilon_Y L_Y(T^*G(f)_\sharp) = \widetilde{E_f}_{G(X)}G(f)_\sharp\varepsilon_Y = G(f)\varepsilon_Y$
Since $\eta_X : F(X) \to T^*(L_X(F(X)))$ and $\varepsilon_X : L_X(T^*G(X)) \to G(X)$ are morphisms in $\mathcal{T}'(\mathcal{D};T_0',\overline{T}^*(E_XF_0))$ and $\mathcal{T}(\mathcal{D};T_0,E_XF_0)$, it follows that η_F and ε_G are morphisms in $\mathrm{Funct}(\mathcal{C}^{op},\mathcal{T}'(\mathcal{D});T_0',\overline{T}^*(F_0))$ and $\mathrm{Funct}(\mathcal{C}^{op},\mathcal{T}'(\mathcal{D});T_0,F_0)$, respectively.

Finally, we verify the equalities $\varepsilon_{L(F)}L(\eta_F) = id_{L(F)}$ and $T^*(\varepsilon_G)\eta_{T^*G} = id_{T^*G}$. Let X be an object of $\mathcal{C}$, then we have $(\varepsilon_{L(F)}L(\eta_F))_X = \varepsilon_X L_X(\eta_X) = id_{L_X(F(X))}$, $(T^*(\varepsilon_G)\eta_{T^*G})_X = T^*(\varepsilon_X)\eta_X = id_{T^*G(X)}$. Therefore L is a left adjoint of $\mathrm{Funct}(id_{\mathcal{C}^{op}},T^*)$. □

Proposition A.11.14 *We use the same notation as in 2) of (A.11.7) and assume that* $\mathrm{Im}\,\sigma' = \tau^{-1}(\mathrm{Im}\,\sigma)$. *Let* F_0 *be an object of* $\mathcal{T}_0(\mathcal{C})$. *If the correspondence* $s \mapsto \beta(s)$ *is bijective, then* $T^* : \mathcal{T}(\mathcal{C};T_0,F_0) \to \mathcal{T}'(\mathcal{C};T_0',\overline{T}^*(F_0))$ *creates limits. In particular, so does the forgetful functor* $\widetilde{U}_{\mathcal{T}} : \mathcal{T}(\mathcal{C};T_0,F_0) \to \mathcal{C}^{k-m}$.

Proof. Let $D : \mathcal{D} \to \mathcal{T}(\mathcal{C};T_0,F_0)$ be a functor and $(L \xrightarrow{p_i} T^*D(i))_{i\in\mathrm{Ob}\,\mathcal{D}}$ a limiting cone of a functor T^*D. Since $D(i) \in \mathcal{T}(\mathcal{C};T_0,F_0)$ for each $i \in \mathrm{Ob}\,\mathcal{D}$ and $L \in \mathrm{Ob},\mathcal{T}'(\mathcal{C};T_0',\overline{T}^*(F_0))$, $D(i)([1]_{\hat\sigma(s)}) = h_{Z_s}$ $(s = 1,2,\dots,m)$ and $L([1]_{\hat\sigma'(s)}) = h_{Z_{\tau_1(s)}}$ $(s = 1,2,\dots,d)$ where $\tau_1 : \{1,2,\dots,d\} \to \{1,2,\dots,m\}$ is the unique map satisfying $\tau_1\tilde\sigma' = \tilde\sigma\tau_0$ and $\hat\sigma\tau_1 = \tau\hat\sigma'$ (A.11.5). We set $D(i)([1]_{\bar\sigma_s}) = h_{X_s^i}$ and $D(\theta) = (D(\theta;1),\dots,D(\theta;k))$ for $\theta \in \mathrm{Mor}\,\mathcal{D}$, then $D(\theta;\hat\sigma(s)) = id_{Z_s}$. We also set $L([1]_{\bar\sigma_s'}) = h_{X_s}$ and $p_i = (p(i;1),\dots,p(i;l))$, then $p(i;\hat\sigma'(s)) = id_{Z_{\tau_1(s)}}$. Then, by (A.4.1),

$$\left(L\left(\coprod_{s=1}^{l-d}[n_s]_{\bar\sigma_s'}\right) = \prod_{s=1}^{l-d}h_{X_s}^{n_s} \xrightarrow{\prod_{s=1}^{l-d}h^{n_s}_{p(i;\bar\sigma_s')}} \prod_{s=1}^{l-d}h^{n_s}_{X^i_{\beta(s)}} = T^*D(i)\left(\coprod_{s=1}^{l-d}[n_s]_{\bar\sigma_s'}\right)\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$$

is a limiting cone of a functor $\mathcal{D} \to \widehat{\mathcal{C}}$ given by $i \mapsto D(i)(\coprod_{s=1}^{l-d}[n_s]_{\bar\sigma_s'}) = \prod_{s=1}^{l-d}h^{n_s}_{X^i_{\beta(s)}}$ and $\theta \mapsto \prod_{s=1}^{l-d}h^{n_s}_{D(\theta;\bar\sigma_{\beta(s)})}$ for $n_1,\dots,n_{l-d} \in \boldsymbol{N}$. By the assumption, there exists a unique $1 \le \gamma(s) \le l-d$ such that $\beta(\gamma(s)) = s$ for each $1 \le s \le k-m$, hence it follows from the above fact that, for each $n_1,\dots,n_k \in \boldsymbol{N}$,

$$\left(\prod_{s=1}^{m}h_{Z_s}^{n_{\hat\sigma(s)}} \times \prod_{s=1}^{k-m}h^{n_{\bar\sigma_s}}_{X_{\gamma(s)}} \xrightarrow{id\times\prod_{s=1}^{k-m}h^{n_{\bar\sigma_s}}_{p(i;\bar\sigma'_{\gamma(s)})}} \prod_{s=1}^{m}h_{Z_s}^{n_{\hat\sigma(s)}} \times \prod_{s=1}^{k-m}h^{n_{\bar\sigma_s}}_{X_s^i}\right)_{i\in\mathrm{Ob}\,\mathcal{D}}$$

is a limiting cone of a functor $\mathcal{D} \to \widehat{\mathcal{C}}$ given by $i \mapsto \prod_{s=1}^{m} h_{Z_s}^{n_{\hat{\sigma}(s)}} \times \prod_{s=1}^{k-m} h_{X_s^i}^{n_{\bar{\sigma}_s}}$ and $\theta \mapsto id \times \prod_{s=1}^{k-m} h_{D(\theta;\bar{\sigma}_s)}^{n_{\bar{\sigma}_s}}$. By (A.4.1), we have a limiting cone $(F \xrightarrow{q_i} D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$ of D, where F is given by $F([1]_{\bar{\sigma}_s}) = h_{X_{\gamma(s)}}$ and $F([1]_{\hat{\sigma}(s)}) = h_{Z_s}$. Then, it is easy to verify that $T^*(F) = FT = L$.

We note that T^* is faithful. In fact, the assumption implies that the functor Π in (A.11.7) is an isomorphism. Since the forgetful functors $\widetilde{U}_{\mathcal{T}}$ and $\widetilde{U}_{\mathcal{T}'}$ are faithful, it follows from (A.11.7) that so is T^*. Suppose that $G : \mathcal{T}^{op} \to \widehat{\mathcal{C}}$ be an object of $\mathcal{T}(\mathcal{C}; T_0, F_0)$ satisfying $T^*(G) = L$. Then, $G([1]_{\bar{\sigma}_s}) = G([1]_{\tau(\bar{\sigma}'_{\gamma(s)})}) = GT([1]_{\bar{\sigma}'_{\gamma(s)}}) = L([1]_{\bar{\sigma}'_{\gamma(s)}}) = h_{X_{\gamma(s)}}$. Thus we see the uniqueness of F satisfying $T^*(F) = L$. Since T^* is faithful, we also have the uniqueness of a cone $(F \xrightarrow{q_i} D(i))_{i \in \mathrm{Ob}\,\mathcal{D}}$ satisfying $T^*(q_i) = p_i$. $\square$

Corollary A.11.15 *If $\mathcal{C}$ has a finite limits (resp. $\mathcal{U}$-limits), so does $\mathcal{T}(\mathcal{C}; T_0, F_0)$ and the forgetful functor $\widetilde{U}_{\mathcal{T}} : \mathcal{T}(\mathcal{C}; T_0, F_0) \to \mathcal{C}^{k-m}$ preserves finite limits (resp. $\mathcal{U}$-limits).*

Proposition A.11.16 *Assume that $\mathrm{Im}\,\sigma' = \tau^{-1}(\mathrm{Im}\,\sigma)$ and the correspondence $s \mapsto \beta(s)$ in (A.11.7) is bijective. If $\mathcal{T}$ is a $\mathcal{U}$-category, then $T^* : \mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}; T_0, F_0) \to \mathcal{T}'(\mathcal{U}\text{-}\mathbf{Ens}; T'_0, \overline{T}^*(F_0))$ has a left adjoint. In particular, the forgetful functor $\widetilde{U}_{\mathcal{T}} : \mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}; T_0, F_0) \to (\mathcal{U}\text{-}\mathbf{Ens})^{k-m}$ has a left adjoint.*

Proof. Since $\mathrm{Ob}\,\mathcal{T}$ is a countable set, the assumption implies that $\mathcal{T}$ is a $\mathcal{U}$-small set. Hence $\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}; T_0, F_0)$ is a $\mathcal{U}$-category by (A.1.3). Since $(\mathcal{U}\text{-}\mathbf{Ens})^{l-d}$ is $\mathcal{U}$-complete, so are $\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}; T_0, F_0)$ and $\mathcal{T}'(\mathcal{U}\text{-}\mathbf{Ens}; T'_0, \overline{T}^*(F_0))$ by (A.11.15). Hence T^* preserves $\mathcal{U}$ limits by (A.11.14). We find a "solution set" for each object H of $\mathcal{T}'(\mathcal{U}\text{-}\mathbf{Ens}; T'_0, \overline{T}^*(F_0))$. First, we note that for a fixed $(Y_1, \ldots, Y_{k-m}) \in \mathrm{Ob}\,(\mathcal{U}\text{-}\mathbf{Ens})^{k-m}$, there is an injection from $\{F \in \mathrm{Ob}\,\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}; T_0, F_0) | \,\widetilde{U}_{\mathcal{T}}(F) = (Y_1, \ldots, Y_{k-m})\}$ to a $\mathcal{U}$-small set

$$\prod_{\theta \in \mathrm{Mor}\,\mathcal{T}} \mathcal{U}\text{-}\mathbf{Ens}\Big(\prod_{s=1}^{m} Z_s^{n(\theta)_{\hat{\sigma}(s)}} \times \prod_{s=1}^{k-m} Y_s^{n(\theta)_{\bar{\sigma}_s}}, \prod_{s=1}^{m} Z_s^{n'(\theta)_{\hat{\sigma}(s)}} \times \prod_{s=1}^{k-m} Y_s^{n'(\theta)_{\bar{\sigma}_s}}\Big),$$

where $F_0([1]_s) = Z_{\tilde{\sigma}(s)}$ ($\sigma = \hat{\sigma}\tilde{\sigma}$ as in (A.11.5)), $\mathrm{dom}(\theta) = \coprod_{s=1}^{k} [n'(\theta)_s]_s$ and $\mathrm{codom}(\theta) = \coprod_{s=1}^{k} [n(\theta)_s]_s$.

Let $f = (f_1, \ldots, f_{k-m}) : H \to T^*(F)$ be a morphism in $\mathcal{T}'(\mathcal{U}\text{-}\mathbf{Ens}; T'_0, \overline{T}^*(F_0))$. Set

$$\widetilde{U}_{\mathcal{T}'}(H) = (X_{\beta(1)}, \ldots, X_{\beta(l-d)}), \quad Y_s = \bigcup_{n \in \mathrm{Ob}\,\mathcal{T}} \bigcup_{\theta \in \mathcal{T}([1]_{\bar{\sigma}_s}, n)} F(\theta)\Big(\prod_{t=1}^{m} Z_t^{n_{\hat{\sigma}(t)}} \times \prod_{t=1}^{k-m} f_t(X_t)^{n_{\bar{\sigma}_t}}\Big),$$

where $n = \coprod_{t=1}^{k} [n_t]_t$. Then $f_s(X_s) \subset Y_s \subset F([1]_{\bar{\sigma}_s})$. For any morphism $\theta : \coprod_{s=1}^{k} [n_s]_s \to \coprod_{s=1}^{k} [n'_s]_s$ in $\mathcal{T}$, $F(\theta)\Big(\prod_{s=1}^{m} Z_s^{n'_{\hat{\sigma}(s)}} \times \prod_{s=1}^{k-m} Y_s^{n'_{\bar{\sigma}_s}}\Big) \subset \prod_{s=1}^{m} Z_s^{n_{\hat{\sigma}(s)}} \times \prod_{s=1}^{k-m} Y_s^{n_{\bar{\sigma}_s}}$. Hence there is a subfunctor $G \in \mathrm{Ob}\,\mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}; T_0, F_0)$ of F such that $\widetilde{U}_{\mathcal{T}}(G) = (Y_1, \ldots, Y_{k-m})$ and that f factors through $T^*(G) \to T^*(F)$. If we set

$$\mathfrak{a} = \mathrm{card}(\mathrm{Mor}\,\mathcal{T})\Big(\sum_{s=1}^{k-m} \mathrm{card}(X_s) + \sum_{s=1}^{m} \mathrm{card}(Z_s)\Big),$$

we have $\mathrm{card}(Y_s) \leq \mathfrak{a}$. Choose a set M such that $\mathrm{card}(M) = \mathfrak{a}$, then $\{G \in \mathcal{T}(\mathcal{U}\text{-}\mathbf{Ens}; T_0, F_0) | \, G([1]_{\bar{\sigma}_s}) \subset M$ *for* $s = 1, 2, \ldots, k-m\}$ is a $\mathcal{U}$-small solution set for H. $\square$

Corollary A.11.17 *Assume that $\mathrm{Im}\,\sigma' = \tau^{-1}(\mathrm{Im}\,\sigma)$ and the correspondence $s \mapsto \beta(s)$ is bijective and that $T_0 : \mathcal{T}_0 \to \mathcal{T}$ satisfies (A.11.6). If $\mathcal{T}$ and $\mathcal{C}$ are $\mathcal{U}$-categories,*

$$T^* : \mathcal{T}(\widehat{\mathcal{C}}_{\mathcal{U}}; T_0, F_0) \to \mathcal{T}'(\widehat{\mathcal{C}}_{\mathcal{U}}; T'_0, \overline{T}^*(F_0))$$

has a left adjoint. In particular, the forgetful functor $\widetilde{U}_{\mathcal{T}} : \mathcal{T}(\widehat{\mathcal{C}}_{\mathcal{U}}; T_0, F_0) \to (\widehat{\mathcal{C}}_{\mathcal{U}})^{k-m}$ has a left adjoint.

Let $\mathcal{T}$ be a k-fold finitary algebraic theory. Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are category with finite products (resp. finite powers if $k = 1$) and $F : \mathcal{C} \to \mathcal{C}'$ is a product preserving functor. Define a functor $F_{\mathcal{T}} : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\mathcal{C}')$ by $F_{\mathcal{T}}(G) = FG$ and $F_{\mathcal{T}}(\varphi : G \to H) = (F(\varphi) : FG \to FH)$. The next assertions are obvious from the definition.

Proposition A.11.18 *1) Let $T : \mathcal{T}' \to \mathcal{T}$ be morphisms of finitary algebraic theories and F as above. Then, the following diagram commutes.*

$$\begin{tikzcd} \mathcal{T}(\mathcal{C}) \arrow[r, "F_{\mathcal{T}}"] \arrow[d, "T^*"] & \mathcal{T}(\mathcal{C}') \arrow[d, "T^*"] \\ \mathcal{T}'(\mathcal{C}) \arrow[r, "F_{\mathcal{T}'}"] & \mathcal{T}'(\mathcal{C}') \end{tikzcd}$$

2) Let $T_0 : \mathcal{T}_0 \to \mathcal{T}$, $T_0' : \mathcal{T}_0' \to \mathcal{T}'$, $T : \mathcal{T}' \to \mathcal{T}$, $\overline{T} : \mathcal{T}_0' \to \mathcal{T}_0$ be morphisms of finitary algebraic theories satisfying $T_0\overline{T} = TT_0'$ and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}$, $\mathcal{B}'$ subcategories of $\mathcal{T}_0(\mathcal{C})$, $\mathcal{T}_0'(\mathcal{C})$, $\mathcal{T}_0(\mathcal{C}')$, $\mathcal{T}_0'(\mathcal{C}')$, respectively. Assume that $\overline{T}^* : \mathcal{T}_0(\mathcal{C}) \to \mathcal{T}_0'(\mathcal{C})$ maps $\mathcal{A}$ into $\mathcal{B}$, $\overline{T}^* : \mathcal{T}_0(\mathcal{C}') \to \mathcal{T}_0'(\mathcal{C}')$ maps $\mathcal{A}'$ into $\mathcal{B}'$, $F_{\mathcal{T}_0} : \mathcal{T}_0(\mathcal{C}) \to \mathcal{T}_0(\mathcal{C}')$ maps $\mathcal{A}$ into $\mathcal{A}'$ and that $F_{\mathcal{T}_0'} : \mathcal{T}_0'(\mathcal{C}) \to \mathcal{T}_0'(\mathcal{C}')$ maps $\mathcal{B}$ into $\mathcal{B}'$. Then we have the following commutative diagram.

$$\begin{tikzcd} \mathcal{T}(\mathcal{C}; T_0, \mathcal{A}) \arrow[r, "F_{\mathcal{T}}"] \arrow[d, "T^*"] & \mathcal{T}(\mathcal{C}'; T_0, \mathcal{A}') \arrow[d, "T^*"] \\ \mathcal{T}'(\mathcal{C}; T_0', \mathcal{B}) \arrow[r, "F_{\mathcal{T}'}"] & \mathcal{T}'(\mathcal{C}'; T_0', \mathcal{B}') \end{tikzcd}$$

3) If F is faithful (resp. fully faithful), so is $F_{\mathcal{T}} : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\mathcal{C}')$. More generally, if F is fully faithful and $F_{\mathcal{T}_0} : \mathcal{T}_0(\mathcal{C}) \to \mathcal{T}_0(\mathcal{C}')$ maps $\mathcal{A}$ fully into $\mathcal{A}'$, so is $F_{\mathcal{T}} : \mathcal{T}(\mathcal{C}; T_0, \mathcal{A}) \to \mathcal{T}(\mathcal{C}'; T_0, \mathcal{A}')$.

Proposition A.11.19 *Suppose that a product preserving functor $F : \mathcal{C} \to \mathcal{C}'$ has a right adjoint $R : \mathcal{C}' \to \mathcal{C}$ with unit $\eta : id_{\mathcal{C}} \to RF$ and counit $\varepsilon : FR \to id_{\mathcal{C}'}$.*

1) R preserves products and $R_{\mathcal{T}} : \mathcal{T}(\mathcal{C}') \to \mathcal{T}(\mathcal{C})$ is a right adjoint of $F_{\mathcal{T}} : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\mathcal{C}')$ with unit $\eta_{\mathcal{T}} : id_{\mathcal{T}(\mathcal{C})} \to R_{\mathcal{T}}F_{\mathcal{T}}$ and counit $\varepsilon_{\mathcal{T}} : F_{\mathcal{T}}R_{\mathcal{T}} \to id_{\mathcal{T}(\mathcal{C}')}$ given by $(\eta_{\mathcal{T}})_G = \eta_G : G \to RFG$, $(\varepsilon_{\mathcal{T}})_H = \varepsilon_H : FRH \to H$ for $G \in \operatorname{Ob}\mathcal{T}(\mathcal{C})$, $H \in \operatorname{Ob}\mathcal{T}(\mathcal{C}')$.

2) Let $T_0 : \mathcal{T}_0 \to \mathcal{T}$ be a morphism of finitary algebraic theories and $\mathcal{A}$, $\mathcal{A}'$ subcategories of $\mathcal{T}_0(\mathcal{C})$, $\mathcal{T}_0(\mathcal{C}')$, satisfying the following conditions.

(a)$F_{\mathcal{T}_0} : \mathcal{T}_0(\mathcal{C}) \to \mathcal{T}_0(\mathcal{C}')$ maps $\mathcal{A}$ into $\mathcal{A}'$ and $R_{\mathcal{T}_0} : \mathcal{T}_0(\mathcal{C}') \to \mathcal{T}_0(\mathcal{C})$ maps $\mathcal{A}'$ into $\mathcal{A}$.

(b)$(\eta_{\mathcal{T}_0})_G : G \to R_{\mathcal{T}_0}F_{\mathcal{T}_0}(G)$ is a morphism in $\mathcal{A}$ for any $G \in \operatorname{Ob}\mathcal{A}$ and $(\varepsilon_{\mathcal{T}_0})_H : F_{\mathcal{T}_0}R_{\mathcal{T}_0}(H) \to H$ is a morphism in $\mathcal{A}'$ for any $H \in \operatorname{Ob}\mathcal{A}'$.
Then, the restrictions of $F_{\mathcal{T}} : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\mathcal{C}')$ and $R_{\mathcal{T}} : \mathcal{T}(\mathcal{C}') \to \mathcal{T}(\mathcal{C})$ to subcategories $\mathcal{T}(\mathcal{C}; T_0, \mathcal{A})$ and $\mathcal{T}(\mathcal{C}'; T_0, \mathcal{A}')$ give functors $F_{\mathcal{T}} : \mathcal{T}(\mathcal{C}; T_0, \mathcal{A}) \to \mathcal{T}(\mathcal{C}'; T_0, \mathcal{A}')$ and $R_{\mathcal{T}} : \mathcal{T}(\mathcal{C}'; T_0, \mathcal{A}') \to \mathcal{T}(\mathcal{C}; T_0, \mathcal{A})$ such that $R_{\mathcal{T}}$ is a right adjoint of $F_{\mathcal{T}}$.

Proof. 1) Since $\eta_G : G \to RFG$ and $\varepsilon_H : FRH \to H$ are natural transformations between product preserving functors, they are morphisms in $\mathcal{T}(\mathcal{C})$ and $\mathcal{T}(\mathcal{C}')$ respectively. It is obvious that $R_{\mathcal{T}}(\varepsilon_{\mathcal{T}})\eta_{\mathcal{T}R_{\mathcal{T}}} = id_{R_{\mathcal{T}}}$ and $\varepsilon_{\mathcal{T}F_{\mathcal{T}}}F_{\mathcal{T}}(\eta_{\mathcal{T}}) = id_{F_{\mathcal{T}}}$ hold.

2) If condition (a) is satisfied, $F_{\mathcal{T}} : \mathcal{T}(\mathcal{C}) \to \mathcal{T}(\mathcal{C}')$ maps $\mathcal{T}(\mathcal{C}; T_0, \mathcal{A})$ into $\mathcal{T}(\mathcal{C}'; T_0, \mathcal{A}')$ and $R_{\mathcal{T}} : \mathcal{T}(\mathcal{C}') \to \mathcal{T}(\mathcal{C})$ maps $\mathcal{T}(\mathcal{C}'; T_0, \mathcal{A}')$ into $\mathcal{T}(\mathcal{C}; T_0, \mathcal{A})$. Suppose that condition (b) is also satisfied. For $G \in \operatorname{Ob}\mathcal{T}(\mathcal{C}; T_0, \mathcal{A})$, $(\eta_{\mathcal{T}})_G : G \to R_{\mathcal{T}}F_{\mathcal{T}}(G)$ is a morphism in $\mathcal{T}(\mathcal{C}; T_0, \mathcal{A})$. In fact, $T_0^*((\eta_{\mathcal{T}})_G) = T_0^*(\eta_G) = \eta_{T_0^*(G)} = (\eta_{\mathcal{T}_0})_{T_0^*(G)} : GT_0 \to RFGT_0$ is a morphism in $\mathcal{A}$. Similarly, for $H \in \operatorname{Ob}\mathcal{T}(\mathcal{C}'; T_0, \mathcal{A}')$, $(\varepsilon_{\mathcal{T}})_G : G \to R_{\mathcal{T}}F_{\mathcal{T}}(G)$ is a morphism in $\mathcal{T}(\mathcal{C}'; T_0, \mathcal{A}')$. □

If $\mathcal{A}$ and $\mathcal{A}'$ are subcategories of $\mathcal{T}_0(\mathcal{C})$ and $\mathcal{T}_0(\mathcal{C}')$ such that $\operatorname{Ob}\mathcal{A} = \{G_0\}$, $\operatorname{Ob}\mathcal{A}' = \{G_0'\}$ and $\operatorname{Mor}\mathcal{A} = \{id_{G_0}\}$, $\operatorname{Mor}\mathcal{A}' = \{id_{G_0'}\}$, the above conditions (a) and (b) reduce to the following.

(a) $G_0' = FG_0$ and $G_0 = RG_0'$.
(b) $\eta_{G_0} = id_{G_0} : G_0 \to RFG_0 = G_0$ or $\varepsilon_{G_0'} = id_{G_0'} : G_0' = FRG_0' \to G_0'$.

Proposition A.11.20 *Suppose that $T_0 : \mathcal{T}_0 \to \mathcal{T}$ satisfies (A.11.6). Let $F : \mathcal{C} \to \mathcal{C}'$ be a product preserving functor and $R : \mathcal{C}' \to \mathcal{C}$ a right adjoint of F with unit $\eta : id_{\mathcal{C}} \to RF$ and counit $\varepsilon : FR \to id_{\mathcal{C}'}$.*

1) If H_0 is an object of $\mathcal{T}_0(\mathcal{C}')$ such that $\varepsilon_{H_0} : FRH_0 \to H_0$ is an isomorphism, then $(\varepsilon_{H_0}^{-1})^\sharp F_{\mathcal{T}} : \mathcal{T}(\mathcal{C}; T_0, RH_0) \to \mathcal{T}(\mathcal{C}'; T_0, H_0)$ is a left adjoint of $R_{\mathcal{T}} : \mathcal{T}(\mathcal{C}'; T_0, H_0) \to \mathcal{T}(\mathcal{C}; T_0, RH_0)$.

2) If G_0 is an object of $\mathcal{T}_0(\mathcal{C})$ such that $\eta_{G_0} : G_0 \to RFG_0$ is an isomorphism, then $\eta_{G_0}^\sharp R_{\mathcal{T}} : \mathcal{T}(\mathcal{C}'; T_0, FG_0) \to \mathcal{T}(\mathcal{C}; T_0, G_0)$ is a right adjoint of $F_{\mathcal{T}} : \mathcal{T}(\mathcal{C}; T_0, G_0) \to \mathcal{T}(\mathcal{C}; T_0, FG_0)$.

Proof. 1) For each $H \in \operatorname{Ob}\mathcal{T}(\mathcal{C}'; T_0, H_0)$ and $G \in \operatorname{Ob}\mathcal{T}(\mathcal{C}; T_0, RH_0)$, we define a morphism $\bar{\varepsilon}_H : (\varepsilon_{H_0}^{-1})^\sharp F_\mathcal{T} R_\mathcal{T}(H) \to H$ in $\mathcal{T}(\mathcal{C}')$ and a morphism $\bar{\eta}_G : G \to R_\mathcal{T}(\varepsilon_{H_0}^{-1})^\sharp F_\mathcal{T}$ in $\mathcal{T}(\mathcal{C})$ to be the following compositions.

$$(\varepsilon_{H_0}^{-1})^\sharp F_\mathcal{T} R_\mathcal{T}(H) = (\varepsilon_{H_0}^{-1})^\sharp (FRH) \xrightarrow{\widetilde{\varepsilon_{H_0}^{-1}}} FRH \xrightarrow{\varepsilon_H} H$$

$$G \xrightarrow{\eta_G} RFG \xrightarrow{R(\widetilde{\varepsilon_{H_0}^{-1}})^{-1}} R(\varepsilon_{H_0}^{-1})^\sharp (FG) = R_\mathcal{T}(\varepsilon_{H_0}^{-1})^\sharp F_\mathcal{T}(G)$$

Using (A.11.9), we can verify that $\bar{\varepsilon}_H$ is a morphism in $\mathcal{T}(\mathcal{C}'; T_0, H_0)$ and $\bar{\eta}_G$ is a morphism in $\mathcal{T}(\mathcal{C}; T_0, RH_0)$ and that $R(\bar{\varepsilon}_H)\bar{\eta}_{R_\mathcal{T}(H)} = id_{R_\mathcal{T}(H)}$, $\bar{\varepsilon}_{(\varepsilon_{H_0}^{-1})^\sharp F_\mathcal{T}(G)}(\varepsilon_{H_0}^{-1})^\sharp F_\mathcal{T}(\bar{\eta}_G) = id_{(\varepsilon_{H_0}^{-1})^\sharp F_\mathcal{T}(G)}$ hold.

2) For each $G \in \operatorname{Ob}\mathcal{T}(\mathcal{C}; T_0, G_0)$ and $H \in \operatorname{Ob}\mathcal{T}(\mathcal{C}'; T_0, FG_0)$, define a morphism $\bar{\eta}_G : G \to \eta_{G_0}^\sharp R_\mathcal{T} F_\mathcal{T}$ in $\mathcal{T}(\mathcal{C})$ and a morphism $\bar{\varepsilon}_H : F_\mathcal{T}\eta_{G_0}^\sharp R_\mathcal{T}(H) \to H$ in $\mathcal{T}(\mathcal{C}')$ to be the following compositions.

$$G \xrightarrow{\eta_G} RFG \xrightarrow{\widetilde{\eta_{G_0}}^{-1}} \eta_{G_0}^\sharp (RFG) = \eta_{G_0}^\sharp R_\mathcal{T} F_\mathcal{T}(G)$$

$$F_\mathcal{T}\eta_{G_0}^\sharp R_\mathcal{T}(H) = F\eta_{G_0}^\sharp (RH) \xrightarrow{F(\widetilde{\eta_{G_0}})} FRH \xrightarrow{\varepsilon_H} H$$

We can easily verify that $\bar{\eta}$ and $\bar{\varepsilon}$ give unit and counit of the adjunction. □

The above adjoints are natural in the following sense.

Proposition A.11.21 *Let $T_0 : \mathcal{T}_0 \to \mathcal{T}$, $T_0' : \mathcal{T}_0' \to \mathcal{T}'$, $T : \mathcal{T}' \to \mathcal{T}$, $\overline{T} : \mathcal{T}_0' \to \mathcal{T}_0$ be morphisms of finitary algebraic theories satisfying $T_0\overline{T} = TT_0'$ and $\operatorname{Im}\sigma' = \tau^{-1}(\operatorname{Im}\sigma)$. In the situation of the preceding result, the following diagrams commute.*

$$\begin{array}{ccc}
\mathcal{T}(\mathcal{C}; T_0, RH_0) & \xrightarrow{(\varepsilon_{H_0}^{-1})^\sharp F_\mathcal{T}} & \mathcal{T}(\mathcal{C}'; T_0, H_0) \\
\downarrow{\scriptstyle T^*} & & \downarrow{\scriptstyle T^*} \\
\mathcal{T}'(\mathcal{C}; T_0', R\overline{T}^*(H_0)) & \xrightarrow{(\varepsilon_{H_0}^{-1})^\sharp F_\mathcal{T}} & \mathcal{T}'(\mathcal{C}'; T_0', \overline{T}^*(H_0))
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{T}(\mathcal{C}; T_0, RH_0) & \xrightarrow{(\varepsilon_{H_0}^{-1})^\sharp F_\mathcal{T}} & \mathcal{T}(\mathcal{C}'; T_0, H_0) \\
\downarrow{\scriptstyle \widetilde{U}_\mathcal{T}} & & \downarrow{\scriptstyle \widetilde{U}_\mathcal{T}} \\
\mathcal{C}^{k-m} & \xrightarrow{F^{k-m}} & \mathcal{C}'^{k-m}
\end{array}$$

$$\begin{array}{ccc}
\mathcal{T}(\mathcal{C}'; T_0, FG_0) & \xrightarrow{\eta_{G_0}^\sharp R_\mathcal{T}} & \mathcal{T}(\mathcal{C}; T_0, G_0) \\
\downarrow{\scriptstyle T^*} & & \downarrow{\scriptstyle T^*} \\
\mathcal{T}'(\mathcal{C}'; T_0', F\overline{T}^*(G_0)) & \xrightarrow{\eta_{G_0}^\sharp R_\mathcal{T}} & \mathcal{T}'(\mathcal{C}; T_0', \overline{T}^*(G_0))
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{T}(\mathcal{C}'; T_0, FG_0) & \xrightarrow{\eta_{G_0}^\sharp R_\mathcal{T}} & \mathcal{T}(\mathcal{C}; T_0, G_0) \\
\downarrow{\scriptstyle \widetilde{U}_\mathcal{T}} & & \downarrow{\scriptstyle \widetilde{U}_\mathcal{T}} \\
\mathcal{C}'^{k-m} & \xrightarrow{R^{k-m}} & \mathcal{C}^{k-m}
\end{array}$$

Proof. A direct consequence of (A.11.9). □

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor. Since $F^* : \widehat{\mathcal{C}'} \to \widehat{\mathcal{C}}$ and $F_* : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}'}$ preserves limits, we have functors $F^*_\mathcal{T} : \mathcal{T}(\widehat{\mathcal{C}'}) \to \mathcal{T}(\widehat{\mathcal{C}})$ and $F_{*\mathcal{T}} : \mathcal{T}(\widehat{\mathcal{C}}) \to \mathcal{T}(\widehat{\mathcal{C}'})$. If $\mathcal{C}$ has finite limits and F preserves them, we also have a functor $F_{!\mathcal{T}} : \mathcal{T}(\widehat{\mathcal{C}}) \to \mathcal{T}(\widehat{\mathcal{C}'})$ ((A.6.12)). The following result is straightforward from (A.11.19) and (A.6.12).

Proposition A.11.22 *1) $F_{*\mathcal{T}} : \mathcal{T}(\widehat{\mathcal{C}}) \to \mathcal{T}(\widehat{\mathcal{C}'})$ is a right adjoint of $F^*_\mathcal{T} : \mathcal{T}(\widehat{\mathcal{C}'}) \to \mathcal{T}(\widehat{\mathcal{C}})$.*

2) If $\mathcal{C}$ has finite limits and F preserves them, $F_{!\mathcal{T}} : \mathcal{T}(\widehat{\mathcal{C}}) \to \mathcal{T}(\widehat{\mathcal{C}'})$ is a left adjoint of $F^*_\mathcal{T} : \mathcal{T}(\widehat{\mathcal{C}'}) \to \mathcal{T}(\widehat{\mathcal{C}})$ and the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{T}(\mathcal{C}) & \xrightarrow{F_\mathcal{T}} & \mathcal{T}(\mathcal{C}') \\
\downarrow{\scriptstyle h_\mathcal{T}} & & \downarrow{\scriptstyle h_\mathcal{T}} \\
\mathcal{T}(\widehat{\mathcal{C}}) & \xrightarrow{F_{!\mathcal{T}}} & \mathcal{T}(\widehat{\mathcal{C}'})
\end{array}$$

Since the forgetful functor $\widetilde{U}_\mathcal{T} : \mathcal{T}(\mathcal{C}; T_0, F_0) \to \mathcal{C}^{k-m}$ is faithful and reflects isomorphisms (A.11.7), we have the following result by (A.4.15).

