A theory of plots

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Contents

1 Plots on a set

We denote by $\mathscr{S}et$ the category of sets and maps. For a category $\mathcal{C}$ and an object X of $\mathcal{C}$, we denote by h_X the presheaf on $\mathcal{C}$ represented by X, that is, $h_X : \mathcal{C}^{op} \to \mathscr{S}et$ is a functor defined by $h_X(U) = \mathcal{C}(U, X)$ and $h_X(f : U \to V) = (f^* : \mathcal{C}(U, X) \to \mathcal{C}(V, X))$. For a morphism $\varphi : X \to Y$ in $\mathcal{C}$, let $h_\varphi : h_X \to h_Y$ be a natural transformation defined by $(h_\varphi)_U = \varphi_* : \mathcal{C}(U, X) \to \mathcal{C}(U, Y)$.

Definition 1.1 *Let $\mathcal{C}$ be a category, $F : \mathcal{C} \to \mathscr{S}et$ a functor and X a set. Define a presheaf F_X on $\mathcal{C}$ to be a composition $\mathcal{C}^{op} \xrightarrow{F^{op}} \mathscr{S}et^{op} \xrightarrow{h_X} \mathscr{S}et$. Here $F^{op} : \mathcal{C}^{op} \to \mathscr{S}et^{op}$ is a functor defined by $F^{op}(U) = F(U)$ for $U \in \operatorname{Ob}\mathcal{C}$ and $F^{op}(f) = F(f)$ for $f \in \operatorname{Mor}\mathcal{C}$. An element of $\coprod_{U \in \operatorname{Ob}\mathcal{C}} F_X(U)$ is called an F-parametrization of X.*

Definition 1.2 *Let $(\mathcal{C}, J)$ be a site, X a set and $F : \mathcal{C} \to \mathscr{S}et$ a functor. Assume that $\mathcal{C}$ has a terminal object $1_\mathcal{C}$ and that $F(1_\mathcal{C})$ consists of a single element $*$. If a subset $\mathscr{D}$ of $\coprod_{U \in \operatorname{Ob}\mathcal{C}} F_X(U)$ satisfies the following conditions, we call $\mathscr{D}$ a the-ology on X with respect to F and $(\mathcal{C}, J)$ or just a the-ology on X for short and call a pair $(X, \mathscr{D})$ a the-ological object. An element of $\mathscr{D}$ is called an F-plot of $(X, \mathscr{D})$.*

(i) $\mathscr{D} \supset F_X(1_\mathcal{C})$

(ii) For a morphism $f : U \to V$ in $\mathcal{C}$, $F_X(f) : F_X(V) \to F_X(U)$ maps $\mathscr{D} \cap F_X(V)$ into $\mathscr{D} \cap F_X(U)$.

(iii) For an object U of $\mathcal{C}$, an element x of $F_X(U)$ belongs to $\mathscr{D} \cap F_X(U)$ if there exists a covering $(U_i \xrightarrow{f_i} U)_{i \in I}$ of U such that $F_X(f_i) : F_X(U) \to F_X(U_i)$ maps x into $\mathscr{D} \cap F_X(U_i)$ for any $i \in I$.

Remark 1.3 *For a subset $\mathscr{D}$ of $\coprod_{U \in \operatorname{Ob}\mathcal{C}} F_X(U)$ and $U \in \operatorname{Ob}\mathcal{C}$, we put $F_\mathscr{D}(U) = \mathscr{D} \cap F_X(U)$.*

(1) $\mathscr{D}$ satisfies condition (i) of (1.2) if and only if $F_\mathscr{D}(1_\mathcal{C}) = F_X(1_\mathcal{C})$.

(2) $\mathscr{D}$ satisfies condition (ii) of (1.2) if and only if a correspondence $U \mapsto F_\mathscr{D}(U)$ defines a subpresheaf $F_\mathscr{D}$ of F_X.

Assume that $\mathscr{D}$ satisfies condition (ii) of (1.2) below. We denote by $j : F_\mathscr{D} \to F_X$ the morphism of presheaves defined from the inclusion maps $F_\mathscr{D}(U) \hookrightarrow F_X(U)$ for $U \in \operatorname{Ob}\mathcal{C}$.

Proposition 1.4 *Condition (iii) of (1.2) is equivalent to the following conditions.*

(iii′) For an object U of $\mathcal{C}$, an element x of $F_X(U)$ belongs to $\mathscr{D} \cap F_X(U)$ if there exists $R \in J(U)$ such that $F_X(f) : F_X(U) \to F_X(\operatorname{dom}(f))$ maps x into $\mathscr{D} \cap F_X(\operatorname{dom}(f))$ for any $f \in R$.

(iii″) The following diagram is cartesian for any object U of $\mathcal{C}$ and covering $(U_i \xrightarrow{f_i} U)_{i \in I}$ of U.

$$\begin{array}{ccc} F_\mathscr{D}(U) & \xrightarrow{\quad j_U \quad} & F_X(U) \\ \Big\downarrow{\scriptstyle (F_\mathscr{D}(f_i))_{i\in I}} & & \Big\downarrow{\scriptstyle (F_X(f_i))_{i\in I}} \\ \prod_{i\in I} F_\mathscr{D}(U_i) & \xrightarrow{\prod_{i\in I} j_{U_i}} & \prod_{i\in I} F_X(U_i) \end{array}$$

Proof. It is clear that (iii') implies (iii) since $R \in J(U)$ is a covering of U. Assume that (iii) is satisfied and that $(U_i \xrightarrow{f_i} U)_{i \in I}$ is a covering of U such that $F_X(f_i) : F_X(U) \to F_X(U_i)$ maps $x \in F_X(U)$ into $\mathscr{D} \cap F_X(U_i)$ for any $i \in I$. Let R be a sieve generated by $(U_i \xrightarrow{f_i} U)_{i \in I}$, which is given by

$$R(V) = \{f \in h_U(V) \,|\, f = f_i g \text{ for some } i \in I \text{ and } g \in \mathcal{C}(V, U_i).\}.$$

Then, for $f \in R$, there exist $i \in I$ and $g : \operatorname{dom}(f) \to U_i$ such that $f = f_i g$. Since $F_X(f_i)(x) \in \mathscr{D} \cap F_X(U_i)$ implies $F_X(f)(x) = F_X(g)F_X(f_i)(x) \in \mathscr{D} \cap F_X(\operatorname{dom}(f))$ by (ii), it follows from (iii') that $x \in \mathscr{D} \cap F_X(U)$.

Suppose that condition (iii) of (1.2) is satisfied. For an object U of $\mathcal{C}$ and covering $(U_i \xrightarrow{f_i} U)_{i \in I}$ of U, if the image of $x \in F_X(U)$ by the map $(F_X(f_i))_{i \in I} : F_X(U) \to \prod_{i \in I} F_X(U_i)$ induced by $F_X(f_i)$'s contained in the image of $\prod_{i \in I} j_{U_i} : \prod_{i \in I} F_\mathscr{D}(U_i) \to \prod_{i \in I} F_X(U_i)$, $F_X(f_i)(x) \in \mathscr{D} \cap F_X(U_i)$ holds for any $i \in I$. Hence $x \in \mathscr{D} \cap F_X(U) = F_\mathscr{D}(U)$ which shows that the above diagram is cartesian. Conversely, suppose that the diagram of (iii'') is cartesian for any object U of $\mathcal{C}$ and covering $(U_i \xrightarrow{f_i} U)_{i \in I}$ of U. For $x \in F_X(U)$, assume that there exists a covering $(U_i \xrightarrow{f_i} U)_{i \in I}$ such that $F_X(f_i) : F_X(U) \to F_X(U_i)$ maps x into $\mathscr{D} \cap F_X(U_i) = F_\mathscr{D}(U)$ for any $i \in I$. Since $(*)$ is cartesian, x is in the image of $j_U : F_\mathscr{D}(U) \to F_X(U)$, namely x belongs to $\mathscr{D} \cap F_X(U)$. $\square$

For a map $\varphi : X \to Y$ and a functor $F : \mathcal{C} \to \mathcal{S}et$, we define a morphism $F_\varphi : F_X \to F_Y$ of presheaves by $(F_\varphi)_U = \varphi_* : F_X(U) = \mathcal{S}et(F(U), X) \to \mathcal{S}et(F(U), Y) = F_Y(U)$.

Definition 1.5 *Let $(\mathcal{C}, J)$ be a site and $F : \mathcal{C} \to \mathcal{S}et$ a functor.*

(1) Let $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ be the-ological objects. If the map $(F_\varphi)_U : F_X(U) \to F_Y(U)$ induced by a map $\varphi : X \to Y$ maps $\mathcal{D} \cap F_X(U)$ into $\mathcal{E} \cap F_Y(U)$ for each $U \in \operatorname{Ob}\mathcal{C}$, we call φ a morphism of the-ological objects. We denote this by $\varphi : (X, \mathcal{D}) \to (Y, \mathcal{E})$.

(2) We define a category $\mathscr{P}_F(\mathcal{C}, J)$ of the-ological objects as follows. Objects of $\mathscr{P}_F(\mathcal{C}, J)$ are the-ological objects and morphisms of $\mathscr{P}_F(\mathcal{C}, J)$ are morphism of the-ological objects.

Remark 1.6 *Let $\varphi : (X, \mathcal{D}) \to (Y, \mathcal{E})$ be a morphism of the-ological objects. It follows from the definition of a morphism of the-ological objects that $(F_\varphi)_U : F_X(U) \to F_Y(U)$ restricts to a map $(\check{F}_\varphi)_U : F_{\mathcal{D}}(U) \to F_{\mathcal{E}}(U)$ which is natural in $U \in \operatorname{Ob}\mathcal{C}$. Thus we have a morphism $\check{F}_\varphi : F_{\mathcal{D}} \to F_{\mathcal{E}}$ of presheaves.*

Definition 1.7 *For the-ologies $\mathcal{D}$ and $\mathcal{E}$ on X, we say that $\mathcal{D}$ is finer than $\mathcal{E}$ and that $\mathcal{E}$ is coarser than $\mathcal{D}$ if $\mathcal{D} \subset \mathcal{E}$.*

Remark 1.8 *We put $\mathcal{D}_{coarse,X} = \coprod_{U \in \operatorname{Ob}\mathcal{C}} F_X(U)$. It is clear that $\mathcal{D}_{coarse,X}$ is the coarsest the-ology on X. For a map $f : Y \to X$ and a the-ologgy $\mathcal{E}$ on Y, $f : (Y, \mathcal{E}) \to (X, \mathcal{D}_{coarse,X})$ is a morphism of the-ologies.*

Proposition 1.9 *Let $(\mathcal{D}_i)_{i \in I}$ be a family of the-ologies on a set X. Then, $\bigcap_{i \in I} \mathcal{D}_i$ is a the-ologgy on X that is the finest the-ology among the-ologies on X which are coarser than $\mathcal{D}_i$ for any $i \in I$.*

Proof. Put $\mathcal{E} = \bigcap_{i \in I} \mathcal{D}_i$. Since $\mathcal{D}_i \supset F_X(1_\mathcal{C})$ for any $i \in I$, $\mathcal{E} \supset F_X(1_\mathcal{C})$ holds. For a morphism $f : U \to V$ of $\mathcal{C}$, since $F_X(f) : F_X(V) \to F_X(U)$ maps $\mathcal{D}_i \cap F_X(V)$ to $\mathcal{D}_i \cap F_X(U)$ for any $i \in I$, $F_X(f)$ maps $\mathcal{E} \cap F_X(V)$ to $\mathcal{E} \cap F_X(U)$. Suppose that there exists a covering $(U_j \xrightarrow{f_j} U)_{j \in J}$ such that $F_X(f_j) : F_X(U) \to F_X(U_j)$ maps $x \in F_X(U)$ into $\mathcal{E} \cap F_X(U_j)$ for any $j \in J$. Hence $F_X(f_j)$ maps x into $\mathcal{D}_i \cap F_X(U_j)$ for any $j \in J$ which implies $x \in \mathcal{D}_i \cap F_X(U)$. Thus we have $x \in \mathcal{E} \cap F_X(U)$. □

For a set X, we denote by $\mathscr{P}_F(\mathcal{C}, J)_X$ a subcategory of $\mathscr{P}_F(\mathcal{C}, J)$ consisting of objects of the form $(X, \mathcal{D})$ and morphisms of the form $id_X : (X, \mathcal{D}) \to (X, \mathcal{E})$. Then, $\mathscr{P}_F(\mathcal{C}, J)_X$ is regarded as an ordered set of the-ologies on X. We often denote by $\mathcal{D}$ an object $(X, \mathcal{D})$ of $\mathscr{P}_F(\mathcal{C}, J)_X$ for short. It follows from (1.8) that $\mathcal{D}_{coarse,X}$ is the maximum (terminal) object of $\mathscr{P}_F(\mathcal{C}, J)_X$.

Corollary 1.10 *$\mathscr{P}_F(\mathcal{C}, J)_X$ is complete as an ordered set.*

Proof. Let Σ be a non-empty subset of $\mathscr{P}_F(\mathcal{C}, J)_X$. Then, $\inf \Sigma = \bigcap_{\mathcal{D} \in \Sigma} \mathcal{D}$ by (1.9). We denote by $\hat{\Sigma}$ a subset of $\mathscr{P}_F(\mathcal{C}, J)_X$ consisting of elements which contain every elements of Σ. Then it follows from (1.9) that $\bigcap_{\mathcal{E} \in \hat{\Sigma}} \mathcal{E}$ is an element of $\mathscr{P}_F(\mathcal{C}, J)_X$. Thus we see $\sup \Sigma = \bigcap_{\mathcal{E} \in \hat{\Sigma}} \mathcal{E}$. □

Proposition 1.11 *Let $\mathcal{S}$ be a subset of $\coprod_{U \in \operatorname{Ob}\mathcal{C}} F_X(U)$ which contains $F_X(1_\mathcal{C})$. For $f \in \operatorname{Mor}\mathcal{C}$, define a subset $\mathcal{S}_f$ of $F_X(\operatorname{dom}(f))$ by $\mathcal{S}_f = F_X(f)(\mathcal{S} \cap F_X(\operatorname{codom}(f)))$. For $U \in \operatorname{Ob}\mathcal{C}$, we define a subset $\mathcal{S}(U)$ of $F_X(U)$ by*

$$\mathcal{S}(U) = \left\{ x \in F_X(U) \,\middle|\, \text{There exists } R \in J(U) \text{ such that } F_X(g)(x) \in \bigcup_{f \in \operatorname{Mor}\mathcal{C}} \mathcal{S}_f \text{ for all } g \in R. \right\}.$$

If we put $\mathcal{G}(\mathcal{S}) = \coprod_{U \in \operatorname{Ob}\mathcal{C}} \mathcal{S}(U)$ and $\Sigma = \{\mathcal{D} \in \mathscr{P}_F(\mathcal{C}, J)_X \,|\, \mathcal{D} \supset \mathcal{S}\}$, then $\mathcal{G}(\mathcal{S}) = \inf \Sigma \in \mathscr{P}_F(\mathcal{C}, J)_X$.

Proof. Since $\mathcal{S}_{id_U} = \mathcal{S} \cap F_X(U)$, $\mathcal{S} \subset \bigcup_{f \in \operatorname{Mor}\mathcal{C}} \mathcal{S}_f$ holds. For $x \in \left(\bigcup_{f \in \operatorname{Mor}\mathcal{C}} \mathcal{S}_f\right) \cap F_X(U)$, there exists $f \in \operatorname{Mor}\mathcal{C}$ such that $\operatorname{dom}(f) = U$ and $x \in \mathcal{S}_f \cap F_X(U)$. Then, we have $x = \alpha F(f)$ for some $\alpha \in \mathcal{S} \cap F_X(\operatorname{codom}(f))$. For $g \in h_U$, since $F_X(g)(x) = F_X(g)(\alpha F(f)) = \alpha F(fg) = F_X(fg)(\alpha) \in F_X(fg)(\mathcal{S} \cap F_X(\operatorname{codom}(f)) = \mathcal{S}_{fg}$ and $h_U \in J(R)$, it follows that $x \in S(U)$. Hence we have $\left(\bigcup_{f \in \operatorname{Mor}\mathcal{C}} \mathcal{S}_f\right) \cap F_X(U) \subset \mathcal{S}(U)$ and $\mathcal{G}(\mathcal{S}) \supset \bigcup_{f \in \operatorname{Mor}\mathcal{C}} \mathcal{S}_f \supset \mathcal{S} \supset F_X(1_\mathcal{C})$.

Let $f : U \to V$ be a morphism in $\mathcal{C}$. For $x \in \mathscr{G}(\mathcal{S}) \cap F_X(V) = \mathcal{S}(V)$, there exists $R \in J(V)$ such that $F_X(g)(x) \in \bigcup_{f \in \operatorname{Mor}\mathcal{C}} \mathcal{S}_f$ for all $g \in R$. Hence there exists $s_g \in \operatorname{Mor}\mathcal{C}$ for each $g \in R$ such that $F_X(g)(x) \in \mathcal{S}_{s_g}$. It follows that there exists $x_g \in \mathcal{S} \cap F_X(\operatorname{codom}(s_g))$ which satisfies $F_X(s_g)(x_g) = F_X(g)(x)$ for each $g \in R$. Define a sieve $h_f^{-1}(R)$ on U by $h_f^{-1}(R) = \{j \in \operatorname{Mor}\mathcal{C} \,|\, \operatorname{codom}(j) = U,\ fj \in R\}$. Then, for $j \in h_f^{-1}(R)$, since $F_X(j)(F_X(f)(x)) = F_X(fj)(x) = F_X(s_{fj})(x_{fj}) \in F_X(s_{fj})(\mathcal{S} \cap F_X(\operatorname{codom}(s_{fj}))) = \mathcal{S}_{fj}$ and $h_f^{-1}(R) \in J(U)$ hold, we have $F_X(f)(x) \in \mathscr{G}(\mathcal{S}) \cap F_X(U) = \mathcal{S}(U)$. Thus $F_X(f) : F_X(V) \to F_X(U)$ maps $\mathscr{G}(\mathcal{S}) \cap F_X(V)$ into $\mathscr{G}(\mathcal{S}) \cap F_X(U)$.

For $U \in \operatorname{Ob}\mathcal{C}$ and $x \in F_X(U)$, suppose that there exists $R \in J(U)$ such that $F_X(f) : F_X(U) \to F_X(\operatorname{dom}(f))$ maps x into $\mathscr{G}(\mathcal{S}) \cap F_X(\operatorname{dom}(f)) = \mathcal{S}(\operatorname{dom}(f))$ for any $f \in R$. Then, there exists $S_f \in J(\operatorname{dom}(f))$ such that

$$F_X(fg)(x) = F_X(g)(F_X(f)(x)) \in \bigcup_{j \in \operatorname{Mor}\mathcal{C}} \mathcal{S}_j \ \cdots (*)$$

holds for any $g \in S_f$. Put $T = \{fg \,|\, f \in R,\ g \in S_f\}$. Since $T \in J(U)$, $(*)$ implies $x \in \mathcal{S}(U) = \mathscr{G}(\mathcal{S}) \cap F_X(U)$. Hence we conclude that $\mathscr{G}(\mathcal{S})$ is a the-ologgy on X.

Suppose that a the-ologgy $\mathscr{D}$ on X contains $\mathcal{S}$. For $f \in \operatorname{Mor}\mathcal{C}$, since

$$\mathcal{S}_f = F_X(f)(\mathcal{S} \cap F_X(\operatorname{codom}(f)) \subset F_X(f)(\mathscr{D} \cap F_X(\operatorname{codom}(f)) \subset \mathscr{D} \cap F_X(\operatorname{dom}(f)),$$

We have $\bigcup_{f \in \operatorname{Mor}\mathcal{C}} \mathcal{S}_f \subset \mathscr{D}$ which implies $\mathcal{S}(U) \subset \mathscr{D}$ for any $U \in \operatorname{Ob}\mathcal{C}$ by (1.4). Hence $\mathscr{G}(\mathcal{S}) \subset \mathscr{D}$ holds. □

Remark 1.12 *(1) For $U \in \operatorname{Ob}\mathcal{C}$, the subset $\mathcal{S}(U)$ of $F_X(U)$ defined in (1.11) coincides with*

$$\left\{x \in F_X(U) \,\middle|\, \text{There exists a covering } (U_i \xrightarrow{g_i} U)_{i \in I} \text{ such that } F_X(g_i)(x) \in \bigcup_{f \in \operatorname{Mor}\mathcal{C}} \mathcal{S}_f \text{ for all } i \in I.\right\}.$$

In fact, since $R \in J(U)$ is a covering of U, $\mathcal{S}(U)$ is contained in the above set. Suppose that, for $x \in F_X(U)$, there exists a covering $(U_i \xrightarrow{g_i} U)_{i \in I}$ such that $F_X(g_i)(x) \in \bigcup_{f \in \operatorname{Mor}\mathcal{C}} \mathcal{S}_f$ for any $i \in I$. We choose $f_i \in \operatorname{Mor}\mathcal{C}$ which satisfies $F_X(g_i)(x) \in \mathcal{S}_{f_i}$ for each $i \in I$. Let R be a sieve on U generated by $(U_i \xrightarrow{g_i} U)_{i \in I}$. For $j \in R$, there exist $i \in I$ and $k \in \mathcal{C}(\operatorname{dom}(j), U_i)$ such that $j = g_i k$. Then we have $F_X(j)(x) = F_X(k)(F_X(g_i)(x))$, which belongs to $F_X(k)(\mathcal{S}_{f_i}) = F_X(f_i k)(\mathcal{S} \cap F_X(\operatorname{codom}(f_i))) = \mathcal{S}_{f_i k}$. Therefore we have $x \in \mathcal{S}(U)$ and the above set is contained in $\mathcal{S}(U)$.

(2) Let Σ be a non-empty subset of $\mathscr{P}_F(\mathcal{C}, J)_X$ and put $\mathcal{S}(\Sigma) = \bigcup_{\mathscr{D} \in \Sigma} \mathscr{D}$. For $f \in \operatorname{Mor}\mathcal{C}$ and $x \in \mathcal{S}(\Sigma)_f$, there exist $\mathscr{D} \in \Sigma$ and $y \in \mathscr{D} \cap F_X(\operatorname{codom}(f))$ such that $x = F_X(f)(y)$ which belongs to $\mathscr{D} \cap F_X(\operatorname{dom}(f))$. It follows that $\bigcup_{f \in \operatorname{Mor}\mathcal{C}} \mathcal{S}(\Sigma)_f \subset \mathcal{S}(\Sigma)$ holds. Since $\mathcal{S}(\Sigma) \subset \bigcup_{f \in \operatorname{Mor}\mathcal{C}} \mathcal{S}(\Sigma)_f$, we have $\mathcal{S}(\Sigma) = \bigcup_{f \in \operatorname{Mor}\mathcal{C}} \mathcal{S}(\Sigma)_f$. Thus, for $U \in \operatorname{Ob}\mathcal{C}$, the following equality holds.

$$\mathcal{S}(\Sigma)(U) = \left\{x \in F_X(U) \,\middle|\, \text{There exists a covering } (U_i \xrightarrow{g_i} U)_{i \in I} \text{ such that } F_X(g_i)(x) \in \bigcup_{\mathscr{D} \in \Sigma} \mathscr{D} \text{ for all } i \in I.\right\}$$

Hence $\sup \Sigma = \mathscr{G}(\mathcal{S}(\Sigma)) = \bigcup_{U \in \mathcal{C}} \mathcal{S}(\Sigma)(U)$.

Definition 1.13 *For a subset $\mathcal{S}$ of $\coprod_{U \in \operatorname{Ob}\mathcal{C}} F_X(U)$ containing $F_X(1_{\mathcal{C}})$, we call $\mathscr{G}(\mathcal{S})$ defined in (1.11) the the-ology generated by $\mathcal{S}$.*

Definition 1.14 *Let $(\mathcal{C}, J)$ be a site and X a set. We put $\mathscr{D}_{disc,X} = \bigcap_{\mathscr{D} \in \operatorname{Ob}\mathscr{P}_F(\mathcal{C},J)_X} \mathscr{D}$ and call this the discrete the-ology on X. $\mathscr{D}_{disc,X}$ is the finest the-ology on X.*

Remark 1.15 *(1) For any map $f : X \to Y$ and a the-ologgy $\mathscr{E}$ on Y, $f : (X, \mathscr{D}_{disc,X}) \to (Y, \mathscr{E})$ is a morphism of the-ologies. In particular, $(X, \mathscr{D}_{disc,X})$ is the minimum (initial) object of $\mathscr{P}_F(\mathcal{C}, J)_X$.*

(2) Since $\mathscr{D}_{disc,X} \supset F_X(1_{\mathcal{C}})$, $\mathscr{D}_{disc,X}$ contains the image of the map $F_X(o_U) : F_X(1_{\mathcal{C}}) \to F_X(U)$ induced by the unique map $o_U : U \to 1_{\mathcal{C}}$ for any $U \in \operatorname{Ob}\mathcal{C}$. Hence every constant map in $F_X(U)$ belongs to $\mathscr{D}_{disc,X}$.

(3) Let $\mathcal{S}_{const}$ be the set of all constant maps in $\coprod_{U \in \operatorname{Ob}\mathcal{C}} F_X(U)$. Then $\mathcal{S}_{const} = \bigcup_{f \in \operatorname{Mor}\mathcal{C}} (\mathcal{S}_{const})_f$. Hence $\mathscr{D}_{disc,X} \cap F_X(U) = \mathscr{D}(\mathcal{S}_{const}) \cap F_X(U)$ coincides with the following set.

$$\{x \in F_X(U) \,|\, \text{There exists a covering } (U_i \xrightarrow{g_i} U)_{i \in I} \text{ such that } F_X(g_i)(x) \text{ is a contant map for all } i \in I.\}$$

Let $\mathcal{A}$ be an abelian category. We assume that there exists a functor $\varPsi : \mathcal{A} \to \mathcal{S}et$ which preserves products and terminal objects. For an object M of $\mathcal{A}$, let $\mathrm{pr}_{M,i} : M \times M \to M$ be the projection to i-th component for $i = 1, 2$. We denote by $\varepsilon_M : 0 \to M$ the unique morphism. Since $\mathcal{A}(M \times M, M)$ has a structure of an abelian group, we put $\alpha_M = \mathrm{pr}_{M,1} + \mathrm{pr}_{M,2} \in \mathcal{A}(M \times M, M)$ and $\iota_M = -id_M \in \mathcal{A}(M, M)$, then $(M, \varepsilon_M, \mu_M, \iota_M)$ is an abelian group object in $\mathcal{A}$. Since $\varPsi$ preserves products, maps $\varPsi(\mathrm{pr}_{M,1}), \varPsi(\mathrm{pr}_{M,2}) : \varPsi(M \times M) \to \varPsi(M)$ induced by the projections define a bijection $(\varPsi(\mathrm{pr}_1), \varPsi(\mathrm{pr}_2)) : \varPsi(M \times M) \to \varPsi(M) \times \varPsi(M)$. We define a map $\alpha^{\varPsi}_M : \varPsi(M) \times \varPsi(M) \to \varPsi(M)$ to be the following composition.

$$\varPsi(M) \times \varPsi(M) \xrightarrow{(\varPsi(\mathrm{pr}_1),\varPsi(\mathrm{pr}_2))^{-1}} \varPsi(M \times M) \xrightarrow{\varPsi(\alpha_M)} \varPsi(M)$$

We denote $\varPsi(0)$ by 0 which a terminal object of $\mathcal{S}et$ by the assumption. Put $\varepsilon^{\varPsi}_M = \varPsi(\varepsilon_M) : 0 \to \varPsi(M)$ and $\iota^{\varPsi}_M = \varPsi(\iota_M) : \varPsi(M) \to \varPsi(M)$. We can verify that $(\varPsi(M), \varepsilon^{\varPsi}_M, \alpha^{\varPsi}_M, \iota^{\varPsi}_M)$ is an abelian group.

We denote by $\mathrm{Ch}(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$. Objects of $\mathrm{Ch}(\mathcal{A})$ are families $(d_i : C_i \to C_{i-1})_{i\in \boldsymbol{Z}}$ of morphisms in $\mathcal{A}$ which satisfy $d_{i-1}d_i = 0$ for any $i \in \boldsymbol{Z}$. Morphisms from an object $(d_i : C_i \to C_{i-1})_{i\in\boldsymbol{Z}}$ to an object $(d'_i : D_i \to D_{i-1})_{i\in\boldsymbol{Z}}$ are families $(f_i : C_i \to D_i)_{i\in\boldsymbol{Z}}$ of morphisms in $\mathcal{A}$ which satisfy $d_i f_i = f_{i-1}d_i$ for any $i \in \boldsymbol{Z}$. For $k \in \boldsymbol{Z}$, let $\Delta_k : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$ be a functor defined by $\Delta_k((d_i : C_i \to C_{i-1})_{i\in\boldsymbol{Z}}) = C_k$ for $(d_i : C_i \to C_{i-1})_{i\in\boldsymbol{Z}} \in \mathrm{Ob}\,\mathrm{Ch}(\mathcal{A})$ and $\Delta_k((f_i : C_i \to D_i)_{i\in\boldsymbol{Z}}) = f_k$ for $(f_i : C_i \to D_i)_{i\in\boldsymbol{Z}} \in \mathrm{Mor}\,\mathrm{Ch}(\mathcal{A})$.

Definition 1.16 *Let $(\mathcal{C}, J)$ be a site and $F : \mathcal{C} \to \mathcal{S}et$, $\Lambda : \mathcal{C}^{op} \to \mathrm{Ch}(\mathcal{A})$ functors. For an object $(X, \mathscr{D})$ of $\mathscr{P}_F(\mathcal{C}, J)$, we consider the presheaf $F_{\mathscr{D}}$ on $\mathcal{C}$ given in (1.3). For an integer k, we call a natural transformation $\omega : F_{\mathscr{D}} \to \varPsi\Delta_k\Lambda$ a Λ-k-form of $(X, \mathscr{D})$. We denote by $\Omega_k((X, \mathscr{D}); \Lambda)$ the set of all Λ-k-forms of $(X, \mathscr{D})$.*

For $\omega, \chi \in \Omega_k((X, \mathscr{D}); \Lambda)$ and $U \in \mathrm{Ob}\,\mathcal{C}$, we can consider the sum $\omega_U + \chi_U$ of $\omega_U, \chi_U : F_{\mathscr{D}}(U) \to \varPsi\Delta_k\Lambda(U)$ by using the structure of an abelian group $\varPsi\Delta_k\Lambda(U)$. Since $\omega_U + \chi_U$ is natural in U, we define $\omega + \chi \in \Omega_k((X, \mathscr{D}); \Lambda)$ by $(\omega + \chi)_U = \omega_U + \chi_U$. Thus $\Omega_k((X, \mathscr{D}); \Lambda)$ has a structure of an abelian group. For $U \in \mathrm{Ob}\,\mathcal{C}$, let us denote by $d^{\Lambda}_{k,U} : \Delta_k\Lambda(U) \to \Delta_{k-1}\Lambda(U)$ the boundary morphism of a chain complex $\Lambda(U)$ in $\mathcal{A}$. Then, we have a homomorphism $\varPsi(d^{\Lambda}_{k,U}) : \varPsi\Delta_k\Lambda(U) \to \varPsi\Delta_{k-1}\Lambda(U)$ of abelian groups which is natural in U. Thus we have a chain complex $(\varPsi(d^{\Lambda}_{k,U}) : \varPsi\Delta_k\Lambda(U) \to \varPsi\Delta_{k-1}\Lambda(U))_{k\in\boldsymbol{Z}}$.

For $\omega \in \Omega_k((X, \mathscr{D}); \Lambda)$, we define $d^{\Lambda}_k(\omega) \in \Omega_{k-1}((X, \mathscr{D}); \Lambda)$ by $d^{\Lambda}_k(\omega)_U = \varPsi(d^{\Lambda}_{k,U})\omega_U : F_{\mathscr{D}}(U) \to \varPsi\Delta_{k-1}\Lambda(U)$ for $U \in \mathrm{Ob}\,\mathcal{C}$. Since $d^{\Lambda}_k(\omega)_U$ is natural in U, we have an element $d^{\Lambda}_k(\omega)$ of $\Omega_{k-1}((X, \mathscr{D}); \Lambda)$ and a correspondence $\omega \mapsto d^{\Lambda}_k(\omega)$ defines a homomorphism $d^{\Lambda}_k : \Omega_k((X, \mathscr{D}); \Lambda) \to \Omega_{k-1}((X, \mathscr{D}); \Lambda)$ of abelian groups which gives a chain complex $\Omega_{\bullet}((X, \mathscr{D}); \Lambda) = (d^{\Lambda}_k : \Omega_k((X, \mathscr{D}); \Lambda) \to \Omega_{k-1}((X, \mathscr{D}); \Lambda))_{k\in\boldsymbol{Z}}$.

Definition 1.17 *Let us denote by $H^k((X, \mathscr{D}); \Lambda)$ the k-dimensional cohomology group of the chain complex $\Omega_{\bullet}((X, \mathscr{D}); \Lambda)$ defined above. We call $H^*((X, \mathscr{D}); \Lambda) = \sum\limits_{k\in\boldsymbol{Z}} H^k((X, \mathscr{D}); \Lambda)$ the Λ-cohomology group of $(X, \mathscr{D})$.*

2 Category of the-ology

For a map $f : X \to Y$ and $(Y, \mathscr{E}) \in \mathrm{Ob}\,\mathscr{P}_F(\mathcal{C}, J)$, we define a the-ologgy $\mathscr{E}^f$ on X to be the coarsest the-ology such that $f : (X, \mathscr{E}^f) \to (Y, \mathscr{E})$ is a morphism of the-ologies.

Proposition 2.1 *For a map $f : X \to Y$ and $(Y, \mathscr{E}) \in \mathrm{Ob}\,\mathscr{P}_F(\mathcal{C}, J)$, $\mathscr{E}^f$ is given by*

$$\mathscr{E}^f = \coprod_{U\in\mathrm{Ob}\,\mathcal{C}} (F_f)^{-1}(\mathscr{E} \cap F_Y(U)) = \coprod_{U\in\mathrm{Ob}\,\mathcal{C}} \big\{\varphi \in F_X(U) \,\big|\, f\varphi \in \mathscr{E} \cap F_Y(U)\big\}.$$

Proof. We put $\bar{\mathscr{E}} = \coprod\limits_{U\in\mathrm{Ob}\,\mathcal{C}} \big\{\varphi \in F_X(U) \,\big|\, f\varphi \in \mathscr{E} \cap F_Y(U)\big\}$. Since $\mathscr{E} \supset F_Y(1_{\mathcal{C}})$, $\bar{\mathscr{E}} \supset F_X(1_{\mathcal{C}})$ holds.

For a morphism $\rho : U \to V$ of $\mathcal{C}$ and $\psi \in \bar{\mathscr{E}} \cap F_X(V)$, then $f\psi \in \mathscr{E} \cap F_Y(V)$ implies that $fF_X(\rho)(\psi) = f\psi\rho_* = F_Y(\rho)(f\psi)$ is contained in $\mathscr{E} \cap F_Y(U)$, which shows that $F_X(\rho)(\psi)$ is contained in $\bar{\mathscr{E}} \cap F_X(U)$. Thus $F_X(\rho) : F_X(V) \to F_X(U)$ maps $\bar{\mathscr{E}} \cap F_X(V)$ to $\bar{\mathscr{E}} \cap F_X(U)$.

For $\varphi \in F_X(U)$, assume that there exists a covering $(U_i \xrightarrow{\rho_i} U)_{i\in I}$ such that $F_X(\rho_i) : F_X(U) \to F_X(U_i)$ maps φ into $\bar{\mathscr{E}} \cap F_X(U_i)$ for any $i \in I$. Then, $F_Y(\rho_i)(f\varphi) = f\varphi\rho_{i*} = fF_X(\rho_i)(\varphi) \in \mathscr{E} \cap F_Y(U_i)$ for any $i \in I$. Hence $f\varphi \in \mathscr{E} \cap F_Y(U)$ which implies $\varphi \in \bar{\mathscr{E}} \cap F_X(U)$. Therefore $\bar{\mathscr{E}}$ is a the-ologgy on X.

Suppose that $\mathscr{D}$ is a the-ologgy on X such that $f : (X, \mathscr{D}) \to (Y, \mathscr{E})$ is a morphism of the-ologies. Then, $(F_f)_U : F_X(U) \to F_Y(U)$ maps $\mathscr{D} \cap F_X(U)$ into $\mathscr{E} \cap F_Y(U)$ for each $U \in \mathrm{Ob}\,\mathcal{C}$. Then $\mathscr{D} \cap F_X(U)$ is contained in $\big\{\varphi \in F_X(U) \,\big|\, f\varphi \in \mathscr{E} \cap F_Y(U)\big\}$. Hence we have $\mathscr{D} \subset \bar{\mathscr{E}}$ which shows $\bar{\mathscr{E}} = \mathscr{E}^f$. □

The following result is straightforward from the definition of $\mathscr{E}^f$.

Proposition 2.2 *Let $(\mathscr{E}_i)_{i\in I}$ a family of the-ologies on a set Y, For a map $f : X \to Y$, $\left(\bigcap_{i\in I}\mathscr{E}_i\right)^f = \bigcap_{i\in I}\mathscr{E}_i^f$ holds.*

Let us define a forgetful functor $\Gamma_F : \mathscr{P}_F(\mathcal{C}, J) \to \mathcal{S}et$ by $\Gamma(X, \mathscr{D}) = X$ for an object $(X, \mathscr{D})$ of $\mathscr{P}_F(\mathcal{C}, J)$ and $\Gamma_F(\varphi : (X, \mathscr{D}) \to (Y, \mathscr{E})) = (\varphi : X \to Y)$ for a morphism $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{E})$ in $\mathscr{P}_F(\mathcal{C}, J)$.

It is clear that Γ_F is faithful. In other words, if we put

$$\mathscr{P}_F(\mathcal{C}, J)_f((X, \mathscr{D}), (Y, \mathscr{E})) = \Gamma_F^{-1}(f) \cap \mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Y, \mathscr{E}))$$

for a map $f : X \to Y$ and $(X, \mathscr{D}), (Y, \mathscr{E}) \in \operatorname{Ob}\mathscr{P}_F(\mathcal{C}, J)$, $\mathscr{P}_F(\mathcal{C}, J)_f((X, \mathscr{D}), (Y, \mathscr{E}))$ has at most one element. We see that $\mathscr{P}_F(\mathcal{C}, J)_f((X, \mathscr{D}), (Y, \mathscr{E}))$ is not empty if and only if $\mathscr{D} \subset \mathscr{E}^f$ which is equivalent that $\mathscr{P}_F(\mathcal{C}, J)_X((X, \mathscr{D}), (X, \mathscr{E}^f))$ is not empty.

Proposition 2.3 *For maps $f : X \to Y$, $g : W \to X$ and an object $(Y, \mathscr{E})$ of $\mathscr{P}_F(\mathcal{C}, J)_Y$, $\mathscr{E}^{fg} = (\mathscr{E}^f)^g$ holds and $\Gamma_F : \mathscr{P}_F(\mathcal{C}, J) \to \mathcal{S}et$ is a fibered category.*

Proof. For $U \in \operatorname{Ob}\mathcal{C}$, $\varphi \in \mathscr{E}^{fg} \cap F_W(U)$ holds if and only if $fg\varphi \in \mathscr{E} \cap F_Y(U)$ which is equivalent to $g\varphi \in \mathscr{E}^f \cap F_X(U)$. Moreover $g\varphi \in \mathscr{E}^f \cap F_X(U)$ holds if and only if $\varphi \in (\mathscr{E}^f)^g \cap F_W(U)$. Thus we have $\mathscr{E}^{fg} = (\mathscr{E}^f)^g$. We put $f^*(Y, \mathscr{E}) = (X, \mathscr{E}^f)$ and let $\alpha_f(Y, \mathscr{E}) : f^*(Y, \mathscr{E}) = (X, \mathscr{E}^f) \to (Y, \mathscr{E})$ be the unique morphism in $\mathscr{P}_F(\mathcal{C}, J)$ that satisfies $\Gamma_F(\alpha_f(Y, \mathscr{E})) = f$. For an object $(X, \mathscr{D})$ of $\mathscr{P}_F(\mathcal{C}, J)_X$, a map

$$\mathscr{P}_F(\mathcal{C}, J)_X((X, \mathscr{D}), (X, \mathscr{E}^f)) \to \mathscr{P}_F(\mathcal{C}, J)_f((X, \mathscr{D}), (Y, \mathscr{E}))$$

which maps φ to $\alpha_f(Y, \mathscr{E})\varphi$ is bijective, namely $\alpha_f(Y, \mathscr{E})$ is a cartesian morphism. The equality $\mathscr{E}^{fg} = (\mathscr{E}^f)^g$ implies that the following composition coincides with $\alpha_{fg}(Y, \mathscr{E})$.

$$(W, \mathscr{E}^{fg}) = (W, (\mathscr{E}^f)^g) \xrightarrow{\alpha_g(X, \mathscr{E}^f)} (X, \mathscr{E}^f) \xrightarrow{\alpha_f(Y, \mathscr{E})} (Y, \mathscr{E})$$

Therefore $\Gamma_F : \mathscr{P}_F(\mathcal{C}, J) \to \mathcal{S}et$ is a fibered category. □

For a map $f : X \to Y$ and $(X, \mathscr{D}) \in \operatorname{Ob}\mathscr{P}_F(\mathcal{C}, J)$, we define a the-ologgy $\mathscr{D}_f$ on Y to be the finest the-ology such that $f : (X, \mathscr{D}) \to (Y, \mathscr{D}_f)$ is a morphism of the-ologies, that is, $\mathscr{D}_f = \bigcap_{\mathscr{E}\in\Sigma}\mathscr{E}$, where

$$\Sigma = \left\{\mathscr{E} \in \operatorname{Ob}\mathscr{P}_F(\mathcal{C}, J)_Y \,\middle|\, \mathscr{E} \supset \coprod_{U\in\operatorname{Ob}\mathcal{C}}(F_f)_U(\mathscr{D} \cap F_X(U))\right\}.$$

Remark 2.4 *We can also describe $\mathscr{D}_f$ by using (1.11) as follows. Consider a subset $\mathcal{S}$ of $\coprod_{U\in\operatorname{Ob}\mathcal{C}} F_Y(U)$ given by $\mathcal{S} = F_Y(1_\mathcal{C})\coprod\Big(\coprod_{U\in\operatorname{Ob}\mathcal{C}, U\neq 1_\mathcal{C}}(F_f)_U(\mathscr{D}\cap F_X(U))\Big)$. Then, if $U \neq 1_\mathcal{C}$, we have $\mathcal{S}\cap F_Y(U) = (F_f)_U(\mathscr{D}\cap F_X(U))$ and the subset $\mathcal{S}_g = F_Y(g)(\mathcal{S}\cap F_Y(\operatorname{codom}(g)))$ of $F_Y(\operatorname{dom}(g))$ for $g \in \operatorname{Mor}\mathcal{C}$ is given by*

$$\mathcal{S}_g = F_Y(g)((F_f)_{\operatorname{codom}(g)}(\mathscr{D}\cap F_X(\operatorname{codom}(g)))) = (F_f)_{\operatorname{dom}(g)}(F_X(g)(\mathscr{D}\cap F_X(\operatorname{codom}(g))))$$

if $\operatorname{codom}(g) \neq 1_\mathcal{C}$. Since $F_X(g) : F_X(\operatorname{codom}(g)) \to F_X(\operatorname{dom}(g))$ maps $\mathscr{D}\cap F_X(\operatorname{codom}(g))$ into $\mathscr{D}\cap F_X(\operatorname{dom}(g))$, the above equality implies $\mathcal{S}_g \subset (F_f)_{\operatorname{dom}(g)}(\mathscr{D}\cap F_X(\operatorname{dom}(g))) = \mathcal{S}_{id_{\operatorname{dom}(g)}}$. If $\operatorname{codom}(g) = 1_\mathcal{C}$, g is the unique morphism $o_V : V \to 1_\mathcal{C}$. Hence we have $\bigcup_{g\in\operatorname{Mor}\mathcal{C}}\mathcal{S}_g = \bigcup_{V\in\operatorname{Ob}\mathcal{C}, V\neq 1_\mathcal{C}}\mathcal{S}_{id_V} \cup \bigcup_{V\in\operatorname{Ob}\mathcal{C}}\mathcal{S}_{o_V}$. It follows that the following equality holds for $V \in \operatorname{Ob}\mathcal{C}$.

$$\Big(\bigcup_{g\in\operatorname{Mor}\mathcal{C}}\mathcal{S}_g\Big)\cap F_Y(V) = \mathcal{S}_{id_V}\cup\mathcal{S}_{o_V} = (F_f)_V(\mathscr{D}\cap F_X(V))\cup F_Y(o_V)(F_Y(1_\mathcal{C}))$$

For $U \in \operatorname{Ob}\mathcal{C}$, the subset $\mathcal{S}(U)$ of $F_Y(U)$ defined in (1.11) is the set of elements y of $F_Y(U)$ which satisfy the following condition $(*)$.

- $(*)$ *There exists $R \in J(U)$ such that, for each $h \in R$, $F_Y(h)(y) : F(\operatorname{dom}(h)) \to Y$ is a constant map or there exists $x \in \mathscr{D}\cap F_X(\operatorname{dom}(h))$ which satisfies $F_Y(h)(y) = (F_f)_{\operatorname{dom}(h)}(x)$.*

We remark that if $f : X \to Y$ is surjective, we can replace the above condition by the following condition.

- $(*')$ *There exists $R \in J(U)$ such that, for each $h \in R$, there exists $x \in \mathscr{D}\cap F_X(\operatorname{dom}(h))$ which satisfies $F_Y(h)(y) = (F_f)_{\operatorname{dom}(h)}(x)$.*

If we put $\mathscr{G}(\mathcal{S}) = \coprod_{U\in\operatorname{Ob}\mathcal{C}}\mathcal{S}(U)$, we have $\mathscr{D}_f = \mathscr{G}(\mathcal{S})$.

Proposition 2.5 *$\Gamma_F : \mathscr{P}_F(\mathcal{C}, J) \to \mathcal{S}et$ is a bifibered category.*

Proof. For a map $f : X \to Y$, we define a functor $f_* : \mathscr{P}_F(\mathcal{C}, J)_X \to \mathscr{P}_F(\mathcal{C}, J)_Y$ as follows. For an object $(X, \mathscr{D})$ of $\mathscr{P}_F(\mathcal{C}, J)_X$, we put $f_*(X, \mathscr{D}) = (Y, \mathscr{D}_f)$. If $\mathscr{D}$ and $\mathscr{D}'$ are the-ologies on X such that $\mathscr{D} \subset \mathscr{D}'$, then $\mathscr{D}_f \subset \mathscr{D}'_f$. Hence we can put $f_*(id_X : (X, \mathscr{D}) \to (X, \mathscr{D}')) = (id_Y : (Y, \mathscr{D}_f) \to (Y, \mathscr{D}'_f))$.

For an object $(X, \mathscr{D})$ of $\mathscr{P}_F(\mathcal{C}, J)_X$ and an object $(Y, \mathscr{E})$ of $\mathscr{P}_F(\mathcal{C}, J)_Y$, $\mathscr{D}_f \subset \mathscr{E}$ holds if and only if $(F_f)_U(\mathscr{D} \cap F_X(U)) \subset \mathscr{E}$ for any $U \in \operatorname{Ob}\mathcal{C}$, which is equivalent to $\mathscr{D} \subset \mathscr{E}^f$. Thus $\mathscr{P}_F(\mathcal{C}, J)_Y(f_*(X, \mathscr{D}), (Y, \mathscr{E}))$ is not empty if and only if $\mathscr{P}_F(\mathcal{C}, J)_X((X, \mathscr{D}), f^*(Y, \mathscr{E}))$ is not empty. It follows that f_* is a left adjoint of f^* and that $\Gamma_F : \mathscr{P}_F(\mathcal{C}, J) \to \mathcal{S}et$ is a bifibered category. □

Remark 2.6 *For $(X, \mathscr{D}) \in \operatorname{Ob}\mathscr{P}_F(\mathcal{C}, J)_X$, $(Y, \mathscr{E}) \in \operatorname{Ob}\mathscr{P}_F(\mathcal{C}, J)_Y$ and a map $f : X \to Y$, $\mathscr{D} \subset (\mathscr{D}_f)^f$ and $(\mathscr{E}^f)_f \subset \mathscr{E}$ hold. Hence the unit $\eta^f : id_{\mathscr{P}_F(\mathcal{C},J)_X} \to f^*f_*$ and the counit $\varepsilon^f : f_*f^* \to id_{\mathscr{P}_F(\mathcal{C},J)_Y}$ of the adjunction $f_* \dashv f^*$ are given by morphisms $\eta^f_{(X,\mathscr{D})} : (X, \mathscr{D}) \to (X, (\mathscr{D}_f)^f)$ and $\varepsilon^f_{(Y,\mathscr{E})} : (Y, (\mathscr{E}^f)_f) \to (Y, \mathscr{E})$ in $\mathscr{P}_F(\mathcal{C}, J)_X$ and $\mathscr{P}_F(\mathcal{C}, J)_Y$, respectively.*

Proposition 2.7 *Let $f : X \to Y$ and $g : Y \to Z$ be maps. For a the-ology $\mathscr{D}$ on X, $(\mathscr{D}_f)_g = \mathscr{D}_{gf}$ holds.*

Proof. Let $p : \mathcal{F} \to \mathcal{E}$ be a bifibered category and $f : X \to Y$, $g : Y \to Z$ morphisms in $\mathcal{E}$. Then, the inverse image functors $f^* : \mathcal{F}_Y \to \mathcal{F}_X$, $g^* : \mathcal{F}_Z \to \mathcal{F}_Y$ and $(gf)^* : \mathcal{F}_X \to \mathcal{F}_Z$ have left adjoints $f_* : \mathcal{F}_X \to \mathcal{F}_Y$, $g_* : \mathcal{F}_Y \to \mathcal{F}_Z$ and $(gf)_* : \mathcal{F}_X \to \mathcal{F}_Z$, respectively. Since $g_*f_* : \mathcal{F}_X \to \mathcal{F}_Z$ is also a left adjoint of $(gf)_* : \mathcal{F}_X \to \mathcal{F}_Z$, there is a natural equivalence $g_*f_* \to (gf)_*$. In the case $\mathcal{F} = \mathscr{P}_F(\mathcal{C}, J)$, $\mathcal{E} = \mathcal{S}et$ and $p = \Gamma_F$, there is an isomorphism $(Z, (\mathscr{D}_f)_g) \to (Z, \mathscr{D}_{gf})$ in $\mathscr{P}_F(\mathcal{C}, J)_Z$. Since $\mathscr{P}_F(\mathcal{C}, J)_Z$ is a partially ordered set, we have $(\mathscr{D}_f)_g = \mathscr{D}_{gf}$. □

Lemma 2.8 *Let $f : (X, \mathscr{D}) \to (Y, \mathscr{E})$ be a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ and $h : X \to Z$ a surjection. If there exists a morphism $g : (Y, \mathscr{E}) \to (Z, \mathscr{D}_h)$ in $\mathscr{P}_F(\mathcal{C}, J)$ which satisfies $gf = h$, we have $\mathscr{E}_g = \mathscr{D}_h$.*

Proof. Since $\mathscr{E}_g$ is the finest the-ology on Z such that $g : (Y, \mathscr{E}) \to (Z, \mathscr{E}_g)$ is a morphism of the-ologies, $\mathscr{E}_g$ is contained in $\mathscr{D}_h$. Let U be an object of $\mathcal{C}$ and take $\alpha \in \mathscr{D}_h \cap F_Z(U)$. It follows from (2.4) that there exists $R \in J(U)$ such that, for each $k \in R$, there exists $\beta \in \mathscr{E} \cap F_X(\operatorname{dom}(k))$ which satisfies $F_Z(k)(\alpha) = (F_h)_{\operatorname{dom}(k)}(\beta)$. Since both $f : (X, \mathscr{D}) \to (Y, \mathscr{E})$ and $g : (Y, \mathscr{E}) \to (Z, \mathscr{E}_g)$ are morphisms in $\mathscr{P}_F(\mathcal{C}, J)$, so is the composition $h = gf : (X, \mathscr{D}) \to (Z, \mathscr{E}_g)$. Hence $F_Z(k)(\alpha) = (F_h)_{\operatorname{dom}(k)}(\beta)$ belongs to $\mathscr{E}_g \cap F_Z(\operatorname{dom}(k))$ for any $k \in R$, which shows that α belongs to $\mathscr{E}_g \cap F_Z(U)$. Thus we have $\mathscr{D}_h \subset \mathscr{E}_g$. □

Let $p : \mathcal{F} \to \mathcal{E}$ be a bifibered category. Suppose that the following diagram in $\mathcal{E}$ is commutative.

$$\begin{array}{ccc} W & \xrightarrow{\ g\ } & Z \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle j} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

We denote by $\eta^f : id_{\mathcal{F}_X} \to f^*f_*$ and $\varepsilon^g : g_*g^* \to id_{\mathcal{F}_Z}$ the unit of the adjunction $f_* \dashv f^*$ and the counit of the adjunction $g_* \dashv g^*$, respectively. For an object M of $\mathcal{F}_X$, we denote by $\Phi_M : g_*i^*(M) \to j^*f_*(M)$ the following composition of morphims in $\mathcal{F}_Z$.

$$\begin{aligned} g_*i^*(M) &\xrightarrow{g_*i^*(\eta^f_M)} g_*i^*f^*f_*(M) \xrightarrow{g_*(c_{f,i}(f_*(M)))} g_*(fi)^*f_*(M) = g_*(jg)^*f_*(M) \\ &\xrightarrow{g_*(c_{j,g}(f_*(M))^{-1})} g_*g^*j^*f_*(M) \xrightarrow{\varepsilon^g_{j^*f_*(M)}} j^*f_*(M) \end{aligned}$$

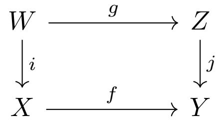
Then, we have a natural transformation $\Phi : g_*i^* \to j^*f_*$.

In the case $\mathcal{E} = \mathcal{S}et$, $\mathcal{F} = \mathscr{P}_F(\mathcal{C}, J)$ and $p = \Gamma_F$, it follows from (2.6) and (2.3) that the above composition for $M = (X, \mathscr{D}) \in \operatorname{Ob}\mathscr{P}_F(\mathcal{C}, J)_X$ coincides with the following composition.

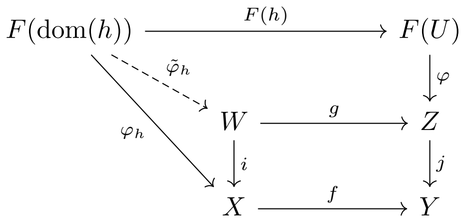
$$(Z, (\mathscr{D}^i)_g) \xrightarrow{g_*i^*(\eta^f_{(X,\mathscr{D})})} (Z, (((\mathscr{D}_f)^f)^i)_g) = (Z, ((\mathscr{D}_f)^{fi})_g) = (Z, ((\mathscr{D}_f)^{jg})_g) = (Z, (((\mathscr{D}_f)^j)^g)_g) \xrightarrow{\varepsilon^g_{j^*f_*(X,\mathscr{D})}} (Z, (\mathscr{D}_f)^j)$$

Thus $\Phi_{(X,\mathscr{D})} : g_*i^*(X, \mathscr{D}) \to j^*f_*(X, \mathscr{D})$ coincides with a morphism $id_Z : (Z, (\mathscr{D}^i)_g) \to (Z, (\mathscr{D}_f)^j)$ in $\mathscr{P}_F(\mathcal{C}, J)_Z$. Namely, $(\mathscr{D}^i)_g$ is contained in $(\mathscr{D}_f)^j$.

Proposition 2.9 *If the following diagram in $\mathcal{S}et$ is cartesian and f is surjective, then $(\mathscr{D}_f)^j = (\mathscr{D}^i)_g$ holds for a the-ology $\mathscr{D}$ on X.*



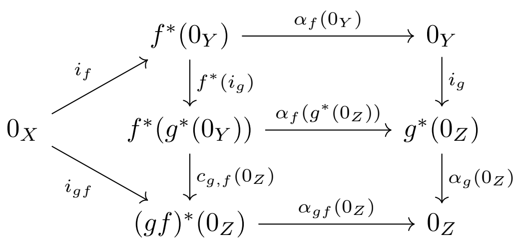
Proof. We have seen that $(\mathscr{D}^i)_g$ is contained in $(\mathscr{D}_f)^j$. Let U be an object of $\mathcal{C}$ and take $\varphi \in (\mathscr{D}_f)^j \cap F_Z(U)$. Since $j\varphi \in \mathscr{D}_f \cap F_Y(U)$, it follows from (2.4) that there exists $R \in J(U)$ such that, for each $h \in R$, there exists $\varphi_h \in \mathscr{D} \cap F_X(\mathrm{dom}(h))$ which satisfies $j\varphi F(h) = F_Y(h)(j\varphi) = (F_f)_{\mathrm{dom}(h)}(\varphi_h) = f\varphi_h$. Hence there exists unique map $\tilde{\varphi}_h : F(\mathrm{dom}(h)) \to W$ that makes the following diagram commute.



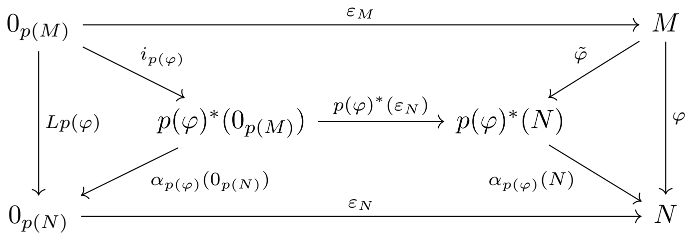
Since $i\tilde{\varphi}_h = \varphi_h \in \mathscr{D} \cap F_X(\mathrm{dom}(h))$ holds, we have $\tilde{\varphi}_h \in \mathscr{D}^i \cap F_W(\mathrm{dom}(h))$, which implies $\varphi \in (\mathscr{D}^i)_g \cap F_Z(U)$ by (2.4). Thus we see that $(\mathscr{D}_f)^j$ is contained in $(\mathscr{D}^i)_g$. $\square$

Proposition 2.10 *Let $p : \mathcal{F} \to \mathcal{E}$ be a prefibered category. If $\mathcal{F}_X$ has an initial object for any object X of $\mathcal{E}$, then p has a left adjoint.*

Proof. We denote by 0_X an initial object of $\mathcal{F}_X$ and define a functor $L : \mathcal{E} \to \mathcal{F}$ as follows. We put $L(X) = 0_X$ for an object X of $\mathcal{E}$. For a morphism $f : X \to Y$ in $\mathcal{E}$ and an object N of $\mathcal{F}_Y$, we denote by $i_f : 0_X \to f^*(0_Y)$ unique morphism in $\mathcal{F}_X$ and by $\alpha_f(N) : f^*(N) \to N$ the cartesian morphism that is mapped to f by p. Put $L(f) = \alpha_f(0_Y)i_f$. Since the identity morphism of 0_X is unique morphism in $\mathcal{E}_X$ from 0_X to 0_X, $L(id_X)$ is the identity morphism of 0_X if $X = Y$. For composable morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{E}$, let $f^*(i_g) : f^*(0_Y) \to f^*(g^*(0_Y))$ and $c_{g,f}(0_Z) : f^*(g^*(0_Y)) \to (gf)^*(0_Z)$ be unique morphisms in $\mathcal{F}_X$ that make the upper and the lower rectangles of the following diagram commutative, respectively.



Since i_f, $f^*(i_g)$, $c_{g,f}(0_Z)$ and i_{gf} are morphisms in $\mathcal{F}_X$, the left triangle of the above diagram is commutative. Hence $L(gf) = L(g)L(f)$ holds, which shows that L is a functor. pL is the identity functor of $\mathcal{E}$ since $p(i_f) = id_X$ and $p(\alpha_f(0_Y)) = f$ hold for any morphism $f : X \to Y$ in $\mathcal{E}$. We denote by $\eta : id_{\mathcal{E}} \to pL$ the identity natural transformation. For an object M of $\mathcal{F}$, let $\varepsilon_M : Lp(M) = 0_{p(M)} \to M$ be unique morphism in $\mathcal{F}_{p(M)}$. For a morphism $\varphi : M \to N$ in $\mathcal{F}$, there exists unique morphism $\tilde{\varphi} : M \to p(\varphi)^*(N)$ in $\mathcal{F}_{p(M)}$ that makes the right triangle of the following diagram commute. The right triangle of the following diagram commutes by the definition of L and the lower trapezoid of the following diagram commutes by the definition of $p(\varphi)^*(\varepsilon_N)$. Since ε_M, $\tilde{\varphi}$, $i_{p(\varphi)}$, $\alpha_{p(\varphi)}(0_{p(N)})$ are morphisms in $\mathcal{F}_{p(M)}$ and $0_{p(M)}$ is an initial object of $\mathcal{F}_{p(M)}$, the upper trapezoid of the following diagram is also commutative.



Thus we have a natural transformation $\varepsilon : Lp \to id_{\mathcal{F}}$. For an object M of $\mathcal{F}$, since $p(\varepsilon_M)$ is the identity morphism of $p(M)$, a composition $p(M) \xrightarrow{\eta_{p(M)}} p(M) = pLp(M) \xrightarrow{p(\varepsilon_M)} p(M)$ is also the identity morphism of M. For an object X of $\mathcal{E}$, since $\varepsilon_{L(X)} : LpL(X) = 0_X \to 0_X = L(X)$ is the identity morphism of 0_X, a composition $L(X) \xrightarrow{L(\eta_X)} LpL(X) \xrightarrow{\varepsilon_{L(X)}} L(X)$ is the identity morphism of $L(X) = 0_X$. Therefore L is a left adjoint of p. □

Corollary 2.11 *Let $p : \mathcal{F} \to \mathcal{E}$ be a bifibered category. If $\mathcal{F}_X$ has a terminal object for any object X of $\mathcal{E}$, then p has a right adjoint.*

Proof. Since $p : \mathcal{F} \to \mathcal{E}$ is a cofibered category, $p^{op} : \mathcal{F}^{op} \to \mathcal{E}^{op}$ is a fibered category. By the assumption, $\mathcal{F}_X^{op}$ has an initial object an it follows from (2.10) that p^{op} has a left adjoint $L : \mathcal{E}^{op} \to \mathcal{F}^{op}$ of p^{op}. Hence $L^{op} : \mathcal{E} \to \mathcal{F}$ is a right adjoint of p. □

Remark 2.12 *Under the assumption of the above corollary, a right adjoint $R : \mathcal{E} \to \mathcal{F}$ of p is given as follows. For an object X of $\mathcal{E}$, we denote by 1_X a terminal object of $\mathcal{F}_X$ and put $R(X) = 1_X$. For each morphism $f : X \to Y$ of $\mathcal{E}$ and an object M of $\mathcal{F}_X$, we choose a right adjoint $f_* : \mathcal{F}_X \to \mathcal{F}_Y$ of the inverse image functor $f^* : \mathcal{F}_Y \to \mathcal{F}_X$ and a cocartesian morphism $\alpha^f(M) : M \to f_*(M)$ which is mapped to f by p. We define $R(f) : 1_X \to 1_Y$ to be a composition $1_X \xrightarrow{\alpha^f(M)} f_*(1_X) \xrightarrow{o_Y} 1_Y$, where o_Y is the unique morphism in $\mathcal{F}_Y$.*

By (2.5) and (2.11), we deduce the following result.

Corollary 2.13 *$\Gamma_F : \mathscr{P}_F(\mathcal{C}, J) \to \mathcal{S}et$ has left and right adjoints.*

Remark 2.14 *A left adjoint $\mathcal{L} : \mathcal{S}et \to \mathscr{P}_F(\mathcal{C}, J)$ and the right adjoint $\mathcal{R} : \mathcal{S}et \to \mathscr{P}_F(\mathcal{C}, J)$ of Γ_F are given by $\mathcal{L}(X) = (X, \mathscr{D}_{disc,X})$, $\mathcal{L}(\varphi : X \to Y) = (\varphi : (X, \mathscr{D}_{disc,X}) \to (Y, \mathscr{D}_{disc,Y}))$ and $\mathcal{R}(X) = (X, \mathscr{D}_{coarse,X})$, $\mathcal{R}(\varphi : X \to Y) = (\varphi : (X, \mathscr{D}_{coarse,X}) \to (Y, \mathscr{D}_{coarse,Y}))$.*

Let $\{(X_i, \mathscr{D}_i)\}_{i\in I}$ be a family of objects of $\mathscr{P}_F(\mathcal{C}, J)$. We denote by $\mathrm{pr}_j : \prod_{i\in I} X_i \to X_j$ the projection to the j-th component and $\iota_j : X_j \to \coprod_{i\in I} X_i$ the inclusion to the i-th summand. Put $\mathscr{D}^I = \bigcap_{i\in I} \mathscr{D}_i^{\mathrm{pr}_i}$. Then, $\mathscr{D}^I$ is the coarsest the-ology such that $\mathrm{pr}_i : \left(\prod_{i\in I} X_i, \mathscr{D}^I\right) \to (X_i, \mathscr{D}_i)$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ for any $i \in I$.

Let $\mathscr{D}_I$ be the finest the-ology on $\coprod_{i\in I} X_i$ such that $\iota_j : (X_j, \mathscr{D}_j) \to \left(\coprod_{i\in I} X_i, \mathscr{D}_I\right)$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ for any $i \in I$. If we put $\mathcal{S}_I = \left\{\mathscr{E} \in \mathrm{Ob}\, \mathscr{P}_F(\mathcal{C}, J)_{\coprod_{i\in I} X_i} \,\middle|\, \mathscr{E} \supset \bigcup_{i\in I} (\mathscr{D}_i)_{\iota_i}\right\}$, then $\mathscr{D}_I = \bigcap_{\mathscr{E}\in\mathcal{S}_I} \mathscr{E}$. It follows (2) of (1.12) that $\mathscr{D}_I \cap F_{\coprod_{i\in I} X_i}(U)$ for $U \in \mathrm{Ob}\,\mathcal{C}$ is given as follows.

$$\left\{x \in F_{\coprod_{i\in I} X_i}(U) \,\middle|\, \text{There exists a covering } (U_j \xrightarrow{g_j} U)_{j\in J} \text{ such that } F_{\coprod_{i\in I} X_i}(g_j)(x) \in \bigcup_{i\in I} (\mathscr{D}_i)_{\iota_i} \text{ for all } j \in J.\right\}$$

Proposition 2.15 *(1) $\left(\left(\prod_{i\in I} X_i, \mathscr{D}^I\right) \xrightarrow{\mathrm{pr}_i} (X_i, \mathscr{D}_i)\right)_{i\in I}$ is a product of $\{(X_i, \mathscr{D}_i)\}_{i\in I}$.*

(2) $\left((X_i, \mathscr{D}_i) \xrightarrow{\iota_i} \left(\coprod_{i\in I} X_i, \mathscr{D}_I\right)\right)_{i\in I}$ is a coproduct of $\{(X_i, \mathscr{D}_i)\}_{i\in I}$.

Proof. (1) Let $\{\varphi_i : (Y, \mathscr{E}) \to (X_i, \mathscr{D}_i)\}_{i\in I}$ be a family of morphisms in $\mathscr{P}_F(\mathcal{C}, J)$. Let $\varphi : Y \to \prod_{i\in I} X_i$ be the unique map that satisfies $\mathrm{pr}_i\varphi = \varphi_i$ for any $i \in I$. For $U \in \mathrm{Ob}\,\mathcal{C}$, $x \in \mathscr{E} \cap F_Y(U)$ and $i \in I$, it follows that $\mathrm{pr}_i(F_\varphi)_U(x) = (F_{\mathrm{pr}_i})_U(F_\varphi)_U(x) = (F_{\varphi_i})_U(x) \in \mathscr{D}_i \cap F_{X_i}(U)$ which shows $(F_\varphi)_U(x) \in \mathscr{D}_i^{\mathrm{pr}_i}$. Thus $(F_\varphi)_U(x) \in \bigcap_{i\in I} \mathscr{D}_i^{\mathrm{pr}_i} = \mathscr{D}^I$ and $\varphi : (Y, \mathscr{E}) \to \left(\prod_{i\in I} X_i, \mathscr{D}^I\right)$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$.

(2) Let $\{\psi_i : (X_i, \mathscr{D}_i) \to (Y, \mathscr{E})\}_{i\in I}$ be a family of morphisms in $\mathscr{P}_F(\mathcal{C}, J)$. Let $\psi : \coprod_{i\in I} X_i \to Y$ be the unique map that satisfies $\psi\iota_i = \psi_i$ for any $i \in I$. We claim that $\mathscr{E}^\psi \supset \bigcup_{i\in I} (\mathscr{D}_i)_{\iota_j}$ which holds if and only if $\mathscr{E}^\psi \supset (F_{\iota_j})_U(\mathscr{D}_j \cap F_{X_j}(U))$ for any $j \in I$ and $U \in \mathrm{Ob}\,\mathcal{C}$. In fact, for $x \in \mathscr{D}_j \cap F_{X_j}(U)$, since we have $\psi(F_{\iota_j})_U(x) = (F_{\psi\iota_j})_U(x) = (F_{\psi_j})_U(x) \in \mathscr{E} \cap F_Y(U)$, $(F_{\iota_j})_U(x)$ belongs to $\mathscr{E}^\psi \cap F_{\coprod_{i\in I} X_i}(U)$. It follows that $\mathscr{E}^\psi$ contains $\mathscr{D}_I$ which implies that $\psi : \left(\coprod_{i\in I} X_i, \mathscr{D}_I\right) \to (Y, \mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. □

Definition 2.16 *We call* $\left(\prod_{i\in I} X_i, \mathscr{D}_I\right)$ *the product the-ology on* $\prod_{i\in I} X_i$ *and denote this by* $\prod_{i\in I}(X_i, \mathscr{D}_i)$*. Similarly, we call* $\left(\coprod_{i\in I} X_i, \mathscr{D}^I\right)$ *the sum the-ology on* $\coprod_{i\in I} X_i$ *and denote this by* $\coprod_{i\in I}(X_i, \mathscr{D}_i)$*.*

Remark 2.17 *Let* $(X, \mathscr{D})$ *and* $(Y, \mathscr{E})$ *be objects of* $\mathscr{P}_F(\mathcal{C}, J)$*. We denote by* $\mathrm{pr}_X : X\times Y \to X$*,* $\mathrm{pr}_Y : X\times Y \to Y$ *the projections and by* $i_y : X \times \{y\} \to X \times Y$ *the inclusion map for* $y \in Y$*. Since* $\mathrm{pr}_Y i_y : X \times \{y\} \to Y$ *is a constant map, we have* $\mathscr{E}^{\mathrm{pr}_Y i_y} = \mathscr{D}_{coarse, X\times\{y\}}$*. Hence* $(\mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y})^{i_y} = \mathscr{D}^{\mathrm{pr}_X i_y} \cap \mathscr{E}^{\mathrm{pr}_Y i_y} = \mathscr{D}^{\mathrm{pr}_X i_y}$ *holds by (2.2) and (2.3). Let* $j_y : X \to X \times \{y\}$ *be a map defined by* $j_y(x) = (x, y)$*. Then* $\mathrm{pr}_X i_y$ *is the inverse of* j_y *and* $j_y : (X, \mathscr{D}) \to (X \times \{y\}, (\mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y})^{i_y})$ *is an isomorphism in* $\mathscr{P}_F(\mathcal{C}, J)$*.*

Lemma 2.18 *Let* $f : X \to Z$*,* $g : Y \to W$ *be surjections and* $\mathscr{D}$*,* $\mathscr{E}$ *the-ologies on* X*,* Y*, respectively. We denote by* $\mathrm{pr}_X : X \times Y \to X$*,* $\mathrm{pr}_Y : X \times Y \to Y$*,* $\mathrm{pr}_Z : Z \times W \to Z$*,* $\mathrm{pr}_W : Z \times W \to W$ *the projections. Consider objects* $(Z, \mathscr{D}_f)$*,* $(W, \mathscr{E}_g)$ *of* $\mathscr{P}_F(\mathcal{C}, J)$ *and form the product* $(Z \times W, (\mathscr{D}_f)^{\mathrm{pr}_Z} \cap (\mathscr{E}_g)^{\mathrm{pr}_W})$ *in* $\mathscr{P}_F(\mathcal{C}, J)$*. Then, we have* $(\mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y})_{f\times g} = (\mathscr{D}_f)^{\mathrm{pr}_Z} \cap (\mathscr{E}_g)^{\mathrm{pr}_W}$*.*

Proof. Since $(\mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y})_{f\times g}$ is the finest the-ology on $Z \times W$ such that

$$f \times g : (X \times Y, \mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y}) \to (Z \times W, (\mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y})_{f\times g})$$

is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ and $f \times g : (X \times Y, \mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y}) \to (Z \times W, (\mathscr{D}_f)^{\mathrm{pr}_Z} \cap (\mathscr{E}_g)^{\mathrm{pr}_W})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, $(\mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y})_{f\times g}$ is contained in $(\mathscr{D}_f)^{\mathrm{pr}_Z} \cap (\mathscr{E}_g)^{\mathrm{pr}_W}$.

For $U \in \mathrm{Ob}\,\mathcal{C}$ and $\alpha \in (\mathscr{D}_f)^{\mathrm{pr}_Z} \cap (\mathscr{E}_g)^{\mathrm{pr}_W} \cap F_{Z\times W}(U)$, since $\mathrm{pr}_Z\alpha \in \mathscr{D}_F \cap F_Z(U)$ and $\mathrm{pr}_W\alpha \in \mathscr{E}_g \cap F_W(U)$, there exist $R, S \in J(U)$ such that for any $h \in R$ and $k \in S$, there exist $\beta_h \in \mathscr{D} \cap F_X(\mathrm{dom}(h))$ and $\gamma_k \in \mathscr{E} \cap F_Y(\mathrm{dom}(k))$ which satisfy $\mathrm{pr}_Z\alpha F(h) = F_Z(h)(\mathrm{pr}_Z\alpha) = f\beta_h$ and $\mathrm{pr}_W\alpha F(k) = F_W(k)(\mathrm{pr}_W\alpha) = g\gamma_k$ by (2.4). Hence, for any $h \in R \cap S$, we have the following equality.

$$F_{Z\times W}(h)(\alpha) = \alpha F(h) = (\mathrm{pr}_Z\alpha F(h), \mathrm{pr}_W\alpha F(h)) = (f\beta_h, g\gamma_h) = (f \times g)(\beta_h, \gamma_h)$$

Since $R \cap S \in J(U)$ and $(\beta_h, \gamma_h) \in \mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y}$, it follows from (2.4) we have $\alpha \in (\mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y})_{f\times g} \cap F_{Z\times W}(U)$. Thus $(\mathscr{D}_f)^{\mathrm{pr}_Z} \cap (\mathscr{E}_g)^{\mathrm{pr}_W}$ is contained in $(\mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y})_{f\times g}$. □

Proposition 2.19 *Let* $f, g : (X, \mathscr{D}) \to (Y, \mathscr{E})$ *be morphisms in* $\mathscr{P}_F(\mathcal{C}, J)$*. Then, equalizers and coequalizers of* f *and* g *exist.*

Proof. Put $Z = \{x \in X \mid f(x) = g(x)\}$ and let $i : Z \to X$ be the inclusion map. Suppose that a morphism $h : (V, \mathscr{F}) \to (X, \mathscr{D})$ in $\mathscr{P}_F(\mathcal{C}, J)$ satisfies $fh = gh$. Let $\tilde{h} : V \to Z$ be the unique map that satisfies $i\tilde{h} = h$. For $U \in \mathrm{Ob}\,\mathcal{C}$ and $\varphi \in \mathscr{F} \cap F_V(U)$, we have $i(F_{\tilde{h}})_U(\varphi) = (F_{i\tilde{h}})_U(\varphi) = (F_h)_U(\varphi) \in \mathscr{D} \cap F_X(U)$, which shows $(F_{\tilde{h}})_U(\varphi) \in \mathscr{D}^i \cap F_Z(U)$. Therefore $\tilde{h} : (V, \mathscr{F}) \to (Z, \mathscr{D}^i)$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ and $i : (Z, \mathscr{D}^i) \to (X, \mathscr{D})$ is an equalizer of f and g.

Let W be the quotient set of Y by an equivalence relation on Y generated by $f(x) \sim g(x)$ for $x \in X$. We denote by $q : Y \to W$ the quotient map. Suppose that a morphism $h : (Y, \mathscr{E}) \to (V, \mathscr{F})$ in $\mathscr{P}_F(\mathcal{C}, J)$ satisfies $hf = hg$. Let $\bar{h} : W \to V$ be the unique map that satisfies $\bar{h}q = h$. For $U \in \mathrm{Ob}\,\mathcal{C}$ and $\psi \in \mathscr{E} \cap F_Y(U)$, since $\bar{h}(F_q)_U(\psi) = (F_{\bar{h}q})_U(\psi) = (F_h)_U(\psi) \in \mathscr{F} \cap F_V(U)$ holds, we have $(F_q)_U(\psi) \in \mathscr{F}^{\bar{h}}$. Hence $\mathscr{F}^{\bar{h}}$ contains $(F_q)_U(\mathscr{E} \cap F_Y(U))$ for any $U \in \mathrm{Ob}\,\mathcal{C}$ which implies that $\mathscr{F}^{\bar{h}} \supset \mathscr{E}_q$ holds and $\bar{h} : (W, \mathscr{E}_q) \to (V, \mathscr{F})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. Thus we see that $q : (Y, \mathscr{E}) \to (W, \mathscr{E}_q)$ is a coequalizer of f and g. □

Remark 2.20 *Suppose that* X *is a set which has only one element and* $\mathscr{D}$ *is a the-ologgy on* X*. Since* $F_X(U)$ *is also a set which has only one element for any* $U \in \mathrm{Ob}\,\mathcal{C}$*, the map* $F_X(o_U) : F_X(1_{\mathcal{C}}) \to F_X(U)$ *induced by the unique morphism* $o_U : U \to 1_{\mathcal{C}}$ *surjective. Since* $F_X(1_{\mathcal{C}}) \subset \mathscr{D}$*, the condition (ii) of (1.3) implies* $F_X(U) \subset \mathscr{D}$*. Thus* $\mathscr{D} = \coprod_{U\in\mathrm{Ob}\,\mathcal{C}} F_X(U)$ *holds, namely* $\mathscr{D}_{coarse,\{1\}}$ *is the only the-ology on* $\{1\}$*. We also remark that* $(\{1\}, \mathscr{D}_{coarse,\{1\}})$ *is a terminal object of* $\mathscr{P}_F(\mathcal{C}, J)$*.*

Proposition 2.21 *Let* $f : (X, \mathscr{D}) \to (Y, \mathscr{E})$ *and* $g : (Z, \mathscr{F}) \to (Y, \mathscr{E})$ *be morphisms in* $\mathscr{P}_F(\mathcal{C}, J)$*. We consider the following cartesian square in* $\mathscr{S}et$*.*

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\tilde{g}} & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

Then, $(Z,\mathscr{F}) \xleftarrow{\tilde{f}} (X\times_Y Z, \mathscr{D}^{\tilde{g}}\cap\mathscr{E}^{\tilde{f}}) \xrightarrow{\tilde{g}} (X,\mathscr{D})$ *is a limit of a diagram* $(X,\mathscr{D}) \xrightarrow{f} (Y,\mathscr{E}) \xleftarrow{g} (Z,\mathscr{F})$ *in* $\mathscr{P}_F(\mathcal{C},J)$.

Proof. We denote by $\mathrm{pr}_X : X\times Z\to X$ and $\mathrm{pr}_Z : X\times Z\to Z$ the projections. Let $j : X\times_Y Z\to X\times Z$ be the inclusion map. Then, j is an equalizer of maps $f\mathrm{pr}_X, g\mathrm{pr}_Z : X\times Z\to Y$ in $\mathcal{S}et$. It follows from (2.19) that

$$j : (X\times_Y Z, (\mathscr{D}^{\mathrm{pr}_X}\cap\mathscr{F}^{\mathrm{pr}_Z})^j) \to (X\times Z, \mathscr{D}^{\mathrm{pr}_X}\cap\mathscr{F}^{\mathrm{pr}_Z})$$

is an equalizer of morphisms $f\mathrm{pr}_X, g\mathrm{pr}_Z : (X\times Z, \mathscr{D}^{\mathrm{pr}_X}\cap\mathscr{F}^{\mathrm{pr}_Z}) \to (Y,\mathscr{E})$ in $\mathscr{P}_F(\mathcal{C},J)$. Now the assertion follows from an equality $(\mathscr{D}^{\mathrm{pr}_X}\cap\mathscr{F}^{\mathrm{pr}_Z})^j = (\mathscr{D}^{\mathrm{pr}_X})^j\cap(\mathscr{F}^{\mathrm{pr}_Z})^j = \mathscr{D}^{\mathrm{pr}_X j}\cap\mathscr{F}^{\mathrm{pr}_Z j} = \mathscr{D}^{\tilde{g}}\cap\mathscr{E}^{\tilde{f}}$ obtained from (2.2) and (2.3). □

For objects $(X,\mathscr{D}), (Y,\mathscr{E})$ of $\mathscr{P}_F(\mathcal{C},J)$, we define a map $\mathrm{ev} : X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))\to Y$ by $\mathrm{ev}(x,f) = f(x)$ and also define a set $\Sigma_{\mathscr{D},\mathscr{E}}$ of the-ologies on $\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))$ by

$$\Sigma_{\mathscr{D},\mathscr{E}} = \{\mathscr{F}\in\mathscr{P}_F(\mathcal{C},J)_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}\,|\,\mathscr{E}^{\mathrm{ev}}\supset\mathscr{D}^{\mathrm{pr}_1}\cap\mathscr{F}^{\mathrm{pr}_2}\}.$$

Here $\mathrm{pr}_1 : X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))\to X$ and $\mathrm{pr}_2 : X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))\to\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))$ are the projections. Then, $\Sigma_{\mathscr{D},\mathscr{E}}$ is the set of the-ology $\mathscr{F}$ on $\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))$ such that

$$\mathrm{ev} : (X,\mathscr{D})\times(\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E})),\mathscr{F})\to(Y,\mathscr{E})$$

is a morphism in $\mathscr{P}_F(\mathcal{C},J)$.

Lemma 2.22 $\Sigma_{\mathscr{D},\mathscr{E}}$ *is not empty.*

Proof. It suffices to show that the discrete the-ology $\mathscr{D}_{disc,\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}$ on $\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))$ belongs to $\Sigma_{\mathscr{D},\mathscr{E}}$. For $U\in\mathrm{Ob}\,\mathcal{C}$ and $f\in\mathscr{D}_{disc,\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}\cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$, there exists a covering $(U_i\xrightarrow{g_i}U)_{i\in I}$ such that $F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(g_i)(f)$ is a constant map for every $i\in I$ by (1.15). We also take $x\in\mathscr{D}\cap F_X(U)$. Then, $(x,f) : F(U)\to X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))$ is regarded as an element of $F_{X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$ which is mapped by

$$F_{X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(g_i) : F_{X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)\to F_{X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U_i)$$

to a map $(F_X(g_i)(x), F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(g_i)(f)) = (xF(g_i), fF(g_i)) : F(U_i)\to X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))$. It follows from the commutativity of a diagram

$$\begin{array}{ccc}
F_{X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U) & \xrightarrow{(F_{\mathrm{ev}})_U} & F_Y(U) \\
\big\downarrow{\scriptstyle F_{X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(g_i)} & & \big\downarrow{\scriptstyle F_Y(g_i)} \\
F_{X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U_i) & \xrightarrow{(F_{\mathrm{ev}})_{U_i}} & F_Y(U_i)
\end{array}$$

that $F_Y(g_i)(F_{\mathrm{ev}})_U$ maps (x,f) to $(F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(g_i)(f))(F_X(g_i)(x)) = (fF(g_i))(xF(g_i))\in F_Y(U_i)$. By the assumption on $(U_i\xrightarrow{g_i}U)_{i\in I}$, $F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(g_i)(f) = fF(g_i) : F(U_i)\to\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))$ is a constant map. Hence if we denote the image of this map by c, $(F_c)_{U_i}$ maps $\mathscr{D}\cap F_X(U_i)$ to $\mathscr{E}\cap F_Y(U_i)$ and we have $(F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(g_i)(f))(F_X(g_i)(x)) = c(xF(g_i))\in\mathscr{E}\cap F_Y(U_i)$ since $xF(g_i)\in\mathscr{D}\cap F_X(U_i)$. Therefore $F_Y(g_i)(F_{\mathrm{ev}})_U(x,f)\in\mathscr{E}\cap F_Y(U_i)$ for any $i\in I$, which shows $(F_{\mathrm{ev}})_U(x,f)$ belongs to $\mathscr{E}\cap F_Y(U)$. Thus $\mathrm{ev} : (X,\mathscr{D})\times(\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E})),\mathscr{D}_{disc,\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))})\to(Y,\mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$. □

For $U\in\mathrm{Ob}\,\mathcal{C}$, we consider the following condition (E) on an element φ of $F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$.

(E) For any $V,W\in\mathrm{Ob}\,\mathcal{C}$, $f\in\mathcal{C}(W,U)$, $g\in\mathcal{C}(W,V)$ and $\psi\in\mathscr{D}\cap F_X(V)$, the following composition belongs to $\mathscr{E}\cap F_Y(W)$.

$$F(W)\xrightarrow{(F(g),F(f))}F(V)\times F(U)\xrightarrow{\psi\times\varphi}X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))\xrightarrow{\mathrm{ev}}Y$$

Define a set $\mathscr{E}^{\mathscr{D}}$ of F-parametrizations of a set $\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))$ so that $\mathscr{E}^{\mathscr{D}}\cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$ is a subset of $F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$ consisting of elements which satisfy the above condition (E).

Proposition 2.23 $\mathscr{E}^{\mathscr{D}}$ *is a the-ologgy on* $\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))$.

Proof. For $\varphi\in F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(1_{\mathcal{C}})$, $V,W\in\mathrm{Ob}\,\mathcal{C}$, $g\in\mathcal{C}(W,V)$ and $\psi\in\mathscr{D}\cap F_X(V)$, a composition

$$F(W)\xrightarrow{(F(g),F(o_W))}F(V)\times F(1_{\mathcal{C}})\xrightarrow{\psi\times\varphi}X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))\xrightarrow{\mathrm{ev}}Y$$

coincides with $(F_{\varphi(*)})_W(F_X(g)(\psi))$. Here $o_W : W \to 1_{\mathcal{C}}$ denotes the unique morphism and $*$ is unique element of $F(1_{\mathcal{C}})$. Since $(F_{\varphi(*)})_W : F_X(W) \to F_Y(W)$ maps $\mathscr{D} \cap F_X(W)$ to $\mathscr{E} \cap F_Y(W)$ and $F_X(g)(\psi)$ belongs to $\mathscr{D} \cap F_X(W)$, $(F_{\varphi(*)})_W(F_X(g)(\psi))$ is an element of $\mathscr{E} \cap F_Y(W)$. Hence $\mathscr{E}^{\mathscr{D}}$ contains $F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(1_{\mathcal{C}})$.

Let $j : Z \to U$ be a morphism in $\mathcal{C}$. For $\varphi \in \mathscr{E}^{\mathscr{D}} \cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$, $V, W \in \operatorname{Ob}\mathcal{C}$, $f \in \mathcal{C}(W, Z)$, $g \in \mathcal{C}(W, V)$ and $\psi \in \mathscr{D} \cap F_X(V)$, since a composition

$$F(W) \xrightarrow{(F(g),F(f))} F(V) \times F(Z) \xrightarrow{\psi \times F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(j)(\varphi)} X \times \mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Y, \mathscr{E})) \xrightarrow{\text{ev}} Y$$

coincides with $F(W) \xrightarrow{(F(g),F(jf))} F(V) \times F(U) \xrightarrow{\psi\times\varphi} X \times \mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Y, \mathscr{E})) \xrightarrow{\text{ev}} Y$ and the latter composition belongs to $\mathscr{E} \cap F_Y(W)$ which shows $F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(j)(\varphi) \in \mathscr{E}^{\mathscr{D}} \cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(Z)$.

Assume that, for $\varphi \in F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$, there exists $R \in J(U)$ such that $F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(j)(\varphi)$ belongs to $\mathscr{E}^{\mathscr{D}} \cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(\operatorname{dom}(j))$ for any $j \in R$. We take $V, W \in \operatorname{Ob}\mathcal{C}$, $f \in \mathcal{C}(W, U)$, $g \in \mathcal{C}(W, V)$ and $\psi \in \mathscr{D} \cap F_X(V)$ and put $h_f^{-1}(R) = \{i \in \operatorname{Mor}\mathcal{C} \mid \operatorname{codom}(i) = W,\ fi \in R\}$. Then, $h_f^{-1}(R) \in J(W)$. For any $i \in h_f^{-1}(R)$, a composition

$$F(\operatorname{dom}(i)) \xrightarrow{F(i)} F(W) \xrightarrow{(F(g),F(f))} F(V) \times F(U) \xrightarrow{\psi\times\varphi} X \times \mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Y, \mathscr{E})) \xrightarrow{\text{ev}} Y$$

coincides with a composition

$$F(\operatorname{dom}(i)) \xrightarrow{(F(gi),F(id_{\operatorname{dom}(i)}))} F(V) \times F(\operatorname{dom}(i)) \xrightarrow{\psi\times F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(fi)(\varphi)} X \times \mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Y, \mathscr{E})) \xrightarrow{\text{ev}} Y$$

which belongs to $\mathscr{E} \cap F_Y(\operatorname{dom}(i))$ since $F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(fi)(\varphi) \in \mathscr{E}^{\mathscr{D}} \cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(\operatorname{dom}(fi))$. Hence we have $F_Y(i)(\text{ev}(\psi \times \varphi)(F(g), F(f))) \in \mathscr{E} \cap F_Y(\operatorname{dom}(i))$ for any $i \in h_f^{-1}(R)$ and this shows that $\text{ev}(\psi \times \varphi)(F(g), F(f))$ belongs to $\mathscr{E} \cap F_Y(W)$. Hence $\varphi \in \mathscr{E}^{\mathscr{D}} \cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$ follows from the definition of $\mathscr{E}^{\mathscr{D}}$. □

We denote by $(Y, \mathscr{E})^{(X,\mathscr{D})}$ an object $(\mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Y, \mathscr{E})), \mathscr{E}^{\mathscr{D}})$ of $\mathscr{P}_F(\mathcal{C}, J)$.

Proposition 2.24 *$\mathscr{E}^{\mathscr{D}}$ is maximum element of $\Sigma_{\mathscr{D},\mathscr{E}}$.*

Proof. For $U \in \operatorname{Ob}\mathcal{C}$ and $\xi \in \mathscr{D}^{\text{pr}_1} \cap (\mathscr{E}^{\mathscr{D}})^{\text{pr}_2} \cap F_{X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$, it follows from $\text{pr}_1\xi \in \mathscr{D} \cap F_X(U)$ and $\text{pr}_2\xi \in \mathscr{E}^{\mathscr{D}} \cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$ that the following composition belongs to $\mathscr{E} \cap F_Y(U)$.

$$F(U) \xrightarrow{(F(id_U),F(id_U))} F(U) \times F(U) \xrightarrow{\text{pr}_1\xi\times\text{pr}_2\xi} X \times \mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Y, \mathscr{E})) \xrightarrow{\text{ev}} Y$$

Since this composition coincides with $\text{ev}\xi$, we see that $\xi \in \mathscr{E}^{\text{ev}}$ holds. Hence we have $\mathscr{E}^{\text{ev}} \supset \mathscr{D}^{\text{pr}_1} \cap (\mathscr{E}^{\mathscr{D}})^{\text{pr}_2}$ and $\mathscr{E}^{\mathscr{D}}$ is an element of $\Sigma_{\mathscr{D},\mathscr{E}}$.

For $\mathscr{F} \in \Sigma_{\mathscr{D},\mathscr{E}}$ and $W \in \operatorname{Ob}\mathcal{C}$, since $\text{ev} : (X, \mathscr{D}) \times (\mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Y, \mathscr{E})), \mathscr{F}) \to (Y, \mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, $(F_{\text{ev}})_W : F_{X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(W) \to F_Y(W)$ maps $\mathscr{D}^{\text{pr}_1} \cap \mathscr{F}^{\text{pr}_2} \cap F_{X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(W)$ into $\mathscr{E} \cap F_Y(W)$. For $\varphi \in \mathscr{F} \cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$, we take $V, W \in \operatorname{Ob}\mathcal{C}$, $f \in \mathcal{C}(W, U)$, $g \in \mathcal{C}(W, V)$ and $\psi \in \mathscr{D} \cap F_X(V)$. Then, we have $\varphi F(f) = F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(f)(\varphi) \in \mathscr{F} \cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(W)$ and $\psi F(g) = F_X(g)(\psi) \in \mathscr{D} \cap F_X(W)$ which implies $(\psi F(g), \varphi F(f)) \in \mathscr{D}^{\text{pr}_1} \cap \mathscr{F}^{\text{pr}_2} \cap F_{X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(W)$. It follows that a composition $F(W) \xrightarrow{(F(g),F(f))} F(V) \times F(U) \xrightarrow{\psi\times\varphi} X \times \mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Y, \mathscr{E})) \xrightarrow{\text{ev}} Y$ belongs to $\mathscr{E} \cap F_Y(W)$. Therefore $\varphi \in \mathscr{E}^{\mathscr{D}}$ holds and this shows $\mathscr{F} \subset \mathscr{E}^{\mathscr{D}}$. Thus $\mathscr{E}^{\mathscr{D}}$ is maximum element of $\Sigma_{\mathscr{D},\mathscr{E}}$. □

Lemma 2.25 *Let $(X, \mathscr{D})$ be an object of $\mathscr{P}_F(\mathcal{C}, J)$ and $\xi : (Y, \mathscr{E}) \to (Z, \mathscr{F})$ a morphism in $\mathscr{P}_F(\mathcal{C}, J)$.*

(1) $id_X \times \xi : X \times Y \to X \times Z$ defines a morphism $id_X \times \xi : (X, \mathscr{D}) \times (Y, \mathscr{E}) \to (X, \mathscr{D}) \times (Z, \mathscr{F})$ in $\mathscr{P}_F(\mathcal{C}, J)$.

(2) A map $\xi_* : \mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Y, \mathscr{E})) \to \mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Z, \mathscr{F}))$ defined by $\xi_*(\alpha) = \xi\alpha$ defines a morphism $\xi_* : (\mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Y, \mathscr{E})), \mathscr{E}^{\mathscr{D}}) \to (\mathscr{P}_F(\mathcal{C}, J)((X, \mathscr{D}), (Z, \mathscr{F})), \mathscr{F}^{\mathscr{D}})$ in $\mathscr{P}_F(\mathcal{C}, J)$.

(3) A map $\xi^* : \mathscr{P}_F(\mathcal{C}, J)((Z, \mathscr{F}), (X, \mathscr{D})) \to \mathscr{P}_F(\mathcal{C}, J)((Y, \mathscr{E}), (X, \mathscr{D}))$ defined by $\xi^*(\alpha) = \alpha\xi$ defines a morphism $\xi^* : (\mathscr{P}_F(\mathcal{C}, J)((Z, \mathscr{F}), (X, \mathscr{D})), \mathscr{D}^{\mathscr{F}}) \to (\mathscr{P}_F(\mathcal{C}, J)((Y, \mathscr{E}), (X, \mathscr{D})), \mathscr{D}^{\mathscr{E}})$ in $\mathscr{P}_F(\mathcal{C}, J)$.

Proof. (1) We denote by $\text{pr}'_X : X \times Z \to X$ and $\text{pr}'_Z : X \times Z \to Z$ the projections. Since $\text{pr}'_X(id_X \times \xi) = \text{pr}_X$ and $\text{pr}'_Z(id_X \times \xi) = \xi\text{pr}_Y$, the following equalities hold for $U \in \operatorname{Ob}\mathcal{C}$ and $\varphi \in \mathscr{D}^{\text{pr}_X} \cap \mathscr{E}^{\text{pr}_Y} \cap F_{X\times Y}(U)$.

$$(F_{\text{pr}'_X})_U(F_{id_X\times\xi})_U(\varphi) = (F_{\text{pr}_X})_U(\varphi) \in \mathscr{D} \cap F_X(U), \quad (F_{\text{pr}'_Z})_U(F_{id_X\times\xi})_U(\varphi) = (F_\xi)_U(F_{\text{pr}_Y})_U(\varphi) \in \mathscr{F} \cap F_Z(U)$$

Hence $(F_{id_X\times\xi})_U : F_{X\times Y}(U) \to F_{X\times Z}(U)$ maps $\mathscr{D}^{\text{pr}_X} \cap \mathscr{E}^{\text{pr}_Y} \cap F_{X\times Y}(U)$ into $\mathscr{D}^{\text{pr}'_X} \cap \mathscr{F}^{\text{pr}'_Z} \cap F_{X\times Z}(U)$. Thus $id_X \times \xi : (X, \mathscr{D}) \times (Y, \mathscr{E}) = (X \times Y, \mathscr{D}^{\text{pr}_X} \cap \mathscr{E}^{\text{pr}_Y}) \to (X \times Z, \mathscr{D}^{\text{pr}'_X} \cap \mathscr{F}^{\text{pr}'_Z}) = (X, \mathscr{D}) \times (Z, \mathscr{F})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$.

(2) For $U \in \operatorname{Ob}\mathcal{C}$ and $\varphi \in \mathscr{E}^{\mathscr{D}} \cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$, we take $V, W \in \operatorname{Ob}\mathcal{C}$, $f \in \mathcal{C}(W,U)$, $g \in \mathcal{C}(W,V)$ and $\psi \in \mathscr{D} \cap F_X(V)$. Since a composition $F(W) \xrightarrow{(F(g),F(f))} F(V) \times F(U) \xrightarrow{\psi\times\varphi} X \times \mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E})) \xrightarrow{\mathrm{ev}} Y$ belongs to $\mathscr{E} \cap F_Y(W)$, and ξ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$, the composition of the upper row of the following diagram belongs to $\mathscr{F} \cap F_Z(W)$ by the commutativity of the diagram.

$$\begin{array}{ccccccc} F(W) & \xrightarrow{(F(g),F(f))} & F(V)\times F(U) & \xrightarrow{\psi\times(F_{\xi_*})_U(\varphi)} & X \times \mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Z,\mathscr{F})) & \xrightarrow{\mathrm{ev}} & Z \\ & & \big\downarrow{\scriptstyle \psi\times\varphi} & & & & \uparrow{\scriptstyle \xi} \\ & & X \times \mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E})) & & \xrightarrow{\mathrm{ev}} & & Y \end{array}$$

Hence $(F_{\xi_*})_U : F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U) \to F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Z,\mathscr{E}))}(U)$ maps $\mathscr{E}^{\mathscr{D}} \cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))}(U)$ into $\mathscr{F}^{\mathscr{D}} \cap F_{\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Z,\mathscr{E}))}(U)$. Thus $\xi_* : (\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E})),\mathscr{E}^{\mathscr{D}}) \to (\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Z,\mathscr{F})),\mathscr{F}^{\mathscr{D}})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$.

(3) For $U \in \operatorname{Ob}\mathcal{C}$ and $\varphi \in \mathscr{D}^{\mathscr{F}} \cap F_{\mathscr{P}_F(\mathcal{C},J)((Z,\mathscr{F}),(X,\mathscr{D}))}(U)$, we take $V, W \in \operatorname{Ob}\mathcal{C}$, $f \in \mathcal{C}(W,U)$, $g \in \mathcal{C}(W,V)$ and $\psi \in \mathscr{E} \cap F_Y(V)$. Since ξ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$, we have $(F_\xi)_V(\psi) \in \mathscr{F} \cap F_Z(V)$ and this implies that a composition $F(W) \xrightarrow{(F(g),F(f))} F(V) \times F(U) \xrightarrow{(F_\xi)_V(\psi)\times\varphi} Z \times \mathscr{P}_F(\mathcal{C},J)((Z,\mathscr{F}),(X,\mathscr{D})) \xrightarrow{\mathrm{ev}} X$ belongs to $\mathscr{D} \cap F_X(W)$. Thus the composition of the upper row of the following diagram belongs to $\mathscr{D} \cap F_X(W)$ by the commutativity of the diagram.

$$\begin{array}{ccccccc} F(W) & \xrightarrow{(F(g),F(f))} & F(V)\times F(U) & \xrightarrow{\psi\times(F_{\xi^*})_U(\varphi)} & Y \times \mathscr{P}_F(\mathcal{C},J)((Y,\mathscr{E}),(X,\mathscr{D})) & \xrightarrow{\mathrm{ev}} & X \\ & & & \searrow{\scriptstyle (F_\xi)_V(\psi)\times\varphi} & & \nearrow{\scriptstyle \mathrm{ev}} & \\ & & & & Y \times \mathscr{P}_F(\mathcal{C},J)((Z,\mathscr{F}),(X,\mathscr{D})) & & \end{array}$$

Hence $(F_{\xi^*})_U : F_{\mathscr{P}_F(\mathcal{C},J)((Z,\mathscr{F}),(X,\mathscr{D}))}(U) \to F_{\mathscr{P}_F(\mathcal{C},J)((Y,\mathscr{E}),(X,\mathscr{D}))}(U)$ maps $\mathscr{D}^{\mathscr{F}} \cap F_{\mathscr{P}_F(\mathcal{C},J)((Z,\mathscr{F}),(X,\mathscr{D}))}(U)$ into $\mathscr{D}^{\mathscr{E}} \cap F_{\mathscr{P}_F(\mathcal{C},J)((Y,\mathscr{E}),(X,\mathscr{D}))}(U)$. Thus $\xi^* : (\mathscr{P}_F(\mathcal{C},J)((Z,\mathscr{F}),(X,\mathscr{D})),\mathscr{D}^{\mathscr{F}}) \to (\mathscr{P}_F(\mathcal{C},J)((Y,\mathscr{E}),(X,\mathscr{D})),\mathscr{D}^{\mathscr{E}})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$. □

For objects $(X,\mathscr{D})$, $(Y,\mathscr{E})$ of $\mathscr{P}_F(\mathcal{C},J)$ and $y \in Y$, we define a map $\iota_y : X \to X \times Y$ by $\iota_y(x) = (x,y)$. We denote by $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ the projections. Since $\mathrm{pr}_X\iota_y$ is the identity map of X and $\mathrm{pr}_Y\iota_y$ is the constant map whose image is $\{y\}$, $(F_{\mathrm{pr}_X})_U(F_{\iota_y})_U : F_X(U) \to F_X(U)$ maps $\mathscr{D} \cap F_X(U)$ to $\mathscr{D} \cap F_X(U)$ and $(F_{\mathrm{pr}_Y})_U(F_{\iota_y})_U : F_X(U) \to F_Y(U)$ maps $\mathscr{D} \cap F_X(U)$ to $\mathscr{E} \cap F_Y(U)$ for any $U \in \operatorname{Ob}\mathcal{C}$. Therefore $(F_{\iota_y})_U : F_X(U) \to F_{X\times Y}(U)$ maps $\mathscr{D} \cap F_X(U)$ to $\mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y} \cap F_{X\times Y}(U)$, that is, ι_y belongs to $\mathscr{P}_F(\mathcal{C},J)((Y,\mathscr{E}),(X\times Y,\mathscr{D}^{\mathrm{pr}_X}\cap\mathscr{E}^{\mathrm{pr}_Y})$. Thus a map $\eta : Y \to \mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(X\times Y,\mathscr{D}^{\mathrm{pr}_X}\cap\mathscr{E}^{\mathrm{pr}_Y}))$ is defined by $\eta(y) = \iota_y$.

Lemma 2.26 *The map $\eta : Y \to \mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(X\times Y,\mathscr{D}^{\mathrm{pr}_X}\cap\mathscr{E}^{\mathrm{pr}_Y})$ defined above defines a morphism $\eta : (Y,\mathscr{E}) \to (X\times Y,\mathscr{D}^{\mathrm{pr}_X}\cap\mathscr{E}^{\mathrm{pr}_Y})^{(X,\mathscr{D})} = ((X,\mathscr{D})\times(Y,\mathscr{E}))^{(X,\mathscr{D})}$ in $\mathscr{P}_F(\mathcal{C},J)$.*

Proof. It suffices to verify that $(F_\eta)_U(\varphi) \in (\mathscr{D}^{\mathrm{pr}_X}\cap\mathscr{E}^{\mathrm{pr}_Y})^{\mathscr{D}}$ holds for any $U \in \operatorname{Ob}\mathcal{C}$ and $\varphi \in \mathscr{E} \cap F_Y(U)$. We take $V, W \in \operatorname{Ob}\mathcal{C}$, $f \in \mathcal{C}(W,U)$, $g \in \mathcal{C}(W,V)$ and $\psi \in \mathscr{D} \cap F_X(V)$. The image of $u \in F(W)$ by the following composition is $\mathrm{ev}(\psi(gu),\iota_{\varphi(fu)}) = (\psi(gu),\varphi(fu)) = (F_X(g)(\psi)(u),F_Y(f)(\varphi)(u))$.

$$F(W) \xrightarrow{(F(g),F(f))} F(V)\times F(U) \xrightarrow{\psi\times(F_\eta)_U(\varphi)} X\times\mathscr{P}_F(\mathcal{C},J)((Y,\mathscr{E}),(X\times Y,\mathscr{D}^{\mathrm{pr}_X}\cap\mathscr{E}^{\mathrm{pr}_Y})) \xrightarrow{\mathrm{ev}} X\times Y$$

Hence the following diagram is commutative.

$$\begin{array}{ccccccccc} X & & \longleftarrow & & \longleftarrow & \xleftarrow{\mathrm{pr}_X} & & & \\ \big\uparrow{\scriptstyle F_X(f)(\varphi)} & & & & & & & & \\ F(W) & \xrightarrow{(F(g),F(f))} & F(V)\times F(U) & \xrightarrow{\psi\times(F_\eta)_U(\varphi)} & X\times\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(X\times Y,\mathscr{D}^{\mathrm{pr}_X}\cap\mathscr{E}^{\mathrm{pr}_Y})) & \xrightarrow{\mathrm{ev}} & X\times Y \\ \big\downarrow{\scriptstyle F_Y(g)(\psi)} & & & & & & & & \\ Y & & \longleftarrow & & \longleftarrow & \xleftarrow{\mathrm{pr}_Y} & & & \end{array}$$

Since $F_X(f)(\varphi) \in \mathscr{D} \cap F_X(W)$ and $F_Y(g)(\psi) \in \mathscr{E} \cap F_Y(W)$, the composition of the middle row of the above map belongs to $\mathscr{D}^{\mathrm{pr}_X}\cap\mathscr{E}^{\mathrm{pr}_Y}\cap F_{X\times Y}(W)$. □

For an object $(X,\mathscr{D})$, we define functors $P_{(X,\mathscr{D})}, E_{(X,\mathscr{D})} : \mathscr{P}_F(\mathcal{C},J) \to \mathscr{P}_F(\mathcal{C},J)$ as follows. We put

$$P_{(X,\mathscr{D})}(Y,\mathscr{E}) = (X,\mathscr{D}) \times (Y,\mathscr{E}) = (X \times Y, \mathscr{D}^{\mathrm{pr}_X} \cap \mathscr{E}^{\mathrm{pr}_Y}) \quad P_{(X,\mathscr{D})}(\xi) = id_X \times \xi$$
$$E_{(X,\mathscr{D})}(Y,\mathscr{E}) = (Y,\mathscr{E})^{(X,\mathscr{D})} = (\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E})), \mathscr{E}^{\mathscr{D}}) \quad E_{(X,\mathscr{D})}(\xi) = \xi_*$$

for an object $(Y,\mathscr{E})$ of $\mathscr{P}_F(\mathcal{C},J)$ and a morphism $\xi : (Y,\mathscr{E}) \to (Z,\mathscr{F})$ in $\mathscr{P}_F(\mathcal{C},J)$. Then, the following maps define natural transformations $\mathrm{ev}_{(X,\mathscr{D})} : P_{(X,\mathscr{D})}E_{(X,\mathscr{D})} \to id_{\mathscr{P}_F(\mathcal{C},J)}$ and $\eta_{(X,\mathscr{D})} : id_{\mathscr{P}_F(\mathcal{C},J)} \to E_{(X,\mathscr{D})}P_{(X,\mathscr{D})}$.

$$\mathrm{ev} = (\mathrm{ev}_{(X,\mathscr{D})})_{(Y,\mathscr{E})} : P_{(X,\mathscr{D})}E_{(X,\mathscr{D})}(Y,\mathscr{E}) = (X,\mathscr{D}) \times (Y,\mathscr{E})^{(X,\mathscr{D})} \to (Y,\mathscr{E})$$
$$\eta = (\eta_{(X,\mathscr{D})})_{(Y,\mathscr{E})} : (Y,\mathscr{E}) \to ((X,\mathscr{D}) \times (Y,\mathscr{E}))^{(X,\mathscr{D})} = E_{(X,\mathscr{D})}P_{(X,\mathscr{D})}(Y,\mathscr{E})$$

Proposition 2.27 *$\mathscr{P}_F(\mathcal{C},J)$ is cartesian closed.*

Proof. Let $(X,\mathscr{D})$ and $(Y,\mathscr{E})$ be objects of $\mathscr{P}_F(\mathcal{C},J)$. It is easy to verify that the following composition is the identity map of $X \times Y$.

$$P_{(X,\mathscr{D})}(Y,\mathscr{E}) \xrightarrow{P_{(X,\mathscr{D})}((\eta_{(X,\mathscr{D})})_{(Y,\mathscr{E})})} P_{(X,\mathscr{D})}E_{(X,\mathscr{D})}P_{(X,\mathscr{D})}(Y,\mathscr{E}) \xrightarrow{(\mathrm{ev}_{(X,\mathscr{D})})_{P_{(X,\mathscr{D})}(Y,\mathscr{E})}} P_{(X,\mathscr{D})}(Y,\mathscr{E})$$

Let $\mathrm{pr}_1 : X \times \mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E})) \to X$ and $\mathrm{pr}_2 : X \times \mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E})) \to \mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E}))$ be the projections. Then, the underlying set of $E_{(X,\mathscr{D})}P_{(X,\mathscr{D})}E_{(X,\mathscr{D})}(Y,\mathscr{E})$ is

$$\mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(X \times \mathscr{P}_F(\mathcal{C},J)((X,\mathscr{D}),(Y,\mathscr{E})), \mathscr{D}^{\mathrm{pr}_1} \cap (\mathscr{E}^{\mathscr{D}})^{\mathrm{pr}_2}).$$

For $\varphi \in E_{(X,\mathscr{D})}(Y,\mathscr{E})$, since $(\mathrm{ev}_{(X,\mathscr{D})})_{(Y,\mathscr{E})}\iota_\varphi : X \to Y$ maps $x \in X$ to $\varphi(x)$, we have $(\mathrm{ev}_{(X,\mathscr{D})})_{(Y,\mathscr{E})}\iota_\varphi = \varphi$, which implies that the following composition is the identity map of $E_{(X,\mathscr{D})}(Y,\mathscr{E})$.

$$E_{(X,\mathscr{D})}(Y,\mathscr{E}) \xrightarrow{(\eta_{(X,\mathscr{D})})_{E_{(X,\mathscr{D})}(Y,\mathscr{E})}} E_{(X,\mathscr{D})}P_{(X,\mathscr{D})}E_{(X,\mathscr{D})}(Y,\mathscr{E}) \xrightarrow{E_{(X,\mathscr{D})}((\mathrm{ev}_{(X,\mathscr{D})})_{(Y,\mathscr{E})})} E_{(X,\mathscr{D})}(Y,\mathscr{E})$$

Therefore, $E_{(X,\mathscr{D})}$ is a right adjoint of $P_{(X,\mathscr{D})}$ with unit $\eta_{(X,\mathscr{D})}$ and counit $\mathrm{ev}_{(X,\mathscr{D})}$. □

3 Locally cartesian closedness

For a category $\mathcal{E}$, let $\mathcal{E}^{(2)}$ be the category of morphisms in $\mathcal{E}$ defined as follows. Put $\mathrm{Ob}\,\mathcal{E}^{(2)} = \mathrm{Mor}\,\mathcal{E}$ and a morphism from $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ to $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ is a pair $\langle \xi : E \to F, f : X \to Y \rangle$ of morphisms in $\mathcal{E}$ which satisfies $\rho\xi = f\pi$. The composition of morphisms $\langle \xi, f \rangle : \boldsymbol{E} \to \boldsymbol{F}$ and $\langle \zeta, g \rangle : \boldsymbol{F} \to \boldsymbol{G}$ is defined to be $\langle \zeta\xi, gf \rangle : \boldsymbol{E} \to \boldsymbol{G}$. We define a functor $\wp_{\mathcal{E}} : \mathcal{E}^{(2)} \to \mathcal{E}$ by $\wp_{\mathcal{E}}(E \xrightarrow{\pi} X) = X$ and $\wp_{\mathcal{E}}(\langle \xi, f \rangle) = f$. For an object X of $\mathcal{E}$, we denote by $\mathcal{E}^{(2)}_X$ a subcategory of $\mathcal{E}^{(2)}$ given as follows. We mention that $\mathcal{E}^{(2)}_X$ is often denoted by $\mathcal{E}/X$ in literatures.

$$\mathrm{Ob}\,\mathcal{E}^{(2)}_X = \{\boldsymbol{E} \in \mathrm{Ob}\,\mathcal{E}^{(2)} \,|\, \wp_{\mathcal{E}}(\boldsymbol{E}) = X\}, \qquad \mathrm{Mor}\,\mathcal{E}^{(2)}_X = \{\boldsymbol{\xi} \in \mathrm{Mor}\,\mathcal{E}^{(2)} \,|\, \wp_{\mathcal{E}}(\boldsymbol{\xi}) = id_X\}$$

For a morphism $f : X \to Y$ in $\mathcal{E}$, an object $\boldsymbol{E}$ of $\mathcal{E}^{(2)}_X$ and an object $\boldsymbol{F}$ of $\mathcal{E}^{(2)}_Y$, we denote by $\mathcal{E}^{(2)}_f(\boldsymbol{E},\boldsymbol{F})$ the set of all morphisms $\boldsymbol{\xi} : \boldsymbol{E} \to \boldsymbol{F}$ in $\mathcal{E}^{(2)}$ such that $\wp_{\mathcal{E}}(\boldsymbol{\xi}) = f$.

If $\mathcal{E}$ has finite limits, $\wp_{\mathcal{E}} : \mathcal{E}^{(2)} \to \mathcal{E}$ is a fibered category as we explain below. For a morphism $f : X \to Y$ in $\mathcal{E}$ and an object $\boldsymbol{F} = (F \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}_Y$, consider the following cartesian square in $\mathcal{E}$.

$$\begin{array}{ccc} F \times_Y X & \xrightarrow{f_\rho} & F \\ \downarrow{\scriptstyle \rho_f} & & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{f} & Y \end{array}$$

We put $f^*(\boldsymbol{F}) = (F \times_Y X \xrightarrow{\rho_f} X)$ and $\boldsymbol{\alpha}_f(\boldsymbol{F}) = \langle f_\rho, f \rangle : f^*(\boldsymbol{F}) \to \boldsymbol{F}$. The following result is straightforward from the definition of cartesian square.

Proposition 3.1 *$\boldsymbol{\alpha}_f(\boldsymbol{F})$ is a cartesian morphism, that is, for any object $\boldsymbol{G}$ of $\mathcal{E}^{(2)}_X$ the map*

$$\boldsymbol{\alpha}_f(\boldsymbol{F})_* : \mathcal{E}^{(2)}_X(\boldsymbol{G}, f^*(\boldsymbol{F})) \to \mathcal{E}^{(2)}_f(\boldsymbol{G},\boldsymbol{F})$$

defined by $\boldsymbol{\alpha}_f(\boldsymbol{F})_*(\boldsymbol{\xi}) = \boldsymbol{\alpha}_f(\boldsymbol{F})\boldsymbol{\xi}$ is bijective.

Remark 3.2 *For the identity morphism id_X of $X \in \mathrm{Ob}\,\mathcal{E}$ and $\boldsymbol{F} \in \mathrm{Ob}\,\mathcal{E}^{(2)}_X$, the identity morphism $id_{\boldsymbol{F}}$ of $\boldsymbol{F}$ is obviously cartesian. In this case, we can regard F as $F \times_X X$ and identify $id^*_X(\boldsymbol{F})$ with $\boldsymbol{F}$. Hence $\boldsymbol{\alpha}_{id_X}(N) : id^*_X(\boldsymbol{F}) \to \boldsymbol{F}$ is the identity morphism of $\boldsymbol{F}$.*

For objects $\boldsymbol{E}$, $\boldsymbol{F}$ of $\mathcal{E}^{(2)}_Y$ and a morphism $\boldsymbol{\varphi} : \boldsymbol{E} \to \boldsymbol{F}$ in $\mathcal{E}^{(2)}_Y$, let $f^*(\boldsymbol{\varphi}) : f^*(\boldsymbol{E}) \to f^*(\boldsymbol{F})$ be the unique morphism in $\mathcal{E}^{(2)}_X$ that is mapped to a composition $f^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{\alpha}_f(\boldsymbol{E})} \boldsymbol{E} \xrightarrow{\boldsymbol{\varphi}} \boldsymbol{F}$ by the following bijection given in (3.1).

$$\boldsymbol{\alpha}_f(\boldsymbol{F})_* : \mathcal{E}^{(2)}_X(f^*(\boldsymbol{E}), f^*(\boldsymbol{F})) \to \mathcal{E}^{(2)}_f(f^*(\boldsymbol{E}), \boldsymbol{F})$$

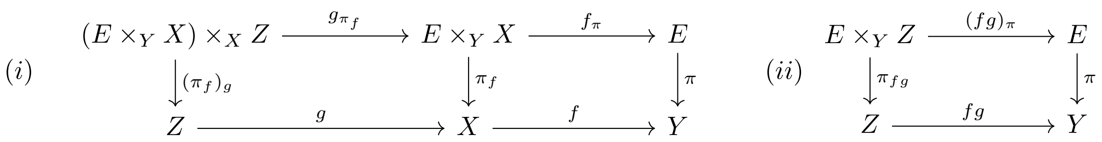
Thus we have the inverse image functor $f^* : \mathcal{E}^{(2)}_Y \to \mathcal{E}^{(2)}_X$ associated with a morphism $f : X \to Y$ in $\mathcal{E}$. It follows from the definition of f^* that the bijection in (3.1) is natural in $\boldsymbol{F}$.

For morphisms $f : X \to Y$, $g : Z \to X$ in $\mathcal{E}$ and an object $\boldsymbol{E}$ of $\mathcal{E}^{(2)}_Y$, let $\boldsymbol{c}_{f,g}(\boldsymbol{E}) : g^*(f^*(\boldsymbol{E})) \to (fg)^*(\boldsymbol{E})$ be the unique morphism in $\mathcal{E}^{(2)}_Z$ that is mapped to a composition $g^*(f^*(\boldsymbol{E})) \xrightarrow{\boldsymbol{\alpha}_g(f^*(\boldsymbol{E}))} f^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{\alpha}_f(\boldsymbol{E})} \boldsymbol{E}$ by the following bijection given in (3.1).

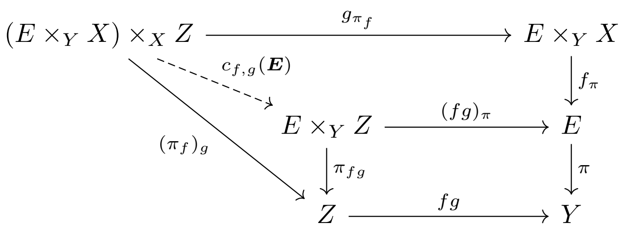
$$\boldsymbol{\alpha}_{fg}(\boldsymbol{E})_* : \mathcal{E}^{(2)}_Z(g^*(f^*(\boldsymbol{E})), (fg)^*(\boldsymbol{E})) \to \mathcal{E}^{(2)}_{fg}(g^*(f^*(\boldsymbol{E})), \boldsymbol{E})$$

Proposition 3.3 *$\boldsymbol{c}_{f,g}(\boldsymbol{E})$ is an isomorphism in $\mathcal{E}^{(2)}_Z$. Hence $\wp_{\mathcal{E}} : \mathcal{E}^{(2)} \to \mathcal{E}$ is a fibered category.*

Proof. We consider the following diagrams in $\mathcal{E}$ such that the left and right rectangles of the left diagram (i) and the right diagram (ii) are cartesian.

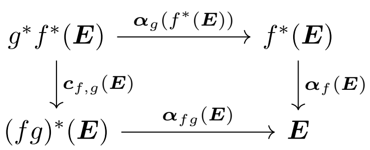


Hence there exists unique morphism $c_{f,g}(\boldsymbol{E}) : (E \times_Y X) \times_X Z \to E \times_Y Z$ that makes the following diagram commute.



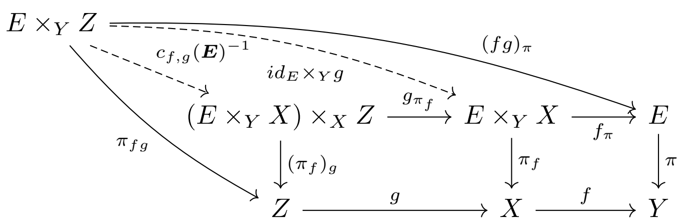
Since the outer rectangle of diagram (i) is also cartesian, it follows that $c_{f,g}(\boldsymbol{E})$ is an isomorphism. Since $\boldsymbol{\alpha}_f(\boldsymbol{E})\boldsymbol{\alpha}_g(f^*(\boldsymbol{E})) = \langle f_\pi g_{\pi_f}, fg\rangle$ and $\boldsymbol{\alpha}_{fg}(\boldsymbol{E}) = \langle (fg)_\pi, fg\rangle$, $\boldsymbol{\alpha}_{fg}(\boldsymbol{E})_*$ maps $\langle c_{f,g}(\boldsymbol{E}), id_Z\rangle$ to $\boldsymbol{\alpha}_f(\boldsymbol{E})\boldsymbol{\alpha}_g(f^*(\boldsymbol{E}))$ by the commutativity of the above diagram. Thus we have $\boldsymbol{c}_{f,g}(\boldsymbol{E}) = \langle c_{f,g}(\boldsymbol{E}), id_Z\rangle$ which is an isomorphism.□

Remark 3.4 *(1) It follows from the definition of $\boldsymbol{c}_{f,g}(\boldsymbol{E})$, the following diagram is commutative.*



Hence we have $\boldsymbol{c}_{f,id_X}(\boldsymbol{E}) = \boldsymbol{c}_{id_Y,f}(\boldsymbol{E}) = id_{f^*(\boldsymbol{E})}$ by (3.2) and the uniqueness of $\boldsymbol{c}_{f,g}(\boldsymbol{E})$.

(2) There exists unique morphisms $id_E \times_Y g : E \times_Y Z \to E \times_Y X$ and $c_{f,g}(\boldsymbol{E})^{-1} : E \times_Y Z \to (E \times_Y X) \times_X Z$ in $\mathcal{E}$ that makes the following diagram commute. The inverse $\boldsymbol{c}_{f,g}(\boldsymbol{E})^{-1} : (fg)^*(\boldsymbol{E}) \to g^*(f^*(\boldsymbol{E}))$ of $\boldsymbol{c}_{f,g}(\boldsymbol{E})$ is given by $\boldsymbol{c}_{f,g}(\boldsymbol{E})^{-1} = \langle c_{f,g}(\boldsymbol{E})^{-1}, id_Z\rangle$.



The following result is easily verified. In fact, this fact holds for the case that $\wp_{\mathcal{E}}$ is a general fibered category ([3]).

Proposition 3.5 *For composable morphisms $f : X \to Y$, $g : Z \to X$ in $\mathcal{E}$ and a morphism $\varphi : \boldsymbol{E} \to \boldsymbol{F}$ in $\mathcal{E}^{(2)}_Y$, the following diagram commutes. In other words, $\mathbf{c}_{f,g}$ gives a natural transformation $g^*f^* \to (fg)^*$ of functors from $\mathcal{E}^{(2)}_Y$ to $\mathcal{E}^{(2)}_Z$.*

$$\begin{array}{ccc} g^*f^*(\boldsymbol{E}) & \xrightarrow{\boldsymbol{c}_{f,g}(\boldsymbol{E})} & (fg)^*(\boldsymbol{E}) \\ \downarrow{\scriptstyle g^*f^*(\varphi)} & & \downarrow{\scriptstyle (fg)^*(\varphi)} \\ g^*f^*(\boldsymbol{F}) & \xrightarrow{\boldsymbol{c}_{f,g}(\boldsymbol{F})} & (fg)^*(\boldsymbol{F}) \end{array}$$

For a morphism $f : X \to Y$ in $\mathcal{E}$, define a functor $f_* : \mathcal{E}^{(2)}_X \to \mathcal{E}^{(2)}_Y$ as follows. We put $f_*(\boldsymbol{E}) = (E \xrightarrow{f\rho} Y)$ for an object $\boldsymbol{E} = (E \xrightarrow{\rho} X)$ of $\mathcal{E}^{(2)}_X$. We put $f_*(\langle \xi, id_X\rangle) = \langle \xi, id_Y\rangle : f_*(\boldsymbol{E}) \to f_*(\boldsymbol{F})$ for a morphism $\langle \xi, id_X\rangle : \boldsymbol{E} \to \boldsymbol{F}$ in $\mathcal{E}^{(2)}_X$.