Proposition A.11.23 *Let $\mathcal{C}$ be a category and G a set of object of $\mathcal{C}$. et $G_U = \{\widetilde{U}_\mathcal{T}(X_1, X_2, \dots, X_{k-m}) | \, X_i \in G\}$. Suppose that the forgetful functor $\widetilde{U}_\mathcal{T} : \mathcal{T}(\mathcal{C}; T_0, F_0) \to \mathcal{C}^{k-m}$ has a left adjoint. If G is a generator of $\mathcal{C}$ by epimorphisms (resp. a generator of $\mathcal{C}$, a generator of $\mathcal{C}$ for monomorphisms, a generator of $\mathcal{C}$ for strict monomorphisms) then so is G_U of $\mathcal{T}(\mathcal{C}; T_0, F_0)$.*

In order to construct finitary algebraic theories, we make some preparations. Let $\mathcal{C}$ be a category with a set Γ of objects of $\mathcal{C}$ having the following properties.

(1) For each finite family $(X_i)_{1\le i\le n}$ of elements of Γ, the coproduct $\coprod_{i=1}^{n} X_i$ exists.

(2) For each $X \in \operatorname{Ob}\mathcal{C}$, there exists a finite family $(X_i)_{1\le i\le n}$ of elements of Γ such that $X = \coprod_{i=1}^{n} X_i$.

(3) If $\coprod_{i=1}^{n} X_i = \coprod_{j=1}^{m} Y_j$ for $X_i, Y_j \in \Gamma$ ($1 \le i \le n$, $1 \le j \le m$), then $n = m$ and there exists a bijection $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ such that $Y_j = X_{\sigma(j)}$ for $1 \le j \le n$.

It follows from (1) and (2) that $\mathcal{C}$ is a small category and it has finite coproducts.

Suppose that a set $M(X, Y)$ is given for each $X \in \Gamma$ and $Y \in \operatorname{Ob}\mathcal{C}$. We put $M^*(X, Y) = M(X, Y)$ if $X \neq Y$ and $M^*(X, X) = M(X, X) \cup \{id_X\}$ (disjoint union). Consider a graph $\mathcal{G}$ defined by $\operatorname{Ob}\mathcal{G} = \operatorname{Ob}\mathcal{C}$ and $\mathcal{G}(X, Y) = \bigcup_{Z\in \operatorname{Ob}\mathcal{C}}(\mathcal{C}(Z, Y) \times M^*(X, Z))$ (disjoint union) if $X \in \Gamma$, $\mathcal{G}(X, Y) = \prod_{i=1}^{n} \mathcal{G}(X_i, Y)$ if $X = \coprod_{i=1}^{n} X_i$ ($X_i \in \Gamma$). We regard $\mathcal{C}(X, Y)$ as a subset of $\mathcal{G}(X, Y)$ by the map $\mathcal{C}(X, Y) \to \mathcal{G}(X, Y)$ $f \mapsto ((f\nu_i, id_{X_i}))_{1\le i\le n}$ if $X = \coprod_{i=1}^{n} X_i$ ($X_i \in \Gamma$), where $\nu_i : X_i \to X$ is the canonical morphism into the i-th summand.

Suppose that $X = \coprod_{i=1}^{n} X_i$, $Y = \coprod_{j=1}^{m} Y_j$ ($X_i, Y_j \in \Gamma$) and $f = ((s_i, t_i))_{1\le i\le n} \in \mathcal{G}(X, Y)$, $g = ((u_j, v_j))_{1\le j\le m} \in \mathcal{G}(Y, Z)$ ($s_i \in \mathcal{C}(Z_i, Y)$, $t_i \in M^*(X_i, Z_i)$, $u_j \in \mathcal{C}(W_j, Z)$, $v_j \in M^*(Y_j, W_j)$). We say that f and g are composable if one of the following conditions is satisfied.

(1) $W_j = Y_j$ and $v_j = id_{Y_j}$ for all $1 \le j \le m$.

(2) For each $1 \le i \le n$, there exists $1 \le j(i) \le m$ such that $X_i = Z_i = Y_{j(i)}$ and $t_i = id_{Y_{j(i)}}$, moreover, $s_i : Y_{j(i)} \to Y$ is the canonical morphism into the $j(i)$-th summand.

If f and g are composable, we define the composition $gf \in \mathcal{G}(X, Z)$ by $gf = ((us_i, t_i))_{1\le i\le n}$ ($u = (u_j)_{1\le j\le m} \in \mathcal{C}(Y, Z)$) if the condition (1) holds, and $gf = ((u_{j(i)}, t_{j(i)}))_{1\le i\le n}$ if the condition (2) holds. In particular, if $Z = Y$, $W_j = Y_j$, $v_j = id_{Y_j}$ and $u_j : Y_j \to Y$ is the canonical morphism into the j-th summand for each $1 \le j \le m$, we put $id_Y = ((u_j, id_{Y_j}))_{1\le j\le m} \in \mathcal{G}(Y, Y)$. Then, for any $f \in \mathcal{G}(X, Y)$, f and id_Y, id_X and f are composable and we have $id_Y f = f id_X = f$.

Let $(X_i)_{1\le i\le n}$ and $(Y_i)_{1\le i\le n}$ be families of objects of $\mathcal{C}$ and Y an object of $\mathcal{C}$. Define maps $\pi : \prod_{i=1}^{n} \mathcal{G}(X_i, Y) \to \mathcal{G}\Big(\coprod_{i=1}^{n} X_i, Y\Big)$ and $\coprod : \prod_{i=1}^{n} \mathcal{G}(X_i, Y_i) \to \mathcal{G}\Big(\coprod_{i=1}^{n} X_i, \coprod_{i=1}^{n} Y_i\Big)$ as follows. Suppose $X_i = \coprod_{j=1}^{m_i} X_i^j$ ($X_i^j \in \Gamma$) and that, for $(f_i)_{1\le i\le n} \in \prod_{i=1}^{n} \mathcal{G}(X_i, Y) = \prod_{i=1}^{n}\prod_{j=1}^{m_i} \mathcal{G}(X_i^j, Y)$, $f_i = ((g_i^j, h_i^j))_{1\le j\le m_i}$ ($g_i^j \in \mathcal{C}(Z_i^j, Y)$, $h_i^j \in M^*(X_i^j, Z_i^j)$). Put $I = \{(i, j) |\, 1 \le i \le n,\ 1 \le j \le m_i\}$ and $\pi((f_i)_{1\le i\le n}) = ((g_i^j, h_i^j))_{(i,j)\in I} \in \prod_{(i,j)\in I} \mathcal{G}(X_i^j, Y) = \mathcal{G}\Big(\coprod_{i=1}^{n} X_i, Y\Big)$. We denote by $\nu_i : Y_i \to \coprod_{i=1}^{n} Y_i$ the canonical morphism into the i-th summand. For $(f_i)_{1\le i\le n} \in \prod_{i=1}^{n} \mathcal{G}(X_i, Y_i) = \prod_{i=1}^{n}\prod_{j=1}^{m_i} \mathcal{G}(X_i^j, Y_i)$, suppose $f_i = ((g_i^j, h_i^j))_{1\le j\le m_i}$ ($g_i^j \in \mathcal{C}(Z_i^j, Y_i)$, $h_i^j \in M^*(X_i^j, Z_i^j)$). We set $\coprod_{i=1}^{n} f_i =$ $\pi((\nu_i f_i)_{1\le i\le n})$, where $\nu_i f_i = ((\nu_i g_i^j, h_i^j))_{1\le j\le m_i}$. Then the assignment $(f_i)_{1\le i\le n} \mapsto \coprod_{i=1}^{n} f_i$ gives a map $\coprod : \prod_{i=1}^{n} \mathcal{G}(X_i, Y_i) \to \mathcal{G}\Big(\coprod_{i=1}^{n} X_i, \coprod_{i=1}^{n} Y_i\Big)$.

For $X \in \Gamma$, $Y \in \operatorname{Ob}\mathcal{G} = \operatorname{Ob}\mathcal{C}$, we set

$$W(X, Y) = \bigcup_{n\ge 1}\Bigg(\bigcup_{X_1,\dots,X_{n-1}\in \operatorname{Ob}\mathcal{G}} \mathcal{G}(X_{n-1}, Y) \times \mathcal{G}(X_{n-2}, X_{n-1}) \times \cdots \times \mathcal{G}(X_1, X_2) \times \mathcal{G}(X, X_1)\Bigg),$$

where both unions are disjoint. An element of $W(X, Y)$ is called a word.

Suppose that a subset $R(X, Y)$ of $W(X, Y) \times W(X, Y)$ is given for each $X \in \Gamma$ and $Y \in \operatorname{Ob}\mathcal{C}$. Let $R^*(X, Y)$ be the smallest equivalence relation on $W(X, Y)$ satisfying the following conditions.

(1) $R(X, Y) \subset R^*(X, Y)$.

(2) If f_{i-1} and f_i are composable for some $2 \leq i \leq n$, $(f_n, \ldots, f_i, f_{i-1}, \ldots, f_1) \in \mathcal{G}(X_{n-1}, Y) \times \cdots \times \mathcal{G}(X_{i-1}, X_i) \times \mathcal{G}(X_{i-2}, X_{i-1}) \times \cdots \times \mathcal{G}(X, X_1)$ is equivalent to $(f_n, \ldots, f_i f_{i-1}, \ldots, f_1) \in \mathcal{G}(X_{n-1}, Y) \times \cdots \times \mathcal{G}(X_{i-2}, X_i) \times \cdots \times \mathcal{G}(X, X_1)$.

(3) Suppose $X_{i-1} = \coprod_{j=1}^{m} Z_j$, $X_{i-2} = \coprod_{j=1}^{m} W_j$ and $f_i = \pi((f_i^j))_{1 \leq j \leq m}$, $f_{i-1} = \coprod_{j=1}^{m} f_{i-1}^j$ ($f_i^j \in \mathcal{G}(Z_j, X_i)$, $f_{i-1}^j \in \mathcal{G}(W_j, Z_j)$). If $f_{i-1}^{j_0}$ and $f_i^{j_0}$ are composable for some $1 \leq j_0 \leq m$, we set $X'_{i-1} = \left(\coprod_{j \neq j_0} Z_j\right) \coprod W_{j_0}$, $X''_{i-1} = \left(\coprod_{j \neq j_0} Z_j\right) \coprod X_i$ and $f'_{i-1} = \left(\coprod_{j \neq j_0} f_{i-1}^j\right) \coprod id_{W_{j_0}} \in \mathcal{G}(X_{i-2}, X'_{i-1})$, $f'_i = \pi((f_i^j, f_i^{j_0} f_{i-1}^{j_0})_{j \neq j_0}) \in \mathcal{G}(X'_{i-1}, X_i)$, $f''_{i-1} = \left(\coprod_{j \neq j_0} f_{i-1}^j\right) \coprod f_i^{j_0} f_{i-1}^{j_0} \in \mathcal{G}(X_{i-2}, X''_{i-1})$, $f''_i = \pi((f_i^j, id_{X_i})_{j \neq j_0}) \in \mathcal{G}(X''_{i-1}, X_i)$. Then, $(f_n, \ldots, f_i, f_{i-1}, \ldots, f_1) \in \mathcal{G}(X_{n-1}, Y) \times \cdots \times \mathcal{G}(X_{i-1}, X_i) \times \mathcal{G}(X_{i-2}, X_{i-1}) \times \cdots \times \mathcal{G}(X, X_1)$ is equivalent to both $(f_n, \ldots, f'_i, f'_{i-1}, \ldots, f_1) \in \mathcal{G}(X_{n-1}, Y) \times \cdots \times \mathcal{G}(X'_{i-1}, X_i) \times \mathcal{G}(X_{i-2}, X'_{i-1}) \times \ldots \times \mathcal{G}(X, X_1)$ and $(f_n, \ldots, f''_i, f''_{i-1}, \ldots, f_1) \in \mathcal{G}(X_{n-1}, Y) \times \cdots \times \mathcal{G}(X''_{i-1}, X_i) \times \mathcal{G}(X_{i-2}, X''_{i-1}) \times \cdots \times \mathcal{G}(X, X_1)$.

(4) Suppose $X_{i-1} = \coprod_{j=1}^{m} Z_j$, $X_{i-2} = \coprod_{j=1}^{m} W_j$, $X_i = \coprod_{j=1}^{m} V_j$ and $f_i = \coprod_{j=1}^{m} f_i^j$, $f_{i-1} = \coprod_{j=1}^{m} f_{i-1}^j$ ($f_i^j \in \mathcal{G}(Z_j, V_j)$, $f_{i-1}^j \in \mathcal{G}(W_j, Z_j)$). If $f_{i-1}^{j_0}$ and $f_i^{j_0}$ are composable for some $1 \leq j_0 \leq m$, we set $X'_{i-1} = \left(\coprod_{j \neq j_0} Z_j\right) \coprod W_{j_0}$, $X''_{i-1} = \left(\coprod_{j \neq j_0} Z_j\right) \coprod V_{j_0}$ and $f'_{i-1} = \left(\coprod_{j \neq j_0} f_{i-1}^j\right) \coprod id_{W_{j_0}} \in \mathcal{G}(X_{i-2}, X'_{i-1})$, $f'_i = \left(\coprod_{j \neq j_0} f_i^j\right) \coprod f_i^{j_0} f_{i-1}^{j_0} \in \mathcal{G}(X'_{i-1}, X_i)$, $f''_{i-1} = \left(\coprod_{j \neq j_0} f_{i-1}^j\right) \coprod f_i^{j_0} f_{i-1}^{j_0} \in \mathcal{G}(X_{i-2}, X''_{i-1})$, $f''_i = \left(\coprod_{j \neq j_0} f_i^j\right) \coprod id_{V_{j_0}} \in \mathcal{G}(X''_{i-1}, X_i)$. Then, $(f_n, \ldots, f_i, f_{i-1}, \ldots, f_1) \in \mathcal{G}(X_{n-1}, Y) \times \cdots \times \mathcal{G}(X_{i-1}, X_i) \times \mathcal{G}(X_{i-2}, X_{i-1}) \times \cdots \times \mathcal{G}(X, X_1)$ is equivalent to both $(f_n, \ldots, f'_i, f'_{i-1}, \ldots, f_1) \in \mathcal{G}(X_{n-1}, Y) \times \cdots \times \mathcal{G}(X'_{i-1}, X_i) \times \mathcal{G}(X_{i-2}, X'_{i-1}) \times \ldots \times \mathcal{G}(X, X_1)$ and $(f_n, \ldots, f''_i, f''_{i-1}, \ldots, f_1) \in \mathcal{G}(X_{n-1}, Y) \times \cdots \times \mathcal{G}(X''_{i-1}, X_i) \times \mathcal{G}(X_{i-2}, X''_{i-1}) \times \cdots \times \mathcal{G}(X, X_1)$.

We construct a category $\mathcal{C}^*$ below. The set of objects of $\mathcal{C}^*$ is the same as that of $\mathcal{C}$. For $X, Y \in \mathrm{Ob}\,\mathcal{C}^*$, we put $\mathcal{C}^*(X, Y) = W(X, Y)/R^*(X, Y)$ if $X \in \Gamma$, and $\mathcal{C}^*(X, Y) = \prod_{i=1}^{n} \mathcal{C}^*(X_i, Y)$ if $X = \coprod_{i=1}^{n} X_i$ ($X_i \in \Gamma$). If $X \in \Gamma$, $Y = \coprod_{j=1}^{m} Y_j$ ($Y_j \in \Gamma$) and $(g, f) \in \mathcal{C}^*(Y, Z) \times \mathcal{C}^*(X, Y)$ ($g = (g_j)_{1 \leq j \leq m}$, $g_j \in \mathcal{C}^*(Y_j, Z)$), let $(f_n, f_{n-1}, \ldots, f_1) \in \mathcal{G}(X_{n-1}, Y) \times \mathcal{G}(X_{n-2}, X_{n-1}) \times \cdots \times \mathcal{G}(X, X_1)$ and $(g^j_{k_j}, g^j_{k_j-1}, \ldots, g^j_1) \in \mathcal{G}(Y^j_{k_j-1}, Z) \times \mathcal{G}(Y^j_{k_j-2}, Y^j_{k_j-1}) \times \cdots \times \mathcal{G}(Y_j, Y^j_1)$ be words representing f and g_j, respectively. By inserting id_Z on the left of the word representing g_j, we may assume that $k_1 = k_2 = \cdots = k_m = k$. We define the composition $gf \in \mathcal{C}^*(X, Z)$ to be the element represented by a word

$$\left(\pi((g^j_k)_{1 \leq j \leq m}), \coprod_{j=1}^{m} g^j_{k-1}, \ldots, \coprod_{j=1}^{m} g^j_1, f_n, f_{n-1}, \ldots, f_1\right) \in \mathcal{G}\left(\coprod_{j=1}^{m} Y^j_{k-1}, Z\right) \times \mathcal{G}\left(\coprod_{j=1}^{m} Y^j_{k-2}, \coprod_{j=1}^{m} Y^j_{k-1}\right) \times \cdots$$
$$\cdots \times \mathcal{G}\left(Y, \coprod_{j=1}^{m} Y^j_1\right) \times \mathcal{G}(X_{n-1}, Y) \times \mathcal{G}(X_{n-2}, X_{n-1}) \times \cdots \times \mathcal{G}(X, X_1).$$

It is easy to see that this composition is well-defined. For a general $X \in \mathrm{Ob}\,\mathcal{C}^*$, we define the composition of $f = (f_l)_{1 \leq l \leq r} \in \mathcal{C}^*(X, Y) = \prod_{l=1}^{r} \mathcal{C}^*(X_l, Y)$ ($X = \coprod_{l=1}^{r} X_l$) and $g \in \mathcal{C}^*(Y, Z)$ componentwise, namely, $gf = (gf_l)_{1 \leq l \leq r} \in \mathcal{C}^*(X, Y) = \prod_{l=1}^{r} \mathcal{C}^*(X_l, Z)$.

Thus we have a category $\mathcal{C}^*$ with finite coproducts and a coproduct preserving functor $\mathcal{C} \to \mathcal{C}^*$ which is the identity map on the set of objects. We apply this construction to define finitary algebraic theories starting from the trivial finitary algebraic theories.

Example A.11.24 *We define 1-fold finitary theories $(\mathcal{T}_{mon}, \omega)$, $(\mathcal{T}_{gr}, \omega)$, $(\mathcal{T}_{ab}, \omega)$ and $(\mathcal{T}_{an}, \omega)$ as follows.*

1) We set $M_{mon}([1], [0]) = \{e\}$, $M_{mon}([1], [2]) = \{\alpha\}$ and $M_{mon}([1], [n]) = \phi$ if $n \neq 0, 2$. We also denote by e, α the elements $(id_{[0]}, e) \in \mathcal{G}([1], [0])$, $(id_{[2]}, \alpha) \in \mathcal{G}([1], [2])$. Let $R_{mon}([1], [n])$ be a set of pairs of words given by $R_{mon}([1], [n]) = \phi$ if $n \neq 1, 3$ and

$R_{mon}([1], [1]) = \{((e \coprod id_{[1]}, \alpha), (id_{[1]})), ((id_{[1]} \coprod e, \alpha), (id_{[1]}))\}$,

$R_{mon}([1], [3]) = \{((\alpha \coprod id_{[1]}, \alpha), (id_{[1]} \coprod \alpha, \alpha))\}$.

We denote by $\mathcal{T}_{mon}$ the category $\mathcal{N}^*$ obtained from the 1-fold trivial finitary algebraic theory $\mathcal{N}$.

2) We set $M_{gr}([1], [1]) = \{\iota\}$ and $M_{gr}([1], [n]) = M_{mon}([1], [n])$ if $n \neq 1$. We also denote by ι the element $(id_{[1]}, \iota) \in \mathcal{G}([1], [1])$. Let $R_{gr}([1], [n])$ be a set of pairs of words given by $R_{gr}([1], [n]) = R_{mon}([1], [n])$ if $n \neq 1$

and $R_{gr}([1],[1]) = R_{mon}([1],[1]) \cup \{((\delta, \iota \coprod id_{[1]}, \alpha), (0, e)), ((\delta, id_{[1]} \coprod \iota, \alpha), (0, e))\}$*, where* $0 : [0] \to [1]$ *and* $\delta : [2] \to [1]$ *are unique morphisms in* $\mathcal{N}$*. We denote by* $\mathcal{T}_{gr}$ *the category* $\mathcal{N}^*$ *obtained from the 1-fold trivial finitary algebraic theory* $\mathcal{N}$*.*

3) We set $M_{ab}([1],[n]) = M_{gr}([1],[n])$ *for* $n \geq 0$*. We denote by* $\tau : [2] \to [2]$ *the morphism in* $\mathcal{N}$ *given by* $\tau(1) = 2$*,* $\tau(2) = 1$*. Let* $R_{ab}([1],[n])$ *be a set of pairs of words given by* $R_{ab}([1],[n]) = R_{gr}([1],[n])$ *if* $n \neq 2$ *and* $R_{ab}([1],[2]) = \{((\tau, \alpha), (\alpha))\}$*. We denote by* $\mathcal{T}_{ab}$ *the category* $\mathcal{N}^*$ *obtained from the 1-fold trivial finitary algebraic theory* $\mathcal{N}$*.*

4) We set $M_{an}([1],[0]) = \{e, u\}$*,* $M_{an}([1],[2]) = \{\alpha, \mu\}$*,* $M_{an}([1],[n]) = M_{ab}([1],[n])$ *for* $n \neq 0, 2$*. We also denote by* u*,* μ *the elements* $(id_{[0]}, u) \in \mathcal{G}([1],[0])$*,* $(id_{[2]}, \mu) \in \mathcal{G}([1],[2])$*. Let* $R_{an}([1],[n])$ *be a set of pairs of words given by* $R_{an}([1],[n]) = R_{ab}([1],[n])$ *if* $n \neq 1, 2, 3$ *and*

$R_{an}([1],[1]) = \{((u \coprod id_{[1]}, \mu), (id_{[1]})), ((id_{[1]} \coprod u, \mu), (id_{[1]}))\} \cup R_{ab}([1],[1])$,

$R_{an}([1],[2]) = \{((\tau, \mu), (\mu))\} \cup R_{ab}([1],[2])$,

$R_{an}([1],[3]) = \{((\mu \coprod id_{[1]}, \mu), (id_{[1]} \coprod \mu, \mu)), ((\sigma, \mu \coprod \mu, \alpha), (id_{[1]} \coprod \alpha, \mu))\} \cup R_{ab}([1],[3])$,

where $\sigma : [4] \to [3]$ *is the morphism in* $\mathcal{N}$ *given by* $\sigma(1) = \sigma(3) = 1$*,* $\sigma(2) = 2$*,* $\sigma(4) = 3$*. We denote by* $\mathcal{T}_{an}$ *the category* $\mathcal{N}^*$ *obtained from the 1-fold trivial finitary algebraic theory* $\mathcal{N}$*.*

For a category $\mathcal{C}$, categories $\mathcal{T}_{mon}(\mathcal{C})$, $\mathcal{T}_{gr}(\mathcal{C})$, $\mathcal{T}_{ab}(\mathcal{C})$ and $\mathcal{T}_{an}(\mathcal{C})$ are called the category of internal monoids, groups, abelian groups and commutative rings in $\mathcal{C}$, respectively.

For a morphism $\theta : \coprod_{s=1}^{k} [p_s]_s \to \coprod_{s=1}^{k} [q_s]_s$ in a k-fold finitary algebraic theory $\mathcal{T}$, we denote by $\theta^n : \coprod_{s=1}^{k} [np_s]_s \to \coprod_{s=1}^{k} [nq_s]_s$ the n-fold coproduct of θ.

Lemma A.11.25 *For any morphism* $\zeta : [m] \to [n]$ *in* $\mathcal{T}_{ab}$*, the following diagrams commutes.*

$$\begin{tikzcd} {[m]} \arrow[r, "\zeta"] \arrow[dr, "e_{[m]}"'] & {[n]} \arrow[d, "e_{[n]}"] \\ & 0 \end{tikzcd} \qquad \begin{tikzcd} {[m]} \arrow[r, "\zeta"] \arrow[d, "\iota_{[m]}"] & {[n]} \arrow[d, "\iota_{[n]}"] \\ {[m]} \arrow[r, "\zeta"] & {[n]} \end{tikzcd} \qquad \begin{tikzcd} {[m]} \arrow[r, "\zeta"] \arrow[d, "\alpha_{[m]}"] & {[n]} \arrow[d, "\alpha_{[n]}"] \\ {[m] \coprod [m]} \arrow[r, "\zeta \coprod \zeta"] & {[n] \coprod [n]} \end{tikzcd}$$

Proof. If the assertion holds for $\zeta : [m] \to [n]$ and $\xi : [n] \to [l]$, it also holds for $\xi\zeta : [m] \to [l]$. Moreover, if the assertion holds for $\zeta\nu_i : [1] \to [n]$ ($\nu_i : [1] \to [m]$ the canonical morphism into the i-th summand) for each $i = 1, 2, \ldots, m$, it holds for $\zeta : [m] \to [n]$. Hence it suffices to show the assertion for a morphism with domain $[1]$ represented by a single word. Since $M_{ab}([1], 0) = \{e\}$, $M_{ab}([1],[1]) = \{\iota\}$, $M_{ab}([1],[2]) = \{\alpha\}$ and $M_{ab}([1],[n]) = \phi$ if $n \neq 0, 2$, it suffices to verify the assertion for $\zeta = e, \iota, \alpha$ and ν_i.

For an object $[n]$ of $\mathcal{T}_{ab}$, let $h_{[n]} : \mathcal{T}_{ab}^{op} \to \mathbf{Ens}$ be the functor represented by $[n]$, then $h_{[n]}([1])$ is an abelian group with unit $h_{[n]}(e) : 1 = h_{[n]}(0) \to h_{[n]}([1])$, addition $h_{[n]}(\alpha) : h_{[n]}([1]) \times h_{[n]}([1]) = h_{[n]}([2]) \to h_{[n]}([1])$, inverse $h_{[n]}(\iota) : h_{[n]}([1]) \to h_{[n]}([1])$. Moreover, $h_{[n]}(\nu_i) : h_{[n]}([1])^m = h_{[n]}([m]) \to h_{[n]}([1])$ is the projection onto the i-th component. Hence the following diagrams commutes for $\zeta = e, \iota, \alpha$ and ν_i, and this proves the assertion.

$$\begin{tikzcd} 1 \arrow[d, "h_{[n]}(e_{[m]})"'] \arrow[dr, "h_{[n]}(e_{[1]})"] & \\ h_{[n]}([m]) \arrow[r, "h_{[n]}(\zeta)"] & h_{[n]}([1]) \end{tikzcd} \qquad \begin{tikzcd} h_{[n]}([m]) \arrow[r, "h_{[n]}(\zeta)"] \arrow[d, "h_{[n]}(\iota_{[m]})"] & h_{[n]}([1]) \arrow[d, "h_{[n]}(\iota_{[1]})"] \\ h_{[n]}([m]) \arrow[r, "h_{[n]}(\zeta)"] & h_{[n]}([1]) \end{tikzcd}$$

$$\begin{tikzcd} h_{[n]}([m]) \times h_{[n]}([m]) \arrow[r, "h_{[n]}(\zeta) \times h_{[n]}(\zeta)"] \arrow[d, "h_{[n]}(\alpha_{[m]})"] & h_{[n]}([1]) \times h_{[n]}([1]) \arrow[d, "h_{[n]}(\alpha_{[1]})"] \\ h_{[n]}([m]) \arrow[r, "h_{[n]}(\zeta)"] & h_{[n]}([1]) \end{tikzcd}$$

□

Example A.11.26 *We construct a 2-fold finitary algebraic theory* $\mathcal{T}_{mod}$ *as follows. Let* $\mathcal{C}$ *be the product of* $\mathcal{T}_{an}$ *and* $\mathcal{T}_{ab}$ *and put* $[n]_1 = ([n], 0)$*,* $[n]_2 = (0, [n])$*,* $u_1 = (u, id_0), e_1 = (e, id_0) \in \mathcal{C}([1]_1, 0)$*,* $\iota_1 = (\iota, id_0) \in \mathcal{C}([1]_1, [1]_1)$*,* $\mu_1 = (\mu, id_0), \alpha_1 = (\alpha, id_0) \in \mathcal{C}([1]_1, [2]_1)$*,* $e_2 = (id_0, e) \in \mathcal{C}([1]_2, 0)$*,* $\iota_2 = (id_0, \iota) \in \mathcal{C}([1]_2, [1]_2)$*,* $\alpha_2 = (id_0, \alpha) \in \mathcal{C}([1]_2, [2]_2)$*. We set* $M_{mod}([1]_2, [1]_1 \coprod [1]_2) = \{\varphi\}$ *and* $M_{mod}([m]_1 \coprod [n]_2, [k]_1 \coprod [l]_2) = \phi$ *for*

$(m,n,k,l) \neq (0,1,1,1)$. We also denote by φ the element $(id_{[1]_1 \coprod [1]_2}, \varphi) \in \mathcal{G}([1]_1, 0)$.
Let $R_{mod}([1]_s, [m]_1 \coprod [n]_2)$ be a set of pairs of words given by

$R_{mod}([1]_2,[1]_2) = \{((u \coprod id_{[1]_2}, \varphi), (id_{[1]_2}))\}$,

$R_{mod}([1]_2,[2]_1 \coprod [1]_2) = \{((id_{[1]_1} \coprod \varphi, \varphi), (\mu_1 \coprod id_{[1]_2}, \varphi)), ((\delta_2, \varphi \coprod \varphi, \alpha_2), (\alpha_1 \coprod id_{[1]_2}, \varphi))\}$,

$R_{mod}([1]_2,[1]_1 \coprod [2]_2) = \{((\delta_1, \varphi \coprod \varphi, \alpha_2), (id_{[1]_1} \coprod \alpha_2, \varphi))\}$,

where $\delta_s \in \mathcal{C}([2]_s, [1]_s)$ $(s = 1, 2)$ are morphisms induced by the unique map $[2] \to [1]$ in $\mathcal{N}$. We denote by $\mathcal{T}_{mod}$ the category $\mathcal{C}^*$.

Define functors $T_{an} : \mathcal{T}_{an} \to \mathcal{T}_{mod}$ and $T_{ab} : \mathcal{T}_{ab} \to \mathcal{T}_{mod}$ to be the compositions $\mathcal{T}_{an} \xrightarrow{T_1} \mathcal{C} \to \mathcal{C}^*$, $\mathcal{T}_{ab} \xrightarrow{T_2} \mathcal{C} \to \mathcal{C}^*$.

Since $\mathcal{G}([1]_1, [p]_1 \coprod [q]_2) = \{\nu_1 T_1(\xi) \mid \xi \in \mathcal{T}_{an}([1],[p])\}$ by the construction, the following fact is easily verified.

Proposition A.11.27 *$T_{an} : \mathcal{T}_{an} \to \mathcal{T}_{mon}$ satisfies the conditions of (A.11.6).*

For a category $\mathcal{C}$ and an internal commutative ring A in $\mathcal{C}$, we call $\mathcal{T}_{mon}(\mathcal{C}; T_{an}, A)$ the category of internal A-modules in $\mathcal{C}$.

Let $T_0 : \mathcal{T}_0 \to \mathcal{T}$ be a morphism of finitary algebraic theories as in (A.11.4) and $\{\bar\sigma_1, \bar\sigma_2, \dots, \bar\sigma_{k-m}\}$ $(\bar\sigma_1 < \bar\sigma_2 < \cdots < \bar\sigma_{k-m})$ the complement of the image of σ. Consider the $(k-m)$-fold product $\mathcal{T}_{ab}^{k-m}$ of $\mathcal{T}_{ab}$ and the morphism $T_s : \mathcal{T}_{ab} \to \mathcal{T}_{ab}^{k-m}$ into the s-th factor (A.11.2). Suppose that there is a morphism $T : \mathcal{T}_{ab}^{k-m} \to \mathcal{T}$ of finitary algebraic theories such that $T([1]_s) = [1]_{\bar\sigma_s}$ $(1 \le s \le k-m)$. We set $e_s = TT_s(e) : [1]_s \to 0$, $\iota_s = TT_s(\iota) : [1]_s \to [1]_s$ and $\alpha_s = TT_s(\alpha) : [1]_s \to [2]_s$ and denote by $\delta_s \in \mathcal{T}([2]_s, [1]_s)$ the morphism $\omega_s(\delta)$ induced by the unique map $\delta : \langle 2 \rangle \to \langle 1 \rangle$ in $\mathcal{N}$. For $n_{\bar\sigma} = \coprod_{s=1}^{k-m} [n_{\bar\sigma_s}]_{\bar\sigma_s} \in \operatorname{Ob}\mathcal{T}$, we denote by $e_{n_{\bar\sigma}} : n_{\bar\sigma} \to 0$, $\iota_{n_{\bar\sigma}} : n_{\bar\sigma} \to n_{\bar\sigma}$ and $\alpha_{n_{\bar\sigma}} : n_{\bar\sigma} \to n_{\bar\sigma} \coprod n_{\bar\sigma}$ the morphisms $\coprod_{s=1}^{k-m} e_s^{n_{\bar\sigma_s}}$, $\coprod_{s=1}^{k-m} \iota_s^{n_{\bar\sigma_s}}$ and $\coprod_{s=1}^{k-m} \alpha_s^{n_{\bar\sigma_s}}$ respectively, where φ^r denotes the r-fold coproduct of φ. Similarly, for $n = \coprod_{r=1}^{k} [n_r]_r \in \operatorname{Ob}\mathcal{T}$, we denote by $\delta_n : n \coprod n \to n$ the morphism $\coprod_{r=1}^{k} \delta_r^{n_r}$.