Proposition 3.6 *$f_* : \mathcal{E}^{(2)}_X \to \mathcal{E}^{(2)}_Y$ is a left adjoint of $f^* : \mathcal{E}^{(2)}_Y \to \mathcal{E}^{(2)}_X$. Hence $\wp_{\mathcal{E}} : \mathcal{E}^{(2)} \to \mathcal{E}$ is a bifibered category.*

Proof. For an object $\boldsymbol{E}$ of $\mathcal{E}^{(2)}_X$ and an object $\boldsymbol{F}$ of $\mathcal{E}^{(2)}_Y$, we define a map $\Phi_{\boldsymbol{E},\boldsymbol{F}} : \mathcal{E}^{(2)}_f(\boldsymbol{E}, \boldsymbol{F}) \to \mathcal{E}^{(2)}_Y(f_*(\boldsymbol{E}), \boldsymbol{F})$ by $\Phi_{\boldsymbol{E},\boldsymbol{F}}(\langle \xi, f\rangle) = \langle \xi, id_Y\rangle$. It is clear that $\Phi_{\boldsymbol{E},\boldsymbol{F}}$ is bijective and natural in $\boldsymbol{E}$ and $\boldsymbol{F}$. It follows from (3.1) that we have a bijection $\Phi_{\boldsymbol{E},\boldsymbol{F}}\boldsymbol{\alpha}_f(\boldsymbol{F})_* : \mathcal{E}^{(2)}_X(\boldsymbol{E}, f^*(\boldsymbol{F})) \to \mathcal{E}^{(2)}_Y(f_*(\boldsymbol{E}), \boldsymbol{F})$ which is natural in $\boldsymbol{E}$ and $\boldsymbol{F}$. □

Remark 3.7 *The unit $\boldsymbol{\eta} : id_{\mathcal{E}^{(2)}_X} \to f^*f_*$ and the counit $\boldsymbol{\varepsilon} : f_*f^* \to id_{\mathcal{E}^{(2)}_Y}$ of the adjunction $f_* \dashv f^*$ are given as follows. For an object $\boldsymbol{E}$ of $\mathcal{E}^{(2)}_X$, there exists unique morphism $\boldsymbol{\eta}_{\boldsymbol{E}} : \boldsymbol{E} \to f^*(f_*(\boldsymbol{E}))$ in $\mathcal{E}^{(2)}_X$ such that $\boldsymbol{\alpha}_f(f_*(\boldsymbol{E}))_* : \mathcal{E}^{(2)}_X(\boldsymbol{E}, f^*(f_*(\boldsymbol{E}))) \to \mathcal{E}^{(2)}_f(\boldsymbol{E}, f_*(\boldsymbol{E}))$ maps $\boldsymbol{\eta}_{\boldsymbol{E}}$ to $(\langle id_E, f\rangle : \boldsymbol{E} \to f_*(\boldsymbol{E})) \in \mathcal{E}^{(2)}_f(\boldsymbol{E}, f_*(\boldsymbol{E}))$ by (3.1). It is easy to verify that $\boldsymbol{\eta}_{\boldsymbol{E}}$ is natural in $\boldsymbol{E}$. For an object $\boldsymbol{F} = (F \xrightarrow{\pi} Y)$ of $\mathcal{E}^{(2)}_Y$, consider the following cartesian square.*

$$\begin{array}{ccc} F \times_Y X & \xrightarrow{f_\pi} & F \\ \downarrow{\scriptstyle \pi_f} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

Then, we have $f_*(f^*(\boldsymbol{F})) = (F \times_Y X \xrightarrow{f\pi_f} Y)$ and define $\boldsymbol{\varepsilon}_{\boldsymbol{F}} : f_*(f^*(\boldsymbol{F})) \to \boldsymbol{F}$ by $\boldsymbol{\varepsilon}_{\boldsymbol{F}} = \langle f_\pi, id_Y\rangle$.

$\mathscr{P}_F(\mathcal{C}, J)$ is complete and cocomplete by (2.15) and (2.19), in particular $\mathscr{P}_F(\mathcal{C}, J)$ has finite limits. Hence we can consider the fibered category $\wp_{\mathscr{P}_F(\mathcal{C},J)} : \mathscr{P}_F(\mathcal{C}, J)^{(2)} \to \mathscr{P}_F(\mathcal{C}, J)$ of morphisms in $\mathscr{P}_F(\mathcal{C}, J)$ by (3.3). It follows from (3.6) that the inverse image functors of this fibered category have left adjoints. We show that the inverse image functors also have right adjoints below.

Let $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{F})$ be a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ and $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (X, \mathscr{D}))$ an object of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}$. For $y \in Y$, we denote by $\iota_y : \varphi^{-1}(y) \to X$ the inclusion map and consider a the-ology $\mathscr{D}^{\iota_y}$ on $\varphi^{-1}(y)$. We define a subset $E(\varphi; y)$ of $\mathscr{P}_F(\mathcal{C}, J)((\varphi^{-1}(y), \mathscr{D}^{\iota_y}), (E, \mathscr{E}))$ by $E(\varphi; y) = \emptyset$ if $\varphi^{-1}(y) = \emptyset$ and

$$E(\varphi; y) = \{\alpha \in \mathscr{P}_F(\mathcal{C}, J)((\varphi^{-1}(y), \mathscr{D}^{\iota_y}), (E, \mathscr{E})) \,|\, \pi\alpha = \iota_y\}$$

if $\varphi^{-1}(y) \neq \emptyset$. Put $E(\varphi) = \coprod_{y \in Y} E(\varphi; y)$ and define map $\varphi_{!\boldsymbol{E}} : E(\varphi) \to Y$ by $\varphi_{!\boldsymbol{E}}(\alpha) = y$ if $\alpha \in E(\varphi; y)$. Note that the image of $\varphi_{!\boldsymbol{E}}$ coincides with the image of φ. We consider the following cartesian square $(*)$ in $\mathcal{S}et$.

$$(*) \qquad \begin{array}{ccc} E(\varphi) \times_Y X & \xrightarrow{\tilde{\varphi}_{\boldsymbol{E}}} & E(\varphi) \\ \downarrow{\scriptstyle \widetilde{\varphi_{!\boldsymbol{E}}}} & & \downarrow{\scriptstyle \varphi_{!\boldsymbol{E}}} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

Define a map $\varepsilon^{\varphi}_{\boldsymbol{E}} : E(\varphi) \times_Y X \to E$ by $\varepsilon^{\varphi}_{\boldsymbol{E}}(\alpha, x) = \alpha(x)$ if $\alpha \in E(\varphi; y)$ and $x \in \varphi^{-1}(y)$ for $y \in Y$. Then, $\varepsilon^{\varphi}_{\boldsymbol{E}}$ makes the following diagram commute.

$$\begin{array}{ccc} E(\varphi)\times_Y X & \xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}} & E \\ & \searrow^{\widetilde{\varphi_{!\boldsymbol{E}}}} & \downarrow{\scriptstyle \pi} \\ & & X \end{array}$$

Let $\Sigma_{\boldsymbol{E},\varphi}$ the set of all the-ology $\mathscr{L}$ on $E(\varphi)$ such that $\mathscr{L}\subset\mathscr{F}^{\varphi_{!\boldsymbol{E}}}$ and $\mathscr{D}^{\widetilde{\varphi_{!\boldsymbol{E}}}}\cap\mathscr{L}^{\tilde{\varphi}_{\boldsymbol{E}}}\subset\mathscr{E}^{\varepsilon^{\varphi}_{\boldsymbol{E}}}$ hold. We note that $\mathscr{L}\in\Sigma_{\boldsymbol{E},\varphi}$ if and only if both $\varphi_{!\boldsymbol{E}}:(E(\varphi),\mathscr{L})\to(Y,\mathscr{F})$ and $\varepsilon^{\varphi}_{\boldsymbol{E}}:(E(\varphi)\times_Y X,\mathscr{D}^{\widetilde{\varphi_{!\boldsymbol{E}}}}\cap\mathscr{L}^{\tilde{\varphi}_{\boldsymbol{E}}})\to(E,\mathscr{E})$ are morphisms in $\mathscr{P}_F(\mathcal{C},J)$.

Proposition 3.8 *$\Sigma_{\boldsymbol{E},\varphi}$ is not empty.*

Proof. It suffices to show that the discrete the-ology $\mathscr{D}_{disc,E(\varphi)}$ on $E(\varphi)$ belongs to $\Sigma_{\boldsymbol{E},\varphi}$. It follows from (1.15) that $\mathscr{D}_{disc,E(\varphi)}\subset\mathscr{F}^{\varphi_{!\boldsymbol{E}}}$ holds. For $U\in\operatorname{Ob}\mathcal{C}$, suppose that $\psi\in\mathscr{D}^{\widetilde{\varphi_{!\boldsymbol{E}}}}\cap\mathscr{D}^{\tilde{\varphi}_{\boldsymbol{E}}}_{disc,E(\varphi)}\cap F_{E(\varphi)\times_Y X}(U)$. Then,we have $\widetilde{\varphi_{!\boldsymbol{E}}}\psi\in\mathscr{D}\cap F_X(U)$ and $\tilde{\varphi}_{\boldsymbol{E}}\psi\in\mathscr{D}_{disc,E(\varphi)}\cap F_{E(\varphi)}(U)$. Hence there exists a covering $(U_i\xrightarrow{g_i}U)_{i\in I}$ such that $F_{E(\varphi)}(g_i)(\tilde{\varphi}_{\boldsymbol{E}}\psi):F_{E(\varphi)}(U_i)\to E(\varphi)$ is a constant map for every $i\in I$ by (1.15). We denote by $\alpha_i\in E(\varphi)$ the image of $F_{E(\varphi)}(g_i)(\tilde{\varphi}_{\boldsymbol{E}}\psi)$ and put $y_i=\varphi_{!\boldsymbol{E}}(\alpha_i)$. Then we have $\alpha_i\in E(\varphi\,;y_i)$ and the image of $F_X(g_i)(\widetilde{\varphi_{!\boldsymbol{E}}}\psi)=\widetilde{\varphi_{!\boldsymbol{E}}}\psi F(g_i):F(U_i)\to X$ is contained in $\varphi^{-1}(y_i)$. Hence we have a map $\xi_i:F(U_i)\to\varphi^{-1}(y_i)$ satisfying $\iota_{y_i}\xi_i=F_X(g_i)(\widetilde{\varphi_{!\boldsymbol{E}}}\psi)\in\mathscr{D}\cap F_X(U_i)$, which shows $\xi_i\in\mathscr{D}^{\iota_{y_i}}\cap F_{\varphi^{-1}(y_i)}(U_i)$. Since we have an equality $F_{E(\varphi)\times_Y X}(g_i)(\psi)=(F_{E(\varphi)}(g_i)(\tilde{\varphi}_{\boldsymbol{E}}\psi),\iota_{y_i}\xi_i):F(U_i)\to E(\varphi)\times_Y X$, it follows that the following equality holds.

$$F_E(g_i)(F_{\varepsilon^{\varphi}_{\boldsymbol{E}}}(\psi))=F_{\varepsilon^{\varphi}_{\boldsymbol{E}}}(F_{E(\varphi)\times_Y X}(g_i)(\psi))=\alpha_i\xi_i=F_{\alpha_i}(\xi_i)$$

Since $\alpha_i:(\varphi^{-1}(y_i),\mathscr{D}^{\iota_{y_i}})\to(E,\mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$, we have $F_{\alpha_i}(\xi_i)\in\mathscr{E}\cap F_E(U_i)$ for any $i\in I$. Therefore $F_{\varepsilon^{\varphi}_{\boldsymbol{E}}}(\psi)\in\mathscr{E}\cap F_E(U)$ holds and we see that $\mathscr{D}^{\widetilde{\varphi_{!\boldsymbol{E}}}}\cap\mathscr{D}^{\tilde{\varphi}_{\boldsymbol{E}}}_{disc,E(\varphi)}\subset\mathscr{E}^{\varepsilon^{\varphi}_{\boldsymbol{E}}}$ holds. □

For $U\in\operatorname{Ob}\mathcal{C}$, we consider the following condition (LE) on an element γ of $F_{E(\varphi)}(U)$.

(LE) If $V,W\in\operatorname{Ob}\mathcal{C}$, $f\in\mathcal{C}(W,U)$, $g\in\mathcal{C}(W,V)$ and $\psi\in\mathscr{D}\cap F_X(V)$ satisfy $\varphi\psi F(g)=\varphi_{!\boldsymbol{E}}\gamma F(f)$, a composition $F(W)\xrightarrow{(\gamma F(f),\,\psi F(g))}E(\varphi)\times_Y X\xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}}E$ belongs to $\mathscr{E}\cap F_E(W)$ and a composition $F(U)\xrightarrow{\gamma}E(\varphi)\xrightarrow{\varphi_{!\boldsymbol{E}}}Y$ belongs to $\mathscr{F}\cap F_Y(U)$.

Define a set $\mathscr{D}_{\boldsymbol{E},\varphi}$ of F-parametrizations of a set $E(\varphi)$ so that $\mathscr{D}_{\boldsymbol{E},\varphi}\cap F_{E(\varphi)}(U)$ is a subset of $F_{E(\varphi)}(U)$ consisting of elements which satisfy the above condition (LE) for any $U\in\operatorname{Ob}\mathcal{C}$.

Proposition 3.9 *$\mathscr{D}_{\boldsymbol{E},\varphi}$ is a the-ologgy on $E(\varphi)$.*

Proof. Suppose that $\gamma\in F_{E(\varphi)}(1_{\mathcal{C}})$, $V,W\in\operatorname{Ob}\mathcal{C}$, $g\in\mathcal{C}(W,V)$ and $\psi\in\mathscr{D}\cap F_X(V)$ satisfy $\varphi\psi F(g)=\varphi_{!\boldsymbol{E}}\gamma F(o_W)$. Put $y_\varphi=\varphi_{!\boldsymbol{E}}(\gamma(*))$. Then, $\gamma(*)\in E(\varphi\,;y_\varphi)$ and $\gamma(*):(\varphi^{-1}(y_\varphi),\mathscr{D}^{\iota_{y_\varphi}})\to(E,\mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$ and $\pi\gamma(*)=\iota_{y_\varphi}$ holds. There exists unique map $\bar{\psi}:F(W)\to\varphi^{-1}(y_\varphi)$ that satisfies $\iota_{y_\varphi}\bar{\psi}=\psi F(g)=F_X(g)(\psi)$. Since $F_X(g)(\psi)\in\mathscr{D}\cap F_X(W)$, we have $\bar{\psi}\in\mathscr{D}^{\iota_{y_\varphi}}\cap F_{\varphi^{-1}(y_\varphi)}(W)$. This implies $(F_{\gamma(*)})_W(\bar{\psi})\in\mathscr{E}\cap F_E(W)$. On the other hand, a composition $F(W)\xrightarrow{(\gamma F(o_W),\,\psi F(g))}E(\varphi)\times_Y X\xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}}E$ coincides with $\gamma(*)\bar{\psi}=(F_{\gamma(*)})_W(\bar{\psi})$ which belongs to $\mathscr{E}\cap F_E(W)$. Moreover we have $\varphi_{!\boldsymbol{E}}\gamma\in F_Y(1_{\mathcal{C}})\subset\mathscr{F}$. Hence $\mathscr{D}_{\boldsymbol{E},\varphi}$ contains $F_{E(\varphi)}(1_{\mathcal{C}})$.

Let $j:Z\to U$ be a morphism in $\mathcal{C}$. For $\gamma\in\mathscr{D}_{\boldsymbol{E},\varphi}\cap F_{E(\varphi)}(U)$, $V,W\in\operatorname{Ob}\mathcal{C}$, $f\in\mathcal{C}(W,Z)$, $g\in\mathcal{C}(W,V)$ and $\psi\in\mathscr{D}\cap F_X(V)$, assume that $\varphi\psi F(g)=\varphi_{!\boldsymbol{E}}F_{E(\varphi)}(j)(\gamma)F(f)$ holds. Since a composition

$$F(W)\xrightarrow{(F_{E(\varphi)}(j)(\gamma)F(f),\,\psi F(g))}E(\varphi)\times_Y X\xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}}E$$

coincides with $F(W)\xrightarrow{(\gamma F(jf),\,\psi F(g))}E(\varphi)\times_Y X\xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}}E$ which belongs to $\mathscr{E}\cap F_E(W)$ since $\gamma\in\mathscr{D}_{\boldsymbol{E},\varphi}\cap F_{E(\varphi)}(U)$. Since $\varphi_{!\boldsymbol{E}}\gamma\in\mathscr{F}\cap F_Y(U)$, $\varphi_{!\boldsymbol{E}}F_{E(\varphi)}(j)(\gamma)=F_Y(j)(\varphi_{!\boldsymbol{E}}\gamma)\in\mathscr{F}\cap F_Y(Z)$ holds. Thus $F_{E(\varphi)}(j)(\gamma)$ belongs to $\mathscr{D}_{\boldsymbol{E},\varphi}\cap F_{E(\varphi)}(Z)$.

Assume that, for $\gamma\in F_{E(\varphi)}(U)$, there exists $R\in J(U)$ such that $F_{E(\varphi)}(j)(\gamma)$ belongs to $\mathscr{D}_{\boldsymbol{E},\varphi}\cap F_{E(\varphi)}(\operatorname{dom}(j))$ for any $j\in R$. Suppose that $\varphi\psi F(g)=\varphi_{!\boldsymbol{E}}\gamma F(f)$ holds for $V,W\in\operatorname{Ob}\mathcal{C}$, $f\in\mathcal{C}(W,U)$, $g\in\mathcal{C}(W,V)$ and $\psi\in\mathscr{D}\cap F_X(V)$. If we put $h_f^{-1}(R)=\{i\in\operatorname{Mor}\mathcal{C}\,|\,\operatorname{codom}(i)=W,\ fi\in R\}$, then we have $h_f^{-1}(R)\in J(W)$ and $F_{E(\varphi)}(fi)(\gamma)\in\mathscr{D}_{\boldsymbol{E},\varphi}\cap F_{E(\varphi)}(\operatorname{dom}(i))$ for any $i\in h_f^{-1}(R)$. Hence the following composition belongs to $\mathscr{E}\cap F_E(\operatorname{dom}(i))$ for any $i\in h_f^{-1}(R)$.

$$F(\operatorname{dom}(i))\xrightarrow{(F_{E(\varphi)}(fi)(\gamma),\,\psi F(gi))}E(\varphi)\times_Y X\xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}}E$$

Since the above composition coincides with a composition $F(\mathrm{dom}(i)) \xrightarrow{F(i)} F(W) \xrightarrow{(\gamma F(f),\, \psi F(g))} X\times_Y E(\varphi) \xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}} E$, it follows that a composition $F(W) \xrightarrow{(\gamma F(f),\, \psi F(g))} X\times_Y E(\varphi) \xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E}\cap F_E(W)$. Since $F_{E(\varphi)}(j)(\gamma)$ belongs to $\mathscr{D}_{\boldsymbol{E},\varphi}\cap F_{E(\varphi)}(\mathrm{dom}(j))$, we have $F_Y(j)(\varphi_{!\boldsymbol{E}}\gamma) = \varphi_{!\boldsymbol{E}}F_{E(\varphi)}(j)(\gamma) \in \mathscr{F}\cap F_Y(\mathrm{dom}(j))$ for any $j\in R$. It follows that $\varphi_{!\boldsymbol{E}}\gamma \in \mathscr{F}\cap F_Y(U)$. Thus we have $\gamma\in\mathscr{D}_{\boldsymbol{E},\varphi}\cap F_{E(\varphi)}(U)$. □

Proposition 3.10 *$\mathscr{D}_{\boldsymbol{E},\varphi}$ is maximum element of $\Sigma_{\boldsymbol{E},\varphi}$.*

Proof. For $U\in\mathrm{Ob}\,\mathcal{C}$ and $\xi\in\mathscr{D}^{\widetilde{\varphi_{!\boldsymbol{E}}}}\cap\mathscr{D}^{\tilde\varphi_{\boldsymbol{E}}}_{\boldsymbol{E},\varphi}\cap F_{E(\varphi)\times_Y X}(U)$, $\varphi\widetilde{\varphi_{!\boldsymbol{E}}}\xi = \varphi_{!\boldsymbol{E}}\tilde\varphi_{\boldsymbol{E}}\xi$ holds and it follows from $\widetilde{\varphi_{!\boldsymbol{E}}}\xi\in\mathscr{D}\cap F_X(U)$ and $\tilde\varphi_{\boldsymbol{E}}\xi\in\mathscr{D}_{\boldsymbol{E},\varphi}\cap F_{E(\varphi)}(U)$ that a composition $F(U)\xrightarrow{(\tilde\varphi_{\boldsymbol{E}}\xi,\,\widetilde{\varphi_{!\boldsymbol{E}}}\xi)} E(\varphi)\times_Y X \xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}} Y$ belongs to $\mathscr{E}\cap F_Y(U)$. Since this composition coincides with $\varepsilon^{\varphi}_{\boldsymbol{E}}\xi$, we see that $\xi\in\mathscr{E}^{\varepsilon^{\varphi}_{\boldsymbol{E}}}$ holds. Hence $\mathscr{D}^{\widetilde{\varphi_{!\boldsymbol{E}}}}\cap\mathscr{D}^{\tilde\varphi_{\boldsymbol{E}}}_{\boldsymbol{E},\varphi}$ is contained in $\mathscr{E}^{\varepsilon^{\varphi}_{\boldsymbol{E}}}$. It is clear from the definition of $\mathscr{D}_{\boldsymbol{E},\varphi}$ that $\mathscr{D}_{\boldsymbol{E},\varphi}$ is contained in $\mathscr{F}^{\varphi_{!\boldsymbol{E}}}$. Thus $\mathscr{D}_{\boldsymbol{E},\varphi}$ is an element of $\Sigma_{\boldsymbol{E},\varphi}$.

For $\mathscr{L}\in\Sigma_{\boldsymbol{E},\varphi}$ and $U\in\mathrm{Ob}\,\mathcal{C}$, suppose that $\gamma\in\mathscr{L}\cap F_{E(\varphi)}(U)$, $V,W\in\mathrm{Ob}\,\mathcal{C}$, $f\in\mathcal{C}(W,U)$, $g\in\mathcal{C}(W,V)$ and that $\psi\in\mathscr{D}\cap F_X(V)$ satisfies $\varphi\psi F(g) = \varphi_{!\boldsymbol{E}}\gamma F(f)$. Since $\mathscr{L}\subset\mathscr{F}^{\varphi_{!\boldsymbol{E}}}$, a composition $F(U)\xrightarrow{\gamma} E(\varphi)\xrightarrow{\varphi_{!\boldsymbol{E}}} Y$ belongs to $\mathscr{F}\cap F_Y(U)$. On the other hand, since $\widetilde{\varphi_{!\boldsymbol{E}}}(\gamma F(f),\psi F(g)) = F_X(g)(\psi)\in\mathscr{D}\cap F_X(W)$ and $\tilde\varphi_{\boldsymbol{E}}(\gamma F(f),\psi F(g)) = F_{E(\varphi)}(\gamma)\in\mathscr{L}\cap F_{E(\varphi)}(W)$ hold, we have $(\gamma F(f),\psi F(g))\in\mathscr{D}^{\widetilde{\varphi_{!\boldsymbol{E}}}}\cap\mathscr{L}^{\tilde\varphi_{\boldsymbol{E}}}\subset\mathscr{E}^{\varepsilon^{\varphi}_{\boldsymbol{E}}}$. It follows that a composition $F(W)\xrightarrow{(\gamma F(f),\,\psi F(g))} E(\varphi)\times_Y X\xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E}\cap F_E(W)$. Therefore $\gamma\in\mathscr{D}_{\boldsymbol{E},\varphi}$ holds and this shows $\mathscr{L}\subset\mathscr{D}_{\boldsymbol{E},\varphi}$. Since $\mathscr{D}_{\boldsymbol{E},\varphi}$ is an element of $\Sigma_{\boldsymbol{E},\varphi}$ by (2.23), $\mathscr{D}_{\boldsymbol{E},\varphi}$ is maximum element of $\Sigma_{\boldsymbol{E},\varphi}$. □

Let $\boldsymbol{E} = ((E,\mathscr{E})\xrightarrow{\pi}(X,\mathscr{D}))$, $\boldsymbol{G} = ((G,\mathscr{G})\xrightarrow{\rho}(X,\mathscr{D}))$ be objects of $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}$ and $\varphi:(X,\mathscr{D})\to(Y,\mathscr{F})$ a morphism in $\mathscr{P}_F(\mathcal{C},J)$. Let $\langle\xi, id_X\rangle:\boldsymbol{E}\to\boldsymbol{G}$ be a morphism in $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}$. If $\alpha\in E(\varphi\,;y)$ for $y\in Y$, we have $\rho\xi\alpha = \pi\alpha = \iota_y$, hence $\xi\alpha\in G(\varphi\,;y)$. Thus we can define a map $\xi_\varphi: E(\varphi)\to G(\varphi)$ by $\xi_\varphi(\alpha) = \xi\alpha$. We consider the following diagram whose outer trapezoid and lower rectangle are cartesian.

$$\begin{array}{ccccc}
E(\varphi)\times_Y X & & \xrightarrow{\quad\tilde\varphi_{\boldsymbol{E}}\quad} & & E(\varphi)\\
 & G(\varphi)\times_Y X & \xrightarrow{\tilde\varphi_{\boldsymbol{G}}} & G(\varphi) & \\
 & \big\downarrow{\scriptstyle\widetilde{\varphi_{!\boldsymbol{G}}}} & & \big\downarrow{\scriptstyle\varphi_{!\boldsymbol{G}}} & \\
 & X & \xrightarrow{\varphi} & Y &
\end{array}$$

Since the right triangle of the above diagram is commutative, there exists unique map

$$\xi_\varphi\times_Y id_X : E(\varphi)\times_Y X\to G(\varphi)\times_Y X$$

that makes the above diagram commutative.

Proposition 3.11 *$\xi_\varphi:(E(\varphi),\mathscr{D}_{\boldsymbol{E},\varphi})\to(G(\varphi),\mathscr{D}_{\boldsymbol{G},\varphi})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$ and the following diagram is commutative.*

$$\begin{array}{ccc}
E(\varphi)\times_Y X & \xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}} & E\\
\big\downarrow{\scriptstyle\xi_\varphi\times_Y id_X} & & \big\downarrow{\scriptstyle\xi}\\
G(\varphi)\times_Y X & \xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{G}}} & G
\end{array}$$

Proof. It is clear from the definitions of $\varepsilon^{\varphi}_{\boldsymbol{E}}$, $\varepsilon^{\varphi}_{\boldsymbol{G}}$ and ξ_φ that the above diagram is commutative. For $U\in\mathrm{Ob}\,\mathcal{C}$ and $\gamma\in\mathscr{D}_{\boldsymbol{E},\varphi}\cap F_{E(\varphi)}(U)$, we take $V,W\in\mathrm{Ob}\,\mathcal{C}$, $f\in\mathcal{C}(W,U)$, $g\in\mathcal{C}(W,V)$ and $\psi\in\mathscr{D}\cap F_X(V)$ satisfy $\varphi\psi F(g) = \varphi_{!\boldsymbol{G}}F_{\xi_\varphi}(\gamma)F(f)$. Since $\varphi_{!\boldsymbol{G}}F_{\xi_\varphi}(\gamma) = F_{\varphi_{!\boldsymbol{G}}\xi_\varphi}(\gamma) = F_{\varphi_{!\boldsymbol{E}}}(\gamma) = \varphi_{!\boldsymbol{E}}\gamma$, $\varphi\psi F(g) = \varphi_{!\boldsymbol{E}}\gamma F(f)$ holds. It follows from the assumption $\gamma\in\mathscr{D}_{\boldsymbol{E},\varphi}\cap F_{E(\varphi)}(U)$ that a composition $F(U)\xrightarrow{F_{\xi_\varphi}(\gamma)} G(\varphi)\xrightarrow{\varphi_{!\boldsymbol{G}}} Y$ belongs to $\mathscr{F}\cap F_Y(U)$ and that a composition $F(W)\xrightarrow{(\gamma F(f),\,\psi F(g))} E(\varphi)\times_Y X\xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E}\cap F_E(W)$. We note that the following diagram is commutative.

$$\begin{array}{ccccc}
F(W) & \xrightarrow{(\gamma F(f),\,\psi F(g))} & E(\varphi)\times_Y X & \xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{E}}} & E\\
 & \searrow{\scriptstyle(F_{\xi_\varphi}(\gamma)F(f),\,\psi F(g))} & \big\downarrow{\scriptstyle\xi_\varphi\times_Y id_X} & & \big\downarrow{\scriptstyle\xi}\\
 & & G(\varphi)\times_Y X & \xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{G}}} & G
\end{array}$$

Since $\xi\colon(E,\mathscr{E})\to(G,\mathscr{G})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$, a composition $F(W)\xrightarrow{(F_{\xi_\varphi}(\gamma)F(f),\,\psi F(g))}G(\varphi)\times_Y X\xrightarrow{\varepsilon^{\varphi}_{\boldsymbol{G}}}E$ belongs to $\mathscr{E}\cap F_G(W)$ by the commutativity of the above diagram. $\square$

Remark 3.12 *We note that* $\boldsymbol{X}=((X,\mathscr{D})\xrightarrow{id_X}(X,\mathscr{D}))$ *is a terminal object of* $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}$. *For* $y\in Y$, *since* $X(\varphi\,;y)=\{\iota_y\}$ *if* $\varphi^{-1}(y)$ *is not empty,* $X(\varphi)$ *is identified with the image* $\varphi(X)$ *of* φ *and* $\varphi_{!\boldsymbol{X}}:X(\varphi)\to Y$ *is identified with the inclusion map* $\varphi(X)\to Y$. *For an object* $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(X,\mathscr{D}))$ *of* $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}$, *the map* $\pi_\varphi:E(\varphi)\to X(\varphi)$ *induced by the unique morphism* $\langle\pi,id_X\rangle:\boldsymbol{E}\to\boldsymbol{X}$ *in* $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}$ *maps* $E(\varphi;y)$ *to* $\{\iota_y\}$ *if* $\varphi^{-1}(y)$ *is not empty.*

Remark 3.13 *Let* $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(X,\mathscr{D}))$, $\boldsymbol{G}=((G,\mathscr{G})\xrightarrow{\rho}(X,\mathscr{D}))$, $\boldsymbol{H}=((H,\mathscr{H})\xrightarrow{\chi}(X,\mathscr{D}))$ *be objects of* $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}$ *and* $\langle\xi,id_X\rangle:\boldsymbol{E}\to\boldsymbol{G}$, $\langle\zeta,id_X\rangle:\boldsymbol{G}\to\boldsymbol{H}$ *be morphisms in* $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}$. *For a morphism* $\varphi:(X,\mathscr{D})\to(Y,\mathscr{F})$, *it follows from the definition of* ξ_φ *that* $(\zeta\xi)_\varphi:E(\varphi)\to H(\varphi)$ *coincides with a composition* $E(\varphi)\xrightarrow{\xi_\varphi}G(\varphi)\xrightarrow{\zeta_\varphi}H(\varphi)$. *We also note that* $(id_E)_\varphi$ *coincides with the identity map of* $E(\varphi)$.

We define a functor $\varphi_!:\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}\to\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(Y,\mathscr{E})}$ by putting $\varphi_!(\boldsymbol{E})=((E(\varphi),\mathscr{D}_{\boldsymbol{E},\varphi})\xrightarrow{\varphi_{!\boldsymbol{E}}}(Y,\mathscr{F}))$ for an object $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(X,\mathscr{D}))$ of $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}$ and $\varphi_!(\langle\xi,id_X\rangle)=\langle\xi_\varphi,id_Y\rangle:\varphi_!(\boldsymbol{E})\to\varphi_!(\boldsymbol{G})$ for a morphism $\langle\xi,id_X\rangle:\boldsymbol{E}\to\boldsymbol{G}$ in $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}$. It follows from (3.10) and (3.11) that we have a natural transformation $\boldsymbol{\varepsilon}^{\varphi}:\varphi^*\varphi_!\to id_{\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}}$ defined by

$$\boldsymbol{\varepsilon}^{\varphi}_{\boldsymbol{E}}=\langle\varepsilon^{\varphi}_{\boldsymbol{E}},id_X\rangle:((E(\varphi)\times_Y X,\mathscr{D}^{\tilde{\varphi}_{\boldsymbol{E}}}_{\boldsymbol{E},\varphi}\cap\mathscr{D}^{\widetilde{\varphi_{!\boldsymbol{E}}}})\xrightarrow{\widetilde{\varphi_{!\boldsymbol{E}}}}(X,\mathscr{D}))\to((E,\mathscr{E})\xrightarrow{\pi}(X,\mathscr{D})).$$

For an object $\boldsymbol{G}=((G,\mathscr{G})\xrightarrow{\rho}(Y,\mathscr{F}))$ of $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(Y,\mathscr{F})}$, we consider the following cartesian square in $\mathscr{P}_F(\mathcal{C},J)$.

$$\begin{array}{ccc}(G\times_Y X,\mathscr{G}^{\varphi_\rho}\cap\mathscr{D}^{\rho_\varphi}) & \xrightarrow{\varphi_\rho} & (G,\mathscr{G})\\ \downarrow{\scriptstyle\rho_\varphi} & & \downarrow{\scriptstyle\rho}\\ (X,\mathscr{D}) & \xrightarrow{\varphi} & (Y,\mathscr{F})\end{array}$$

Then, $\varphi^*(\boldsymbol{G})=((G\times_Y X,\mathscr{G}^{\varphi_\rho}\cap\mathscr{D}^{\rho_\varphi})\xrightarrow{\rho_\varphi}(X,\mathscr{D}))$ and $(G\times_Y X)(\varphi)$ is described as a set as follows.

$$\begin{aligned}(G\times_Y X)(\varphi)&=\coprod_{y\in Y}(G\times_Y X)(\varphi\,;y)=\coprod_{y\in Y}\bigl\{\alpha\in\mathscr{P}_F(\mathcal{C},J)((\varphi^{-1}(y),\mathscr{D}^{\iota_y}),(G\times_Y X,\mathscr{D}^{\rho_\varphi}\cap\mathscr{G}^{\varphi_\rho}))\bigm|\rho_\varphi\alpha=\iota_y\bigr\}\\ &=\coprod_{y\in Y}\bigl\{(\lambda,\iota_y)\in\mathscr{P}_F(\mathcal{C},J)((\varphi^{-1}(y),\mathscr{D}^{\iota_y}),(G\times_Y X,\mathscr{D}^{\rho_\varphi}\cap\mathscr{G}^{\varphi_\rho}))\bigm|\lambda\colon\varphi^{-1}(y)\to G\text{ satisfies }\rho\lambda=\varphi\iota_y\bigr\}\\ &=\coprod_{y\in Y}\bigl\{(\lambda,\iota_y)\in\mathscr{P}_F(\mathcal{C},J)((\varphi^{-1}(y),\mathscr{D}^{\iota_y}),(G\times_Y X,\mathscr{D}^{\rho_\varphi}\cap\mathscr{G}^{\varphi_\rho}))\bigm|\lambda\colon\varphi^{-1}(y)\to G\text{ satisfies }\lambda(\varphi^{-1}(y))\subset\rho^{-1}(y)\bigr\}\end{aligned}$$

For $v\in G$, let us denote by $c_v:\varphi^{-1}(\rho(v))\to G$ the constant map whose image is $\{v\}$. Then we have $c_v(\varphi^{-1}(\rho(v)))=\{v\}\subset\rho^{-1}(\rho(v))$ which implies $(c_v,\iota_{\rho(v)})\in(G\times_Y X)(\varphi)$. Define a map $\eta^{\varphi}_{\boldsymbol{G}}:G\to(G\times_Y X)(\varphi)$ by $\eta^{\varphi}_{\boldsymbol{G}}(v)=(c_v,\iota_{\rho(v)})$. Then, $\eta^{\varphi}_{\boldsymbol{G}}$ makes the following diagram commute.

$$\begin{array}{ccc}G & \xrightarrow{\eta^{\varphi}_{\boldsymbol{G}}} & (G\times_Y X)(\varphi)\\ & \searrow^{\rho} & \downarrow{\scriptstyle\varphi_{!\varphi^*(\boldsymbol{G})}}\\ & & Y\end{array}$$

Proposition 3.14 $\eta^{\varphi}_{\boldsymbol{G}}:(G,\mathscr{G})\to((G\times_Y X)(\varphi),\mathscr{D}_{\varphi^*(\boldsymbol{G}),\varphi})$ *is a morphism in* $\mathscr{P}_F(\mathcal{C},J)$.

Proof. For $U\in\operatorname{Ob}\mathcal{C}$ and $\gamma\in\mathscr{G}\cap F_G(U)$, we take $V,W\in\operatorname{Ob}\mathcal{C}$, $f\in\mathcal{C}(W,U)$, $g\in\mathcal{C}(W,V)$ and $\psi\in\mathscr{D}\cap F_X(V)$ such that $\varphi\psi F(g)=\varphi_{!\varphi^*(\boldsymbol{G})}F_{\eta^{\varphi}_{\boldsymbol{G}}}(\gamma)F(f)$ holds. Since $F_{\eta^{\varphi}_{\boldsymbol{G}}}(\gamma)=\eta^{\varphi}_{\boldsymbol{G}}\gamma$, a composition

$$F(U)\xrightarrow{F_{\eta^{\varphi}_{\boldsymbol{G}}}(\gamma)}(G\times_Y X)(\varphi)\xrightarrow{\varphi_{!\varphi^*(\boldsymbol{G})}}Y$$

coincides with $\rho\gamma=F_\rho(\gamma)$ which belongs to $\mathscr{F}\cap F_Y(U)$. On the other hand, it follows from the definitions of

$\varepsilon^{\varphi}_{\varphi^*(\boldsymbol{G})}$ and $\eta^{\varphi}_{\boldsymbol{G}}$ that the following composition coincides with a map $(\gamma F(f), \psi F(g)) : F(W) \to G \times_Y X$.

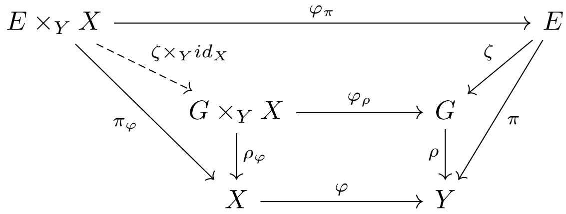
$$F(W) \xrightarrow{(F_{\eta^{\varphi}_{\boldsymbol{G}}}(\gamma)F(f),\, \psi F(g))} (G \times_Y X)(\varphi) \times_Y X \xrightarrow{\varepsilon^{\varphi}_{\varphi^*(\boldsymbol{G})}} G \times_Y X$$

Since $\gamma \in \mathscr{G} \cap F_G(U)$ and $\psi \in \mathscr{D} \cap F_X(V)$, $(\gamma F(f), \psi F(g)) = (F_G(f)(\gamma), F_X(g)(\psi)) \in \mathscr{G}^{\varphi_\rho} \cap \mathscr{D}^{\rho_\varphi} \cap F_{G\times_Y X}(W)$ holds. It follows that $F_{\eta^{\varphi}_{\boldsymbol{G}}}(\gamma)$ belongs to $\mathscr{D}_{\varphi^*(\boldsymbol{G}),\varphi} \cap F_{(G\times_Y X)(\varphi)}(U)$. □

For objects $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (Y, \mathscr{F}))$, $\boldsymbol{G} = ((G, \mathscr{G}) \xrightarrow{\rho} (Y, \mathscr{F}))$ of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(Y,\mathscr{F})}$ and a morphism $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{F})$ in $\mathscr{P}_F(\mathcal{C}, J)$, we consider the following cartesian squares in $\mathscr{P}_F(\mathcal{C}, J)$.

$$\begin{array}{ccc} (E \times_Y X, \mathscr{E}^{\varphi_\pi} \cap \mathscr{D}^{\pi_\varphi}) & \xrightarrow{\varphi_\pi} & (E, \mathscr{E}) \\ \downarrow{\scriptstyle \pi_\varphi} & & \downarrow{\scriptstyle \pi} \\ (X, \mathscr{D}) & \xrightarrow{\varphi} & (Y, \mathscr{F}) \end{array} \qquad \begin{array}{ccc} (G \times_Y X, \mathscr{G}^{\varphi_\rho} \cap \mathscr{D}^{\rho_\varphi}) & \xrightarrow{\varphi_\rho} & (G, \mathscr{G}) \\ \downarrow{\scriptstyle \rho_\varphi} & & \downarrow{\scriptstyle \rho} \\ (X, \mathscr{D}) & \xrightarrow{\varphi} & (Y, \mathscr{F}) \end{array}$$

Let $\langle \zeta, id_Y \rangle : \boldsymbol{E} \to \boldsymbol{G}$ be a morphism in $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(Y,\mathscr{F})}$. Since $\rho\zeta = \pi$ holds, there exists unique morphism $\zeta \times_Y id_X : (E \times_Y X, \mathscr{E}^{\varphi_\pi} \cap \mathscr{D}^{\pi_\varphi}) \to (G \times_Y X, \mathscr{G}^{\varphi_\rho} \cap \mathscr{D}^{\rho_\varphi})$ in $\mathscr{P}_F(\mathcal{C}, J)$ that makes the following diagram commutative.



The following result is easily verified from the definitions of $\eta^{\varphi}_{\boldsymbol{E}}$, $\eta^{\varphi}_{\boldsymbol{G}}$ and $(\zeta \times_Y id_X)_\varphi$.

Proposition 3.15 *For a morphism $\langle \zeta, id_Y \rangle : ((E, \mathscr{E}) \xrightarrow{\pi} (Y, \mathscr{F})) \to ((G, \mathscr{G}) \xrightarrow{\rho} (Y, \mathscr{F}))$ in $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(Y,\mathscr{F})}$, the following diagram is commutative.*

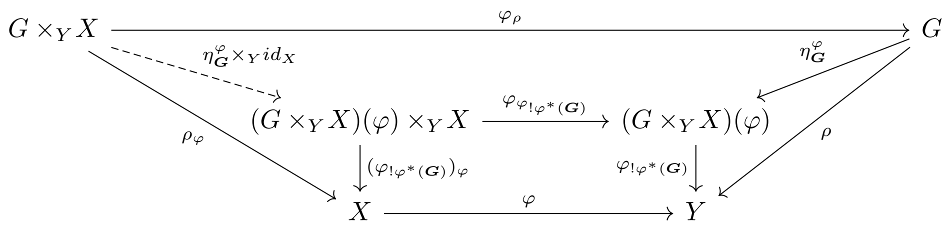
$$\begin{array}{ccc} E & \xrightarrow{\eta^{\varphi}_{\boldsymbol{E}}} & (E \times_Y X)(\varphi) \\ \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle (\zeta \times_Y id_X)_\varphi} \\ G & \xrightarrow{\eta^{\varphi}_{\boldsymbol{G}}} & (G \times_Y X)(\varphi) \end{array}$$

It follows from (3.14) and (3.15) that there is a natural transformation $\boldsymbol{\eta}^{\varphi} : id_{\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(Y,\mathscr{F})}} \to \varphi_!\varphi^*$ defined by

$$\boldsymbol{\eta}^{\varphi}_{\boldsymbol{G}} = \langle \eta^{\varphi}_{\boldsymbol{G}}, id_Y \rangle : ((G, \mathscr{G}) \xrightarrow{\rho} (Y, \mathscr{F})) \to (((G \times_Y X)(\varphi), \mathscr{D}_{\varphi^*(\boldsymbol{G}),\varphi}) \xrightarrow{\varphi_!\varphi^*(\boldsymbol{G})} (Y, \mathscr{F}))$$

for an object $\boldsymbol{G} = ((G, \mathscr{G}) \xrightarrow{\rho} (Y, \mathscr{F}))$ of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(Y,\mathscr{F})}$.

Consider the following diagram, where the outer trapezoid and the lower rectangle are cartesian.



Since the right triangle of the above diagram is commutative, there exists unique map $\eta^{\varphi}_{\boldsymbol{G}} \times_Y id_X : G \times_Y X \to (G \times_Y X)(\varphi) \times_Y X$ that makes the above diagram commute.

Lemma 3.16 *For an objects $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (X, \mathscr{D}))$, $\boldsymbol{G} = ((G, \mathscr{G}) \xrightarrow{\rho} (Y, \mathscr{F}))$ of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}$ and a morphism $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{F})$ in $\mathscr{P}_F(\mathcal{C}, J)$, the following compositions are both identity maps.*

$$E(\varphi) \xrightarrow{\eta^{\varphi}_{\varphi_!(\boldsymbol{E})}} (E(\varphi) \times_Y X)(\varphi) \xrightarrow{(\varepsilon^{\varphi}_{\boldsymbol{E}})_\varphi} E(\varphi), \qquad G \times_Y X \xrightarrow{\eta^{\varphi}_{\boldsymbol{G}} \times_Y id_X} (G \times_Y X)(\varphi) \times_Y X \xrightarrow{\varepsilon^{\varphi}_{\varphi^*(\boldsymbol{G})}} G \times_Y X$$

Proof. For $\alpha \in E(\varphi)$, suppose $\alpha \in E(\varphi\,;y)$ for $y \in Y$, then the following equality holds for $x \in \varphi^{-1}(y)$.

$$\big((\varepsilon^{\varphi}_{\boldsymbol{E}})_{\varphi}\eta^{\varphi}_{\varphi_!(\boldsymbol{E})}(\alpha)\big)(x) = \big((\varepsilon^{\varphi}_{\boldsymbol{E}})_{\varphi}(c_{\alpha}, \iota_y)\big)(x) = \varepsilon^{\varphi}_{\boldsymbol{E}}(\alpha, x) = \alpha(x)$$

For $(v, x) \in G \times_Y X$, then we have $\rho(v) = \varphi(x)$ and $v \in \rho^{-1}(\varphi(x))$. Hence we have the following equality.

$$\varepsilon^{\varphi}_{\varphi^*(\boldsymbol{G})}(\eta^{\varphi}_{\boldsymbol{G}} \times_Y id_X)(v, x) = \varepsilon^{\varphi}_{\varphi^*(\boldsymbol{G})}((c_v, \iota_y), x) = (c_v, \iota_y)(x) = (v, x)$$

Thus the assertion follows. □

For an object $\boldsymbol{G} = ((G, \mathscr{G}) \xrightarrow{\rho} (Y, \mathscr{F}))$ of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(Y,\mathscr{F})}$ and an object $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (X, \mathscr{D}))$ of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(X,\mathscr{D})}$, since compositions

$$\varphi_!(\boldsymbol{E}) \xrightarrow{\boldsymbol{\eta}^{\varphi}_{\varphi_!(\boldsymbol{E})}} \varphi_!\varphi^*\varphi_!(\boldsymbol{E}) \xrightarrow{\varphi_!(\boldsymbol{\varepsilon}^{\varphi}_{\boldsymbol{E}})} \varphi_!(\boldsymbol{E}), \qquad \varphi^*(\boldsymbol{G}) \xrightarrow{\varphi^*(\boldsymbol{\eta}^{\varphi}_{\boldsymbol{G}})} \varphi^*\varphi_!\varphi^*(\boldsymbol{G}) \xrightarrow{\boldsymbol{\varepsilon}^{\varphi}_{\varphi^*(\boldsymbol{G})}} \varphi^*(\boldsymbol{G})$$

are both identity morphisms by (3.16), we have the following result.

Proposition 3.17 *$\varphi_!$: is a right adjoint of φ^*. Hence $\mathscr{P}_F(\mathcal{C}, J)$ is locally cartesian closed.*

Remark 3.18 *Let $\boldsymbol{E} = ((Y, \mathscr{E}) \xrightarrow{\pi} (X, \mathscr{D}))$, $\boldsymbol{F} = ((Z, \mathscr{F}) \xrightarrow{\rho} (X, \mathscr{D}))$ and $\boldsymbol{G} = ((W, \mathscr{G}) \xrightarrow{\chi} (X, \mathscr{D}))$ be objects of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(X,\mathscr{D})}$. It follows from (2.11) and (3.17) that there exist natural bijections*

$$\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(X,\mathscr{D})}(\rho_*\rho^*(\boldsymbol{E}), \boldsymbol{G}) \to \mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(Z,\mathscr{F})}(\rho^*(\boldsymbol{E}), \rho^*(\boldsymbol{G})),$$
$$\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(Z,\mathscr{F})}(\rho^*(\boldsymbol{E}), \rho^*(\boldsymbol{G})) \to \mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(X,\mathscr{D})}(\boldsymbol{E}, \rho_!\rho^*(\boldsymbol{G})).$$

We note that the product $\boldsymbol{E} \times \boldsymbol{F}$ of $\boldsymbol{E}$ and $\boldsymbol{F}$ is given by $\boldsymbol{E} \times \boldsymbol{F} = \rho_*\rho^*(\boldsymbol{E})$. Hence if we put $\boldsymbol{G}^{\boldsymbol{F}} = \rho_!\rho^*(\boldsymbol{G})$, we have a natural bijection

$$\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(X,\mathscr{D})}(\boldsymbol{E} \times \boldsymbol{F}, \boldsymbol{G}) \to \mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(X,\mathscr{D})}(\boldsymbol{E}, \boldsymbol{G}^{\boldsymbol{F}}).$$

This shows that $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(X,\mathscr{D})}$ is cartesian closed.

4 Strong subobject classifier

Definition 4.1 *Let $\mathcal{E}$ be a category.*

(1) Two morphisms $p : X \to Y$ and $i : Z \to W$ in $\mathcal{E}$ are said to be orthogonal if the following left diagram is commutative, there exits unique morphism $s : Y \to Z$ that makes the following right diagram commute.

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle i} \\ Y & \xrightarrow{v} & W \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{u} & Z \\ \big\downarrow{\scriptstyle p} & \overset{s}{\nearrow} & \big\downarrow{\scriptstyle i} \\ Y & \xrightarrow{v} & W \end{array}$$

If p and i are orthogonal, we denote this by $p \perp i$.

(2) For a class C of morphisms in $\mathcal{E}$, we put

$$C^{\perp} = \{i \in \operatorname{Mor}\mathcal{E}\,|\, p \perp i \text{ if } p \in C\}, \qquad {}^{\perp}C = \{p \in \operatorname{Mor}\mathcal{E}\,|\, p \perp i \text{ if } i \in C\}.$$

(3) Let E be the class of all epimorphisms in $\mathcal{E}$. A monomorphism $i : Z \to W$ in $\mathcal{E}$ is called a strong monomorphism if i belongs to $E^{\perp}$.

(4) Let M be the class of all monomorphisms in $\mathcal{E}$. An epimorphism $p : X \to Y$ in $\mathcal{E}$ is called a strong epimorphism if p belongs to ${}^{\perp}M$.

Proposition 4.2 *Let C be a class of morphisms in $\mathcal{E}$.*

(1) If D is a class of morphisms in $\mathcal{E}$ which contains C, then $C^{\perp} \supset D^{\perp}$ and ${}^{\perp}C \supset {}^{\perp}D$.

(2) $C \subset {}^{\perp}(C^{\perp})$ and $C \subset ({}^{\perp}C)^{\perp}$ hold.

(3) $({}^{\perp}(C^{\perp}))^{\perp} = C^{\perp}$ and ${}^{\perp}(({}^{\perp}C)^{\perp}) = {}^{\perp}C$ hold.

Proof. (1) Since $f \in C$ implies $f \in D$, the assertion is straightforward from the definition (4.1).

(2) For $p \in C$, we have $p \perp j$ for any $j \in C^{\perp}$, which shows $p \in {}^{\perp}(C^{\perp})$. Thus we have $C \subset {}^{\perp}(C^{\perp})$. For $i \in C$, we have $p \perp i$ for any $p \in {}^{\perp}C$, which shows $i \in ({}^{\perp}C)^{\perp}$. Thus we have $C \subset ({}^{\perp}C)^{\perp}$.

(3) It follows from (1) and (2) that we have $({}^{\perp}(C^{\perp}))^{\perp} \subset C^{\perp}$ and ${}^{\perp}(({}^{\perp}C)^{\perp}) \subset {}^{\perp}C$. Suppose that $i \in C^{\perp}$ and $p \in {}^{\perp}(C^{\perp})$. Then, $p \perp j$ for any $j \in C^{\perp}$ in particular, we have $p \perp i$. Hence $p \perp i$ holds for any $p \in {}^{\perp}(C^{\perp})$,

which implies $i \in ({}^\perp(C^\perp))^\perp$. Thus we have $C^\perp \subset ({}^\perp(C^\perp))^\perp$. Suppose that $i \in {}^\perp C$ and $p \in ({}^\perp C)^\perp$. Then, $p \perp j$ for any $j \in {}^\perp C$ in particular, we have $p \perp i$. Hence $p \perp i$ holds for any $p \in ({}^\perp C)^\perp$, which implies $i \in (({}^\perp C)^\perp)^\perp$. Thus we have ${}^\perp C \subset (({}^\perp C)^\perp)^\perp$. □

Proposition 4.3 *(1) If $i : Z \to W$ is an equalizer of $f, g : W \to V$, then i is a strong monomorphism.*
(2) If $p : X \to Y$ is a coequalizer of $f, g : U \to X$, then p is a strong epimorphism.

Proof. (1) Suppose that the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{v} & W \end{array}$$

Then, we have $fvp = fiu = giu = gvp$. Hence if p is an epimorphism, it follows that $fv = gv$. Since i is an equalizer of $f, g : W \to V$, there exists unique $s : Y \to Z$ that satisfies $v = is$. Then, $isp = vp = iu$ which implies $sp = u$ since i is a monomorphism.

(2) Suppose that the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{u} & Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{v} & W \end{array}$$

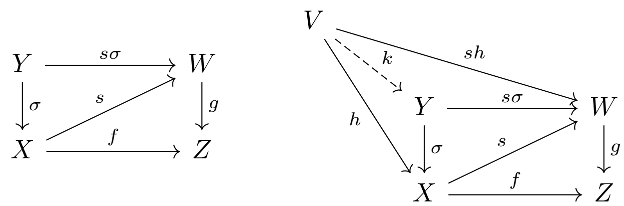
Then, we have $iuf = vpf = vpg = iug$. Hence if i is a monomorphism, it follows that $uf = ug$. Since p is a coequalizer of $f, g : U \to X$, there exists unique $s : Y \to Z$ that satisfies $u = sp$. Then, $isp = iu = vp$ which implies $is = v$ since p is an epimorphism. □

Definition 4.4 *Let $\mathcal{E}$ be a category with a terminal object $1_{\mathcal{E}}$. If a morphism $t : 1_{\mathcal{E}} \to \Omega$ satisfies the following condition, we call t a strong subobject classifier of $\mathcal{E}$.*

(∗) *For each strong monomorphism $\sigma : Y \rightarrowtail X$ in $\mathcal{E}$, there exists unique morphism $\phi_\sigma : X \to \Omega$ that makes the following square cartesian.*

$$\begin{array}{ccc} Y & \xrightarrow{o_Y} & 1_{\mathcal{E}} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle t} \\ X & \xrightarrow{\phi_\sigma} & \Omega \end{array}$$

Remark 4.5 *Assume that the outer rectangle of the following left diagram is cartesian. If $h : V \to X$ satisfies $fh = gsh$, then there exists unique morphism $k : V \to Y$ that satisfies $\sigma k = h$ by the assumption.*



Hence if $\sigma : Y \to X$ is a monomorphism, σ is an equalizer of $f, gs : X \to Z$. It follows that if $\mathcal{E}$ has a strong subobject classifier, each strong monomorphism in $\mathcal{E}$ is an equalizer of a certain pair of morphisms.

Proposition 4.6 *A morphism $i : (Y, \mathscr{E}) \to (X, \mathscr{D})$ in $\mathscr{P}_F(\mathcal{C}, J)$ is a monomorphism if and only if $i : Y \to X$ is injective.*

Proof. It is clear that $i : (Y, \mathscr{E}) \to (X, \mathscr{D})$ in $\mathscr{P}_F(\mathcal{C}, J)$ is a monomorphism if $i : Y \to X$ is injective. Suppose that $i : (Y, \mathscr{E}) \to (X, \mathscr{D})$ is a monomorphism in $\mathscr{P}_F(\mathcal{C}, J)$ and that $i(a) = i(b)$ holds for $a, b \in Y$. Define maps $f, g : \{1\} \to Y$ by $f(1) = a$ and $g(1) = b$. Then $f, g : (\{1\}, \mathscr{D}_{disc,\{1\}}) \to (Y, \mathscr{E})$ are morphisms in $\mathscr{P}_F(\mathcal{C}, J)$ which satisfy $if = ig$. Thus we have $f = g$ which implies $a = b$. □

Proposition 4.7 *Let $\sigma : (Y, \mathscr{F}) \to (X, \mathscr{D})$ be a strong monomorphism in $\mathscr{P}_F(\mathcal{C}, J)$ and denote by $i : \sigma(Y) \to X$ the inclusion map. Then there is a surjection $\tilde{\sigma} : Y \to \sigma(Y)$ which satisfies $i\tilde{\sigma} = \sigma$. This map gives an isomorphism $\tilde{\sigma} : (Y, \mathscr{F}) \to (\sigma(Y), \mathscr{D}^i)$ in $\mathscr{P}_F(\mathcal{C}, J)$.*

Proof. Since $\sigma : Y \to X$ is injective by (4.6), $\tilde{\sigma}$ is bijective. Since $(F_\sigma)_U = (F_i)_U(F_{\tilde{\sigma}})_U : F_Y(U) \to F_X(U)$ maps $\mathscr{F} \cap F_Y(U)$ into $\mathscr{D} \cap F_X(U)$, $(F_{\tilde{\sigma}})_U : F_Y(U) \to F_X(U)$ maps $\mathscr{F} \cap F_Y(U)$ into $(F_i)_U^{-1}(\mathscr{D} \cap F_X(U)) = \mathscr{D}^i \cap F_{\sigma(Y)}(U)$ for $U \in \mathrm{Ob}\,\mathcal{C}$. Hence $\tilde{\sigma} : (Y, \mathscr{F}) \to (\sigma(Y), \mathscr{D}^i)$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. Consider the following left commutative diagram.

$$\begin{array}{ccc} (Y, \mathscr{F}) & \xrightarrow{id_Y} & (Y, \mathscr{F}) \\ \downarrow{\scriptstyle \tilde{\sigma}} & & \downarrow{\scriptstyle \sigma} \\ (S(Y), \mathscr{D}^i) & \xrightarrow{i} & (X, \mathscr{D}) \end{array} \qquad \begin{array}{ccc} (Y, \mathscr{F}) & \xrightarrow{id_Y} & (Y, \mathscr{F}) \\ \downarrow{\scriptstyle \tilde{\sigma}} & \overset{s}{\nearrow} & \downarrow{\scriptstyle \sigma} \\ (S(Y), \mathscr{D}^i) & \xrightarrow{i} & (X, \mathscr{D}) \end{array}$$

Since $\tilde{\sigma} : (Y, \mathscr{F}) \to (\sigma(Y), \mathscr{D}^i)$ is an epimorphism in $\mathscr{P}_F(\mathcal{C}, J)$ and $\sigma : (Y, \mathscr{F}) \to (X, \mathscr{D})$ is a strong monomorphism in $\mathscr{P}_F(\mathcal{C}, J)$, there exists a morphism $s : (S(Y), \mathscr{D}^i) \to (Y, \mathscr{F})$ in $\mathscr{P}_F(\mathcal{C}, J)$ which makes the above right diagram commute. Hence we have $s\tilde{\sigma} = id_Y$ and $i\tilde{\sigma}s = \sigma s = i$. Since i is a monomorphism, the latter equality implies $\tilde{\sigma}s = id_{s(Y)}$. Therefore $\tilde{\sigma} : (Y, \mathscr{F}) \to (\sigma(Y), \mathscr{D}^i)$ is an isomorphism in $\mathscr{P}_F(\mathcal{C}, J)$. □

Let $t : \{1\} \to \{0, 1\}$ be an inclusion map. Then, $t : (\{1\}, \mathscr{D}_{coarse,\{1\}}) \to (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$.

Proposition 4.8 *Let $(X, \mathscr{D})$ be an object of $\mathscr{P}_F(\mathcal{C}, J)$ and Y a subset of X. We denote by $\sigma : Y \to X$ the inclusion map and define a map $\phi_\sigma : X \to \{0, 1\}$ by $\phi_\sigma(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases}$. Then, the following diagram is a cartesian square in $\mathscr{P}_F(\mathcal{C}, J)$.*

$$\begin{array}{ccc} (Y, \mathscr{D}^\sigma) & \xrightarrow{o_Y} & (\{1\}, \mathscr{D}_{coarse,\{1\}}) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle t} \\ (X, \mathscr{D}) & \xrightarrow{\phi_\sigma} & (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}}) \end{array}$$

Proof. Let $f : (W, \mathscr{F}) \to (X, \mathscr{D})$ be a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ which stisfies $\phi_\sigma f = to_W$. Then, we have $\phi_\sigma f(W) \subset \{1\}$ which shows $f(W) \subset Y$. Hence there is unique map $\tilde{f} : W \to Y$ which satisfies $\sigma\tilde{f} = f$. For each $U \in \mathrm{Ob}\,\mathcal{C}$, since $(F_\sigma)_U(F_{\tilde{f}})_U = (F_f)_U : F_W(U) \to F_X(U)$ maps $\mathscr{F} \cap F_W(U)$ into $\mathscr{D} \cap F_X(U)$, it follows that $(F_{\tilde{f}})_U : F_W(U) \to F_Y(U)$ maps $\mathscr{F} \cap F_W(U)$ into $(F_\sigma)_U^{-1}(\mathscr{D} \cap F_X(U)) = \mathscr{D}^\sigma \cap F_Y(U)$. Thus $\tilde{f} : (W, \mathscr{F}) \to (Y, \mathscr{D}^\sigma)$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. □

Remark 4.9 *The morphism $\sigma : (Y, \mathscr{D}^\sigma) \to (X, \mathscr{D})$ is an equalizer of $\phi_\sigma : (X, \mathscr{D}) \to (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}})$ and a composition $(X, \mathscr{D}) \xrightarrow{o_X} (\{1\}, \mathscr{D}_{coarse,\{1\}}) \xrightarrow{t} (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}})$ by (4.5). In particular, $\sigma : (Y, \mathscr{D}^\sigma) \to (X, \mathscr{D})$ is a strong monomorphism in $\mathscr{P}_F(\mathcal{C}, J)$ by (4.3).*

Proposition 4.10 *$t : (\{1\}, \mathscr{D}_{coarse,\{1\}}) \to (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}})$ is a strong subobject classifier in $\mathscr{P}_F(\mathcal{C}, J)$.*

Proof. Let $\sigma : (Y, \mathscr{F}) \to (X, \mathscr{D})$ be a strong monomorphism in $\mathscr{P}_F(\mathcal{C}, J)$. We denote by $i : \sigma(Y) \to X$ the inclusion map. It follows from (4.8) that there exists a morphism $\phi_\sigma : (X, \mathscr{D}) \to (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}})$ such that the following diagram is cartesian.

$$\begin{array}{ccc} (\sigma(Y), \mathscr{D}^i) & \xrightarrow{o_{\sigma(Y)}} & (\{1\}, \mathscr{D}_{coarse,\{1\}}) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle t} \\ (X, \mathscr{D}) & \xrightarrow{\phi_\sigma} & (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}}) \end{array}$$

Then, the following diagram is also cartesian by (4.7).

$$\begin{array}{ccc} (Y, \mathscr{F}) & \xrightarrow{o_Y} & (\{1\}, \mathscr{D}_{coarse,\{1\}}) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle t} \\ (X, \mathscr{D}) & \xrightarrow{\phi_\sigma} & (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}}) \end{array}$$

Suppose that a map $\psi : (X, \mathscr{D}) \to (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}})$ also makes the following diagram cartesian.

$$\begin{array}{ccc} (Y, \mathscr{F}) & \xrightarrow{\quad o_Y \quad} & (\{1\}, \mathscr{D}_{coarse,\{1\}}) \\ \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle t} \\ (X, \mathscr{D}) & \xrightarrow{\quad \psi \quad} & (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}}) \end{array}$$

Since the forgetful functor $\varGamma_F : \mathscr{P}_F(\mathcal{C}, J) \to \mathcal{S}et$ has a left adjoint, $\varGamma_F$ preserves limits. Hence

$$\begin{array}{ccc} Y & \xrightarrow{\quad o_Y \quad} & \{1\} \\ \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle t} \\ X & \xrightarrow{\quad \psi \quad} & \{0, 1\} \end{array}$$

is a cartesian square in $\mathcal{S}et$. Since $\psi\sigma = to_Y$, we have $\psi(x) = 1$ if $x \in \sigma(Y)$. If $\psi(x) = 1$ for $x \in X$, we define a map $f : \{1\} \to X$ by $f(1) = x$. Then we have $\psi f = tid_{\{1\}}$ which implies that there exists a map $\bar{f} : \{1\} \to Y$ which satisfies $\sigma\bar{f} = f$. Thus $x = f(1) = \sigma(\bar{f}(1)) \in \sigma(Y)$. Therefore $\psi = \phi_\sigma$ holds and this shows the uniqueness of ϕ_σ. □

By (2.15), (2.19), (3.17) and (4.10), we have the following result.

Theorem 4.11 *$\mathscr{P}_F(\mathcal{C}, J)$ is a quasitopos.*

Proposition 4.12 *$\pi : (X, \mathscr{D}) \to (Y, \mathscr{E})$ is an epimorphism in $\mathscr{P}_F(\mathcal{C}, J)$ if and only if $\pi : X \to Y$ is surjective.*

Proof. It is clear that $\pi : (X, \mathscr{D}) \to (Y, \mathscr{E})$ is an epimorphism in $\mathscr{P}_F(\mathcal{C}, J)$ if $\pi : X \to Y$ is surjective. Assume that $\pi : (X, \mathscr{D}) \to (Y, \mathscr{E})$ is an epimorphism in $\mathscr{P}_F(\mathcal{C}, J)$. We denote by $\sigma : \pi(X) \to Y$ the inclusion map. Since $\sigma : (\pi(X), \mathscr{E}^\sigma) \to (Y, \mathscr{E})$ is a strong monomorphism by (4.9), there exists a morphism $\phi_\sigma : (Y, \mathscr{E}) \to (\{0, 1\}, \mathscr{D}_{disc,\{0,1\}})$ such that the following left diagram is cartesian.

$$\begin{array}{ccc} (\pi(X), \mathscr{E}^\sigma) & \xrightarrow{o_{\pi(X)}} & (\{1\}, \mathscr{D}_{coarse,\{1\}}) \\ \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle t} \\ (Y, \mathscr{E}) & \xrightarrow{\quad \phi_\sigma \quad} & (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}}) \end{array} \qquad \begin{array}{ccccc} & & (\pi(X), \mathscr{E}^\sigma) & \xrightarrow{o_{\pi(X)}} & (\{1\}, \mathscr{D}_{coarse,\{1\}}) \\ & \nearrow{\scriptstyle \bar{\pi}} & \Big\downarrow{\scriptstyle \sigma} & \nearrow{\scriptstyle o_Y} & \Big\downarrow{\scriptstyle t} \\ (X, \mathscr{D}) & \xrightarrow{\quad \pi \quad} & (Y, \mathscr{E}) & \xrightarrow{\quad \phi_\sigma \quad} & (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}}) \end{array}$$

Let $\bar{\pi} : X \to \pi(X)$ be the surjection induced by π. Then $\bar{\pi} : (X, \mathscr{D}) \to (\pi(X), \mathscr{E}^\sigma)$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. We consider a composition $to_Y : (Y, \mathscr{E}) \to (\{0, 1\}, \mathscr{D}_{coarse,\{0,1\}})$ which is a constant map whose image is $\{1\}$. Since $\phi_\sigma\pi = \phi_\sigma\sigma\bar{\pi} = to_{\pi(X)}\bar{\pi}$, $\phi_\sigma\pi$ is also a constant map to $\{1\}$. Thus we have $\phi_\sigma\pi = to_Y\pi$. Since π is an epimorphism, we have $\phi_\sigma = to_Y$, in other words, ϕ_σ is a contant map to $\{1\}$. Therefore $\pi(X) = \phi_\sigma^{-1}(\{1\}) = Y$ and π is surjective. □

5 Comparison of categories of plots

Definition 5.1 *Let $(\mathcal{C}, J)$ and $(\mathcal{C}', J')$ be sites and $T : \mathcal{C}' \to \mathcal{C}$ a functor.*

(1) We say that T preserves coverings if, for any object U of $\mathcal{C}'$ and any covering $(U_i \xrightarrow{f_i} U)_{i\in I}$ of U, $(T(U_i) \xrightarrow{T(f_i)} T(U))_{i\in I}$ is a covering of $T(U)$.

(2) For $U \in \mathrm{Ob}\,\mathcal{C}'$ and a sieve R on $T(U)$, we set $R^T = \{f \in h_U \,|\, T(f) \in R(T(\mathrm{dom}(f)))\}$. We say that T is cocontinuous if $R^T \in J'(U)$ for any $U \in \mathrm{Ob}\,\mathcal{C}'$ and $R \in J(T(U))$.

For $U \in \mathrm{Ob}\,\mathcal{C}'$ and a sieve R on U, we denote by $T(R)$ a sieve on $T(U)$ generated by $\{T(f) \in h_{T(U)} \,|\, f \in R\}$.

Proposition 5.2 *$T : \mathcal{C}' \to \mathcal{C}$ preserves coverings if and only if following condition is satisfied.*

(∗) *For $U \in \mathrm{Ob}\,\mathcal{C}'$ and $R \in J'(U)$, $T(R) \in J(T(U))$ holds.*

Proof. Let U be an object of $\mathcal{C}$. For $R \in J'(U)$, since $(f : \mathrm{dom}(f) \to U)_{f\in R}$ is a covering of U, $(T(f) : T(\mathrm{dom}(f)) \to T(U))_{f\in R}$ is a covering of $T(U)$ if T preserves coverings. Hence $T(R) \in J(T(U))$. Conversely, we assume condition $(*)$. For a covering $(U_i \xrightarrow{f_i} U)_{i\in I}$ of U, let R be the sieve generated by $(U_i \xrightarrow{f_i} U)_{i\in I}$ and R' the sieve generated by $(T(U_i) \xrightarrow{T(f_i)} T(U))_{i\in I}$. Since $(T(U_i) \xrightarrow{T(f_i)} T(U)) \in T(R)$ for any $i \in I$, R' is contained in $T(R)$. If $f \in T(R)$, there exist $(g : \mathrm{dom}(g) \to U) \in R$ and a morphism $k : \mathrm{dom}(f) \to T(\mathrm{dom}(g))$ in $\mathcal{C}$ such that $f = T(g)k$. Since R be the sieve generated by $(U_i \xrightarrow{f_i} U)_{i\in I}$, there exist $i \in I$ and a morphism $l : \mathrm{dom}(g) \to U_i$ such that $g = f_i l$. Thus we have $f = T(f_i)T(l)k$ which shows $f \in R'$ and $T(R)$ is contained in R'. Hence $T(R) = R'$ and $(T(U_i) \xrightarrow{T(f_i)} T(U))_{i\in I}$ is a covering of $T(U)$. □

Let $(\mathcal{C}, J)$ and $(\mathcal{C}', J')$ be sites and $T : \mathcal{C}' \to \mathcal{C}$, $F : \mathcal{C} \to \mathcal{S}et$ functors. Assume that $\mathcal{C}$ and $\mathcal{C}'$ have terminal objects $1_{\mathcal{C}}$ and $1_{\mathcal{C}'}$, respectively and that $F(1_{\mathcal{C}})$ is a set consists of a single element. We note that, for $U \in \mathrm{Ob}\,\mathcal{C}'$ and a set X, $(FT)_X(U) = \mathcal{S}et(FT(U), X) = F_X(T(U))$ holds. Let X be a set and $\mathcal{S}$ a subset of $\coprod_{V\in \mathrm{Ob}\,\mathcal{C}} F_X(V)$. We define a subset $T^*(\mathcal{S})$ of $\coprod_{U\in \mathrm{Ob}\,\mathcal{C}'} (FT)_X(U)$ by $T^*(\mathcal{S}) = \coprod_{U\in \mathrm{Ob}\,\mathcal{C}'} \mathcal{S} \cap F_X(T(U))$.

Proposition 5.3 *Let $\mathscr{D}$ be a the-ology on a set X with respect to F and $(\mathcal{C}, J)$. $T^*(\mathscr{D})$ satisfies condition (ii) of (1.2) for FT. If T satisfies $T(1_{\mathcal{C}'}) = 1_{\mathcal{C}}$, $T^*(\mathscr{D})$ satisfies condition (i) of (1.2) for FT. If T preserves coverings, $T^*(\mathscr{D})$ satisfies condition (iii) of (1.2) for FT and $(\mathcal{C}', J')$.*

Proof. For a morphism $f : U \to V$ in $\mathcal{C}'$, since $F_X(T(f)) : F_X(T(V)) \to F_X(T(U))$ maps $\mathscr{D} \cap F_X(T(V))$ into $\mathscr{D} \cap F(T(U))$, $T^*(\mathscr{D})$ satisfies condition (ii) of (1.2) for $FT : \mathcal{C}' \to \mathcal{S}et$.

Assume that T satisfies $T(1_{\mathcal{C}'}) = 1_{\mathcal{C}}$. Since $\mathscr{D} \supset F_X(1_{\mathcal{C}})$, we have

$$T^*(\mathscr{D}) \supset \mathscr{D} \cap F_X(T(1_{\mathcal{C}'})) = \mathscr{D} \cap F_X(1_{\mathcal{C}}) = F_X(1_{\mathcal{C}}) = (FT)_X(1_{\mathcal{C}'}).$$

Thus $T^*(\mathscr{D})$ satisfies condition (i) of (1.2) for FT.

Assume that T preserves coverings. For an object U of $\mathcal{C}'$ and an element x of $(FT)_X(U)$, suppose that there exists a covering $(U_i \xrightarrow{f_i} U)_{i\in I}$ of U such that $(FT)_X(f_i)(x) \in T^*(\mathscr{D}) \cap (FT)_X(U_i)$ for any $i \in I$. Since $(T(U_i) \xrightarrow{T(f_i)} T(U))_{i\in I}$ is a covering of $T(U)$ and $F_X(T(f_i))(x) \in \mathscr{D} \cap F_X(T(U_i))$ for any $i \in I$, x belongs to $\mathscr{D} \cap F_X(T(U)) = T^*(\mathscr{D}) \cap (FT)_X(U)$. Hence $T^*(\mathscr{D})$ satisfies condition (iii) of (1.2) for FT. □

We assume that satisfies $T(1_{\mathcal{C}'}) = 1_{\mathcal{C}}$ and that T preserves coverings below. We define a functor $T^* : \mathscr{P}_F(\mathcal{C}, J) \to \mathscr{P}_{FT}(\mathcal{C}', J')$ as follows. Put $T^*(X, \mathscr{D}) = (X, T^*(\mathscr{D}))$ for $(X, \mathscr{D}) \in \mathrm{Ob}\,\mathcal{C}$. For a morphism $f : (X, \mathscr{D}) \to (Y, \mathscr{E})$ in $\mathscr{P}_F(\mathcal{C}, J)$ and an object U of $\mathcal{C}$, if $\alpha \in T^*(\mathscr{D}) \cap (FT)_X(U)$, then $\alpha \in \mathscr{D} \cap F_X(T(U))$ hence $f\alpha = (F_f)_{T(U)}(\alpha)$ belongs to $\mathscr{E} \cap F_Y(T(U)) = T^*(\mathscr{E}) \cap (FT)_Y(U)$. It follows that $f : (X, T^*(\mathscr{D})) \to (Y, T^*(\mathscr{E}))$ is a morphism in $\mathscr{P}_{FT}(\mathcal{C}', J')$. We define $T^*(f : (X, \mathscr{D}) \to (Y, \mathscr{E}))$ to be $f : (X, T^*(\mathscr{D})) \to (Y, T^*(\mathscr{E}))$.

Proposition 5.4 *Let $f : X \to Y$ be a map.*

(1) For a the-ology $\mathscr{E}$ on Y with respect to F and $(\mathcal{C}, J)$, a the-ology $T^*(\mathscr{E}^f)$ on X with respect to FT and $(\mathcal{C}', J')$ coincides with $T^*(\mathscr{E})^f$.

(2) For a the-ology $\mathscr{D}$ on X with respect to F and $(\mathcal{C}, J)$, a the-ology $T^*(\mathscr{D}_f)$ on Y with respect to FT and $(\mathcal{C}', J')$ is coarser than $T^*(\mathscr{D})_f$. If T is cocontinuous, $T^*(\mathscr{D}_f)$ coincides with $T^*(\mathscr{D})_f$.

Proof. Let U be an object of $\mathcal{C}'$.

(1) The following equality shows $T^*(\mathscr{E}^f) = T^*(\mathscr{E})^f$.

$$\begin{aligned} T^*(\mathscr{E}^f) \cap (FT)_X(U) &= \mathscr{E}^f \cap F_X(T(U)) = \{\varphi \in F_X(T(U)) \,|\, f\varphi \in \mathscr{E}\} \\ &= \{\varphi \in (FT)_X(U) \,|\, f\varphi \in T^*(\mathscr{E})\} = T^*(\mathscr{E})^f \cap (FT)_X(U) \end{aligned}$$

(2) Since $T^*(f) : (X, T^*(\mathscr{D})) \to (Y, T^*(\mathscr{D}_f))$ is a morphism in $\mathscr{P}_{FT}(\mathcal{C}'J)$ and $T^*(f) = f$ in $\mathcal{S}et$, we have $T^*(\mathscr{D})_f \subset T^*(\mathscr{D}_f)$. Assume that T is cocontinuous. For $\varphi \in T^*(\mathscr{D}_f) \cap (FT)_Y(U) = \mathscr{D}_f \cap F_Y(T(U))$, there exists $R \in J(T(U))$ such that, for each $h \in R$, $\varphi F(h) : F(\mathrm{dom}(h)) \to Y$ is a constant map or there exists $\psi \in \mathscr{D} \cap F_X(\mathrm{dom}(h))$ which satisfies $\varphi F(h) = f\psi$ by (2.4). Then, $R^T \in J'(U)$ and, for any $k \in R^T$, since $T(k) \in R(T(\mathrm{dom}(k)))$, $\varphi F(T(k)) : FT(\mathrm{dom}(k)) \to Y$ is a constant map or there exists $\rho \in \mathscr{D} \cap F_X(T(\mathrm{dom}(k)))$ which satisfies $\varphi F(T(k)) = f\rho$. Since $\mathscr{D} \cap F_X(T(\mathrm{dom}(k))) = T^*(\mathscr{D}) \cap (FT)_X(\mathrm{dom}(k))$, it follows from (2.4) that $\varphi \in T^*(\mathscr{D})_f \cap (FT)_Y(U)$. Thus $T^*(\mathscr{D}_f)$ coincides with $T^*(\mathscr{D})_f$. □

Proposition 5.5 *For a family $(\mathscr{D}_i)_{i\in I}$ of the-ologies on a set X, $T^*\Big(\bigcap_{i\in I} \mathscr{D}_i\Big) = \bigcap_{i\in I} T^*(\mathscr{D}_i)$ holds.*

Proof. For an object U of $\mathcal{C}'$, we have the following equality.

$$T^*\Big(\bigcap_{i\in I}\mathscr{D}_i\Big)\cap(FT)_X(U)=\Big(\bigcap_{i\in I}\mathscr{D}_i\Big)\cap F_X(T(U))=\bigcap_{i\in I}(\mathscr{D}_i\cap F_X(T(U)))=\bigcap_{i\in I}(T^*(\mathscr{D}_i)\cap(FT)_X(U))$$
$$=\Big(\bigcap_{i\in I}T^*(\mathscr{D}_i)\Big)\cap(FT)_X(U)$$

Hence the result follows. □

Proposition 5.6 *$T^*:\mathscr{P}_F(\mathcal{C},J)\to\mathscr{P}_{FT}(\mathcal{C}',J')$ preserves limits. If T is cocontinuous, T^* preserves colimits.*

Proof. Let $f,g:(X,\mathscr{D})\to(Y,\mathscr{E})$ be morphisms in $\mathscr{P}_F(\mathcal{C},J)$. Put $Z=\{x\in X\,|\,f(x)=g(x)\}$ and denote by $e:Z\to X$ the inclusion map. Then $e:(Z,\mathscr{D}^e)\to(X,\mathscr{D})$ is an equalizer of f and g in $\mathscr{P}_F(\mathcal{C},J)$ by (2.19). Since $T^*(\mathscr{D}^e)=T^*(\mathscr{D})^e$ by (5.4), it follows that $T^*(e)=e:(Z,T^*(\mathscr{D}^e))\to(X,T^*(\mathscr{D}))$ is an equalizer of $T^*(f)=f:(X,T^*(\mathscr{D}))\to(Y,T^*(\mathscr{E}))$ and $T^*(g)=g:(X,T^*(\mathscr{D}))\to(Y,T^*(\mathscr{E}))$.

Let $\{(X_i,\mathscr{D}_i)\}_{i\in I}$ be a family of objects of $\mathscr{P}_F(\mathcal{C},J)$ and denote by $\mathrm{pr}_j:\prod_{i\in I}X_i\to X_j$ the projection to the j-th component. Then, $\Big(\Big(\prod_{i\in I}X_i,\bigcap_{i\in I}\mathscr{D}_i^{\mathrm{pr}_i}\Big)\xrightarrow{\mathrm{pr}_i}(X_i,\mathscr{D}_i)\Big)_{i\in I}$ is a product of $\{(X_i,\mathscr{D}_i)\}_{i\in I}$ by (2.15). Since $T^*\Big(\bigcap_{i\in I}\mathscr{D}_i^{\mathrm{pr}_i}\Big)=\bigcap_{i\in I}T^*(\mathscr{D}_i^{\mathrm{pr}_i})=\bigcap_{i\in I}T^*(\mathscr{D}_i)^{\mathrm{pr}_i}$ by (5.5) and (5.4), $T^*\Big(\prod_{i\in I}X_i,\bigcap_{i\in I}\mathscr{D}_i^{\mathrm{pr}_i}\Big)=\Big(\prod_{i\in I}X_i,\bigcap_{i\in I}T^*(\mathscr{D}_i)^{\mathrm{pr}_i}\Big)$ holds, which shows that $T^*:\mathscr{P}_F(\mathcal{C},J)\to\mathscr{P}_{FT}(\mathcal{C}',J')$ preserves products.

Assume that T is cocontinuous. For morphisms $f,g:(X,\mathscr{D})\to(Y,\mathscr{E})$ in $\mathscr{P}_F(\mathcal{C},J)$, let $q:Y\to W$ be a coequalizer of f and g in $\mathcal{S}et$. Then $q:(Y,\mathscr{E})\to(W,\mathscr{E}_q)$ is a coequalizer of f and g in $\mathscr{P}_F(\mathcal{C},J)$ by (2.19). Since $\varGamma_{FT}(T^*(h))=\varGamma_F(h)$ for any morphism h in $\mathscr{P}_F(\mathcal{C},J)$, $q:(Y,T^*(\mathscr{E}))\to(W,T^*(\mathscr{E})_q)$ is a coequalizer of $T^*(f)$ and $T^*(g)$. Since $T^*(\mathscr{E})_q=T^*(\mathscr{E}_q)$ by (5.4), it follows that $T^*(q):(Y,T^*(\mathscr{E}))\to(W,T^*(\mathscr{E}_q))$ is a coequalizer of $T^*(f)$ and $T^*(g)$. Thus T^* preserves coequalizers.

Let $(X_i,\mathscr{D}_i)$ $(i\in i)$ be objects of $\mathscr{P}_F(\mathcal{C},J)$. We denote by $\iota_j:X_j\to\coprod_{i\in I}X_i$ the inclusion to the i-th summand. Let $\mathscr{D}_I$ be the finest the-ology with respect to F and $(\mathcal{C},J)$ on $\coprod_{i\in I}X_j$ such that $\iota_j:(X_j,\mathscr{D}_j)\to\Big(\coprod_{i\in I}X_i,\mathscr{D}_I\Big)$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$ for any $j\in I$. Similarly, let $T^*(\mathscr{D})_I$ be the finest the-ology with respect to FT and $(\mathcal{C}',J')$ on $\coprod_{i\in I}X_j$ such that $T^*(\iota_j):(X_j,T^*(\mathscr{D}_j))\to\Big(\coprod_{i\in I}X_i,T^*(\mathscr{D})_I\Big)$ is a morphism in $\mathscr{P}_{FT}(\mathcal{C}',J')$ for any $j\in I$. Since $T^*(\iota_j):(X_j,T^*(\mathscr{D}_j))\to\Big(\coprod_{i\in I}X_i,T^*(\mathscr{D}_I)\Big)$ is a morphism in $\mathscr{P}_{FT}(\mathcal{C}',J')$ for any $j\in I$, we have $T^*(\mathscr{D})_I\subset T^*(\mathscr{D}_I)$. For $U\in\mathrm{Ob}\,\mathcal{C}'$ and $x\in T^*(\mathscr{D}_I)\cap(FT)_{\coprod_{i\in I}X_i}(U)=\mathscr{D}_I\cap F_{\coprod_{i\in I}X_i}(T(U))$, there exists $R\in J(T(U))$ such that, for any $g\in R$, $F_{\coprod_{i\in I}X_i}(g)(x)\in(\mathscr{D}_i)_{\iota_i}$ holds for some $i\in I$. Since T is cocontinuous, R^T belongs to $J'(U)$. For any $f\in R^T$, since $T(f)\in R$, we have $F_{\coprod_{i\in I}X_i}(T(f))(x)\in(\mathscr{D}_i)_{\iota_i}\cap F_{\coprod_{i\in I}X_i}(T(\mathrm{dom}(f)))$ for some $i\in I$. Since $F_{\coprod_{i\in I}X_i}(T(f))(x)=x(FT)(f)=(FT)_{\coprod_{i\in I}X_i}(f)(x)$ and $T^*(\mathscr{D}_i)_{\iota_i}=T^*((\mathscr{D}_i)_{\iota_i})$ by (2) of (5.2), it follows that $(FT)_{\coprod_{i\in I}X_i}(f)(x)$ belongs to

$$(\mathscr{D}_i)_{\iota_i}\cap F_{\coprod_{i\in I}X_i}(T(\mathrm{dom}(f)))=T^*((\mathscr{D}_i)_{\iota_i})\cap(FT)_{\coprod_{i\in I}X_i}(\mathrm{dom}(f))=T^*(\mathscr{D}_i)_{\iota_i}\cap(FT)_{\coprod_{i\in I}X_i}(\mathrm{dom}(f)).$$

Therefore we have $x\in T^*(\mathscr{D})_I\cap(FT)_{\coprod_{i\in I}X_i}(U)$ and we conclude that $T^*(\mathscr{D})_I=T^*(\mathscr{D}_I)$, that is, T^* preserves coproducts. □

For a set X, let $T^*_X:\mathscr{P}_F(\mathcal{C},J)_X\to\mathscr{P}_{FT}(\mathcal{C}',J')_X$ be the functor obtained from $T^*:\mathscr{P}_F(\mathcal{C},J)\to\mathscr{P}_{FT}(\mathcal{C}',J')$ by restricting the source and the target.

Proposition 5.7 *$T^*_X:\mathscr{P}_F(\mathcal{C},J)_X\to\mathscr{P}_{FT}(\mathcal{C}',J')_X$ preserves the terminal object. If T is cocontinuous, it also preserves the initial object.*

Proof. We denote by $\mathscr{D}'_{coarse,X}$ the terminal object of $\mathscr{P}_{FT}(\mathcal{C}',J')_X$. It follows from the definition of T^* that we have the following equality which shows that T^*_X preserves the terminal object.

$$T^*(\mathscr{D}_{coarse,X})=\coprod_{U\in\mathrm{Ob}\,\mathcal{C}'}\Big(\coprod_{V\in\mathrm{Ob}\,\mathcal{C}}F_X(V)\Big)\cap F_X(T(U))=\coprod_{U\in\mathrm{Ob}\,\mathcal{C}'}F_X(T(U))=\coprod_{U\in\mathrm{Ob}\,\mathcal{C}'}(FT)_X(U)=\mathscr{D}'_{coarse,X}$$

Let us denote by $\mathscr{D}'_{disc,X}$ the initial object of $\mathscr{P}_{FT}(\mathcal{C}',J')_X$. Then, we have $\mathscr{D}'_{disc,X}\subset T^*(\mathscr{D}_{disc,X})$. For

$U \in \mathrm{Ob}\,\mathcal{C}'$, $T^*(\mathscr{D}_{disc,X}) \cap (FT)_X(U) = \mathscr{D}_{disc,X} \cap F_X(T(U))$ coincides with the following set by (1.14).

$$\big\{x \in F_X(T(U)) \,\big|\, \text{There exists } R \in J(T(U)) \text{ such that } F_X(g)(x) \text{ is a contant map for all } g \in R.\big\}$$

For $x \in T^*(\mathscr{D}_{disc,X}) \cap (FT)_X(U)$, there exists $R \in J(T(U))$ such that $F_X(g)(x)$ is a contant map for all $g \in R$. If we assume that T is cocontinuous, then $R^T \in J'(U)$ and for any $h \in R^T$, $(FT)_X(h)(x) = F_X(T(h))(x)$ is a constant map since $T(h) \in R(T(\mathrm{dom}(h)))$. Thus we see that $T^*(\mathscr{D}_{disc,X}) \cap (FT)_X(U)$ is contained in $\mathscr{D}'_{disc,X} \cap (FT)_X(U)$ for any $U \in \mathrm{Ob}\,\mathcal{C}$. □

Since $T^*(\{1\}, \mathscr{D}_{coarse,\{1\}}) = (\{1\}, \mathscr{D}'_{coarse,\{1\}})$ and $T^*(\{0,1\}, \mathscr{D}_{coarse,\{0,1\}}) = (\{0,1\}, \mathscr{D}'_{coarse,\{0,1\}})$ by (5.8), we have the following result by (4.10).

Corollary 5.8 *$T^* : \mathscr{P}_F(\mathcal{C}, J) \to \mathscr{P}_{FT}(\mathcal{C}', J')$ preserves strong subobject classifiers.*

For a functor $\Psi : \mathcal{E} \to \mathcal{D}$, we define a functor $\Psi^{(2)} : \mathcal{E}^{(2)} \to \mathcal{D}^{(2)}$ by $\Psi^{(2)}(\boldsymbol{E}) = (\Psi(E) \xrightarrow{\Psi(\pi)} \Psi(X))$ for an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathrm{Ob}\,\mathcal{E}^{(2)}$ and $\Psi^{(2)}(\boldsymbol{\varphi}) = \langle \Psi(\xi) : \Psi(E) \to \Psi(D), \Psi(\varphi) : \Psi(X) \to \Psi(Y)\rangle$ for objects $\boldsymbol{E} = (E \xrightarrow{\pi} X)$, $\boldsymbol{D} = (D \xrightarrow{\rho} Y)$ of $\mathcal{C}^{(2)}$ and a morphism $\boldsymbol{\varphi} = \langle \xi : E \to D, \varphi : X \to Y\rangle : \boldsymbol{E} \to \boldsymbol{D}$ in $\mathcal{E}^{(2)}$. For an object X of $\mathcal{E}$, we denote by $\Psi^{(2)}_X : \mathcal{E}^{(2)}_X \to \mathcal{D}^{(2)}_{\Psi(X)}$ a functor obtained from $\Psi^{(2)}$ by by restricting the source and the target.

Suppose that $\mathcal{E}$ and $\mathcal{D}$ are categories with finite limits. For an object $\boldsymbol{D} = (D \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}$ and a morphism $\varphi : X \to Y$ in $\mathcal{E}$, we consider the following cartesian squares.

$$\begin{array}{ccc} D \times_Y X & \xrightarrow{\varphi_\rho} & D \\ \big\downarrow{\scriptstyle \rho_\varphi} & & \big\downarrow{\scriptstyle \rho} \\ X & \xrightarrow{\varphi} & Y \end{array} \qquad\qquad \begin{array}{ccc} \Psi(D) \times_{\Psi(Y)} \Psi(X) & \xrightarrow{\Psi(\varphi)_{\Psi(\rho)}} & \Psi(D) \\ \big\downarrow{\scriptstyle \Psi(\rho)_{\Psi(\varphi)}} & & \big\downarrow{\scriptstyle \Psi(\rho)} \\ \Psi(X) & \xrightarrow{\Psi(\varphi)} & \Psi(Y) \end{array}$$

We note that $\varphi^*(\boldsymbol{D}) = (D \times_Y X \xrightarrow{\rho_\varphi} X)$ and $\Psi(\varphi)^*(\Psi^{(2)}_Y(\boldsymbol{D})) = (\Psi(D) \times_{\Psi(Y)} \Psi(X) \xrightarrow{\Psi(\rho)_{\Psi(\varphi)}} \Psi(X))$ holds. If we put $\boldsymbol{X} = (X \xrightarrow{\varphi} Y)$, a product $\boldsymbol{D} \times \boldsymbol{X}$ of $\boldsymbol{D}$ and $\boldsymbol{X}$ in $\mathcal{E}^{(2)}_Y$ and a product $\Psi^{(2)}_Y(\boldsymbol{D}) \times \Psi^{(2)}_Y(\boldsymbol{X})$ of $\Psi^{(2)}_Y(\boldsymbol{D})$ and $\Psi^{(2)}_Y(\boldsymbol{X})$ in $\mathcal{D}^{(2)}_{\Psi(Y)}$ are given as follows.

$$\boldsymbol{D} \times \boldsymbol{X} = (D \times_Y X \xrightarrow{\varphi\rho_\varphi} Y), \quad \Psi^{(2)}_Y(\boldsymbol{D}) \times \Psi^{(2)}_Y(\boldsymbol{X}) = (\Psi(D) \times_{\Psi(Y)} \Psi(X) \xrightarrow{\Psi(\varphi)\Psi(\rho)_{\Psi(\varphi)}} \Psi(Y))$$

The unique morphism $(\Psi(\varphi_\rho), \Psi(\rho_\varphi)) : \Psi(D \times_Y X) \to \Psi(D) \times_{\Psi(Y)} \Psi(X)$ in $\mathcal{D}$ that makes the following diagram commute defines morphisms $(\Psi_\varphi)_{\boldsymbol{D}} : \Psi^{(2)}_X \varphi^*(\boldsymbol{D}) \to \Psi(\varphi)^* \Psi^{(2)}_Y(\boldsymbol{D})$ and $\Psi^\times_{\boldsymbol{D},\boldsymbol{X}} : \Psi^{(2)}_Y(\boldsymbol{D} \times \boldsymbol{X}) \to \Psi^{(2)}_Y(\boldsymbol{D}) \times \Psi^{(2)}_Y(\boldsymbol{X})$ in $\mathcal{D}^{(2)}_{\Psi(X)}$ and $\mathcal{D}^{(2)}_{\Psi(Y)}$, respectively.

$$\begin{array}{cccc} \Psi(D \times_Y X) & & \xrightarrow{\qquad\qquad\Psi(\varphi_\rho)\qquad\qquad} & \\ & \overset{(\Psi(\varphi_\rho),\Psi(\rho_\varphi))}{\dashrightarrow} \; \Psi(D) \times_{\Psi(Y)} \Psi(X) & \xrightarrow{\Psi(\varphi)_{\Psi(\rho)}} & \Psi(D) \\ {\scriptstyle \Psi(\rho_\varphi)}\searrow & \big\downarrow{\scriptstyle \Psi(\rho)_{\Psi(\varphi)}} & & \big\downarrow{\scriptstyle \Psi(\rho)} \\ & \Psi(X) & \xrightarrow{\Psi(\varphi)} & \Psi(Y) \end{array}$$

For a category $\mathcal{C}$ with products, we denote by $P_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a functor given by $P_{\mathcal{C}}(X, Y) = X \times Y$ for $(X, Y) \in \mathrm{Ob}\,(\mathcal{C} \times \mathcal{C})$ and $P_{\mathcal{C}}(f, g) = f \times g$ and $(f, g) \in \mathrm{Mor}\,(\mathcal{C} \times \mathcal{C})$. Then, we have natural transformations $\Psi_\varphi : \Psi^{(2)}_X \varphi^* \to \Psi(\varphi)^* \Psi^{(2)}_Y$ and $\Psi^\times_Y : \Psi^{(2)}_Y P_{\mathcal{E}^{(2)}_Y} \to P_{\mathcal{D}^{(2)}_{\Psi(Y)}}(\Psi^{(2)}_Y \times \Psi^{(2)}_Y)$.

If Ψ preserves finite limits, then $(\Psi(\varphi_\rho), \Psi(\rho_\varphi)) : \Psi(D \times_Y X) \to \Psi(D) \times_{\Psi(Y)} \Psi(X)$ is an isomorphism which implies that $\Psi_\varphi : \Psi^{(2)}_X \varphi^* \to \Psi(\varphi)^* \Psi^{(2)}_Y$ and $\Psi^\times_Y : \Psi^{(2)}_Y P_{\mathcal{E}^{(2)}_Y} \to P_{\mathcal{D}^{(2)}_{\Psi(Y)}}(\Psi^{(2)}_Y \times \Psi^{(2)}_Y)$ are natural equivalences.

We assume that Ψ preserves finite limits below. Suppose that the inverse image functors $\varphi^* : \mathcal{E}^{(2)}_Y \to \mathcal{E}^{(2)}_X$ and $\Psi(\varphi)^* : \mathcal{D}^{(2)}_{\Psi(Y)} \to \mathcal{D}^{(2)}_{\Psi(X)}$ have right adjoints $\varphi_! : \mathcal{E}^{(2)}_X \to \mathcal{E}^{(2)}_Y$ and $\Psi(\varphi)_! : \mathcal{D}^{(2)}_{\Psi(X)} \to \mathcal{D}^{(2)}_{\Psi(Y)}$, respectively. We denote by $\varepsilon^\varphi : \varphi^* \varphi_! \to id_{\mathcal{E}^{(2)}_X}$ the counit of the adjunction $\varphi^* \dashv \varphi_!$. For an object $\boldsymbol{E} = (E \xrightarrow{\pi} X)$ of $\mathcal{E}^{(2)}_X$, let us define a morphism $\Psi^\varphi_{\boldsymbol{E}} : \Psi^{(2)}_Y \varphi_!(\boldsymbol{E}) \to \Psi(\varphi)_! \Psi^{(2)}_X(\boldsymbol{E})$ to the adjoint of a composition

$$\Psi(\varphi)^*\Psi^{(2)}_Y\varphi_!(\boldsymbol{E}) \xrightarrow{(\Psi_\varphi)^{-1}_{\varphi_!(\boldsymbol{E})}} \Psi^{(2)}_X\varphi^*\varphi_!(\boldsymbol{E}) \xrightarrow{\Psi^{(2)}_X(\varepsilon^\varphi_{\boldsymbol{E}})} \Psi^{(2)}_X(\boldsymbol{E})$$

with respect to the adjunction $\Psi(\varphi)^* \dashv \Psi(\varphi)_!$. Since $\Psi^\varphi_{\boldsymbol{E}}$ is natural in $\boldsymbol{E}$, we have a natural transformation $\Psi^\varphi : \Psi^{(2)}_Y\varphi_! \to \Psi(\varphi)_!\Psi^{(2)}_X$.

For an object $\boldsymbol{D} = (D \xrightarrow{\rho} Y)$ of $\mathcal{E}^{(2)}_Y$, we define a morphism $\tilde{\Psi}^\varphi_{\boldsymbol{D}} : \Psi^{(2)}_Y\varphi_!\varphi^*(\boldsymbol{D}) \to \Psi(\varphi)_!\Psi(\varphi)^*\Psi^{(2)}_Y(\boldsymbol{D})$ to be the adjoint of a composition

$$\Psi(\varphi)^*\Psi^{(2)}_Y\varphi_!\varphi^*(\boldsymbol{D}) \xrightarrow{(\Psi_\varphi)^{-1}_{\varphi_!\varphi^*(\boldsymbol{D})}} \Psi^{(2)}_X\varphi^*\varphi_!\varphi^*(\boldsymbol{D}) \xrightarrow{\Psi^{(2)}_X(\varepsilon^\varphi_{\varphi^*(\boldsymbol{D})})} \Psi^{(2)}_X\varphi^*(\boldsymbol{D}) \xrightarrow{(\Psi_\varphi)_{\boldsymbol{D}}} \Psi(\varphi)^*\Psi^{(2)}_Y(\boldsymbol{D})$$

with respect to the adjunction $\Psi(\varphi)^* \dashv \Psi(\varphi)_!$. Since $\tilde{\Psi}^\varphi_{\boldsymbol{D}}$ is natural in $\boldsymbol{D}$, we have a natural transformation $\tilde{\Psi}^\varphi : \Psi^{(2)}_Y\varphi_!\varphi^* \to \Psi(\varphi)_!\Psi(\varphi)^*\Psi^{(2)}_Y$. The following diagram is commutative by the naturality of the adjunction $\Psi(\varphi)^* \dashv \Psi(\varphi)_!$.

$$\begin{array}{ccc}
\mathcal{D}^{(2)}_{\Psi(X)}(\Psi(\varphi)^*\Psi^{(2)}_Y\varphi_!\varphi^*(\boldsymbol{D}), \Psi^{(2)}_X\varphi^*(\boldsymbol{D})) & \xrightarrow[\cong]{adjunction\ \Psi(\varphi)^*\dashv\Psi(\varphi)_!} & \mathcal{D}^{(2)}_{\Psi(Y)}(\Psi^{(2)}_Y\varphi_!\varphi^*(\boldsymbol{D}), \Psi(\varphi)_!\Psi^{(2)}_X\varphi^*(\boldsymbol{D})) \\
\downarrow{\scriptstyle (\Psi_\varphi)_{\boldsymbol{D}*}} & & \downarrow{\scriptstyle \Psi(\varphi)_!((\Psi_\varphi)_{\boldsymbol{D}})_*} \\
\mathcal{D}^{(2)}_{\Psi(X)}(\Psi(\varphi)^*\Psi^{(2)}_Y\varphi_!\varphi^*(\boldsymbol{D}), \Psi(\varphi)^*\Psi^{(2)}_Y(\boldsymbol{D})) & \xrightarrow[\cong]{adjunction\ \Psi(\varphi)^*\dashv\Psi(\varphi)_!} & \mathcal{D}^{(2)}_{\Psi(Y)}(\Psi^{(2)}_Y\varphi_!\varphi^*(\boldsymbol{D}), \Psi(\varphi)_!\Psi(\varphi)^*\Psi^{(2)}_Y(\boldsymbol{D}))
\end{array}$$

It follows from the commutativity of the above diagram that we have $\tilde{\Psi}^\varphi_{\boldsymbol{D}} = \Psi(\varphi)_!((\Psi_\varphi)_{\boldsymbol{D}})\Psi^\varphi_{\varphi^*(\boldsymbol{D})}$. Since Ψ_φ is a natural equivalence, we have the following result.

Proposition 5.9 *If $\Psi^\varphi : \Psi^{(2)}_Y\varphi_! \to \Psi(\varphi)_!\Psi^{(2)}_X$ is a natural equivalence, so is $\tilde{\Psi}^\varphi : \Psi^{(2)}_Y\varphi_!\varphi^* \to \Psi(\varphi)_!\Psi(\varphi)^*\Psi^{(2)}_Y$.*

We are going to apply the above argument to the case $\mathcal{E} = \mathscr{P}_F(\mathcal{C}, J)$, $\mathcal{D} = \mathscr{P}_{FT}(\mathcal{C}', J)$ and $\Psi = T^*$.

Lemma 5.10 *Let $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{F})$ be a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ and $\boldsymbol{E}$ an object of $\mathscr{P}_F(\mathcal{C}, J)_{(Y,\mathscr{F})}$. Then, a morphism $(T^*_\varphi)_{\boldsymbol{E}} : T^{*(2)}_{(X,\mathscr{D})}\varphi^*(\boldsymbol{E}) \to T^*(\varphi)^*T^{*(2)}_{(Y,\mathscr{F})}(\boldsymbol{E})$ in $\mathscr{P}_{FT}(\mathcal{C}', J')_{T^*(X,\mathscr{D})}$ is the identity morphism of $T^{*(2)}_{(X,\mathscr{D})}\varphi^*(\boldsymbol{E})$.*

Proof. Put $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\rho} (Y, \mathscr{F}))$. We consider the following cartesian diagram in $\mathcal{S}et$.

$$\begin{array}{ccc}
E \times_Y X & \xrightarrow{\varphi_\rho} & E \\
\downarrow{\scriptstyle \rho_\varphi} & & \downarrow{\scriptstyle \rho} \\
X & \xrightarrow{\varphi} & Y
\end{array}$$

Then, we have $T^{*(2)}_{(X,\mathscr{D})}\varphi^*(\boldsymbol{E}) = ((E \times_Y X, T^*(\mathscr{E}^{\varphi_\rho} \cap \mathscr{D}^{\rho_\varphi}) \xrightarrow{\rho_\varphi} (X, T^*(\mathscr{D})))$. The following diagram in $\mathscr{P}_{FT}(\mathcal{C}, J')$ is commutative and the lower rectangle is cartesian.

$$\begin{array}{ccccc}
(E \times_Y X, T^*(\mathscr{E}^{\varphi_\rho} \cap \mathscr{D}^{\rho_\varphi})) & & & & \\
& \searrow{\scriptstyle (\varphi_\rho, \rho_\varphi) = id_{E\times_Y X}} & & & \\
& & (E \times_Y X, T^*(\mathscr{E})^{\varphi_\rho} \cap T^*(\mathscr{D})^{\rho_\varphi}) & \xrightarrow{\varphi_\rho} & (E, T^*(\mathscr{E})) \\
& & \downarrow{\scriptstyle \rho_\varphi} & & \downarrow{\scriptstyle \rho} \\
& & (X, T^*(\mathscr{D})) & \xrightarrow{\varphi} & (Y, T^*(\mathscr{F}))
\end{array}$$

It follows that $T^*(\varphi)^*T^{*(2)}_{(Y,\mathscr{F})}(\boldsymbol{E}) = ((E \times_Y X, T^*(\mathscr{E})^{\varphi_\rho} \cap T^*(\mathscr{D})^{\rho_\varphi}) \xrightarrow{\rho_\varphi} (X, T^*(\mathscr{D})))$ holds. Hence the assertion from an equality $T^*(\mathscr{E}^{\varphi_\rho} \cap \mathscr{D}^{\rho_\varphi}) = T^*(\mathscr{E})^{\varphi_\rho} \cap T^*(\mathscr{D})^{\rho_\varphi}$ which is a consequence of (5.4) and (5.5). □

Let us define a "foregetful" functor $\mathit{\Gamma}^{(2)}_{FT} : \mathscr{P}_{FT}(\mathcal{C}', J')^{(2)} \to \mathcal{S}et^{(2)}$ by $\mathit{\Gamma}^{(2)}_{FT}((E, \mathscr{E}) \xrightarrow{\pi} (X, \mathscr{D})) = (E \xrightarrow{\pi} X)$ and $\mathit{\Gamma}^{(2)}_{FT}(\langle\xi, f\rangle : ((E, \mathscr{E}) \xrightarrow{\pi} (X, \mathscr{X})) \to ((F, \mathscr{F}) \xrightarrow{\rho} (Y, \mathscr{Y}))) = (\langle\xi, f\rangle : (E \xrightarrow{\pi} X) \to (F \xrightarrow{\rho} Y))$.

For a category $\mathcal{E}$, we denote by $\wp'_{\mathcal{E}} : \mathcal{E}^{(2)} \to \mathcal{E}$ a functor defined by $\wp'_{\mathcal{E}}(E \xrightarrow{\pi} B) = E$ and $\wp'_{\mathcal{E}}(\langle\xi, f\rangle) = \xi$.

Proposition 5.11 *Let $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{F})$ be a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, and $\boldsymbol{E}$ an object of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}$. $\mathit{\Gamma}^{(2)}_{FT}(T^{*\varphi}_{\boldsymbol{E}}) : \mathit{\Gamma}^{(2)}_{FT}T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E}) \to \mathit{\Gamma}^{(2)}_{FT}T^*(\varphi)_!T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{E})$ is the identity morphism of $\mathit{\Gamma}^{(2)}_{FT}T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E})$.*

Proof. We use the same notation as in section 3, where we denote by $\boldsymbol{\varepsilon}^{\varphi} : \varphi^*\varphi_! \to id_{\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}}$ the counit of the adjunction $\varphi^* \dashv \varphi_!$. We also denote by $\boldsymbol{\eta}^{T(\varphi)} : id_{\mathscr{P}_{FT}(\mathcal{C}',J')^{(2)}_{T^*(Y,\mathscr{F})}} \to T^*(\varphi)_! T^*(\varphi)^*$ the unit of the adjunction $T(\varphi)^* \dashv T(\varphi)_!$. Let $\boldsymbol{E} = ((E,\mathscr{E}) \xrightarrow{\pi} (X,\mathscr{D}))$ be an object of $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{D})}$. It follows from the definition of $T^{*\varphi} : T^{*(2)}_{(Y,\mathscr{F})}\varphi_! \to T^*(\varphi)_! T^{*(2)}_{(X,\mathscr{D})}$, $T^{*\varphi}_{\boldsymbol{E}} : T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E}) \to T^*(\varphi)_! T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{E})$ is the the following composition.

$$T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E}) \xrightarrow{\boldsymbol{\eta}^{T(\varphi)}_{T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E})}} T^*(\varphi)_! T^*(\varphi)^* T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E}) \xrightarrow{T^*(\varphi)_!((T^*_{\varphi})^{-1}_{\varphi_!(\boldsymbol{E})})} T^*(\varphi)_! T^{*(2)}_{(X,\mathscr{D})}\varphi^*\varphi_!(\boldsymbol{E})$$
$$\xrightarrow{T^*(\varphi)_! T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{\varepsilon}^{\varphi}_{\boldsymbol{E}})} T^*(\varphi)_! T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{E})$$

Recall that $\varphi_!(\boldsymbol{E})$ is define to be $((E(\varphi),\mathscr{D}_{\boldsymbol{E},\varphi}) \xrightarrow{\varphi_{!\boldsymbol{E}}} (Y,\mathscr{F}))$. Hence we have the following equality

$$T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E}) = ((E(\varphi), T^*(\mathscr{D}_{\boldsymbol{E},\varphi})) \xrightarrow{\varphi_{!\boldsymbol{E}}} (Y, T^*(\mathscr{F})))$$

The following diagram in $\mathscr{P}_{FT}(\mathcal{C},J')$ is cartesian.

$$\begin{array}{ccc}
(E(\varphi)\times_Y X, T^*(\mathscr{D}_{\boldsymbol{E},\varphi})^{\tilde{\varphi}_{\boldsymbol{E}}} \cap T^*(\mathscr{D})^{\widetilde{\varphi_{!\boldsymbol{E}}}}) & \xrightarrow{\tilde{\varphi}_{\boldsymbol{E}}} & (E(\varphi), T^*(\mathscr{D}_{\boldsymbol{E},\varphi})) \\
\downarrow{\scriptstyle \widetilde{\varphi_{!\boldsymbol{E}}}} & & \downarrow{\scriptstyle \varphi_{!\boldsymbol{E}}} \\
(X, T^*(\mathscr{D})) & \xrightarrow{\varphi} & (Y, T^*(\mathscr{F}))
\end{array}$$

Thus we have $T^*(\varphi)^* T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E}) = \big(((E(\varphi)\times_Y X, T^*(\mathscr{D}_{\boldsymbol{E},\varphi})^{\tilde{\varphi}_{\boldsymbol{E}}} \cap T^*(\mathscr{D})^{\widetilde{\varphi_{!\boldsymbol{E}}}}) \xrightarrow{\widetilde{\varphi_{!\boldsymbol{E}}}} (X, T^*(\mathscr{D}))\big)$ and the image of $T^*(\varphi)^* T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E})$ by $T^*(\varphi)_! : \mathscr{P}_{FT}(\mathcal{C}',J)_{T^*(X,\mathscr{D})} \to \mathscr{P}_{FT}(\mathcal{C}',J)_{T^*(Y,\mathscr{F})}$ is given by

$$T^*(\varphi)_! T^*(\varphi)^* T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E}) = \Big(((E(\varphi)\times_Y X)(\varphi), \mathscr{D}_{T^*(\varphi)^* T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E}),\varphi}) \xrightarrow{\varphi_{!T^*(\varphi)^* T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E})}} (Y, T^*(\mathscr{F}))\Big).$$

Since $T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{E}) = ((E, T^*(\mathscr{E})) \xrightarrow{\pi} (X, T^*(\mathscr{D})))$, $T^*(\varphi)_! T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{E})$ is given by

$$T^*(\varphi)_! T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{E}) = ((E(\varphi), \mathscr{D}_{T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{E}),\varphi}) \xrightarrow{\varphi_{!\boldsymbol{E}}} (Y, T^*(\mathscr{F})))$$

We note that $T^*(\varphi)_!((T^*_{\varphi})^{-1}_{\varphi_!(\boldsymbol{E})}) : T^*(\varphi)_! T^*(\varphi)^* T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E}) \to T^*(\varphi)_! T^{*(2)}_{(X,\mathscr{D})}\varphi^*\varphi_!(\boldsymbol{E})$ is the identity morphism of $T^*(\varphi)_! T^*(\varphi)^* T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E})$ by (5.10). We have the following equalities.

$$\wp'_{\mathscr{P}_F(\mathcal{C},J)}\Big(\boldsymbol{\eta}^{T(\varphi)}_{T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E})}\Big) = \Big(\eta^{\varphi}_{T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E})} : (E(\varphi), T^*(\mathscr{D}_{\boldsymbol{E},\varphi})) \to ((E(\varphi)\times_Y X)(\varphi), \mathscr{D}_{T^*(\varphi)^* T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E}),\varphi})\Big)$$
$$\wp'_{\mathscr{P}_F(\mathcal{C},J)}\big(T^*(\varphi)_! T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{\varepsilon}^{\varphi}_{\boldsymbol{E}})\big) = \Big((\varepsilon^{\varphi}_{\boldsymbol{E}})_{\varphi} : ((E(\varphi)\times_Y X)(\varphi), \mathscr{D}_{T^{*(2)}_{(X,\mathscr{D})}\varphi^*\varphi_!(\boldsymbol{E}),\varphi}) \to \big(E(\varphi), \mathscr{D}_{T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{E}),\varphi}\big)\Big)$$

Hence a morphism $\wp'_{\mathscr{P}_F(\mathcal{C},J)}(T^{*\varphi}_{\boldsymbol{E}}) : \wp'_{\mathscr{P}_F(\mathcal{C},J)} T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E}) \to \wp'_{\mathscr{P}_F(\mathcal{C},J)} T^*(\varphi)_! T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{E})$ is a composition.

$$(E(\varphi), T^*(\mathscr{D}_{\boldsymbol{E},\varphi})) \xrightarrow{\eta^{\varphi}_{T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E})}} \big((E(\varphi)\times_Y X)(\varphi), \mathscr{D}_{T^*(\varphi)^* T^{*(2)}_{(Y,\mathscr{F})}\varphi_!(\boldsymbol{E}),\varphi}\big) \xrightarrow{(\varepsilon^{\varphi}_{\boldsymbol{E}})_{\varphi}} \big(E(\varphi), \mathscr{D}_{T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{E}),\varphi}\big).$$

It follows from (3.16) that the image of the above composition by the forgetful functor $\varGamma_{FT} : \mathscr{P}_{FT}(\mathcal{C},,J') \to \mathcal{S}et$ is the identity map of $E(\varphi)$. Since $\wp'_{\mathcal{S}et}\varGamma^{(2)}_{FT} = \varGamma_{FT}\wp'_{\mathscr{P}_F(\mathcal{C},J)}$, the assertion follows. □

Remark 5.12 *It follows from the above result the the-ology $\mathscr{D}_{T^{*(2)}_{(X,\mathscr{D})}(\boldsymbol{E}),\varphi}$ on $E(\varphi)$ is coarser than $T^*(\mathscr{D}_{\boldsymbol{E},\varphi})$.*

Let $F, F' : \mathcal{C} \to \mathcal{S}et$ be functors and $\Phi : F \to F'$ be a natural transformation. We assume that both $F(1_{\mathcal{C}})$ and $F'(1_{\mathcal{C}})$ consist of single element. For a the-ology $\mathscr{D}$ on a set X with respect to F and $(\mathcal{C},J)$, we define a subset $\Phi_*(\mathscr{D})$ of $\coprod_{U\in \mathrm{Ob}\,\mathcal{C}} F'_X(U)$ by $\Phi_*(\mathscr{D}) \cap F'_X(U) = \{x \in F'_X(U) \,|\, x\Phi_U \in \mathscr{D}\cap F_X(U)\}$.

Proposition 5.13 *$\Phi_*(\mathscr{D})$ is a the-ology on X with respect to F' and $(\mathcal{C},J)$. For a morphism $\varphi : (X,\mathscr{D}) \to (Y,\mathscr{E})$ in $\mathscr{P}_F(\mathcal{C},J)$, $\varphi : (X,\Phi_*(\mathscr{D})) \to (Y,\Phi_*(\mathscr{E}))$ is a morphism in $\mathscr{P}_{F'}(\mathcal{C},J)$.*

Proof. Since $\mathscr{D} \supset F_X(1_{\mathcal{C}})$, we have $\Phi_*(\mathscr{D}) \cap F'_X(1_{\mathcal{C}}) = \{x \in F'_X(1_{\mathcal{C}}) \,|\, x\Phi_{1_{\mathcal{C}}} \in F_X(1_{\mathcal{C}})\} = F'_X(1_{\mathcal{C}})$. Hence $\Phi_*(\mathscr{D})$ contains $F'_X(1_{\mathcal{C}})$. For a morphism $f : U \to V$ in $\mathcal{C}$ and $x \in \Phi_*(\mathscr{D}) \cap F'_X(V)$, we have

$$F'_X(f)(x)\Phi_U = xF'(f)\Phi_U = x\Phi_V F(f) = F_X(f)(x\Phi_V) \in \mathscr{D} \cap F_X(U)$$

since $x\Phi_V \in \mathscr{D} \cap F_X(V)$ and $\mathscr{D}$ is a the-ology on X with respect to F and $(\mathcal{C}, J)$. Thus $F'_X(f)(x)$ belongs to $\Phi_*(\mathscr{D}) \cap F'_X(U)$. For $U \in \mathrm{Ob}\,\mathcal{C}$ and $x \in \Phi_*(\mathscr{D}) \cap F'_X(U)$, suppose that there exists a covering $(U_i \xrightarrow{f_i} U)_{i \in I}$ of U such that $F'_X(f_i)(x) \in \Phi_*(\mathscr{D}) \cap F'_X(U_i)$ for any $i \in I$. Then, we have

$$F_X(f_i)(x\Phi_U) = x\Phi_U F(f_i) = xF'(f_i)\Phi_{U_i} = F'_X(f_i)(x)\Phi_{U_i} \in \mathscr{D} \cap F_X(U_i)$$

for any $i \in I$. Since $\mathscr{D}$ is a the-ology on X, $x\Phi_U$ belongs to $\mathscr{D} \cap F_X(U)$, hence $x \in \Phi_*(\mathscr{D}) \cap F'_X(U)$. Therefore $\Phi_*(\mathscr{D})$ is a the-ology on X with respect to F' and $(\mathcal{C}, J)$.

For $U \in \mathrm{Ob}\,\mathcal{C}$ and $x \in \Phi_*(\mathscr{D}) \cap F'_X(U)$, since $x\Phi_U \in \mathscr{D} \cap F_X(U)$ and $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, $\varphi x\Phi_U = (F_\varphi)_U(x\Phi_U) \in \mathscr{E} \cap F_Y(U)$ holds. Hence we have $(F'_\varphi)_U(x) = \varphi x \in \Phi_*(\mathscr{E}) \cap F'_Y(U)$ and $\varphi : (X, \Phi_*(\mathscr{D})) \to (Y, \Phi_*(\mathscr{E}))$ is a morphism in $\mathscr{P}_{F'}(\mathcal{C}, J)$. □

It follows from (5.13) that we can define a functor $\Phi_* : \mathscr{P}_F(\mathcal{C}, J) \to \mathscr{P}_{F'}(\mathcal{C}, J)$ by $\Phi_*(X, \mathscr{D}) = (X, \Phi_*(\mathscr{D}))$ and $\Phi_*(\varphi : (X, \mathscr{D}) \to (Y, \mathscr{E})) = (\varphi : (X, \Phi_*(\mathscr{D})) \to (Y, \Phi_*(\mathscr{E})))$.

Proposition 5.14 *Let $f : X \to Y$ be a map. For a the-ology $\mathscr{E}$ on Y with respect to F and $(\mathcal{C}, J)$, a the-ology $\Phi_*(\mathscr{E}^f)$ on X with respect to F' and $(\mathcal{C}, J)$ coincides with $\Phi_*(\mathscr{E})^f$.*

Proof. Let U be an object of $\mathcal{C}$. The following equality shows $\Phi_*(\mathscr{E}^f) = \Phi_*(\mathscr{E})^f$.

$$\begin{aligned}\Phi_*(\mathscr{E}^f) \cap F'_X(U) &= \{x \in F'_X(U) \,|\, x\Phi_U \in \mathscr{E}^f \cap F_X(U)\} = \{x \in F'_X(U) \,|\, fx\Phi_U \in \mathscr{E} \cap F_Y(U)\}\\ &= \{x \in F'_X(U) \,|\, fx \in \Phi_*(\mathscr{E})\} = \Phi_*(\mathscr{E})^f \cap F'_X(U)\end{aligned}$$

□

Proposition 5.15 *For a family $(\mathscr{D}_i)_{i \in I}$ of the-ologies on a set X, $\Phi_*\Big(\bigcap_{i \in I} \mathscr{D}_i\Big) = \bigcap_{i \in I} \Phi_*(\mathscr{D}_i)$ holds.*

Proof. For an object U of $\mathcal{C}$, we have the following equality.

$$\begin{aligned}\Phi_*\Big(\bigcap_{i \in I} \mathscr{D}_i\Big) \cap F'_X(U) &= \{x \in F'_X(U) \,|\, x\Phi_U \in \mathscr{D}_i \cap F_X(U) \text{ for any } i \in I.\}\\ &= \{x \in F'_X(U) \,|\, x \in \Phi_*(\mathscr{D}_i) \text{ for any } i \in I.\} = \Big(\bigcap_{i \in I} \Phi_*(\mathscr{D}_i)\Big) \cap F'_X(U)\end{aligned}$$

Hence the result follows. □

Proposition 5.16 *$\Phi_* : \mathscr{P}_F(\mathcal{C}, J) \to \mathscr{P}_{F'}(\mathcal{C}, J)$ preserves limits.*

Proof. Let $f, g : (X, \mathscr{D}) \to (Y, \mathscr{E})$ be morphisms in $\mathscr{P}_F(\mathcal{C}, J)$. Put $Z = \{x \in X \,|\, f(x) = g(x)\}$ and denote by $e : Z \to X$ the inclusion map. Then $e : (Z, \mathscr{D}^e) \to (X, \mathscr{D})$ is an equalizer of f and g in $\mathscr{P}_F(\mathcal{C}, J)$ by (2.19). Since $\Phi_*(\mathscr{D}^e) = \Phi_*(\mathscr{D})^e$ by (5.14), it follows that $\Phi_*(e) = e : (Z, \Phi_*(\mathscr{D}^e)) \to (X, \Phi_*(\mathscr{D}))$ is an equalizer of $\Phi_*(f) = f : (X, \Phi_*(\mathscr{D})) \to (Y, \Phi_*(\mathscr{E}))$ and $\Phi_*(g) = g : (X, \Phi_*(\mathscr{D})) \to (Y, \Phi_*(\mathscr{E}))$.

Let $\{(X_i, \mathscr{D}_i)\}_{i \in I}$ be a family of objects of $\mathscr{P}_F(\mathcal{C}, J)$ and denote by $\mathrm{pr}_j : \prod_{i \in I} X_i \to X_j$ the projection to the j-th component. Then, $\Big(\Big(\prod_{i \in I} X_i, \bigcap_{i \in I} \mathscr{D}_i^{\mathrm{pr}_i}\Big) \xrightarrow{\mathrm{pr}_i} (X_i, \mathscr{D}_i)\Big)_{i \in I}$ is a product of $\{(X_i, \mathscr{D}_i)\}_{i \in I}$ by (2.15). Since $\Phi_*\Big(\bigcap_{i \in I} \mathscr{D}_i^{\mathrm{pr}_i}\Big) = \bigcap_{i \in I} \Phi_*(\mathscr{D}_i^{\mathrm{pr}_i}) = \bigcap_{i \in I} \Phi_*(\mathscr{D}_i)^{\mathrm{pr}_i}$ by (5.15) and (5.5), $\Phi_*\Big(\prod_{i \in I} X_i, \bigcap_{i \in I} \mathscr{D}_i^{\mathrm{pr}_i}\Big) = \Big(\prod_{i \in I} X_i, \bigcap_{i \in I} \Phi_*(\mathscr{D}_i)^{\mathrm{pr}_i}\Big)$ holds, which shows that $\Phi_* : \mathscr{P}_F(\mathcal{C}, J) \to \mathscr{P}_{F'}(\mathcal{C}, J)$ preserves products. □

6 Groupoids associated with epimorphisms

Let $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ be an object $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(B,\mathscr{B})}$ such that π is an epimorphism. Then, π is surjective by (4.12), hence $\pi^{-1}(x)$ is not an empty set for any $x \in B$. We denote by $i_x : \pi^{-1}(x) \to E$ the inclusion map. We define a set $G_1(\boldsymbol{E})(x, y)$ for $x, y \in B$ by

$$G_1(\boldsymbol{E})(x, y) = \{\varphi \in \mathscr{P}_F(\mathcal{C}, J)((\pi^{-1}(x), \mathscr{E}^{i_x}), (\pi^{-1}(y), \mathscr{E}^{i_y})) \,|\, \varphi \text{ is an isomorphism.}\}$$

Put $G_1(\boldsymbol{E}) = \coprod_{x,y \in B} G_1(\boldsymbol{E})(x, y)$ and define maps $\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}} : G_1(\boldsymbol{E}) \to B$, $\iota_{\boldsymbol{E}} : G_1(\boldsymbol{E}) \to G_1(\boldsymbol{E})$ and $\varepsilon_{\boldsymbol{E}} : B \to G_1(\boldsymbol{E})$ by $\sigma_{\boldsymbol{E}}(\varphi) = x$, $\tau_{\boldsymbol{E}}(\varphi) = y$, $\iota_{\boldsymbol{E}}(\varphi) = \varphi^{-1}$ if $\varphi \in G_1(\boldsymbol{E})(x, y)$ and $\varepsilon_{\boldsymbol{E}}(x) = id_{\pi^{-1}(x)}$. Let

$$\begin{array}{ccc} G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E}) & \xrightarrow{\mathrm{pr}_2} & G_1(\boldsymbol{E}) \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \sigma_{\boldsymbol{E}}} \\ G_1(\boldsymbol{E}) & \xrightarrow{\tau_{\boldsymbol{E}}} & B \end{array}$$

be a cartesian square. In other words, $G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E})$ is given by

$$G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E}) = \{(\varphi,\psi) \in G_1(\boldsymbol{E}) \times G_1(\boldsymbol{E}) \,|\, \tau_{\boldsymbol{E}}(\varphi) = \sigma_{\boldsymbol{E}}(\psi)\}$$

as a set. We define a map $\mu_{\boldsymbol{E}} : G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E}) \to G_1(\boldsymbol{E})$ by $\mu_{\boldsymbol{E}}(\varphi,\psi) = \psi\varphi$.