Proposition A.11.28 *Suppose that T_0 satisfies the conditions of (A.11.6) and that there is a morphism $T : \mathcal{T}_{ab}^{k-k_0} \to \mathcal{T}$ of finitary algebraic theories such that $T([1]_s) = [1]_{\bar\sigma_s}$ $(1 \le s \le k-k_0)$ and the following diagrams commute for any $n = \coprod_{r=1}^{k} [n_r]_r \in \operatorname{Ob}\mathcal{T}$, $1 \le s \le k-k_0$ and morphism $\theta : [1]_{\bar\sigma_s} \to n$.*

$$\begin{tikzcd}
{[1]_{\bar\sigma_s}} \arrow[r, "\theta"] \arrow[d, "e_s"] & T_0(n_\sigma) \coprod n_{\bar\sigma_s} \arrow[d, "id_{T_0(n_\sigma)} \coprod e_{n_{\bar\sigma}}"] \\
0 \arrow[r] & T_0(n_\sigma) \coprod 0
\end{tikzcd}
\qquad
\begin{tikzcd}
{[1]_{\bar\sigma_s}} \arrow[r, "\theta"] \arrow[d, "\iota_s"] & T_0(n_\sigma) \coprod n_{\bar\sigma_s} \arrow[d, "id_{T_0(n_\sigma)} \coprod \iota_{n_{\bar\sigma}}"] \\
{[1]_{\bar\sigma_s}} \arrow[r, "\theta"] & T_0(n_\sigma) \coprod n_{\bar\sigma_s}
\end{tikzcd}$$

$$\begin{tikzcd}
{[1]_{\bar\sigma_s}} \arrow[r, "\alpha_s"] \arrow[d, "\theta"] & {[1]_{\bar\sigma_s} \coprod [1]_{\bar\sigma_s}} \arrow[r, "\theta \coprod \theta"] & (T_0(n_\sigma) \coprod n_{\bar\sigma_s}) \coprod (T_0(n_\sigma) \coprod n_{\bar\sigma_s}) \arrow[d, "\delta_{T_0(n_\sigma)} \coprod id_{n_{\bar\sigma}} \coprod id_{n_{\bar\sigma}}"] \\
T_0(n_\sigma) \coprod n_{\bar\sigma_s} \arrow[rr, "id_{T_0(n_\sigma)} \coprod \alpha_{n_{\bar\sigma}}"] & & T_0(n_\sigma) \coprod (n_{\bar\sigma_s} \coprod n_{\bar\sigma_s})
\end{tikzcd}$$

Here we put $n_\sigma = \coprod_{s=1}^{k_0} [n_{\sigma(s)}]_s \in \operatorname{Ob}\mathcal{T}_0$ $n_{\bar\sigma} = \coprod_{s=1}^{k-k_0} [n_{\bar\sigma_s}]_{\bar\sigma_s} \in \operatorname{Ob}\mathcal{T}$. Then, for a category $\mathcal{C}$ with finite product and an object F_0 of in $\mathcal{T}_0(\mathcal{C})$, $\mathcal{T}(\mathcal{C}; T_0, F_0)$ is an additive category.

Proof. For objects F, G of $\operatorname{Ob}\mathcal{T}(\mathcal{C}; T_0, F_0)$, we define a structure of an abelian group on $\mathcal{T}(\mathcal{C}; T_0, F_0)(F, G)$ as follows.

Let $0 : F \to G$ be the morphism given by $0_{[1]_{\sigma(s)}} = id_{F_0([1]_s)}$, $0_{[1]_{\bar\sigma_s}} = G(e_s)F(o)$ and $0_{n \coprod m} = 0_n \times 0_m$, where $o : 0 \to [1]_{\bar\sigma_s}$ denotes the unique morphism. For morphisms $f, g \in \mathcal{T}(\mathcal{C}; T_0, F_0)(F, G)$, define $f + g, -f : F \to G$ by $(f+g)_{[1]_{\sigma(s)}} = id_{F_0([1]_s)}$, $(f+g)_{[1]_{\bar\sigma_s}} = G(\alpha_s)(f_{[1]_{\bar\sigma_s}} \times g_{[1]_{\bar\sigma_s}})F(\delta_s)$, $(f+g)_{m \coprod n} = (f+g)_m \times (f+g)_n$ and $(-f)_{[1]_{\sigma(s)}} = id_{F_0([1]_s)}$, $(-f)_{[1]_{\bar\sigma_s}} = G(\iota_s) f_{[1]_{\bar\sigma_s}}$, $(-f)_{m \coprod n} = (-f)_m \times (-f)_n$.

We have to verify the naturality of 0, $f + g$ and $-f$. It suffices to show that the following diagrams commute for a morphism $\theta : [1]_t \to n$ and $t = 1, 2, \dots, k$.

$$\begin{array}{ccc}
\begin{array}{ccc} F(n) & \xrightarrow{F(\theta)} & F([1]_t) \\ \downarrow 0_n & & \downarrow 0_{[1]_t} \\ G(n) & \xrightarrow{G(\theta)} & G([1]_t) \end{array}
&
\begin{array}{ccc} F(n) & \xrightarrow{F(\theta)} & F([1]_t) \\ \downarrow (f+g)_n & & \downarrow (f+g)_{[1]_t} \\ G(n) & \xrightarrow{G(\theta)} & G([1]_t) \end{array}
&
\begin{array}{ccc} F(n) & \xrightarrow{F(\theta)} & F([1]_t) \\ \downarrow (-f)_n & & \downarrow (-f)_{[1]_t} \\ G(n) & \xrightarrow{G(\theta)} & G([1]_t) \end{array}
\end{array}$$

For $n = \coprod_{r=1}^{k} [n_r]_r \in \operatorname{Ob}\mathcal{T}$, we set $n_\sigma = \coprod_{s=1}^{k_0} [n_{\sigma(s)}]_s \in \operatorname{Ob}\mathcal{T}_0$ and $n_{\bar\sigma} = \coprod_{s=1}^{k-k_0} [n_{\bar\sigma_s}]_{\bar\sigma_s}$ then $n = T_0(n_\sigma) \coprod n_{\bar\sigma_s}$. If $t = \sigma(s)$ for some $1 \leq s \leq k_0$, there exists a unique $\theta' : [1]_s \to n_\sigma$ such that $\theta = \nu T_0(\theta')$, where $\nu : T_0(n_\sigma) \to T_0(n_\sigma) \coprod n_{\bar\sigma_s}$ is the canonical morphism in $\mathcal{T}$. Since $0_{T_0(n_\sigma)} = (f+g)_{T_0(n_\sigma)} = (-f)_{T_0(n_\sigma)} = id_{F_0(n_\sigma)}$, $0_{[1]_{\sigma(s)}} = (f+g)_{[1]_{\sigma(s)}} = (-f)_{[1]_{\sigma(s)}} = id_{F_0([1]_s)}$, $FT_0(\theta') = GT_0(\theta') = F_0(\theta')$ and $F(\nu)$, $G(\nu)$ are identified with the projections onto the first factor, the following diagrams are commutative.

$$\begin{array}{ccccccc}
F(T_0(n_\sigma)\coprod n_{\bar\sigma_s}) & = & F(T_0(n_\sigma))\times F(n_{\bar\sigma_s}) & \xrightarrow{F(\nu)} & F(T_0(n_\sigma)) & \xrightarrow{FT_0(\theta')} & F([1]_{\sigma(s)}) \\
\downarrow 0_{T_0(n_\sigma)\coprod n_{\bar\sigma_s}} & & \downarrow 0_{T_0(n_\sigma)}\times 0_{n_{\bar\sigma_s}} & & \downarrow 0_{T_0(n_\sigma)} & & \downarrow 0_{[1]_{\sigma(s)}} \\
G(T_0(n_\sigma)\coprod n_{\bar\sigma_s}) & = & G(T_0(n_\sigma))\times G(n_{\bar\sigma_s}) & \xrightarrow{G(\nu)} & G(T_0(n_\sigma)) & \xrightarrow{GT_0(\theta')} & G([1]_{\sigma(s)})
\end{array}$$

$$\begin{array}{ccccccc}
F(T_0(n_\sigma)\coprod n_{\bar\sigma_s}) & = & F(T_0(n_\sigma))\times F(n_{\bar\sigma_s}) & \xrightarrow{F(\nu)} & F(T_0(n_\sigma)) & \xrightarrow{FT_0(\theta')} & F([1]_{\sigma(s)}) \\
\downarrow (f+g)_{T_0(n_\sigma)\coprod n_{\bar\sigma_s}} & & \downarrow (f+g)_{T_0(n_\sigma)}\times (f+g)_{n_{\bar\sigma_s}} & & \downarrow (f+g)_{T_0(n_\sigma)} & & \downarrow (f+g)_{[1]_{\sigma(s)}} \\
G(T_0(n_\sigma)\coprod n_{\bar\sigma_s}) & = & G(T_0(n_\sigma))\times G(n_{\bar\sigma_s}) & \xrightarrow{G(\nu)} & G(T_0(n_\sigma)) & \xrightarrow{GT_0(\theta')} & G([1]_{\sigma(s)})
\end{array}$$

$$\begin{array}{ccccccc}
F(T_0(n_\sigma)\coprod n_{\bar\sigma_s}) & = & F(T_0(n_\sigma))\times F(n_{\bar\sigma_s}) & \xrightarrow{F(\nu)} & F(T_0(n_\sigma)) & \xrightarrow{FT_0(\theta')} & F([1]_{\sigma(s)}) \\
\downarrow (-f)_{T_0(n_\sigma)\coprod n_{\bar\sigma_s}} & & \downarrow (-f)_{T_0(n_\sigma)}\times (-f)_{n_{\bar\sigma_s}} & & \downarrow (-f)_{T_0(n_\sigma)} & & \downarrow (-f)_{[1]_{\sigma(s)}} \\
G(T_0(n_\sigma)\coprod n_{\bar\sigma_s}) & = & G(T_0(n_\sigma))\times G(n_{\bar\sigma_s}) & \xrightarrow{G(\nu)} & G(T_0(n_\sigma)) & \xrightarrow{GT_0(\theta')} & G([1]_{\sigma(s)})
\end{array}$$

Suppose $t = \bar\sigma_s$ for some $1 \leq s \leq k - k_0$. We note that the following diagram commutes.

$$\begin{array}{ccc}
[1]_{\bar\sigma_s}\coprod [1]_{\bar\sigma_s} & \xrightarrow{\theta\coprod\theta} & n\coprod n \\
\downarrow \delta_{[1]_{\bar\sigma_s}} & & \downarrow \delta_n \\
[1]_{\bar\sigma_s} & \xrightarrow{\theta} & n
\end{array}$$

It follows from the assumption that the following diagrams commute.

$$\begin{array}{ccc}
F(T_0(n_\sigma)\coprod n_{\bar\sigma}) & \xrightarrow{F(\theta)} & F([1]_{\bar\sigma_s}) \\
\downarrow id_{FT_0(n_\sigma)}\times F(o) & & \downarrow F(o) \\
F(T_0(n_\sigma))\times F(0) & \xrightarrow{\mathrm{pr}_2} & F(0) \\
\| & & \| \\
F(T_0(n_\sigma))\times G(0) & \xrightarrow{\mathrm{pr}_2} & G(0) \\
\downarrow id_{GT_0(n_\sigma)}\times G(e_{n_{\bar\sigma}}) & & \downarrow G(e_s) \\
G(T_0(n_\sigma)\coprod n_{\bar\sigma}) & \xrightarrow{G(\theta)} & G([1]_{\bar\sigma_s})
\end{array}
\qquad
\begin{array}{ccc}
F(T_0(n_\sigma)\coprod n_{\bar\sigma}) & \xrightarrow{F(\theta)} & F([1]_{\bar\sigma_s}) \\
\downarrow id_{FT_0(n_\sigma)}\times f_{n_{\bar\sigma}} & & \\
F(T_0(n_\sigma))\times G(n_{\bar\sigma}) & & \downarrow f_{[1]_{\bar\sigma_s}} \\
\| & & \\
G(T_0(n_\sigma))\times G(n_{\bar\sigma}) & \xrightarrow{G(\theta)} & G([1]_{\bar\sigma_s}) \\
\downarrow id_{GT_0(n_\sigma)}\times G(\iota_{n_{\bar\sigma}}) & & \downarrow G(\iota_s) \\
G(T_0(n_\sigma)\coprod n_{\bar\sigma}) & \xrightarrow{G(\theta)} & G([1]_{\bar\sigma_s})
\end{array}$$

$$\begin{tikzcd}
F(T_0(n_\sigma)\coprod n_{\bar\sigma}) \arrow[d, "id_{F_0(n_\sigma)}\times F(\delta_{n_{\bar\sigma}})"'] \arrow[dr, "F_0(\delta_{n_\sigma})\times F(\delta_{n_{\bar\sigma}})"] & \\
F_0(n_\sigma)\times F(n_{\bar\sigma})\times F(n_{\bar\sigma}) \arrow[r, "F_0(\delta_{n_\sigma})\times id_{F(n_{\bar\sigma})\times F(n_{\bar\sigma})}"] \arrow[d, "id_{F_0(n_\sigma)}\times f_{n_{\bar\sigma}}\times g_{n_{\bar\sigma}}"'] & F_0(n_\sigma)\times F_0(n_\sigma)\times F(n_{\bar\sigma})\times F(n_{\bar\sigma}) \arrow[d, "id_{F_0(n_\sigma)\times F_0(n_\sigma)}\times f_{n_{\bar\sigma}}\times g_{n_{\bar\sigma}}"] \\
F_0(n_\sigma)\times G(n_{\bar\sigma})\times G(n_{\bar\sigma}) \arrow[r, "F_0(\delta_{n_\sigma})\times id_{G(n_{\bar\sigma})\times G(n_{\bar\sigma})}"] & F_0(n_\sigma)\times F_0(n_\sigma)\times G(n_{\bar\sigma})\times G(n_{\bar\sigma})
\end{tikzcd}$$

$$
\begin{tikzcd}
F(T_0(n_\sigma)\coprod n_{\bar\sigma}) \arrow[r, "F(\theta)"] \arrow[d, "F_0(\delta_{n_\sigma})\times F(\delta_{n_{\bar\sigma}})"] & F([1]_{\bar\sigma_s}) \arrow[d, "F(\delta_{[1]_{\bar\sigma_s}})"] \\
F_0(n_\sigma)\times F_0(n_\sigma)\times F(n_{\bar\sigma})\times F(n_{\bar\sigma}) \arrow[r, "F(\theta)\times F(\theta)"] \arrow[d, "id_{F_0(n_\sigma)\times F_0(n_\sigma)}\times f_{n_{\bar\sigma}}\times g_{n_{\bar\sigma}}"] & F([1]_{\bar\sigma_s})\times F([1]_{\bar\sigma_s}) \arrow[d, "f_{[1]_{\bar\sigma_s}}\times g_{[1]_{\bar\sigma_s}}"] \\
F_0(n_\sigma)\times F_0(n_\sigma)\times G(n_{\bar\sigma})\times G(n_{\bar\sigma}) \arrow[r, "G(\theta)\times G(\theta)"] & G([1]_{\bar\sigma_s})\times G([1]_{\bar\sigma_s}) \arrow[d, "G(\alpha_s)"] \\
F_0(n_\sigma)\times G(n_{\bar\sigma})\times G(n_{\bar\sigma}) \arrow[u, "F_0(\delta_{n_\sigma})\times id_{G(n_{\bar\sigma})\times G(n_{\bar\sigma})}"] \arrow[d, "id_{F_0(n_\sigma)}\times G(\alpha_{n_{\bar\sigma}})"] & G([1]_{\bar\sigma_s}) \\
G(T_0(n_\sigma)\coprod n_{\bar\sigma}) \arrow[ur, "G(\theta)"] &
\end{tikzcd}
$$

The fact that the operations in $\mathcal{T}(\mathcal{C};T_0,F_0)(F,G)$ gives a structure of an abelian group is easily verified. It is also easy to verify that the composition map is biadditive.

A zero object 0 of $\mathcal{T}(\mathcal{C};T_0,F_0)$ is a product preserving functor given as follows. $0([1]_{\sigma(s)}) = F_0([1]_s)$, $0([1]_{\bar\sigma_s}) = 1$, where 1 denotes the terminal object of $\mathcal{C}$. Let $\theta : [1]_t \to T(n_\sigma)\coprod n_{\bar\sigma}$ be a morphism in $\mathcal{T}$. If $t = \sigma(s)$ for some s, there exists a unique morphism $\theta' : [1]_s \to n_\sigma$ in $\mathcal{T}_0$ such that $\theta = \nu T_0(\theta')$ as before. We define $0(\theta) : F(T_0(n_\sigma)\coprod n_{\bar\sigma}) \to F_0([1]_s)$ to be a composition $F_0(n_\sigma)\times F(n_{\bar\sigma}) \xrightarrow{\mathrm{pr}_1} F_0(n_\sigma) \xrightarrow{F_0(\theta')} F_0([1]_s)$. If $t = \bar\sigma_s$ for some s, $F(\theta)$ is the unique morphism to the terminal object.

Since $\mathcal{C}$ has finite products, $\mathcal{T}(\mathcal{C};T_0,F_0)$ also has finite product. Thus we have shown that $\mathcal{T}(\mathcal{C};T_0,F_0)$ is an additive category. $\square$

Let $T_0 : \mathcal{T}_0 \to \mathcal{T}$ be a morphism of finitary algebraic theories and let $T_0^{op} : \mathcal{T}_0^{op} \to \mathcal{T}^{op}$ be the same functor as T_0. Then, the identity functor $id_{\mathcal{T}} : \mathcal{T}^{op} \to \mathcal{T}^{op}$ is an object of $\mathcal{T}(\mathcal{T}^{op};T_0,T_0^{op})$ and it is a "universal example" of $\mathcal{T}$-models in the following sense.

If $\mathcal{C}$ is a category with finite products and F_0 is an object of $\mathcal{T}_0(\mathcal{C})$, then for any object F of $\mathcal{T}(\mathcal{C};T_0,F_0)$, we have $F = F_{\mathcal{T}}(id_{\mathcal{T}})$.

For an object n of $\mathcal{T}$, let $h_n : \mathcal{T}^{op} \to \mathbf{Ens}$ be the functor represented by n, that is, $h_n(m) = \mathcal{T}(m,n)$. Then, morphisms $\theta, \xi : m \to m'$ in $\mathcal{T}$ are equal if and only if $h_n(\theta) = h_n(\xi)$ for any $n \in \mathrm{Ob}\,\mathcal{T}$.

Let $\theta : [k]_1\coprod[l]_2 \to [m]_1\coprod[n]_2$ be a morphism in $\mathcal{T}_{mod}$. It follows from (A.11.27) that $\theta\nu_1 = \nu_1' T_{an}(\xi_\theta)$ for a unique morphism $\xi_\theta : [k] \to [m]$ in $\mathcal{T}_{an}$, where $\nu_1 : [k]_1 \to [k]_1\coprod[l]_2$ and $\nu_1' : [m]_1 \to [m]_1\coprod[n]_2$ are the canonical morphisms into the first components.

Lemma A.11.29 *The following diagrams in $\mathcal{T}_{mod}$ commute.*

$$
\begin{tikzcd}
{[k]_1\coprod[l]_2} \arrow[r, "\theta"] \arrow[d, "id_{[k]_1}\coprod e_{[l]_2}"] & {[m]_1\coprod[n]_2} \arrow[d, "id_{[m]_1}\coprod e_{[n]_2}"] \\
{[k]_1\coprod 0} \arrow[r, "T_{an}(\xi_\theta)\coprod id_0"] & {[m]_1\coprod 0}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[k]_1\coprod[l]_2} \arrow[r, "\theta"] \arrow[d, "id_{[k]_1}\coprod \iota_{[l]_2}"] & {[m]_1\coprod[n]_2} \arrow[d, "id_{[m]_1}\coprod \iota_{[n]_2}"] \\
{[k]_1\coprod[l]_2} \arrow[r, "\theta"] & {[m]_1\coprod[n]_2}
\end{tikzcd}
$$

$$
\begin{tikzcd}
{[k]_1\coprod[l]_2} \arrow[r, "\iota_{k1}\coprod\alpha_{[l]_2}"] \arrow[d, "\theta"] & {([k]_1\coprod[l]_2)\coprod([k]_1\coprod[l]_2)} \arrow[r, "\theta\coprod\theta"] & {([m]_1\coprod[n]_2)\coprod([m]_1\coprod[n]_2)} \arrow[d, "\delta_{[m]_1}\coprod id_{[n]_2}\coprod id_{[n]_2}"] \\
{[m]_1\coprod[n]_2} \arrow[rr, "id_{[m]_1}\coprod\alpha_{[n]_2}"] & & {[m]_1\coprod([n]_2\coprod[n]_2)}
\end{tikzcd}
$$

Here $\iota_{k1} : [k]_1 \to [k]_1\coprod[k]_1$ denotes the canonical morphism into the first summand.

Proof. Suppose that the assertion holds for morphisms $\theta : [k]_1\coprod[l]_2 \to [m]_1\coprod[n]_2$ and $\psi : [m]_1\coprod[n]_2 \to [p]_1\coprod[q]_2$ in $\mathcal{T}_{mod}$. By the uniqueness of $\xi_{\psi\theta}$, we have $\xi_{\psi\theta} = \xi_\psi\xi_\theta$ and it follows that the upper left diagram commute for $\psi\theta$. It is obvious that the upper right diagram commute for $\psi\theta$. We define a morphism $\bar\psi : [m]_1\coprod([n]_2\coprod[n]_2) \to [p]_1\coprod([q]_2\coprod[q]_2)$ as follows. We denote by $\nu_{1i} : [p]_1\coprod[q]_2 \to [p]_1\coprod([q]_2\coprod[q]_2)$ $(i = 2,3)$ the canonical morphism into the first and the i-th summands. We also denote by $\bar\nu_1 : [m]_1 \to [m]_1\coprod([n]_2\coprod[n]_2)$, $\bar\nu_i : [n]_2 \to [m]_1\coprod([n]_2\coprod[n]_2)$ $(i = 2,3)$ the canonical morphisms into the first, the i-th summands, respectively.

Define $\bar{\psi}$ to be the morphism satisfying $\bar{\psi}\bar{\nu}_1 = \nu_{12}\nu_1'' T_{an}(\xi_\psi)$ and $\bar{\psi}\bar{\nu}_i = \nu_{1i}\psi\nu_2'$ $(i = 2, 3)$, where $\nu_1'' : [p]_1 \to [p]_1 \coprod [q]_2$ and $\nu_2' : [n]_1 \to [m]_1 \coprod [m]_2$ are the canonical morphisms. Then, it is easy to verify that the following diagram commutes.

$$\begin{array}{ccc} ([m]_1 \coprod [n]_2) \coprod ([m]_1 \coprod [n]_2) & \xrightarrow{\psi \coprod \psi} & ([p]_1 \coprod [q]_2) \coprod ([p]_1 \coprod [q]_2) \\ \downarrow{\scriptstyle \delta_{[m]_1} \coprod id_{[n]_2} \coprod id_{[n]_2}} & & \downarrow{\scriptstyle \delta_{[p]_1} \coprod id_{[q]_2} \coprod id_{[q]_2}} \\ [m]_1 \coprod ([n]_2 \coprod [n]_2) & \xrightarrow{\bar{\psi}} & [p]_1 \coprod ([q]_2 \coprod [q]_2) \end{array}$$

Thus we have
$(id_{[p]_1} \coprod \alpha_{[q]_2})\psi\theta = (\delta_{[p]_1} \coprod id_{[q]_2} \coprod id_{[q]_2})(\psi \coprod \psi)(\iota_{m1} \coprod \alpha_{[n]_2})\theta$
$= \bar{\psi}(\delta_{[m]_1} \coprod id_{[n]_2} \coprod id_{[n]_2})(\iota_{m1} \coprod \alpha_{[n]_2})\theta = \bar{\psi}(id_{[m]_1} \coprod \alpha_{[n]_2})\theta$
$= \bar{\psi}(\delta_{[m]_1} \coprod id_{[n]_2} \coprod id_{[n]_2})(\theta \coprod \theta)(\iota_{k1} \coprod \alpha_{[l]_2}) = (\delta_{[p]_1} \coprod id_{[q]_2} \coprod id_{[q]_2})(\psi\theta \coprod \psi\theta)(\iota_{k1} \coprod \alpha_{[l]_2})$
and the lower diagram commutes for $\psi\theta$.

Moreover, if the assertion holds for $\theta\nu_i : [1]_1 \to [m]_1 \coprod [n]_2$ $(1 \le i \le k)$ and $\theta\nu_i : [1]_2 \to [m]_1 \coprod [n]_2$ $(k+1 \le i \le k+l)$, where ν_i's are the canonical morphism into the i-th summand), it holds for $\theta : [m] \to [n]$. Hence it suffices to show the assertion for morphisms with domain $[1]_s$ $(s = 1, 2)$ represented by a single word.

It follows from (A.11.25) that the diagrams commute if $\theta = T_{an}(\xi) \coprod T_{ab}(\zeta)$ for any $\xi \in \mathcal{T}_{an}([k], [m])$, $\zeta \in \mathcal{T}_{ab}([l], [n])$, in other words, the diagrams commute if θ is a morphism in a subcategory $\mathcal{T}_{an} \times \mathcal{T}_{ab}$ of $\mathcal{T}_{mod}$. Since $M_{mod}([1]_2, [1]_1 \coprod [1]_2) = \{\varphi\}$ and $M_{mod}([m]_1 \coprod [n]_2, [k]_1 \coprod [l]_2) = \phi$ if $(m, n, k, l) \neq (0, 1, 1, 1)$, it suffices to verify the assertion only for $\theta = \varphi$. The commutativity for $\theta = \varphi$ of the lower diagram follows from the construction of $\mathcal{T}_{mod}$. We can easily show that the upper diagrams commute as in the proof of (A.11.25). □

In particular, it follows from the above fact that $T_{ab} : \mathcal{T}_{ab} \to \mathcal{T}_{mod}$ satisfies the condition of (A.11.28). Hence if $\mathcal{C}$ is a category with finite products and A is an object of $\mathcal{T}_{an}(\mathcal{C})$, $\mathcal{T}_{mod}(\mathcal{C}; T_{an}, A)$ is an additive category.

A.12 Abelian category

Let $\mathcal{U}$ be a fixed universe. Recall that an abelian category $\mathcal{A}$ is a $\mathcal{U}$-category satisfying the following axioms.

A1) For any $A, B \in \operatorname{Ob}\mathcal{A}$, $\mathcal{A}(A, B)$ is an abelian group.
A2) For any $A, B, C \in \operatorname{Ob}\mathcal{A}$, the composition map $\mathcal{A}(A, B) \times \mathcal{A}(B, C) \to \mathcal{A}(A, C)$ is biadditive.
A3) $\mathcal{A}$ has a null object, that is, there is an object 0 such that $\mathcal{A}(0, 0)$ is the zero group.
A4) For $A, B \in \operatorname{Ob}\mathcal{A}$, there exists an object $C \in \mathcal{A}$ and morphisms $i_1 : A \to C$, $i_2 : B \to C$, $p_1 : C \to A$ and $p_2 : C \to B$ which satisfy the identities $p_1 i_1 = id_A$, $p_2 i_2 = id_B$ and $i_1 p_1 + i_2 p_2 = id_C$.
A5) For any morphism $f : A \to B$ in $\mathcal{A}$, a kernel and a cokernel of f exist.
A6) Every monomorphism is a kernel of a homomorphism and every epimorphism is a cokernel of a homomorphism.

It follows from the above axioms that each morphism $f : A \to B$ of $\mathcal{A}$ has a kernel and a cokernel and that the canonical morphism $\operatorname{Im} f \to \operatorname{Coim} f$ is an isomorphism, where $\operatorname{Im} f \to B$ is a kernel of the canonical epimorphism $B \to \operatorname{Coker} f$ and $A \to \operatorname{Coim} f$ is a cokernel of the canonical monomorphism $\operatorname{Ker} f \to A$.

For $A, B \in \operatorname{Ob}\mathcal{A}$, the object C in A4 is unique up to isomorphism. We usually denote this by $A \oplus B$. Note that $A \xleftarrow{p_1} A \oplus B \xrightarrow{p_2} B$ is a product of A and B, $A \xrightarrow{i_1} A \oplus B \xleftarrow{i_2} B$ is a coproduct of A and B.

Proposition A.12.1 *Let A_1 and A_2 be subobjects of A. Then, the supremum and infimum of A_1 and A_2 exist in the ordered set* $\operatorname{Sub}(A)$ *of all subobjects of A.*

Proof. Let $\iota : A_1 \oplus A_2 \to A$ be the morphism induced by the inclusion morphisms $j_1 : A_1 \to A$, $j_2 : A_2 \to A$. Then, the image of ι is the supremum of A_1 and A_2. Let $\pi : A \to \operatorname{Coker} j_1 \oplus \operatorname{Coker} j_2$ be the morphism induced by the canonical morphisms $A \to \operatorname{Coker} j_1$, $A_2 \to \operatorname{Coker} j_2$. Then, the kernel of π is the infimum of A_1 and A_2. □

We denote by $A_1 + A_2$ (resp. $A_1 \cap A_2$) the supremum (resp. infimum) A_1 and A_2.

Let $(A_i)_{i \in I}$ be a family of objects of an abelian category $\mathcal{A}$. We call a coproduct of $(A_i)_{i \in I}$ a direct sum of $(A_i)_{i \in I}$, which is denoted by $\bigoplus_{i \in I} A_i$. Consider the following axiom.

A7) If $(A_i)_{i \in I}$ is a family of objects of $\mathcal{A}$ and I is a $\mathcal{U}$-small set, a direct sum of $(A_i)_{i \in I}$ exists.

The following results are consequences of A7).

Proposition A.12.2 *Assume that an abelian category $\mathcal{A}$ satisfies A7).*

1) For $A \in \mathrm{Ob}\,\mathcal{A}$, if M is a $\mathcal{U}$-small subset of $\mathrm{Sub}(A)$, the supremum of M exists. For a family of subobjects $(A_i)_{i\in I}$, we denote by $\sum_{i\in I} A_i$ the supremum of $\{A_i|\, i \in I\}$.

2) Let $\mathcal{D}$ be a $\mathcal{U}$-small category and $D : \mathcal{D} \to \mathcal{A}$ a functor. We denote by $\iota_i : D(i) \to \bigoplus_{i\in \mathrm{Ob}\,\mathcal{D}} D(i)$ ($i \in \mathrm{Ob}\,\mathcal{D}$) the canonical morphism. Put $N_D = \sum_{\theta\in \mathrm{Mor}\,\mathcal{D}} \mathrm{Im}(\iota_{\mathrm{dom}(\theta)} - \iota_{\mathrm{codom}(\theta)} D(\theta))$ and let $\pi : \bigoplus_{i\in \mathrm{Ob}\,\mathcal{D}} D(i) \to \bigoplus_{i\in \mathrm{Ob}\,\mathcal{D}} D(i)/N_D$ be the cokernel of the inclusion morphism $N_D \to \bigoplus_{i\in \mathrm{Ob}\,\mathcal{D}} D(i)$. Then, $(D(i) \xrightarrow{\pi\iota_i} \bigoplus_{i\in \mathrm{Ob}\,\mathcal{D}} D(i)/N_D)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of D.

3) With the same notations as above, let $(D(i) \xrightarrow{\alpha_i} L)_{i\in \mathrm{Ob}\,\mathcal{D}}$ be a colimiting cone, $(D(i) \xrightarrow{\beta_i} A)_{i\in \mathrm{Ob}\,\mathcal{D}}$ a cone and $f : L \to A$ the unique morphism such that $f\alpha_i = \beta_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$. Then, $\mathrm{Im}\, f = \sum_{i\in \mathrm{Ob}\,\mathcal{D}} \mathrm{Im}\,\beta_i$.

4) Let A be an object of $\mathcal{A}$ and $D : \mathcal{D} \to \mathrm{Sub}(A)$ a functor. We denote by Quot(A) the category of quotient objects of A. Define a functor $D' : \mathcal{D} \to \mathit{Quot}(A)$ by $D'(i) = A/D(i)$ and $D'(\alpha) = (A/D(i) \to A/D(j)) =$ (the morphism induced by$A \to A/D(j)$) for $\alpha \in \mathcal{D}(i,j)$. Let $\rho_i : A/D(i) \to A/\sum_{i\in \mathrm{Ob}\,\mathcal{D}} D(i)$ be the canonical morphism. Then $(A/D(i) \xrightarrow{\rho_i} A/\sum_{i\in \mathrm{Ob}\,\mathcal{D}} D(i))_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a colimiting cone of D'.

Proof. 1) Let $(A_i)_{i\in I}$ be a $\mathcal{U}$-small family of subobjects of A and denote by $\rho : \bigoplus_{i\in I} A_i \to A$ the morphism induced by the inclusion morphisms $A_i \to A$ ($i \in I$). Then, the image of ρ is the supremum of $\{A_i|\, i \in I\}$ in $\mathrm{Sub}(A)$.

2) Let $(D(i) \xrightarrow{\lambda_i} M)_{i\in \mathrm{Ob}\,\mathcal{D}}$ be a cone and $\lambda : \bigoplus_{i\in \mathrm{Ob}\,\mathcal{D}} D(i) \to M$ the morphism satisfying $\lambda_i = \lambda\iota_i$ for each $i \in \mathrm{Ob}\,\mathcal{D}$. Then $\lambda(\iota_{\mathrm{dom}(\theta)} - \iota_{\mathrm{codom}(\theta)} D(\theta)) = \lambda_{\mathrm{dom}(\theta)} - \lambda_{\mathrm{codom}(\theta)} D(\theta) = 0$. Hence there exists a unique morphism $\mu : \bigoplus_{i\in \mathrm{Ob}\,\mathcal{D}} D(i)/N_D \to M$ such that $\lambda = \mu\pi$. Thus $\lambda_i = \mu\pi\iota_i$ for every $i \in \mathrm{Ob}\,\mathcal{D}$.

3) Put $B = \sum_{i\in \mathrm{Ob}\,\mathcal{D}} \mathrm{Im}\,\beta_i$. Since $f\alpha_i = \beta_i$ for any $i \in \mathrm{Ob}\,\mathcal{D}$, we have $\mathrm{Im}\, f \supset \mathrm{Im}\,\beta_i$, hence $\mathrm{Im}\, f \supset B$. Let $\iota : B \to A$ be the inclusion morphism and $\beta'_i : D(i) \to B$ the unique morphism such that $\iota\beta'_i = \beta_i$. Since ι is a monomorphism, it follows that $(D(i) \xrightarrow{\beta'_i} B)_{i\in \mathrm{Ob}\,\mathcal{D}}$ is a cone and we have a unique morphism $f' : L \to B$ such that $f'\alpha_i = \beta'_i$. Thus we have $\iota f'\alpha_i = \beta_i = f\alpha_i$ which implies $\iota f' = f$. Therefore $\mathrm{Im}\, f \subset \mathrm{Im}\,\iota = B$.