We consider the following cartesian squares.

$$\begin{array}{ccc} E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) & \xrightarrow{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}} & G_1(\boldsymbol{E}) \\ \downarrow{\scriptstyle \mathrm{pr}^{\sigma}_E} & & \downarrow{\scriptstyle \sigma_{\boldsymbol{E}}} \\ E & \xrightarrow{\pi} & B \end{array} \qquad \begin{array}{ccc} E \times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E}) & \xrightarrow{\mathrm{pr}^{\tau}_{G_1(\boldsymbol{E})}} & G_1(\boldsymbol{E}) \\ \downarrow{\scriptstyle \mathrm{pr}^{\tau}_E} & & \downarrow{\scriptstyle \tau_{\boldsymbol{E}}} \\ E & \xrightarrow{\pi} & B \end{array}$$

Hence $E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})$ and $E \times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E})$ are given as follows as sets.

$$E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) = \{(e,\varphi) \in E \times G_1(\boldsymbol{E}) \,|\, \pi(e) = \sigma_{\boldsymbol{E}}(\varphi)\}, \quad E \times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E}) = \{(e,\varphi) \in E \times G_1(\boldsymbol{E}) \,|\, \pi(e) = \tau_{\boldsymbol{E}}(\varphi)\}$$

There exists unique map $id_E \times_B \iota_{\boldsymbol{E}} : E \times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \to E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})$ that makes the following diagram commute.

$$\begin{array}{ccccc} E \times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E}) & & \xrightarrow{\quad \mathrm{pr}^{\tau}_{G_1(\boldsymbol{E})} \quad} & & G_1(\boldsymbol{E}) \\ & \searrow^{id_E \times_B \iota_{\boldsymbol{E}}} & & \swarrow^{\iota_{\boldsymbol{E}}} & \\ & E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) & \xrightarrow{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}} & G_1(\boldsymbol{E}) & \\ {\scriptstyle \mathrm{pr}^{\tau}_E} \searrow & \downarrow{\scriptstyle \mathrm{pr}^{\sigma}_E} & & \downarrow{\scriptstyle \sigma_{\boldsymbol{E}}} & \swarrow {\scriptstyle \tau_{\boldsymbol{E}}} \\ & E & \xrightarrow{\pi} & B & \end{array}$$

We define a map $\hat{\xi}_{\boldsymbol{E}} : E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \to E$ by $\hat{\xi}_{\boldsymbol{E}}(e,\varphi) = i_{\tau_{\boldsymbol{E}}(\varphi)}\varphi(e)$. Let $\Sigma_{\boldsymbol{E}}$ the set of all the-ologies $\mathscr{L}$ on $G_1(\boldsymbol{E})$ which satisfy $\mathscr{E}^{\mathrm{pr}^{\sigma}_E} \cap \mathscr{L}^{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}} \subset \mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}}$, $\mathscr{E}^{\mathrm{pr}^{\tau}_E} \cap \mathscr{L}^{\mathrm{pr}^{\tau}_{G_1(\boldsymbol{E})}} \subset \mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}(id_E \times_B \iota_{\boldsymbol{E}})}$ and $\mathscr{L} \subset \mathscr{B}^{\sigma_{\boldsymbol{E}}} \cap \mathscr{B}^{\tau_{\boldsymbol{E}}}$. We note that the $\mathscr{L} \in \Sigma_{\boldsymbol{E}}$ if and only if following maps are morphisms in $\mathscr{P}_F(\mathcal{C}, J)$.

$$\hat{\xi}_{\boldsymbol{E}} : \left(E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^{\sigma}_E} \cap \mathscr{L}^{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}}\right) \to (E, \mathscr{E})$$
$$\hat{\xi}_{\boldsymbol{E}}(id_E \times_B \iota_{\boldsymbol{E}}) : \left(E \times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^{\tau}_E} \cap \mathscr{L}^{\mathrm{pr}^{\tau}_{G_1(\boldsymbol{E})}}\right) \to (E, \mathscr{E})$$
$$\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}} : (G_1(\boldsymbol{E}), \mathscr{L}) \to (B, \mathscr{B})$$

Proposition 6.1 *$\Sigma_{\boldsymbol{E}}$ is not empty.*

Proof. It suffices to show that the discrete the-ology $\mathscr{D}_{disc,G_1(\boldsymbol{E})}$ on $G_1(\boldsymbol{E})$ belongs to $\Sigma_{\boldsymbol{E}}$. It follows from (1.15) that $\mathscr{D}_{disc,G_1(\boldsymbol{E})} \subset \mathscr{B}^{\sigma_{\boldsymbol{E}}} \cap \mathscr{B}^{\tau_{\boldsymbol{E}}}$ holds. For $U \in \mathrm{Ob}\,\mathcal{C}$, suppose that $\psi \in \mathscr{E}^{\mathrm{pr}^{\sigma}_E} \cap \mathscr{D}_{disc,G_1(\boldsymbol{E})}^{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}} \cap F_{E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})}(U)$. Then,we have $\mathrm{pr}^{\sigma}_E\psi \in \mathscr{E} \cap F_E(U)$ and $\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}\psi \in \mathscr{D}_{disc,G_1(\boldsymbol{E})} \cap F_{G_1(\boldsymbol{E})}(U)$. Hence there exists a covering $(U_j \xrightarrow{g_j} U)_{i \in J}$ such that $F_{G_1(\boldsymbol{E})}(g_j)(\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}\psi) : F(U_j) \to G_1(\boldsymbol{E})$ is a constant map for every $i \in J$ by (1.15). Let us denote by $\alpha_j \in G_1(\boldsymbol{E})$ the image of $F_{G_1(\boldsymbol{E})}(g_j)(\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}\psi)$ and put $x_j = \sigma_{\boldsymbol{E}}(\alpha_j)$, $y_j = \tau_{\boldsymbol{E}}(\alpha_j)$. Then we have $\alpha_j \in G_1(\boldsymbol{E})(x_j, y_j)$ and the image of $F_E(g_j)(\mathrm{pr}^{\sigma}_E\psi) = \mathrm{pr}^{\sigma}_E\psi F(g_j) : F(U_j) \to E$ is contained in $\pi^{-1}(x_j)$. Hence we have a map $\zeta_j : F(U_j) \to \pi^{-1}(x_j)$ satisfying $i_{x_j}\zeta_j = F_E(g_j)(\mathrm{pr}^{\sigma}_E\psi) \in \mathscr{E} \cap F_E(U_j)$, which shows $\zeta_j \in \mathscr{E}^{i_{x_j}} \cap F_{\pi^{-1}(x_j)}(U_j)$. Since we have an equality

$$F_{E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})}(g_j)(\psi) = (i_{x_j}\zeta_j, F_{G_1(\boldsymbol{E})}(g_j)(\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}\psi)) : F(U_j) \to E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}),$$

it follows that the following equality holds.

$$F_E(g_j)(F_{\hat{\xi}_{\boldsymbol{E}}}(\psi)) = F_{\hat{\xi}_{\boldsymbol{E}}}(F_{E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})}(g_j)(\psi)) = \hat{\xi}_{\boldsymbol{E}}(i_{x_j}\zeta_j, F_{G_1(\boldsymbol{E})}(g_j)(\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}\psi)) = i_{y_j}\alpha_j\zeta_j = F_{i_{y_j}}(F_{\alpha_j}(\zeta_j))$$

Since $\alpha_j : (\pi^{-1}(x_j), \mathscr{E}^{i_{x_j}}) \to (\pi^{-1}(y_j), \mathscr{E}^{i_{y_j}})$ and $i_{y_j} : (\pi^{-1}(y_j), \mathscr{E}^{i_{y_j}}) \to (E, \mathscr{E})$ are morphisms in $\mathscr{P}_F(\mathcal{C}, J)$, we have $F_{i_{y_j}}(F_{\alpha_j}(\zeta_j)) \in \mathscr{E} \cap F_E(U_j)$ for any $i \in J$. Therefore $F_{\hat{\xi}_{\boldsymbol{E}}}(\psi) \in \mathscr{E} \cap F_E(U)$ holds and we see that $\mathscr{E}^{\mathrm{pr}^{\sigma}_E} \cap \mathscr{D}_{disc,G_1(\boldsymbol{E})}^{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}} \subset \mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}}$ holds.

For $U \in \mathrm{Ob}\,\mathcal{C}$, suppose that $\psi \in \mathscr{E}^{\mathrm{pr}^\tau_E} \cap \mathscr{D}^{\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}}_{disc,G_1(\boldsymbol{E})} \cap F_{E\times^{\tau_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})}(U)$. Then, we have $\mathrm{pr}^\tau_E\psi \in \mathscr{E} \cap F_E(U)$ and $\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}\psi \in \mathscr{D}_{disc,G_1(\boldsymbol{E})} \cap F_{G_1(\boldsymbol{E})}(U)$. Hence there exists a covering $(U_j \xrightarrow{g_j} U)_{i\in J}$ such that $F_{G_1(\boldsymbol{E})}(g_j)(\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}\psi) : F(U_j) \to G_1(\boldsymbol{E})$ is a constant map for every $i \in J$ by (1.15). We denote by $\alpha_j \in G_1(\boldsymbol{E})$ the image of $F_{G_1(\boldsymbol{E})}(g_j)(\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}\psi)$ and put $x_j = \sigma_{\boldsymbol{E}}(\alpha_j)$, $y_j = \tau_{\boldsymbol{E}}(\alpha_j)$. Then we have $\alpha_j \in G_1(\boldsymbol{E})(x_j, y_j)$ and the image of $F_E(g_j)(\mathrm{pr}^\tau_E\psi) = \mathrm{pr}^\tau_E\psi F(g_j) : F(U_j) \to E$ is contained in $\pi^{-1}(y_j)$. Hence we have a map $\zeta_j : F(U_j) \to \pi^{-1}(y_j)$ satisfying $i_{y_j}\zeta_j = F_E(g_j)(\mathrm{pr}^\tau_E\psi) \in \mathscr{E} \cap F_E(U_j)$, which shows $\zeta_j \in \mathscr{E}^{i_{y_j}} \cap F_{\pi^{-1}(y_j)}(U_j)$. Since we have an equality

$$F_{E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})}(g_j)(\psi) = (i_{y_j}\zeta_j, F_{G_1(\boldsymbol{E})}(g_j)(\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}\psi)) : F(U_j) \to E \times^{\tau_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}),$$

it follows that the following equality holds.

$$\begin{aligned} F_E(g_j)(F_{\hat{\xi}_{\boldsymbol{E}}(id_E\times_B\iota_{\boldsymbol{E}})}(\psi)) &= F_{\hat{\xi}_{\boldsymbol{E}}(id_E\times_B\iota_{\boldsymbol{E}})}(F_{E\times^{\tau_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})}(g_j)(\psi)) = \hat{\xi}_{\boldsymbol{E}}(id_E\times_B\iota_{\boldsymbol{E}})(i_{y_j}\zeta_j, F_{G_1(\boldsymbol{E})}(g_j)(\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}\psi)) \\ &= \hat{\xi}_{\boldsymbol{E}}(i_{y_j}\zeta_j, \iota_{\boldsymbol{E}}F_{G_1(\boldsymbol{E})}(g_j)(\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}\psi)) = i_{x_j}\alpha_j^{-1}\zeta_j = F_{i_{x_j}}(F_{\alpha_j^{-1}}(\zeta_j)) \end{aligned}$$

Since $\alpha_j^{-1} : (\pi^{-1}(y_j), \mathscr{E}^{i_{y_j}}) \to (\pi^{-1}(x_j), \mathscr{E}^{i_{x_j}})$ and $i_{x_j} : (\pi^{-1}(x_j), \mathscr{E}^{i_{x_j}}) \to (E, \mathscr{E})$ are morphisms in $\mathscr{P}_F(\mathcal{C}, J)$, we have $F_{i_{x_j}}(F_{\alpha_j^{-1}}(\zeta_j)) \in \mathscr{E} \cap F_E(U_j)$ for any $i \in J$. Therefore $F_{\hat{\xi}_{\boldsymbol{E}}(id_E\times_B\iota_{\boldsymbol{E}})}(\psi) \in \mathscr{E} \cap F_E(U)$ holds and we see that $\mathscr{E}^{\mathrm{pr}^\tau_E} \cap \mathscr{D}^{\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}}_{disc,G_1(\boldsymbol{E})} \subset \mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}(id_E\times_B\iota_{\boldsymbol{E}})}$ holds. □

For $U \in \mathrm{Ob}\,\mathcal{C}$, we consider the following conditions $(G1)$, $(G2)$, $(G3)$ on an element γ of $F_{G_1(\boldsymbol{E})}(U)$.

$(G1)$ If $V, W \in \mathrm{Ob}\,\mathcal{C}$, $f \in \mathcal{C}(W, U)$, $g \in \mathcal{C}(W, V)$ and $\lambda \in \mathscr{E} \cap F_E(V)$ satisfy $\pi\lambda F(g) = \sigma_{\boldsymbol{E}}\gamma F(f)$, a composition $F(W) \xrightarrow{(\lambda F(g),\, \gamma F(f))} E \times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E} \cap F_E(W)$.

$(G2)$ If $V, W \in \mathrm{Ob}\,\mathcal{C}$, $f \in \mathcal{C}(W, U)$, $g \in \mathcal{C}(W, V)$ and $\lambda \in \mathscr{E} \cap F_E(V)$ satisfy $\pi\lambda F(g) = \tau_{\boldsymbol{E}}\gamma F(f)$, a composition $F(W) \xrightarrow{(\lambda F(g),\, \iota_{\boldsymbol{E}}\gamma F(f))} E \times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E} \cap F_E(W)$.

$(G3)$ Compositions $F(U) \xrightarrow{\gamma} G_1(\boldsymbol{E}) \xrightarrow{\sigma_{\boldsymbol{E}}} B$ and $F(U) \xrightarrow{\gamma} G_1(\boldsymbol{E}) \xrightarrow{\tau_{\boldsymbol{E}}} B$ belong to $\mathscr{B} \cap F_B(U)$.

Define a set $\mathscr{G}_{\boldsymbol{E}}$ of F-parametrizations of a set $G_1(\boldsymbol{E})$ so that $\mathscr{G}_{\boldsymbol{E}} \cap F_{G_1(\boldsymbol{E})}(U)$ is a subset of $F_{G_1(\boldsymbol{E})}(U)$ consisting of elements which satisfy the above conditions $(G1)$, $(G2)$ and $(G3)$ for any $U \in \mathrm{Ob}\,\mathcal{C}$.

Remark 6.2 *The conditions $(G1)$, $(G2)$ and $(G3)$ on $\gamma \in F_{G_1(\boldsymbol{E})}(U)$ above are equivalent to the following conditions $(G1')$, $(G2')$ and $(G3')$, respectively.*

$(G1')$ *If $V, W \in \mathrm{Ob}\,\mathcal{C}$, $f \in \mathcal{C}(W, U)$, $g \in \mathcal{C}(W, V)$ and $\lambda \in \mathscr{E} \cap F_E(V)$ satisfy $\pi\lambda F(g) = \sigma_{\boldsymbol{E}}\gamma F(f)$, then γ satisfies $((\lambda F(g), \gamma F(f)) : F(W) \to E \times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})) \in \mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}} \cap F_{E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})}(W)$.*

$(G2')$ *If $V, W \in \mathrm{Ob}\,\mathcal{C}$, $f \in \mathcal{C}(W, U)$, $g \in \mathcal{C}(W, V)$ and $\lambda \in \mathscr{E} \cap F_E(V)$ satisfy $\pi\lambda F(g) = \tau_{\boldsymbol{E}}\gamma F(f)$, then γ satisfies $((\lambda F(g), \gamma F(f)) : F(W) \to E \times^{\tau_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})) \in \mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}(id_E\times_B\iota_{\boldsymbol{E}})} \cap F_{E\times^{\tau_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})}(W)$.*

$(G3')$ *$\gamma \in \mathscr{B}^{\sigma_{\boldsymbol{E}}} \cap \mathscr{B}^{\tau_{\boldsymbol{E}}} \cap F_{G_1(\boldsymbol{E})}(U)$*

Proposition 6.3 *$\mathscr{G}_{\boldsymbol{E}}$ is a the-ologgy on $G_1(\boldsymbol{E})$.*

Proof. For $\gamma \in F_{G_1(\boldsymbol{E})}(1_{\mathcal{C}})$, put $s = \sigma_{\boldsymbol{E}}(\gamma(*))$, $t = \tau_{\boldsymbol{E}}(\gamma(*))$. We take $V, W \in \mathrm{Ob}\,\mathcal{C}$, $o_W \in \mathcal{C}(W, 1_{\mathcal{C}})$, $g \in \mathcal{C}(W, V)$. Assume that $\lambda \in \mathscr{E} \cap F_E(V)$ satisfies $\pi\lambda F(g) = \sigma_{\boldsymbol{E}}\gamma F(o_W)$. Then, the image of $\lambda F(g) : F(W) \to E$ is contained in $\pi^{-1}(s)$ hence there exists a map $\zeta : F(W) \to \pi^{-1}(s)$ which satisfies $\lambda F(g) = i_s\zeta$. Since $\lambda F(g) \in \mathscr{E} \cap F_E(W)$, we have $\zeta \in \mathscr{E}^{i_s} \cap F_{\pi^{-1}(s)}(W)$. We note that $\gamma(*) : (\pi^{-1}(s), \mathscr{E}^{i_s}) \to (\pi^{-1}(t), \mathscr{E}^{i_t})$ and $i_t : (\pi^{-1}(t), \mathscr{E}^{i_t}) \to (E, \mathscr{E})$ are morphisms in $\mathscr{P}_F(\mathcal{C}, J)$. It follows that a composition $F(W) \xrightarrow{(\lambda F(g),\, \gamma F(o_W))} E \times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ coincides with a composition $F(W) \xrightarrow{\zeta} \pi^{-1}(s) \xrightarrow{\gamma(*)} \pi^{-1}(t) \xrightarrow{i_t} E$ which belongs to $\mathscr{E} \cap F_E(W)$. Therefore γ satisfies $(G1)$. Assume that $\lambda \in \mathscr{E} \cap F_E(V)$ satisfies $\pi\lambda F(g) = \tau_{\boldsymbol{E}}\gamma F(o_W)$. Then, the image of $\lambda F(g) : F(W) \to E$ is contained in $\pi^{-1}(t)$ hence there exists a map $\zeta : F(W) \to \pi^{-1}(t)$ which satisfies $\lambda F(g) = i_t\zeta$. Since $\lambda F(g) \in \mathscr{E} \cap F_E(W)$, we have $\zeta \in \mathscr{E}^{i_t} \cap F_{\pi^{-1}(t)}(W)$. Note that $\iota_{\boldsymbol{E}}(\gamma(*)) : (\pi^{-1}(t), \mathscr{E}^{i_t}) \to (\pi^{-1}(s), \mathscr{E}^{i_s})$ and $i_t : (\pi^{-1}(t), \mathscr{E}^{i_t}) \to (E, \mathscr{E})$ are morphisms in $\mathscr{P}_F(\mathcal{C}, J)$. It follows that a composition $F(W) \xrightarrow{(\lambda F(g),\, \iota_{\boldsymbol{E}}\gamma F(o_W))} E \times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ coincides with a composition $F(W) \xrightarrow{\zeta} \pi^{-1}(t) \xrightarrow{\iota_{\boldsymbol{E}}(\gamma(*))} \pi^{-1}(s) \xrightarrow{i_s} E$ which belongs to $\mathscr{E} \cap F_E(W)$. Therefore γ satisfies $(G2)$. Since $F_{\sigma_{\boldsymbol{E}}}(\gamma), F_{\tau_{\boldsymbol{E}}}(\gamma) \in F_B(1_{\mathcal{C}}) \subset \mathscr{B}$, we have $\gamma \in \mathscr{B}^{\sigma_{\boldsymbol{E}}} \cap \mathscr{B}^{\tau_{\boldsymbol{E}}}$. Hence γ satisfies $(G3)$. Thus we have $\mathscr{G}_{\boldsymbol{E}} \supset F_{G_1(\boldsymbol{E})}(1_{\mathcal{C}})$.

Let $h : Z \to U$ be a morphism in $\mathcal{C}$. For $\gamma \in \mathscr{G}_{\boldsymbol{E}} \cap F_{G_1(\boldsymbol{E})}(U)$, we take $V, W \in \mathrm{Ob}\,\mathcal{C}$, $f \in \mathcal{C}(W, Z)$ and $g \in \mathcal{C}(W, V)$. Assume that $\lambda \in \mathscr{E} \cap F_E(V)$ satisfies $\pi\lambda F(g) = \sigma_{\boldsymbol{E}}F_{G_1(\boldsymbol{E})}(h)(\gamma)F(f)$. Since $\pi\lambda F(g) = \sigma_{\boldsymbol{E}}\gamma F(hf)$

and γ satisfies $(G1)$, a composition $F(W) \xrightarrow{(\lambda F(g),\gamma F(hf))} E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E}\cap F_E(W)$. This shows that $F_{G_1(\boldsymbol{E})}(h)(\gamma)$ satisfies $(G1)$. Assume that $\lambda \in \mathscr{E}\cap F_E(V)$ satisfy $\pi\lambda F(g) = \tau_{\boldsymbol{E}} F_{G_1(\boldsymbol{E})}(h)(\gamma)F(f)$. Since $\pi\lambda F(g) = \tau_{\boldsymbol{E}}\gamma F(hf)$ and γ satisfies $(G2)$, a composition $F(W) \xrightarrow{(\lambda F(g),\iota_{\boldsymbol{E}}\gamma F(hf))} E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E}\cap F_E(W)$. This shows that $F_{G_1(\boldsymbol{E})}(h)(\gamma)$ satisfies $(G2)$. Since γ satisfies $(G2)$, compositions $F(Z) \xrightarrow{\gamma F(h)} G_1(\boldsymbol{E}) \xrightarrow{\sigma_{\boldsymbol{E}}} B$ and $F(U) \xrightarrow{\gamma F(h)} G_1(\boldsymbol{E}) \xrightarrow{\tau_{\boldsymbol{E}}} B$ belong to $\mathscr{B}\cap F_B(U)$, which implies that $F_{G_1(\boldsymbol{E})}(h)(\gamma) = \gamma F(h)$ satisfies $(G3)$. Thus we have $F_{G_1(\boldsymbol{E})}(h)(\gamma) = \gamma F(h) \in \mathscr{G}_{\boldsymbol{E}}\cap F_{G_1(\boldsymbol{E})}(Z)$.

For $\gamma\in F_{G_1(\boldsymbol{E})}(U)$, suppose that there exists $R\in J(U)$ such that $F_{G_1(\boldsymbol{E})}(j)(\gamma)\in\mathscr{G}_{\boldsymbol{E}}\cap F_{G_1(\boldsymbol{E})}(\mathrm{dom}(j))$ for any $j\in R$. We take $V,W\in\mathrm{Ob}\,\mathcal{C}$, $f\in\mathcal{C}(W,U)$ and $g\in\mathcal{C}(W,V)$. If we put

$$h_f^{-1}(R)=\{k\in\mathrm{Mor}\,\mathcal{C}\,|\,\mathrm{codom}(k)=W,\ fk\in R\},$$

then we have $h_f^{-1}(R)\in J(W)$ and $F_{G_1(\boldsymbol{E})}(fk)(\gamma)\in\mathscr{G}_{\boldsymbol{E}}\cap F_{G_1(\boldsymbol{E})}(\mathrm{dom}(k))$ for any $k\in h_f^{-1}(R)$. Assume that $\lambda\in\mathscr{E}\cap F_E(V)$ satisfies $\pi\lambda F(g) = \sigma_{\boldsymbol{E}}\gamma F(f)$. Hence the following composition belongs to $\mathscr{E}\cap F_E(W)$ for any $k\in h_f^{-1}(R)$.

$$F(\mathrm{dom}(k)) \xrightarrow{(\lambda F(gk),\, F_{G_1(\boldsymbol{E})}(fk)(\gamma))} E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$$

Since the above composition coincides with the following composition

$$F(\mathrm{dom}(k)) \xrightarrow{F(k)} F(W) \xrightarrow{(\lambda F(g),\,\gamma F(f))} E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$$

for any $k\in h_f^{-1}(R)$, it follows that a composition $F(W) \xrightarrow{(\lambda F(g),\,\gamma F(f))} E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E}\cap F_E(W)$, namely γ satisfies $(G1)$. Assume that $\lambda\in\mathscr{E}\cap F_E(V)$ satisfies $\pi\lambda F(g) = \tau_{\boldsymbol{E}}\gamma F(f)$. Hence the following composition belongs to $\mathscr{E}\cap F_E(W)$ for any $k\in h_f^{-1}(R)$.

$$F(\mathrm{dom}(k)) \xrightarrow{(\lambda F(gk),\,\iota_{\boldsymbol{E}} F_{G_1(\boldsymbol{E})}(fk)(\gamma))} E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$$

Since the above composition coincides with the following composition

$$F(\mathrm{dom}(k)) \xrightarrow{F(k)} F(W) \xrightarrow{(\lambda F(g),\,\iota_{\boldsymbol{E}}\gamma F(f))} E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$$

for any $k\in h_f^{-1}(R)$, it follows that a composition $F(W) \xrightarrow{(\lambda F(g),\,\iota_{\boldsymbol{E}}\gamma F(f))} E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E}\cap F_E(W)$, namely γ satisfies $(G2)$. Since $F_{G_1(\boldsymbol{E})}(j)(\gamma)\in\mathscr{G}_{\boldsymbol{E}}\cap F_{G_1(\boldsymbol{E})}(\mathrm{dom}(j))$ for any $j\in R$, compositions $F(\mathrm{dom}(j)) \xrightarrow{F_{G_1(\boldsymbol{E})}(j)(\gamma)} G_1(\boldsymbol{E}) \xrightarrow{\sigma_{\boldsymbol{E}}} B$ and $F(\mathrm{dom}(j)) \xrightarrow{F_{G_1(\boldsymbol{E})}(j)(\gamma)} G_1(\boldsymbol{E}) \xrightarrow{\tau_{\boldsymbol{E}}} B$ belong to $\mathscr{B}\cap F_B(\mathrm{dom}(j))$. Since the above compositions coincides with compositions $F(\mathrm{dom}(j)) \xrightarrow{F(j)} F(U) \xrightarrow{\gamma} G_1(\boldsymbol{E}) \xrightarrow{\sigma_{\boldsymbol{E}}} B$ and $F(\mathrm{dom}(j)) \xrightarrow{F(j)} F(U) \xrightarrow{\gamma} G_1(\boldsymbol{E}) \xrightarrow{\tau_{\boldsymbol{E}}} B$ respectively for any $j\in R$, it follows that compositions $F(U) \xrightarrow{\gamma} G_1(\boldsymbol{E}) \xrightarrow{\sigma_{\boldsymbol{E}}} B$ and $F(U) \xrightarrow{\gamma} G_1(\boldsymbol{E}) \xrightarrow{\tau_{\boldsymbol{E}}} B$ belong to $\mathscr{B}\cap F_B(U)$. Hence γ satisfies $(G3)$ and we have $\gamma\in\mathscr{G}_{\boldsymbol{E}}\cap F_{G_1(\boldsymbol{E})}(U)$. □

Proposition 6.4 *$\mathscr{G}_{\boldsymbol{E}}$ is maximum element of $\Sigma_{\boldsymbol{E}}$.*

Proof. For $U\in\mathrm{Ob}\,\mathcal{C}$ and $\delta\in\mathscr{E}^{\mathrm{pr}_E^\sigma}\cap\mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}}\cap F_{E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E})}(U)$, $\pi\mathrm{pr}_E^\sigma\delta = \sigma_{\boldsymbol{E}}\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}\delta$ holds and it follows from $\mathrm{pr}_E^\sigma\delta\in\mathcal{E}\cap F_E(U)$ and $\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}\delta\in\mathscr{G}_{\boldsymbol{E}}\cap F_{G_1(\boldsymbol{E})}(U)$ that the following composition belongs to $\mathscr{E}\cap F_E(U)$.

$$F(U) \xrightarrow{\delta = (\mathrm{pr}_E^\sigma\delta,\,\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}\delta)} E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$$

That is, we have $\delta\in\mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}}\cap F_{E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E})}(U)$. It follows that $\mathscr{E}^{\mathrm{pr}_E^\sigma}\cap\mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}}\subset\mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}}$ holds. For $U\in\mathrm{Ob}\,\mathcal{C}$ and $\delta'\in\mathscr{E}^{\mathrm{pr}_E^\tau}\cap\mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}}\cap F_{E\times_B^{\tau_{\boldsymbol{E}}}G_1(\boldsymbol{E})}(U)$, $\pi\mathrm{pr}_E^\tau\delta' = \tau_{\boldsymbol{E}}\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}\delta'$ holds and it follows from $\mathrm{pr}_E^\tau\delta'\in\mathcal{E}\cap F_E(U)$ and $\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}\delta'\in\mathscr{G}_{\boldsymbol{E}}\cap F_{G_1(\boldsymbol{E})}(U)$ that the following composition belongs to $\mathscr{E}\cap F_E(U)$.

$$F(U) \xrightarrow{(id_E\times_B\iota_{\boldsymbol{E}})\delta' = (\mathrm{pr}_E^\tau\delta',\,\iota_{\boldsymbol{E}}\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}\delta')} E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$$

That is, we have $\delta'\in\mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}(id_E\times_B\iota_{\boldsymbol{E}})}\cap F_{E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E})}(U)$. It follows that $\mathscr{E}^{\mathrm{pr}_E^\tau}\cap\mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}}\subset\mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}(id_E\times_B\iota_{\boldsymbol{E}})}$ holds. $\mathscr{G}_{\boldsymbol{E}}\subset\mathscr{B}^{\sigma_{\boldsymbol{E}}}\cap\mathscr{B}^{\tau_{\boldsymbol{E}}}$ holds by $(G3')$ of (6.2). Therefore $\mathscr{G}_{\boldsymbol{E}}$ belongs to $\Sigma_{\boldsymbol{E}}$.

Let $\mathscr{L}$ be an element of $\Sigma_{\boldsymbol{E}}$. For $U\in\mathrm{Ob}\,\mathcal{C}$ and $\gamma\in\mathscr{L}\cap F_{G_1(\boldsymbol{E})}(U)$, we take $V,W\in\mathrm{Ob}\,\mathcal{C}$, $f\in\mathcal{C}(W,U)$ and $g\in\mathcal{C}(W,V)$. Assume that $\lambda\in\mathscr{E}\cap F_E(V)$ satisfies $\pi\lambda F(g) = \sigma_{\boldsymbol{E}}\gamma F(f)$ and put $\delta = (\lambda F(g),\gamma F(f))$. Then we have $\mathrm{pr}_E^\sigma\delta = \lambda F(g)\in\mathscr{E}\cap F_E(W)$ and $\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}\delta = \gamma F(f)\in\mathscr{L}\cap F_{G_1(\boldsymbol{E})}(W)$. It follows that we have $\delta\in\mathscr{E}^{\mathrm{pr}_E^\sigma}\cap\mathscr{L}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}}\cap F_{E\times_B G_1(\boldsymbol{E})}(W)\subset\mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}}\cap F_{E\times_B G_1(\boldsymbol{E})}(W)$, which shows that γ satisfies $(G1)$. Assume that

$\lambda \in \mathscr{E} \cap F_E(V)$ satisfies $\pi\lambda F(g) = \tau_{\boldsymbol{E}}\gamma F(f)$ and put $\delta' = (\lambda F(g), \gamma F(f))$. Then we have $\mathrm{pr}^{\tau}_{E}\delta' = \lambda F(g) \in \mathscr{E} \cap F_E(W)$ and $\mathrm{pr}^{\tau}_{G_1(\boldsymbol{E})}\delta' = \gamma F(f) \in \mathscr{L} \cap F_{G_1(\boldsymbol{E})}(W)$. It follows that δ' belongs to $\mathscr{E}^{\mathrm{pr}^{\tau}_{E}} \cap \mathscr{L}^{\mathrm{pr}^{\tau}_{G_1(\boldsymbol{E})}} \cap F_{E\times_B G_1(\boldsymbol{E})}(W)$ which is contained in $\mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}(id_E\times_B \iota_{\boldsymbol{E}})} \cap F_{E\times_B G_1(\boldsymbol{E})}(W)$. This implies that γ satisfies $(G2)$. Since $\mathscr{L} \subset \mathscr{B}^{\sigma_{\boldsymbol{E}}} \cap \mathscr{B}^{\tau_{\boldsymbol{E}}}$, γ satisfies $(G3)$. Thus we have $\gamma \in \mathscr{G}_{\boldsymbol{E}}$ which implies $\mathscr{L} \subset \mathscr{G}_{\boldsymbol{E}}$. □

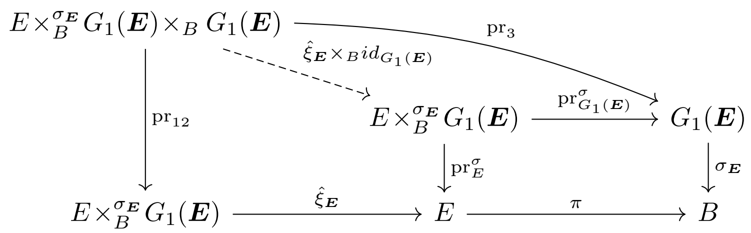
We consider the following cartesian square.

$$\begin{array}{ccc} E\times^{\sigma_{\boldsymbol{E}}}_{B} G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) & \xrightarrow{\mathrm{pr}_{12}} & E\times^{\sigma_{\boldsymbol{E}}}_{B} G_1(\boldsymbol{E}) \\ \Big\downarrow \mathrm{pr}_3 & & \Big\downarrow \tau_{\boldsymbol{E}}\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})} \\ G_1(\boldsymbol{E}) & \xrightarrow{\sigma_{\boldsymbol{E}}} & B \end{array} \qquad (i)$$

Then, we have $E\times^{\sigma_{\boldsymbol{E}}}_{B} G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) = \{(e,\varphi,\psi) \in E\times G_1(\boldsymbol{E})\times G_1(\boldsymbol{E}) \,|\, \pi(e) = \sigma_{\boldsymbol{E}}(\varphi),\ \tau_{\boldsymbol{E}}(\varphi) = \sigma_{\boldsymbol{E}}(\psi)\}$ as a set. It follows from the definition of $\hat{\xi}_{\boldsymbol{E}}$ that the following diagram is commutative.

$$\begin{array}{ccc} E\times^{\sigma_{\boldsymbol{E}}}_{B} G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E \\ \Big\downarrow \mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})} & & \Big\downarrow \pi \\ G_1(\boldsymbol{E}) & \xrightarrow{\tau_{\boldsymbol{E}}} & B \end{array} \qquad (ii)$$

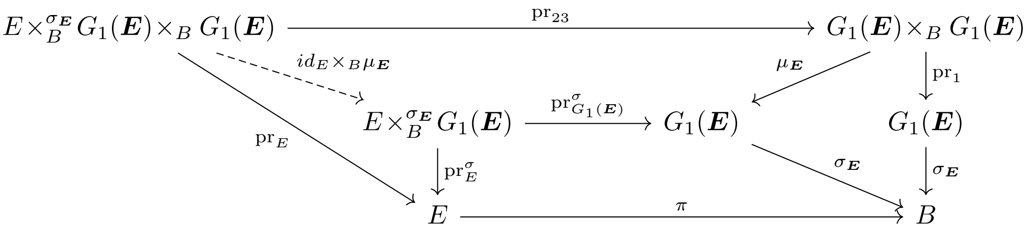
There exists unique map $\hat{\xi}_{\boldsymbol{E}} \times_B id_{G_1(\boldsymbol{E})} : E\times^{\sigma_{\boldsymbol{E}}}_{B} G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) \to E\times^{\sigma_{\boldsymbol{E}}}_{B} G_1(\boldsymbol{E})$ that makes the following diagram commute by the commutativity of diagrams (i) and (ii) above.



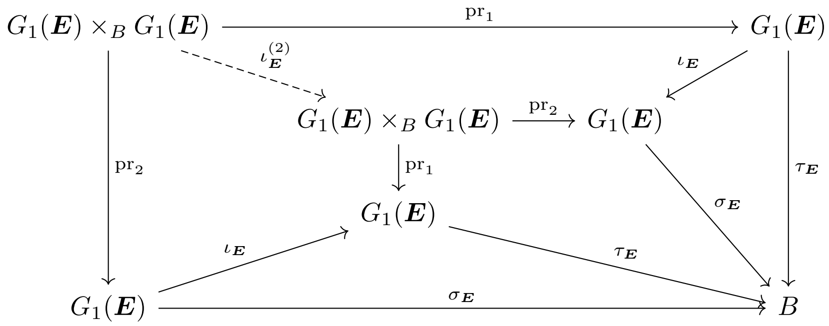
We define maps $\mathrm{pr}_{23} : E\times^{\sigma_{\boldsymbol{E}}}_{B} G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) \to G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E})$ and $\mathrm{pr}_E : E\times^{\sigma_{\boldsymbol{E}}}_{B} G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) \to E$ by $\mathrm{pr}_{23}(e,\varphi,\psi) = (\varphi,\psi)$ and $\mathrm{pr}_E(e,\varphi,\psi) = e$, respectively. Then, there exists unique map

$$id_E \times_B \mu_{\boldsymbol{E}} : E\times^{\sigma_{\boldsymbol{E}}}_{B} G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) \to E\times^{\sigma_{\boldsymbol{E}}}_{B} G_1(\boldsymbol{E})$$

that makes the following diagram commute.



Let $\iota^{(2)}_{\boldsymbol{E}} : G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E}) \to G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E})$ be the unique map that makes the following diagram commute.



We note that $\iota^{(2)}_{\boldsymbol{E}}$ maps $(\varphi,\psi)\in G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E})$ to $(\iota_{\boldsymbol{E}}(\psi),\iota_{\boldsymbol{E}}(\varphi))$. It is easy to verify the following fact.

Lemma 6.5 *The following diagrams are commutative.*

$$\begin{array}{ccc} E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) & \xrightarrow{id_E\times_B\mu_{\boldsymbol{E}}} & E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}) \\ \downarrow{\scriptstyle \hat{\xi}_{\boldsymbol{E}}\times_B id_{G_1(\boldsymbol{E})}} & & \downarrow{\scriptstyle \hat{\xi}_{\boldsymbol{E}}} \\ E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{id_E} & E \\ \downarrow{\scriptstyle (id_E,\,\varepsilon_{\boldsymbol{E}}\pi)} & \nearrow{\scriptstyle \hat{\xi}_{\boldsymbol{E}}} & \\ E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}) & & \end{array} \qquad \begin{array}{ccc} G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) & \xrightarrow{\mu_{\boldsymbol{E}}} & G_1(\boldsymbol{E}) \\ \downarrow{\scriptstyle \iota^{(2)}_{\boldsymbol{E}}} & & \downarrow{\scriptstyle \iota_{\boldsymbol{E}}} \\ G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) & \xrightarrow{\mu_{\boldsymbol{E}}} & G_1(\boldsymbol{E}) \end{array}$$

Proposition 6.6 *The structure maps $\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}}:(G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})\to(B,\mathscr{B})$, $\varepsilon_{\boldsymbol{E}}:(B,\mathscr{B})\to(G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})$, $\mu_{\boldsymbol{E}}:\big(G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}),\mathscr{G}^{\mathrm{pr}_1}_{\boldsymbol{E}}\cap\mathscr{G}^{\mathrm{pr}_2}_{\boldsymbol{E}}\big)\to(G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})$ and $\iota_{\boldsymbol{E}}:(G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})\to(G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})$ of the groupoid $(B,G_1(\boldsymbol{E}))$ are morphisms in $\mathscr{P}_F(\mathcal{C},J)$.*

Proof. It follows from $(G3)$ that $\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}}:(G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})\to(B,\mathscr{B})$ are morphisms in $\mathscr{P}_F(\mathcal{C},J)$. For $U\in\mathrm{Ob}\,\mathcal{C}$ and $x\in\mathscr{B}\cap F_B(U)$, we take $V,W\in\mathrm{Ob}\,\mathcal{C}$, $f\in\mathcal{C}(W,U)$ and $g\in\mathcal{C}(W,V)$. Assume that $\lambda\in\mathscr{E}\cap F_E(V)$ satisfies $\pi\lambda F(g)=\sigma_{\boldsymbol{E}}(F_{\varepsilon_{\boldsymbol{E}}})_U(x)F(f)$. It follows from the definitions of $\varepsilon_{\boldsymbol{E}}$ and $\hat{\xi}_{\boldsymbol{E}}$ that the composition

$$F(W)\xrightarrow{(\lambda F(g),\,(F_{\varepsilon_{\boldsymbol{E}}})_U(x)F(f))}E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\xrightarrow{\hat{\xi}_{\boldsymbol{E}}}E$$

coincides with $\lambda F(g)$ which belongs to $\mathscr{E}\cap F_E(W)$. Hence $(F_{\varepsilon_{\boldsymbol{E}}})_U(x)$ satisfies $(G1)$. Assume that $\lambda\in\mathscr{E}\cap F_E(V)$ satisfies $\pi\lambda F(g)=\tau_{\boldsymbol{E}}(F_{\varepsilon_{\boldsymbol{E}}})_U(x)F(f)$. It follows from the definitions of $\varepsilon_{\boldsymbol{E}}$ and $\hat{\xi}_{\boldsymbol{E}}$ that the composition

$$F(W)\xrightarrow{(\lambda F(g),\,(F_{\varepsilon_{\boldsymbol{E}}})_U(x)F(f))}E\times^{\tau_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\xrightarrow{id_E\times_B\iota_{\boldsymbol{E}}}E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\xrightarrow{\hat{\xi}_{\boldsymbol{E}}}E$$

coincides with $\lambda F(g)$ which belongs to $\mathscr{E}\cap F_E(W)$. It follows that $(F_{\varepsilon_{\boldsymbol{E}}})_U(x)$ satisfies $(G2)$. Since we have $\sigma_{\boldsymbol{E}}(F_{\varepsilon_{\boldsymbol{E}}})_U(x)=\tau_{\boldsymbol{E}}(F_{\varepsilon_{\boldsymbol{E}}})_U(x)=x\in\mathscr{B}\cap F_B(U)$, $(F_{\varepsilon_{\boldsymbol{E}}})_U(x)$ satisfies $(G3)$. Therefore $(F_{\varepsilon_{\boldsymbol{E}}})_U(x)$ belongs to $\mathscr{G}_{\boldsymbol{E}}\cap F_{G_1(\boldsymbol{E})}(U)$ and $\varepsilon_{\boldsymbol{E}}:(B,\mathscr{B})\to(G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$.

For $U\in\mathrm{Ob}\,\mathcal{C}$ and $\gamma\in\mathscr{G}_{\boldsymbol{E}}\cap F_{G_1(\boldsymbol{E})}(U)$, we take $V,W\in\mathrm{Ob}\,\mathcal{C}$, $f\in\mathcal{C}(W,U)$ and $g\in\mathcal{C}(W,V)$. Assume that $\lambda\in\mathscr{E}\cap F_E(V)$ satisfies $\pi\lambda F(g)=\sigma_{\boldsymbol{E}}(F_{\iota_{\boldsymbol{E}}})_U(\gamma)F(f)$. Then, $\pi\lambda F(g)=\tau_{\boldsymbol{E}}\gamma F(f)$ holds and a composition

$$F(W)\xrightarrow{(\lambda F(g),\,(F_{\iota_{\boldsymbol{E}}})_U(\gamma)F(f))}E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\xrightarrow{\hat{\xi}_{\boldsymbol{E}}}E$$

coincides with $F(W)\xrightarrow{(\lambda F(g),\,\iota_{\boldsymbol{E}}\gamma F(f))}E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\xrightarrow{\hat{\xi}_{\boldsymbol{E}}}E$ which belongs to $\mathscr{E}\cap F_E(W)$ since γ satisfies $(G2)$. Hence $(F_{\iota_{\boldsymbol{E}}})_U(\gamma)$ satisfies $(G1)$. Assume that $\lambda\in\mathscr{E}\cap F_E(V)$ satisfies $\pi\lambda F(g)=\tau_{\boldsymbol{E}}(F_{\iota_{\boldsymbol{E}}})_U(\gamma)F(f)$. Then, $\pi\lambda F(g)=\sigma_{\boldsymbol{E}}\gamma F(f)$ holds and a composition $F(W)\xrightarrow{(\lambda F(g),\,\iota_{\boldsymbol{E}}(F_{\iota_{\boldsymbol{E}}})_U(\gamma)F(f))}E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\xrightarrow{\hat{\xi}_{\boldsymbol{E}}}E$ coincides with

$$F(W)\xrightarrow{(\lambda F(g),\,\gamma F(f))}E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\xrightarrow{\hat{\xi}_{\boldsymbol{E}}}E$$

which belongs to $\mathscr{E}\cap F_E(W)$ since γ satisfies $(G1)$. Hence $(F_{\iota_{\boldsymbol{E}}})_U(\gamma)$ satisfies $(G2)$. Since γ satisfies $(G3)$, we have $\sigma_{\boldsymbol{E}}(F_{\iota_{\boldsymbol{E}}})_U(\gamma)=\tau_{\boldsymbol{E}}\in\mathscr{B}\cap F_B(U)$ and $\tau_{\boldsymbol{E}}(F_{\iota_{\boldsymbol{E}}})_U(\gamma)=\sigma_{\boldsymbol{E}}\in\mathscr{B}\cap F_B(U)$. Thus $(F_{\iota_{\boldsymbol{E}}})_U(\gamma)$ also satisfies $(G3)$ and $(F_{\iota_{\boldsymbol{E}}})_U(\gamma)\in\mathscr{G}_{\boldsymbol{E}}\cap F_{G_1(\boldsymbol{E})}(U)$. Therefore $\iota_{\boldsymbol{E}}:(G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})\to(G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$.

For $U\in\mathrm{Ob}\,\mathcal{C}$ and $(\alpha,\beta)\in\mathscr{G}^{\mathrm{pr}_1}_{\boldsymbol{E}}\cap\mathscr{G}^{\mathrm{pr}_2}_{\boldsymbol{E}}\cap F_{G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E})}(U)$, we take $V,W\in\mathrm{Ob}\,\mathcal{C}$, $f\in\mathcal{C}(W,U)$ and $g\in\mathcal{C}(W,V)$. We note that $\alpha,\beta\in\mathscr{G}_{\boldsymbol{E}}\cap F_{G_1(\boldsymbol{E})}(U)$ and that $\tau_{\boldsymbol{E}}\alpha=\sigma_{\boldsymbol{E}}\beta$ holds. Assume that $\lambda\in\mathscr{E}\cap F_E(V)$ satisfies $\pi\lambda F(g)=\sigma_{\boldsymbol{E}}(F_{\mu_{\boldsymbol{E}}})_U((\alpha,\beta))F(f)$. Since $(F_{\mu_{\boldsymbol{E}}})_U((\alpha,\beta))F(f)=\mu_{\boldsymbol{E}}(\alpha,\beta)F(f)$ holds, a composition

$$F(W)\xrightarrow{(\lambda F(g),\,(F_{\mu_{\boldsymbol{E}}})_U((\alpha,\beta))F(f))}E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\xrightarrow{\hat{\xi}_{\boldsymbol{E}}}E$$

coincides with the following composition.

$$F(W)\xrightarrow{(\lambda F(g),\,\alpha F(f),\,\beta F(f))}E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E})\xrightarrow{id_E\times_B\mu_{\boldsymbol{E}}}E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\xrightarrow{\hat{\xi}_{\boldsymbol{E}}}E$$

By the commutativity of the left diagram of (6.5), the above composition coincides with a composition

$$F(W)\xrightarrow{((F_{\hat{\xi}_{\boldsymbol{E}}})_W(\lambda F(g),\,\alpha F(f)),\,\beta F(f))}E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\xrightarrow{\hat{\xi}_{\boldsymbol{E}}}E.$$

Since $\hat{\xi}_{\boldsymbol{E}}:\big(E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}),\mathscr{E}^{\mathrm{pr}^\sigma_E}\cap\mathscr{G}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}}_{\boldsymbol{E}}\big)\to(E,\mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$ and $(\lambda F(g),\alpha F(f))$ belongs to $\mathscr{E}^{\mathrm{pr}^\sigma_E}\cap\mathscr{G}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}}_{\boldsymbol{E}}\cap F_{E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})}(W)$, the above composition belongs to $\mathscr{E}\cap F_E(W)$. Hence $(F_{\mu_{\boldsymbol{E}}})_U((\alpha,\beta))$ satisfies $(G1)$.

Assume that $\lambda\in\mathscr{E}\cap F_E(V)$ satisfies $\pi\lambda F(g)=\tau_{\boldsymbol{E}}(F_{\mu_{\boldsymbol{E}}})_U((\alpha,\beta))F(f)$. Since an equality

$$\iota_{\boldsymbol{E}}(F_{\mu_{\boldsymbol{E}}})_U((\alpha,\beta))F(f) = \iota_{\boldsymbol{E}}\mu_{\boldsymbol{E}}(\alpha,\beta)F(f) = \mu_{\boldsymbol{E}}\iota^{(2)}_{\boldsymbol{E}}(\alpha,\beta)F(f) = \mu_{\boldsymbol{E}}(\iota_{\boldsymbol{E}}\beta,\iota_{\boldsymbol{E}}\alpha)F(f)$$

holds by the commutativity of the left diagram of (6.5), Then, a composition

$$F(W) \xrightarrow{(\lambda F(g),\,\iota_{\boldsymbol{E}}(F_{\mu_{\boldsymbol{E}}})_U((\alpha,\beta))F(f))} E \times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E \quad \cdots (*)$$

coincides with the following composition.

$$F(W) \xrightarrow{(\lambda F(g),\,\iota_{\boldsymbol{E}}\beta F(f),\iota_{\boldsymbol{E}}\alpha F(f))} E \times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E}) \xrightarrow{id_E \times_B \mu_{\boldsymbol{E}}} E \times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$$

The following diagram is commutative by the commutativity of the left diagram of (6.5).

$$\begin{array}{ccccc}
F(W) & \xrightarrow{(\lambda F(g),\,\iota_{\boldsymbol{E}}\beta F(f)\,,\iota_{\boldsymbol{E}}\alpha F(f))} & E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})\times_B G_1(\boldsymbol{E}) & \xrightarrow{id_E\times_B\mu_{\boldsymbol{E}}} & E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}) \\
& \searrow^{((F_{\hat{\xi}_{\boldsymbol{E}}})_W(\lambda F(g),\,(F_{\iota_{\boldsymbol{E}}})_W(\beta F(f)))\,,(F_{\iota_{\boldsymbol{E}}})_W(\alpha F(f)))} & \downarrow{\scriptstyle \hat{\xi}_{\boldsymbol{E}}\times_B id_{G_1(\boldsymbol{E})}} & & \downarrow{\scriptstyle \hat{\xi}_{\boldsymbol{E}}} \\
& & E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E
\end{array}$$

Since $\iota_{\boldsymbol{E}} : (G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}}) \to (G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$, $(F_{\iota_{\boldsymbol{E}}})_W(\beta F(f)))$ and $(F_{\iota_{\boldsymbol{E}}})_W(\alpha F(f))$ belongs to $\mathscr{G}_{\boldsymbol{E}} \cap F_{G_1(\boldsymbol{E})}(W)$. Thus we have $(\lambda F(g),(F_{\iota_{\boldsymbol{E}}})_W(\beta F(f))) \in \mathscr{E}^{\mathrm{pr}^\sigma_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}} \cap F_{E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})}(W)$. Since $\hat{\xi}_{\boldsymbol{E}} : \big(E \times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^\sigma_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}}\big) \to (E,\mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$, $(F_{\hat{\xi}_{\boldsymbol{E}}})_W(\lambda F(g),(F_{\iota_{\boldsymbol{E}}})_W(\beta F(f)))$ belongs to $\boldsymbol{E} \cap F_E(W)$. Then, it follows that $((F_{\hat{\xi}_{\boldsymbol{E}}})_W(\lambda F(g),(F_{\iota_{\boldsymbol{E}}})_W(\beta F(f))),(F_{\iota_{\boldsymbol{E}}})_W(\alpha F(f)))$ also belongs to $\mathscr{E}^{\mathrm{pr}^\sigma_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}} \cap F_{E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})}(W)$. Finally, the image of $((F_{\hat{\xi}_{\boldsymbol{E}}})_W(\lambda F(g),(F_{\iota_{\boldsymbol{E}}})_W(\beta F(f))),(F_{\iota_{\boldsymbol{E}}})_W(\alpha F(f)))$ by $(F_{\hat{\xi}_{\boldsymbol{E}}})_W : F_{E\times^{\sigma_{\boldsymbol{E}}}_B G_1(\boldsymbol{E})}(W) \to F_E(W)$ belongs to $\mathscr{E} \cap F_E(W)$. Therefore the composition $(*)$ belongs to $\mathscr{E} \cap F_E(W)$ and $(F_{\mu_{\boldsymbol{E}}})_U((\alpha,\beta))$ satisfies $(G2)$.

Since both α and β satisfy $(G3)$, it follows that both $\sigma_{\boldsymbol{E}}(F_{\mu_{\boldsymbol{E}}})_U((\alpha,\beta)) = \sigma_{\boldsymbol{E}}\alpha$ and $\tau_{\boldsymbol{E}}(F_{\mu_{\boldsymbol{E}}})_U((\alpha,\beta)) = \tau_{\boldsymbol{E}}\beta$ belongs to $\mathscr{B} \cap F_B(U)$, which shows that $(F_{\mu_{\boldsymbol{E}}})_U((\alpha,\beta))$ satisfies $(G3)$. Hence $\mu_{\boldsymbol{E}}$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$.□

Definition 6.7 *Let $\boldsymbol{E} = ((E,\mathscr{E}) \xrightarrow{\pi} (B,\mathscr{B}))$ be an object of $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(B,\mathscr{B})}$ such that π is an epimorphism. We call the groupoid $((B,\mathscr{B}),(G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}});\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}},\varepsilon_{\boldsymbol{E}},\mu_{\boldsymbol{E}},\iota_{\boldsymbol{E}})$ in $\mathscr{P}_F(\mathcal{C},J)$ the groupoid associated with $\boldsymbol{E}$ and denote this groupoid by $\boldsymbol{G}(\boldsymbol{E})$.*

Let us denote by $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C},J))$ a subcategory of $\mathscr{P}_F(\mathcal{C},J)^{(2)}$ whose objects are epimorphisms in $\mathscr{P}_F(\mathcal{C},J)$ and morphisms are cartesian morphisms in the fibered category $\wp_{\mathscr{P}_F(\mathcal{C},J)} : \mathscr{P}_F(\mathcal{C},J)^{(2)} \to \mathscr{P}_F(\mathcal{C},J)$ of morphisms in $\mathscr{P}_F(\mathcal{C},J)$.

Example 6.8 *For an object $(X,\mathscr{X})$ of $\mathscr{P}_F(\mathcal{C},J)$, we denote by $o_X : (X,\mathscr{X}) \to (\{1\},\mathscr{D}_{coarse,\{1\}})$ the unique morphism in $\mathscr{P}_F(\mathcal{C},J)$. Since o_X is an epimorphism, we regard this as an object $\boldsymbol{O}_X$ of $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C},J))$. The groupoid $\boldsymbol{G}(\boldsymbol{O}_X) = ((\{1\},\mathscr{D}_{coarse,\{1\}}),(G_1(\boldsymbol{O}_X),\mathscr{G}_{\boldsymbol{O}_X});\sigma_{\boldsymbol{O}_X},\tau_{\boldsymbol{O}_X},\varepsilon_{\boldsymbol{O}_X},\mu_{\boldsymbol{O}_X},\iota_{\boldsymbol{O}_X})$ is given as follows.*

We put $\mathrm{End}(X,\mathscr{X}) = \mathscr{P}_F(\mathcal{C},J)((X,\mathscr{X}),(X,\mathscr{X}))$ and define a subset $\mathrm{Aut}(X,\mathscr{X})$ of $\mathrm{End}(X,\mathscr{X})$ by

$$\mathrm{Aut}(X,\mathscr{X}) = \{\varphi \in \mathrm{End}(X,\mathscr{X}) \,|\, \varphi \text{ is an isomorphism.}\}.$$

Then, $G_1(\boldsymbol{O}_X)$ is identified with $\mathrm{Aut}(X,\mathscr{X})$ as a set. The source $\sigma_{\boldsymbol{O}_X}$ and the target $\tau_{\boldsymbol{O}_X}$ are the unique map $G_1(\boldsymbol{O}_X) \to \{1\}$. The unit $\varepsilon_{\boldsymbol{O}_X} : \{1\} \to G_1(\boldsymbol{O}_X)$ maps 1 to id_X. The composition $\mu_{\boldsymbol{O}_X} : G_1(\boldsymbol{O}_X) \times G_1(\boldsymbol{O}_X) \to G_1(\boldsymbol{O}_X)$ maps (φ,ψ) to $\psi\varphi$ and the inverse $\iota_{\boldsymbol{O}_X} : G_1(\boldsymbol{O}_X) \to G_1(\boldsymbol{O}_X)$ maps φ to φ^{-1}.

We denote by $\alpha_X : X \times G_1(\boldsymbol{O}_X) \to X$ the map defined by $\alpha_X(x,\varphi) = \varphi(x)$. Then, the the-ology $\mathscr{G}_{\boldsymbol{O}_X}$ on $G_1(\boldsymbol{O}_X) = \mathrm{Aut}(X,\mathscr{X})$ is described as follows.

For $U \in \mathrm{Ob}\,\mathcal{C}$, $\mathscr{G}_{\boldsymbol{O}_X} \cap F_{G_1(\boldsymbol{O}_X)}(U)$ is a subset of $F_{G_1(\boldsymbol{O}_X)}(U)$ consisting of elements γ which satisfy the following condition (G).

(G) For $V,W \in \mathrm{Ob}\,\mathcal{C}$, $f \in \mathcal{C}(W,U)$, $g \in \mathcal{C}(W,V)$ and $\lambda \in \mathscr{X} \cap F_X(V)$, the following compositions belong to $\mathscr{X} \cap F_X(W)$.

$$F(W) \xrightarrow{(\lambda F(g),\,\gamma F(f))} X \times G_1(\boldsymbol{O}_X) \xrightarrow{\alpha_X} X \qquad F(W) \xrightarrow{(\lambda F(g),\,\iota_{\boldsymbol{O}_X}\gamma F(f))} X \times G_1(\boldsymbol{O}_X) \xrightarrow{\alpha_X} X$$

Let $((G,\mathscr{G});\varepsilon,\mu,\iota)$ be a group object in $\mathscr{P}_F(\mathcal{C},J)$ with structure morphisms $\varepsilon : (\{1\},\mathscr{D}_{disc,\{1\}}) \to (G,\mathscr{G})$, $\mu : (G \times G,\mathscr{G}^{p_1} \cap \mathscr{G}^{p_2}) \to (G,\mathscr{G})$ and $\iota : (G,\mathscr{G}) \to (G,\mathscr{G})$ in $\mathscr{P}_F(\mathcal{C},J)$ which make the following diagrams commute. Here, $p_i : G \times G \to G$ denotes the projection onto the i-th component for $i = 1,2$.

$$\begin{array}{ccc} G\times G\times G & \xrightarrow{\mu\times id_G} & G\times G \\ \downarrow{\scriptstyle id_X\times\mu} & & \downarrow{\scriptstyle \mu} \\ G\times G & \xrightarrow{\mu} & G \end{array} \qquad \begin{array}{ccccc} G\times\{1\} & \xrightarrow{id_G\times\varepsilon} & G\times G & \xleftarrow{\varepsilon\times id_G} & \{1\}\times G \\ \uparrow{\scriptstyle (id_G,\,o_G)} & & \downarrow{\scriptstyle \mu} & & \uparrow{\scriptstyle (o_G,\,id_G)} \\ G & \xrightarrow{id_G} & G & \xleftarrow{id_G} & G \end{array} \qquad \begin{array}{ccccc} G & \xrightarrow{(id_G,\,\iota)} & G\times G & \xleftarrow{(\iota,\,id_G)} & G \\ \downarrow{\scriptstyle o_G} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle o_G} \\ \{1\} & \xrightarrow{\varepsilon} & G & \xleftarrow{\varepsilon} & \{1\} \end{array}$$

For an object $(B,\mathscr{B})$ of $\mathscr{P}_F(\mathcal{C},J)$, we define a groupoid $\boldsymbol{G}_{G,B}$ in $\mathscr{P}_F(\mathcal{C},J)$ as follows. Put $G_1 = B\times G\times B$ and let $\sigma_{G,B},\tau_{G,B} : G_1\to B$ and $\mathrm{pr}_G : G_1\to G$ be the projections given by $\sigma_{G,B}(x,g,y)=x$, $\tau_{G,B}(x,g,y)=y$ and $\mathrm{pr}_G(x,g,y)=g$. Define maps $\varepsilon_{G,B} : B\to G_1$ by $\varepsilon_{G,B}(x)=(x,\varepsilon(1),x)$. Consider a cartesian square

$$\begin{array}{ccc} G_1\times_B G_1 & \xrightarrow{\mathrm{pr}_2} & G_1 \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \sigma_{G,B}} \\ G_1 & \xrightarrow{\tau_{G,B}} & B \end{array}.$$

Then, $G_1\times_B G_1 = \{((x,g,y),(z,h,w))\in G_1\times G_1\,|\,y=z\}$ holds as a set. Define maps $\mu_{G,B} : G_1\times_B G_1\to G_1$ and $\iota_{G,B} : G_1\to G_1$ by $\mu_{G,B}((x,g,y),(z,h,w)) = (x,\mu(g,h),w)$ and $\iota_{G,B}(x,g,y)=(y,\iota(g),x)$, respectively. It is clear that $\sigma_{G,B},\tau_{G,B} : (G_1,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}})\to(B,\mathscr{B})$ and $\mathrm{pr}_G : (G_1,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}})\to(G,\mathscr{G})$ are morphisms in $\mathscr{P}_F(\mathcal{C},J)$. Since $\sigma_{G,B}\varepsilon_{G,B}=\tau_{G,B}\varepsilon_{G,B}=id_X$ and the following diagram is commutative, it follows that $\varepsilon_{G,B} : (B,\mathscr{B})\to(G_1,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}})$ is also a morphism in $\mathscr{P}_F(\mathcal{C},J)$.

$$\begin{array}{ccc} (B,\mathscr{B}) & \xrightarrow{\varepsilon_{G,B}} & (G_1,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}}) \\ \downarrow{\scriptstyle o_B} & & \downarrow{\scriptstyle \mathrm{pr}_G} \\ (\{1\},\mathscr{D}_{disc,\{1\}}) & \xrightarrow{\varepsilon} & (G,\mathscr{G}) \end{array}$$

We note that $\sigma_{G,B}\mu_{G,B}=\sigma_{G,B}\mathrm{pr}_1$ and $\tau_{G,B}\mu_{G,B}=\tau_{G,B}\mathrm{pr}_2$ hold and that the following diagram commutes.

$$\begin{array}{ccc} G_1\times_B G_1 & \xrightarrow{(\mathrm{pr}_G,\mathrm{pr}_G)} & G\times G \\ \downarrow{\scriptstyle \mu_{G,B}} & & \downarrow{\scriptstyle \mu} \\ G_1 & \xrightarrow{\mathrm{pr}_G} & G \end{array}$$

Since $\sigma_{G,B}$, $\tau_{G,B}$, $(\mathrm{pr}_G,\mathrm{pr}_G)$ and μ are morphisms in $\mathscr{P}_F(\mathcal{C},J)$, it follows that

$$\mu_{G,B} : (G_1\times_B G_1,(\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}})^{\mathrm{pr}_1}\cap(\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}})^{\mathrm{pr}_2})\to(G_1,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}})$$

is a morphism in $\mathscr{P}_F(\mathcal{C},J)$. We also have $\sigma_{G,B}\iota_{G,B}=\tau_{G,B}$, $\tau_{G,B}\iota_{G,B}=\sigma_{G,B}$ and $\mathrm{pr}_G\iota_{G,B}=\iota\mathrm{pr}_G$ which imply that $\iota_{G,B} : (G_1,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}})\to(G_1,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$. It is easy to verify that $((B,\mathscr{B}),(B\times G\times B,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{B}^{\tau_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G});\sigma_{G,B},\tau_{G,B},\varepsilon_{G,B},\mu_{G,B},\iota_{G,B})$ is a groupoid in $\mathscr{P}_F(\mathcal{C},J)$.