4) Let $(A/D(i) \xrightarrow{\iota_i} C)_{i\in \mathrm{Ob}\,\mathcal{D}}$ be a cone of D'. We denote by $\pi_i : A \to A/D(i)$ the canonical projection. Since the composition $A \xrightarrow{\pi_i} A/D(i) \xrightarrow{\iota_i} C$ does not depend on $i \in \mathrm{Ob}\,\mathcal{D}$, we denote this morphism by λ. For each $i \in \mathrm{Ob}\,\mathcal{D}$, we have $\mathrm{Ker}\,\lambda \supset \mathrm{Ker}\,\pi_i = D(i)$. Hence $\mathrm{Ker}\,\lambda \supset \sum_{i\in \mathrm{Ob}\mathcal{D}} D(i)$ and λ induces a unique morphism $\lambda' : A/\sum_{i\in \mathrm{Ob}\,\mathcal{D}} D(i) \to C$ such that $\lambda = \lambda'\pi'$ where $\pi' : A \to A/\sum_{i\in \mathrm{Ob}\,\mathcal{D}} D(i)$ is the canonical projection. Since $\pi' = \rho_i\pi_i$ and π_i is an epimorphism, it follows from $\iota_i\pi_i = \lambda'\rho_i\pi_i$ that $\iota_i = \lambda'\rho_i$. □

The next axiom plays an important role in proving the existence of enough injectives.

A8) The axiom A7) is satisfied and if $(A_i)_{i\in I}$ is a family of subobjects of A such that I is a $\mathcal{U}$-small directed set and that $i \leq j$ implies $A_i \subset A_j$, then we have $(\sum_{i\in I} A_i) \cap B = \sum_{i\in I}(A_i \cap B)$ for each subobject B of A.

Proposition A.12.3 *Let $(f_i : A_i \to B_i)_{i\in I}$ be a family of monomorphisms in $\mathcal{A}$. If the axiom A8) is satisfied in $\mathcal{A}$, then, $\bigoplus_{i\in I} f_i : \bigoplus_{i\in I} A_i \to \bigoplus_{i\in I} B_i$ is a monomorphism.*

Proof. Let $(J, \subset)$ be the directed set of all finite subsets of I. Define a functor $D : J \to \mathcal{A}$ by $D(S) = \bigoplus_{i\in S} A_i$ and $D(S \subset T) : \bigoplus_{i\in S} A_i \xrightarrow{i_1} \bigoplus_{i\in S} A_i \oplus \bigoplus_{i\in T-S} A_i = \bigoplus_{i\in T} A_i$. Then, $\bigoplus_{i\in I} A_i$ is a colimit of D. Let $K \to \bigoplus_{i\in I} A_i$ be a kernel of $\bigoplus_{i\in I} f_i$ and $\iota_S : \bigoplus_{i\in S} A_i \to \bigoplus_{i\in I} A_i$, $\kappa_S : \bigoplus_{i\in S} B_i \to \bigoplus_{i\in I} B_i$ denote the canonical morphisms . Then, $(\mathrm{Im}\,\iota_S)_{S\in J}$ is a directed set of subobjects of $\bigoplus_{i\in I} A_i$ and $\bigoplus_{i\in I} A_i = \sum_{S\in J} \mathrm{Im}\,\iota_S$ by (A.12.2). Hence $K = (\sum_{S\in J} \mathrm{Im}\,\iota_S) \cap K = \sum_{S\in J}(\mathrm{Im}\,\iota_S \cap K)$. Since κ_S has a left inverse, it is a monomorphism. Note that he following diagram commutes and $\bigoplus_{i\in S} f_i$ is a monomorphism.

$$\begin{tikzcd}
\bigoplus_{i\in S} A_i \arrow[r, "\oplus_{i\in S} f_i"] \arrow[d, "\iota_S"] & \bigoplus_{i\in S} B_i \arrow[d, "\kappa_S"] \\
\bigoplus_{i\in I} A_i \arrow[r, "\oplus_{i\in I} f_i"] & \bigoplus_{i\in I} B_i
\end{tikzcd}$$

Thus we have $\kappa_S(\bigoplus_{i\in S} f_i)\iota_S^{-1}(\mathrm{Im}\,\iota_S \cap K) = (\bigoplus_{i\in I} f_i)(\mathrm{Im}\,\iota_S \cap K) = 0$ and this implies that $\mathrm{Im}\,\iota_S \cap K = 0$. Therefore we have $K = 0$. □

Lemma A.12.4 *Let $\mathcal{A}$ be an abelian category.*

1) Let $A \xrightarrow{i} B \xrightarrow{p} C$ *and* $D \xrightarrow{j} B \xrightarrow{q} E$ *be exact sequences in* $\mathcal{A}$. *There is an isomorphism* $\operatorname{Im} p/\operatorname{Im} pj \cong \operatorname{Im} q/\operatorname{Im} qi$.

2) If
$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d] & B' \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$
is a pull-back, the morphism $\bar{g} : \operatorname{Coker} f' \to \operatorname{Coker} f$ *induced by* g *is a monomorphism onto* $(\operatorname{Im} f + \operatorname{Im} g)/\operatorname{Im} f$. *In particular, if* f *is an epimorphism, so is* f'.

Proof. 1) We may assume that p and q are epimorphisms by replacing C, E by $\operatorname{Im} p$, $\operatorname{Im} q$ and that i and j are monomorphisms by replacing A, D by $\operatorname{Ker} p$, $\operatorname{Ker} q$. Then, $B \xrightarrow{q} E \to \operatorname{Coker} qi$ induces $q' : C \to \operatorname{Coker} qi$. q' factors through $C \to \operatorname{Coker} pj$ and we have a morphism $\bar{q} : \operatorname{Coker} pj \to \operatorname{Coker} qi$. By symmetry, we also have a morphism $\bar{p} : \operatorname{Coker} qi \to \operatorname{Coker} pj$. $\bar{q}$ and $\bar{p}$ are inverses each other.

2) Apply the above result to exact sequences $0 \to A' \to A \oplus B' \xrightarrow{fp_1 - gp_2} B$ and $0 \to A \xrightarrow{i_1} A \oplus B' \xrightarrow{p_2} B' \to 0$. □

Let $\mathcal{A}$ and $\mathcal{D}$ be categories and $\mathcal{R}$ a subcategory of $\mathcal{D}$ such that $\operatorname{Ob} \mathcal{D} = \operatorname{Ob} \mathcal{R}$. We denote by $\operatorname{Funct}(\mathcal{D}, \mathcal{A}; \mathcal{R})$ a full subcategory of $\operatorname{Funct}(\mathcal{A}, \mathcal{D})$ such that $\operatorname{Ob} \operatorname{Funct}(\mathcal{D}, \mathcal{A}; \mathcal{R}) = \{F \in \operatorname{Ob} \operatorname{Funct}(\mathcal{D}, \mathcal{A}) |\, F(\varphi) = F(\psi)$ *if* $\operatorname{dom}(\varphi) = \operatorname{dom}(\psi)$, $\operatorname{codom}(\varphi) = \operatorname{codom}(\psi)$ *and* $\varphi, \psi \in \operatorname{Mor} \mathcal{R}\}$.

Proposition A.12.5 *Let* $\mathcal{D}$ *be a* $\mathcal{U}$*-small category and* $\mathcal{A}$ *a* $\mathcal{U}$*-category. If* $\mathcal{A}$ *is an abelian category, so is* $\operatorname{Funct}(\mathcal{D}, \mathcal{A}; \mathcal{R})$. *If* $\mathcal{A}$ *satisfies A7 or A8, so does* $\operatorname{Funct}(\mathcal{D}, \mathcal{A}; \mathcal{R})$.

Proof. We put $\mathcal{F} = \operatorname{Funct}(\mathcal{D}, \mathcal{A}; \mathcal{R})$. It follows from (A.1.3) that $\mathcal{F}$ is a $\mathcal{U}$-category. For $F, G \in \operatorname{Ob} \mathcal{F}$ and $\varphi, \psi \in \mathcal{F}(F, G)$, define $\varphi + \psi \in \mathcal{F}(F, G)$ by $(\varphi + \psi)_i = \varphi_i + \psi_i \in \mathcal{A}(F(i), G(i))$. Then, A1 and A2 is obvious. A null object 0 in $\mathcal{F}$ is a constant functor $i \mapsto 0$. For a morphism $\varphi : F \to G$ in $\mathcal{F}$, $\operatorname{Ker} \varphi \to F$ and $G \to \operatorname{Coker} \varphi$ is given by $(\operatorname{Ker} \varphi)_i = \operatorname{Ker} \varphi_i \to F(i)$ and $G(i) \to \operatorname{Coker} \varphi_i = (\operatorname{Coker} \varphi)_i$. Then, we can verify A5 and A6. In particular, a morphism $\varphi : F \to G$ is a monomorphism (resp. epimorphism) in $\mathcal{F}$ if and only if $\varphi_i : F(i) \to G(i)$ is a monomorphism (resp. epimorphism) for each $i \in I$.

If $\mathcal{A}$ has $\mathcal{U}$-small coproducts, so does $\operatorname{Funct}(\mathcal{D}, \mathcal{A})$ ((A.4.1)). Coproducts of objects of $\mathcal{F}$ in $\operatorname{Funct}(\mathcal{D}, \mathcal{A})$ belong to $\mathcal{F}$. Thus $\mathcal{F}$ satisfies A7. Suppose that $\mathcal{A}$ satisfies A8. Let $(F_\lambda)\lambda \in I$ be a family of subobjects of $F \in \operatorname{Ob} \mathcal{F}$ such that I is a directed set and $\lambda \leq \mu$ implies $F_\lambda \subset F_\mu$. Then, since $\sum_{\lambda \in I} F_\lambda$ is given by $(\sum_{\lambda \in I} F_\lambda)(i) = \sum_{\lambda \in I} F_\lambda(i)$, we have $((\sum_{\lambda \in I} F_\lambda) \cap G)(i) = (\sum_{\lambda \in I} F_\lambda(i)) \cap G(i) = \sum_{\lambda \in I}(F_\lambda(i) \cap G(i)) = \sum_{\lambda \in I}(F_\lambda \cap G)(i) = (\sum_{\lambda \in I}(F_\lambda \cap G))(i)$ for a subobject G of F and $i \in \operatorname{Ob} \mathcal{F}$. □

Proposition A.12.6 *Let* $F : \mathcal{A} \to \mathcal{B}$ *be a functor between abelian categories.*

If F *has a right (resp. left) adjoint,* F *is additive and right (resp. left) exact.*

Proof. If F has a right (resp. left) adjoint, F preserves colimits (resp. limits) ((A.3.13)), in particular, finite coproducts (resp. finite products). Hence F is additive. Let $G : \mathcal{B} \to \mathcal{A}$ be a right (resp. left) adjoint of F. If $A \to B \to C \to 0$ (resp. $0 \to A \to B \to C$) is an exact sequence in $\mathcal{A}$ and M is an object of $\mathcal{B}$, it follows from the following diagrams that F is right (resp. left) exact.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{B}(F(C), M) \arrow[r] \arrow[d, "\cong"] & \mathcal{B}(F(B), M) \arrow[r] \arrow[d, "\cong"] & \mathcal{B}(F(A), M) \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathcal{A}(C, G(M)) \arrow[r] & \mathcal{A}(B, G(M)) \arrow[r] & \mathcal{A}(A, G(M)) \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{B}(M, F(A)) \arrow[r] \arrow[d, "\cong"] & \mathcal{B}(M, F(B)) \arrow[r] \arrow[d, "\cong"] & \mathcal{B}(M, F(C)) \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathcal{A}(G(M), A) \arrow[r] & \mathcal{A}(G(M), B) \arrow[r] & \mathcal{A}(G(M), C) \end{tikzcd}$$

□

Let $\mathcal{D}$ be a $\mathcal{U}$-small category. If $\mathcal{A}$ is an abelian category satisfying A7), we have a functor $\varinjlim : \operatorname{Funct}(\mathcal{D}, \mathcal{A}) \to \mathcal{A}$ by (A.12.2). If $\mathcal{A}$ is an abelian category with $\mathcal{U}$-small products, we have a functor $\varprojlim : \operatorname{Funct}(\mathcal{D}, \mathcal{A}) \to \mathcal{A}$. Let $\Delta : \mathcal{A} \to \operatorname{Funct}(\mathcal{D}, \mathcal{A})$ be the diagonal functor, that is, $\Delta(A) : \mathcal{D} \to \mathcal{A}$ $(A \in \operatorname{Ob} \mathcal{A})$ is the constant functor taking value A and $\Delta(f) : \Delta(A) \to \Delta(B)$ $(f \in \mathcal{A}(A, B))$ is given by $\Delta(f)_i = f : A \to B$ for every $i \in \operatorname{Ob} \mathcal{D}$. Then, $\varinjlim$ is a left adjoint of Δ and $\varprojlim$ is a right adjoint of Δ. Thus we have the following result by (A.12.6).

Proposition A.12.7 *Let $\mathcal{A}$ be an abelian category satisfying A7 and $\mathcal{D}$ a $\mathcal{U}$-small category. Then, $\varinjlim$: $\mathrm{Funct}(\mathcal{D},\mathcal{A}) \to \mathcal{A}$ is right exact. If $\mathcal{A}$ has $\mathcal{U}$-small products, $\varprojlim : \mathrm{Funct}(\mathcal{D},\mathcal{A}) \to \mathcal{A}$ is left exact.*

Proposition A.12.8 *Let $\mathcal{A}$ be an abelian category satisfying A7. Then the following conditions are equivalent.*

1) $\mathcal{A}$ satisfies A8.

2) If $(\iota_i : A_i \to A)_{i\in I}$ is a family of subobjects of A such that I is a $\mathcal{U}$-small directed set and that $i \le j$ implies $A_i \subset A_j$, then $(\iota_i : A_i \to A)_{i\in I}$ is a colimiting cone of a diagram $(A_i \to A_j)_{i\le j}$.

3) Let $(A_i)_{i\in I}$ be a $\mathcal{U}$-small family of objects of $\mathcal{A}$ and $(J,\subset)$ the directed set of all finite subsets of I. For $S \in J$, we denote by $\iota_S : \bigoplus_{i\in S} A_i \to \bigoplus_{i\in I} A_i$ the canonical morphism. Then, for any subobject B of $\bigoplus_{i\in I} A_i$, we have $B = \sum_{S\in J}(\mathrm{Im}\,\iota_S \cap B)$.

Proof. 1) ⇒2) Let $(A_i \xrightarrow{\kappa_i} L)_{i\in I}$ be a colimiting cone. □

A.13 Crude tripleability theorem

Let $\mathcal{C}$ and $\mathcal{A}$ be categories. Suppose that a functor $F : \mathcal{C} \to \mathcal{A}$ has a right adjoint $G : \mathcal{A} \to \mathcal{C}$. Put $T = GF$ and let us denote by $\eta : 1_{\mathcal{C}} \to GF = T$ and $\varepsilon : FG \to 1_{\mathcal{A}}$ the unit and the counit of the adjunction, respectively. We consider the monad $\boldsymbol{T} = (T, \eta, G(\varepsilon_F))$ associated with this adjunction. Let $\mathcal{C}^{\boldsymbol{T}}$ be the category of $\boldsymbol{T}$-algebras and $K : \mathcal{A} \to \mathcal{C}^{\boldsymbol{T}}$ the comparison functor defined by $K(A) = \langle G(A), G(\varepsilon_A)\rangle$.

Lemma A.13.1 *If $\mathcal{A}$ has coequalizers of reflexive pairs, the comparison functor K has a left adjoint $M : \mathcal{C}^{\boldsymbol{T}} \to \mathcal{A}$.*

Proof. For each $\langle X, h\rangle \in \mathrm{Ob}\,\mathcal{C}^{\boldsymbol{T}}$, since $F(\eta_X) : F(X) \to FGF(X)$ splits both $F(h)$ and $\varepsilon_{F(X)}$,

$$FGF(X) \underset{\varepsilon_{F(X)}}{\overset{F(h)}{\rightrightarrows}} F(X)$$

is a reflexive pair. Let $e : F(X) \to M\langle X, h\rangle$ be the coequalizer of this pair. For a morphism $f : \langle X, h\rangle \to \langle Y, k\rangle$ of $\mathcal{C}^{\boldsymbol{T}}$, since the following diagrams commute,

$$\begin{tikzcd} FGF(X) \arrow[r, "F(h)"] \arrow[d, "FGF(f)"] & F(X) \arrow[d, "F(f)"] \\ FGF(X) \arrow[r, "F(h)"] & F(X) \end{tikzcd} \qquad \begin{tikzcd} FGF(X) \arrow[r, "\varepsilon_{F(X)}"] \arrow[d, "FGF(f)"] & F(X) \arrow[d, "F(f)"] \\ FGF(X) \arrow[r, "\varepsilon_{F(X)}"] & F(X) \end{tikzcd}$$

f induces a morphism of $\boldsymbol{T}$-algebras $M(f) : M\langle X, h\rangle \to M\langle Y, k\rangle$. Thus we have a functor $M : \mathcal{C}^{\boldsymbol{T}} \to \mathcal{A}$.

For $g \in \mathcal{A}(F(X), A)$ and $\langle X, h\rangle \in \mathrm{Ob}\,\mathcal{C}^{\boldsymbol{T}}$, we note that a composite $X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(g)} G(A)$ gives a morphism of $\boldsymbol{T}$-algebras $\langle X, h\rangle \to K(A)$ if and only if $gF(h) = g\varepsilon_{F(X)} : FGF(X) \to A$. We define $\psi : \mathcal{A}(M\langle X, h\rangle, A) \to \mathcal{C}^{\boldsymbol{T}}(\langle X, h\rangle, K(A))$ as follows. For $f \in \mathcal{A}(M\langle X, h\rangle\ A)$, $\psi(f) \in \mathcal{C}^{\boldsymbol{T}}(\langle X, h\rangle, K(A))$ is the adjoint of a composite $F(X) \xrightarrow{e} M\langle X, h\rangle \xrightarrow{f} A$, that is, a composite $X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(e)} GM\langle X, h\rangle \xrightarrow{G(f)} G(A)$. Since $ef(h) = e\varepsilon_{F(X)}$, $\psi(f)$ is a morphism of $\boldsymbol{T}$-algebras. The inverse of ψ is given as follows. For $f \in \mathcal{C}^{\boldsymbol{T}}(\langle X, h\rangle, K(A))$, since the following diagram commutes,

$$\begin{tikzcd} F(X) \arrow[d, "F(f)"] & FGF(X) \arrow[l, "\varepsilon_{F(X)}"'] \arrow[r, "F(h)"] \arrow[d, "FGF(f)"] & F(X) \arrow[d, "F(f)"] \\ FG(A) \arrow[dr, "\varepsilon_A"] & FGFG(A) \arrow[l, "\varepsilon_{FG(A)}"'] \arrow[r, "FG(\varepsilon_A)"] & FG(A) \arrow[dl, "\varepsilon_A"'] \\ & A & \end{tikzcd}$$

we have $\varepsilon_A F(f)F(h) = \varepsilon_A F(f)\varepsilon_{F(X)}$. $\psi^{-1}(f) : M\langle X, h\rangle \to A$ is the unique morphism satisfying $\psi^{-1}(f)e = \varepsilon_A F(f)$. □

Lemma A.13.2 *If the functor $G : \mathcal{A} \to \mathcal{C}$ of the above lemma preserves coequalizers of reflexive pairs, the unit $\eta_{\langle X,h\rangle} = \psi(1_{M\langle X,h\rangle}) : \langle X, h\rangle \to KM\langle X, h\rangle$ of the above adjunction is an isomorphism for any $\boldsymbol{T}$-algebra $\langle X, h\rangle$.*

Proof. We note that $GFGF(X) \underset{G(\varepsilon_{F(X)})}{\overset{GF(h)}{\rightrightarrows}} GF(X) \xrightarrow{h} X$ is a split fork. In fact $G(\varepsilon_{F(X)})\eta_{GF(X)} = 1_{GF(X)}$, $h\eta_X = 1_X$ and $GF(h)\eta_{GF(X)} = \eta_X h$ hold. Since G preserves a coequalizer $FGF(X) \underset{\varepsilon_{F(X)}}{\overset{F(h)}{\rightrightarrows}} F(X) \xrightarrow{e} M\langle X, h\rangle$, both $G(e)$ and h are coequalizers of the same pair. Hence $\psi(1_{M\langle X,h\rangle}) : \langle X, h\rangle \to KM\langle X, h\rangle$ is an isomorphism which satisfies $\psi(1_{M\langle X,h\rangle})h = G(e)$. □

Lemma A.13.3 *Moreover, if the functor $G : \mathcal{A} \to \mathcal{C}$ reflects isomorphisms, the counit $\varepsilon_A = \psi^{-1}(1_{K(A)}) : MK(A) \to A$ of the above adjunction is an isomorphism for any $A \in \operatorname{Ob}\mathcal{A}$.*

Proof. We note that $GFGFG(A) \underset{G(\varepsilon_{FG(A)})}{\overset{GFG(\varepsilon_A)}{\rightrightarrows}} GFG(A) \xrightarrow{G(\varepsilon_A)} G(A)$ is a split fork. Since

$$FGFG(A) \underset{\varepsilon_{FG(A)}}{\overset{FG(\varepsilon_A)}{\rightrightarrows}} FG(A) \xrightarrow{e} MK(A)$$

is a coequalizer and it is preserved by G, both $G(e)$ and $G(\varepsilon_A)$ are coequalizers of the same pair. On the other hand, since $G(\psi^{-1}(1_{K(A)}))G(e) = G(\varepsilon_A)$, $G(\psi^{-1}(1_{K(A)}))$ is an isomorphism. Therefore $\psi^{-1}(1_{K(A)})$ is an isomorphism by assumption. □

Combining the above lemmas, we have the following result.

Theorem A.13.4 *(Crude Tripleability Theorem) Let $F : \mathcal{C} \to \mathcal{A}$ be a functor and $G : \mathcal{A} \to \mathcal{C}$ a right adjoint of F. If $\mathcal{A}$ has coequalizers of reflexive pairs, G preserves them and G reflects isomorphisms, then the comparison functor $K : \mathcal{A} \to \mathcal{C}^{\boldsymbol{T}}$ is an equivalence of categories.*

We say that a functor $F : \mathcal{C} \to \mathcal{A}$ is monadic if it has a right adjoint and $\mathcal{A}$ is equivalent to the category of algebras for the monad in $\mathcal{C}$ defined by the adjunction via the comparison functor.

A.14 Monads and comonads

Let $G_i : \mathcal{A} \to \mathcal{C}$ $(i = 1, 2)$ be a right adjoint of $F_i : \mathcal{C} \to \mathcal{A}$ with adjunction $a_i : \mathcal{A}(F_i(X), A) \to \mathcal{C}(X, G_i(A))$.

Proposition A.14.1 *For a natural transformation $\varphi : F_2 \to F_1$, there is a unique natural transformation $\psi : G_1 \to G_2$ such that the following diagram commute.*

$$\begin{tikzcd}
\mathcal{A}(F_1(X), A) \arrow[r, "a_1"] \arrow[d, "\varphi_X^*"] & \mathcal{C}(X, G_1(A)) \arrow[d, "\psi_{A*}"] \\
\mathcal{A}(F_2(X), A) \arrow[r, "a_2"] & \mathcal{C}(X, G_2(A))
\end{tikzcd}$$

Conversely, for a natural transformation $\psi : G_1 \to G_2$, there is a unique natural transformation $\varphi : F_2 \to F_1$ such that the above diagram commute. Hence there is a bijection between the set of natural transformations from F_2 to F_1 and the set of natural transformations from G_1 to G_2.

Proof. If such ψ exist, the commutativity of the diagram implies $\psi_A = \psi_A(id_{G_1(A)}) = a_2(a_1^{-1}(1_{G_1(A)})\varphi_{G_1(A)})$. This shows the uniqueness of ψ. It is easy to check that ψ defined above satisfies the requirement. Similarly, for given $\psi : G_1 \to G_2$, φ defined by $\varphi_X = a_2^{-1}(\psi_{F_1(X)}a_1(id_{F_1(X)}))$ is the unique natural transformation that makes the above diagram commute. □

Let $\boldsymbol{F} = (F, \eta, \mu)$ be a monad on $\mathcal{C}$ and $G : \mathcal{C} \to \mathcal{C}$ be a right adjoint of F with adjunction $a : \mathcal{C}(F(X), Y) \to \mathcal{C}(X, G(Y))$. We denote by a^2 a composition $\mathcal{C}(F^2(X), Y) \xrightarrow{a} \mathcal{C}(F(X), G(Y)) \xrightarrow{a} \mathcal{C}(X, G^2(Y))$.

Proposition A.14.2 *Let $\varepsilon : G \to 1_{\mathcal{C}}$ and $\delta : G \to G^2$ be the unique morphisms making the following diagrams commute.*

$$\begin{tikzcd}
\mathcal{C}(F(X), A) \arrow[rr, "a"] \arrow[dr, "\eta_X^*"'] & & \mathcal{C}(X, G(A)) \arrow[dl, "\varepsilon_{A*}"] \\
& \mathcal{C}(X, A) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(F(X), A) \arrow[r, "a"] \arrow[d, "\mu_X^*"] & \mathcal{C}(X, G(A)) \arrow[d, "\delta_{A*}"] \\
\mathcal{C}(F^2(X), A) \arrow[r, "a^2"] & \mathcal{C}(X, G^2(A))
\end{tikzcd}$$

Then, $\boldsymbol{G} = (G, \varepsilon, \delta)$ is a comonad on $\mathcal{C}$.

Proof. We have the following commutative diagrams

$$\begin{tikzcd}
\mathcal{C}(F(X),A) \arrow[rr, "a"] \arrow[d, "\mu_X^*"] & & \mathcal{C}(X,G(A)) \arrow[d, "\delta_{A*}"] \\
\mathcal{C}(F^2(X),A) \arrow[r, "a"] \arrow[dr, "\eta_{F(X)}^*"'] & \mathcal{C}(F(X),G(A)) \arrow[r, "a"] \arrow[d, "\varepsilon_{A*}"] & \mathcal{C}(X,G^2(A)) \arrow[d, "G(\varepsilon_A)_*"] \\
& \mathcal{C}(F(X),A) \arrow[r, "a"] & \mathcal{C}(X,G(A))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}(F(X),A) \arrow[rr, "a"] \arrow[d, "\mu_X^*"] & & \mathcal{C}(X,G(A)) \arrow[d, "\delta_{A*}"] \\
\mathcal{C}(F^2(X),A) \arrow[r, "a"] \arrow[d, "F(\eta_X)^*"] & \mathcal{C}(F(X),G(A)) \arrow[r, "a"] \arrow[d, "\eta_X^*"] & \mathcal{C}(X,G^2(A)) \arrow[dl, "\varepsilon_{G(A)*}"] \\
\mathcal{C}(F(X),A) \arrow[r, "a"] & \mathcal{C}(X,G(A)) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}(F(X),A) \arrow[rr, "a"] \arrow[d, "\mu_X^*"] & & \mathcal{C}(X,G(A)) \arrow[d, "\delta_{A*}"] \\
\mathcal{C}(F^2(X),A) \arrow[r, "a"] \arrow[d, "F(\mu_X)^*"] & \mathcal{C}(F(X),G(A)) \arrow[r, "a"] \arrow[d, "\mu_X^*"] & \mathcal{C}(X,G^2(A)) \arrow[d, "\delta_{G(A)*}"] \\
\mathcal{C}(F^3(X),A) \arrow[r, "a"] & \mathcal{C}(F^2(X),G(A)) \arrow[r, "a^2"] & \mathcal{C}(X,G^3(A))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}(F(X),A) \arrow[rr, "a"] \arrow[d, "\mu_X^*"] & & \mathcal{C}(X,G(A)) \arrow[d, "\delta_{A*}"] \\
\mathcal{C}(F^2(X),A) \arrow[r, "a"] \arrow[d, "\mu_{F(X)}^*"] & \mathcal{C}(F(X),G(A)) \arrow[r, "a"] \arrow[d, "\delta_{A*}"] & \mathcal{C}(X,G^2(A)) \arrow[d, "G(\delta_A)_*"] \\
\mathcal{C}(F^3(X),A) \arrow[r, "a^2"] & \mathcal{C}(F(X),G^2(A)) \arrow[r, "a"] & \mathcal{C}(X,G^3(A))
\end{tikzcd}$$

The first diagram shows that $G(\varepsilon_A)\delta_A = 1_{G(A)}$ if and only if $\mu_X\eta_{F(X)} = 1_{F(X)}$. The second diagram shows that $\varepsilon_{G(A)}\delta_A = 1_{G(A)}$ if and only if $\mu_X F(\eta_X) = 1_{F(X)}$. The third and fourth diagrams show that $\delta_{G(A)}\delta_A = G(\delta_A)\delta$ if and only if $\mu_X F(\mu_X) = \mu_X\mu_{F(X)}$. □

We denote by $\mathcal{C}_{\boldsymbol{G}}$ the category of $\boldsymbol{G}$-coalgebras and $U : \mathcal{C}^{\boldsymbol{F}} \to \mathcal{C}$, $V : \mathcal{C}_{\boldsymbol{G}} \to \mathcal{C}$ denote the forgetful functors. For $h \in \mathcal{C}(F(X), X)$, it follows from the definition that $h\eta_X = \varepsilon_X a(h)$, $a^2(h\mu_X) = \delta_X a(h)$ and $a^2(F(h)h) = G(a(h))a(h)$ hold. Hence $\langle X, h\rangle \in \mathcal{C}^{\boldsymbol{F}}$ if and only if $\langle X, a(h)\rangle \in \mathcal{C}_{\boldsymbol{G}}$. Therefore, $\langle X, h\rangle \mapsto \langle X, a(h)\rangle$ gives an isomorphism $\Phi : \mathcal{C}^{\boldsymbol{F}} \to \mathcal{C}_{\boldsymbol{G}}$ of categories. Thus we have shown the following.

Proposition A.14.3 *There is an isomorphism of categories $\Phi : \mathcal{C}^{\boldsymbol{F}} \to \mathcal{C}_{\boldsymbol{G}}$ such that $V\Phi = U$.*

Let $\boldsymbol{G} = (G, \eta, \mu)$ and $\boldsymbol{H} = (H, \iota, \nu)$ be monads on $\mathcal{C}$ and $\mathcal{D}$, respectively. $U_{\boldsymbol{G}} : \mathcal{C}^{\boldsymbol{G}} \to \mathcal{C}$, $U_{\boldsymbol{H}} : \mathcal{D}^{\boldsymbol{H}} \to \mathcal{D}$ denote the forgetful functors and $F_{\boldsymbol{G}} : \mathcal{C} \to \mathcal{C}^{\boldsymbol{G}}$, $F_{\boldsymbol{H}} : \mathcal{D} \to \mathcal{D}^{\boldsymbol{H}}$ denote the free functors defined by $F_{\boldsymbol{G}}(X) = \langle G(X), \mu_X\rangle$, $F_{\boldsymbol{H}}(Y) = \langle H(Y), \nu_Y\rangle$, respectively. Then $F_{\boldsymbol{G}}$ (resp. $F_{\boldsymbol{H}}$) is a left adjoint of $U_{\boldsymbol{G}}$ (resp. $U_{\boldsymbol{H}}$).

Proposition A.14.4 *For a functor $T : \mathcal{C} \to \mathcal{D}$, the functors $\overline{T} : \mathcal{C}^{\boldsymbol{G}} \to \mathcal{D}^{\boldsymbol{H}}$ which makes*

$$\begin{tikzcd}
\mathcal{C}^{\boldsymbol{G}} \arrow[r, "\overline{T}"] \arrow[d, "U_{\boldsymbol{G}}"] & \mathcal{D}^{\boldsymbol{H}} \arrow[d, "U_{\boldsymbol{H}}"] \\
\mathcal{C} \arrow[r, "T"] & \mathcal{D}
\end{tikzcd}$$

commute are in 1–1 correspondence with natural transformations $\lambda : HT \to TG$ such that $\lambda\iota_T = T(\eta)$ and the following diagram commute.

$$\begin{tikzcd}
H^2T \arrow[r, "H(\lambda)"] \arrow[d, "\nu_T"] & HTG \arrow[r, "\lambda_G"] & TG^2 \arrow[d, "T(\mu)"] \\
HT \arrow[rr, "\lambda"] & & TG
\end{tikzcd}$$

Proof. Suppose that a natural transformation $\lambda : HT \to TG$ is given, satisfying the above conditions. Define $\overline{T_\lambda}$ by $\overline{T_\lambda}\langle X, h\rangle = \langle T(X), T(h)\lambda_X\rangle$ for $\langle X, h\rangle \in \mathcal{C}^{\boldsymbol{G}}$ and $\overline{T_\lambda}(f) = T(f)$ for $f \in \mathcal{C}^{\boldsymbol{G}}(\langle X, h\rangle, \langle Y, g\rangle)$. Then, the following diagrams commute

$$\begin{tikzcd}
H^2T(X) \arrow[rr, "\nu_{T(X)}"] \arrow[d, "\lambda_X"] & & HT(X) \arrow[d, "H(\lambda_X)"] \\
HTG(X) \arrow[r, "\lambda_{G(X)}"] \arrow[d, "HT(h)"] & TG^2(X) \arrow[r, "T(\mu_X)"] \arrow[d, "TG(h)"] & TG(X) \arrow[d, "T(h)"] \\
HT(X) \arrow[r, "\lambda_X"] & TG(X) \arrow[r, "T(h)"] & T(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
& T(X) \arrow[dl, "\iota_{T(X)}"'] \arrow[dr, "T(\eta_X)"] & \\
HT(X) \arrow[r, "\lambda_X"] & TG(X) \arrow[r, "T(h)"] & T(X)
\end{tikzcd}$$

$$\begin{tikzcd}
HT(X) \arrow[r, "\lambda_X"] \arrow[d, "HT(f)"] & TG(X) \arrow[r, "T(h)"] \arrow[d, "TG(f)"] & T(X) \arrow[d, "T(f)"] \\
HT(Y) \arrow[r, "\lambda_Y"] & TG(Y) \arrow[r, "T(g)"] & T(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^{\boldsymbol{G}} \arrow[r, "\overline{T_\lambda}"] \arrow[d, "U_{\boldsymbol{G}}"] & \mathcal{D}^{\boldsymbol{H}} \arrow[d, "U_{\boldsymbol{H}}"] \\
\mathcal{C} \arrow[r, "T_\lambda"] & \mathcal{D}
\end{tikzcd}$$

It follows from the above diagrams that $\langle T(X), T(h)\lambda_X\rangle \in \mathcal{D}^{\boldsymbol{H}}$ and that $\overline{T_\lambda}(f)$ is a morphism of $\boldsymbol{H}$-algebras.
Conversely, suppose that a functor $\overline{T} : \mathcal{C}^{\boldsymbol{G}} \to \mathcal{D}^{\boldsymbol{H}}$ satisfying $U_{\boldsymbol{H}}\overline{T} = TU_{\boldsymbol{G}}$ is given. We put $\overline{T}F_{\boldsymbol{G}}(X) = \langle TG(X), \psi_X\rangle$. Since $\overline{T}F_{\boldsymbol{G}}(f)$ is a morphism of $\boldsymbol{H}$-algebras for a morphism $f : X \to Y$ of $\mathcal{C}$, ψ_X is natural in X and we have a natural transformation $\psi : HTG \to TG$. Define $\lambda(\overline{T}) : HT \to TG$ by $\lambda(\overline{T})_X = \psi_X HT(\eta_X)$. We claim that $\psi_X HT(\mu_X) = T(\mu_X)\psi_{G(X)}$ hold. In fact, a morphism of $\boldsymbol{G}$-algebras $\mu_X : F_{\boldsymbol{G}}(G(X)) \to F_{\boldsymbol{G}}(X)$ induces a morphism of $\boldsymbol{H}$-algebras $\overline{T}(\mu_X) : \overline{T}F_{\boldsymbol{G}}(G(X)) \to \overline{T}F_{\boldsymbol{G}}(X)$. Hence $U_{\boldsymbol{H}}\overline{T}(\mu_X) = T(\mu_X)$ satisfies the above equality. Since ψ_X is a structure map of $\boldsymbol{H}$-algebra, we have the following commutative diagrams.

$$\begin{tikzcd}
T \arrow[r, "\iota_T"] \arrow[d, "T(\eta)"] & HT \arrow[d, "HT(\eta)"] \\
TG \arrow[r, "\iota_{TG}"] \arrow[dr, "id_{TG}"'] & HTG \arrow[d, "\psi"] \\
& TG
\end{tikzcd}
\qquad
\begin{tikzcd}
& & & HTG \arrow[dl, "id_{HTG}"'] \arrow[d, "HT(\eta_G)"] \\
H^2T \arrow[r, "H^2T(\eta)"] \arrow[d, "\nu_T"] & H^2TG \arrow[r, "H(\psi)"] \arrow[d, "\nu_{TG}"] & HTG \arrow[d, "\psi"] & HTG^2 \arrow[l, "HT(\mu)"'] \arrow[d, "\psi_G"] \\
HT \arrow[r, "HT(\eta)"] & HTG \arrow[r, "\psi"] & TG & TG^2 \arrow[l, "T(\mu)"']
\end{tikzcd}$$

Thus $\lambda(\overline{T})$ satisfies the required condition.
The commutativity of the following diagram implies $\lambda(\overline{T_\lambda}) = \lambda$.

$$\begin{tikzcd}
& HT \arrow[dl, "HT(\eta)"'] \arrow[d, "H(\iota_T)"] \arrow[dr, "id_{HT}"] & \\
HTG \arrow[dr, "\lambda_G"'] & HT^2 \arrow[l, "H(\lambda)"'] \arrow[r, "\nu_T"] & HT \arrow[d, "\lambda"] \\
& TG^2 \arrow[r, "T(\mu)"] & TG
\end{tikzcd}$$

On the other hand, for given functor $\overline{T} : \mathcal{C}^{\boldsymbol{G}} \to \mathcal{D}^{\boldsymbol{H}}$ satisfying $U_{\boldsymbol{H}}\overline{T} = TU_{\boldsymbol{G}}$, put $\overline{T_\lambda}\langle X, h\rangle = \langle T(X), \tilde{h}\rangle$. Since $h : G(X) \to X$ gives a morphism of $\boldsymbol{G}$-algebras $F_{\boldsymbol{G}}(X) \to \langle X, h\rangle$, $T(h) : TG(X) \to T(X)$ gives a morphism of $\boldsymbol{H}$-algebras $\overline{T} : \langle TG(X), \psi_X\rangle \to \langle T(X), \tilde{h}\rangle$. Hence we have $T(h)\psi_X HT(\eta_X) = \tilde{h}HT(h)HT(\eta_X) = \tilde{h}$ and this means $\overline{T_{\lambda(\overline{T})}} = \overline{T}$. □

It follows from the above proof that $\lambda : HT \to TG$ defines a natural transformation $\bar{\lambda} : F_{\boldsymbol{H}}T \to \overline{T_\lambda}F_{\boldsymbol{G}}$ such that $U_{\boldsymbol{H}}(\bar{\lambda}) = \lambda$.

Proposition A.14.5 *Let T, $\overline{T}$, λ be as above. λ is an isomorphism if and only if there is a natural equivalence $\kappa : F_{\boldsymbol{H}}T \to \overline{T}F_{\boldsymbol{G}}$ such that $U_{\boldsymbol{H}}(\kappa) : HT \to TG$ satisfies $U_{\boldsymbol{H}}(\kappa)\iota_T = T(\eta)$.*

Proof. If λ is an isomorphism, $\lambda_X : HT(X) \to TG(X)$ gives an isomorphism $\bar{\lambda} : F_{\boldsymbol{H}}T(X) \to \overline{T}F_{\boldsymbol{G}}(X)$ of $\boldsymbol{H}$-algebras satisfying $U_{\boldsymbol{H}}(\bar{\lambda})\iota_T = T(\eta)$.

Suppose that there is an isomorphism $\kappa : F_{\boldsymbol{H}}T \to \overline{T}F_{\boldsymbol{G}}$ satisfying $U_{\boldsymbol{H}}(\kappa)\iota_T = T(\eta)$. Put $\overline{T}F_{\boldsymbol{G}}(X) = \langle TG(X), \psi_X\rangle$, then $\lambda = \psi HT(\eta) = \psi H(U_{\boldsymbol{H}}(\kappa))H(\iota_T) = U_{\boldsymbol{H}}(\kappa)\nu_T H(\iota_T) = U_{\boldsymbol{H}}(\kappa)$. Hence λ is an isomorphism. □

Let $\boldsymbol{G} = (G, \eta, \mu)$ and $\boldsymbol{H} = (H, \iota, \nu)$ be monads on $\mathcal{C}$. A morphism $\lambda : \boldsymbol{H} \to \boldsymbol{G}$ is a natural transformation $\lambda : H \to G$ satisfying $\lambda\iota = \eta$ and $\lambda\nu = \mu\lambda_G H(\lambda)$. Let us denote by $\mathbf{Mon}(\mathcal{C})$ the category of monads on $\mathcal{C}$. Consider a category $\mathrm{ad}(\mathcal{C})$ with object $(U : \mathcal{X} \to \mathcal{C}, F : \mathcal{C} \to \mathcal{X})$ such that F is a left adjoint of U, whose morphism $\overline{T} : (U : \mathcal{X} \to \mathcal{C}, F : \mathcal{C} \to \mathcal{X}) \to (V : \mathcal{Y} \to \mathcal{C}, L : \mathcal{C} \to \mathcal{Y})$ is a functor $\overline{T} : \mathcal{X} \to \mathcal{Y}$ satisfying $VT = U$. Define a functor $\Psi : \mathbf{Mon}(\mathcal{C})^{op} \to \mathrm{ad}(\mathcal{C})$ by $\Psi(\boldsymbol{G}) = (U_{\boldsymbol{G}} : \mathcal{C}^{\boldsymbol{G}} \to \mathcal{C}, F_{\boldsymbol{G}} : \mathcal{C} \to \mathcal{C}^{\boldsymbol{G}})$ and $\Psi(\lambda) = \overline{T}_\lambda$. (A.14.4) immediately implies the following.

Proposition A.14.6 *Ψ is fully faithful. Hence if there is an isomorphism $\overline{T} : \mathcal{C}^{\boldsymbol{G}} \to \mathcal{C}^{\boldsymbol{H}}$ satisfying $U_{\boldsymbol{H}}\overline{T} = U_{\boldsymbol{G}}$, $\boldsymbol{H}$ is isomorphic to $\boldsymbol{G}$.*

We note that a morphism of monads $\lambda : \boldsymbol{H} \to \boldsymbol{G}$ defines a natural transformation $\bar{\lambda} : F_{\boldsymbol{H}} \to \overline{T_\lambda}F_{\boldsymbol{G}}$ such that $U_{\boldsymbol{H}}(\bar{\lambda}) = \lambda$.

We state the dual assertions of (A.14.4), (A.14.5) and (A.14.6).

Let $\boldsymbol{G} = (G, \varepsilon, \delta)$ and $\boldsymbol{H} = (H, \rho, \phi)$ be comonads on $\mathcal{C}$ and $\mathcal{D}$, respectively. $V_{\boldsymbol{G}} : \mathcal{C}_{\boldsymbol{G}} \to \mathcal{C}$, $V_{\boldsymbol{H}} : \mathcal{D}_{\boldsymbol{H}} \to \mathcal{D}$ denote the forgetful functors and $F_{\boldsymbol{G}} : \mathcal{C} \to \mathcal{C}_{\boldsymbol{G}}$, $F_{\boldsymbol{H}} : \mathcal{D} \to \mathcal{D}_{\boldsymbol{H}}$ denote the free functors defined by $F_{\boldsymbol{G}}(X) = \langle G(X), \delta_X\rangle$, $F_{\boldsymbol{H}}(Y) = \langle H(Y), \phi_Y\rangle$, respectively. Then $F_{\boldsymbol{G}}$ (resp. $F_{\boldsymbol{H}}$) is a right adjoint of $V_{\boldsymbol{G}}$ (resp. $V_{\boldsymbol{H}}$).

Proposition A.14.7 *For a functor $T : \mathcal{C} \to \mathcal{D}$, the functors $\overline{T} : \mathcal{C}_{\boldsymbol{G}} \to \mathcal{D}_{\boldsymbol{H}}$ which makes*

$$\begin{tikzcd}
\mathcal{C}_{\boldsymbol{G}} \arrow[r, "\overline{T}"] \arrow[d, "V_{\boldsymbol{G}}"] & \mathcal{D}_{\boldsymbol{H}} \arrow[d, "V_{\boldsymbol{H}}"] \\
\mathcal{C} \arrow[r, "T"] & \mathcal{D}
\end{tikzcd}$$

commute are in 1–1 correspondence with natural transformations $\lambda : TG \to HT$ such that $\rho_T\lambda = T(\varepsilon)$ and the following diagram commute.

$$\begin{tikzcd}
TG \arrow[rr, "\lambda"] \arrow[d, "T(\delta)"] & & HT \arrow[d, "\phi_T"] \\
TG^2 \arrow[r, "\lambda_G"] & HTG \arrow[r, "H(\lambda)"] & H^2T
\end{tikzcd}$$

It follows that $\lambda : TG \to HT$ defines a natural transformation $\bar{\lambda} : \overline{T_\lambda}F_{\boldsymbol{G}} \to F_{\boldsymbol{H}}T$ such that $U_{\boldsymbol{H}}(\bar{\lambda}) = \lambda$.

Proposition A.14.8 *Let T, $\overline{T}$, λ be as above. λ is an isomorphism if and only if there is a natural equivalence $\kappa : \overline{T}F_{\boldsymbol{G}} \to F_{\boldsymbol{H}}T$ such that $V_{\boldsymbol{H}}(\kappa) : TG \to HT$ satisfies $\rho_T V_{\boldsymbol{H}}(\kappa) = T(\varepsilon)$.*

Let $\boldsymbol{G} = (G, \varepsilon, \delta)$ and $\boldsymbol{H} = (H, \rho, \phi)$ be comonads on $\mathcal{C}$. A morphism $\lambda : \boldsymbol{G} \to \boldsymbol{H}$ is a natural transformation $\lambda : G \to H$ satisfying $\rho\lambda = \varepsilon$ and $\phi\lambda = H(\lambda)\lambda_G\delta$ (or $\phi\lambda = \lambda_H G(\lambda)\delta$).

Let us denote by $\mathrm{Comon}(\mathcal{C})$ the category of comonads on $\mathcal{C}$. Consider a category $\mathrm{ad}(\mathcal{C})$ with object $(V : \mathcal{X} \to \mathcal{C}, F : \mathcal{C} \to \mathcal{X})$ such that F is a right adjoint of V, whose morphism $\overline{T} : (V : \mathcal{X} \to \mathcal{C}, F : \mathcal{C} \to \mathcal{X}) \to (U : \mathcal{Y} \to \mathcal{C}, L : \mathcal{C} \to \mathcal{Y})$ is a functor $\overline{T} : \mathcal{X} \to \mathcal{Y}$ satisfying $UT = V$. Define a functor $\Psi : \mathrm{Comon}(\mathcal{C}) \to \mathrm{ad}(\mathcal{C})$ by $\Psi(\boldsymbol{G}) = (V_{\boldsymbol{G}} : \mathcal{C}_{\boldsymbol{G}} \to \mathcal{C}, F_{\boldsymbol{G}} : \mathcal{C} \to \mathcal{C}_{\boldsymbol{G}})$ and $\Psi(\lambda) = \overline{T}_\lambda$. (A.14.7) immediately implies the following.

Proposition A.14.9 *Ψ is fully faithful. Hence if there is an isomorphism $\overline{T} : \mathcal{C}_{\boldsymbol{G}} \to \mathcal{C}_{\boldsymbol{H}}$ satisfying $V_{\boldsymbol{H}}\overline{T} = V_{\boldsymbol{G}}$, $\boldsymbol{H}$ is isomorphic to $\boldsymbol{G}$.*

We note that a morphism of comonads $\lambda : \boldsymbol{G} \to \boldsymbol{H}$ defines a natural transformation $\bar{\lambda} : \overline{T_\lambda}F_{\boldsymbol{G}} \to F_{\boldsymbol{H}}$ such that $U_{\boldsymbol{H}}(\bar{\lambda}) = \lambda$.

Let $F : \mathcal{C} \to \mathcal{A}$ and $M : \mathcal{D} \to \mathcal{B}$ be functors. Suppose that F and M has right adjoints G and H with units $\eta : 1_{\mathcal{C}} \to GF$, $\iota : 1_{\mathcal{D}} \to HM$ and counits $\varepsilon : FG \to 1_{\mathcal{A}}$, $\rho : MH \to 1_{\mathcal{B}}$, respectively. Consider the monads $\boldsymbol{G} = (GF, \eta, G(\varepsilon_F))$ and $\boldsymbol{H} = (HM, \iota, H(\rho_M))$ defined from the adjunctions. We denote by $K_{\boldsymbol{G}} : \mathcal{A} \to \mathcal{C}^{\boldsymbol{G}}$ and $K_{\boldsymbol{H}} : \mathcal{B} \to \mathcal{D}^{\boldsymbol{H}}$ the comparison functors.

Proposition A.14.10 *Let $T : \mathcal{C} \to \mathcal{D}$ and $\widetilde{T} : \mathcal{A} \to \mathcal{B}$ be functors. Suppose that natural transformations $\alpha : MT \to \widetilde{T}F$ and $\beta : H\widetilde{T} \to TG$ are given such that the following diagrams commute.*

$$\begin{tikzcd}
T \arrow[r, "T(\eta)"] \arrow[d, "\iota_T"] & TGF \\
HMT \arrow[r, "H(\alpha)"] & H\widetilde{T}F \arrow[u, "\beta_F"']
\end{tikzcd}
\qquad
\begin{tikzcd}
MH\widetilde{T} \arrow[r, "M(\beta)"] \arrow[d, "\rho_{\widetilde{T}}"] & MTG \arrow[d, "\alpha_G"] \\
\widetilde{T} & \widetilde{T}FG \arrow[l, "\widetilde{T}(\varepsilon)"']
\end{tikzcd}$$

Then, $\lambda = \beta_F H(\alpha) : HMT \to TGF$ satisfies the conditions of (A.14.4). Moreover, let $\overline{T} : \mathcal{C}^{\boldsymbol{G}} \to \mathcal{D}^{\boldsymbol{H}}$ be the lifting of T defined from λ. Then, $\beta : H\widetilde{T} \to TG$ lifts to a natural transformation $\bar{\beta} : K_{\boldsymbol{H}}\widetilde{T} \to \overline{T}K_{\boldsymbol{G}}$.

Proof. The commutativity of the above left diagram implies $\lambda\iota_T = T(\eta)$. By the naturality of ρ, β, it follows from the assumption that the following diagram commutes.

$$\begin{tikzcd}
HMHMT \arrow[r, "HMH(\alpha)"] \arrow[d, "H(\rho_{MT})"] & HMH\widetilde{T}F \arrow[r, "HM(\beta_F)"] \arrow[d, "H(\rho_{\widetilde{T}F})"] & HMTGF \arrow[d, "H(\alpha_{GF})"] \\
HMT \arrow[r, "H(\alpha)"] & H\widetilde{T}F \arrow[d, "\beta_F"] & H\widetilde{T}FGF \arrow[l, "H\widetilde{T}(\varepsilon_F)"'] \arrow[d, "\beta_{FGF}"] \\
& TGF & TGFGF \arrow[l, "TG(\varepsilon_F)"']
\end{tikzcd}$$

Thus λ satisfies the conditions of (A.14.4).

For any object X of $\mathcal{A}$, the commutativity of the following diagram implies that $\beta_X : H\widetilde{T}(X) \to TG(X)$ gives a morphism of $\boldsymbol{H}$-algebras $K_{\boldsymbol{H}}\widetilde{T}(X){=}\langle H\widetilde{T}(X), H(\rho_{\widetilde{T}(A)})\rangle \to \overline{T}K_{\boldsymbol{G}}(X){=}\langle TG(X), TG(\varepsilon_X)\beta_{FG(X)}H(\alpha_{TG(X)})\rangle$.

$$\begin{tikzcd}
HMH\widetilde{T}(X) \arrow[r, "H(\rho_{\widetilde{T}(X)})"] \arrow[d, "HM(\beta_X)"] & H\widetilde{T}(X) \arrow[r, "\beta_X"] & TG(X) \\
HMTG(X) \arrow[r, "\alpha_{G(X)}"] & H\widetilde{T}FG(X) \arrow[r, "\beta_{FG(X)}"] \arrow[u, "H\widetilde{T}(\varepsilon_X)"'] & TGFG(X) \arrow[u, "TG(\varepsilon_X)"']
\end{tikzcd}$$

□

Proposition A.14.11 *Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$ such that a product $Z \times X$ exists for each object Z of $\mathcal{C}$. Define a comonad $\boldsymbol{H} = (H, \rho, \phi)$ on $\mathcal{C}$ by $H(Z) = Z \times X$, $\rho_Z = \mathrm{pr}_1 : H(Z) \to Z$ and $\phi_Z = ((\mathrm{pr}_1, \mathrm{pr}_2), \mathrm{pr}_2) : H(Z) \to H^2(Z)$. Then there is a unique isomorphism $\varXi : \mathcal{C}_{\boldsymbol{H}} \to \mathcal{C}/X$ satisfying $\varSigma_X \varXi = V_{\boldsymbol{H}}$ and $\varXi F_{\boldsymbol{H}} = X^*$, where $V_{\boldsymbol{H}} : \mathcal{C}_{\boldsymbol{H}} \to \mathcal{C}$ is the forgetful functor and $F_{\boldsymbol{H}} : \mathcal{C} \to \mathcal{C}_{\boldsymbol{H}}$ is the free functor.*

Proof. It is easily seen that $\psi : Z \to Z \times X$ is a structure map of an $\boldsymbol{H}$-coalgebra if and only if $\mathrm{pr}_1\psi = id_Z$, in other words, $\psi = (id_Z, f)$ for some $f : Z \to X$. We define $\varXi$ is defined by $\varXi\langle Z, \psi\rangle = (Z \xrightarrow{\mathrm{pr}_2\psi} X)$ and $\varXi(\varphi) = \varphi$ for $\varphi : \langle Z, \psi\rangle \to \langle W, \zeta\rangle$. Then $\varXi$ satisfies $\varSigma_X \varXi = V_{\boldsymbol{H}}$ and $\varXi F_{\boldsymbol{H}} = X^*$, and the inverse $\varXi^{-1}$ is given by $\varXi^{-1}(Z \xrightarrow{f} X) = \langle Z, (id_Z, f)\rangle$ and $\varXi^{-1}(\alpha) = \alpha$ for $\alpha : (Z \xrightarrow{f} X) \to (W \xrightarrow{g} X)$.

Since $V_{\boldsymbol{H}}$ and $\varSigma_X$ are faithful, it follows from $\varSigma_X \varXi = V_{\boldsymbol{H}}$ that $\varXi$ is uniquely determined on the set of morphisms. Put $\varXi\langle Z, \psi\rangle = (Z \xrightarrow{g} X)$, then $\psi : \langle Z, \psi\rangle \to \langle Z \times X, \phi_Z\rangle = F_{\boldsymbol{H}}(Z)$ is a morphism of $\boldsymbol{H}$-coalgebras. Applying $\varXi$ to ψ, $\psi : (Z \xrightarrow{g} X) \to \varXi F_{\boldsymbol{H}}(Z) = X^*(Z) = (Z \times X \xrightarrow{\mathrm{pr}_2} X)$ is a morphism of $\mathcal{C}/X$. Therefore $g = \mathrm{pr}_2\psi$ and the uniqueness of $\varXi$ follows. □

Let $R : \mathcal{C} \to \mathcal{D}$ and $L : \mathcal{D} \to \mathcal{C}$ be functors such that L is a left adjoint of R. We denote by $\varepsilon : LR \to id_{\mathcal{C}}$ and $\eta : id_{\mathcal{D}} \to RL$ the counit and the unit of this adjunction. Consider the comonad $\boldsymbol{G} = (LR, \varepsilon, L(\eta_R))$ on $\mathcal{C}$ obtained from the adjunction.

Proposition A.14.12 *Suppose that $\mathcal{D}$ has a terminal object $1_{\mathcal{D}}$ such that a product $Z \times L(1_{\mathcal{D}})$ exists for each object Z of $\mathcal{C}$. Let $\boldsymbol{H} = (H, \rho, \phi)$ be a comonad on $\mathcal{C}$ given in (A.14.11) for $X = L(1_{\mathcal{D}})$. A natural transformation $\lambda : LR \to H$ defined by $\lambda_Z = (\varepsilon_Z, L(R(Z) \to 1_{\mathcal{D}})) : LR(Z) \to Z \times X$ is a morphism of comonads.*

We note that if $\mathcal{C}$ has a terminal object $1_{\mathcal{C}}$, $R(1_{\mathcal{C}})$ is a terminal object of $\mathcal{D}$.

Proof. It is obvious from the definition that $\rho\lambda = \varepsilon$. We examine each component of

$$\lambda_{Z\times X}LR(\lambda_Z)L(\eta_{R(Z)}) : LR(Z) \to (Z\times X)\times X.$$

$$\begin{aligned}
\mathrm{pr}_1\lambda_{Z\times X}LR(\lambda_Z)L(\eta_{R(Z)}) &= \mathrm{pr}_1\varepsilon_{Z\times X}LR(\lambda_Z)L(\eta_{R(Z)}) = \varepsilon_Z LR(\mathrm{pr}_1)LR(\lambda_Z)L(\eta_{R(Z)})\\
&= \varepsilon_Z LR(\varepsilon_Z)L(\eta_{R(Z)}) = \varepsilon_Z,\\
\mathrm{pr}_2\lambda_{Z\times X}LR(\lambda_Z)L(\eta_{R(Z)}) &= \mathrm{pr}_2\varepsilon_{Z\times X}LR(\lambda_Z)L(\eta_{R(Z)}) = \varepsilon_X LR(\mathrm{pr}_2)LR(\lambda_Z)L(\eta_{R(Z)})\\
&= \varepsilon_X LRL(R(Z)\to 1_{\mathcal{D}})L(\eta_{R(Z)}) = \varepsilon_X L(\eta_{1_{\mathcal{D}}})L(R(Z)\to 1_{\mathcal{D}}) = L(R(Z)\to 1_{\mathcal{D}}),\\
\mathrm{pr}_3\lambda_{Z\times X}LR(\lambda_Z)L(\eta_{R(Z)}) &= L(R(Z\times X)\to 1_{\mathcal{D}})LR(\lambda_Z)L(\eta_{R(Z)}) = L(RLR(Z)\to 1_{\mathcal{D}})L(\eta_{R(Z)})\\
&= L(R(Z)\to 1_{\mathcal{D}}).
\end{aligned}$$

On the other hand, we have $\phi_Z\lambda_Z = (\varepsilon_Z, L(R(Z)\to 1_{\mathcal{D}}), L(R(Z)\to 1_{\mathcal{D}}))$. □

A.15 Adjoint lifting theorems

Theorem A.15.1 *Let T, $\overline{T}$, λ be as in (A.14.4) and suppose that $\mathcal{C}^{\boldsymbol{G}}$ has coequalizer of reflexive pairs. If T has a left adjoint, so has $\overline{T}$.*

Proof. Let $L : \mathcal{D} \to \mathcal{C}$ be a left adjoint of T and denote by $\alpha : 1_{\mathcal{D}} \to TL$, $\beta : LT \to 1_{\mathcal{C}}$ the unit, counit of the adjunction. We define $\gamma : F_{\boldsymbol{G}}LH \to F_{\boldsymbol{G}}L$ so that the following diagram $(*)$ commute for any $Y \in \mathcal{D}$ and $\langle X, h\rangle \in \mathcal{C}^{\boldsymbol{G}}$.

$$\begin{tikzcd}
\mathcal{C}^{\boldsymbol{G}}(F_{\boldsymbol{G}}L(Y), \langle X,h\rangle) \arrow[r, "a_1"] \arrow[d, "\gamma_Y^*"] & \mathcal{C}(L(Y), X) \arrow[r, "a_2"] & \mathcal{D}(Y, T(X)) \arrow[r, "a_3"] & \mathcal{D}^{\boldsymbol{H}}(F_{\boldsymbol{H}}(Y), \overline{T}\langle X,h\rangle) \arrow[d, "\nu_Y^*"] \\
\mathcal{C}^{\boldsymbol{G}}(F_{\boldsymbol{G}}LH(Y), \langle X,h\rangle) \arrow[r, "a_1"] & \mathcal{C}(LH(Y), X) \arrow[r, "a_2"] & \mathcal{D}(H(Y), T(X)) \arrow[r, "a_3"] & \mathcal{D}^{\boldsymbol{H}}(F_{\boldsymbol{H}}H(Y), \overline{T}\langle X,h\rangle)
\end{tikzcd} \qquad (*)$$

Here the horizontal arrows of the above diagram are natural bijections. Hence we have the following.

$$\begin{aligned}
\gamma_Y &= a_1^{-1}a_2^{-1}a_3^{-1}\nu_Y^* a_3a_2a_1(id_{F_{\boldsymbol{G}}LH(Y)})\\
\nu_Y^* a_3a_2a_1(id_{F_{\boldsymbol{G}}LH(Y)}) &= \nu_Y^* a_3a_2(\eta_{L(Y)}) = \nu_Y^* a_3(T(\eta_{L(Y)})\alpha_Y) = \nu_Y^*(T(\mu_{L(Y)})\lambda_{GL(Y)}HT(\eta_{L(Y)})H(\alpha_Y))\\
&= T(\mu_{L(Y)})\lambda_{GL(Y)}HT(\eta_{L(Y)})H(\alpha_Y)\nu_Y = T(\mu_{L(Y)})\lambda_{GL(Y)}HT(\eta_{L(Y)})\nu_{TL(Y)}H^2(\alpha_Y)\\
&= T(\mu_{L(Y)})\lambda_{GL(Y)}\nu_{TGL(Y)}H^2T(\eta_{L(Y)})H^2(\alpha_Y)\\
&= T(\mu_{L(Y)})T(\mu_{GL(Y)})\lambda_{G^2L(Y)}H(\lambda_{GL(Y)})H^2T(\eta_{L(Y)})H^2(\alpha_Y)\\
&= T(\mu_{L(Y)})TG(\mu_{L(Y)})\lambda_{G^2L(Y)}HTG(\eta_{L(Y)})H(\lambda_{L(Y)})H^2(\alpha_Y)\\
&= T(\mu_{L(Y)})TG(\mu_{L(Y)})TG^2(\eta_{L(Y)})\lambda_{GL(Y)}H(\lambda_{L(Y)})H^2(\alpha_Y)\\
&= T(\mu_{L(Y)})\lambda_{GL(Y)}H(\lambda_{L(Y)})H^2(\alpha_Y) = \lambda_{L(Y)}\nu_{TL(Y)}H^2(\alpha_Y) = \lambda_{L(Y)}H(\alpha_Y)\nu_Y
\end{aligned}$$

Thus we have the following equality which shows $\gamma = \mu_LG(\beta_{GL})GL(\lambda_L)GLH(\alpha)$.

$$\begin{aligned}
a_1^{-1}a_2^{-1}a_3^{-1}(\lambda_{L(Y)}H(\alpha_Y)\nu_Y) &= a_1^{-1}a_2^{-1}(\lambda_{L(Y)}H(\alpha_Y)\nu_Y\iota_{H(Y)}) = a_1^{-1}a_2^{-1}(\lambda_{L(Y)}H(\alpha_Y))\\
&= a_1^{-1}(\beta_{GL(Y)}L(\lambda_{L(Y)})LH(\alpha_Y)) = \mu_{L(Y)}G(\beta_{GL(Y)})GL(\lambda_{L(Y)})GLH(\alpha_Y).
\end{aligned}$$

For $\langle Y, g\rangle \in \mathcal{D}^{\boldsymbol{H}}$, $GL(g)$ and γ_Y give a reflexive pair of morphisms $F_{\boldsymbol{G}}LH(X) \rightrightarrows F_{\boldsymbol{G}}L(X)$ of $\boldsymbol{G}$-algebras whose common right inverse is $GL(\iota_Y)$. In fact, since g is a structure map of $\boldsymbol{H}$-algebras, $GL(g)GL(\iota_Y) = id_{GL(Y)}$ is obvious and $\gamma_Y GL(\iota) = id_{GL(Y)}$ follows from $\beta_L L(\alpha) = id_L$ and the commutativity of the following diagram.

$$\begin{tikzcd}
 & & GL \arrow[rr, "GL(\alpha)"] \arrow[dll, "GL(\iota)"'] & & GLTL \arrow[rr, "G(\beta_{L(Y)})"] \arrow[dll, "GL(\iota_{TL})"'] \arrow[d, "GLT(\eta_L)"] & & GL \arrow[drr, "G(\eta_L)"] & & \\
GLH \arrow[rr, "GLH(\alpha)"] & & GLHTL \arrow[rr, "GL(\lambda_L)"] & & GLTGL \arrow[rr, "G(\beta_{GL(Y)})"] & & G^2L \arrow[rr, "\mu_L"] & & GL
\end{tikzcd}$$

Define $\overline{L} : \mathcal{D}^{\boldsymbol{H}} \to \mathcal{C}^{\boldsymbol{G}}$ as follows. Let $\overline{L}\langle Y, g\rangle$ be the coequalizer of $F_{\boldsymbol{G}}LH(X) \underset{\gamma_Y}{\overset{GL(g)}{\rightrightarrows}} F_{\boldsymbol{G}}L(X)$ and if $\varphi : \langle Y, g\rangle \to \langle Z, k\rangle$ is a morphism of $\boldsymbol{H}$-algebras, $\overline{L}(\varphi)$ is the map induced by $F_{\boldsymbol{G}}LH(\varphi)$ and $F_{\boldsymbol{G}}L(\varphi)$. In fact, the following diagrams commute.

$$\begin{tikzcd}
F_{\boldsymbol{G}}LH(Y) \arrow[r, "GL(g)"] \arrow[d, "F_{\boldsymbol{G}}LH(\varphi)"] & F_{\boldsymbol{G}}L(Y) \arrow[d, "F_{\boldsymbol{G}}L(\varphi)"] \\
F_{\boldsymbol{G}}LH(Z) \arrow[r, "GL(k)"] & F_{\boldsymbol{G}}L(Z)
\end{tikzcd}
\qquad
\begin{tikzcd}
F_{\boldsymbol{G}}LH(Y) \arrow[r, "\gamma_Y"] \arrow[d, "F_{\boldsymbol{G}}LH(\varphi)"] & F_{\boldsymbol{G}}L(Y) \arrow[d, "F_{\boldsymbol{G}}L(\varphi)"] \\
F_{\boldsymbol{G}}LH(Z) \arrow[r, "\gamma_Y"] & F_{\boldsymbol{G}}L(Z)
\end{tikzcd}$$

For any $\boldsymbol{G}$-algebra $\langle X, h\rangle$, we have an exact sequence

$$\mathcal{C}^{\boldsymbol{G}}(\overline{L}\langle Y, g\rangle, \langle X, h\rangle) \xrightarrow{e^*} \mathcal{C}^{\boldsymbol{G}}(F_{\boldsymbol{G}}L(Y), \langle X, h\rangle) \underset{\gamma_Y^*}{\overset{GL(g)^*}{\rightrightarrows}} \mathcal{C}^{\boldsymbol{G}}(F_{\boldsymbol{G}}LH(Y), \langle X, h\rangle) \ .$$

On the other hand, $F_{\boldsymbol{H}}H(Y) \underset{\nu_Y}{\overset{H(g)}{\rightrightarrows}} F_{\boldsymbol{H}}(Y) \xrightarrow{g} \langle Y, g\rangle$ is a coequalizer, we also have an exact sequence

$$\mathcal{D}^{\boldsymbol{H}}(\langle Y, g\rangle, \overline{T}\langle X, h\rangle) \xrightarrow{g^*} \mathcal{D}^{\boldsymbol{H}}(F_{\boldsymbol{H}}(Y), \overline{T}\langle X, h\rangle) \underset{\nu_Y^*}{\overset{H(g)^*}{\rightrightarrows}} \mathcal{D}^{\boldsymbol{H}}(F_{\boldsymbol{H}}H(Y), \overline{T}\langle X, h\rangle) \ .$$

The following diagram $(**)$ commute.

$$\begin{tikzcd}
\mathcal{C}^{\boldsymbol{G}}(F_{\boldsymbol{G}}L(Y), \langle X, h\rangle) \arrow[r, "a_1"] \arrow[d, "GL(g)^*"] & \mathcal{C}(L(Y), X) \arrow[r, "a_2"] & \mathcal{D}(Y, T(X)) \arrow[r, "a_3"] & \mathcal{D}^{\boldsymbol{H}}(F_{\boldsymbol{H}}(Y), \overline{T}\langle X, h\rangle) \arrow[d, "H(g)^*"] \\
\mathcal{C}^{\boldsymbol{G}}(F_{\boldsymbol{G}}LH(Y), \langle X, h\rangle) \arrow[r, "a_1"] & \mathcal{C}(LH(Y), X) \arrow[r, "a_2"] & \mathcal{D}(H(Y), T(X)) \arrow[r, "a_3"] & \mathcal{D}^{\boldsymbol{H}}(F_{\boldsymbol{H}}H(Y), \overline{T}\langle X, h\rangle)
\end{tikzcd} \qquad (**)$$

In order to verify the commutativity of the above diagram, it suffices to consider the generic case $\langle X, h\rangle = F_{\boldsymbol{G}}LH(Y)$ and show $a_3a_2a_1GL(g)^*(id_{F_{\boldsymbol{G}}LH(Y)}) = H(g)^*a_3a_2a_1(id_{F_{\boldsymbol{G}}LH(Y)})$.