Definition 6.9 *The groupoid* $((B,\mathscr{B}),(B\times G\times B,\mathscr{B}^{\sigma_{G,B}}\cap\mathscr{G}^{\mathrm{pr}_G}\cap\mathscr{B}^{\tau_{G,B}});\sigma_{G,B},\tau_{G,B},\varepsilon_{G,B},\mu_{G,B},\iota_{G,B})$ *in* $\mathscr{P}_F(\mathcal{C},J)$ *constructed above is called the trivial groupoid associated with* $((G,\mathscr{G});\varepsilon,\mu,\iota)$ *and* $(B,\mathscr{B})$.

Let $(X,\mathscr{X})$ and $(B,\mathscr{B})$ be objects of $\mathscr{P}_F(\mathcal{C},J)$. Let us denote by $\mathrm{pr}_X : X\times B\to X$ and $\mathrm{pr}_B : X\times B\to B$ the projections. Then we have an object $\boldsymbol{X} = ((X\times B,\mathscr{X}^{\mathrm{pr}_X}\cap\mathscr{B}^{\mathrm{pr}_B})\xrightarrow{\mathrm{pr}_B}(B,\mathscr{B}))$ of $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C},J))$. We also have a group object $G_1(\boldsymbol{O}_X)=\mathrm{Aut}(X,\mathscr{X})$ in $\mathscr{P}_F(\mathcal{C},J)$ with unit $\varepsilon_{\boldsymbol{O}_X} : \{1\}\to G_1(\boldsymbol{O}_X)$, product $\mu_{\boldsymbol{O}_X} : G_1(\boldsymbol{O}_X)\times G_1(\boldsymbol{O}_X)\to G_1(\boldsymbol{O}_X)$ and inverse $\iota_{\boldsymbol{O}_X} : G_1(\boldsymbol{O}_X)\to G_1(\boldsymbol{O}_X)$ as we considered in (6.8).

Proposition 6.10 *The groupoid* $\boldsymbol{G}(\boldsymbol{X}) = ((B,\mathscr{B}),(G_1(\boldsymbol{X}),\mathscr{G}_{\boldsymbol{X}});\sigma_{\boldsymbol{X}},\tau_{\boldsymbol{X}},\varepsilon_{\boldsymbol{X}},\mu_{\boldsymbol{X}},\iota_{\boldsymbol{X}})$ *in* $\mathscr{P}_F(\mathcal{C},J)$ *associated with* $\boldsymbol{X}$ *is isomorphic to the trivial groupoid associated with* $((G_1(\boldsymbol{O}_X),\mathscr{G}_{\boldsymbol{O}_X});\varepsilon_{\boldsymbol{O}_X},\mu_{\boldsymbol{O}_X},\iota_{\boldsymbol{O}_X})$ *and* $(B,\mathscr{B})$.

Proof. We denote by $i_x : \mathrm{pr}_B^{-1}(x)\to X\times B$ the inclusion map for $x\in B$. Then, $\mathrm{pr}_X i_x : \mathrm{pr}_B^{-1}(x)\to X$ is a bijection and $\mathrm{pr}_B i_x : \mathrm{pr}_B^{-1}(x)\to B$ is a contant map to $\{x\}$. Hence we have $\mathscr{B}^{\mathrm{pr}_B i_x} = \mathscr{D}_{disc,pr_B^{-1}(x)}$ and the following equality.

$$(\mathscr{X}^{\mathrm{pr}_X}\cap\mathscr{B}^{\mathrm{pr}_B})^{i_x} = \mathscr{X}^{\mathrm{pr}_X i_x}\cap\mathscr{B}^{\mathrm{pr}_B i_x} = \mathscr{X}^{\mathrm{pr}_X i_x}\cap\mathscr{D}_{disc,pr_B^{-1}(x)} = \mathscr{X}^{\mathrm{pr}_X i_x}.$$

Therefore $\mathrm{pr}_X i_x : (\mathrm{pr}_B^{-1}(x),(\mathscr{X}^{\mathrm{pr}_X}\cap\mathscr{B}^{\mathrm{pr}_B})^{i_x})\to(X,\mathscr{X})$ is an isomorphism in $\mathscr{P}_F(\mathcal{C},J)$.

We put $G=G_1(\boldsymbol{O}_X)=\mathrm{Aut}(X,\mathscr{X})$ and $G_1=B\times G\times B$ for short and define a map $\zeta_1 : G_1\to G_1(\boldsymbol{X})$ by $\zeta_1(x,y,\psi) = (\mathrm{pr}_X i_y)^{-1}\psi(\mathrm{pr}_X i_x)$. Then, ζ_1 is bijective. In fact, the inverse $\zeta_1^{-1} : G_1(\boldsymbol{X})\to G_1$ of ζ_1 is given by $\zeta_1^{-1}(\varphi) = (\sigma_{\boldsymbol{X}}(\varphi),\tau_{\boldsymbol{X}}(\varphi),(\mathrm{pr}_X i_{\tau_{\boldsymbol{X}}(\varphi)})\varphi(\mathrm{pr}_X i_{\sigma_{\boldsymbol{X}}(\varphi)})^{-1})$. The following diagrams are commutative, hence $(id_B,\zeta_1) : (B,G_1)\to(B,G_1(\boldsymbol{X}))$ is a morphism of groupoids. Here $\zeta_1\times_B\zeta_1 : G_1\times_B G_1\to G_1(\boldsymbol{X})\times_B G_1(\boldsymbol{X})$ maps (φ,ψ) to $(\zeta_1(\varphi),\zeta_1(\psi))$.

$$\begin{array}{ccccc} B & \xleftarrow{\sigma_{G,B}} & G_1 & \xrightarrow{\tau_{G,B}} & B \\ \downarrow{\scriptstyle id_B} & & \downarrow{\scriptstyle \zeta_1} & & \downarrow{\scriptstyle id_B} \\ B & \xleftarrow{\sigma_{\boldsymbol{X}}} & G_1(\boldsymbol{X}) & \xrightarrow{\tau_{\boldsymbol{X}}} & B \end{array} \quad \begin{array}{ccc} B & \xrightarrow{\varepsilon_{G,B}} & G_1 \\ \downarrow{\scriptstyle id_B} & & \downarrow{\scriptstyle \zeta_1} \\ B & \xrightarrow{\varepsilon_{\boldsymbol{X}}} & G_1(\boldsymbol{X}) \end{array} \quad \begin{array}{ccc} G_1 \times_B G_1 & \xrightarrow{\mu_{G,B}} & G_1 \\ \downarrow{\scriptstyle \zeta_1 \times_B \zeta_1} & & \downarrow{\scriptstyle \zeta_1} \\ G_1(\boldsymbol{X}) \times_B G_1(\boldsymbol{X}) & \xrightarrow{\mu_{\boldsymbol{X}}} & G_1(\boldsymbol{X}) \end{array} \quad \begin{array}{ccc} G_1 & \xrightarrow{\iota_{G,B}} & G_1 \\ \downarrow{\scriptstyle \zeta_1} & & \downarrow{\scriptstyle \zeta_1} \\ G_1(\boldsymbol{X}) & \xrightarrow{\iota_{\boldsymbol{X}}} & G_1(\boldsymbol{X}) \end{array}$$

It remains to show that $\zeta_1 : (G_1, \mathscr{B}^{\sigma_{G,B}} \cap \mathscr{B}^{\tau_{G,B}} \cap \mathscr{G}^{\mathrm{pr}_G}_{\boldsymbol{O}_X}) \to (G_1(\boldsymbol{X}), \mathscr{G}_{\boldsymbol{X}})$ and its inverse are morphisms in $\mathscr{P}_F(\mathcal{C}, J)$. We consider the following cartesian squares.

$$\begin{array}{ccc} (X \times B) \times_B G_1 & \xrightarrow{\mathrm{pr}_{G_1}} & G_1 \\ \downarrow{\scriptstyle \mathrm{pr}_{X \times B}} & & \downarrow{\scriptstyle \sigma_{G,B}} \\ X \times B & \xrightarrow{\mathrm{pr}_B} & B \end{array} \qquad \begin{array}{ccc} (X \times B) \times^{\sigma_{\boldsymbol{X}}}_B G_1(\boldsymbol{X}) & \xrightarrow{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{X})}} & G_1(\boldsymbol{X}) \\ \downarrow{\scriptstyle \mathrm{pr}^{\sigma}_{X \times B}} & & \downarrow{\scriptstyle \sigma_{\boldsymbol{X}}} \\ X \times B & \xrightarrow{\mathrm{pr}_B} & B \end{array}$$

Then $(X \times B) \times_B G_1$ is given by $(X \times B) \times_B G_1 = \{((u,z),(x,y,\psi)) \in (X \times B) \times G_1 \,|\, z = x\}$ as a set. Define maps $\hat{\alpha}_X : (X \times B) \times_B G_1 \to X \times B$ and $id_{X \times B} \times_B \zeta_1 : (X \times B) \times_B G_1 \to (X \times B) \times^{\sigma_{\boldsymbol{X}}}_B G_1(\boldsymbol{X})$ by $\hat{\alpha}_X((u,x),(x,y,\psi)) = (\psi(u), y)$ and $(id_{X \times B} \times_B \zeta_1)((u,x),(x,y,\psi)) = ((u,x), \gamma_1(x,y,\psi))$, respectively. Since projections $\mathrm{pr}_{X \times B}$, pr_{G_1}, pr_X, pr_G, $\tau_{G,B}$ and the right G-action α_X on X are morphisms in $\mathscr{P}_F(\mathcal{C}, J)$, it fillows that $\hat{\alpha}_X = (\alpha_X(\mathrm{pr}_X \mathrm{pr}_{X \times B}, \mathrm{pr}_G \mathrm{pr}_{G_1}), \tau_{G,B}\mathrm{pr}_{G_1})$ is also a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. Let U be an object of $\mathcal{C}$ and $\gamma \in \mathscr{B}^{\sigma_{G,B}} \cap \mathscr{B}^{\tau_{G,B}} \cap \mathscr{G}^{\mathrm{pr}_G}_{\boldsymbol{O}_X} \cap F_{G_1}(U)$. We take $V, W \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathcal{C}(W,U)$, $g \in \mathcal{C}(W,V)$.

Assume that $\lambda \in \mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B} \cap F_{X \times B}(V)$ satisfies $\mathrm{pr}_B \lambda F(g) = \sigma_{\boldsymbol{X}}(F_{\zeta_1})_U(\gamma)F(f)$. Then, we have $\mathrm{pr}_B \lambda F(g) = \sigma_{\boldsymbol{X}} \zeta_1 \gamma F(f) = \sigma_{G,B}\gamma F(f)$, hence there exists a map $(\lambda F(g), \gamma F(f)) : F(W) \to (X \times B) \times_B G_1$ such that the following diagram is commutative. Here $id_{X \times B} \times_B \zeta_1 : (X \times B) \times_B G_1 \to (X \times B) \times^{\sigma_{\boldsymbol{X}}}_B G_1(\boldsymbol{X})$ is given by $(id_{X \times B} \times_B \zeta_1)((u,x),\alpha) = ((u,x), \zeta_1(\alpha))$.

$$\begin{array}{ccccc} F(W) & \xrightarrow{(\lambda F(g),\, \gamma F(f))} & (X \times B) \times_B G_1 & \xrightarrow{\hat{\alpha}_X} & X \times B \\ & \searrow{\scriptstyle (\lambda F(g),\, \zeta_1 \gamma F(f))} & \downarrow{\scriptstyle id_{X \times B} \times_B \zeta_1} & & \downarrow{\scriptstyle id_{X \times B}} \\ & & (X \times B) \times^{\sigma_{\boldsymbol{X}}}_B G_1(\boldsymbol{X}) & \xrightarrow{\hat{\xi}_{\boldsymbol{X}}} & X \times B \end{array}$$

Since $\hat{\alpha}_X$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, $F(W) \xrightarrow{(\lambda F(g),\, \zeta_1 \gamma F(f))} (X \times B) \times^{\sigma_{\boldsymbol{X}}}_B G_1(\boldsymbol{X}) \xrightarrow{\hat{\xi}_{\boldsymbol{X}}} X \times B$ belongs to $\mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B} \cap F_{X \times B}(W)$ by the commutativity of the above diagram. This shows that γ satisfies $(G1)$.

Assume that $\lambda \in \mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B} \cap F_{X \times B}(V)$ satisfies $\mathrm{pr}_B \lambda F(g) = \tau_{\boldsymbol{X}}(F_{\zeta_1})_U(\gamma)F(f)$. Then, we have $\mathrm{pr}_B \lambda F(g) = \tau_{\boldsymbol{X}} \zeta_1 \gamma F(f) = \sigma_{G,B}\iota_{G,B}\gamma F(f)$ and there exists a map $(\lambda F(g), \iota_{G,B}\gamma F(f)) : F(W) \to (X \times B) \times_B G_1$ such that the following diagram is commutative.

$$\begin{array}{ccccc} F(W) & \xrightarrow{(\lambda F(g),\, \iota_{G,B} \gamma F(f))} & (X \times B) \times_B G_1 & \xrightarrow{\hat{\alpha}_X} & X \times B \\ & \searrow{\scriptstyle (\lambda F(g),\, \iota_{\boldsymbol{X}} \zeta_1 \gamma F(f))} & \downarrow{\scriptstyle id_{X \times B} \times_B \zeta_1} & & \downarrow{\scriptstyle id_{X \times B}} \\ & & (X \times B) \times^{\sigma_{\boldsymbol{X}}}_B G_1(\boldsymbol{X}) & \xrightarrow{\hat{\xi}_{\boldsymbol{X}}} & X \times B \end{array}$$

Since $\hat{\alpha}_X$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, $F(W) \xrightarrow{(\lambda F(g),\, \iota_{\boldsymbol{X}} \zeta_1 \gamma F(f))} (X \times B) \times^{\sigma_{\boldsymbol{X}}}_B G_1(\boldsymbol{X}) \xrightarrow{\hat{\xi}_{\boldsymbol{X}}} X \times B$ belongs to $\mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B} \cap F_{X \times B}(W)$ by the commutativity of the above diagram. This shows that γ satisfies $(G2)$.

Since $\gamma \in \mathscr{B}^{\sigma_{G,B}} \cap \mathscr{B}^{\tau_{G,B}} \cap \mathscr{G}^{\mathrm{pr}_G}_{\boldsymbol{O}_X} \cap F_{G_1}(U)$, both $\sigma_X \zeta_1 \gamma = \sigma_{G,B}\gamma$ and $\tau_X \zeta_1 \gamma = \tau_{G,B}\gamma$ belong to $\mathscr{B}$. Thus γ satisfies $(G3)$ and ζ_1 is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$.

For $\gamma \in \mathscr{G}_{\boldsymbol{X}} \cap F_{G_1(\boldsymbol{X})}(U)$, both $\sigma_{G,B}((F_{\zeta_1^{-1}})_U(\gamma)) = \sigma_{\boldsymbol{X}}\gamma$ and $\tau_{G,B}((F_{\zeta_1^{-1}})_U(\gamma)) = \tau_{\boldsymbol{X}}\gamma$ belong to $\mathscr{B} \cap F_B(U)$ since γ satisfies $(G3)$. We put $\gamma' = \mathrm{pr}_G((F_{\zeta_1^{-1}})_U(\gamma))$ and take $U, W \in \mathrm{Ob}\,\mathcal{C}$, $f \in \mathcal{C}(W,U)$, $g \in \mathcal{C}(W,V)$ and $\lambda \in \mathscr{X} \cap F_X(V)$. Define $\lambda' \in \mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B} \cap F_{X \times B}(W)$ by $\lambda' = (\lambda F(g), \sigma_{\boldsymbol{X}}\gamma F(f))$. Then we have $\mathrm{pr}_B \lambda' F(id_W) = \sigma_{\boldsymbol{X}}\gamma F(f)$ and the following diagram is commutative.

$$\begin{array}{ccccc} & \nearrow{\scriptstyle (\lambda' F(id_W),\, \gamma F(f))} & (X \times B) \times^{\sigma_{\boldsymbol{X}}}_B G_1(\boldsymbol{X}) & \xrightarrow{\hat{\xi}_{\boldsymbol{X}}} & X \times B \\ & & \downarrow{\scriptstyle id_{X \times B} \times_B \zeta_1^{-1}} & & \downarrow{\scriptstyle id_{X \times B}} \\ F(W) & \xrightarrow{(\lambda' F(id_W),\, \zeta_1^{-1}\gamma F(f))} & (X \times B) \times_B G_1 & \xrightarrow{\hat{\alpha}_X} & X \times B \\ & & \downarrow{\scriptstyle (\mathrm{pr}_X \mathrm{pr}_{X \times B},\, \mathrm{pr}_G \mathrm{pr}_{G_1})} & & \downarrow{\scriptstyle \mathrm{pr}_X} \\ & \searrow{\scriptstyle (\lambda F(g),\, \gamma' F(f))} & X \times G & \xrightarrow{\alpha_X} & X \end{array}$$

Since γ satisfies $(G1)$ for $\boldsymbol{E} = \boldsymbol{X}$, it follows from the commutativity of the above diagram that a composition $F(W) \xrightarrow{(\lambda F(g),\, \gamma' F(f))} X \times G \xrightarrow{\alpha_X} X$ belong to $\mathscr{X} \cap F_X(W)$.

Define $\lambda'' \in \mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B} \cap F_{X\times B}(W)$ by $\lambda'' = (\lambda F(g), \tau_{\boldsymbol{X}}\gamma F(f))$. Then we have $\mathrm{pr}_B \lambda'' F(id_W) = \tau_{\boldsymbol{X}}\gamma F(f)$ and the following diagram is commutative.

$$\begin{array}{ccccc}
 & \overset{(\lambda'' F(id_W),\, \iota_{\boldsymbol{X}}\gamma F(f))}{\nearrow} & (X\times B)\times_B^{\sigma_{\boldsymbol{X}}} G_1(\boldsymbol{X}) & \xrightarrow{\hat{\xi}_{\boldsymbol{X}}} & X \times B \\
 & & \downarrow{\scriptstyle id_{X\times B}\times_B \zeta_1^{-1}} & & \downarrow{\scriptstyle id_{X\times B}} \\
F(W) & \xrightarrow{(\lambda'' F(id_W),\, \iota_{G,B}\zeta_1^{-1}\gamma F(f))} & (X\times B)\times_B G_1 & \xrightarrow{\hat{\alpha}_X} & X\times B \\
 & & \downarrow{\scriptstyle (\mathrm{pr}_X \mathrm{pr}_{X\times B},\, \mathrm{pr}_G \mathrm{pr}_{G_1})} & & \downarrow{\scriptstyle \mathrm{pr}_X} \\
 & \underset{(\lambda F(g),\, \iota_{\boldsymbol{O}_X}\gamma' F(f))}{\searrow} & X\times G & \xrightarrow{\alpha_X} & X
\end{array}$$

Since γ satisfies $(G2)$ for $\boldsymbol{E} = \boldsymbol{X}$, it follows from the commutativity of the above diagram that a composition $F(W) \xrightarrow{(\lambda F(g),\, \iota_{\boldsymbol{O}_X}\gamma' F(f))} X \times G \xrightarrow{\alpha_X} X$ belong to $\mathscr{X} \cap F_X(W)$. Therefore γ' satisfies condition (G) in (6.8) which implies that $\gamma' = \mathrm{pr}_G((F_{\zeta_1^{-1}})_U(\gamma))$ belongs to $\mathscr{G}_{\boldsymbol{O}_X} \cap F_{G_1(\boldsymbol{O}_X)}(U)$. We conclude that $(F_{\zeta_1^{-1}})_U(\gamma) = \zeta_1^{-1}\gamma$ belongs to $\mathscr{B}^{\sigma_{\boldsymbol{X}}} \cap \mathscr{B}^{\tau_{\boldsymbol{X}}} \cap \mathscr{G}_{\boldsymbol{O}_X}^{\mathrm{pr}_G} \cap F_{G_1}(U)$. Thus ζ_1^{-1} is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. □

Let $\boldsymbol{D} = ((D,\mathscr{D}) \xrightarrow{\rho} (A,\mathscr{A}))$ and $\boldsymbol{E} = ((E,\mathscr{E}) \xrightarrow{\pi} (B,\mathscr{B}))$ be objects of $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C},J))$ and $\boldsymbol{\xi} = \langle \xi, f\rangle : \boldsymbol{D} \to \boldsymbol{E}$ a morphism in $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C},J))$. For $x \in A$ and $y \in B$, we denote by $j_x : \rho^{-1}(x) \to D$ and $i_y : \pi^{-1}(y) \to E$ the inclusion maps, respectively. Let $\xi_x : \rho^{-1}(x) \to \pi^{-1}(f(x))$ be the map obtained from $\xi : D \to E$ by restricting the source and the target, namely ξ_x is the unique map that makes the following diagram commute.

$$\begin{array}{ccc}
\rho^{-1}(x) & \xrightarrow{\xi_x} & \pi^{-1}(f(x)) \\
\downarrow{\scriptstyle j_x} & & \downarrow{\scriptstyle i_{f(x)}} \\
D & \xrightarrow{\xi} & E
\end{array}$$

Lemma 6.11 *$\xi_x : (\rho^{-1}(x), \mathscr{D}^{j_x}) \to (\pi^{-1}(f(x)), \mathscr{E}^{i_{f(x)}})$ is an isomorphism in $\mathscr{P}_F(\mathcal{C},J)$.*

Proof. We consider the inverse image $f^*(\boldsymbol{E}) = ((A\times_B E, \mathscr{A}^{\pi_f} \cap \mathscr{E}^{f_\pi}) \xrightarrow{\pi_f} (A,\mathscr{A}))$ of $\boldsymbol{E}$ by f which is also an object of $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C},J))$. We have a natural cartesian morphism $\boldsymbol{\alpha}_f(\boldsymbol{E}) = \langle f_\pi, f\rangle : f^*(\boldsymbol{E}) \to \boldsymbol{E}$.

$$\begin{array}{ccc}
A\times_B E & \xrightarrow{f_\pi} & E \\
\downarrow{\scriptstyle \pi_f} & & \downarrow{\scriptstyle \pi} \\
A & \xrightarrow{f} & B
\end{array}$$

For $x \in A$, we denote by $i_x^f : \pi_f^{-1}(x) \to A\times_B E$ the inclusion map. Since we have $\pi_f^{-1}(x) = \{x\}\times \pi^{-1}(f(x))$ in $A\times_B E$, there is a bijection $f_x : \pi_f^{-1}(x) \to \pi^{-1}(f(x))$ which makes the following diagram commute.

$$\begin{array}{ccc}
\pi_f^{-1}(x) & \xrightarrow{f_x} & \pi^{-1}(f(x)) \\
\downarrow{\scriptstyle i_x^f} & & \downarrow{\scriptstyle i_{f(x)}} \\
A\times_B E & \xrightarrow{f_\pi} & E
\end{array}$$

Since $\pi_f i_x^f : \pi_f^{-1}(x) \to A$ is a constant map to $\{x\}$, $\mathscr{A}^{\pi_f i_x^f}$ coincides with $\mathscr{D}_{coarse,\pi_f^{-1}(x)}$. Therefore we have $(\mathscr{A}^{\pi_f} \cap \mathscr{E}^{f_\pi})^{i_x^f} = \mathscr{A}^{\pi_f i_x^f} \cap \mathscr{E}^{f_\pi i_x^f} = \mathscr{E}^{i_{f(x)} f_x}$ and it follows that $f_x : (\pi_f^{-1}(x), (\mathscr{A}^{\pi_f} \cap \mathscr{E}^{f_\pi})^{i_x^f}) \to (\pi^{-1}(f(x)), \mathscr{E}^{i_{f(x)}})$ is an isomorphism in $\mathscr{P}_F(\mathcal{C},J)$.

Since $\boldsymbol{\xi}$ is cartesian, $(\rho,\xi) : (D,\mathscr{D}) \to (A\times_B E, \mathscr{A}^{\pi_f} \cap \mathscr{E}^{f_\pi})$ is an isomorphism in $\mathscr{P}_F(\mathcal{C},J)$. Put $\xi_f = (\rho,\xi)$ and we have an isomorphism $\boldsymbol{\xi}_f = \langle \xi_f, id_A\rangle : \boldsymbol{D} \to f^*(\boldsymbol{E})$ in $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{X})}$ that satisfies $\boldsymbol{\alpha}_f(\boldsymbol{E})\boldsymbol{\xi}_f = \boldsymbol{\xi}$. Then $\pi_f\xi_f = \rho$ holds and we have an isomorphism $\xi_{f,x} : (\rho^{-1}(x), \mathscr{D}^{j_x}) \to (\pi_f^{-1}(x), (\mathscr{A}^{\pi_f} \cap \mathscr{E}^{f_\pi})^{i_x^f})$ for each $x \in A$ by restricting the source and the target of ξ_f. Since $\xi = f_\pi \xi_f$, we have $\xi_x = f_x \xi_{f,x}$ which implies that $\xi_x : (\rho^{-1}(x), \mathscr{D}^{j_x}) \to (\pi^{-1}(f(x)), \mathscr{E}^{i_{f(x)}})$ is an isomorphism in $\mathscr{P}_F(\mathcal{C},J)$. □

Remark 6.12 *Since $\xi_f : (D,\mathscr{D}) \to (A\times_B E, \mathscr{A}^{\pi_f} \cap \mathscr{E}^{f_\pi})$ is an isomorphism in $\mathscr{P}_F(\mathcal{C},J)$ which satisfies $\pi_f\xi_f = \rho$ and $f_\pi\xi_f = \xi$, $\mathscr{D} = (\mathscr{A}^{\pi_f} \cap \mathscr{E}^{f_\pi})^{\xi_f} = \mathscr{A}^{\pi_f\xi_f} \cap \mathscr{E}^{f_\pi\xi_f} = \mathscr{A}^{\rho} \cap \mathscr{E}^{\xi}$ holds.*

By (6.11), we can define a bijection $\xi_{x,y} : G_1(\boldsymbol{D})(x,y) \to G_1(\boldsymbol{E})(f(x), f(y))$ by $\xi_{x,y}(\varphi) = \xi_y \varphi \xi_x^{-1}$ for $x, y \in A$. We also define a map $\xi_1 : G_1(\boldsymbol{D}) \to G_1(\boldsymbol{E})$ by $\xi_1(\varphi) = \xi_{x,y}(\varphi)$ where $x = \sigma_{\boldsymbol{D}}(\varphi)$ and $y = \tau_{\boldsymbol{D}}(\varphi)$. Note that a pair (f, ξ_1) of maps is a morphism $\boldsymbol{G}(\boldsymbol{D}) \to \boldsymbol{G}(\boldsymbol{E})$ of groupoids, that is, the following diagrams are commutative. Here, $\xi_1 \times_f \xi_1 : G_1(\boldsymbol{D}) \times_A G_1(\boldsymbol{D}) \to G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E})$ maps (φ, ψ) to $(\xi_1(\varphi), \xi_1(\psi))$.

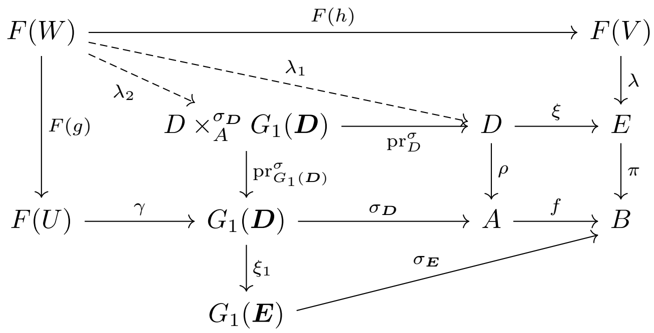
$$\begin{array}{ccccc} A & \xleftarrow{\sigma_{\boldsymbol{D}}} & G_1(\boldsymbol{D}) & \xrightarrow{\tau_{\boldsymbol{D}}} & A \\ \downarrow f & & \downarrow \xi_1 & & \downarrow f \\ B & \xleftarrow{\sigma_{\boldsymbol{E}}} & G_1(\boldsymbol{E}) & \xrightarrow{\tau_{\boldsymbol{E}}} & B \end{array} \quad \begin{array}{ccc} A & \xrightarrow{\varepsilon_{\boldsymbol{D}}} & G_1(\boldsymbol{D}) \\ \downarrow f & & \downarrow \xi_1 \\ B & \xrightarrow{\varepsilon_{\boldsymbol{E}}} & G_1(\boldsymbol{E}) \end{array} \quad \begin{array}{ccc} G_1(\boldsymbol{D}) \times_A G_1(\boldsymbol{D}) & \xrightarrow{\mu_{\boldsymbol{D}}} & G_1(\boldsymbol{D}) \\ \downarrow \xi_1 \times_f \xi_1 & & \downarrow \xi_1 \\ G_1(\boldsymbol{E}) \times_B G_1(\boldsymbol{E}) & \xrightarrow{\mu_{\boldsymbol{E}}} & G_1(\boldsymbol{E}) \end{array} \quad \begin{array}{ccc} G_1(\boldsymbol{D}) & \xrightarrow{\iota_{\boldsymbol{D}}} & G_1(\boldsymbol{D}) \\ \downarrow \xi_1 & & \downarrow \xi_1 \\ G_1(\boldsymbol{E}) & \xrightarrow{\iota_{\boldsymbol{E}}} & G_1(\boldsymbol{E}) \end{array}$$

Define a map $\xi \times_f \xi_1 : D \times_A^{\sigma_{\boldsymbol{D}}} G_1(\boldsymbol{D}) \to E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})$ by $(\xi \times_f \xi_1)(e, \varphi) = (\xi(e), \xi_1(\varphi))$. Then, the following diagram is commutative.

$$\begin{array}{ccc} D \times_A^{\sigma_{\boldsymbol{D}}} G_1(\boldsymbol{D}) & \xrightarrow{\hat{\xi}_{\boldsymbol{D}}} & D \\ \downarrow \xi \times_f \xi_1 & & \downarrow \xi \\ E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E \end{array}$$

Lemma 6.13 *$\xi_1 : (G_1(\boldsymbol{D}), \mathscr{G}_{\boldsymbol{D}}) \to (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. It follows that a pair of morphisms $(f, \xi_1) : \boldsymbol{G}(\boldsymbol{D}) \to \boldsymbol{G}(\boldsymbol{E})$ is a morphism of groupoids in $\mathscr{P}_F(\mathcal{C}, J)$.*

Proof. For $U \in \operatorname{Ob}\mathcal{C}$ and $\gamma \in \mathscr{G}_{\boldsymbol{D}} \cap F_{G_1(\boldsymbol{D})}(U)$, we verify that $(F_{\xi_1})_U(\gamma) = \xi_1\gamma$ satisfies the conditions $(G1)$, $(G2)$ and $(G3)$. We take objects V, W of $\mathcal{C}$ and morphisms $g : W \to U$ and $h : W \to V$ in $\mathcal{C}$. Assume that $\lambda \in \mathscr{E} \cap F_E(V)$ satisfies $\pi\lambda F(h) = \sigma_{\boldsymbol{E}}\xi_1\gamma F(g)$. Since the outer rectangle of the following diagram is commutative and the lower right rectangle is cartesian in $\mathscr{P}_F(\mathcal{C}, J)$, there exists unique F-plot $\lambda_1 \in \mathscr{D} \cap F_D(W)$ that satisfies $\rho\lambda_1 = \sigma_{\boldsymbol{D}}\gamma F(g)$ and $\xi\lambda_1 = \lambda F(h)$.



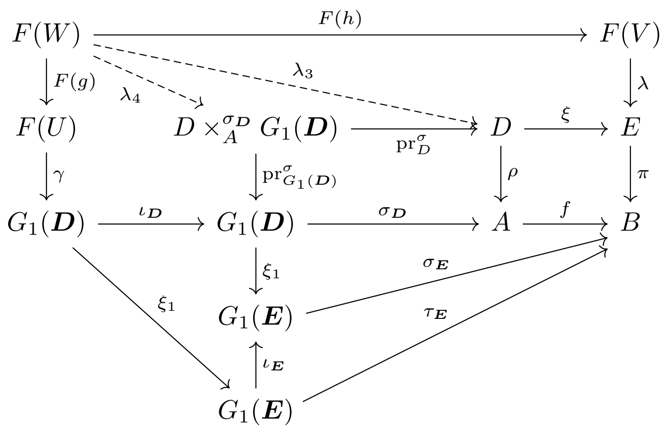
Since γ satisfies $(G1)$ for $\boldsymbol{D}$, the following composition belongs to $\mathscr{D} \cap F_D(W)$.

$$F(W) \xrightarrow{\lambda_2 = (\lambda_1 F(id_W), \gamma F(g))} D \times_A^{\sigma_{\boldsymbol{D}}} G_1(\boldsymbol{D}) \xrightarrow{\hat{\xi}_{\boldsymbol{D}}} D$$

Since $\xi : (D, \mathscr{D}) \to (E, \mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ and the following diagram is commutative, a composition $F(W) \xrightarrow{(\lambda F(h), \xi_1\gamma F(g))} E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E} \cap F_E(W)$. Hence $\xi_1\gamma$ satisfies $(G1)$.

$$\begin{array}{ccccc} F(W) & \xrightarrow{(\lambda_1 F(id_W), \gamma F(g))} & D \times_A^{\sigma_{\boldsymbol{D}}} G_1(\boldsymbol{D}) & \xrightarrow{\hat{\xi}_{\boldsymbol{D}}} & D \\ & \searrow^{(\lambda F(h), \xi_1\gamma F(g))} & \downarrow \xi \times_f \xi_1 & & \downarrow \xi \\ & & E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E \end{array}$$

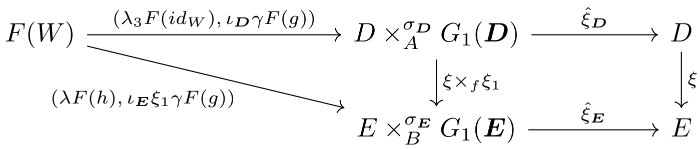
Assume that $\lambda \in \mathscr{E} \cap F_E(V)$ satisfies $\pi\lambda F(h) = \tau_{\boldsymbol{E}}\xi_1\gamma F(g)$. Since the outer rectangle of the following diagram is commutative and the lower right rectangle is cartesian in $\mathscr{P}_F(\mathcal{C}, J)$, there exists unique F-plot $\lambda_3 \in \mathscr{D} \cap F_D(W)$ that satisfies $\rho\lambda_3 = \sigma_{\boldsymbol{D}}\iota_{\boldsymbol{D}}\gamma F(g)$ and $\xi\lambda_3 = \lambda F(h)$.



Since γ satisfies $(G2)$ for $\boldsymbol{D}$, the following composition belongs to $\mathscr{D} \cap F_D(W)$.

$$F(W) \xrightarrow{\lambda_4 = (\lambda_3 F(id_W),\, \iota_{\boldsymbol{D}} \gamma F(g))} D \times_A^{\sigma_{\boldsymbol{D}}} G_1(\boldsymbol{D}) \xrightarrow{\hat{\xi}_{\boldsymbol{D}}} D$$

Since $\xi : (D, \mathscr{D}) \to (E, \mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ and the following diagram is commutative, a composition $F(W) \xrightarrow{(\lambda F(h),\, \iota_{\boldsymbol{E}} \xi_1 \gamma F(g))} E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E} \cap F_E(W)$. Hence $\xi_1\gamma$ satisfies $(G2)$.



Since γ satisfies $(G3)$ for $\boldsymbol{D}$, $\sigma_{\boldsymbol{D}}\gamma, \tau_{\boldsymbol{D}}\gamma \in F_A(U)$ belong to $\mathscr{A} \cap F_A(U)$. Since $f : (A, \mathscr{A}) \to (B, \mathscr{B})$ is a morphism in $\mathscr{P}_E(\mathcal{C}, J)$, $(F_f)_U(\sigma_{\boldsymbol{D}}\gamma)$ and $(F_f)_U(\tau_{\boldsymbol{D}}\gamma)$ belong to $\mathscr{B} \cap F_B(U)$. On the other hand, since $(F_f)_U(\sigma_{\boldsymbol{D}}\gamma) = f\sigma_{\boldsymbol{D}}\gamma = \sigma_{\boldsymbol{E}}\xi_1\gamma$ and $(F_f)_U(\tau_{\boldsymbol{D}}\gamma) = f\tau_{\boldsymbol{D}}\gamma = \tau_{\boldsymbol{E}}\xi_1\gamma$ hold, $\xi_1\gamma$ satisfies $(G3)$. □

We denote by $\mathrm{Grp}(\mathscr{P}_F(\mathcal{C}, J))$ the category of groupopids in $\mathscr{P}_F(\mathcal{C}, J)$. That is, objects of $\mathrm{Grp}(\mathscr{P}_F(\mathcal{C}, J))$ are groupopids in $\mathscr{P}_F(\mathcal{C}, J)$ and morphisms of $\mathrm{Grp}(\mathscr{P}_F(\mathcal{C}, J))$ are morphisms of groupopids. Define a functor

$$\mathbf{Gr} : \mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C}, J)) \to \mathrm{Grp}(\mathscr{P}_F(\mathcal{C}, J))$$

as follows. For an object $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ of $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C}, J))$, let $\mathbf{Gr}(\boldsymbol{E})$ be the groupoid $\boldsymbol{G}(\boldsymbol{E})$ associated with $\boldsymbol{E}$ as we defined in (6.7). For a morphism $\boldsymbol{\xi} = \langle \xi, f \rangle : \boldsymbol{D} \to \boldsymbol{E}$ in $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C}, J))$, we put $\mathbf{Gr}(\boldsymbol{\xi}) = (f, \xi_1) : \boldsymbol{G}(\boldsymbol{D}) \to \boldsymbol{G}(\boldsymbol{E})$. Then $\mathbf{Gr}(\boldsymbol{\xi})$ is a morphism in $\mathrm{Grp}(\mathscr{P}_F(\mathcal{C}, J))$ by (6.13).

Let $\boldsymbol{C} = ((C, \mathscr{C}) \xrightarrow{\chi} (H, \mathscr{H}))$ and $\boldsymbol{D} = ((D, \mathscr{D}) \xrightarrow{\rho} (A, \mathscr{A}))$ be objects of $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C}, J))$ and $\boldsymbol{\zeta} = \langle \zeta, g \rangle : \boldsymbol{C} \to \boldsymbol{D}$ a morphism in $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C}, J))$. We denote by $k_x : \chi^{-1}(x) \to C$, $j_y : \rho^{-1}(y) \to D$ the inclusion maps for $x \in H$ and $y \in A$. We have an isomorphism $\zeta_x : (\chi^{-1}(x), \mathscr{C}^{k_x}) \to (\rho^{-1}(g(x)), \mathscr{D}^{j_{g(x)}})$ in $\mathscr{P}_F(\mathcal{C}, J)$ such that the following diagram is commutative.

$$\begin{array}{ccccc}
\chi^{-1}(x) & \xrightarrow{\zeta_x} & \rho^{-1}(g(x)) & \xrightarrow{\xi_{g(x)}} & \pi^{-1}(f(g(x))) \\
\downarrow{\scriptstyle k_x} & & \downarrow{\scriptstyle j_{g(x)}} & & \downarrow{\scriptstyle i_{f(g(x))}} \\
C & \xrightarrow{\zeta} & D & \xrightarrow{\xi} & E
\end{array}$$

We put $\mathbf{Gr}(\boldsymbol{\zeta}) = (g, \zeta_1)$ and $\mathbf{Gr}(\boldsymbol{\xi\zeta}) = (fg, (\xi\zeta)_1)$. Then, $(\xi\zeta)_1 : G_1(\boldsymbol{C}) \to G_1(\boldsymbol{E})$ maps $\varphi \in G_1(\boldsymbol{C})(x, y)$ to $(\xi_{g(y)}\zeta_y)\varphi(\xi_{g(x)}\zeta_x)^{-1} = \xi_{g(y)}(\zeta_y\varphi\zeta_x^{-1})\xi_{g(x)}^{-1} = \xi_1(\zeta_1(\varphi))$ by the commutativity of the above diagram. It follows that $\mathbf{Gr}(\boldsymbol{\xi\zeta}) = \mathbf{Gr}(\boldsymbol{\xi})\mathbf{Gr}(\boldsymbol{\zeta})$ holds. If $\boldsymbol{id_E}$ is the identity morphism of $\boldsymbol{E}$, it is clear that $\mathbf{Gr}(\boldsymbol{id_E})$ is the identity morphism of $\boldsymbol{G}(\boldsymbol{E})$. Thus we verified that $\mathbf{Gr}$ is a functor from $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C}, J))$ to $\mathrm{Grp}(\mathscr{P}_F(\mathcal{C}, J))$.

Proposition 6.14 *Let $\boldsymbol{D} = ((D, \mathscr{D}) \xrightarrow{\rho} (B, \mathscr{B}))$ and $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ be objects of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(B,\mathscr{B})}$ such that ρ and π are epimorphisms. For a morphism $\boldsymbol{\zeta} : \boldsymbol{D} \to \boldsymbol{E}$ in $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(B,\mathscr{B})}$, we put $\boldsymbol{\zeta} = \langle \zeta, id_B \rangle$. Assume that $\zeta : D \to E$ satisfies the following conditions.*

(i) $\zeta : D \to E$ is surjective and $\mathscr{E}$ coincides with $\mathscr{D}_\zeta$.

(*ii*) *For each* $x \in B$*, if* $a, b \in \rho^{-1}(x)$ *satisfy* $\zeta(a) = \zeta(b)$*, then* $\zeta(\varphi(a)) = \zeta(\varphi(b))$ *holds for any* $\varphi \in G_1(\boldsymbol{D})$ *which satisfies* $\boldsymbol{\sigma_D}(\varphi) = x$*.*

There exists a morphism $\zeta_1 : (G_1(\boldsymbol{D}), \mathscr{G}_{\boldsymbol{D}}) \to (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}})$ *in* $\mathscr{P}_F(\mathcal{C}, J)$ *such that* $(id_B, \zeta_1) : \boldsymbol{G}(\boldsymbol{D}) \to \boldsymbol{G}(\boldsymbol{E})$ *is a morphism of groupoids and the following digram is commutative.*

$$\begin{array}{ccc} D \times_B^{\sigma_D} G_1(\boldsymbol{D}) & \xrightarrow{\hat{\xi}_{\boldsymbol{D}}} & D \\ \downarrow{\scriptstyle \zeta \times_B \zeta_1} & & \downarrow{\scriptstyle \zeta} \\ E \times_B^{\sigma_E} G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E \end{array} \quad \cdots (*)$$

Proof. We denote by $i_x : \rho^{-1}(x) \to D$ and $j_x : \pi^{-1}(x) \to E$ the inclusion maps. Since $\pi\zeta = \rho$ holds, $\zeta : D \to E$ maps $\rho^{-1}(x)$ to $\pi^{-1}(x)$ for any $x \in B$. Let $\zeta_x : \rho^{-1}(x) \to \pi^{-1}(x)$ be the map obtained by restricting the domain of ζ. It follows from $\zeta^{-1}(\pi^{-1}(x)) = \rho^{-1}(x)$ that the following diagram is cartesian in $\mathscr{S}et$.

$$\begin{array}{ccc} \rho^{-1}(x) & \xrightarrow{\zeta_x} & \pi^{-1}(x) \\ \downarrow{\scriptstyle i_x} & & \downarrow{\scriptstyle j_x} \\ D & \xrightarrow{\zeta} & E \end{array}$$

Thus ζ_x is surjective and $(\mathscr{D}^{i_x})_{\zeta_x} = (\mathscr{D}_\zeta)^{j_x}$ holds in $\mathscr{P}_F(\mathcal{C}, J)_{\pi^{-1}(x)}$ by (2.9).

For $x, y \in B$ and $\varphi \in G_1(\boldsymbol{D})(x, y)$, there exists unique map $\varphi_\zeta : \pi^{-1}(x) \to \pi^{-1}(y)$ that makes the following diagram commute by condition (*ii*).

$$\begin{array}{ccc} \rho^{-1}(x) & \xrightarrow{\varphi} & \rho^{-1}(y) \\ \downarrow{\scriptstyle \zeta_x} & & \downarrow{\scriptstyle \zeta_y} \\ \pi^{-1}(x) & \xrightarrow{\varphi_\zeta} & \pi^{-1}(y) \end{array}$$

Let U be an object of $\mathcal{C}$ and take $\alpha \in (\mathscr{D}_\zeta)^{j_x} \cap F_{\pi^{-1}(x)}(U)$. Since $(\mathscr{D}^{i_x})_{\zeta_x} = (\mathscr{D}_\zeta)^{j_x}$, there exists $R \in J(U)$ such that, for each $f \in R$, there exists $\alpha_f \in \mathscr{D}^{i_x} \cap F_{\rho^{-1}(x)}(\mathrm{dom}(f))$ which makes the following diagram commute.

$$\begin{array}{ccccc} F(\mathrm{dom}(f)) & \xrightarrow{\alpha_f} & \rho^{-1}(x) & \xrightarrow{\varphi} & \rho^{-1}(y) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle \zeta_x} & & \downarrow{\scriptstyle \zeta_y} \\ F(U) & \xrightarrow{\alpha} & \pi^{-1}(x) & \xrightarrow{\varphi_\zeta} & \pi^{-1}(y) \end{array}$$

Since $\varphi : (\rho^{-1}(x), \mathscr{D}^{i_x}) \to (\rho^{-1}(y), \mathscr{D}^{i_y})$ and $\zeta_y : (\rho^{-1}(y), \mathscr{D}^{i_y}) \to (\pi^{-1}(y), (\mathscr{D}^{i_y})_{\zeta_y})$ are morphisms in $\mathscr{P}_F(\mathcal{C}, J)$, we have $F_{\pi^{-1}(y)}(f)((F_{\varphi_\zeta})_U(\alpha)) = \varphi_\zeta \alpha F(f) = \zeta_y \varphi \alpha_f = (F_{\zeta_y \varphi})_{\mathrm{dom}(f)}(\alpha_f) \in (\mathscr{D}^{i_y})_{\zeta_y} \cap F_{\pi^{-1}(y)}(\mathrm{dom}(f))$. Since $(\mathscr{D}^{i_y})_{\zeta_y} = (\mathscr{D}_\zeta)^{j_y}$, $F_{\pi^{-1}(y)}(f)((F_{\varphi_\zeta})_U(\alpha))$ belongs to $(\mathscr{D}_\zeta)^{j_y} \cap F_{\pi^{-1}(y)}(\mathrm{dom}(f))$ for any $f \in R$. Thus we see that $(F_{\varphi_\zeta})_U(\alpha) = \varphi_\zeta \alpha \in (\mathscr{D}_\zeta)^{j_y} \cap F_{\pi^{-1}(y)}(U)$. Therefore $\varphi_\zeta : (\pi^{-1}(x), (\mathscr{D}_\zeta)^{j_x}) \to (\pi^{-1}(y), (\mathscr{D}_\zeta)^{j_y})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. For $x, y, z \in B$, $\varphi \in G_1(\boldsymbol{D})(x, y)$ and $\psi \in G_1(\boldsymbol{D})(y, z)$, it follows from the uniqueness of $(\psi\varphi)_\zeta$ and $(id_{\rho^{-1}(x)})_\zeta$ that we have $(\psi\varphi)_\zeta = \psi_\zeta \varphi_\zeta$ and $(id_{\rho^{-1}(x)})_\zeta = id_{\pi^{-1}(x)}$. It follows that $\varphi_\zeta \in G_1(\boldsymbol{E})(x, y)$. We define a map $\zeta_1 : G_1(\boldsymbol{D}) \to G_1(\boldsymbol{E})$ by $\zeta_1(\varphi) = \varphi_\zeta$. It also follows from $(\psi\varphi)_\zeta = \psi_\zeta \varphi_\zeta$ and $(id_{\rho^{-1}(x)})_\zeta = id_{\pi^{-1}(x)}$ that we have equalities $\zeta_1 \mu_{\boldsymbol{D}}(\varphi, \psi) = \mu_{\boldsymbol{E}}(\zeta_1(\varphi), \zeta_1(\psi))$, $\zeta_1(\varepsilon_{\boldsymbol{D}}(x)) = \varepsilon_{\boldsymbol{E}}(x)$ and $\varphi_\zeta^{-1} = (\varphi^{-1})_\zeta$ which implies $\iota_{\boldsymbol{E}} \zeta_1(\varphi) = \zeta_1 \iota_{\boldsymbol{D}}(\varphi)$. It is clear that $\boldsymbol{\sigma_E} \zeta_1 = \boldsymbol{\sigma_D}$ and $\boldsymbol{\tau_E} \zeta_1 = \boldsymbol{\tau_D}$ hold. Hence (id_B, ζ_1) is a morphism of groupoids. For $(d, \varphi) \in D \times_B^{\sigma_D} G_1(\boldsymbol{D})$, since $d \in \rho^{-1}(\boldsymbol{\sigma_E}(\varphi))$, we have the following equality.

$$\hat{\xi}_{\boldsymbol{E}}(\zeta \times_B \zeta_1)(d, \varphi) = \hat{\xi}_{\boldsymbol{E}}(\zeta(d), \varphi_\zeta) = j_{\boldsymbol{\tau_E}(\varphi_\zeta)}(\varphi_\zeta(\zeta_{\boldsymbol{\sigma_E}(\varphi)}(d))) = j_{\boldsymbol{\tau_E}(\varphi_\zeta)}(\zeta_{\boldsymbol{\tau_E}(\varphi)} \varphi(d)) = \zeta(i_{\boldsymbol{\tau_D}(\varphi)} \varphi(d)) = \zeta \hat{\xi}_{\boldsymbol{D}}(d, \varphi)$$

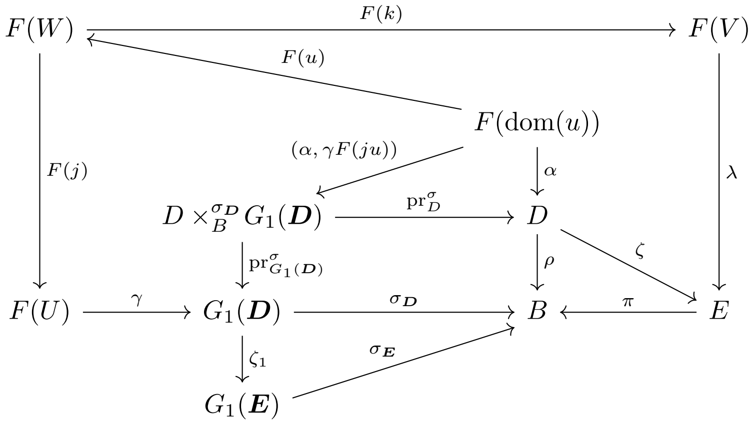
Thus diagram $(*)$ is commutative.

For an object U of $\mathcal{C}$, and $\gamma \in \mathscr{G}_{\boldsymbol{D}} \cap F_{G_1(\boldsymbol{D})}(U)$, we verify that $(F_{\zeta_1})_U(\gamma) = \zeta_1 \gamma$ satisfies the conditions $(G1)$, $(G2)$ and $(G3)$. Since γ satisfies $(G3)$ for $\boldsymbol{D}$ and equalities $\boldsymbol{\sigma_E} \zeta_1 \gamma = \boldsymbol{\sigma_D} \gamma$, $\boldsymbol{\tau_E} \zeta_1 \gamma = \boldsymbol{\tau_D} \gamma$ hold, both $\boldsymbol{\sigma_E} \zeta_1 \gamma$ and $\boldsymbol{\tau_E} \zeta_1 \gamma$ belongs to $\mathscr{B} \cap F_B(U)$, namely $\zeta_1 \gamma$ satisfies $(G3)$.

We take objects V, W of $\mathcal{C}$ and morphisms $j : W \to U$ and $k : W \to V$ in $\mathcal{C}$. Assume that $\lambda \in \mathscr{D}_\zeta \cap F_E(V)$ satisfies $\pi \lambda F(k) = \boldsymbol{\sigma_E} \zeta_1 \gamma F(j)$. It follows from (2.4) that there exists $R \in J(V)$ such that, for each $g \in R$, there exists $\alpha \in \mathscr{D} \cap F_D(\mathrm{dom}(g))$ which satisfies $F_E(g)(\lambda) = (F_\zeta)_{\mathrm{dom}(g)}(\alpha)$. We put

$$h_k^{-1}(R) = \{u \in \mathrm{Mor}\, \mathcal{C} \,|\, \mathrm{codom}(u) = W,\, ku \in R\}.$$

Then, we have $h_k^{-1}(R) \in J(W)$ and for any $u \in h_k^{-1}(R)$, there exists $\alpha \in \mathscr{D} \cap F_D(\mathrm{dom}(k))$ which satisfies $F_E(ku)(\lambda) = (F_\zeta)_{\mathrm{dom}(u)}(\alpha)$. Thus we have the following commutative diagram.



Since γ satisfies $(G1)$ for $\boldsymbol{D}$, the following composition belongs to $\mathscr{D} \cap F_D(\mathrm{dom}(u))$.

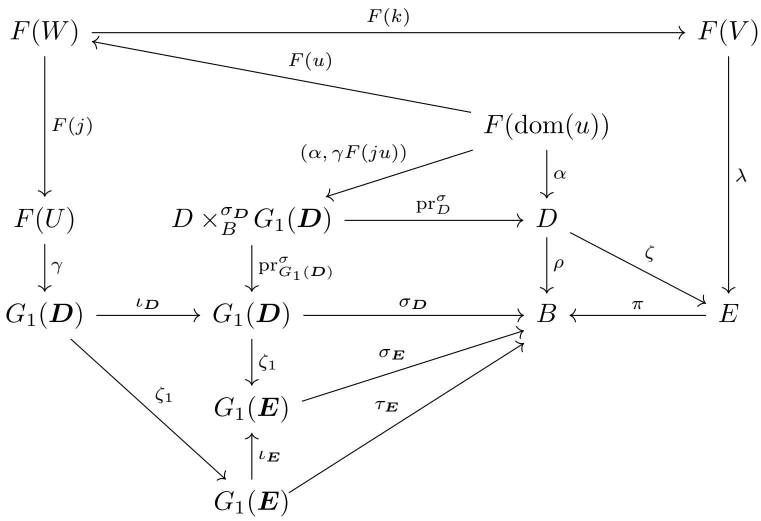
$$F(\mathrm{dom}(u)) \xrightarrow{(\alpha F(id_{\mathrm{dom}(u)}),\, \gamma F(ju))} D \times_B^{\sigma_{\boldsymbol{D}}} G_1(\boldsymbol{D}) \xrightarrow{\hat{\xi}_{\boldsymbol{D}}} D$$

Since $\zeta : (D, \mathscr{D}) \to (E, \mathscr{D}_\zeta)$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ and the following diagram is commutative, a composition $F(\mathrm{dom}(u)) \xrightarrow{F(u)} F(W) \xrightarrow{(\lambda F(k),\, \zeta_1\gamma F(j))} E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{D}_\zeta \cap F_E(\mathrm{dom}(u))$.

$$\begin{array}{ccccc}
F(\mathrm{dom}(u)) & \xrightarrow{(\alpha F(id_{\mathrm{dom}(u)}),\, \gamma F(ju))} & D \times_B^{\sigma_{\boldsymbol{D}}} G_1(\boldsymbol{D}) & \xrightarrow{\hat{\xi}_{\boldsymbol{D}}} & D \\
\big\downarrow F(u) & & \big\downarrow \zeta \times_B \zeta_1 & & \big\downarrow \zeta \\
F(W) & \xrightarrow{(\lambda F(k),\, \zeta_1\gamma F(j))} & E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E
\end{array}$$

Since $h_k^{-1}(R) \in J(W)$ and $u \in h_k^{-1}(R)$ is arbitrary, a composition $F(W) \xrightarrow{(\lambda F(k),\, \zeta_1\gamma F(j))} E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{D}_\zeta \cap F_E(W)$. Hence $\zeta_1\gamma$ satisfies $(G1)$.

Assume that $\lambda \in \mathscr{D}_\zeta \cap F_E(V)$ satisfies $\pi\lambda F(k) = \tau_{\boldsymbol{E}}\zeta_1\gamma F(j)$. It follows from (2.4) that there exists $R \in J(V)$ such that, for each $g \in R$, there exists $\alpha \in \mathscr{D} \cap F_D(\mathrm{dom}(g))$ which satisfies $F_E(g)(\lambda) = (F_\zeta)_{\mathrm{dom}(g)}(\alpha)$. We put $h_k^{-1}(R) = \{u \in \mathrm{Mor}\,\mathcal{C} \,|\, \mathrm{codom}(u) = W,\ ku \in R\}$. Then, we have $h_k^{-1}(R) \in J(W)$ and for any $u \in h_k^{-1}(R)$, there exists $\alpha \in \mathscr{D} \cap F_D(\mathrm{dom}(k))$ which satisfies $F_E(ku)(\lambda) = (F_\zeta)_{\mathrm{dom}(u)}(\alpha)$. Thus we have the following commutative diagram.



Since γ satisfies $(G2)$ for $\boldsymbol{D}$, the following composition belongs to $\mathscr{D} \cap F_D(\mathrm{dom}(u))$.

$$F(\mathrm{dom}(u)) \xrightarrow{(\alpha F(id_{\mathrm{dom}(u)}),\, \iota_{\boldsymbol{D}}\gamma F(ju))} D \times_B^{\sigma_{\boldsymbol{D}}} G_1(\boldsymbol{D}) \xrightarrow{\hat{\xi}_{\boldsymbol{D}}} D$$

Since $\zeta : (D, \mathscr{D}) \to (E, \mathscr{D}_\zeta)$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ and the following diagram is commutative, a composition $F(\mathrm{dom}(u)) \xrightarrow{F(u)} F(W) \xrightarrow{(\lambda F(k),\, \iota_{\boldsymbol{E}}\zeta_1\gamma F(j))} E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{D}_\zeta \cap F_E(\mathrm{dom}(u))$.

$$\begin{array}{ccccc} F(\mathrm{dom}(u)) & \xrightarrow{(\alpha F(id_{\mathrm{dom}(u)}),\, \iota_{\boldsymbol{D}}\gamma F(ju))} & D \times_B^{\sigma_{\boldsymbol{D}}} G_1(\boldsymbol{D}) & \xrightarrow{\hat{\xi}_{\boldsymbol{D}}} & D \\ \downarrow{\scriptstyle F(u)} & & \downarrow{\scriptstyle \zeta \times_B \zeta_1} & & \downarrow{\scriptstyle \zeta} \\ F(W) & \xrightarrow{(\lambda F(k),\, \iota_{\boldsymbol{E}}\zeta_1\gamma F(j))} & E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E \end{array}$$

Since $h_k^{-1}(R) \in J(W)$ and $u \in h_k^{-1}(R)$ is arbitrary, a composition $F(W) \xrightarrow{(\lambda F(k),\, \iota_{\boldsymbol{E}}\zeta_1\gamma F(j))} E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{D}_\zeta \cap F_E(W)$. Hence $\zeta_1\gamma$ satisfies (G2).

Therefore we have a morphism $\zeta_1 : (G_1(\boldsymbol{D}), \mathscr{G}_{\boldsymbol{D}}) \to (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}})$ in $\mathscr{P}_F(\mathcal{C}, J)$. □

7 Fibrations

Definition 7.1 *Let $\boldsymbol{G} = ((G_0, \mathscr{G}_0), (G_1, \mathscr{G}_1); \sigma, \tau, \varepsilon, \mu, \iota)$ be a groupoid in $\mathscr{P}_F(\mathcal{C}, J)$. We denote by $\mathrm{pr}_\sigma, \mathrm{pr}_\tau : G_0 \times G_0 \to G_0$ the projections given by $\mathrm{pr}_\sigma(x, y) = x$ and $\mathrm{pr}_\tau(x, y) = y$. If a map $(\sigma, \tau) : G_1 \to G_0 \times G_0$ given by $(\sigma, \tau)(\varphi) = (\sigma(\varphi), \tau(\varphi))$ is an epimorphism and the the-ology $(\mathscr{G}_1)_{(\sigma,\tau)}$ on $G_0 \times G_0$ coincides with $\mathscr{G}_0^{\mathrm{pr}_\sigma} \cap \mathscr{G}_0^{\mathrm{pr}_\tau}$, we say that $\boldsymbol{G}$ is fibrating ([6], 8.4). Let $\boldsymbol{E}$ be an object of $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C}, J))$. If the groupoid $\boldsymbol{G}(\boldsymbol{E})$ associated with $\boldsymbol{E}$ (6.7) is fibrating, we call $\boldsymbol{E}$ a fibration ([6],8.8).*

Remark 7.2 *If $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ is a fibration, then, since $(\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}}) : G_1(\boldsymbol{E}) \to B \times B$ is surjective, $G_1(\boldsymbol{E})(x, y)$ is not empty for any $x, y \in B$. Hence fibers $(\pi^{-1}(x), \mathscr{E}^{i_x})$ of π are all isomorphic.*

Proposition 7.3 *Let $\boldsymbol{G} = ((G_0, \mathscr{G}_0), (G_1, \mathscr{G}_1); \sigma, \tau, \varepsilon, \mu, \iota)$, $\boldsymbol{H} = ((H_0, \mathscr{H}_0), (H_1, \mathscr{H}_1); \sigma', \tau', \varepsilon', \mu', \iota')$ be groupoids in $\mathscr{P}_F(\mathcal{C}, J)$ and $(f_0, f_1) : \boldsymbol{G} \to \boldsymbol{H}$ a morphism of groupoids in $\mathscr{P}_F(\mathcal{C}, J)$ such that $f_0 : G_0 \to H_0$ is surjective and $\mathscr{H}_0 = (\mathscr{G}_0)_{f_0}$. If $\boldsymbol{G}$ is fibrating, so is $\boldsymbol{H}$.*

Proof. Since $(f_0, f_1) : \boldsymbol{G} \to \boldsymbol{H}$ is a morphism of groupoids, the following diagram is commutative.

$$\begin{array}{ccc} G_1 & \xrightarrow{(\sigma, \tau)} & G_0 \times G_0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0 \times f_0} \\ H_1 & \xrightarrow{(\sigma', \tau')} & H_0 \times H_0 \end{array}$$

Since $(\sigma, \tau) : G_1 \to G_0 \times G_0$ and $f_0 \times f_0 : G_0 \times G_0 \to H_0 \times H_0$ are surjective, so is $(\sigma', \tau') : H_1 \to H_0 \times H_0$. It follows from (2.7), (2.8), (2.18) and the assumption that we have the following equality.

$$\begin{aligned} (\mathscr{H}_1)_{(\sigma',\tau')} &= (\mathscr{G}_1)_{(\sigma',\tau')f_1} = (\mathscr{G}_1)_{(f_0\times f_0)(\sigma,\tau)} = ((\mathscr{G}_1)_{(\sigma,\tau)})_{f_0\times f_0} = (\mathscr{G}_0^{\mathrm{pr}_\sigma} \cap \mathscr{G}_0^{\mathrm{pr}_\tau})_{f_0\times f_0} \\ &= ((\mathscr{G}_0)_{f_0})^{\mathrm{pr}_{\sigma'}} \cap ((\mathscr{G}_0)_{f_0})^{\mathrm{pr}_{\tau'}} = \mathscr{H}_0^{\mathrm{pr}_{\sigma'}} \cap \mathscr{H}_0^{\mathrm{pr}_{\tau'}} \end{aligned}$$

Therefore $\boldsymbol{H}$ is fibrating. □

Proposition 7.4 *Under the assumptions of (6.14), if $\boldsymbol{D}$ is a fibration, so is $\boldsymbol{E}$.*

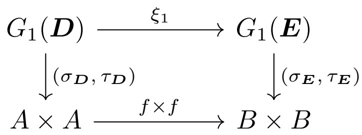
Proof. Since there is a morphism $(id_B, \zeta_1) : \boldsymbol{G}(\boldsymbol{D}) \to \boldsymbol{G}(\boldsymbol{E})$ of groupoids and $\boldsymbol{G}(\boldsymbol{D})$ is fibrating, $\boldsymbol{G}(\boldsymbol{E})$ is also fibrating by (7.3). Hence $\boldsymbol{E}$ is a fibration. □

Lemma 7.5 *Let $(X, \mathscr{X})$ and $(B, \mathscr{B})$ be objects of $\mathscr{P}_F(\mathcal{C}, J)$. We denote the projections by $\mathrm{pr}_X : X \times B \to X$ and $\mathrm{pr}_B : X \times B \to B$. Then $\mathscr{B}$ coincides with $(\mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B})_{\mathrm{pr}_B}$.*

Proof. Since $\mathrm{pr}_B : (X \times B, \mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B}) \to (B, \mathscr{B})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, we have $(\mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B})_{\mathrm{pr}_B} \subset \mathscr{B}$. We choose $a \in X$. For $U \in \mathrm{Ob}\,\mathcal{C}$ and $\gamma \in \mathscr{B} \cap F_B(U)$, define $\bar{\gamma} : F(U) \to X \times B$ by $\bar{\gamma}(x) = (a, \gamma(x))$. Since $\mathrm{pr}_X\bar{\gamma}$ is a constant map and $\mathrm{pr}_Y\bar{\gamma} = \gamma$, we have $\bar{\gamma} \in \mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B} \cap F_{X\times B}(U)$. Hence, for any $h \in h_U$, $\bar{\gamma}F(h) \in \mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B} \cap F_{X\times B}(\mathrm{dom}(h))$ satisfies $F_B(h)(\gamma) = (F_{\mathrm{pr}_B})_{\mathrm{dom}(h)}(\bar{\gamma}F(h))$. This implies that γ belongs to $(\mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B})_{\mathrm{pr}_B}$ by (2.4). Thus we conclude that $(\mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B})_{\mathrm{pr}_B} = \mathscr{B}$ holds. □

Proposition 7.6 *Let $\boldsymbol{\xi} : \boldsymbol{D} \to \boldsymbol{E}$ be a morphism in $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C}, J))$. If $\boldsymbol{E}$ is a fibration, so is $\boldsymbol{D}$.*

Proof. We put $\boldsymbol{D} = ((D, \mathscr{D}) \xrightarrow{\rho} (A, \mathscr{A}))$, $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ and $\boldsymbol{\xi} = \langle \xi, f \rangle : \boldsymbol{D} \to \boldsymbol{E}$. It follows from (6.13) that ξ induces a morphism $\mathbf{Gr}(\boldsymbol{\xi}) = (f, \xi_1) : \boldsymbol{G}(\boldsymbol{D}) \to \boldsymbol{G}(\boldsymbol{E})$ of groupoids. Then, the following diagram is commutative.