$$\begin{aligned}
a_3a_2a_1GL(g)^*(id_{F_{\boldsymbol{G}}LH(Y)}) &= a_3a_2a_1(GL(g)) = a_3a_2(GL(g)\eta_{LH(Y)}) = a_3(TGL(g)T(\eta_{LH(Y)})\alpha_{H(Y)}) \\
&= T(\mu_{L(Y)})\lambda_{GL(Y)}HTGL(g)HT(\eta_{LH(Y)})H(\alpha_{H(Y)}) \\
&= T(\mu_{L(Y)})TG^2L(g)\lambda_{GLH(Y)}HT(\eta_{LH(Y)})H(\alpha_{H(Y)}) \\
&= TGL(g)T(\mu_{LH(Y)})\lambda_{GLH(Y)}HT(\eta_{LH(Y)})H(\alpha_{H(Y)}) \\
&= TGL(g)T(\mu_{LH(Y)})TG(\eta_{LH(Y)})\lambda_{LH(Y)}H(\alpha_{H(Y)}) \\
&= TGL(g)\lambda_{LH(Y)}H(\alpha_{H(Y)}) = \lambda_{L(Y)}HTL(g)H(\alpha_{H(Y)}) \\
&= \lambda_{L(Y)}H(\alpha_Y)H(g) = H(g)^*(\lambda_{L(Y)}H(\alpha_Y)) \\
&= H(g)^*(T(\mu_{L(Y)})TG(\eta_{L(Y)})\lambda_{L(Y)}H(\alpha_Y)) \\
&= H(g)^*(T(\mu_{L(Y)})\lambda_{GL(Y)}HT(\eta_{L(Y)})H(\alpha_Y)) \\
&= H(g)^*a_3(T(\eta_{L(Y)})\alpha_Y) = H(g)^*a_3a_2(\eta_{L(Y)}) = H(g)^*a_3a_2a_1(id_{F_{\boldsymbol{G}}LH(Y)}).
\end{aligned}$$

It follows from the commutativity of diagrams $(*)$ and $(**)$, $a_3a_2a_1$ induces a natural bijection

$$\mathcal{C}^{\boldsymbol{G}}(\overline{L}\langle Y, g\rangle, \langle X, h\rangle) \to \mathcal{D}^{\boldsymbol{H}}(\langle Y, g\rangle, \overline{T}\langle X, h\rangle). \qquad \square$$

Theorem A.15.2 *Let T, $\overline{T}$, λ be as in (A.14.5). If T has a right adjoint, so has $\overline{T}$.*

Proof. Let $R : \mathcal{D} \to \mathcal{C}$ be a right adjoint of T and denote by $\alpha : 1_{\mathcal{C}} \to RT$, $\beta : TR \to 1_{\mathcal{D}}$ the unit, counit of the adjunction. Since λ is an isomorphism, we define $\theta : GR \to RH$ to be $RH(\beta)R(\lambda_R^{-1})\alpha_{GR}$. It follows from $\lambda\iota_T = T(\eta)$ that the following diagram commutes.

$$\begin{tikzcd}
GR \arrow[r, "\alpha_{GR}"] & RTGR & \\
R \arrow[u, "\eta_R"] \arrow[r, "\alpha_R"] \arrow[dr, "id_R"'] & RTR \arrow[u, "RT(\eta_R)"] \arrow[r, "R(\iota_{TR})"] \arrow[d, "R(\beta)"] & RHTR \arrow[ul, "R(\lambda_R)"'] \arrow[d, "RH(\beta)"] \\
& R \arrow[r, "R(\iota)"] & RH
\end{tikzcd}$$

Hence θ satisfies $\theta\eta_R = R(\iota)$. Moreover, the following diagram commutes.

$$\begin{tikzcd}
G^2R \arrow[r, "G(\alpha_{GR})"] \arrow[d, "\alpha_{G^2R}"] & GRTGR \arrow[d, "\alpha_{GRTGR}"] & GRHTR \arrow[l, "GR(\lambda_R)"'] \arrow[r, "GRH(\beta)"] \arrow[d, "\alpha_{GRHTR}"] & GRH \arrow[d, "\alpha_{GRH}"] \\
RTG^2R \arrow[r, "RTG(\alpha_{GR})"] & RTGRTGR & RTGRHTR \arrow[l, "RTGR(\lambda_R)"'] \arrow[r, "RTGRH(\beta)"] & RTGRH \\
RHTGR \arrow[u, "R(\lambda_{GR})"] \arrow[r, "RHT(\alpha_{GR})"] \arrow[dr, "id_{RHTGR}"'] & RHTRTGR \arrow[u, "R(\lambda_{RTGR})"] \arrow[d, "RH(\beta_{TGR})"] & RHTRHTR \arrow[l, "RHTR(\lambda_R)"'] \arrow[r, "RHTRH(\beta)"] \arrow[u, "R(\lambda_{RHTR})"] \arrow[d, "RH(\beta_{HTR})"] & RHTRH \arrow[u, "R(\lambda_{RH})"] \arrow[d, "RH(\beta_H)"] \\
 & RHTGR & RH^2TR \arrow[l, "RH(\lambda_R)"'] \arrow[r, "RH^2(\beta)"] & RH^2
\end{tikzcd}$$

Thus we see $\theta_H G(\theta) = RH^2(\beta)RH(\lambda_R^{-1})R(\lambda_{GR}^{-1})\alpha_{G^2R}$. By the commutativity of the diagram in (A.14.4), the following diagram commutes

$$\begin{tikzcd}
 & GRH \arrow[rr, "\theta_H"] & & RH^2 \arrow[dr, "R(\nu)"] & \\
G^2R \arrow[ur, "G(\theta)"] \arrow[r, "\alpha_{G^2R}"] \arrow[d, "\mu_R"] & RTG^2R \arrow[d, "RT(\mu_R)"] & RHTGR \arrow[l, "R(\lambda_{GR})"'] & RH^2TR \arrow[l, "RH(\lambda_R)"'] \arrow[u, "RH^2(\beta)"] \arrow[d, "R(\nu_{TR})"] & RH \\
GR \arrow[r, "\alpha_{GR}"] & RTG^2R & & RHTR \arrow[ll, "R(\lambda_R)"'] \arrow[ur, "RH(\beta)"'] &
\end{tikzcd}$$

Therefore θ satisfies the conditions of (A.14.4) and we have a functor $\overline{R} : \mathcal{D}^{\boldsymbol{H}} \to \mathcal{C}^{\boldsymbol{G}}$ such that $U_{\boldsymbol{G}}\overline{R} = RU_{\boldsymbol{H}}$.

For $\langle X, h\rangle \in \operatorname{Ob}\mathcal{C}^{\boldsymbol{G}}$ and $\langle Y, g\rangle \in \operatorname{Ob}\mathcal{D}^{\boldsymbol{H}}$, we claim that $\varphi : X \to R(Y)$ gives a morphism of $\boldsymbol{G}$-algebras $\langle X, h\rangle \to \overline{R}\langle Y, g\rangle$ if and only if $\beta_Y T(\varphi) : T(X) \to Y$ gives a morphism of $\boldsymbol{H}$-algebras $\overline{T}\langle X, h\rangle \to \langle Y, g\rangle$. Suppose that φ is a morphism of $\boldsymbol{G}$-algebras. Then we have $\varphi h = R(g)\theta_Y G(\varphi) = R(g)RH(\beta_Y)R(\lambda_{R(Y)}^{-1})\alpha_{GR(Y)}G(\varphi)$.

$$\begin{aligned}
\beta_Y T(\varphi)T(h)\lambda_X &= TR(g)TRH(\beta_Y)TR(\lambda_{R(Y)}^{-1})T(\alpha_{GR(Y)})TG(\varphi)\lambda_X \\
&= \beta_Y TR(g)TRH(\beta_Y)TR(\lambda_{R(Y)}^{-1})T(\alpha_{GR(Y)})\lambda_{R(Y)}HT(\varphi) \\
&= g\beta_{H(Y)}TRH(\beta_Y)TR(\lambda_{R(Y)}^{-1})T(\alpha_{GR(Y)})\lambda_{R(Y)}HT(\varphi) \\
&= gH(\beta_Y)\beta_{HTR(Y)}TR(\lambda_{R(Y)}^{-1})T(\alpha_{GR(Y)})\lambda_{R(Y)}HT(\varphi) \\
&= gH(\beta_Y)\lambda_{R(Y)}^{-1}\beta_{TGR(Y)}T(\alpha_{GR(Y)})\lambda_{R(Y)}HT(\varphi) \\
&= gH(\beta_Y)\lambda_{R(Y)}^{-1}\lambda_{R(Y)}HT(\varphi) = gH(\beta_Y)HT(\varphi)
\end{aligned}$$

Thus we see that $\beta_Y T(\varphi)$ gives a morphism of $\boldsymbol{H}$-algebras $\overline{T}\langle X, h\rangle \to \langle Y, g\rangle$. Conversely, suppose that $\beta_Y T(\varphi)$ gives a morphism of $\boldsymbol{H}$-algebras. Then we have $\beta_Y T(\varphi)T(h)\lambda_X = gH(\beta_Y)HT(\varphi)$.

$$\begin{aligned}
R(g)\theta_Y G(\varphi) &= R(g)RH(\beta_Y)R(\lambda_{R(Y)}^{-1})\alpha_{GR(Y)}G(\varphi) = R(g)RH(\beta_Y)R(\lambda_{R(Y)}^{-1})RTG(\varphi)\alpha_{G(X)} \\
&= R(g)RH(\beta_Y)RHT(\varphi)R(\lambda_X^{-1})\alpha_{G(X)} = R(\beta_Y)RT(\varphi)RT(h)R(\lambda_X)R(\lambda_X^{-1})\alpha_{G(X)} \\
&= R(\beta_Y)RT(\varphi)RT(h)\alpha_{G(X)} = R(\beta_Y)RT(\varphi)\alpha_X h = R(\beta_Y)\alpha_Y\varphi h = \varphi h
\end{aligned}$$

Hence $\varphi : X \to R(Y)$ gives a morphism of $\boldsymbol{G}$-algebras $\langle X, h\rangle \to \overline{R}\langle Y, g\rangle$ and $\overline{R}$ is a right adjoint of $\overline{T}$. □

In the situation of (A.14.10), we have the following corollaries.

Corollary A.15.3 *Suppose that $\mathcal{C}^{\boldsymbol{G}}$ has coequalizer of reflexive pairs, $\beta : H\widetilde{T} \to TG$ is an equivalence, $K_{\boldsymbol{G}}$ has a left adjoint and that $K_{\boldsymbol{H}}$ is fully faithful. If T has a left adjoint, so has $\widetilde{T}$.*

Proof. It follows from (A.15.1) that $\overline{T}$ has a left adjoint $\overline{L}$. Let $N_{\boldsymbol{G}} : \mathcal{C}^{\boldsymbol{G}} \to \mathcal{A}$ be a left adjoint of $K_{\boldsymbol{G}}$ and put $\widetilde{L} = N_{\boldsymbol{G}}\overline{L}K_{\boldsymbol{H}}$. Since β lifts to an equivalence $\bar{\beta} : K_{\boldsymbol{H}}\widetilde{T} \to \overline{T}K_{\boldsymbol{G}}$, we have a chain of natural bijections.

$$\begin{aligned}
\mathcal{A}(\widetilde{L}(X), Y) = \mathcal{A}(N_{\boldsymbol{G}}\overline{L}K_{\boldsymbol{H}}(X), Y) &\cong \mathcal{C}^{\boldsymbol{G}}(\overline{L}K_{\boldsymbol{H}}(X), K_{\boldsymbol{G}}(Y)) \cong \mathcal{D}^{\boldsymbol{H}}(K_{\boldsymbol{H}}(X), \overline{T}K_{\boldsymbol{G}}(Y)) \\
&\xrightarrow[\cong]{\bar{\beta}_*^{-1}} \mathcal{D}^{\boldsymbol{H}}(K_{\boldsymbol{H}}(X), K_{\boldsymbol{H}}\widetilde{T}(Y)) \cong \mathcal{B}(X, \widetilde{T}(Y))
\end{aligned}$$

Hence $\widetilde{T}$ has a left adjoint $\widetilde{L}$. □

Corollary A.15.4 *Suppose that $\alpha : MT \to \widetilde{T}F$ and $\beta : H\widetilde{T} \to TG$ are equivalences, $K_{\boldsymbol{G}}$ has a right adjoint and that $K_{\boldsymbol{H}}$ is fully faithful. If T has a right adjoint, so has $\widetilde{T}$.*

Proof. Since $\lambda = \beta_F H(\alpha)$ is an equivalence, we can apply (A.15.2) and have a right adjoint $\overline{R}$ of $\overline{T}$. Let $N_{\boldsymbol{G}} : \mathcal{C}^{\boldsymbol{G}} \to \mathcal{A}$ be a right adjoint of $K_{\boldsymbol{G}}$ and put $\widetilde{R} = N_{\boldsymbol{G}}\overline{R}K_{\boldsymbol{H}}$. Then, we have a chain of natural bijections.

$$\begin{aligned}\mathcal{A}(X, \widetilde{R}(Y)) = \mathcal{A}(X, N_{\boldsymbol{G}}\overline{R}K_{\boldsymbol{H}}(Y)) &\cong \mathcal{C}^{\boldsymbol{G}}(K_{\boldsymbol{G}}(X), \overline{R}K_{\boldsymbol{H}}(Y)) \cong \mathcal{D}^{\boldsymbol{H}}(\overline{T}K_{\boldsymbol{G}}(X), K_{\boldsymbol{H}}(Y)) \\ &\xrightarrow[\cong]{\bar{\beta}^*} \mathcal{D}^{\boldsymbol{H}}(K_{\boldsymbol{H}}\widetilde{T}(X), K_{\boldsymbol{H}}(X)) \cong \mathcal{B}(\widetilde{T}(X), Y)\end{aligned}$$

Hence $\widetilde{T}$ has a right adjoint $\widetilde{R}$. □

A.16 Cartesian closed category

Let $\mathcal{C}$ be a $\mathcal{U}$-category with finite products. For $Y, Z \in \operatorname{Ob}\mathcal{C}$, define a $\mathcal{U}$-presheaf $P_{Y,Z} : \mathcal{C}^{op} \to \mathcal{U}$-**Ens** by $P_{Y,Z}(X) = \mathcal{C}(X \times Y, Z)$.

Definition A.16.1 *If $P_{Y,Z}$ is representable for any $Y, Z \in \operatorname{Ob}\mathcal{C}$, $\mathcal{C}$ is called a cartesian closed category. We denote by Z^Y the object of $\mathcal{C}$ that represents $P_{Y,Z}$ and by $\exp_{X,Y,Z} : \mathcal{C}(X \times Y, Z) \to \mathcal{C}(X, Z^Y)$ the natural bijection.*

For morphisms $f : X \times Y \to Z$ and $g : X \to Z^Y$, if $g = \exp_{X,Y,Z}(f)$ holds, g (resp. f) is called the (exponential) transpose of f (resp. g). It follows from the results of the second section of this appendix that the functor $\mathcal{C} \to \mathcal{C}$ given by $X \mapsto X \times Y$ and $(f : X \to X') \mapsto (f \times id_Y : X \times Y \to X' \times Y)$ has a left adjoint which assigns Z^Y to $Z \in \operatorname{Ob}\mathcal{C}$ for any $Y \in \operatorname{Ob}\mathcal{C}$. For a morphism $f : X \to Z$ in $\mathcal{C}$, we denote by $f^Y : X^Y \to Z^Y$ the morphism induced by f.

If we denote the unit (resp. counit) of this adjunction by $\eta_X^Y : X \to (X \times Y)^Y$ (resp. $\varepsilon_Y^X : Y^X \times X \to Y$), the exponential transpose of $f : X \times Y \to Z$ (resp. $g : X \to Z^Y$) is the composition $X \xrightarrow{\eta_X^Y} (X \times Y)^Y \xrightarrow{f^Y} Z^Y$ (resp. $X \times Y \xrightarrow{g \times id_Y} Z^Y \times Y \xrightarrow{\varepsilon_Z^Y} Z$).

The naturality of the adjunction implies the following fact.

Proposition A.16.2 *Let $f : V \to X$ and $g : Z \to W$ be morphisms in $\mathcal{C}$. Then the following diagrams commute.*

$$\begin{tikzcd} \mathcal{C}(X \times Y, Z) \arrow[r, "\exp_{X,Y,Z}"] \arrow[d, "(f \times id_Y)^*"] & \mathcal{C}(X, Z^Y) \arrow[d, "f^*"] \\ \mathcal{C}(V \times Y, Z) \arrow[r, "\exp_{V,Y,Z}"] & \mathcal{C}(V, Z^Y) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(X \times Y, Z) \arrow[r, "\exp_{X,Y,Z}"] \arrow[d, "g_*"] & \mathcal{C}(X, Z^Y) \arrow[d, "g_*^Y"] \\ \mathcal{C}(X \times Y, W) \arrow[r, "\exp_{X,Y,W}"] & \mathcal{C}(X, W^Y) \end{tikzcd}$$

Hence if $\beta : X \to Z^Y$ is the exponential transpose of $\alpha : X \times Y \to Z$, the exponential transpose of the compositions $\alpha(f \times id_Y) : V \times Y \to Z$ and $g\alpha : X \times Y \to W$ are βf and $g^Y\beta$, respectively. In particular, $g^Y : Z^Y \to W^Y$ is the exponential transpose of $g\varepsilon_Z^Y$.

Let $g : Y \to W$ be a morphism in $\mathcal{C}$. We define a morphism $X^g : X^W \to X^Y$ to be the exponential transpose of the composite $X^W \times Y \xrightarrow{id_{X^W} \times g} X^W \times W \xrightarrow{\varepsilon_X^W} X$.

Proposition A.16.3 *For a morphism $g : Y \to W$ in $\mathcal{C}$, the following diagram commutes.*

$$\begin{tikzcd} X^W \times Y \arrow[r, "X^g \times id_Y"] \arrow[d, "id_{X^W} \times g"] & X^Y \times Y \arrow[d, "\varepsilon_X^Y"] \\ X^W \times W \arrow[r, "\varepsilon_X^W"] & X \end{tikzcd}$$

Proof. The following diagram commutes by (A.16.2).

$$\begin{tikzcd}
\mathcal{C}(X^Y \times Y, X) \arrow[r, "\exp_{X^Y,Y,X}"] \arrow[d, "(X^g \times id_Y)^*"] & \mathcal{C}(X^Y, X^Y) \arrow[d, "(X^g)^*"] \\
\mathcal{C}(X^W \times Y, X) \arrow[r, "\exp_{X^W,Y,X}"] & \mathcal{C}(X^W, X^Y)
\end{tikzcd}$$

Hence we have

$$\begin{aligned}
\exp_{X^W,Y,X}(\varepsilon_X^Y(X^g \times id_Y)) &= \exp_{X^W,Y,X}((X^g \times id_Y)^*(\varepsilon_X^Y)) = (X^g)^*(\exp_{X^Y,Y,X}(\varepsilon_X^Y)) = X^g \\
&= \exp_{X^W,Y,X}(\varepsilon_X^W(id_{X^W} \times g)).
\end{aligned}$$

Since $\exp_{X^W,Y,X}$ is bijective, the result follows. □

Proposition A.16.4 *For morphisms $f : X \to Z$ and $g : Y \to W$ in $\mathcal{C}$, the following diagram commutes.*

$$\begin{tikzcd}
X^W \arrow[r, "X^g"] \arrow[d, "f^W"] & X^Y \arrow[d, "f^Y"] \\
Z^W \arrow[r, "Z^g"] & Z^Y
\end{tikzcd}$$

Proof. It follows from (A.16.2) that $f^Y X^g = f_*^Y(X^g)$ is the transpose of $f\varepsilon_X^W(id_{X^W} \times g)$ and that $Z^g f^W = (f^W)^*(Z^g)$ is the transpose of $\varepsilon_Z^W(id_{Z^W} \times g)(f^W \times id_Y)$. By the naturality of the counit, the following diagram commutes.

$$\begin{tikzcd}
X^W \times Y \arrow[r, "id_{X^W} \times g"] \arrow[d, "f^W \times id_Y"] & X^W \times W \arrow[r, "\varepsilon_X^W"] \arrow[d, "f^W \times id_W"] & X \arrow[d, "f"] \\
Z^W \times Y \arrow[r, "id_{Z^W} \times g"] & Z^W \times W \arrow[r, "\varepsilon_Z^W"] & Z
\end{tikzcd}$$

Thus we have $f\varepsilon_X^W(id_{X^W} \times g) = \varepsilon_Z^W(id_{Z^W} \times g)(f^W \times id_Y)$ and the result follows. □

We put $f^g = f^Y X^g = Z^g f^W : X^W \to Z^Y$. The above result implies that the correspondences $(X, Y) \mapsto X^Y$ and $(f : X \to Z, g : Y \to W) \mapsto (f^g : X^W \to Z^Y)$ defines a functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ and we call this an exponential functor.

Proposition A.16.5 *For a morphism $f : X \to Y$ of $\mathcal{C}$, the following diagram is commutative for any object W of $\mathcal{C}$.*

$$\begin{tikzcd}
\mathcal{C}(W \times Y, Z) \arrow[r, "\exp_{W,Y,Z}"] \arrow[d, "(id_W \times f)^*"] & \mathcal{C}(W, Z^Y) \arrow[d, "Z_*^f"] \\
\mathcal{C}(W \times X, Z) \arrow[r, "\exp_{W,X,Z}"] & \mathcal{C}(W, Z^X)
\end{tikzcd}$$

Proof. For any morphism $\alpha : W \times Y \to Z$, the following is commutative by (A.2.1) and the naturality of the counit.

$$\begin{tikzcd}
W \times X \arrow[r, "\eta_W^Y \times id_X"] \arrow[d, "id_W \times f"] & (W \times Y)^Y \times X \arrow[r, "\alpha^Y \times id_X"] \arrow[d, "id_{(W \times Y)^Y} \times f"] & Z^Y \times X \arrow[d, "id_{Z^Y} \times f"] \\
W \times Y \arrow[r, "\eta_W^Y \times id_Y"] \arrow[dr, "id_{W \times Y}"'] & (W \times Y)^Y \times Y \arrow[r, "\alpha^Y \times id_Y"] \arrow[d, "\varepsilon_{W \times Y}^Y"] & Z^Y \times Y \arrow[d, "\varepsilon_Z^Y"] \\
 & W \times Y \arrow[r, "\alpha"] & Z
\end{tikzcd}$$

Hence we have

$$\begin{aligned}
\exp_{W,X,Z}((id_W \times f)^*(\alpha)) &= \exp_{W,X,Z}(\alpha(id_W \times f)) = \exp_{W,X,Z}(\varepsilon_Z^Y(id_{Z^Y} \times f)(\alpha^Y \eta_W^Y \times id_X)) \\
&= \exp_{W,X,Z}((\alpha^Y \eta_W^Y \times id_X)^*(\varepsilon_Z^Y(id_{Z^Y} \times f)) = (\alpha^Y \eta_W^Y)^*(\exp_{Z^Y,X,Z}(\varepsilon_Z^Y(id_{Z^Y} \times f))) \\
&= Z^f \alpha^Y \eta_W^Y = Z_*^f(\exp_{W,Y,Z}(\alpha)).
\end{aligned}$$

□

For an object Z of a cartesian closed category $\mathcal{C}$, let us denote by $P_Z : \mathcal{C}^{op} \to \mathcal{C}$ the functor defined by $P_Z X = Z^X$ and $P_Z(f) = Z^f : Z^Y \to Z^X$ for an object X and a morphism $f : X \to Y$. By (A.16.4), $g : Z \to W$ defines a natural transformation $P_g : P_Z \to P_W$.

We write P_{Z*} for the same data considered as a functor $\mathcal{C} \to \mathcal{C}^{op}$.

Lemma A.16.6 *P_{Z*} is a left adjoint of P_Z*

Proof. $\mathcal{C}^{op}(P_{Z*}X, Y) = \mathcal{C}(Y, Z^X) \cong \mathcal{C}(Y \times X, Z) \cong \mathcal{C}(X \times Y, Z) \cong \mathcal{C}(X, Z^Y) = \mathcal{C}(X, P_Z Y)$. □

The unit $\eta(Z) : 1_{\mathcal{C}} \to P_Z P_{Z*}$ and the counit $\varepsilon(Z) : P_{Z*}P_Z \to 1_{\mathcal{C}^{op}}$ are given as follows. $\eta(Z)_X : X \to P_Z P_{Z*} X = Z^{Z^X}$ is the transpose of $X \times Z^X \xrightarrow{T} Z^X \times X \xrightarrow{\varepsilon_Z^X} Z$, where T is the switching map and $\varepsilon(Z)_X : P_{Z*}P_Z X \to X$ is the same morphism regarded as a morphism in $\mathcal{C}^{op}$.

Lemma A.16.7 *The following diagram is commutative.*

$$\begin{tikzcd}
X \times Z^X \arrow[r, "T"] \arrow[d, "\eta(Z)_X \times id_{Z^X}"] & Z^X \times X \arrow[d, "\varepsilon_Z^X"] \\
Z^{Z^X} \times Z^X \arrow[r, "\varepsilon_Z^{Z^X}"] & Z
\end{tikzcd}$$

Proof. The exponential transpose of $\varepsilon_Z^X T$ is $\eta(Z)_X$ by definition and that of $\varepsilon_Z^{Z^X}(\eta(X)_X \times id_{Z^X})$ is also $\eta(Z)_X$ by (A.16.2). □

Let $\gamma_{X,Y,Z} : Z^Y \times Y^X \to Z^X$ be the exponential transpose of the following composition.

$$Z^Y \times Y^X \times X \xrightarrow{id_{Z^Y} \times \varepsilon_Y^X} Z^Y \times Y \xrightarrow{\varepsilon_Z^Y} Z$$

Proposition A.16.8 *The following diagram commutes.*

$$\begin{tikzcd}
Z^Y \times Y^X \times X \arrow[r, "id_{Z^Y} \times \varepsilon_Y^X"] \arrow[d, "\gamma_{X,Y,Z} \times id_X"] & Z^Y \times Y \arrow[d, "\varepsilon_Z^Y"] \\
Z^X \times X \arrow[r, "\varepsilon_Z^X"] & Z
\end{tikzcd}$$

Proof. By (A.16.2), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}(Z^X \times X, Z) \arrow[r, "\exp_{Z^X, X, Z}"] \arrow[d, "(\gamma_{X,Y,Z} \times id_X)^*"] & \mathcal{C}(Z^X, Z^X) \arrow[d, "\gamma_{X,Y,Z}^*"] \\
\mathcal{C}(Z^Y \times Y^X \times X, Z) \arrow[r, "\exp_{Z^Y \times Y^X, X, Z}"] & \mathcal{C}(Z^Y \times Y^X, Z^Y)
\end{tikzcd}$$

Since $\varepsilon_Z^X \in \mathcal{C}(Z^X \times X, Z)$ is the exponential transpose of the identity morphism of Z^X, $\varepsilon_Z^X(\gamma_{X,Y,Z} \times id_X)$ is also the exponential transpose of $\gamma_{X,Y,Z}$ by the commutativity of the above diagram. □

Proposition A.16.9 *The following diagram commutes.*

$$\begin{tikzcd}
W^Z \times Z^Y \times Y^X \arrow[r, "id_{W^Z} \times \gamma_{X,Y,Z}"] \arrow[d, "\gamma_{Y,Z,W} \times id_{Y^X}"] & W^Z \times Z^X \arrow[d, "\gamma_{X,Z,W}"] \\
W^Y \times Y^X \arrow[r, "\gamma_{X,Y,W}"] & W^X
\end{tikzcd}$$

Proof. By (A.16.2), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}(W^Z \times Z^X \times X, W) \arrow[r, "(id_{W^Z} \times \gamma_{X,Y,Z} \times id_X)^*"] \arrow[d, "\exp_{W^Z \times Z^X, X, W}"] & \mathcal{C}(W^Z \times Z^Y \times Y^X \times X, W) \arrow[d, "\exp_{W^Z \times Z^Y \times Y^X, X, W}"] & \mathcal{C}(W^Y \times Y^X \times X, W) \arrow[l, "(\gamma_{Y,Z,W} \times id_{Y^X} \times id_X)^*"'] \arrow[d, "\exp_{W^Y \times Y^X, X, W}"'] \\
\mathcal{C}(W^Z \times Z^X, W^X) \arrow[r, "(id_{W^Z} \times \gamma_{X,Y,Z})^*"] & \mathcal{C}(W^Z \times Z^Y \times Y^X, W^X) & \mathcal{C}(W^Y \times Y^X, W^X) \arrow[l, "(\gamma_{Y,Z,W} \times id_{Y^X})^*"']
\end{tikzcd}$$

Since $\gamma_{X,Z,W} = \exp_{W^Z\times Z^X,X,W}(\varepsilon^Z_W(id_{W^Z}\times\varepsilon^X_Z))$ and $\gamma_{X,Y,W} = \exp_{W^Y\times Y^X,X,W}(\varepsilon^Y_W(id_{W^Y}\times\varepsilon^X_Y))$, it follows from the commutativity of the diagram of (A.16.8) and the above diagram that

$$\begin{aligned}
\exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon^Z_W(id_{W^Z}\times\varepsilon^Y_Z(id_{Z^Y}\times\varepsilon^X_Y)) &= \exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon^Z_W(id_{W^Z}\times\varepsilon^X_Z(\gamma_{X,Y,Z}\times id_X))\\
&= \exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon^Z_W(id_{W^Z}\times\varepsilon^X_Z)(id_{W^Z}\times\gamma_{X,Y,Z}\times id_X))\\
&= \gamma_{X,Z,W}(id_{W^Z}\times\gamma_{X,Y,Z})\\
\exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon^Z_W(id_{W^Z}\times\varepsilon^Y_Z(id_{Z^Y}\times\varepsilon^X_Y)) &= \exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon^Z_W(id_{W^Z}\times\varepsilon^Y_Z)(id_{W^Z}\times id_{Z^Y}\times\varepsilon^X_Y))\\
&= \exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon^Y_W(\gamma_{Y,Z,W}\times id_Y)(id_{W^Z\times Z^Y}\times\varepsilon^X_Y))\\
&= \exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon^Y_W(\gamma_{Y,Z,W}\times\varepsilon^X_Y))\\
&= \exp_{W^Z\times Z^Y\times Y^X,X,W}(\varepsilon^Y_W(id_{W^Y}\times\varepsilon^X_Y)(\gamma_{Y,Z,W}\times id_{Y^X}\times id_X))\\
&= \gamma_{X,Y,W}(\gamma_{Y,Z,W}\times id_{Y^X}).
\end{aligned}$$

□

We denote by 1 a terminal object of $\mathcal{C}$ and define $\epsilon_X : 1 \to X^X$ to be the exponential transpose of the projection $\mathrm{pr}_2 : 1\times X \to X$ to the second factor.

Proposition A.16.10 *The following diagrams commute.*

$$\begin{tikzcd}
1\times X \arrow[d, "\epsilon_X\times id_X"] \arrow[dr, "\mathrm{pr}_2"] & \\
X^X\times X \arrow[r, "\epsilon_X"] & X
\end{tikzcd}$$

Proof. By (A.16.2), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}(X^X\times X, X) \arrow[r, "\exp_{X^X,X,X}"] \arrow[d, "(\epsilon_X\times id_X)^*"] & \mathcal{C}(X^X, X^X) \arrow[d, "\epsilon_X^*"] \\
\mathcal{C}(1\times X, X) \arrow[r, "\exp_{1,X,X}"] & \mathcal{C}(1, X^X)
\end{tikzcd}$$

Since the exponential transpose of id_{X^X} is $\varepsilon^X_X \in \mathcal{C}(X^X\times X, X)$ and the exponential transpose of ϵ_X is $\mathrm{pr}_2 \in \mathcal{C}(1\times X, X)$, we have $\varepsilon^X_X(\epsilon_X\times id_X) = \mathrm{pr}_2$ by the commutaivity of the above diagram. □

Proposition A.16.11 *The following diagrams commute.*

$$\begin{tikzcd}
Y^X\times 1 \arrow[r, "id_{Y^X}\times\epsilon_X"] \arrow[dr, "\mathrm{pr}_1"'] & Y^X\times X^X \arrow[d, "\gamma_{X,X,Y}"] \\
& Y^X
\end{tikzcd}
\qquad
\begin{tikzcd}
1\times Y^X \arrow[r, "\epsilon_Y\times id_{Y^X}"] \arrow[dr, "\mathrm{pr}_2"'] & Y^Y\times Y^X \arrow[d, "\gamma_{X,Y,Y}"] \\
& Y^X
\end{tikzcd}$$

Proof. We first claim that the following diagram commutes.

$$\begin{tikzcd}
1\times X \arrow[r, "\epsilon_X\times id_X"] \arrow[dr, "\mathrm{pr}_2"'] & X^X\times X \arrow[d, "\varepsilon^X_X"] \\
& X
\end{tikzcd} \quad \cdots\ (ii)$$

By the naturarity of exp, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}(X^X\times X, X) \arrow[r, "\exp_{X^X,X,X}"] \arrow[d, "(\epsilon_X\times id_X)^*"] & \mathcal{C}(X^X, X^X) \arrow[d, "\epsilon_X^*"] \\
\mathcal{C}(1\times X, X) \arrow[r, "\exp_{1,X,X}"] & \mathcal{C}(1, X^X)
\end{tikzcd} \quad \cdots\ (iii)$$

Since $\epsilon_X = \exp_{1,X,X}(\mathrm{pr}_2)$ and $\exp_{X^X,X,X}(\varepsilon^X_X) = id_{X^X}$, it follows from the commutativity of (iii) that

$$\exp_{1,X,X}(\varepsilon^X_X(\epsilon_X \times id_X)) = \exp_{X^X,X,X}(\varepsilon^X_X)\epsilon_X = \exp_{1,X,X}(\mathrm{pr}_2).$$

Thus we have $\varepsilon^X_X(\epsilon_X \times id_X) = \mathrm{pr}_2$. By (A.16.2), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}(Y^X \times X^X \times X, Y) \arrow[r, "(id_{Y^X}\times\epsilon_X\times id_X)^*"] \arrow[d, "\exp_{Y^X\times X^X,X,Y}"] & \mathcal{C}(Y^X \times 1 \times X, Y) \arrow[d, "\exp_{Y^X\times 1,X,Y}"] & \mathcal{C}(Y^X \times X, Y) \arrow[l, "(\mathrm{pr}_1\times id_X)^*"'] \arrow[d, "\exp_{Y^X,X,Y}"] \\
\mathcal{C}(Y^X \times X^X, Y^X) \arrow[r, "(id_{Y^X}\times\epsilon_X)^*"] & \mathcal{C}(Y^X \times 1, X^Y) & \mathcal{C}(Y^X, Y^X) \arrow[l, "\mathrm{pr}_1^*"']
\end{tikzcd}$$

Since $\exp_{Y^X\times X^X,X,Y}(\varepsilon^X_Y(id_{Y^X}\times\varepsilon^X_X)) = \gamma_{X,X,Y}$ and $\exp_{Y^X,X,Y}(\varepsilon^X_Y) = id_{Y^X}$, it follows from the commutativity of (ii) and the above diagram that

$$\begin{aligned}
\mathrm{pr}_1 &= \exp_{Y^X,X,Y}(\varepsilon^X_Y)\mathrm{pr}_1 = \exp_{Y^X\times 1,X,Y}(\varepsilon^X_Y(id_{Y^X}\times\mathrm{pr}_2)) = \exp_{Y^X\times 1,X,Y}(\varepsilon^X_Y(\mathrm{pr}_1\times id_X)) \\
&= \exp_{Y^X\times 1,X,Y}(\varepsilon^X_Y(id_{Y^X}\times\varepsilon^X_X(\epsilon_X\times id_X))) = \exp_{Y^X\times 1,X,Y}(\varepsilon^X_Y(id_{Y^X}\times\varepsilon^X_X)(id_{Y^X}\times\epsilon_X\times id_X)) \\
&= \exp_{Y^X\times X^X,X,Y}(\varepsilon^X_Y(id_{Y^X}\times\varepsilon^X_X))(id_{Y^X}\times\epsilon_X) = \gamma_{X,X,Y}(id_{Y^X}\times\epsilon_X)
\end{aligned}$$

By (A.16.2), the following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}(Y^Y \times Y^X \times X, Y) \arrow[r, "(\epsilon_Y\times id_{Y^X}\times id_X)^*"] \arrow[d, "\exp_{Y^Y\times Y^X,X,Y}"] & \mathcal{C}(1 \times Y^X \times X, Y) \arrow[d, "\exp_{1\times Y^X,X,Y}"] & \mathcal{C}(Y^X \times X, Y) \arrow[l, "(\mathrm{pr}_2\times id_X)^*"] \arrow[d, "\exp_{Y^X,X,Y}"] \\
\mathcal{C}(Y^Y \times Y^X, Y^X) \arrow[r, "(\epsilon_Y\times id_{Y^X})^*"] & \mathcal{C}(1 \times Y^X, X^Y) & \mathcal{C}(Y^X, Y^X) \arrow[l, "\mathrm{pr}_2^*"]
\end{tikzcd}$$

Since $\exp_{Y^Y\times Y^X,X,Y}(\varepsilon^Y_Y(id_{Y^Y}\times\varepsilon^X_Y)) = \gamma_{X,Y,Y}$ and $\varepsilon^X_Y = \exp_{Y^X,X,Y}(id_{Y^X})$, it follows from the commutativity of (iii) and the above diagram that

$$\begin{aligned}
\mathrm{pr}_2 &= \exp_{Y^X,X,Y}(\varepsilon^X_Y)\mathrm{pr}_2 = \exp_{1\times Y^X,X,Y}(\varepsilon^X_Y(\mathrm{pr}_2\times id_X)) = \exp_{1\times Y^X,X,Y}(\mathrm{pr}_2(id_1\times\varepsilon^X_Y)) \\
&= \exp_{1\times Y^X,X,Y}(\varepsilon^Y_Y(\epsilon_Y\times id_Y)(id_1\times\varepsilon^X_Y)) = \exp_{1\times Y^X,X,Y}(\varepsilon^Y_Y(id_{Y^Y}\times\varepsilon^X_Y)(\epsilon_Y\times id_{Y^X}\times id_X)) \\
&= \exp_{Y^Y\times Y^X,X,Y}(\varepsilon^Y_Y(id_{Y^Y}\times\varepsilon^X_Y))(\epsilon_Y\times id_{Y^X}) = \gamma_{X,Y,Y}(\epsilon_Y\times id_{Y^X})
\end{aligned}$$

□

Let $\mathcal{C}$ and $\mathcal{D}$ be cartesian closed categories and $F : \mathcal{C} \to \mathcal{D}$ a functor which preserves finite products. The exponential transpose of a composition $F(Y^X)\times F(X) \xrightarrow{\cong} F(Y^X\times X) \xrightarrow{F(\varepsilon^X_Y)} F(Y)$ gives a morphism $\xi^X_Y : F(Y^X) \to F(Y)^{F(X)}$. If this morphism is an isomorphism for any objects X and Y, we say that F preserves exponentials.

Proposition A.16.12 *Let $\mathcal{C}$ and $\mathcal{D}$ be cartesian closed categories and $F : \mathcal{C} \to \mathcal{D}$ a functor which preserves finite products.*

1) $\xi^X_Y : F(Y^X) \to F(Y)^{F(X)}$ is the unique morphism that makes the following diagram commute.

$$\begin{tikzcd}
F(Y^X \times X) \arrow[r, "(F(\mathrm{pr}_1),F(\mathrm{pr}_2))"] \arrow[d, "F(\varepsilon^X_Y)"] & F(Y^X)\times F(X) \arrow[d, "\xi^X_Y\times id_{F(X)}"] \\
F(Y) & F(Y)^{F(X)}\times F(X) \arrow[l, "\varepsilon^{F(X)}_{F(Y)}"']
\end{tikzcd}$$

2) The following diagram commutes.

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta^Y_X)"] \arrow[d, "\eta^{F(Y)}_{F(X)}"] & F((X\times Y)^Y) \arrow[d, "\xi^Y_{X\times Y}"] \\
(F(X)\times F(Y))^{F(Y)} & F(X\times Y)^{F(Y)} \arrow[l, "(F(\mathrm{pr}_1),F(\mathrm{pr}_2))^{F(Y)}"']
\end{tikzcd}$$

3) Let $f : X \to Z$ and $g : Y \to W$ be morphisms in $\mathcal{C}$. Then, the following diagram commutes.

$$\begin{tikzcd} F(X^W) \arrow[r, "\xi_X^W"] \arrow[d, "F(f^g)"] & F(X)^{F(W)} \arrow[d, "F(f)^{F(g)}"] \\ F(Z^Y) \arrow[r, "\xi_Z^Y"] & F(Z)^{F(Y)} \end{tikzcd}$$

4) The following diagram is commutative for any objects X, Y, Z of $\mathcal{C}$.

$$\begin{tikzcd} \mathcal{C}(X\times Y, Z) \arrow[r, "F"] \arrow[d, "\exp_{X,Y,Z}"] & \mathcal{D}(F(X\times Y), F(Z)) \arrow[r, "((F(\mathrm{pr}_1), F(\mathrm{pr}_2))^{-1})^*"] & \mathcal{D}(F(X)\times F(Y), F(Z)) \arrow[d, "\exp_{F(X),F(Y),F(Z)}"] \\ \mathcal{C}(X, Z^Y) \arrow[r, "F"] & \mathcal{D}(F(X), F(Z^Y)) \arrow[r, "(\xi_Z^Y)_*"] & \mathcal{D}(F(X), F(Z)^{F(Y)}) \end{tikzcd}$$

Proof. 1) The first assertion is straightforward from the definition of ξ_Z^Y and (A.16.2).

2) The following diagram commutes by (A.16.2).

$$\begin{tikzcd} \mathcal{D}(F((X\times Y)^Y)\times F(Y), F(X\times Y)) \arrow[r, "\exp_{F((X\times Y)^Y), F(Y), F(X\times Y)}"] \arrow[d, "(F(\eta_X^Y)\times id_{F(Y)})^*"] & \mathcal{D}(F((X\times Y)^Y), F(X\times Y)^{F(Y)}) \arrow[d, "F(\eta_X^Y)^*"] \\ \mathcal{D}(F(X)\times F(Y), F(X\times Y)) \arrow[r, "\exp_{F(X)\times F(Y), F(Y), F(X\times Y)}"] \arrow[d, "(F(\mathrm{pr}_1), F(\mathrm{pr}_2))_*"] & \mathcal{D}(F(X), F(X\times Y)^{F(Y)}) \arrow[d, "(F(\mathrm{pr}_1), F(\mathrm{pr}_2))_*^{F(Y)}"] \\ \mathcal{D}(F(X)\times F(Y), F(X)\times F(Y)) \arrow[r, "\exp_{F(X)\times F(Y), F(Y), F(X)\times F(Y)}"] & \mathcal{D}(F(X)\times F(Y), (F(X)\times F(Y))^{F(Y)}) \end{tikzcd}$$

Hence the transpose of $(F(\mathrm{pr}_1), F(\mathrm{pr}_2))^{F(Y)}\xi_{X\times Y}^Y F(\eta_X^Y)$ is the following composition.

$$F(X)\times F(Y) \xrightarrow{F(\eta_X^Y)\times id_{F(Y)}} F((X\times Y)^Y)\times F(Y) \xrightarrow{\cong} F((X\times Y)^Y\times Y)$$
$$\xrightarrow{F(\varepsilon_{X\times Y}^Y)} F(X\times Y) \xrightarrow{(F(\mathrm{pr}_1), F(\mathrm{pr}_2))} F(X)\times F(Y)$$

Then, the assertion follows from the commutativity of the diagram below.

$$\begin{tikzcd} F(X)\times F(Y) \arrow[r, "(F(\mathrm{pr}_1), F(\mathrm{pr}_2))^{-1}"] \arrow[d, "F(\eta_X^Y)\times id_{F(Y)}"] & F(X\times Y) \arrow[r, "(F(\mathrm{pr}_1), F(\mathrm{pr}_2))"] \arrow[d, "F(\eta_X^Y\times id_Y)"] & F(X)\times F(Y) \\ F((X\times Y)^Y)\times F(Y) \arrow[r, "\cong"] & F((X\times Y)^Y\times Y) \arrow[r, "F(\varepsilon_{X\times Y}^Y)"] & F(X\times Y) \arrow[u, "(F(\mathrm{pr}_1), F(\mathrm{pr}_2))"] \end{tikzcd}$$

3) It suffices to show that the following diagrams commute.

$$\begin{tikzcd} F(X^W) \arrow[r, "\xi_X^W"] \arrow[d, "F(f^W)"] & F(X)^{F(W)} \arrow[d, "F(f)^{F(W)}"] \\ F(Z^W) \arrow[r, "\xi_Z^W"] & F(Z)^{F(W)} \end{tikzcd} \qquad \begin{tikzcd} F(Z^W) \arrow[r, "\xi_Z^W"] \arrow[d, "F(Z^g)"] & F(Z)^{F(W)} \arrow[d, "F(Z)^{F(g)}"] \\ F(Z^Y) \arrow[r, "\xi_Z^Y"] & F(Z)^{F(Y)} \end{tikzcd}$$

The following diagram commutes by (A.16.2).

$$\begin{tikzcd} \mathcal{D}(F(X^W)\times F(W), F(X)) \arrow[r, "F(f)_*"] \arrow[d, "\exp_{F(X^W), F(W), F(X)}"] & \mathcal{D}(F(X^W)\times F(W), F(Z)) \arrow[d, "\exp_{F(X^W), F(W), F(Z)}"] & \mathcal{D}(F(X^W)\times F(W), F(Z)) \arrow[l, "(F(f^W)\times id_{F(W)})^*"'] \arrow[d, "\exp_{F(X^W), F(W), F(Z)}"'] \\ \mathcal{D}(F(X^W), F(X)^{F(W)}) \arrow[r, "F(f)_*^{F(W)}"] & \mathcal{D}(F(X^W), F(Z)^{F(W)}) & \mathcal{D}(F(X^W), F(Z)^{F(W)}) \arrow[l, "F(f^W)^*"'] \end{tikzcd}$$

It follows that the transposes of $F(f)^{F(W)}\xi_X^W$ and $\xi_Z^W F(f^W)$ are

$$F(X^W)\times F(W) \xrightarrow{\cong} F(X^W\times W) \xrightarrow{F(\varepsilon_X^W)} F(X) \xrightarrow{F(f)} F(Z) \text{ and}$$

$$F(X^W)\times F(W) \xrightarrow{F(f^W)\times id_{F(W)}} F(Z^W)\times F(W) \xrightarrow{\cong} F(Z^W\times W) \xrightarrow{F(\varepsilon_Z^W)} F(Z) \text{, respectively.}$$

Since the following diagram commutes, the above compositions coincides. Hence $F(f)^{F(W)}\xi_X^W = \xi_Z^W F(f^W)$.

$$\begin{tikzcd}
F(X^W)\times F(W) \arrow[r, "\cong"] \arrow[d, "F(f^W)\times id_{F(W)}"] & F(X^W\times W) \arrow[r, "F(\varepsilon_X^W)"] \arrow[d, "F(f^W\times id_W)"] & F(X) \arrow[d, "F(f)"] \\
F(Z^W)\times F(W) \arrow[r, "\cong"] & F(Z^W\times W) \arrow[r, "F(\varepsilon_Z^W)"] & F(Z)
\end{tikzcd}$$

The following diagram commutes by (A.16.2).

$$\begin{tikzcd}
\mathcal{D}(F(Z^W)\times F(W),F(Z)) \arrow[r, "(id_{F(Z^W)}\times F(g))^*"] \arrow[d, "\exp_{F(Z^W),F(W),F(Z)}"] & \mathcal{D}(F(Z^W)\times F(Y),F(Z)) \arrow[d, "\exp_{F(Z^W),F(Y),F(Z)}"] & \mathcal{D}(F(Z^Y)\times F(Y),F(Z)) \arrow[l, "(F(Z^g)\times id_{F(Y)})^*"'] \arrow[d, "\exp_{F(Z^Y),F(Y),F(Z)}"'] \\
\mathcal{D}(F(Z^W),F(Z)^{F(W)}) \arrow[r, "F(Z)^{F(g)}_*"] & \mathcal{D}(F(Z^W),F(Z)^{F(Y)}) & \mathcal{D}(F(Z^Y),F(Z)^{F(Y)}) \arrow[l, "F(Z^g)^*"']
\end{tikzcd}$$

The transpose of $F(Z)^{F(g)}\xi_Z^W$ is $F(Z^W)\times F(Y) \xrightarrow{id_{F(Z^W)}\times F(g)} F(Z^W)\times F(W) \xrightarrow{\cong} F(Z^W\times W) \xrightarrow{F(\varepsilon_Z^W)} F(Z)$ and the transpose of $\xi_Z^Y F(Z^g)$ is $F(Z^W)\times F(Y) \xrightarrow{F(Z^g)\times id_{F(Y)}} F(Z^Y)\times F(Y) \xrightarrow{\cong} F(Z^Y\times Y) \xrightarrow{F(\varepsilon_Z^Y)} F(Z)$. Since the following diagram commutes by (A.16.3), the assertion follows.

$$\begin{tikzcd}
F(Z^W)\times F(W) \arrow[r, "\cong"] & F(Z^W\times W) \arrow[dr, "F(\varepsilon_Z^W)"] & \\
F(Z^W)\times F(Y) \arrow[r, "\cong"] \arrow[u, "id_{F(Z^W)}\times F(g)"] \arrow[d, "F(Z^g)\times id_{F(Y)}"] & F(Z^W\times Y) \arrow[u, "F(id_{Z^W}\times g)"] \arrow[d, "F(Z^g\times id_Y)"] & F(Z) \\
F(Z^Y)\times F(Y) \arrow[r, "\cong"] & F(Z^Y\times Y) \arrow[ur, "F(\varepsilon_Z^Y)"'] &
\end{tikzcd}$$

4) For $f\in\mathcal{C}(X\times Y,Z)$, we put $\bar f=\exp_{X,Y,Z}(f)$. By the commutativity of

$$\begin{tikzcd}
\mathcal{D}(F(Z^Y)\times F(Y),F(Z)) \arrow[r, "\exp_{F(Z^Y),F(Y),F(Z)}"] \arrow[d, "(F(\bar f)\times id_{F(Y)})^*"] & \mathcal{D}(F(Z^Y),F(Z)^{F(Y)}) \arrow[d, "F(\bar f)^*"] \\
\mathcal{D}(F(X)\times F(Y),F(Z)) \arrow[r, "\exp_{F(X),F(Y),F(Z)}"] & \mathcal{D}(F(X),F(Z)^{F(Y)})
\end{tikzcd}$$

the transpose of $\xi_Z^Y F(\bar f): F(X)\to F(Z)^{F(Y)}$ is the composite

$$F(X)\times F(Y) \xrightarrow{F(\bar f)\times id_{F(Y)}} F(Z^Y)\times F(Y) \xrightarrow{\cong} F(Z^Y\times Y) \xrightarrow{F(\varepsilon_Z^Y)} F(Z).$$

Then, the result follows from the commutativity of the following diagram and the equality $\varepsilon_Z^Y(\bar f\times id_Y)=f$.

$$\begin{tikzcd}
F(X\times Y) \arrow[r, "F(\bar f\times id_Y)"] \arrow[d, "(F(\mathrm{pr}_1),F(\mathrm{pr}_2))", "\cong"'] & F(Z^Y\times Y) \arrow[d, "(F(\mathrm{pr}_1),F(\mathrm{pr}_2))", "\cong"'] \\
F(X)\times F(Y) \arrow[r, "F(\bar f)\times id_{F(Y)}"] & F(Z^Y)\times F(Y)
\end{tikzcd}$$

□

Lemma A.16.13 *Let $\mathcal{C}$ and $\mathcal{D}$ be cartesian closed categories and $F:\mathcal{C}\to\mathcal{D}$ a functor which preserves finite products. The following diagram commutes.*

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta(Z)_X)"] \arrow[d, "\eta(F(Z))_{F(X)}"] & F(Z^{Z^X}) \arrow[d, "\xi_Z^{Z^X}"] \\
F(Z)^{F(Z)^{F(X)}} \arrow[r, "F(Z)^{\xi_Z^X}"] & F(Z)^{F(Z^X)}
\end{tikzcd}$$

Proof. Since the following diagram commutes by (A.16.5),

$$\begin{tikzcd}
\mathcal{D}(F(X)\times F(Z)^{F(X)}, F(Z)) \arrow[r, "(1\times\xi_Z^X)^*"] \arrow[d, "\exp_{F(X),F(Z)^{F(X)},F(Z)}"] & \mathcal{D}(F(X)\times F(Z^X), F(Z)) \arrow[d, "\exp_{F(X),F(Z^X),F(Z)}"] \\
\mathcal{D}(F(X), F(Z)^{F(Z)^{F(X)}}) \arrow[r, "F(Z)_*^{\xi_Z^X}"] & \mathcal{D}(F(X), F(Z)^{F(Z^X)})
\end{tikzcd}$$

the transpose of $F(Z)^{\xi_Z^X}\eta(F(Z))_{F(X)}$ is the following composite.

$$F(X)\times F(Z^X) \xrightarrow{id_{F(X)}\times\xi_Z^X} F(X)\times F(Z)^{F(X)} \xrightarrow{T} F(Z)^{F(X)}\times F(X) \xrightarrow{\varepsilon_{F(Z)}^{F(X)}} F(Z)$$

We also have the following commutative diagram.

$$\begin{tikzcd}
 & F(X\times Z^X) \arrow[dr, "F(T)"] & \\
F(X)\times F(Z^X) \arrow[ur, "\cong"] \arrow[r, "T"] \arrow[d, "id_{F(X)}\times\xi_Z^X"] & F(Z^X)\times F(X) \arrow[r, "\cong"] \arrow[d, "\xi_Z^X\times id_{F(X)}"] & F(Z^X\times X) \arrow[d, "F(\varepsilon_Z^X)"] \\
F(X)\times F(Z)^{F(X)} \arrow[r, "T"] & F(Z)^{F(X)}\times F(X) \arrow[r, "\varepsilon_{F(Z)}^{F(X)}"] & F(Z)
\end{tikzcd}$$

Thus the transpose of $F(Z)^{\xi_Z^X}\eta(F(Z))_{F(X)}$ is the composite

$$F(X)\times F(Z^X) \xrightarrow{\cong} F(X\times Z^X) \xrightarrow{F(T)} F(Z^X\times X) \xrightarrow{F(\varepsilon_Z^X)} F(Z).$$

On the other hand, since

$$\begin{tikzcd}
\mathcal{D}\Big(F\Big(Z^{Z^X}\Big)\times F(Z^X), F(Z)\Big) \arrow[r, "\exp_{F(Z^{Z^X}),F(Z^X),F(Z)}"] \arrow[d, "(F(\eta(Z)_X)\times id_{F(Z^X)})^*"] & \mathcal{D}\Big(F\Big(Z^{Z^X}\Big), F(Z)^{F(Z^X)}\Big) \arrow[d, "F(\eta(Z)_X)^*"] \\
\mathcal{D}(F(X)\times F(Z^X), F(Z)) \arrow[r, "\exp_{F(X),F(Z^X),F(Z)}"] & \mathcal{D}\big(F(X), F(Z)^{F(Z^X)}\big)
\end{tikzcd}$$

is commutative, the transpose of $\xi_Z^{Z^X}F(\eta(Z)_X)$ is the composite

$$F(X)\times F(Z^X) \xrightarrow{F(\eta(Z)_X)\times id_{F(Z^X)}} F(Z^{Z^X})\times F(Z^X) \xrightarrow{\cong} F(Z^{Z^X}\times Z^X) \xrightarrow{F(\varepsilon_Z^{Z^X})} F(Z)$$

Applying F to the diagram of (A.16.7), we have a commutative diagram

$$\begin{tikzcd}
F(X)\times F(Z^X) \arrow[r, "\cong"] \arrow[d, "F(\eta(Z)_X)\times id_{F(Z^X)}"] & F(X\times Z^X) \arrow[r, "F(T)"] \arrow[d, "F(\eta(Z)_X\times id_{Z^X})"] & F(Z^X\times X) \arrow[d, "F(\varepsilon_Z^X)"] \\
F(Z^{Z^X})\times F(Z^X) \arrow[r, "\cong"] & F(Z^{Z^X}\times Z^X) \arrow[r, "F(\varepsilon_Z^{Z^X})"] & F(Z)
\end{tikzcd}$$

and this proves the assertion. □

Proposition A.16.14 *Under the assumption of (A.14.12), suppose that $\mathcal{C}$ and $\mathcal{D}$ are cartesian closed categories with terminal objects $1_{\mathcal{C}}$, $1_{\mathcal{D}} = R(1_{\mathcal{C}})$ and that R preserves exponentials. Then, $\lambda : \boldsymbol{G}\to\boldsymbol{H}$ is an isomorphism of comonads. Hence we have an isomorphism $\overline{T_\lambda} : \mathcal{C}_{\boldsymbol{G}}\to\mathcal{C}_{\boldsymbol{H}}$ of categories satisfying $V_{\boldsymbol{H}}\overline{T_\lambda} = V_{\boldsymbol{G}}$ and a natural equivalence $\bar{\lambda} : \overline{T_\lambda}F_{\boldsymbol{G}}\to F_{\boldsymbol{H}}$.*

Proof. Define $\psi_Z : Z\times L(1_{\mathcal{D}})\to LR(Z)$ to be the following composition.

$$\begin{aligned}
Z\times L(1_{\mathcal{D}}) &\xrightarrow{T} L(1_{\mathcal{D}})\times Z \xrightarrow{L\left(\eta_{1_{\mathcal{D}}}^{R(Z)}\right)\times id_Z} L((1_{\mathcal{D}}\times R(Z))^{R(Z)})\times Z \xrightarrow{L(\mathrm{pr}_2^{R(Z)})\times id_Z} L(R(Z)^{R(Z)})\times Z \\
&\xrightarrow{L\left((\eta_{R(Z)})^{R(Z)}\right)\times id_Z} L(RLR(Z)^{R(Z)})\times Z \xrightarrow{L(\xi_{LR(Z)}^Z)^{-1}\times id_Z} LR(LR(Z)^Z)\times Z \\
&\xrightarrow{\varepsilon_{LR(Z)^Z}\times id_Z} LR(Z)^Z\times Z \xrightarrow{\varepsilon_{LR(Z)}^Z} LR(Z)
\end{aligned}$$

First, we show $\psi_Z\lambda_Z = id_{LR(Z)}$. We denote by o the unique morphism $Z \to 1_{\mathcal{C}}$.

$$\begin{aligned}\psi_Z\lambda_Z &= \varepsilon^Z_{LR(Z)}\Big(\varepsilon_{LR(Z)^Z}L(\xi^Z_{LR(Z)})^{-1}L((\eta_{R(Z)})^{R(Z)})L(\mathrm{pr}_2^{R(Z)})L(\eta_{1_{\mathcal{D}}}^{R(Z)}) \times id_Z\Big)\,(LR(o), \varepsilon_Z)\\ &= \varepsilon^Z_{LR(Z)}(\varepsilon_{LR(Z)^Z} \times \varepsilon_Z)\Big(L\Big((\xi^Z_{LR(Z)})^{-1}(\eta_{R(Z)})^{R(Z)}\mathrm{pr}_2^{R(Z)}\eta_{1_{\mathcal{D}}}^{R(Z)}R(o)\Big), id_{LR(Z)}\Big) \;\cdots\; (1)\end{aligned}$$

Since $\mathrm{pr}_2(R(o) \times id_{R(Z)}) = \mathrm{pr}_2 : R(Z) \times R(Z) \to R(Z)$, the composition

$$R(Z) \xrightarrow{R(o)} 1_{\mathcal{D}} \xrightarrow{\eta_{1_{\mathcal{D}}}^{R(Z)}} (1_{\mathcal{D}} \times R(Z))^{R(Z)} \xrightarrow{\mathrm{pr}_2^{R(Z)}} R(Z)^{R(Z)}$$

is the transpose of $\mathrm{pr}_2 : R(Z) \times R(Z) \to R(Z)$. Hence $(\eta_{R(Z)})^{R(Z)}\mathrm{pr}_2^{R(Z)}\eta_{1_{\mathcal{D}}}^{R(Z)}R(o)$ is the transpose of $\eta_{R(Z)}\mathrm{pr}_2$. We put $\zeta = (\eta_{R(Z)})^{R(Z)}\mathrm{pr}_2^{R(Z)}\eta_{1_{\mathcal{D}}}^{R(Z)}R(o)$ and $\omega = (R(\mathrm{pr}_1), R(\mathrm{pr}_2)) : R(LR(Z)^Z \times Z) \xrightarrow{\cong} R(LR(Z)^Z) \times R(Z)$. Then, we have $R(\varepsilon^Z_{LR(Z)})\omega^{-1}((\xi^Z_{LR(Z)})^{-1} \times id_{R(Z)})(\zeta, id_{R(Z)}) = \varepsilon^{R(Z)}_{RLR(Z)}(\zeta \times id_{R(Z)})\Delta = \eta_{R(Z)}\mathrm{pr}_2\Delta = \eta_{R(Z)}$ by 1) of (A.16.12). By the naturarity of counit, we have the following commutative diagram.

$$\begin{tikzcd}
L(RLR(Z)^{R(Z)}) \times LR(Z) \arrow[d, "L(\xi^Z_{LR(Z)})^{-1} \times id_{LR(Z)}"] & L(RLR(Z)^{R(Z)} \times R(Z)) \arrow[l, "(L(\mathrm{pr}_1), L(\mathrm{pr}_2))"'] \arrow[d, "L((\xi^Z_{LR(Z)})^{-1} \times id_{R(Z)})"] & LR(Z) \arrow[l, "L(\zeta, id_{R(Z)})"'] \\
LR(LR(Z)^Z) \times LR(Z) \arrow[d, "\varepsilon_{LR(Z)^Z} \times \varepsilon_Z"] & L(R(LR(Z)^Z) \times R(Z)) \arrow[l, "(L(\mathrm{pr}_1), L(\mathrm{pr}_2))"'] & \\
LR(Z)^Z \times Z \arrow[d, "\varepsilon^Z_{LR(Z)}"] & LR(LR(Z)^Z \times Z) \arrow[l, "\varepsilon_{LR(Z)^Z \times Z}"'] \arrow[u, "L(\omega)"'] \arrow[d, "LR(\varepsilon^Z_{LR(Z)})"] & \\
LR(Z) & LRLR(Z) \arrow[l, "\varepsilon_{LR(Z)}"'] &
\end{tikzcd}$$

It follows from the above diagram that

$$\begin{aligned}(1) &= \varepsilon^Z_{LR(Z)}(\varepsilon_{LR(Z)^Z} \times \varepsilon_Z)(L(\xi^Z_{LR(Z)})^{-1} \times id_{LR(Z)})(L(\zeta), id_{LR(Z)})\\ &= \varepsilon_{LR(Z)}LR(\varepsilon^Z_{LR(Z)})L(\omega)^{-1}L((\xi^Z_{LR(Z)})^{-1} \times id_{R(Z)})L(\zeta, id_{R(Z)}) = \varepsilon_{LR(Z)}L(\eta_{R(Z)}) = id_{R(Z)}\end{aligned}$$

In order to show $\lambda_Z\psi_Z = id_{Z \times L(1_{\mathcal{D}})}$, we examine $\mathrm{pr}_1\lambda_Z\psi_Z$ and $\mathrm{pr}_2\lambda_Z\psi_Z$.

$$\begin{aligned}\mathrm{pr}_1\lambda_Z\psi_Z &= \varepsilon_Z\varepsilon^Z_{LR(Z)}\Big(\Big(\varepsilon_{LR(Z)^Z}L(\xi^Z_{LR(Z)})^{-1}L((\eta_{R(Z)})^{R(Z)})L(\mathrm{pr}_2^{R(Z)})L(\eta_{1_{\mathcal{D}}}^{R(Z)})\Big) \times id_Z\Big)T\\ &= \varepsilon^Z_Z\Big(\Big((\varepsilon_Z)^Z\varepsilon_{LR(Z)^Z}L(\xi^Z_{LR(Z)})^{-1}L((\eta_{R(Z)})^{R(Z)})L(\mathrm{pr}_2^{R(Z)})L(\eta_{1_{\mathcal{D}}}^{R(Z)})\Big) \times id_Z\Big)T\\ &= \varepsilon^Z_Z\Big(\Big(\varepsilon_{Z^Z}LR((\varepsilon_Z)^Z)L(\xi^Z_{LR(Z)})^{-1}L((\eta_{R(Z)})^{R(Z)})L(\mathrm{pr}_2^{R(Z)})L(\eta_{1_{\mathcal{D}}}^{R(Z)})\Big) \times id_Z\Big)T\\ &= \varepsilon^Z_Z\Big(\Big(\varepsilon_{Z^Z}L(\xi^Z_Z)^{-1}L(R(\varepsilon_Z)^{R(Z)})L((\eta_{R(Z)})^{R(Z)})L(\mathrm{pr}_2^{R(Z)})L(\eta_{1_{\mathcal{D}}}^{R(Z)})\Big) \times id_Z\Big)T\\ &= \varepsilon^Z_Z\Big(\Big(\varepsilon_{Z^Z}L(\xi^Z_Z)^{-1}L(\mathrm{pr}_2^{R(Z)})L(\eta_{1_{\mathcal{D}}}^{R(Z)})\Big) \times id_Z\Big)T = \varepsilon^Z_Z\Big(\Big(\varepsilon_{Z^Z}L((\xi^Z_Z)^{-1}\mathrm{pr}_2^{R(Z)}\eta_{1_{\mathcal{D}}}^{R(Z)})\Big) \times id_Z\Big)T \cdots (2)\end{aligned}$$

The following diagram commutes by 2) of (A.16.12), we have $(\xi^Z_Z)^{-1}\mathrm{pr}_2^{R(Z)}\eta_{1_{\mathcal{D}}}^{R(Z)} = R(\mathrm{pr}_2^Z\eta^Z_{1_{\mathcal{C}}})$.

$$\begin{tikzcd}
1_{\mathcal{D}} \arrow[r, "\eta_{1_{\mathcal{D}}}^{R(Z)}"] \arrow[d, equal] & (1_{\mathcal{D}} \times R(Z))^{R(Z)} \arrow[dr, "\mathrm{pr}_2^{R(Z)}"] & \\
R(1_{\mathcal{C}}) \arrow[dr, "R(\eta^Z_{1_{\mathcal{C}}})"'] & R(1_{\mathcal{C}} \times Z)^{R(Z)} \arrow[u, "(R(\mathrm{pr}_1), R(\mathrm{pr}_2))^{R(Z)}"'] \arrow[r, "R(\mathrm{pr}_2)^{R(Z)}"] & R(Z)^{R(Z)} \\
& R((1_{\mathcal{C}} \times Z)^Z) \arrow[u, "\xi^Z_{1_{\mathcal{C}} \times Z}"'] \arrow[r, "R(\mathrm{pr}_2^Z)"] & R(Z^Z) \arrow[u, "\xi^Z_Z"']
\end{tikzcd}$$

Therefore we have

$$\begin{aligned}(2) &= \varepsilon^Z_Z\big((\varepsilon_{Z^Z}LR(\mathrm{pr}_2^Z\eta^Z_{1_{\mathcal{C}}})) \times id_Z\big)T = \varepsilon^Z_Z\big((\mathrm{pr}_2^Z\varepsilon_{(1_{\mathcal{C}} \times Z)^Z}LR(\eta^Z_{1_{\mathcal{C}}})) \times id_Z\big)T = \varepsilon^Z_Z\big((\mathrm{pr}_2^Z\eta^Z_{1_{\mathcal{C}}}\varepsilon_{1_{\mathcal{C}}}) \times id_Z\big)T\\ &= \mathrm{pr}_2\varepsilon^Z_{1_{\mathcal{C}} \times Z}(\eta^Z_{1_{\mathcal{C}}} \times id_Z)(\varepsilon_{1_{\mathcal{C}}} \times id_Z)T = \mathrm{pr}_2(\varepsilon_{1_{\mathcal{C}}} \times id_Z)T = \mathrm{pr}_2T = \mathrm{pr}_1.\end{aligned}$$

Next, we examine $\mathrm{pr}_2\lambda_Z\psi_Z$. We note that $\psi_Z : Z \times L(1_{\mathcal{D}}) \to LR(Z)$ is the image of $\eta_{R(Z)} \in \mathcal{D}(R(Z), RLR(Z))$ by compositions of the following isomorphisms

$$\mathcal{D}(R(Z), RLR(Z)) \xrightarrow{\mathrm{pr}_2^*} \mathcal{D}(1_{\mathcal{D}} \times R(Z), RLR(Z)) \xrightarrow{\exp_{1_{\mathcal{D}},R(Z),RLR(Z)}} \mathcal{D}(1_{\mathcal{D}}, RLR(Z)^{R(Z)}) \xrightarrow{(\xi^Z_{LR(Z)})_*^{-1}}$$
$$\mathcal{D}(1_{\mathcal{D}}, R(LR(Z)^Z)) \xrightarrow{adj^{-1}} \mathcal{C}(L(1_{\mathcal{D}}), LR(Z)^Z) \xrightarrow{\exp^{-1}_{L(1_{\mathcal{D}}),Z,LR(Z)}} \mathcal{C}(L(1_{\mathcal{D}}) \times Z, LR(Z)) \xrightarrow{T^*} \mathcal{C}(Z \times L(1_{\mathcal{D}}), LR(Z)).$$