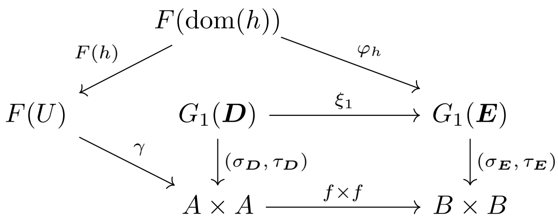


For $x, y \in A$, since $(\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}}) : G_1(\boldsymbol{E}) \to B \times B$ is surjective, there exists $\varphi \in G_1(\boldsymbol{E})$ which satisfies $\sigma_{\boldsymbol{E}}(\varphi) = f(x)$ and $\tau_{\boldsymbol{E}}(\varphi) = f(y)$. Since there is a bijection $\xi_{x,y} : G_1(\boldsymbol{D})(x, y) \to G_1(\boldsymbol{E})(f(x), f(y))$ by (6.11), there exists $\psi \in G_1(\boldsymbol{D})(x, y)$ which satisfies $\sigma_{\boldsymbol{D}}(\psi) = x$ and $\tau_{\boldsymbol{D}}(\psi) = y$. Hence $(\sigma_{\boldsymbol{D}}, \tau_{\boldsymbol{D}}) : G_1(\boldsymbol{E}) \to A \times A$ is surjective.

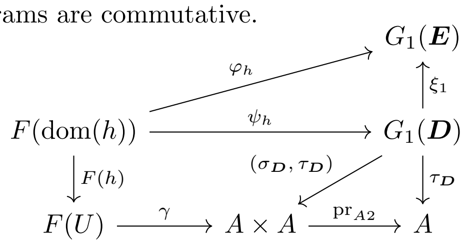
We denote by $\mathrm{pr}_{Ai} : A \times A \to A$ and $\mathrm{pr}_{Bi} : B \times B \to B$ the projections onto the i-th component. Since $\sigma_{\boldsymbol{D}}, \tau_{\boldsymbol{D}} : (G_1(\boldsymbol{D}), \mathscr{G}_{\boldsymbol{D}}) \to (A, \mathscr{A})$ are morphisms in $\mathscr{P}_F(\mathcal{C}, J)$, $(\sigma_{\boldsymbol{D}}, \tau_{\boldsymbol{D}}) : (G_1(\boldsymbol{D}), \mathscr{G}_{\boldsymbol{D}}) \to (A \times A, \mathscr{A}^{\mathrm{pr}_{A1}} \cap \mathscr{A}^{\mathrm{pr}_{A2}})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. On the other hand, since $(\mathscr{G}_{\boldsymbol{D}})_{(\sigma_{\boldsymbol{D}}, \tau_{\boldsymbol{D}})}$ is the finest the-ology on $A \times A$ such that $(\sigma_{\boldsymbol{D}}, \tau_{\boldsymbol{D}}) : (G_1(\boldsymbol{D}), \mathscr{G}_{\boldsymbol{D}}) \to (A \times A, (\mathscr{G}_{\boldsymbol{D}})_{(\sigma_{\boldsymbol{D}}, \tau_{\boldsymbol{D}})})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, $(\mathscr{G}_{\boldsymbol{D}})_{(\sigma_{\boldsymbol{D}}, \tau_{\boldsymbol{D}})} \subset \mathscr{A}^{\mathrm{pr}_{A1}} \cap \mathscr{A}^{\mathrm{pr}_{A2}}$ holds. For $U \in \mathrm{Ob}\,\mathcal{C}$ and $\gamma \in \mathscr{A}^{\mathrm{pr}_{A1}} \cap \mathscr{A}^{\mathrm{pr}_{A2}} \cap F_{A \times A}(U)$, since

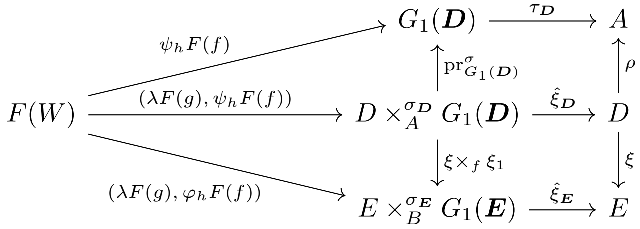
$$f \times f : (A \times A, \mathscr{A}^{\mathrm{pr}_{A1}} \cap \mathscr{A}^{\mathrm{pr}_{A2}}) \to (B \times B, \mathscr{B}^{\mathrm{pr}_{B1}} \cap \mathscr{B}^{\mathrm{pr}_{B2}})$$

is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, $(F_{f \times f})_U(\gamma) \in \mathscr{B}^{\mathrm{pr}_{B1}} \cap \mathscr{B}^{\mathrm{pr}_{B2}} \cap F_{B \times B}(U)$. Since $\mathscr{B}^{\mathrm{pr}_{B1}} \cap \mathscr{B}^{\mathrm{pr}_{B2}} = (\mathscr{G}_{\boldsymbol{E}})_{(\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}})}$ by the assumption, we have $(F_{f \times f})_U(\gamma) \in (\mathscr{G}_{\boldsymbol{E}})_{(\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}})} \cap F_{B \times B}(U)$. It follows from (2.4) that there exists $R \in J(U)$ such that, for any $h \in R$, there exists $\varphi_h \in \mathscr{G}_{\boldsymbol{E}} \cap F_{G_1(\boldsymbol{E})}(\mathrm{dom}(h))$ which makes the following diagram commute.



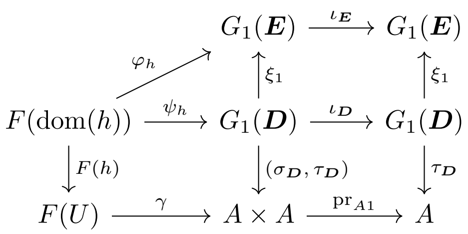
We define a map $\psi_h : F(\mathrm{dom}(h)) \to G_1(\boldsymbol{D})$ as follows. For $u \in F(\mathrm{dom}(h))$, put $F_{A \times A}(h)(\gamma) = (x, y)$. It follows from the commutativity of the above diagram that $\varphi_h(u)$ belongs to $G_1(\boldsymbol{E})(f(x), f(y))$. It follows from (6.11) that we can define $\psi_h(u) \in G_1(\boldsymbol{D})(x, y)$ by $\psi_h(u) = \xi_y^{-1} \varphi_h(u) \xi_x$. In order to show that ψ_h belongs to $\mathscr{G}_{\boldsymbol{D}} \cap F_{G_1(\boldsymbol{D})}(\mathrm{dom}(h))$, we take $V, W \in \mathrm{Ob}\,\mathcal{C}$, $f \in \mathcal{C}(W, \mathrm{dom}(h))$ and $g \in \mathcal{C}(W, V)$. Assume that $\lambda \in \mathscr{D} \cap F_D(V)$ satisfies $\rho\lambda F(g) = \sigma_{\boldsymbol{D}} \psi_h F(f)$. Since $(\sigma_{\boldsymbol{D}}, \tau_{\boldsymbol{D}})\psi_h = \gamma F(h)$ and $\xi_1 \psi_h = \varphi_h$, the following diagrams are commutative.

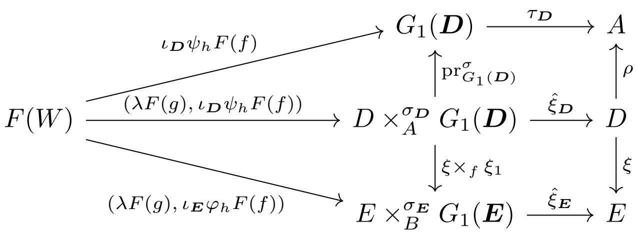




Since $(F_{\tau_{\boldsymbol{D}}})_{\mathrm{dom}(u)}(\psi_h) = (F_{\mathrm{pr}_{A2}})_{\mathrm{dom}(u)}(F_{A \times A}(h)(\gamma))$ and $F_{A \times A}(h)(\gamma) \in \mathscr{A}^{\mathrm{pr}_{A1}} \cap \mathscr{A}^{\mathrm{pr}_{A2}} \cap F_{A \times A}(\mathrm{dom}(h))$, it follows from the commutativity of the above diagram that $(F_{\hat{\xi}_{\boldsymbol{D}}})_W((\lambda F(g), \psi_h F(f)))$ belongs to $\mathscr{A}^{\rho} \cap F_D(W)$. On the other hand, since $\lambda \in \mathscr{D} \cap F_D(V)$, $\varphi_h \in \mathscr{G}_{\boldsymbol{E}} \cap F_{G_1(\boldsymbol{E})}(\mathrm{dom}(h))$, $(\lambda F(g), \varphi_h F(f)) : F(W) \to E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})$ belongs to $\mathscr{E}^{\mathrm{pr}^{\sigma}_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}} \cap F_{E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})}(W)$. Since $\hat{\xi}_E : (E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^{\sigma}_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}}) \to (E, \mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, $(F_{\hat{\xi}_{\boldsymbol{D}}})_W((\lambda F(g), \psi_h F(f)))$ belongs to $\mathscr{E}^{\xi} \cap F_D(W)$ by the commutativity of the above diagram. Thus we have $(F_{\hat{\xi}_{\boldsymbol{D}}})_W((\lambda F(g), \psi_h F(f))) \in \mathscr{A}^{\rho} \cap \mathscr{E}^{\xi} \cap F_D(W) = \mathscr{D} \cap F_D(W)$ by (6.12) and ψ_h satisfies (G1).

Assume that $\lambda \in \mathscr{D} \cap F_D(V)$ satisfies $\rho\lambda F(g) = \tau_{\boldsymbol{D}} \psi_h F(f)$. Since $(\sigma_{\boldsymbol{D}}, \tau_{\boldsymbol{D}})\psi_h = \gamma F(h)$ and $\xi_1 \psi_h = \varphi_h$, the following diagrams are commutative.





Since $(F_{\tau_{\boldsymbol{D}}})_{\mathrm{dom}(u)}(\iota_{\boldsymbol{D}}\psi_h) = (F_{\mathrm{pr}_{A1}})_{\mathrm{dom}(u)}(F_{A\times A}(h)(\gamma))$ and $F_{A\times A}(h)(\gamma) \in \mathscr{A}^{\mathrm{pr}_{A1}} \cap \mathscr{A}^{\mathrm{pr}_{A2}} \cap F_{A\times A}(\mathrm{dom}(h))$, it follows from the commutativity of the above diagram that $(F_{\hat{\xi}_{\boldsymbol{D}}})_W((\lambda F(g), \psi_h F(f)))$ belongs to $\mathscr{A}^{\rho} \cap F_D(W)$. Since $\lambda \in \mathscr{D} \cap F_D(V)$, $\iota_{\boldsymbol{E}}\varphi_h \in \mathscr{G}_{\boldsymbol{E}} \cap F_{G_1(\boldsymbol{E})}(\mathrm{dom}(h))$, $(\lambda F(g), \iota_{\boldsymbol{E}}\varphi_h F(f)) : F(W) \to E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})$ belongs to $\mathscr{E}^{\mathrm{pr}_E^{\sigma}} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}_{G_1(\boldsymbol{E})}^{\sigma}} \cap F_{E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})}(W)$. Since $\hat{\xi}_E : (E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}_E^{\sigma}} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}_{G_1(\boldsymbol{E})}^{\sigma}}) \to (E, \mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, $(F_{\hat{\xi}_{\boldsymbol{D}}})_W((\lambda F(g), \iota_{\boldsymbol{D}}\psi_h F(f)))$ belongs to $\mathscr{E}^{\xi} \cap F_D(W)$ by the commutativity of the above diagram. Thus we have $(F_{\hat{\xi}_{\boldsymbol{D}}})_W((\lambda F(g), \iota_{\boldsymbol{D}}\psi_h F(f))) \in \mathscr{A}^{\rho} \cap \mathscr{E}^{\xi} \cap F_D(W) = \mathscr{D} \cap F_D(W)$ by (6.12) and ψ_h satisfies $(G2)$.

By $(\sigma_{\boldsymbol{D}}, \tau_{\boldsymbol{D}})\psi_h = \gamma F(h)$, $\sigma_{\boldsymbol{D}}\psi_h = (F_{\mathrm{pr}_{A1}})_{\mathrm{dom}(h)}(F_{A\times A}(h)(\gamma))$ and $\tau_{\boldsymbol{D}}\psi_h = (F_{\mathrm{pr}_{A2}})_{\mathrm{dom}(h)}(F_{A\times A}(h)(\gamma))$ hold. Since $F_{A\times A}(h)(\gamma) \in \mathscr{A}^{\mathrm{pr}_{A1}} \cap \mathscr{A}^{\mathrm{pr}_{A2}} \cap F_{A\times A}(\mathrm{dom}(h))$, we have $(F_{\mathrm{pr}_{Ai}})_{\mathrm{dom}(h)}(F_{A\times A}(h)(\gamma)) \in \mathscr{A} \cap F_A(\mathrm{dom}(h))$ for $i = 1, 2$. Hence both $\sigma_{\boldsymbol{D}}\psi_h$ and $\tau_{\boldsymbol{D}}\psi_h$ belong to $\mathscr{A} \cap F_A(\mathrm{dom}(h))$, which shows that ψ_h satisfies $(G3)$. Therefore we have $\phi_h \in \mathscr{G}_{\boldsymbol{E}} \cap F_{G_1(\boldsymbol{D})}(\mathrm{dom}(h))$ and it follows from (2.4) and $F_{A\times A}(h)(\gamma) = (F_{(\sigma_{\boldsymbol{D}}, \tau_{\boldsymbol{D}})})_{\mathrm{dom}(h)}(\psi_h)$ that γ belongs to $(\mathscr{G}_{\boldsymbol{E}})_{(\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}})} \cap F_{A\times A}(U)$. Thus we conclude that $(\mathscr{G}_{\boldsymbol{D}})_{(\sigma_{\boldsymbol{D}}, \tau_{\boldsymbol{D}})} = \mathscr{A}^{\mathrm{pr}_{A1}} \cap \mathscr{A}^{\mathrm{pr}_{A2}}$ holds. □

Example 7.7 *Let $((G, \mathscr{G}); \varepsilon, \mu, \iota)$ be a group in $\mathscr{P}_F(\mathcal{C}, J)$ and $(B, \mathscr{B})$ an object of $\mathscr{P}_F(\mathcal{C}, J)$. Consider the trivial groupoid $\boldsymbol{G}_{G,B} = ((B, \mathscr{B}), (B \times G \times B, \mathscr{B}^{\sigma_{G,B}} \cap \mathscr{B}^{\tau_{G,B}} \cap \mathscr{G}^{\mathrm{pr}_G}); \sigma_{G,B}, \tau_{G,B}, \varepsilon_{G,B}, \mu_{G,B}, \iota_{G,B})$ in $\mathscr{P}_F(\mathcal{C}, J)$ associated with $((G, \mathscr{G}); \varepsilon, \mu, \iota)$ and $(B, \mathscr{B})$. Since $(\sigma_{G,B}, \tau_{G,B}) : B \times G \times B \to B \times B$ is a projection, it follows from (7.5) that $\boldsymbol{G}_{G,B}$ is fibrating. Hence $\boldsymbol{X} = ((X \times B, \mathscr{X}^{\mathrm{pr}_X} \cap \mathscr{B}^{\mathrm{pr}_B}) \xrightarrow{\mathrm{pr}_B} (B, \mathscr{B}))$ is a fibration by (6.10). We call $\boldsymbol{X}$ a product fibration.*

Definition 7.8 *Let $\mathcal{C}$ be a category with a terminal object $1_{\mathcal{C}}$. For an object U of $\mathcal{C}$, we say that a functor $F : \mathcal{C} \to \mathcal{S}et$ is U-pointed if $F : \mathcal{C}(1_{\mathcal{C}}, U) \to \mathcal{S}et(F(1_{\mathcal{C}}), F(U))$ is surjective. If F is U-pointed for any object U of $\mathcal{C}$, we say that F is pointed.*

Proposition 7.9 *If a category $\mathcal{C}$ has a terminal object $1_{\mathcal{C}}$, then the functor $h^{1_{\mathcal{C}}} : \mathcal{C} \to \mathcal{S}et$ defined by $h^{1_{\mathcal{C}}}(U) = \mathcal{C}(1_{\mathcal{C}}, U)$ and $h^{1_{\mathcal{C}}}(f : U \to V) = (f_* : \mathcal{C}(1_{\mathcal{C}}, U) \to \mathcal{C}(1_{\mathcal{C}}, V))$ is pointed.*

Proof. For an object U of $\mathcal{C}$ and $\alpha \in \mathcal{S}et(h^{1_{\mathcal{C}}}(1_{\mathcal{C}}), h^{1_{\mathcal{C}}}(U))$, put $f = \alpha(id_{1_{\mathcal{C}}}) \in h^{1_{\mathcal{C}}}(U) = \mathcal{C}(1_{\mathcal{C}}, U)$. Then, we have $h^{1_{\mathcal{C}}}(f)(id_{1_{\mathcal{C}}}) = id_{1_{\mathcal{C}}} f = f = \alpha(id_{1_{\mathcal{C}}})$ which shows $h^{1_{\mathcal{C}}}(f) = \alpha$. Hence $h^{1_{\mathcal{C}}}$ is pointed. □

Definition 7.10 *Let $(\mathcal{C}, J)$ be a site. For an object U of $\mathcal{C}$, we say that a functor $F : \mathcal{C} \to \mathcal{S}et$ is U-local if F satisfies the following condition (L). If F is U-local for any object U of $\mathcal{C}$, we say that F is local.*

(L) For an object V of $\mathcal{C}$ and a map $\alpha : F(V) \to F(U)$, if there exists a covering $(V_i \xrightarrow{f_i} V)_{i\in I}$ of V such that $F(f_i)^* : \mathcal{S}et(F(V), F(U)) \to \mathcal{S}et(F(V_i), F(U))$ maps α into the image of $F : \mathcal{C}(V_i, U) \to \mathcal{S}et(F(V_i), F(U))$ for any $i \in I$, then α belongs to the image of $F : \mathcal{C}(V, U) \to \mathcal{S}et(F(V), F(U))$.

Remark 7.11 *Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{S}et$ a functor. For an object U of $\mathcal{C}$, we define a subset $\mathscr{F}_U$ of $\coprod_{V\in\mathrm{Ob}\,\mathcal{C}} F_{F(U)}(V)$ by $\mathscr{F}_U = \coprod_{V\in\mathrm{Ob}\,\mathcal{C}} \mathrm{Im}(F : \mathcal{C}(V, U) \to \mathcal{S}et(F(V), F(U)) = F_{F(U)}(V))$. Then, it is easy to verify that $\mathscr{F}_U$ satisfies condition (ii) of (1.2).*

(1) Assume that $\mathcal{C}$ has a terminal object $1_{\mathcal{C}}$. Since $\mathscr{F}_U \cap F_{F(U)}(1_{\mathcal{C}}) = \mathrm{Im}(F : \mathcal{C}(1_{\mathcal{C}}, U) \to F_{F(U)}(1_{\mathcal{C}}))$, F is U-pointed if and only if $\mathscr{F}_U$ satisfies condition (i) of (1.2).

(2) For a site $(\mathcal{C}, J)$, F is U-local if and only if $\mathscr{F}_U$ satisfies condition (iii) of (1.2).

Thus $\mathscr{F}_U$ is a the-ologgy on $F(U)$ if and only if F is U-pointed and U-local. Assume that F is pointed and local. For an object V of $\mathcal{C}$, a morphism $f : U \to W$ in $\mathcal{C}$ and $\varphi \in \mathscr{F}_U \cap F_{F(U)}(V)$, since there exists $g \in \mathcal{C}(V, U)$ such that $F(g) = \varphi$, we have $(F_{F(f)})_V(\varphi) = F(f)\varphi = F(f)F(g) = F(fg) \in \mathscr{F}_U \cap F_{F(W)}(V)$. It follows that $(F_{F(f)})_V : F_{F(U)}(V) \to F_{F(W)}(V)$ maps $\mathscr{F}_U \cap F_{F(U)}(V)$ into $\mathscr{F}_W \cap F_{F(W)}(V)$. We define a functor $\bar{F} : \mathcal{C} \to \mathscr{P}_F(\mathcal{C}, J)$ by $\bar{F}(U) = (F(U), \mathscr{F}_U)$ for $U \in \mathrm{Ob}\,\mathcal{C}$ and $\bar{F}(f : U \to W) = (F(f) : (F(U), \mathscr{F}_U) \to (F(W), \mathscr{F}_W))$ for a morphism $f : U \to W$ in $\mathcal{C}$. Then $\varGamma_F \bar{F} = F$ holds.

Example 7.12 *Define a category $\mathcal{C}^{\infty}$ as follows. Objects of $\mathcal{C}^{\infty}$ are open sets of n dimensional Euclidean space $\boldsymbol{R}^n$ for some $n \geqq 0$. Morphisms of $\mathcal{C}^{\infty}$ are C^{∞}-maps. For $U \in \mathrm{Ob}\,\mathcal{C}^{\infty}$, let $P_{\infty}(U)$ be the set of families $(U_i \xrightarrow{f_i} U)_{i\in I}$ of open embeddings such that $U = \bigcup_{i\in I} f_i(U_i)$. It is easy to verify that P_{∞} is a pretopology on $\mathcal{C}^{\infty}$. We give a Grothendieck topology J_{∞} on $\mathcal{C}^{\infty}$ generated by P_{∞}. Then, the forgetful functor $F : \mathcal{C}^{\infty} \to \mathcal{S}et$ is pointed and local. For a set X, a the-ology on X with respect to F and $(\mathcal{C}^{\infty}, J_{\infty})$ is usually called a diffeology on X and a the-ological object with respect to F and $(\mathcal{C}^{\infty}, J_{\infty})$ is called a diffeological space.*

Example 7.13 *Let k be an algebraically closed field. We denote by $\mathcal{Aff}_k$ the category of affine varieties over k. For $V \in \operatorname{Ob}\mathcal{Aff}_k$, let $P_{\mathcal{Aff}_k}(V)$ be the set of families $(V_i \xrightarrow{f_i} V)_{i\in I}$ of Zariski open embeddings such that $V = \bigcup_{i\in I} f_i(V_i)$. It is easy to verify that $P_{\mathcal{Aff}_k}$ is a pretopology on $\mathcal{Aff}_k$. We give a Grothendieck topology $J_{\mathcal{Aff}_k}$ on $\mathcal{Aff}_k$ generated by $P_{\mathcal{Aff}_k}$. Then, the forgetful functor $F : \mathcal{Aff}_k \to \mathcal{S}et$ is pointed and local.*

Proposition 7.14 *Let $(X, \mathscr{X})$ be an object of $\mathscr{P}_F(\mathcal{C}, J)$. Suppose that $F : \mathcal{C} \to \mathcal{S}et$ is U-pointed and U-local for an object U of $\mathcal{C}$. Then, a map $\varphi : F(U) \to X$ is an F-plot if and only if $\varphi : (F(U), \mathscr{F}_U) \to (X, \mathscr{X})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$.*

Proof. Assume that $\varphi : F(U) \to X$ is an F-plot, namely, $\varphi \in \mathscr{D}\cap F_X(U)$. For $V \in \operatorname{Ob}\mathcal{C}$ and $\psi \in \mathscr{F}_U\cap F_{F(U)}(V)$, there exists $f \in \mathcal{C}(V, U)$ such that $F(f) = \psi$. Then, we have $(F_\varphi)_V(\psi) = \varphi F(f) = F_X(f)(\varphi) \in \mathscr{D} \cap F_X(V)$, which shows that $\varphi : (F(U), \mathscr{F}_U) \to (X, \mathscr{X})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$.

Conversely, assume that $\varphi : (F(U), \mathscr{F}_U) \to (X, \mathscr{X})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. Since $id_{F(U)} = F(id_U)$ belongs to $\mathscr{F}_U \cap F_{F(U)}(U)$, we have $\varphi = \varphi id_{F(U)} = (F_\varphi)_U(id_{F(U)}) \in \mathscr{D} \cap F_X(U)$. Hence φ is an F-plot. □

Lemma 7.15 *For an object $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}$, the following diagram in $\mathscr{P}_F(\mathcal{C}, J)$ is cartesian.*

$$\begin{array}{ccc} \big(E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^\sigma_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}}\big) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & (E, \mathscr{E}) \\ \Big\downarrow{\scriptstyle \mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}} & & \Big\downarrow{\scriptstyle \pi} \\ (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}}) & \xrightarrow{\tau_{\boldsymbol{E}}} & (B, \mathscr{B}) \end{array}$$

Proof. Since $\pi\hat{\xi}_{\boldsymbol{E}} = \tau_{\boldsymbol{E}}\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}$ holds, we have $\pi\hat{\xi}_{\boldsymbol{E}}(id_E\times_B\iota_{\boldsymbol{E}}) = \tau_{\boldsymbol{E}}\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}(id_E\times_B\iota_{\boldsymbol{E}}) = \tau_{\boldsymbol{E}}\iota_{\boldsymbol{E}}\mathrm{pr}^\tau_{G_1(\boldsymbol{E})} = \sigma_{\boldsymbol{E}}\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}$. Hence there exist morphisms

$$\kappa : \big(E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^\sigma_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}}\big) \to \big(E \times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^\tau_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}}\big)$$
$$\lambda : \big(E \times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^\tau_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}}\big) \to \big(E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^\sigma_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}}\big)$$

in $\mathscr{P}_F(\mathcal{C}, J)$ that make the following diagrams commute.

$$\begin{array}{ccccc} \big(E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^\sigma_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}}\big) & & \xrightarrow{\qquad\hat{\xi}_{\boldsymbol{E}}\qquad} & & (E, \mathscr{E}) \\ & \overset{\kappa}{\searrow} & & & \\ & & \big(E \times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^\tau_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}}\big) & \xrightarrow{\mathrm{pr}^\tau_E} & (E, \mathscr{E}) \\ & {\scriptstyle \mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}}\searrow & \Big\downarrow{\scriptstyle \mathrm{pr}^\tau_{G_1(\boldsymbol{E})}} & & \Big\downarrow{\scriptstyle \pi} \\ & & (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}}) & \xrightarrow{\tau_{\boldsymbol{E}}} & (B, \mathscr{B}) \end{array}$$

$$\begin{array}{ccccc} \big(E \times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^\tau_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\tau_{G_1(\boldsymbol{E})}}\big) & & \xrightarrow{\qquad\hat{\xi}_{\boldsymbol{E}}(id_E\times_B\iota_{\boldsymbol{E}})\qquad} & & (E, \mathscr{E}) \\ & \overset{\lambda}{\searrow} & & & \\ & & \big(E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}^\sigma_E} \cap \mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}}\big) & \xrightarrow{\mathrm{pr}^\sigma_E} & (E, \mathscr{E}) \\ & {\scriptstyle \mathrm{pr}^\tau_{G_1(\boldsymbol{E})}}\searrow & \Big\downarrow{\scriptstyle \mathrm{pr}^\sigma_{G_1(\boldsymbol{E})}} & & \Big\downarrow{\scriptstyle \pi} \\ & & (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}}) & \xrightarrow{\sigma_{\boldsymbol{E}}} & (B, \mathscr{B}) \end{array}$$

Since κ maps $(x, \varphi) \in E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})$ to $(\varphi(x), \varphi) \in E \times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E})$ and λ maps $(y, \psi) \in E \times_B^{\tau_{\boldsymbol{E}}} G_1(\boldsymbol{E})$ to $(\psi^{-1}(y), \psi) \in E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})$, λ is the inverse of κ. It follows that κ is an isomorphism in $\mathscr{P}_F(\mathcal{C}, J)$. Since the lower rectangle of the upper diagram is cartesian, the assertion follows. □

Let $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ be a fibration. For $b \in B$, define a map $\iota_b : B \to B \times B$ by $\iota_b(x) = (b, x)$. We denote by $\mathrm{pr}_{Bi} : B \times B \to B$ the projection onto the i-th component for $i = 1, 2$. Since $\mathrm{pr}_{B1}\iota_b$ is a constant map and $\mathrm{pr}_{B2}\iota_b$ is the identity map of B, $\iota_b : (B, \mathscr{B}) \to (B \times B, \mathscr{B}^{\mathrm{pr}_{B1}} \cap \mathscr{B}^{\mathrm{pr}_{B2}})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. For $U \in \operatorname{Ob}\mathcal{C}$ and $\gamma \in \mathscr{B} \cap F_B(U)$, since $(F_{\iota_b})_U(\gamma) \in \mathscr{B}^{\mathrm{pr}_{B1}} \cap \mathscr{B}^{\mathrm{pr}_{B2}} = (\mathscr{G}_{\boldsymbol{E}})_{(\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}})}$, it follows from (2.4) that there exists $R \in J(U)$ such that, for each $h \in R$, there exists $\gamma_h \in \mathscr{G}_{\boldsymbol{E}} \cap F_{G_1(\boldsymbol{E})}(\mathrm{dom}(h))$ which satisfies $F_{B\times B}(h)((F_{\iota_b})_U(\gamma)) = (F_{(\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}})})_{\mathrm{dom}(h)}(\gamma_h)$. For $u \in F(\mathrm{dom}(h))$, since $\gamma_h(u)$ belongs to $G_1(\boldsymbol{E})(b, \gamma(F(h)(u)))$ by the commutativity of the following diagram, $\pi((\gamma_h(u))(e)) = \gamma(F(h)(u))$ holds for $e \in \pi^{-1}(b)$.

$$\begin{array}{ccccc} F(\mathrm{dom}(h)) & & \xrightarrow{\quad\gamma_h\quad} & & G_1(\boldsymbol{E}) \\ \Big\downarrow{\scriptstyle F(h)} & & & & \Big\downarrow{\scriptstyle (\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}})} \\ F(U) & \xrightarrow{\gamma} & B & \xrightarrow{\iota_b} & B\times B \end{array}$$

We denote by $\mathrm{pr}_{\pi^{-1}(b)} : \pi^{-1}(b)\times F(\mathrm{dom}(h)) \to \pi^{-1}(b)$ and $\mathrm{pr}_{F(\mathrm{dom}(h))} : \pi^{-1}(b)\times F(\mathrm{dom}(h)) \to F(\mathrm{dom}(h))$ the projections onto the first and second components, respectively. We also denote by $i_b : \pi^{-1}(b) \to E$ the inclusion map. For $(e,u) \in \pi^{-1}(b)\times F(\mathrm{dom}(h))$, since $\pi(e) = b = \sigma_{\boldsymbol{E}}\gamma_h(u)$ by the commutativity of the above diagram, we have a map $(i_b\mathrm{pr}_{\pi^{-1}(b)}, \gamma_h\mathrm{pr}_{F(\mathrm{dom}(h))}) : \pi^{-1}(b)\times F(\mathrm{dom}(h)) \to E \times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E})$. Let us denote by $\bar{\gamma}_h : \pi^{-1}(b)\times F(\mathrm{dom}(h)) \to E$ a composition $\pi^{-1}(b)\times F(\mathrm{dom}(h)) \xrightarrow{(i_b\mathrm{pr}_{\pi^{-1}(b)},\gamma_h\mathrm{pr}_{F(\mathrm{dom}(h))})} E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$. Then $\bar{\gamma}_h(e,u) = (\gamma_h(u))(e)$ holds for $(e,u)\in\pi^{-1}(b)\times F(\mathrm{dom}(h))$.

Lemma 7.16 *The following diagram is cartesian in the category of sets.*

$$\begin{array}{ccc} \pi^{-1}(b)\times F(\mathrm{dom}(h)) & \xrightarrow{\quad\bar{\gamma}_h\quad} & E \\ \Big\downarrow{\scriptstyle \mathrm{pr}_{F(\mathrm{dom}(h))}} & & \Big\downarrow{\scriptstyle \pi} \\ F(\mathrm{dom}(h)) & \xrightarrow{\quad\gamma F(h)\quad} & B \end{array}$$

Proof. We note that $\pi\bar{\gamma}_h = \gamma F(h)\mathrm{pr}_{F(\mathrm{dom}(h))}$ holds by the definition of $\bar{\gamma}_h$. Assume that $(e,u)\in E\times F(\mathrm{dom}(h))$ satisfies $\gamma F(h)(u) = \pi(e)$, namely $e\in\pi^{-1}(\gamma F(h)(u))$. Since $\gamma_h(u) : \pi^{-1}(b)\to\pi^{-1}(\gamma F(h)(u))$ is surjective, there exists $e'\in\pi^{-1}(b)$ which maps to e by $\gamma_h(u)$. Hence we have $\bar{\gamma}_h(e',u) = (\gamma_h(u))(e') = e$. Suppose that $(e'',u')\in\pi^{-1}(b)\times F(\mathrm{dom}(h))$ satisfies $\mathrm{pr}_{F(\mathrm{dom}(h))}(e'',u') = u$ and $\bar{\gamma}_h(e'',u') = e$. It is clear that $u' = u$, hence we have $(\gamma_h(u))(e'') = \bar{\gamma}_h(e'',u') = e = (\gamma_h(u))(e')$. Since $\gamma_h(u) : \pi^{-1}(b)\to\pi^{-1}(\gamma F(h)(u))$ is injective, it follows that $e'' = e'$. Thus the assertion follows. □

Lemma 7.17 *If $F : \mathcal{C}\to\mathcal{S}et$ is pointed and local, the following diagram is cartesian in $\mathscr{P}_F(\mathcal{C},J)$.*

$$\begin{array}{ccc} \left(\pi^{-1}(b)\times F(\mathrm{dom}(h)), (\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\right) & \xrightarrow{\quad\bar{\gamma}_h\quad} & (E,\mathscr{E}) \\ \Big\downarrow{\scriptstyle \mathrm{pr}_{F(\mathrm{dom}(h))}} & & \Big\downarrow{\scriptstyle \pi} \\ (F(\mathrm{dom}(h)),\mathscr{F}_{\mathrm{dom}(h)}) & \xrightarrow{\quad\gamma F(h)\quad} & (B,\mathscr{B}) \end{array}$$

Proof. Since γ is an F-plot, so is $\gamma F(h)$, hence $\gamma F(h) : (F(\mathrm{dom}(h)),\mathscr{F}_{\mathrm{dom}(h)}) \to (B,\mathscr{B})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$ by (7.14). Since γ_h is an F-plot, $\gamma_h : (F(\mathrm{dom}(h)),\mathscr{F}_{\mathrm{dom}(h)}) \to (G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}}))$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$ hence so is $\gamma_h\mathrm{pr}_{F(\mathrm{dom}(h))} : \left(\pi^{-1}(b)\times F(\mathrm{dom}(h)), (\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\right) \to (G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})$. $i_b\mathrm{pr}_{\pi^{-1}(b)} : \left(\pi^{-1}(b)\times F(\mathrm{dom}(h)), (\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\right) \to (E,\mathscr{E})$ is also a morphism in $\mathscr{P}_F(\mathcal{C},J)$. Thus $(i_b\mathrm{pr}_{\pi^{-1}(b)},\gamma_h\mathrm{pr}_{F(\mathrm{dom}(h))}) : \left(\pi^{-1}(b)\times F(\mathrm{dom}(h)), (\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\right) \to (E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}_E}\cap\mathscr{G}_{\boldsymbol{E}}^{\sigma_{\boldsymbol{E}}})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$. Since $\hat{\xi}_{\boldsymbol{E}} : (E\times_B^{\sigma_{\boldsymbol{E}}} G_1(\boldsymbol{E}), \mathscr{E}^{\mathrm{pr}_E}\cap\mathscr{G}_{\boldsymbol{E}}^{\sigma_{\boldsymbol{E}}}) \to (E,\mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$, we see that $\bar{\gamma}_h = \hat{\xi}_{\boldsymbol{E}}(i_b\mathrm{pr}_{\pi^{-1}(b)},\gamma_h\mathrm{pr}_{F(\mathrm{dom}(h))}) : \left(\pi^{-1}(b)\times F(\mathrm{dom}(h)), (\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\right) \to (E,\mathscr{E})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$. It is clear that the following projection is a morphism in $\mathscr{P}_F(\mathcal{C},J)$.

$$\mathrm{pr}_{F(\mathrm{dom}(h))} : \left(\pi^{-1}(b)\times F(\mathrm{dom}(h)), (\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\right) \to (F(\mathrm{dom}(h)),\mathscr{F}_{\mathrm{dom}(h)})$$

Hence $(\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}$ is contained in $\mathscr{E}^{\bar{\gamma}_h}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}$.

For $U\in\mathrm{Ob}\,\mathcal{C}$ and $\alpha\in\mathscr{E}^{\bar{\gamma}_h}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\cap F_{\pi^{-1}(b)\times F(\mathrm{dom}(h))}(U)$, put $\alpha_1 = \mathrm{pr}_{\pi^{-1}(b)}\alpha$ and $\alpha_2 = \mathrm{pr}_{F(\mathrm{dom}(h))}\alpha$. Since $\hat{\xi}_{\boldsymbol{E}}(i_b\alpha_1,\gamma_h\alpha_2) = \bar{\gamma}_h\alpha\in\mathscr{E}\cap F_E(U)$, we have $(i_b\alpha_1,\gamma_h\alpha_2)\in\mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}}\cap F_{E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E})}(U)$. On the other hand, since $\gamma_h\alpha_2 = (F_{\gamma_h\mathrm{pr}_{F(\mathrm{dom}(h))}})_U(\alpha)\in\mathscr{G}_{\boldsymbol{E}}$, we also have $(i_b\alpha_1,\gamma_h\alpha_2)\in\mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}}\cap F_{E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E})}(U)$. Therefore $(i_b\alpha_1,\gamma_h\alpha_2)$ belongs to $\mathscr{E}^{\hat{\xi}_{\boldsymbol{E}}}\cap\mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}}\cap F_{E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E})}(U) = \mathscr{E}^{\mathrm{pr}^{\sigma}_E}\cap\mathscr{G}_{\boldsymbol{E}}^{\mathrm{pr}^{\sigma}_{G_1(\boldsymbol{E})}}\cap F_{E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E})}(U)$ by (7.15). Thus we have $i_b\alpha_1 = \mathrm{pr}^{\sigma}_E(i_b\alpha_1,\gamma_h\alpha_2)\in\mathscr{E}\cap F_E(U)$ which implies $\alpha_1\in\mathscr{E}^{i_b}\cap F_{\pi^{-1}(b)}(U)$. It follows that α belongs to $(\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\cap F_{\pi^{-1}(b)\times F(\mathrm{dom}(h))}(U)$ and that $\mathscr{E}^{\bar{\gamma}_h}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\subset(\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}$ holds. We conclude that $\mathscr{E}^{\bar{\gamma}_h}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}$ coincides with $(\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}}\cap\mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}$. Since a diagram

$$\begin{array}{ccc} \left(\pi^{-1}(b)\times F(\mathrm{dom}(h)), \mathscr{E}^{\bar{\gamma}_h}\cap \mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\right) & \xrightarrow{\bar{\gamma}_h} & (E,\mathscr{E}) \\ \downarrow{\scriptstyle \mathrm{pr}_{F(\mathrm{dom}(h))}} & & \downarrow{\scriptstyle \pi} \\ (F(\mathrm{dom}(h)), \mathscr{F}_{\mathrm{dom}(h)}) & \xrightarrow{\gamma F(h)} & (B,\mathscr{B}) \end{array}$$

is cartesian by (7.16), the assertion follows. □

Assume that the lower right rectangle of the following diagram is cartesian. Then, there exists unique map $\hat{\gamma}_h : \pi^{-1}(b)\times F(\mathrm{dom}(h)) \to F(U)\times_B E$ that makes the following diagram commute.

$$\begin{array}{ccccc} \pi^{-1}(b)\times F(\mathrm{dom}(h)) & \overset{\hat{\gamma}_h}{\dashrightarrow} & & \xrightarrow{\bar{\gamma}_h} & E \\ & & F(U)\times_B E & \xrightarrow{\gamma_\pi} & E \\ \downarrow{\scriptstyle \mathrm{pr}_{F(\mathrm{dom}(h))}} & & \downarrow{\scriptstyle \pi_\gamma} & & \downarrow{\scriptstyle \pi} \\ F(\mathrm{dom}(h)) & \xrightarrow{F(h)} & F(U) & \xrightarrow{\gamma} & B \end{array}$$

Proposition 7.18 *We assume that $F:\mathcal{C}\to \mathcal{S}et$ is pointed and local. Consider objects*

$$\gamma^*(\boldsymbol{E}) = ((F(U)\times_B E, \mathscr{F}_U^{\pi_\gamma}\cap \mathscr{E}^{\gamma_\pi}) \xrightarrow{\pi_\gamma} (F(U),\mathscr{F}_U))$$
$$\boldsymbol{G} = \left(\left(\pi^{-1}(b)\times F(\mathrm{dom}(h)), (\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}}\cap \mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\right) \xrightarrow{\mathrm{pr}_{F(\mathrm{dom}(h))}} (F(\mathrm{dom}(h)),\mathscr{F}_{\mathrm{dom}(h)})\right)$$

of $\mathscr{P}_F(\mathcal{C},J)$. Then, $\boldsymbol{\gamma}_h = \langle \hat{\gamma}_h, F(h)\rangle : \boldsymbol{G}\to \gamma^*(\boldsymbol{E})$ is a cartesian morphism in $\mathscr{P}_F(\mathcal{C},J)^{(2)}$.

Proof. Since $\bar{\gamma}_h = \gamma_\pi\hat{\gamma}_h$, the outer rectangle of the following diagram is cartesian by (7.17). Since the right rectangle of the following diagram is also cartesian, it follows that the left rectangle of the following diagram is cartesian.

$$\begin{array}{ccccc} (\pi^{-1}(b)\times F(\mathrm{dom}(h)), (\mathscr{E}^{i_b})^{\mathrm{pr}_{\pi^{-1}(b)}}\cap \mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}) & \xrightarrow{\hat{\gamma}_h} & (F(U)\times_B E, \mathscr{F}_U^{\pi_\gamma}\cap\mathscr{E}^{\gamma_\pi}) & \xrightarrow{\gamma_\pi} & (E,\mathscr{E}) \\ \downarrow{\scriptstyle \mathrm{pr}_{F(\mathrm{dom}(h))}} & & \downarrow{\scriptstyle \pi_\gamma} & & \downarrow{\scriptstyle \pi} \\ (F(\mathrm{dom}(h)),\mathscr{F}_{\mathrm{dom}(h)}) & \xrightarrow{F(h)} & (F(U),\mathscr{F}_U) & \xrightarrow{\gamma} & (B,\mathscr{B}) \end{array}$$

□

Let $\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2 : \boldsymbol{D}\to\boldsymbol{E}$ be morphisms in $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C},J))$. Put $\boldsymbol{D} = ((D,\mathscr{D}) \xrightarrow{\rho} (A,\mathscr{A}))$, $\boldsymbol{E} = ((E,\mathscr{E})\xrightarrow{\pi} (B,\mathscr{B}))$ and $\boldsymbol{\zeta}_k = \langle \zeta_k, f_k\rangle$ for $k=1,2$. For $a\in A$ and $b\in B$, we denote by $j_a:\rho^{-1}(a)\to D$, $i_b:\pi^{-1}(b)\to E$ the inclusion maps. It follows from (6.11) that the morphisms $\zeta_{k,x} : (\rho^{-1}(x),\mathscr{D}^{j_x}) \to (\pi^{-1}(f_k(x)), \mathscr{E}^{i_{f_k(x)}})$ $(k=1,2)$ obtained by restricting $\zeta_k : (D,\mathscr{D})\to(E,\mathscr{E})$ are isomorphisms in $\mathscr{P}_F(\mathcal{C},J)$. Thus we have an isomorphism $\zeta_{2,x}\zeta_{1,x}^{-1} : (\pi^{-1}(f_1(x)),\mathscr{E}^{i_{f_1(x)}}) \to (\pi^{-1}(f_2(x)),\mathscr{E}^{i_{f_2(x)}})$ in $\mathscr{P}_F(\mathcal{C},J)$. We define a map $\tilde{\zeta} : A\to G_1(\boldsymbol{E})$ by $\tilde{\zeta}(x) = \zeta_{2,x}\zeta_{1,x}^{-1}$. Then, $\sigma_{\boldsymbol{E}}\tilde{\zeta}(x) = f_1(x)$ and $\tau_{\boldsymbol{E}}\tilde{\zeta}(x) = f_2(x)$ hold and the following diagram is commutative.

$$\begin{array}{ccc} & \overset{\tilde{\zeta}}{\nearrow} & G_1(\boldsymbol{E}) \\ & & \downarrow{\scriptstyle (\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}})} \\ A & \xrightarrow{(f_1,f_2)} & B\times B \end{array}$$

Lemma 7.19 *$\tilde{\zeta} : (A,\mathscr{A})\to (G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$.*

Proof. We denote by $f_j^*(\boldsymbol{E}) = ((A\times_B^{f_j} E, \mathscr{A}^{\pi_{f_j}}\cap\mathscr{E}^{(f_j)_\pi}) \xrightarrow{\pi_{f_j}} (A,\mathscr{A}))$ the inverse image of $\boldsymbol{E}$ by f_j. Then, the following left diagram is cartesian and the right one is also cartesian by the assumption.

$$\begin{array}{ccc} (A\times_B^{f_j} E, \mathscr{A}^{\pi_{f_j}}\cap\mathscr{E}^{(f_j)_\pi}) & \xrightarrow{(f_j)_\pi} & (E,\mathscr{E}) \\ \downarrow{\scriptstyle \pi_{f_j}} & & \downarrow{\scriptstyle \pi} \\ (A,\mathscr{A}) & \xrightarrow{f_j} & (B,\mathscr{B}) \end{array} \qquad \begin{array}{ccc} (D,\mathscr{D}) & \xrightarrow{\zeta_j} & (E,\mathscr{E}) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \pi} \\ (A,\mathscr{A}) & \xrightarrow{f_j} & (B,\mathscr{B}) \end{array}$$

Hence there exists unique isomorphism $(\rho,\zeta_j):(D,\mathscr{D})\to(A\times_B^{f_j}E,\mathscr{A}^{\pi_{f_j}}\cap\mathscr{E}^{(f_j)_\pi})$ in $\mathscr{P}_F(\mathcal{C},J)$ that makes the following diagram commute.

$$
\begin{array}{ccccc}
(D,\mathscr{D}) & & \xrightarrow{\quad\zeta_j\quad} & & (E,\mathscr{E}) \\
 & \overset{(\rho,\zeta_j)}{\searrow} & & & \\
 & & (A\times_B^{f_j}E,\mathscr{A}^{\pi_{f_j}}\cap\mathscr{E}^{(f_j)_\pi}) & \xrightarrow{(f_j)_\pi} & (E,\mathscr{E}) \\
 & \underset{\rho}{\searrow} & \Big\downarrow{\scriptstyle \pi_{f_j}} & & \Big\downarrow{\scriptstyle \pi} \\
 & & (A,\mathscr{A}) & \xrightarrow{\quad f_j\quad} & (B,\mathscr{B})
\end{array}
$$

We put $\psi_j=(\rho,\zeta_j)$, then $\psi_j(x)=(\rho(x),\zeta_{j,\rho(x)}(x))$ holds for $x\in D$ and the inverse

$$\psi_j^{-1}:(A\times_B^{f_j}E,\mathscr{A}^{\pi_{f_j}}\cap\mathscr{E}^{(f_j)_\pi})\to(D,\mathscr{D})$$

of ψ_j is given by $\psi_j^{-1}(a,e)=\zeta_{j,a}^{-1}(e)$. Hence $\psi_k\psi_j^{-1}:(A\times_B^{f_1}E,\mathscr{A}^{\pi_{f_k}}\cap\mathscr{E}^{(f_j)_\pi})\to(A\times_B^{f_k}E,\mathscr{A}^{\pi_{f_k}}\cap\mathscr{E}^{(f_k)_\pi})$ for $(j,k)=(1,2),(2,1)$ are given by $\psi_k\psi_j^{-1}(a,e)=\psi_k(\zeta_{j,a}^{-1}(e))=\big(\rho(\zeta_{j,a}^{-1}(e)),\zeta_{k,\rho(\zeta_{j,a}^{-1}(e))}(\zeta_{j,a}^{-1}(e))\big)=(a,\zeta_{k,a}\zeta_{j,a}^{-1}(e))$. Thus we have $\psi_2\psi_1^{-1}(a,e)=(a,\tilde{\zeta}(a)(e))=(a,\hat{\xi}_{\boldsymbol{E}}(e,\tilde{\zeta}(a)))$ and $\psi_1\psi_2^{-1}(a,e)=(a,\tilde{\zeta}(a)^{-1}(e))=(a,\hat{\xi}_{\boldsymbol{E}}(e,(\iota_{\boldsymbol{E}}\tilde{\zeta})(a)))$. We note that $\pi(f_1)_\pi=f_1\pi_{f_1}=\sigma_{\boldsymbol{E}}\tilde{\zeta}\pi_{f_1}$ and $\pi(f_2)_\pi=f_2\pi_{f_2}=\tau_{\boldsymbol{E}}\tilde{\zeta}\pi_{f_2}=\sigma_{\boldsymbol{E}}\iota_{\boldsymbol{E}}\tilde{\zeta}\pi_{f_2}$ holds and that the following diagrams are commutative.

$$
\begin{array}{ccc}
A\times_B^{f_1}E & \xrightarrow{\psi_2\psi_1^{-1}} & A\times_B^{f_2}E \\
\Big\downarrow{\scriptstyle ((f_1)_\pi,\tilde{\zeta}\pi_{f_1})} & & \Big\downarrow{\scriptstyle (f_2)_\pi} \\
E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E
\end{array}
\qquad
\begin{array}{ccc}
A\times_B^{f_2}E & \xrightarrow{\psi_1\psi_2^{-1}} & A\times_B^{f_1}E \\
\Big\downarrow{\scriptstyle ((f_2)_\pi,\iota_{\boldsymbol{E}}\tilde{\zeta}\pi_{f_2})} & & \Big\downarrow{\scriptstyle (f_1)_\pi} \\
E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E
\end{array}
$$

Since compositions

$$(A\times_B^{f_1}E,\mathscr{A}^{\pi_{f_1}}\cap\mathscr{E}^{(f_1)_\pi})\xrightarrow{\psi_2\psi_1^{-1}}(A\times_B^{f_2}E,\mathscr{A}^{\pi_{f_2}}\cap\mathscr{E}^{(f_2)_\pi})\xrightarrow{(f_2)_\pi}(E,\mathscr{E}),$$
$$(A\times_B^{f_2}E,\mathscr{A}^{\pi_{f_2}}\cap\mathscr{E}^{(f_2)_\pi})\xrightarrow{\psi_1\psi_2^{-1}}(A\times_B^{f_1}E,\mathscr{A}^{\pi_{f_1}}\cap\mathscr{E}^{(f_1)_\pi})\xrightarrow{(f_1)_\pi}(E,\mathscr{E})$$

are morphisms in $\mathscr{P}_F(\mathcal{C},J)$, so are the following.

$$\hat{\xi}_{\boldsymbol{E}}((f_1)_\pi,\tilde{\zeta}\pi_{f_1}):(A\times_B^{f_1}E,\mathscr{A}^{\pi_{f_1}}\cap\mathscr{E}^{(f_1)_\pi})\to(E,\mathscr{E}),\quad \hat{\xi}_{\boldsymbol{E}}((f_2)_\pi,\iota_{\boldsymbol{E}}\tilde{\zeta}\pi_{f_2}):(A\times_B^{f_2}E,\mathscr{A}^{\pi_{f_2}}\cap\mathscr{E}^{(f_2)_\pi})\to(E,\mathscr{E})$$

For $U\in\operatorname{Ob}\mathcal{C}$ and $\gamma\in\mathscr{A}\cap F_A(U)$, we verify that $(F_{\tilde{\zeta}})_U(\gamma)=\tilde{\zeta}\gamma$ satisfies the conditions $(G1)$, $(G2)$ and $(G3)$. We take $V,W\in\operatorname{Ob}\mathcal{C}$, $h\in\mathcal{C}(W,U)$, $k\in\mathcal{C}(W,V)$. Assume that $\lambda\in\mathscr{E}\cap F_E(V)$ satisfies $\pi\lambda F(k)=\sigma_{\boldsymbol{E}}\tilde{\zeta}\gamma F(h)$. Then, $f_1\gamma F(h)=\sigma_{\boldsymbol{E}}\tilde{\zeta}\gamma F(h)=\pi\lambda F(k)$ holds and the following diagram is commutative.

$$
\begin{array}{cccc}
 & A\times_B^{f_1}E & \xrightarrow{\psi_2\psi_1^{-1}} & A\times_B^{f_2}E \\
 \overset{(\gamma F(h),\lambda F(k))}{\nearrow} & \Big\downarrow{\scriptstyle ((f_1)_\pi,\tilde{\zeta}\pi_{f_1})} & & \Big\downarrow{\scriptstyle (f_2)_\pi} \\
F(W)\xrightarrow{(\lambda F(k),\tilde{\zeta}\gamma F(h))} & E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E
\end{array}
$$

Since $(\gamma F(h),\lambda F(k)):F(W)\to A\times_B^{f_1}E$ belongs to $\mathscr{A}^{\pi_{f_1}}\cap\mathscr{E}^{(f_1)_\pi}\cap F_{A\times_B^{f_1}E}(W)$ and $\hat{\xi}_{\boldsymbol{E}}((f_1)_\pi,\tilde{\zeta}\pi_{f_1})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$, a composition $F(W)\xrightarrow{(\lambda F(k),\tilde{\zeta}\gamma F(h))}E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E})\xrightarrow{\hat{\xi}_{\boldsymbol{E}}}E$ belongs to $\mathscr{E}\cap F_E(W)$ by the commutativity of the above diagram. Thus $\tilde{\zeta}\gamma$ satisfies the condition $(G1)$.

Assume that $\lambda\in\mathscr{E}\cap F_E(V)$ satisfies $\pi\lambda F(k)=\tau_{\boldsymbol{E}}\tilde{\zeta}\gamma F(h)$. Then, $f_2\gamma F(h)=\tau_{\boldsymbol{E}}\tilde{\zeta}\gamma F(h)=\pi\lambda F(k)$ holds and the following diagram is commutative.

$$
\begin{array}{cccc}
 & A\times_B^{f_2}E & \xrightarrow{\psi_1\psi_2^{-1}} & A\times_B^{f_1}E \\
 \overset{(\gamma F(h),\lambda F(k))}{\nearrow} & \Big\downarrow{\scriptstyle ((f_2)_\pi,\iota_{\boldsymbol{E}}\tilde{\zeta}\pi_{f_2})} & & \Big\downarrow{\scriptstyle (f_1)_\pi} \\
F(W)\xrightarrow{(\lambda F(k),\iota_{\boldsymbol{E}}\tilde{\zeta}\gamma F(h))} & E\times_B^{\sigma_{\boldsymbol{E}}}G_1(\boldsymbol{E}) & \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} & E
\end{array}
$$

Since $(\gamma F(h),\lambda F(k)) : F(W) \to A \times^{f_2}_{B} E$ belongs to $\mathscr{A}^{\pi_{f_2}} \cap \mathscr{E}^{(f_2)_\pi} \cap F_{A\times^{f_2}_B E}(W)$ and $\hat{\xi}_{\boldsymbol{E}}((f_2)_\pi, \iota_{\boldsymbol{E}}\tilde{\zeta}\pi_{f_2})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, a composition $F(W) \xrightarrow{(\lambda F(k),\,\iota_{\boldsymbol{E}}\tilde{\zeta}\gamma F(h))} E \times^{\sigma_{\boldsymbol{E}}}_{B} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E} \cap F_E(W)$ by the commutativity of the above diagram. Thus $\tilde{\zeta}\gamma$ satisfies the condition $(G2)$.

Since we have $\sigma_{\boldsymbol{E}}\tilde{\zeta} = f_1$ and $\tau_{\boldsymbol{E}}\tilde{\zeta} = f_2$ and $f_1, f_2 : (A,\mathscr{A}) \to (B,\mathscr{B})$ are morphisms in $\mathscr{P}_F(\mathcal{C}, J)$, compositions $F(U) \xrightarrow{\tilde{\zeta}\gamma} G_1(\boldsymbol{E}) \xrightarrow{\sigma_{\boldsymbol{E}}} B$ and $F(U) \xrightarrow{\tilde{\zeta}\gamma} G_1(\boldsymbol{E}) \xrightarrow{\tau_{\boldsymbol{E}}} B$ belong to $\mathscr{B} \cap F_B(U)$. Hence $\tilde{\zeta}\gamma$ satisfies the condition $(G3)$. □

Proposition 7.20 *([6], 8.9) We assume that $F : \mathcal{C} \to \mathcal{S}et$ is pointed and local. An object $\boldsymbol{E} = ((E,\mathscr{E}) \xrightarrow{\pi} (B,\mathscr{B}))$ of $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C}, J))$ is a fibration if and only if the following condition (P) is satisfied.*

(P) There exists an object $(T,\mathscr{T})$ of $\mathscr{P}_F(\mathcal{C}, J)$ such that, for any $U \in \mathrm{Ob}\,\mathcal{C}$ and $\gamma \in \mathscr{B} \cap F_B(U)$, there exists a covering $(U_i \xrightarrow{f_i} U)_{i\in U}$ of U such that the inverse image $(\gamma F(f_i))^*(\boldsymbol{E})$ of $\boldsymbol{E}$ by $\gamma F(f_i) : F(U_i) \to B$ is isomorphic to a product fibration $\mathrm{pr}_{F(U_i)} : (T \times F(U_i), \mathscr{T}^{\mathrm{pr}_T} \cap \mathscr{F}_{U_i}^{\mathrm{pr}_{F(U_i)}}) \to (F(U_i), \mathscr{F}_{U_i})$ for any $i \in I$. Here $\mathrm{pr}_T : T \times F(U_i) \to T$ and $\mathrm{pr}_{F(U_i)} : T \times F(U_i) \to F(U_i)$ denote the projections.

Proof. If $\boldsymbol{E}$ is a fibration, the condition (P) follows from (7.2) and (7.18).

Suppose that $\boldsymbol{E}$ satisfies the condition (P). Since $(\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}}) : (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}}) \to (B \times B, \mathscr{B}^{\mathrm{pr}_{B1}} \cap \mathscr{B}^{\mathrm{pr}_{B2}})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$ and $(\mathscr{G}_{\boldsymbol{E}})_{(\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}})}$ is the finest the-ology on $B \times B$, $(\mathscr{G}_{\boldsymbol{E}})_{(\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}})}$ is contained in $\mathscr{B}^{\mathrm{pr}_{B1}} \cap \mathscr{B}^{\mathrm{pr}_{B2}}$. For $U \in \mathrm{Ob}\,\mathcal{C}$, assume that $\gamma \in \mathscr{B}^{\mathrm{pr}_{B1}} \cap \mathscr{B}^{\mathrm{pr}_{B2}} \cap F_{B\times B}(U)$. We put $\gamma_j = \mathrm{pr}_{Bj}\gamma \in \mathscr{B} \cap F_B(U)$ for $j = 1, 2$. There exist coverings $(U_{ji} \xrightarrow{f_{ji}} U)_{i\in I_j}$ of U for $j = 1, 2$ such that, for any $i \in I_j$, the inverse image $(\gamma_j F(f_{ji}))^*(\boldsymbol{E})$ of $\boldsymbol{E}$ by $\gamma_j F(f_{ji}) : F(U_{ji}) \to B$ is isomorphic to the following product fibration by (P).

$$\mathrm{pr}_{F(U_{ji})} : \left(T \times F(U_{ji}), \mathscr{T}^{\mathrm{pr}_T} \cap \mathscr{F}_{U_{ji}}^{\mathrm{pr}_{F(U_{ji})}}\right) \to (F(U_{ji}), \mathscr{F}_{U_{ji}})$$

Let $R_j \in J(U)$ be the sieve generated by $(U_{ji} \xrightarrow{f_{ji}} U)_{i\in I_j}$ and put $R = R_1 \cap R_2$. Then $R \in J(U)$ and, for any $h \in R$ and $j = 1, 2$, there exists $i \in I_j$ and $g_{ji} \in \mathcal{C}(\mathrm{dom}(h), U_{ji})$ which satisfies $h = f_{ji}g_{ji}$. Since the inverse image of a product fibration is also a product fibration, the inverse image $(\gamma_j F(h))^*(\boldsymbol{E})$ of $\boldsymbol{E}$ by $\gamma_j F(h) : F(\mathrm{dom}(h)) \to B$ is isomorphic to the following product fibration for any $h \in R$ and $j = 1, 2$.

$$\boldsymbol{P}_h = \left(\left(T \times F(\mathrm{dom}(h)), \mathscr{T}^{\mathrm{pr}_T} \cap \mathscr{F}_{\mathrm{dom}(h)}^{\mathrm{pr}_{F(\mathrm{dom}(h))}}\right) \xrightarrow{\mathrm{pr}_{F(\mathrm{dom}(h))}} (F(\mathrm{dom}(h)), \mathscr{F}_{\mathrm{dom}(h)})\right)$$

Hence there exists a cartesian morphism $\boldsymbol{\gamma}_{h,j} = \langle \gamma_{h,j}, \gamma_j F(h)\rangle : \boldsymbol{P}_h \to \boldsymbol{E}$. We apply (7.19) to these cartesian morphisms $\boldsymbol{\gamma}_{h,1}$ and $\boldsymbol{\gamma}_{h,2}$. Then, we have a map $\tilde{\gamma}_h : F(\mathrm{dom}(h)) \to G_1(\boldsymbol{E})$ which makes the following diagram commute.

$$\begin{array}{ccc} F(\mathrm{dom}(h)) & \xrightarrow{\tilde{\gamma}_h} & G_1(\boldsymbol{E}) \\ \big\downarrow{\scriptstyle F(h)} & & \big\downarrow{\scriptstyle (\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}})} \\ F(U) & \xrightarrow{\gamma} & B \times B \end{array}$$

In particular, if $\gamma : F(U) \to B \times B$ is a constant map to (b_1, b_2), then γ is an F-plot of $B \times B$ and we have $(\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}})\tilde{\gamma}_h(x) = \gamma F(h) = (b_1, b_2)$, hence $(\sigma_{\boldsymbol{E}}, \tau_{\boldsymbol{E}}) : G_1(\boldsymbol{E}) \to B \times B$ is surjective. It follows from (7.19) that $\tilde{\gamma}_h : (F(\mathrm{dom}(h)), \mathscr{F}_{\mathrm{dom}(h)}) \to (G_1(\boldsymbol{E}), \mathscr{G}_{\boldsymbol{E}})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$, hence it belongs to $\mathscr{G}_{\boldsymbol{E}} \cap F_{G_1(\boldsymbol{E})}(\mathrm{dom}(h))$ by (7.14). This implies that γ belongs to $(\mathscr{G}_{\boldsymbol{E}})_{(\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}})}$ by (2.4). Therefore we conclude that $(\mathscr{G}_{\boldsymbol{E}})_{(\sigma_{\boldsymbol{E}},\tau_{\boldsymbol{E}})}$ coincides with $\mathscr{B}^{\mathrm{pr}_{B1}} \cap \mathscr{B}^{\mathrm{pr}_{B2}}$ and that $\boldsymbol{E}$ is a fibration. □

8 F-topology

Let $\mathcal{T}op$ be the category of topological spaces and continuous maps. We denote by $\mathcal{U} : \mathcal{T}op \to \mathcal{S}et$ the forgetful functor. For a functor $F : \mathcal{C} \to \mathcal{S}et$, we assume in this section that there exists a functor $F_{\mathcal{T}} : \mathcal{C} \to \mathcal{T}op$ which satisfies $F = \mathcal{U}F_{\mathcal{T}}$.