The following diagram commutes.

$$\begin{tikzcd}
\mathcal{C}(Z \times L(1_{\mathcal{D}}), LR(Z)) \arrow[r, "LR(o)_*"] \arrow[d, "T^*"] & \mathcal{C}(Z \times L(1_{\mathcal{D}}), L(1_{\mathcal{D}})) \arrow[d, "T^*"] & \\
\mathcal{C}(L(1_{\mathcal{D}})\times Z, LR(Z)) \arrow[r, "LR(o)_*"] \arrow[d, "\exp_{L(1_{\mathcal{D}}),Z,LR(Z)}"] & \mathcal{C}(L(1_{\mathcal{D}})\times Z, L(1_{\mathcal{D}})) \arrow[d, "\exp_{L(1_{\mathcal{D}}),Z,L(1_{\mathcal{D}})}"] & \mathcal{C}(L(1_{\mathcal{D}})\times 1_{\mathcal{C}}, L(1_{\mathcal{D}})) \arrow[l, "(id\times o)^*"'] \arrow[d, "\exp_{L(1_{\mathcal{D}}),1_{\mathcal{C}},L(1_{\mathcal{D}})}"] \\
\mathcal{C}(L(1_{\mathcal{D}}), LR(Z)^Z) \arrow[r, "LR(o)^Z_*"] \arrow[d, "adj"] & \mathcal{C}(L(1_{\mathcal{D}}), L(1_{\mathcal{D}})^Z) \arrow[d, "adj"] & \mathcal{C}(L(1_{\mathcal{D}}), L(1_{\mathcal{D}})^{1_{\mathcal{C}}}) \arrow[l, "L(1_{\mathcal{D}})^o_*"'] \arrow[d, "adj"] \\
\mathcal{D}(1_{\mathcal{D}}, R(LR(Z)^Z)) \arrow[r, "R(LR(o)^Z)_*"] \arrow[d, "(\xi^Z_{LR(Z)})_*"] & \mathcal{D}(1_{\mathcal{D}}, R(L(1_{\mathcal{D}})^Z)) \arrow[d, "(\xi^Z_{LR(1_{\mathcal{C}})})_*"] & \mathcal{D}(1_{\mathcal{D}}, R(L(1_{\mathcal{D}})^{1_{\mathcal{C}}})) \arrow[l, "R(L(1_{\mathcal{D}})^o)_*"'] \arrow[d, "(\xi^{1_{\mathcal{C}}}_{LR(1_{\mathcal{C}})})_*"] \\
\mathcal{D}(1_{\mathcal{D}}, RLR(Z)^{R(Z)}) \arrow[r, "RLR(o)^{R(Z)}_*"] \arrow[d, "\exp^{-1}_{1_{\mathcal{D}},R(Z),RLR(Z)}"] & \mathcal{D}(1_{\mathcal{D}}, RL(1_{\mathcal{D}})^{R(Z)}) \arrow[d, "\exp^{-1}_{1_{\mathcal{D}},R(Z),RL(1_{\mathcal{D}})}"] & \mathcal{D}(1_{\mathcal{D}}, RL(1_{\mathcal{D}})^{1_{\mathcal{D}}}) \arrow[l, "RL(1_{\mathcal{D}})^{R(o)}_*"'] \arrow[d, "\exp^{-1}_{1_{\mathcal{D}},1_{\mathcal{D}},RL(1_{\mathcal{D}})}"] \\
\mathcal{D}(1_{\mathcal{D}}\times R(Z), RLR(Z)) \arrow[r, "RLR(o)_*"] \arrow[d, "(o,id_{R(Z)})^*"] & \mathcal{D}(1_{\mathcal{D}}\times R(Z), RL(1_{\mathcal{D}})) \arrow[d, "(o,id_{R(Z)})^*"] & \mathcal{D}(1_{\mathcal{D}}\times 1_{\mathcal{D}}, RL(1_{\mathcal{D}})) \arrow[l, "(id_{1_{\mathcal{D}}}\times R(o))^*"'] \arrow[d, "\Delta^*"] \\
\mathcal{D}(R(Z), RLR(Z)) \arrow[r, "RLR(o)_*"] & \mathcal{D}(R(Z), RL(1_{\mathcal{D}})) & \mathcal{D}(1_{\mathcal{D}}, RL(1_{\mathcal{D}})) \arrow[l, "R(o)^*"']
\end{tikzcd}$$

Let us denote by $\overline{pr}_1 : L(1_{\mathcal{D}}) \to L(1_{\mathcal{D}})^{1_{\mathcal{C}}}$ the transpose of $\mathrm{pr}_1 : L(1_{\mathcal{D}}) \times 1_{\mathcal{C}} \to L(1_{\mathcal{D}})$. By the commutativity of

$$\begin{tikzcd}
R(L(1_{\mathcal{D}}) \times 1_{\mathcal{C}}) \arrow[rr, "\cong"] \arrow[d, "R(\mathrm{pr}_1)"] \arrow[dr, "R(\overline{pr}_1\times id_{1_{\mathcal{C}}})"] & & RL(1_{\mathcal{D}}) \times R(1_{\mathcal{C}}) \arrow[d, "R(\overline{pr}_1)\times id_{1_{\mathcal{C}}}"] \\
RL(1_{\mathcal{D}}) & R(L(1_{\mathcal{D}})^{1_{\mathcal{C}}} \times 1_{\mathcal{C}}) \arrow[l, "R(\varepsilon^{1_{\mathcal{C}}}_{L(1_{\mathcal{D}})})"'] \arrow[r, "\cong"] & R(L(1_{\mathcal{D}})^{1_{\mathcal{C}}}) \times R(1_{\mathcal{C}})
\end{tikzcd}$$

and (A.16.2), the transpose of composition $RL(1_{\mathcal{D}}) \xrightarrow{R(\overline{pr}_1)} R(L(1_{\mathcal{D}})^{1_{\mathcal{C}}}) \xrightarrow{\xi^{1_{\mathcal{C}}}_{L(1_{\mathcal{D}})}} RL(1_{\mathcal{D}})^{R(1_{\mathcal{C}})}$ is the transpose $\bar{p}_1 : RL(1_{\mathcal{D}}) \to RL(1_{\mathcal{D}})^{R(1_{\mathcal{C}})}$ of the projection $p_1 : RL(1_{\mathcal{D}}) \times R(1_{\mathcal{C}}) \to RL(1_{\mathcal{D}})$. Hence $\mathrm{pr}_1 \in \mathcal{C}(L(1_{\mathcal{D}}) \times 1_{\mathcal{C}}, L(1_{\mathcal{D}}))$ maps to $\bar{p}_1\eta_{1_{\mathcal{D}}} \in \mathcal{D}(1_{\mathcal{D}}, RL(1_{\mathcal{D}})^{1_{\mathcal{D}}})$ by the following compositions of isomorphisms.

$$\mathcal{C}(L(1_{\mathcal{D}}) \times 1_{\mathcal{C}}, L(1_{\mathcal{D}})) \xrightarrow{\exp_{L(1_{\mathcal{D}}),1_{\mathcal{C}},L(1_{\mathcal{D}})}} \mathcal{C}(L(1_{\mathcal{D}}), L(1_{\mathcal{D}})^{1_{\mathcal{C}}}) \xrightarrow{adj} \mathcal{D}(1_{\mathcal{D}}, R(L(1_{\mathcal{D}})^{1_{\mathcal{C}}})) \xrightarrow{\left(\xi^{1_{\mathcal{C}}}_{L(1_{\mathcal{D}})}\right)_*} \mathcal{D}(1_{\mathcal{D}}, RL(1_{\mathcal{D}})^{1_{\mathcal{D}}})$$

Moreover, the transpose of $\bar{p}_1\eta_{1_{\mathcal{D}}}$ is $p_1(\eta_{1_{\mathcal{D}}} \times id_{R(1_{\mathcal{C}})})$, which maps to $\eta_{1_{\mathcal{D}}}$ by $\Delta^* : \mathcal{D}(1_{\mathcal{D}} \times 1_{\mathcal{D}}, RL(1_{\mathcal{D}})) \to \mathcal{D}(1_{\mathcal{D}}, RL(1_{\mathcal{D}}))$. Therefore $\mathrm{pr}_2 \in \mathcal{C}(Z \times L(1_{\mathcal{D}}), L(1_{\mathcal{D}}))$ maps to $RLR(o)\eta_{R(Z)} = \eta_{1_{\mathcal{D}}}R(o) \in \mathcal{D}(R(Z), RL(1_{\mathcal{D}}))$ by the following compositions of isomorphisms.

$$\mathcal{C}(Z \times L(1_{\mathcal{D}}), L(1_{\mathcal{D}})) \xrightarrow{T^*} \mathcal{C}(L(1_{\mathcal{D}}) \times Z, L(1_{\mathcal{D}})) \xrightarrow{\exp_{L(1_{\mathcal{D}}),Z,L(1_{\mathcal{D}})}} \mathcal{C}(L(1_{\mathcal{D}}), L(1_{\mathcal{D}})^Z) \xrightarrow{adj} \mathcal{D}(1_{\mathcal{D}}, R(L(1_{\mathcal{D}})^Z))$$
$$\xrightarrow{\left(\xi^Z_{L(1_{\mathcal{D}})}\right)_*} \mathcal{D}(1_{\mathcal{D}}, RL(1_{\mathcal{D}})^{R(Z)}) \xrightarrow{\exp^{-1}_{1_{\mathcal{D}},R(Z),RL(1_{\mathcal{D}})}} \mathcal{D}(1_{\mathcal{D}} \times R(Z), RL(1_{\mathcal{D}})) \xrightarrow{(o,id_{R(Z)})^*} \mathcal{D}(R(Z), RL(1_{\mathcal{D}}))$$

This shows $\mathrm{pr}_2\lambda_Z\psi_Z = LR(o)\psi_Z = \mathrm{pr}_2$. Thus we have shown $\lambda_Z\psi_Z = id_{Z\times L(1_{\mathcal{D}})}$. □

Theorem A.16.15 *Let $R : \mathcal{C} \to \mathcal{D}$ be a functor between cartesian closed categories preserving exponentials. If R has a left adjoint $L : \mathcal{D} \to \mathcal{C}$ which is comonadic, then there exist an object X of $\mathcal{C}$ (unique up to isomorphism) and an equivalence $\Psi : \mathcal{D} \to \mathcal{C}/X$ such that $\Sigma_X\Psi = L$ and that ΨR is naturally equivalent to X^*.*

Proof. Let us denote by $\boldsymbol{G} = (LR, \varepsilon, L(\eta_R))$ the comomand on $\mathcal{C}$ obtained from the adjunction. By the assumption, the comparison functor $K_{\boldsymbol{G}} : \mathcal{D} \to \mathcal{C}_{\boldsymbol{G}}$ is an equivalence. Put $X = LR(1_{\mathcal{C}})$ and let $\boldsymbol{H}$ be the comonad given in (A.14.11). We denote by $F_{\boldsymbol{G}} : \mathcal{C} \to \mathcal{C}_{\boldsymbol{G}}$ and $F_{\boldsymbol{H}} : \mathcal{C} \to \mathcal{C}_{\boldsymbol{H}}$ the free functors and by $V_{\boldsymbol{G}} : \mathcal{C}_{\boldsymbol{G}} \to \mathcal{C}$ and $F_{\boldsymbol{H}} : \mathcal{C}_{\boldsymbol{H}} \to \mathcal{C}$ the forgetful functors. It follows from (A.14.12) and (A.16.14) that there exist an isomorphism $T : \mathcal{C}_{\boldsymbol{G}} \to \mathcal{C}_{\boldsymbol{H}}$ satisfying $V_{\boldsymbol{H}}T = V_{\boldsymbol{G}}$ and a natural equivalence $\bar{\lambda} : TF_{\boldsymbol{G}} \to F_{\boldsymbol{H}}$. We also have an isomorphism $\varXi : \mathcal{C}_{\boldsymbol{H}} \to \mathcal{C}/X$ satisfying $\varSigma_X \varXi = V_{\boldsymbol{H}}$ and $\varXi F_{\boldsymbol{H}} = X^*$ by (A.14.11). Set $\varPsi = \varXi T K_{\boldsymbol{G}}$ and $\varphi = \varXi(\bar{\lambda})$. Since $V_{\boldsymbol{G}} K_{\boldsymbol{G}} = L$ and $K_{\boldsymbol{G}} R = F_{\boldsymbol{G}}$, we have $\varSigma_X \varPsi = L$ and $\varphi : \varPsi R \to X^*$ is a natural equivalence.

Suppose that there exist an object Y of $\mathcal{C}$ and an equivalence $\varPhi : \mathcal{D} \to \mathcal{C}/Y$ such that $\varSigma_Y \varPhi = L$ and that $\varPhi R$ is naturally equivalent to Y^*. Since $R(1_{\mathcal{C}})$ is a terminal object of $\mathcal{D}$ and an equivalence $\varPhi$ preserves the terminal object, $\varPhi R(1_{\mathcal{C}})$ is a terminal object of $\mathcal{C}/Y$, hence it is isomorphic to $id_Y : Y \to Y$. If we put $\varPhi R(1_{\mathcal{C}}) = (p : W \to Y)$, p is an isomorphism. On the other hand, since $W = \varSigma_Y \varPhi R(1_{\mathcal{C}}) = LR(1_{\mathcal{C}}) = X$, it follows that Y is isomorphic to X. □

Proposition A.16.16 *Let $\mathcal{C}$ be a category with finite limits. $\mathcal{C}$ is cartesian closed if and only if $X^* : \mathcal{C} \to \mathcal{C}/X$ has a right adjoint for any object X.*

Proof. Suppose that $\mathcal{C}$ is cartesian closed. For an object $Y \xrightarrow{f} X$ of $\mathcal{C}/X$, define $\Pi_X(f)$ by the pull-back

$$\begin{tikzcd} \Pi_X(f) \arrow[r, "o"] \arrow[d] & 1_{\mathcal{C}} \arrow[d, "\bar{p}_2"] \\ Y^X \arrow[r, "f^X"] & X^X \end{tikzcd},$$

where $\bar{p}_2$ is the transpose of the projection $p_2 : 1_{\mathcal{C}} \times X \to X$. We note that the projection $\mathrm{pr}_2 : Z \times X \to X$ factors through p_2, in fact $\mathrm{pr}_2 = p_2(o \times id_X)$, where $o : Z \to 1_{\mathcal{C}}$ is the unique morphism. Hence the transpose $\overline{\mathrm{pr}_2} : Z \to X^X$ is a composite $\bar{p}_2 o$. If $g : Z \times X \to Y$ is a morphism and $\bar{g} : Z \to Y^X$ is its transpose, then $fg = \mathrm{pr}_2$ holds if and only if $f^X \bar{g} = \bar{p}_2 o$. Thus we have a natural bijection $\mathcal{C}/X(X^*(Z), (Y \xrightarrow{f} X)) \to \mathcal{C}(Z, \Pi_X(f))$ and this defines a right adjoint Π_X of X^*.

Conversely, suppose that X^* has a right adjoint Π_X for any X. We set $X^Y = \Pi_X X^*(Y)$ for an object Y. Then $\mathcal{C}(Z, X^Y)$ is naturally isomorphic to $\mathcal{C}/X(X^*(Z), X^*(Y))$ by the assumption. It follows from (A.3.9) that $\mathcal{C}/X(X^*(Z), X^*(Y))$ is naturally isomorphic to $\mathcal{C}(\varSigma_X X^*(Z), Y) = \mathcal{C}(Z \times X, Y)$. □

Lemma A.16.17 *Let $\mathcal{C}$ be a cartesian closed category.*

1) Let $1_{\mathcal{C}}$ denote a terminal object of $\mathcal{C}$, then $X \times 1_{\mathcal{C}} \cong X^{1_{\mathcal{C}}} \cong X$ and $1_{\mathcal{C}}^X \cong 1_{\mathcal{C}}$ hold for any object X.

2) If an initial object $0_{\mathcal{C}}$ of $\mathcal{C}$ exists, $X \times 0_{\mathcal{C}}$ is an initial object and $X^{0_{\mathcal{C}}}$ is a terminal object for any object X.

3) If an initial object is isomorphic to a terminal object, every object of $\mathcal{C}$ is isomorphic to a terminal object.

Proof. 1) We denote by $o : X \to 1_{\mathcal{C}}$ the unique morphism. Then, $(id_X, o) : X \to X \times 1_{\mathcal{C}}$ is an isomorphism with inverse $\mathrm{pr}_1 : X \times 1_{\mathcal{C}} \to X$. Let $\overline{\mathrm{pr}_1} : X \to X^{1_{\mathcal{C}}}$ be the transpose of pr_1. For a morphism $f : Y \to X$, the transpose of $\overline{\mathrm{pr}_1} f$ is $\mathrm{pr}_1(f \times id_1) = f p_1$, where $p_1 : Y \times 1_{\mathcal{C}} \to Y$ is the projection. Hence the composite

$$\mathcal{C}(Y, X) \xrightarrow{\overline{\mathrm{pr}_1}_*} \mathcal{C}(Y, X^{1_{\mathcal{C}}}) \xrightarrow{ex} \mathcal{C}(Y \times 1_{\mathcal{C}}, X)$$

coincides with p_1^*. Since p_1 is an isomorphism, it follows that $\overline{\mathrm{pr}_1}_*$ is a bijection. By (A.3.8), $\overline{\mathrm{pr}_1}$ is an isomorphism.

2) For any object X and Y, there are bijections $\mathcal{C}(Y, X^{0_{\mathcal{C}}}) \cong \mathcal{C}(Y \times 0_{\mathcal{C}}, X) \cong \mathcal{C}(0_{\mathcal{C}} \times Y, X) \cong \mathcal{C}(0_{\mathcal{C}}, X^Y)$ and $\mathcal{C}(0_{\mathcal{C}}, X^Y)$ consists of a single element.

3) By 1) and 2), $X \cong X^{1_{\mathcal{C}}} \cong X^{0_{\mathcal{C}}} \cong 1_{\mathcal{C}}$. □

Definition A.16.18 *An initial object 0 of a category is called a strict initial object if every morphism whose codomain is 0 is an isomorphism. Dually, a terminal object 1 of a category is called a strict terminal object if every morphism whose domain is 1 is an isomorphism.*

Proposition A.16.19 *An initial object of a cartesian closed category is a strict initial object.*

Proof. Let $\mathcal{C}$ be a cartesian closed category and $0_{\mathcal{C}}$ an initial object of $\mathcal{C}$. For an object X, let $\mathrm{pr}_1 : X \times 0_{\mathcal{C}} \to X$ and $\mathrm{pr}_2 : X \times 0_{\mathcal{C}} \to 0_{\mathcal{C}}$ the projections. Since $X \times 0_{\mathcal{C}}$ is an initial object of $\mathcal{C}$ by (A.16.17), pr_2 is an isomorphism. Hence $\mathrm{pr}_1 \mathrm{pr}_2^{-1} : 0_{\mathcal{C}} \to X$ coincides with unique morphism $\iota_X : 0_{\mathcal{C}} \to X$. Suppose that there is a morphism $f : X \to 0_{\mathcal{C}}$. Then, $f\iota_X : 0_{\mathcal{C}} \to 0_{\mathcal{C}}$ is the identity morphism of $0_{\mathcal{C}}$. On the other hand, $\iota_X f = \mathrm{pr}_1 \mathrm{pr}_2^{-1} f = id_X$ by the commutativity of the following diagram.

$$\begin{tikzcd} & X \arrow[dl, "id_X"'] \arrow[d, "(id_X, f)"] \arrow[dr, "f"] & \\ X & X \times 0_{\mathcal{C}} \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] & 0_{\mathcal{C}} \end{tikzcd}$$

Hence f is an isomorphism. $\square$

Proposition A.16.20 *Let $f : X \to Z$ and $g : Y \to W$ be morphisms in a cartesian closed category $\mathcal{C}$.*
1) If f is a monomorphism, so is $f^Y : X^Y \to Z^Y$.
2) If f and g are epimorphisms, so is $f \times g : X \times Y \to Z \times W$.
3) If f is an epimorphism, so is $Y^f : Y^Z \to Y^X$.

Proof. Let W be an object of $\mathcal{C}$.
1) Since the following diagram commutes by (A.16.2) and f_* is injective, f_*^Y is injective.

$$\begin{tikzcd} \mathcal{C}(W \times Y, X) \arrow[r, "\exp_{W,Y,X}"] \arrow[d, "f_*"] & \mathcal{C}(W, X^Y) \arrow[d, "f_*^Y"] \\ \mathcal{C}(W \times Y, Z) \arrow[r, "\exp_{W,Y,Z}"] & \mathcal{C}(W, Z^Y) \end{tikzcd}$$

Hence f^Y is a monomorphism.
2) Since the following diagram commutes by (A.16.2) and f^* is injective, $(f \times id_Y)^*$ is injective.

$$\begin{tikzcd} \mathcal{C}(Z \times Y, W) \arrow[r, "\exp_{Z,Y,W}"] \arrow[d, "(f \times id_Y)^*"] & \mathcal{C}(Z, W^Y) \arrow[d, "f^*"] \\ \mathcal{C}(X \times Y, W) \arrow[r, "\exp_{X,Y,W}"] & \mathcal{C}(X, W^Y) \end{tikzcd}$$

Hence $f \times id_Y$ is an epimorphism. Let $T_1 : Z \times Y \to Y \times Z$ and $T_2 : W \times Z \to Z \times W$ be the switching maps. Then $id_Z \times g = T_2(g \times id_Z)T_1$ is an epimorphism. Thus $f \times g = (id_Z \times g)(f \times id_Y)$ is an epimorphism.
3) Since the following diagram commutes by (A.16.5) and f^* is injective, $(id_W \times f)^*$ is injective by (2).

$$\begin{tikzcd} \mathcal{C}(W \times Z, Y) \arrow[r, "\exp_{W,Z,Y}"] \arrow[d, "(id_W \times f)^*"] & \mathcal{C}(W, Y^Z) \arrow[d, "Y_*^f"] \\ \mathcal{C}(W \times X, Y) \arrow[r, "\exp_{W,X,Y}"] & \mathcal{C}(W, Y^X) \end{tikzcd}$$

Hence Z^f is an epimorphism. $\square$

Definition A.16.21 *Let $\mathcal{E}$ be a category with finite limits. If $\mathcal{E}_X^{(2)}$ is a cartesian closed category for any object X of $\mathcal{E}$, we call $\mathcal{E}$ a locally cartesian closed category.*

Proposition A.16.22 *$\mathcal{E}$ a locally cartesian closed category if and only if the inverse image functor $f^* : \mathcal{E}_Y^{(2)} \to \mathcal{E}_X^{(2)}$ has a right adjoint for any morphism $f : X \to Y$ in $\mathcal{E}$.*

Proof. Suppose that $\mathcal{E}_X^{(2)}$ is a cartesian closed category for any object X of $\mathcal{E}$. Let $f : X \to Y$ be a morphism in $\mathcal{E}$. We regard f as an object $\boldsymbol{X} = (X \xrightarrow{f} Y)$ of $\mathcal{E}_Y^{(2)}$. For an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathcal{E}_X^{(2)}$, consider an object $f_*(\boldsymbol{E}) = (E \xrightarrow{f\pi} Y)$ and a morphism $\boldsymbol{\pi} = \langle \pi, id_X \rangle : f_*(\boldsymbol{E}) \to \boldsymbol{X}$ in $\mathcal{E}_Y^{(2)}$. Since $\mathcal{E}_Y^{(2)}$ is cartesian closed, there exist objects $f_*(\boldsymbol{E})^{\boldsymbol{X}}$, $\boldsymbol{X}^{\boldsymbol{X}}$ of $\mathcal{E}_Y^{(2)}$ and bijections $\Phi_{\boldsymbol{F}} : \mathcal{E}_Y^{(2)}(\boldsymbol{F} \times \boldsymbol{X}, f_*(\boldsymbol{E})) \to \mathcal{E}_Y^{(2)}(\boldsymbol{F}, f_*(\boldsymbol{E})^{\boldsymbol{X}})$ and $\Phi'_{\boldsymbol{F}} : \mathcal{E}_Y^{(2)}(\boldsymbol{F} \times \boldsymbol{X}, \boldsymbol{X}) \to \mathcal{E}_Y^{(2)}(\boldsymbol{F}, \boldsymbol{X}^{\boldsymbol{X}})$ which are natural in $\boldsymbol{F} = (F \xrightarrow{\rho} Y) \in \mathrm{Ob}\, \mathcal{E}_Y^{(2)}$. We denote by $\mathbf{1}_Y$ a terminal object $(Y \xrightarrow{id_Y} Y)$ of $\mathcal{E}_Y^{(2)}$ and define a morphism $\tilde{\boldsymbol{p}}_2 : \mathbf{1}_Y \to \boldsymbol{X}^{\boldsymbol{X}}$ to be the exponential transpose of the projection $\boldsymbol{p}_2 : \mathbf{1}_Y \times \boldsymbol{X} \to \boldsymbol{X}$. Consider the following cartesian square in $\mathcal{E}_Y^{(2)}$.

$$\begin{tikzcd} \mathbf{1}_Y \times_{\boldsymbol{X}^{\boldsymbol{X}}} f_*(\boldsymbol{E})^{\boldsymbol{X}} \arrow[r, "\tilde{\boldsymbol{p}}_2"] \arrow[d, "\tilde{\boldsymbol{p}}_1"] & f_*(\boldsymbol{E})^{\boldsymbol{X}} \arrow[d, "\boldsymbol{\pi}^{\boldsymbol{X}}"] \\ \mathbf{1}_Y \arrow[r, "\tilde{\boldsymbol{p}}_2"] & \boldsymbol{X}^{\boldsymbol{X}} \end{tikzcd}$$

For an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathcal{E}_Y^{(2)}$, we denote by $\boldsymbol{o_F} = \langle \rho, id_X \rangle : \boldsymbol{F} \to \boldsymbol{1}_Y$ the unique morphism. We have

$$\mathcal{E}_Y^{(2)}(\boldsymbol{F}, \boldsymbol{1}_Y \times_{\boldsymbol{X}^{\boldsymbol{X}}} f_*(\boldsymbol{E})^{\boldsymbol{X}}) = \{(\boldsymbol{o_F}, \boldsymbol{\xi}) \,|\, \pi^{\boldsymbol{X}}\boldsymbol{\xi} = \bar{\boldsymbol{p}}_2\boldsymbol{o_F},\ \boldsymbol{\xi} \in \mathcal{E}_Y^{(2)}(\boldsymbol{F}, f_*(\boldsymbol{E})^{\boldsymbol{X}})\}.$$

$\Phi_{\boldsymbol{F}}'^{-1} : \mathcal{E}_Y^{(2)}(\boldsymbol{F}, \boldsymbol{X}^{\boldsymbol{X}}) \to \mathcal{E}_Y^{(2)}(\boldsymbol{F} \times \boldsymbol{X}, \boldsymbol{X})$ maps $\bar{\boldsymbol{p}}_2\boldsymbol{o_F}$ to a composition $\boldsymbol{F} \times \boldsymbol{X} \xrightarrow{\boldsymbol{o_F} \times id_{\boldsymbol{X}}} \boldsymbol{1}_Y \times \boldsymbol{X} \xrightarrow{\boldsymbol{p}_2} \boldsymbol{X}$ which coincides with the projection $\mathrm{pr}_2 : \boldsymbol{F} \times \boldsymbol{X} \to \boldsymbol{X}$. For $\boldsymbol{\xi} \in \mathcal{E}_Y^{(2)}(\boldsymbol{F}, f_*(\boldsymbol{E})^{\boldsymbol{X}})$, $\Phi_{\boldsymbol{F}}'^{-1}$ maps $\pi^{\boldsymbol{X}}\boldsymbol{\xi}$ to a composition $\boldsymbol{F} \times \boldsymbol{X} \xrightarrow{\Phi_{\boldsymbol{F}}^{-1}(\boldsymbol{\xi})} f_*(\boldsymbol{E}) \xrightarrow{\boldsymbol{\pi}} \boldsymbol{X}$. Define a subset M of $\mathcal{E}_Y^{(2)}(\boldsymbol{F} \times \boldsymbol{X}, f_*(\boldsymbol{E}))$ by $M = \{\boldsymbol{\zeta} \in \mathcal{E}_Y^{(2)}(\boldsymbol{F} \times \boldsymbol{X}, f_*(\boldsymbol{E})) \,|\, \boldsymbol{\pi\zeta} = \mathrm{pr}_2\}$. Let $\gamma : M \to \mathcal{E}_Y^{(2)}(\boldsymbol{F}, \boldsymbol{1}_Y \times_{\boldsymbol{X}^{\boldsymbol{X}}} f_*(\boldsymbol{E})^{\boldsymbol{X}})$ be a map defined by $\gamma(\boldsymbol{\zeta}) = (\boldsymbol{o_F}, \Phi_{\boldsymbol{F}}(\boldsymbol{\zeta}))$. Then, γ is bijective. Let

$$\begin{tikzcd} F \times_Y X \arrow[r, "f_\rho"] \arrow[d, "\rho_f"] & F \arrow[d, "\rho"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} E \times_Y X \arrow[r, "f_{f\pi}"] \arrow[d, "(f\pi)_f"] & E \arrow[d, "f\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be cartesian squares. Then, we have $f^*(\boldsymbol{F}) = (F \times_Y X \xrightarrow{\rho_f} X)$, $f^*f_*(\boldsymbol{E}) = (E \times_Y X \xrightarrow{(f\pi)_f} X)$ and equalities

$$\begin{aligned} \mathcal{E}_X^{(2)}(f^*(\boldsymbol{F}), \boldsymbol{E}) &= \{\langle \varphi, id_X \rangle \,|\, \varphi \in \mathcal{E}(F \times_Y X, E),\ \pi\varphi = \rho_f\} \\ \mathcal{E}_X^{(2)}(f^*(\boldsymbol{F}), f^*f_*(\boldsymbol{E})) &= \{\langle \psi, id_X \rangle \,|\, \psi \in \mathcal{E}(F \times_Y X, E \times_Y X),\ (f\pi)_f\psi = \rho_f\} \\ &= \{\langle (\varphi, \rho_f), id_X \rangle \,|\, \varphi \in \mathcal{E}(F \times_Y X, E),\ f\pi\varphi = f\rho_f\}. \end{aligned}$$

Since $\boldsymbol{F} \times \boldsymbol{X} = (F \times_Y X \xrightarrow{f\rho_f} Y) = f_*f^*(\boldsymbol{F})$ and $f_* : \mathcal{E}_X^{(2)} \to \mathcal{E}_Y^{(2)}$ is a left adjoint of the inverse image functor f^* by (8.2.13), there is a bijection $\Psi_{\boldsymbol{F}} : \mathcal{E}_Y^{(2)}(\boldsymbol{F} \times \boldsymbol{X}, f_*(\boldsymbol{E})) = \mathcal{E}_Y^{(2)}(f_*f^*(\boldsymbol{F}), f_*(\boldsymbol{E})) \to \mathcal{E}_X^{(2)}(f^*(\boldsymbol{F}), f^*f_*(\boldsymbol{E}))$. For $\varphi \in \mathcal{E}(F \times_Y X, E)$, $\boldsymbol{\varphi} = \langle \varphi, id_Y \rangle \in M$ if and only if φ satisfies $\pi\varphi = \rho_f$ since $\mathrm{pr}_2 = \langle \rho_f, id_Y \rangle : \boldsymbol{F} \times \boldsymbol{X} \to \boldsymbol{X}$. Hence $\Psi_{\boldsymbol{F}}(M) = \{\langle (\varphi, \rho_f), id_X \rangle \,|\, \varphi \in \mathcal{E}(F \times_Y X, E),\ \pi\varphi = \rho_f\}$ holds and a map $\delta : \mathcal{E}_X^{(2)}(f^*(\boldsymbol{F}), \boldsymbol{E}) \to \Psi_{\boldsymbol{F}}(M)$ defined by $\delta(\langle \varphi, id_X \rangle) = \langle (\varphi, \rho_f), id_X \rangle$ is a bijection. Thus we have the following chain of natural bijections.

$$\mathcal{E}_Y^{(2)}(\boldsymbol{F}, \boldsymbol{1}_Y \times_{\boldsymbol{X}^{\boldsymbol{X}}} f_*(\boldsymbol{E})^{\boldsymbol{X}}) \xrightarrow{\gamma^{-1}} M \xrightarrow{\Psi_{\boldsymbol{F}}} \Psi_{\boldsymbol{F}}(M) \xrightarrow{\delta^{-1}} \mathcal{E}_X^{(2)}(f^*(\boldsymbol{F}), \boldsymbol{E})$$

Therefore a functor $f_! : \mathcal{E}_X^{(2)} \to \mathcal{E}_Y^{(2)}$ defined by $f_!(\boldsymbol{E}) = \boldsymbol{1}_Y \times_{\boldsymbol{X}^{\boldsymbol{X}}} f_*(\boldsymbol{E})^{\boldsymbol{X}}$ and $f_!(\boldsymbol{\xi}) = id_{\boldsymbol{1}_Y} \times_{\boldsymbol{X}^{\boldsymbol{X}}} f_*(\boldsymbol{\xi})^{\boldsymbol{X}}$ is a right adjoint of f^*.

Conversely, assume that the inverse image functor $f^* : \mathcal{E}_Y^{(2)} \to \mathcal{E}_X^{(2)}$ has a right adjoint $f_! : \mathcal{E}_X^{(2)} \to \mathcal{E}_Y^{(2)}$ for any morphism $f : X \to Y$ in $\mathcal{E}$. For objects $\boldsymbol{E}$, $\boldsymbol{F}$ and $\boldsymbol{G}$ of $\mathcal{E}_X^{(2)}$, it follows from the assumption and (8.2.13) that there are natural bijections $\mathcal{E}_X^{(2)}(\boldsymbol{G}, \rho_!\rho^*(\boldsymbol{E})) \to \mathcal{E}_Y^{(2)}(\rho^*(\boldsymbol{G}), \rho^*(\boldsymbol{E}))$ and $\mathcal{E}_Y^{(2)}(\rho^*(\boldsymbol{G}), \rho^*(\boldsymbol{E})) \to \mathcal{E}_X^{(2)}(\rho_*\rho^*(\boldsymbol{G}), \boldsymbol{E})$ if $\boldsymbol{F} = (F \xrightarrow{\rho} X)$. Since $\rho_*\rho^*(\boldsymbol{G}) = \boldsymbol{G} \times \boldsymbol{F}$, $\boldsymbol{E}^{\boldsymbol{F}}$ is defined to be $\rho_!\rho^*(\boldsymbol{E})$. $\square$

To be continued