Definition 8.1 *For an object $(X,\mathscr{D})$ of $\mathscr{P}_F(\mathcal{C}, J)$, we define a set $\mathcal{O}_{(X,\mathscr{D})}$ of subsets of X by*

$$\mathcal{O}_{(X,\mathscr{D})} = \{O \subset X \mid \alpha^{-1}(O) \text{ is an open set of } F_{\mathcal{T}}(U) \text{ for any } U \in \mathrm{Ob}\,\mathcal{C} \text{ and } \alpha \in \mathscr{D} \cap F_X(U)\}.$$

It is easy to verify that $\mathcal{O}_{(X,\mathscr{D})}$ is a topology on X. In fact, $\mathcal{O}_{(X,\mathscr{D})}$ is the coarsest topology on X such that $\alpha : F_{\mathcal{T}}(U) \to X$ is continuous for any $U \in \mathrm{Ob}\,\mathcal{C}$ and $\alpha \in \mathscr{D} \cap F_X(U)$. We call $\mathcal{O}_{(X,\mathscr{D})}$ the F-topology on X associated with $\mathscr{D}$.

Let $\varphi : (X, \mathscr{D}) \to (Y, \mathscr{E})$ be a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. For $O \in \mathcal{O}_{(Y,\mathscr{E})}$ and $U \in \mathrm{Ob}\,\mathcal{C}$, $\alpha \in \mathscr{D} \cap F_X(U)$, since $\varphi\alpha = (F_\varphi)_U(\alpha) \in \mathscr{E} \cap F_Y(U)$ holds, we have $\alpha^{-1}(\varphi^{-1}(O)) = (\varphi\alpha)^{-1}(O)$ which is an open set of $F_{\mathcal{T}}(U)$. Hence we have $\varphi^{-1}(O) \in \mathcal{O}_{(X,\mathscr{D})}$ and $\varphi : (X, \mathcal{O}_{(X,\mathscr{D})}) \to (Y, \mathcal{O}_{(Y,\mathscr{E})})$ is a continuous map. Define a functor $\mathcal{T} : \mathscr{P}_F(\mathcal{C}, J) \to \mathcal{T}op$ by $\mathcal{T}((X, \mathscr{D})) = (X, \mathcal{O}_{(X,\mathscr{D})})$ and $\mathcal{T}(\varphi : (X, \mathscr{D}) \to (Y, \mathscr{E})) = (\varphi : (X, \mathcal{O}_{(X,\mathscr{D})}) \to (Y, \mathcal{O}_{(Y,\mathscr{E})}))$.

Definition 8.2 *For a topological space $(X, \mathcal{O})$, we define a set $\mathscr{D}_{(X,\mathcal{O})}$ of F-parametrizations as follows.*

$$\mathscr{D}_{(X,\mathcal{O})} = \coprod_{U \in \mathrm{Ob}\,\mathcal{C}} \{\alpha \in F_X(U) \,|\, \alpha : F_{\mathcal{T}}(U) \to X \text{ is continuous.}\}$$

If $\mathscr{D}_{(X,\mathcal{O})}$ is a the-ologgy on X, we call an element of $\mathscr{D}_{(X,\mathcal{O})}$ an F-$(X, \mathcal{O})$-plot.

The following proposition gives a sufficient condition for $\mathscr{D}_{(X,\mathcal{O})}$ being a the-ologgy on X.

Proposition 8.3 *Let $(X, \mathcal{O})$ be a topological space. If the following condition (C) is satisfied for $(X, \mathcal{O})$, then $\mathscr{D}_{(X,\mathcal{O})}$ is a the-ology on X.*

- (C) *For any $U \in \mathrm{Ob}\,\mathcal{C}$, a map $\alpha : F_{\mathcal{T}}(U) \to X$ is continuous if there exists a covering $(U_i \xrightarrow{f_i} U)_{i \in I}$ of U such that compositions $F_{\mathcal{T}}(U_i) \xrightarrow{F_{\mathcal{T}}(f_i)} F_{\mathcal{T}}(U) \xrightarrow{\alpha} X$ are continuous for any $i \in I$.*

Proof. Since $F(1_{\mathcal{C}})$ has only one element, every map from $F_{\mathcal{T}}(1_{\mathcal{C}})$ to X is continuous. Hence $\mathscr{D}_{(X,\mathcal{O})} \supset F_X(1_{\mathcal{C}})$ holds. For a morphism $f : U \to V$ in $\mathcal{C}$ and $\alpha \in \mathscr{D}_{(X,\mathcal{O})} \cap F_X(V)$, since $F_{\mathcal{T}}(f) : F_{\mathcal{T}}(V) \to F_{\mathcal{T}}(U)$ is continuous, so is $F_X(f)(\alpha) = \alpha F_{\mathcal{T}}(f) : F_{\mathcal{T}}(U) \to X$. It follows that $F_X(f)(\alpha) \in \mathscr{D}_{(X,\mathcal{O})} \cap F_X(U)$. For an object U of $\mathcal{C}$, suppose that there exists a covering $(U_i \xrightarrow{f_i} U)_{i \in I}$ such that $F_X(f_i) : F_X(U) \to F_X(U_i)$ maps $\alpha \in F_X(U)$ into $\mathscr{D}_{(X,\mathcal{O})} \cap F_X(U_i)$ for any $i \in I$. Then, $\alpha F_{\mathcal{T}}(f_i) = F_X(f_i)(\alpha) : F_{\mathcal{T}}(U_i) \to X$ is continuous for any $i \in I$. Hence $\alpha : F_{\mathcal{T}}(U) \to X$ is continuous and belongs to $\mathscr{D}_{(X,\mathcal{O})} \cap F_X(U)$. □

Remark 8.4 *We consider the following condition (Q) on $F_{\mathcal{T}} : \mathcal{C} \to \mathcal{T}op$.*

- (Q) *For any $U \in \mathrm{Ob}\,\mathcal{C}$, there exists a covering $(U_i \xrightarrow{f_i} U)_{i \in I}$ of U such that the map $\coprod_{i \in I} F_{\mathcal{T}}(U_i) \to F_{\mathcal{T}}(U)$ induced by the family $\big(F_{\mathcal{T}}(U_i) \xrightarrow{F_{\mathcal{T}}(f_i)} F_{\mathcal{T}}(U)\big)_{i \in I}$ of maps is a quotient map.*

If the condition (Q) is satisfied, the condition (C) of (8.3) is satisfied for any topological space $(X, \mathcal{O})$.

Lemma 8.5 *Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ and $(Z, \mathcal{O}_Z)$ be topological spaces. For continuous maps $f : X \to Y$ and $g : Y \to Z$, if $gf : X \to Z$ is a quotient map, so is g.*

Proof. For an open set O of Z, assume that $g^{-1}(O)$ is an open set of Y. Then, $f^{-1}(g^{-1}(O)) = (gf)^{-1}(O)$ is an open set by the continuity of f. It follows from the assumption that O is an open set of Z. □

Proposition 8.6 *For an object U of $\mathcal{C}$, suppose that there exists a covering R of U such that the map $\rho : \coprod_{f \in R} F_{\mathcal{T}}(\mathrm{dom}(f)) \to F_{\mathcal{T}}(U)$ induced by the family $\big(F_{\mathcal{T}}(\mathrm{dom}(f)) \xrightarrow{F_{\mathcal{T}}(f)} F_{\mathcal{T}}(U)\big)_{f \in R}$ of maps is a quotient map. Let $\bar{R}$ be the sieve on U generated by R. Then, the map $\bar{\rho} : \coprod_{u \in \bar{R}} F_{\mathcal{T}}(\mathrm{dom}(u)) \to F_{\mathcal{T}}(U)$ induced by the family $\big(F_{\mathcal{T}}(\mathrm{dom}(u)) \xrightarrow{F_{\mathcal{T}}(u)} F_{\mathcal{T}}(U)\big)_{u \in \bar{R}}$ of maps is a quotient map.*

Proof. For $u \in \bar{R}$, there exist $f_u \in R$ and $g_u \in \mathrm{Mor}\,\mathcal{C}$ such that $\mathrm{codom}(g_u) = \mathrm{dom}(f_u)$ and $u = f_u g_u$. We put $X = \coprod_{f \in R} F_{\mathcal{T}}(\mathrm{dom}(f))$ and $Y = \coprod_{u \in \bar{R} - R} F_{\mathcal{T}}(\mathrm{dom}(u))$, then we have $X \coprod Y = \coprod_{u \in \bar{R}} F_{\mathcal{T}}(\mathrm{dom}(u))$. Let $\rho' : \coprod_{u \in \bar{R} - R} F_{\mathcal{T}}(\mathrm{dom}(u)) \to F_{\mathcal{T}}(U)$ be the map induced by the family $\big(F_{\mathcal{T}}(\mathrm{dom}(u)) \xrightarrow{F_{\mathcal{T}}(u)} F_{\mathcal{T}}(U)\big)_{u \in \bar{R} - R}$ of maps. We denote by $\iota_X : X \to X \coprod Y$ and $\iota_Y : Y \to X \coprod Y$ the inclusion maps. Then $\bar{\rho} : X \coprod Y \to F_{\mathcal{T}}(U)$ is the unique map that satisfy $\bar{\rho}\iota_X = \rho$ and $\bar{\rho}\iota_Y = \rho'$. Since ρ is a quotient map, so is $\bar{\rho}$ by (8.5). □

Thus we have the following result.

Proposition 8.7 *The condition (Q) in (8.4) is equivalent to the following condition.*

- (Q') *For any $U \in \mathrm{Ob}\,\mathcal{C}$, there exists $R \in J(U)$ such that the map $\coprod_{f \in R} F_{\mathcal{T}}(\mathrm{dom}(f)) \to F_{\mathcal{T}}(U)$ induced by the family $\big(F_{\mathcal{T}}(\mathrm{dom}(f)) \xrightarrow{F_{\mathcal{T}}(f)} F_{\mathcal{T}}(U)\big)_{f \in R}$ of maps is a quotient map.*

Proposition 8.8 *(1) For an object* $(X, \mathscr{D})$ *of* $\mathscr{P}_F(\mathcal{C}, J)$*, we have* $\mathscr{D} \subset \mathscr{D}_{(X,\mathcal{O}_{(X,\mathscr{D})})}$.
(2) For a topological space $(X, \mathcal{O})$*,* $\mathcal{O} \subset \mathcal{O}_{(X,\mathscr{D}_{(X,\mathcal{O})})}$ *holds.*

Proof. (1) For $U \in \operatorname{Ob}\mathcal{C}$ and $\alpha \in \mathscr{D} \cap F_X(U)$, since $\alpha : F_{\mathcal{T}}(U) \to X$ is continuous map with respect to the topology $\mathcal{O}_{(X,\mathscr{D})}$ on X, it follows $\alpha \in \mathscr{D}_{(X,\mathcal{O}_{(X,\mathscr{D})})} \cap F_X(U)$. Therefore $\mathscr{D} \subset \mathscr{D}_{(X,\mathcal{O}_{(X,\mathscr{D})})}$ holds.

(2) For $U \in \operatorname{Ob}\mathcal{C}$ and $\alpha \in \mathscr{D}_{(X,\mathcal{O})} \cap F_X(U)$, since $\alpha : F_{\mathcal{T}}(U) \to X$ is continuous, $\alpha^{-1}(O)$ is an open set of $F_{\mathcal{T}}(U)$ for any $O \in \mathcal{O}$. By the definition of $\mathcal{O}_{(X,\mathscr{D}_{(X,\mathcal{O})})}$, we have $\mathcal{O} \subset \mathcal{O}_{(X,\mathscr{D}_{(X,\mathcal{O})})}$. □

Assume that $(X, \mathscr{D}_{(X,\mathcal{O})})$ is an object of $\mathscr{P}_F(\mathcal{C}, J)$ for any topological space $(X, \mathcal{O})$. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces and $f : X \to Y$ a continuous map. Then $f : (X, \mathscr{D}_{(X,\mathcal{O}_X)}) \to (Y, \mathscr{D}_{(Y,\mathcal{O}_Y)})$ is a morphism in $\mathscr{P}_F(\mathcal{C}, J)$. In fact, for $U \in \operatorname{Ob}\mathcal{C}$ and $\alpha \in \mathscr{D}_{(X,\mathcal{O}_X)} \cap F_X(U)$, since $(F_f)_U(\alpha) = f\alpha : F_{\mathcal{T}}(U) \to Y$ is continuous, $(F_f)_U(\alpha) \in \mathscr{D}_{(Y,\mathcal{O}_Y)} \cap F_Y(U)$ holds. We define a functor $\mathcal{P} : \mathcal{T}op \to \mathscr{P}_F(\mathcal{C}, J)$ by $\mathcal{P}((X, \mathcal{O})) = (X, \mathscr{D}_{(X,\mathcal{O})})$ for an object $(X, \mathcal{O})$ of $\mathcal{T}op$ and $\mathcal{P}(f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)) = (f : (X, \mathscr{D}_{(X,\mathcal{O}_X)}) \to (Y, \mathscr{D}_{(Y,\mathcal{O}_Y)}))$ for a continuous map $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. We remark that $\mathit{\Gamma}_F\mathcal{P} = \mathcal{U}$ and $\mathcal{U}\mathcal{T} = \mathit{\Gamma}_F$ hold and that both $\mathcal{P}$ and $\mathcal{T}$ are faithful.

Proposition 8.9 *Suppose that* $(X, \mathscr{D}_{(X,\mathcal{O})})$ *is an object of* $\mathscr{P}_F(\mathcal{C}, J)$ *for any topological space* $(X, \mathcal{O})$*. Then,* $\mathcal{P} : \mathcal{T}op \to \mathscr{P}_F(\mathcal{C}, J)$ *is a right adjoint of* $\mathcal{T} : \mathscr{P}_F(\mathcal{C}, J) \to \mathcal{T}op$*.*

Proof. It follows from (1) of (8.8) that we have a morphism $\eta_{(X,\mathscr{D})} : (X, \mathscr{D}) \to (X, \mathscr{D}_{(X,\mathcal{O}_{(X,\mathscr{D})})}) = \mathcal{P}\mathcal{T}((X, \mathscr{D}))$ in $\mathscr{P}_F(\mathcal{C}, J)$ which is natural in $(X, \mathscr{D}) \in \operatorname{Ob}\mathscr{P}_F(\mathcal{C}, J)$. It follows from (2) of (8.8) that we have a continuous bijection $\varepsilon_{(X,\mathcal{O})} : \mathcal{T}\mathcal{P}((X, \mathcal{O})) = (X, \mathcal{O}_{(X,\mathscr{D}_{(X,\mathcal{O})})}) \to (X, \mathcal{O})$ which is natural in $(X, \mathcal{O}) \in \operatorname{Ob}\mathcal{T}op$. Then, $\eta : id_{\mathscr{P}_F(\mathcal{C},J)} \to \mathcal{P}\mathcal{T}$ and $\varepsilon : \mathcal{T}\mathcal{P} \to id_{\mathcal{T}op}$ are the unit and the counit of the adjunction $\mathcal{T} \dashv \mathcal{P}$, respectively. □

For a topological space $(Y, \mathcal{O}_Y)$ and a map $f : X \to Y$, we put $\mathcal{O}^f = \{O \subset X \mid O = f^{-1}(V) \text{ for some } V \in \mathcal{O}_Y\}$. Then $\mathcal{O}^f$ is the coarsest topology on X such that $f : X \to Y$ is a continuous map.

Proposition 8.10 *For a map* $f : X \to Y$ *and an object* $(Y, \mathscr{E})$ *of* $\mathscr{P}_F(\mathcal{C}, J)$*, consider the the-ology* $\mathscr{E}^f$ *on* X*. Then, the* F*-topology* $\mathcal{O}_{(X,\mathscr{E}^f)}$ *on* X *associated with* $\mathscr{E}^f$ *is finer than* $\mathcal{O}^f_{(Y,\mathscr{E})}$*.*

Proof. For $V \in \mathcal{O}_{(Y,\mathscr{E})}$, $U \in \operatorname{Ob}\mathcal{C}$ and $\alpha \in \mathscr{E}^f \cap F_X(U)$, since $\alpha^{-1}(f^{-1}(V)) = (f\alpha)^{-1}(V)$ and $f\alpha \in \mathscr{E} \cap F_Y(U)$, $\alpha^{-1}(f^{-1}(V))$ is an open set of $F_{\mathcal{T}}(U)$. Hence we have $f^{-1}(V) \in \mathcal{O}_{(X,\mathscr{E}^f)}$ which implies $\mathcal{O}^f_{(Y,\mathscr{E})} \subset \mathcal{O}_{(X,\mathscr{E}^f)}$. □

For a topological space $(X, \mathcal{O}_X)$ and a map $f : X \to Y$, we put $\mathcal{O}_f = \{O \subset Y \mid f^{-1}(O) \in \mathcal{O}_X\}$. Then $\mathcal{O}_f$ is the finest topology on Y such that $f : X \to Y$ is a continuous map.

Proposition 8.11 *For a map* $f : X \to Y$ *and an object* $(X, \mathscr{D})$ *of* $\mathscr{P}_F(\mathcal{C}, J)$*, consider the the-ology* $\mathscr{D}_f$ *on* Y*. Then, the* F*-topology* $\mathcal{O}_{(Y,\mathscr{D}_f)}$ *on* Y *associated with* $\mathscr{D}_f$ *is coarser than* $(\mathcal{O}_{(X,\mathscr{D})})_f$*. If* $F_{\mathcal{T}} : \mathcal{C} \to \mathcal{T}op$ *satisfies the following condition* (Q'')*,* $\mathcal{O}_{(Y,\mathscr{D}_f)}$ *coincides with* $(\mathcal{O}_{(X,\mathscr{D})})_f$*.*

(Q'') *For any* $U \in \operatorname{Ob}\mathcal{C}$ *and* $R \in J(U)$*, the map* $\coprod_{f \in R} F_{\mathcal{T}}(\operatorname{dom}(f)) \to F_{\mathcal{T}}(U)$ *induced by the family*

$$\left(F_{\mathcal{T}}(\operatorname{dom}(h)) \xrightarrow{F_{\mathcal{T}}(h)} F_{\mathcal{T}}(U)\right)_{h \in R} \text{ of maps is a quotient map.}$$

Proof. For $O \in \mathcal{O}_{(Y,\mathscr{D}_f)}$, $U \in \operatorname{Ob}\mathcal{C}$ and $\alpha \in \mathscr{D} \cap F_X(U)$, since $\alpha^{-1}(f^{-1}(O)) = (f\alpha)^{-1}(O)$ and $f\alpha = (F_f)_U(\alpha)$ belongs to $\mathscr{D}_f \cap F_Y(U)$, $\alpha^{-1}(f^{-1}(O))$ is an open set of $F_{\mathcal{T}}(U)$. Hence we have $f^{-1}(O) \in \mathcal{O}_{(X,\mathscr{D})}$ which shows $O \in (\mathcal{O}_{(X,\mathscr{D})})_f$. Therefore $\mathcal{O}_{(Y,\mathscr{D}_f)} \subset (\mathcal{O}_{(X,\mathscr{D})})_f$ holds.

Assume that $F_{\mathcal{T}}$ satisfies (Q''). We take $O \in (\mathcal{O}_{(X,\mathscr{D})})_f$, $U \in \operatorname{Ob}\mathcal{C}$ and $\alpha \in \mathscr{D}_f \cap F_Y(U)$. There exists $R \in J(U)$ such that $F_Y(h)(\alpha) \in \bigcup_{g \in \operatorname{Mor}\mathcal{C}} \mathcal{S}_g$ for all $h \in R$. Then, $F_Y(h)(\alpha) \in \mathcal{S}_{g_h}$ for some $g_h \in \operatorname{Mor}\mathcal{C}$ such that $\operatorname{dom}(g_h) = \operatorname{dom}(h)$. Assume that $\operatorname{codom}(g_h) \neq 1_{\mathcal{C}}$. Since $\mathcal{S}_{g_h} = (F_f)_{\operatorname{dom}(g_h)}(F_X(g_h)(\mathscr{D} \cap F_X(\operatorname{codom}(g_h))))$ by (2.4), there exists $j_h \in \mathscr{D} \cap F_X(\operatorname{codom}(g_h))$ such that $F_Y(h)(\alpha) = (F_f)_{\operatorname{dom}(g_h)}(F_X(g_h)(j_h))$. Thus we have the following commutative diagram.

$$\begin{array}{ccccc}
F(\operatorname{dom}(g_h)) & = & F(\operatorname{dom}(h)) & \xrightarrow{F(h)} & F(U) \\
\downarrow{\scriptstyle F(g_h)} & & & & \downarrow{\scriptstyle \alpha} \\
F(\operatorname{codom}(g_h)) & \xrightarrow{j_h} & X & \xrightarrow{f} & Y
\end{array}$$

Since $j_h \in \mathscr{D}$ and $f^{-1}(O) \in \mathcal{O}_{(X,\mathscr{D})}$, $j_h^{-1}(f^{-1}(O))$ is an open set of $F_{\mathcal{T}}(\mathrm{codom}(g_h))$. Then the continuity of $F(g_h)$ implies that $F(h)^{-1}(\alpha^{-1}(O)) = F(g_h)^{-1}(j_h^{-1}(f^{-1}(O)))$ is an open set of $F(\mathrm{dom}(h))$. Consider the case $\mathrm{codom}(g_h) = 1_{\mathcal{C}}$. Then, $\mathcal{S}_{g_h} = F_Y(g_h)(F_Y(1_{\mathcal{C}}))$ by (2.4) and there exists a constant map $j_h \in F_Y(1_{\mathcal{C}})$ such that $\alpha F(h) = F_Y(h)(\alpha) = F_Y(g_h)(j_h) = j_h F(g_h)$ which is a constant map. It follows that $F(h)^{-1}(\alpha^{-1}(O))$ concides with $F(\mathrm{dom}(h))$ if O contains the image of j_h and that $F(h)^{-1}(\alpha^{-1}(O))$ is empty otherwise. Therefore $F(h)^{-1}(\alpha^{-1}(O))$ is an open set of $F_{\mathcal{T}}(\mathrm{dom}(h))$ for any $h \in R$. It follows from (Q'') that $\alpha^{-1}(O)$ is an open set of $F_{\mathcal{T}}(U)$ for any $\alpha \in \mathscr{D}_f \cap F_Y(U)$. Hence $O \in \mathcal{O}_{(Y,\mathscr{D}_f)}$ holds and we have $(\mathcal{O}_{(X,\mathscr{D})})_f \subset \mathcal{O}_{(Y,\mathscr{D}_f)}$. □

9 Representations of groupoids in the category of plots

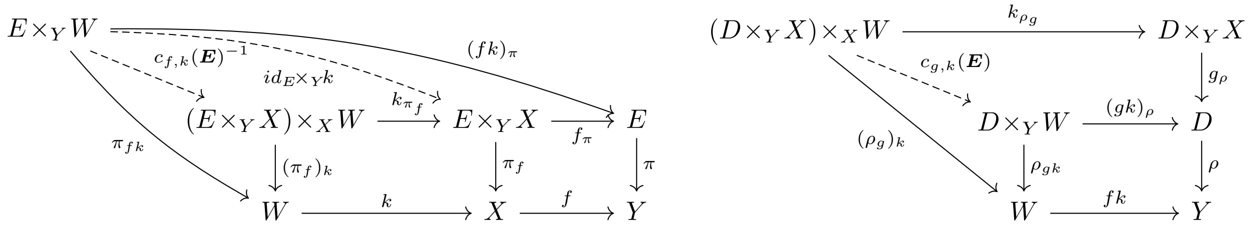
Let $f : (X,\mathscr{X}) \to (Y,\mathscr{Y})$, $g : (X,\mathscr{X}) \to (Z,\mathscr{Z})$, $k : (W,\mathscr{W}) \to (X,\mathscr{X})$ be morphisms in $\mathscr{P}_F(\mathcal{C},J)$ and $\boldsymbol{E} = ((E,\mathscr{E}) \xrightarrow{\pi} (Y,\mathscr{Y}))$, $\boldsymbol{D} = ((D,\mathscr{D}) \xrightarrow{\rho} (Z,\mathscr{Z}))$ objects of $\mathscr{P}_F(\mathcal{C},J)^{(2)}$. It follows from (3.3) that there are isomorphisms $\boldsymbol{c}_{f,k}(\boldsymbol{E})^{-1} : (fk)^*(\boldsymbol{E}) \to k^*(f^*(\boldsymbol{E}))$ and $\boldsymbol{c}_{g,k}(\boldsymbol{D}) : k^*(g^*(\boldsymbol{D})) \to (gk)^*(\boldsymbol{D})$ in $\mathscr{P}_F(\mathcal{C},J)^{(2)}$. Consider the following diagrams whose rectangles are all cartesian.

$$\begin{array}{ccccccccc}
(E\times_Y X)\times_X W & \xrightarrow{k_{\pi_f}} & E\times_Y X & \xrightarrow{f_\pi} & E & \qquad & E\times_Y W & \xrightarrow{(fk)_\pi} & E \\
\downarrow{\scriptstyle (\pi_f)_k} & & \downarrow{\scriptstyle \pi_f} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{fk}} & & \downarrow{\scriptstyle \pi} \\
W & \xrightarrow{k} & X & \xrightarrow{f} & Y & & W & \xrightarrow{fk} & Y \\
(D\times_Z X)\times_X W & \xrightarrow{k_{\rho_g}} & D\times_Z X & \xrightarrow{g_\rho} & D & & D\times_Z W & \xrightarrow{(gk)_\rho} & D \\
\downarrow{\scriptstyle (\rho_g)_k} & & \downarrow{\scriptstyle \rho_g} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho_{gk}} & & \downarrow{\scriptstyle \rho} \\
W & \xrightarrow{k} & X & \xrightarrow{g} & Z & & W & \xrightarrow{gk} & Z
\end{array}$$

It follows from (3.3) and (3.4) that we have unique isomorphisms in $\mathscr{P}_F(\mathcal{C},J)$

$$c_{f,k}(\boldsymbol{E})^{-1} : (E\times_Y W, \mathscr{E}^{(fk)_\pi} \cap \mathscr{W}^{\pi_{fk}}) \to ((E\times_Y X)\times_X W, (\mathscr{E}^{f_\pi} \cap \mathscr{X}^{\pi_f})^{k_{\pi_f}} \cap \mathscr{W}^{(\pi_f)_k})$$
$$c_{g,k}(\boldsymbol{E}) : ((D\times_Z X)\times_X W, (\mathscr{D}^{g_\rho} \cap \mathscr{X}^{\rho_g})^{k_{\rho_g}} \cap \mathscr{W}^{(\rho_g)_k}) \to (D\times_Z W, \mathscr{D}^{(gk)_\rho} \cap \mathscr{W}^{\rho_{gk}})$$

that make following diagram commute.



We note that $\boldsymbol{c}_{f,k}(\boldsymbol{E})^{-1} = \langle c_{f,k}(\boldsymbol{E})^{-1}, id_W\rangle$ and $\boldsymbol{c}_{g,k}(\boldsymbol{D}) = \langle c_{g,k}(\boldsymbol{D}), id_W\rangle$ hold. The following fact follows from the above diagrams.

Proposition 9.1 *$c_{f,k}(\boldsymbol{E})^{-1}$ and $c_{g,k}(\boldsymbol{D})$ are given by $c_{f,k}(\boldsymbol{E})^{-1}(u,w) = (u,k(w),w)$ for $(u,w) \in E\times_Y W$ and $c_{g,k}(\boldsymbol{D})(v,x,w) = (v,w)$ for $(v,x,w) \in (D\times_Z X)\times_X W$, respectively.*

For a morphism $\boldsymbol{\xi} : f^*(\boldsymbol{E}) \to g^*(\boldsymbol{D})$ in $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(X,\mathscr{X})}$, we define a morphism $\boldsymbol{\xi}_k : (fk)^*(\boldsymbol{E}) \to (gk)^*(\boldsymbol{D})$ in $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(W,\mathscr{W})}$ to be a composition $(fk)^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{c}_{f,k}(\boldsymbol{E})^{-1}} k^*(f^*(\boldsymbol{E})) \xrightarrow{k^*(\boldsymbol{\xi})} k^*(g^*(\boldsymbol{D})) \xrightarrow{\boldsymbol{c}_{g,k}(\boldsymbol{D})} (gk)^*(\boldsymbol{D})$. We put $\boldsymbol{\xi} = \langle \xi, id_X\rangle$, where $\xi : (E\times_Y X, \mathscr{E}^{f_\pi} \cap \mathscr{X}^{\pi_f}) \to (D\times_Z X, \mathscr{D}^{g_\rho} \cap \mathscr{X}^{\rho_g})$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$ which satisfies $\rho_g \xi = \pi_f$. Then, there exists unique morphism

$$\xi \times_X id_W : ((E\times_Y X)\times_X W, (\mathscr{E}^{f_\pi} \cap \mathscr{X}^{\pi_f})^{k_{\pi_f}} \cap \mathscr{W}^{(\pi_f)_k}) \to ((D\times_Z X)\times_X W, (\mathscr{D}^{g_\rho} \cap \mathscr{X}^{\rho_g})^{k_{\rho_g}} \cap \mathscr{W}^{(\rho_g)_k})$$

that makes the following diagram commute.

$$\begin{array}{ccccc}
W & \xleftarrow{(\pi_f)_k} & (E\times_Y X)\times_X W & \xrightarrow{k_{\pi_f}} & E\times_Y X \\
\downarrow{\scriptstyle id_W} & & \downarrow{\scriptstyle \xi\times_X id_W} & & \downarrow{\scriptstyle \xi} \\
W & \xleftarrow{(\rho_g)_k} & (D\times_Z X)\times_X W & \xrightarrow{k_{\rho_g}} & D\times_Z X
\end{array}$$

Then, we have $k^*(\boldsymbol{\xi}) = \langle \xi \times_X id_W, id_W\rangle$. We denote by $\xi_k : (E\times_Y W, \mathscr{E}^{(fk)_\pi}\cap \mathscr{W}^{\pi_{fk}}) \to (D\times_Z X, \mathscr{D}^{(gk)_\rho}\cap \mathscr{W}^{\rho_{gk}})$ the following composition.

$$(E\times_Y W, \mathscr{E}^{(fk)_\pi}\cap \mathscr{W}^{\pi_{fk}}) \xrightarrow{c_{f,k}(\boldsymbol{E})^{-1}} ((E\times_Y X)\times_X W, (\mathscr{E}^{f_\pi}\cap \mathscr{X}^{\pi_f})^{k_{\pi_f}}\cap \mathscr{W}^{(\pi_f)_k}) \xrightarrow{\xi\times_X id_W} ((D\times_Z X)\times_X W, (\mathscr{D}^{g_\rho}\cap \mathscr{X}^{\rho_g})^{k_{\rho_g}}\cap \mathscr{W}^{(\rho_g)_k}) \xrightarrow{c_{g,k}(\boldsymbol{D})} (D\times_Z W, \mathscr{D}^{(gk)_\rho}\cap \mathscr{W}^{\rho_{gk}})$$

It follows from the definition of $\boldsymbol{\xi}_k : (fk)^*(\boldsymbol{E}) \to (gk)^*(\boldsymbol{D})$ that $\boldsymbol{\xi}_k = \langle \xi_k, id_W\rangle$. Since $\rho_g\xi = \pi_f$, we have $\xi(u,x) = (g_\rho\xi(u,x), x)$ for $(u,x)\in E\times_Y X$. Thus we have the following result.

Proposition 9.2 *ξ_k maps $(u,w)\in E\times_Y W$ to $(g_\rho\xi(u,k(w)),w)\in D\times_Y W$.*

Let $\boldsymbol{G} = ((G_0,\mathscr{G}_0),(G_1,\mathscr{G}_1);\sigma,\tau,\varepsilon,\mu,\iota)$ be a groupoid in $\mathscr{P}_F(\mathcal{C},J)$ and $\boldsymbol{E} = ((E,\mathscr{E}) \xrightarrow{\pi} (G_0,\mathscr{G}_0))$ be an object of $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(G_0,\mathscr{G}_0)}$. Recall that we consider the following cartesian square.

$$\begin{array}{ccc} G_1\times_{G_0} G_1 & \xrightarrow{\mathrm{pr}_2} & G_1 \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \sigma} \\ G_1 & \xrightarrow{\tau} & G_0 \end{array}$$

Definition 9.3 *We call a pair $(\boldsymbol{E},\boldsymbol{\xi})$ of object $\boldsymbol{E}$ of $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(G_0,\mathscr{G}_0)}$ and a morphism $\boldsymbol{\xi} : \sigma^*(\boldsymbol{E}) \to \tau^*(\boldsymbol{E})$ in $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(G_1,\mathscr{G}_1)}$ a representation of $\boldsymbol{G}$ on $\boldsymbol{E}$ if $\boldsymbol{\xi}$ satisfies the following conditions.*

(A) The following diagram is commutative.

$$\begin{array}{ccc} (\sigma\mathrm{pr}_1)^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{\xi}_{\mathrm{pr}_1}} (\tau\mathrm{pr}_1)^*(\boldsymbol{E}) = (\sigma\mathrm{pr}_2)^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{\xi}_{\mathrm{pr}_2}} (\tau\mathrm{pr}_2)^*(\boldsymbol{E}) \\ \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ (\sigma\mu)^*(\boldsymbol{E}) \xrightarrow{\qquad\qquad\boldsymbol{\xi}_\mu\qquad\qquad} (\tau\mu)^*(\boldsymbol{E}) \end{array}$$

(U) $\boldsymbol{\xi}_\varepsilon : id^*_{G_0}(\boldsymbol{E}) = (\sigma\varepsilon)^*(\boldsymbol{E}) \to (\tau\varepsilon)^*(\boldsymbol{E}) = id^*_{G_0}(\boldsymbol{E})$ coincides with the identity morphism of $id^*_{G_0}(\boldsymbol{E}) = \boldsymbol{E}$.

Definition 9.4 *Let $(\boldsymbol{E},\boldsymbol{\xi})$ and $(\boldsymbol{D},\boldsymbol{\zeta})$ be representations of $\boldsymbol{G}$ on $\boldsymbol{E}$ and $\boldsymbol{D}$, respectively. If a morphism $\boldsymbol{\varphi} : \boldsymbol{E}\to\boldsymbol{D}$ in $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(G_0,\mathscr{G}_0)}$ makes the following diagram commute, we call $\boldsymbol{\varphi}$ a morphism of representations.*

$$\begin{array}{ccc} \sigma^*(\boldsymbol{E}) & \xrightarrow{\boldsymbol{\xi}} & \tau^*(\boldsymbol{E}) \\ \downarrow{\scriptstyle \sigma^*(\boldsymbol{\varphi})} & & \downarrow{\scriptstyle \tau^*(\boldsymbol{\varphi})} \\ \sigma^*(\boldsymbol{D}) & \xrightarrow{\boldsymbol{\zeta}} & \tau^*(\boldsymbol{D}) \end{array}$$

We denote by $\mathrm{Rep}(\boldsymbol{G})$ *the category whose objects are representations of $\boldsymbol{G}$ and morphisms are morphisms of representations. We call* $\mathrm{Rep}(\boldsymbol{G})$ *the category of representations of $\boldsymbol{G}$.*

Let $\boldsymbol{G} = ((G_0,\mathscr{G}_0),(G_1,\mathscr{G}_1);\sigma,\tau,\varepsilon,\mu,\iota)$, $\boldsymbol{H} = ((H_0,\mathscr{H}_0),(H_1,\mathscr{H}_1);\sigma',\tau',\varepsilon',\mu',\iota')$ be groupoids in $\mathscr{P}_F(\mathcal{C},J)$ and $\boldsymbol{f} = (f_0,f_1) : \boldsymbol{H}\to\boldsymbol{G}$ a morphism of groupoids. For a representation $(\boldsymbol{E},\boldsymbol{\xi})$ of $\boldsymbol{G}$ on $\boldsymbol{E}$, we define a morphism $\boldsymbol{\xi_f} : \sigma'^*(f_0^*(\boldsymbol{E})) \to \tau'^*(f_0^*(\boldsymbol{E}))$ in $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(H_1,\mathscr{H}_1)}$ to be the following composition.

$$\sigma'^*(f_0^*(\boldsymbol{E})) \xrightarrow{c_{f_0,\sigma'}(\boldsymbol{E})} (f_0\sigma')^*(\boldsymbol{E}) = (\sigma f_1)^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{\xi}_{f_1}} (\tau f_1)^*(\boldsymbol{E}) = (f_0\tau')^*(\boldsymbol{E}) \xrightarrow{c_{f_0,\tau'}(\boldsymbol{E})^{-1}} \tau'^*(f_0^*(\boldsymbol{E}))$$

Proposition 9.5 *([10],[11]) $(f_0^*(\boldsymbol{E}),\boldsymbol{\xi_f})$ is a representation of $\boldsymbol{H}$ on $f_0^*(\boldsymbol{E})$.*

Proposition 9.6 *([10], [11]) Let $(\boldsymbol{E},\boldsymbol{\xi})$ and $(\boldsymbol{D},\boldsymbol{\zeta})$ be objects of* $\mathrm{Rep}(\boldsymbol{G})$ *and $\boldsymbol{\varphi} : (\boldsymbol{E},\boldsymbol{\xi}) \to (\boldsymbol{D},\boldsymbol{\zeta})$ a morphism in* $\mathrm{Rep}(\boldsymbol{G})$*. For a morphism $\boldsymbol{f} = (f_0,f_1) : \boldsymbol{H}\to\boldsymbol{G}$ of groupoids in $\mathscr{P}_F(\mathcal{C},J)$, $f_0^*(\boldsymbol{\varphi}) : f_0^*(\boldsymbol{E}) \to f_0^*(\boldsymbol{D})$ defines a morphism $f_0^*(\boldsymbol{\varphi}) : (f_0^*(\boldsymbol{E}),\boldsymbol{\xi_f}) \to (f_0^*(\boldsymbol{D}),\boldsymbol{\zeta_f})$ in* $\mathrm{Rep}(\boldsymbol{H})$.

(9.4) and (9.5) enable us to define the notion of restriction functor.

Definition 9.7 *Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be groupoids in $\mathscr{P}_F(\mathcal{C},J)$. For a morphism $\boldsymbol{f}=(f_0,f_1):\boldsymbol{H}\to\boldsymbol{G}$ of groupoids in $\mathscr{P}_F(\mathcal{C},J)$, define a functor $\boldsymbol{f}^\bullet:\mathrm{Rep}(\boldsymbol{G})\to\mathrm{Rep}(\boldsymbol{D})$ by $\boldsymbol{f}^\bullet(\boldsymbol{E},\boldsymbol{\xi})=(f_0^*(\boldsymbol{E}),\boldsymbol{\xi_f})$ for an object $(\boldsymbol{E},\boldsymbol{\xi})$ of $\mathrm{Rep}(\boldsymbol{G})$ and $\boldsymbol{f}^\bullet(\boldsymbol{\varphi})=f_0^*(\boldsymbol{\varphi})$ for a morphism $\boldsymbol{\varphi}:(\boldsymbol{E},\boldsymbol{\xi})\to(\boldsymbol{D},\boldsymbol{\zeta})$ in $\mathrm{Rep}(\boldsymbol{G})$. We call $(f_0^*(\boldsymbol{E}),\boldsymbol{\xi_f})$ the restriction of $(\boldsymbol{E},\boldsymbol{\xi})$ along $\boldsymbol{f}$ and $\boldsymbol{f}^\bullet$ the restriction functor associated with $\boldsymbol{f}$.*

We consider the following diagrams whose rectangles are cartesian.

$$\begin{array}{ccccc}
(E\times_{G_0}H_0)\times^{\sigma'}_{H_0}H_1 & \xrightarrow{\sigma'_{\pi_{f_0}}} & E\times_{G_0}H_0 & \xrightarrow{(f_0)_\pi} & E\\
\downarrow{\scriptstyle (\pi_{f_0})_{\sigma'}} & & \downarrow{\scriptstyle \pi_{f_0}} & & \downarrow{\scriptstyle\pi}\\
H_1 & \xrightarrow{\sigma'} & H_0 & \xrightarrow{f_0} & G_0
\end{array}
\qquad
\begin{array}{ccccc}
(E\times_{G_0}H_0)\times^{\tau'}_{H_0}H_1 & \xrightarrow{\tau'_{\pi_{f_0}}} & E\times_{G_0}H_0 & \xrightarrow{(f_0)_\pi} & E\\
\downarrow{\scriptstyle (\pi_{f_0})_{\tau'}} & & \downarrow{\scriptstyle \pi_{f_0}} & & \downarrow{\scriptstyle\pi}\\
H_1 & \xrightarrow{\tau'} & H_0 & \xrightarrow{f_0} & G_0
\end{array}$$

$$\begin{array}{ccccc}
(E\times^{\sigma}_{G_0}G_1)\times_{G_1}H_1 & \xrightarrow{(f_1)_{\pi_\sigma}} & E\times^{\sigma}_{G_0}G_1 & \xrightarrow{\sigma_\pi} & E\\
\downarrow{\scriptstyle (\pi_\sigma)_{f_1}} & & \downarrow{\scriptstyle \pi_\sigma} & & \downarrow{\scriptstyle\pi}\\
H_1 & \xrightarrow{f_1} & G_1 & \xrightarrow{\sigma} & G_0
\end{array}
\qquad
\begin{array}{ccccc}
(E\times^{\tau}_{G_0}G_1)\times_{G_1}H_1 & \xrightarrow{(f_1)_{\pi_\tau}} & E\times^{\tau}_{G_0}G_1 & \xrightarrow{\tau_\pi} & E\\
\downarrow{\scriptstyle (\pi_\tau)_{f_1}} & & \downarrow{\scriptstyle \pi_\tau} & & \downarrow{\scriptstyle\pi}\\
H_1 & \xrightarrow{f_1} & G_1 & \xrightarrow{\tau} & G_0
\end{array}$$

The following result can be verified from the definition of $\boldsymbol{\xi_f}$.

Proposition 9.8 *We put $\boldsymbol{\xi_f}=\langle\xi_{\boldsymbol f},id_{H_0}\rangle$ for a morphism*

$$\xi_{\boldsymbol f}:((E\times_{G_0}H_0)\times^{\sigma'}_{H_0}H_1,(\mathscr{E}^{(f_0)_\pi}\cap\mathscr{H}_0^{\pi_{f_0}})^{\sigma'_{\pi_{f_0}}}\cap\mathscr{H}_1^{(\pi_{f_0})_{\sigma'}})\to((E\times_{G_0}H_0)\times^{\tau'}_{H_0}H_1,(\mathscr{E}^{(f_0)_\pi}\cap\mathscr{H}_0^{\pi_{f_0}})^{\tau'_{\pi_{f_0}}}\cap\mathscr{H}_1^{(\pi_{f_0})_{\tau'}})$$

in $\mathscr{P}_F(\mathcal{C},J)$. Then, $\xi_{\boldsymbol f}$ maps $((u,x),y)\in(E\times_{G_0}H_0)\times^{\sigma'}_{H_0}H_1$ to $((\tau_\pi\xi(u,f_1(y)),\tau'(y)),y)\in(E\times_{G_0}H_0)\times^{\tau'}_{H_0}H_1$.

Let $\boldsymbol f=(f_0,f_1),\boldsymbol g=(g_0,g_1):\boldsymbol H\to\boldsymbol G$ be morphisms of groupoids in $\mathscr{P}_F(\mathcal{C},J)$. Suppose that a morphism $\chi:(H_0,\mathscr{H}_0)\to(G_1,\mathscr{G}_1)$ in $\mathscr{P}_F(\mathcal{C},J)$ makes the following diagrams commute.

$$\begin{array}{ccccc}
G_0 & \xleftarrow{f_0} & H_0 & \xrightarrow{g_0} & G_0\\
 & {\scriptstyle\sigma}\nwarrow & \downarrow{\scriptstyle\chi} & \nearrow{\scriptstyle\tau} & \\
 & & G_1 & &
\end{array}
\qquad
\begin{array}{ccc}
H_1 & \xrightarrow{(f_1,\chi\tau')} & G_1\times_{G_0}G_1\\
\downarrow{\scriptstyle(\chi\sigma',g_1)} & & \downarrow{\scriptstyle\mu}\\
G_1\times_{G_0}G_1 & \xrightarrow{\mu} & G_1
\end{array}$$

For a representation $(\boldsymbol E,\boldsymbol\xi)$ of $\boldsymbol G$, we define a morphism $\chi^\bullet_{(\boldsymbol E,\boldsymbol\xi)}:f_0^*(\boldsymbol E)\to g_0^*(\boldsymbol E)$ in $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(H_0,\mathscr{H}_0)}$ to be $\boldsymbol\xi_\chi:f_0^*(\boldsymbol E)=(\sigma\chi)^*(\boldsymbol E)\to(\tau\chi)^*(\boldsymbol E)=g_0^*(\boldsymbol E)$.

Proposition 9.9 *([10], [11]) $\chi^\bullet_{(\boldsymbol E,\boldsymbol\xi)}$ defines a morphism of representations $\chi^\bullet_{(\boldsymbol E,\boldsymbol\xi)}:(f_0^*(\boldsymbol E),\boldsymbol{\xi_f})\to(g_0^*(\boldsymbol E),\boldsymbol{\xi_g})$ and the following diagram in $\mathrm{Rep}(\boldsymbol H)$ commutes for a morphism $\varphi:(\boldsymbol E,\boldsymbol\xi)\to(\boldsymbol D,\boldsymbol\zeta)$ of representations of $\boldsymbol G$.*

$$\begin{array}{ccc}
(f_0^*(\boldsymbol E),\boldsymbol{\xi_f}) & \xrightarrow{f^*(\varphi)} & (f_0^*(\boldsymbol D),\boldsymbol{\zeta_f})\\
\downarrow{\scriptstyle\boldsymbol\xi_\chi} & & \downarrow{\scriptstyle\boldsymbol\zeta_\chi}\\
(g_0^*(\boldsymbol E),\boldsymbol{\xi_g}) & \xrightarrow{g^*(\varphi)} & (g_0^*(\boldsymbol D),\boldsymbol{\zeta_g})
\end{array}$$

Thus we have a natural transformation $\chi^\bullet:\boldsymbol f^\bullet\to\boldsymbol g^\bullet$.

Let $f:(X,\mathscr{X})\to(Y,\mathscr{Y})$, $g:(X,\mathscr{X})\to(Z,\mathscr{Z})$ and $k:(V,\mathscr{V})\to(X,\mathscr{X})$ be morphisms in $\mathscr{P}_F(\mathcal{C},J)$ and $\boldsymbol E=((E,\mathscr{E})\xrightarrow{\pi}(Y,\mathscr{Y}))$ an object of $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(Y,\mathscr{Y})}$. We consider the following commutative diagram in $\mathscr{P}_F(\mathcal{C},J)$ whose outer trapezoid and lower rectangle are cartesian.

$$\begin{array}{ccccc}
(E\times_Y V,\mathscr{E}^{(fk)_\pi}\cap\mathscr{V}^{\pi_{fk}}) & \overset{id_E\times_Y k}{\dashrightarrow} & (E\times_Y X,\mathscr{E}^{f_\pi}\cap\mathscr{X}^{\pi_f}) & \xrightarrow{f_\pi} & (E,\mathscr{E})\\
\downarrow{\scriptstyle\pi_{fk}} & & \downarrow{\scriptstyle\pi_f} & & \downarrow{\scriptstyle\pi}\\
(V,\mathscr{V}) & \xrightarrow{k} & (X,\mathscr{X}) & \xrightarrow{f} & (Y,\mathscr{Y})
\end{array}$$

There exists unique morphism $id_E \times_Y k : (E \times_Y V, \mathscr{E}^{(fk)_\pi} \cap \mathscr{V}^{\pi_{fk}}) \to (E \times_Y X, \mathscr{E}^{f_\pi} \cap \mathscr{X}^{\pi_f})$ that makes the above diagram commute. Since objects $(gk)_*(fk)^*(\boldsymbol{E})$ and $g_*f^*(\boldsymbol{E})$ of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(Z,\mathscr{Z})}$ are given by

$$\begin{aligned}(gk)_*(fk)^*(\boldsymbol{E}) &= ((E \times_Y V, \mathscr{E}^{(fk)_\pi} \cap \mathscr{V}^{\pi_{fk}}) \xrightarrow{gk\pi_{fk}} (Z, \mathscr{Z}))\\ g_*f^*(\boldsymbol{E}) &= ((E \times_Y X, \mathscr{E}^{f_\pi} \cap \mathscr{X}^{\pi_f}) \xrightarrow{g\pi_f} (Z, \mathscr{Z})),\end{aligned}$$

we define a morphism $\boldsymbol{E}_k : (gk)_*(fk)^*(\boldsymbol{E}) \to g_*f^*(\boldsymbol{E})$ in $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(Z,\mathscr{Z})}$ by $\boldsymbol{E}_k = \langle id_E \times_Y k, id_Z\rangle$. It is easy to verify the following fact.

Proposition 9.10 *For a morphism $j : (U, \mathscr{U}) \to (V, \mathscr{V})$ in $\mathscr{P}_F(\mathcal{C}, J)$, a composition*

$$(gkj)_*(fkj)^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{E}_j} (gk)_*(fk)^*(\boldsymbol{E}) \xrightarrow{\boldsymbol{E}_k} g_*f^*(\boldsymbol{E})$$

coincides with $\boldsymbol{E}_{kj} : (gkj)_*(fkj)^*(\boldsymbol{E}) \to g_*f^*(\boldsymbol{E})$. Moreover, $\boldsymbol{E}_k$ is natural in $\boldsymbol{E}$, that is, for a morphism $\boldsymbol{\varphi} : \boldsymbol{E} \to \boldsymbol{D}$ in $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(Y,\mathscr{Y})}$, the following diagram is commutative.

$$\begin{array}{ccc}(gk)_*(fk)^*(\boldsymbol{E}) & \xrightarrow{\boldsymbol{E}_k} & g_*f^*(\boldsymbol{E})\\ \Big\downarrow{\scriptstyle (gk)_*(fk)^*(\boldsymbol{\varphi})} & & \Big\downarrow{\scriptstyle g_*f^*(\boldsymbol{\varphi})}\\ (gk)_*(fk)^*(\boldsymbol{D}) & \xrightarrow{\boldsymbol{D}_k} & g_*f^*(\boldsymbol{D})\end{array}$$

Let $f : (X, \mathscr{X}) \to (Y, \mathscr{Y})$, $g : (X, \mathscr{X}) \to (Z, \mathscr{Z})$, $h : (V, \mathscr{V}) \to (Z, \mathscr{Z})$ and $i : (V, \mathscr{V}) \to (W, \mathscr{W})$ be morphisms in $\mathscr{P}_F(\mathcal{C}, J)$. We consider the following cartesian square in $\mathscr{P}_F(\mathcal{C}, J)$.

$$\begin{array}{ccc}(X \times_Z V, \mathscr{X}^{h_g} \cap \mathscr{V}^{g_h}) & \xrightarrow{g_h} & (V, \mathscr{V})\\ \Big\downarrow{\scriptstyle h_g} & & \Big\downarrow{\scriptstyle h}\\ (X, \mathscr{X}) & \xrightarrow{g} & (Z, \mathscr{Z})\end{array}$$

For an object $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (Y, \mathscr{Y}))$ of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(Y,\mathscr{Y})}$, we consider the following commutative diagrams in $\mathscr{P}_F(\mathcal{C}, J)$ whose rectangles are all cartesian.

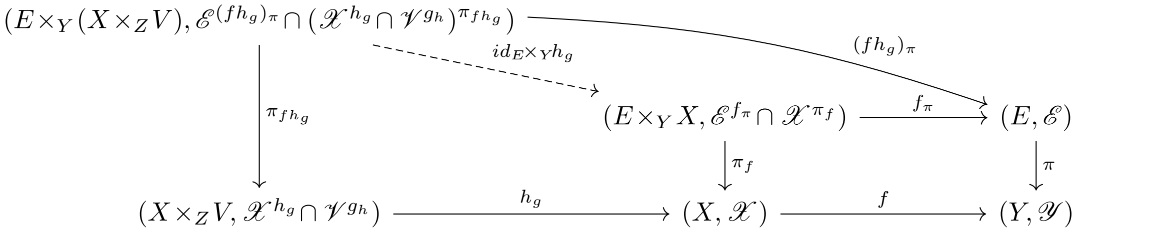
$$\begin{array}{ccc}(E \times_Y (X \times_Z V), \mathscr{E}^{(fh_g)_\pi} \cap (\mathscr{X}^{h_g} \cap \mathscr{V}^{g_h})^{\pi_{fh_g}}) & \xrightarrow{(fh_g)_\pi} & (E, \mathscr{E})\\ \Big\downarrow{\scriptstyle \pi_{fh_g}} & & \Big\downarrow{\scriptstyle \pi}\\ (X \times_Z V, \mathscr{X}^{h_g} \cap \mathscr{V}^{g_h}) & \xrightarrow{fh_g} & (Y, \mathscr{Y})\end{array}$$

$$\begin{array}{ccccc}((E \times_Y X) \times_Z V, (\mathscr{E}^{f_\pi} \cap \mathscr{X}^{\pi_f})^{h_{g\pi_f}} \cap \mathscr{V}^{(g\pi_f)_h}) & \xrightarrow{h_{g\pi_f}} & (E \times_Y X, \mathscr{E}^{f_\pi} \cap \mathscr{X}^{\pi_f}) & \xrightarrow{f_\pi} & (E, \mathscr{E})\\ \Big\downarrow & & \Big\downarrow{\scriptstyle \pi_f} & & \Big\downarrow{\scriptstyle \pi}\\ \Big\downarrow{\scriptstyle (g\pi_f)_h} & & (X, \mathscr{X}) & \xrightarrow{f} & (Y, \mathscr{Y})\\ \Big\downarrow & & \Big\downarrow{\scriptstyle g} & & \\ (V, \mathscr{V}) & \xrightarrow{h} & (Z, \mathscr{Z}) & & \end{array}$$

Thus we have the following equalities.

$$\begin{aligned}(ig_h)_*(fh_g)^*(\boldsymbol{E}) &= ((E \times_Y (X \times_Z V), \mathscr{E}^{(fh_g)_\pi} \cap (\mathscr{X}^{h_g} \cap \mathscr{V}^{g_h})^{\pi_{fh_g}}) \xrightarrow{ig_h\pi_{fh_g}} (W, \mathscr{W}))\\ i_*h^*g_*f^*(\boldsymbol{E}) &= (((E \times_Y X) \times_Z V, (\mathscr{E}^{f_\pi} \cap \mathscr{X}^{\pi_f})^{h_{g\pi_f}} \cap \mathscr{V}^{(g\pi_f)_h}) \xrightarrow{i(g\pi_f)_h} (W, \mathscr{W}))\end{aligned}$$

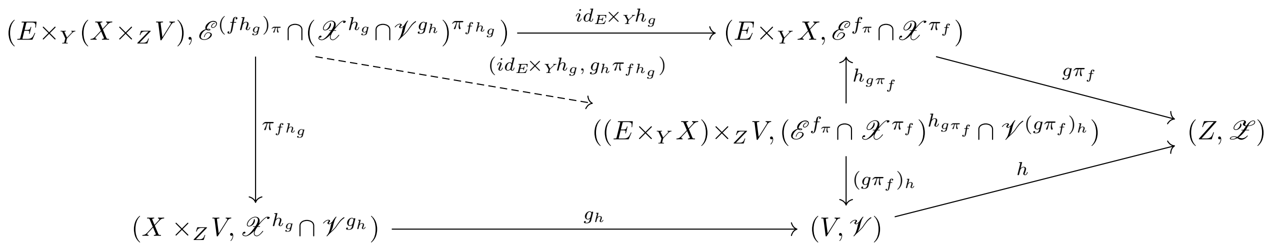
There exists unique morphism $id_E \times_Y h_g : (E \times_Y (X \times_Z V), \mathscr{E}^{(fh_g)_\pi} \cap (\mathscr{X}^{h_g} \cap \mathscr{V}^{g_h})^{\pi_{fh_g}}) \to (E \times_Y X, \mathscr{E}^{f_\pi} \cap \mathscr{X}^{\pi_f})$ that makes the following diagram commute.



There exists unique morphism

$(id_E\times_Y h_g, g_h\pi_{fh_g}) : (E\times_Y(X\times_Z V), \mathscr{E}^{(fh_g)_\pi}\cap(\mathscr{X}^{h_g}\cap\mathscr{V}^{g_h})^{\pi_{fh_g}}) \to ((E\times_Y X)\times_Z V, (\mathscr{E}^{f_\pi}\cap\mathscr{X}^{\pi_f})^{h_{g\pi_f}}\cap\mathscr{V}^{(g\pi_f)_h})$

that makes the following diagram commute.



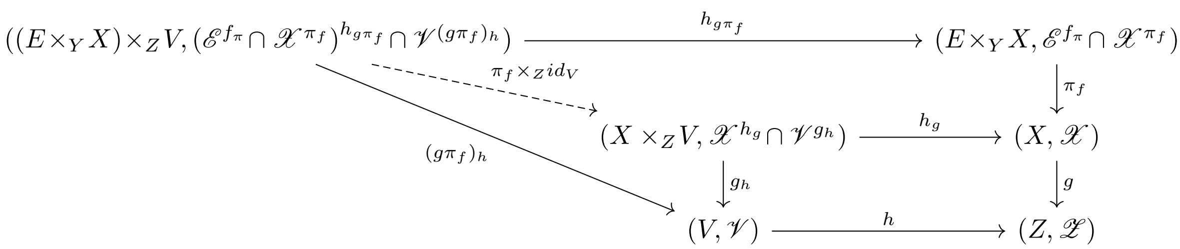
Thus we have a morphism $\langle(id_E\times_Y h_g, g_h\pi_{fh_g}), id_W\rangle : (ig_h)_*(fh_g)^*(\boldsymbol{E}) \to i_*h^*g_*f^*(\boldsymbol{E})$ in $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(W,\mathscr{W})}$ which we denote by $\theta_{f,g,h,i}(\boldsymbol{E})$ below.

Proposition 9.11 *([11] Proposition 2.4.15)* $\theta_{f,g,h,i}(\boldsymbol{E}) : (ig_h)_*(fh_g)^*(\boldsymbol{E}) \to i_*h^*g_*f^*(\boldsymbol{E})$ *is an isomorphism which is natural in* $\boldsymbol{E}$.

Proof. There exists unique morphism

$$\pi_f\times_Z id_V : ((E\times_Y X)\times_Z V, (\mathscr{E}^{f_\pi}\cap\mathscr{X}^{\pi_f})^{h_{g\pi_f}}\cap\mathscr{V}^{(g\pi_f)_h}) \to (E\times_Y X, \mathscr{E}^{f_\pi}\cap\mathscr{X}^{\pi_f})$$

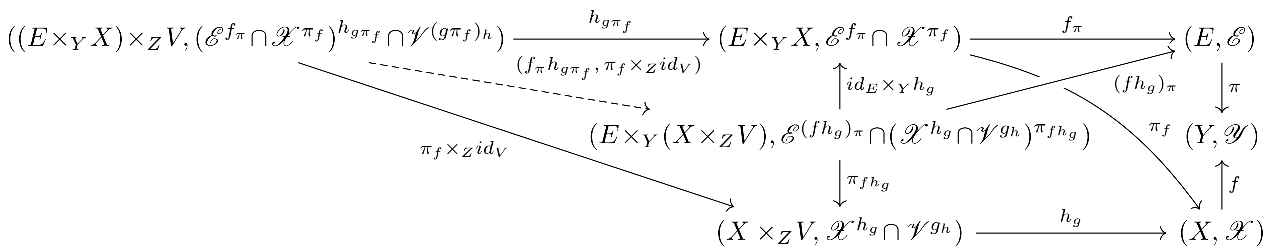
in $\mathscr{P}_F(\mathcal{C}, J)$ that makes the following diagram commute.



Hence here exists unique morphism

$(f_\pi h_{g\pi_f}, \pi_f\times_Z id_V) : ((E\times_Y X)\times_Z V, (\mathscr{E}^{f_\pi}\cap\mathscr{X}^{\pi_f})^{h_{g\pi_f}}\cap\mathscr{V}^{(g\pi_f)_h}) \to (E\times_Y(X\times_Z V), \mathscr{E}^{(fh_g)_\pi}\cap(\mathscr{X}^{h_g}\cap\mathscr{V}^{g_h})^{\pi_{fh_g}})$

in $\mathscr{P}_F(\mathcal{C}, J)$ that makes the following diagram commute.



Thus we have a morphism $\langle(f_\pi h_{g\pi_f}, \pi_f\times_Z id_V), id_W\rangle : i_*h^*g_*f^*(\boldsymbol{E}) \to (ig_h)_*(fh_g)^*(\boldsymbol{E})$ in $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(W,\mathscr{W})}$ which is the inverse of $\theta_{f,g,h,i}(\boldsymbol{E})$. The naturality of $\theta_{f,g,h,i}(\boldsymbol{E})$ in $\boldsymbol{E}$ is clear from the definition of $\theta_{f,g,h,i}(\boldsymbol{E})$. □

Remark 9.12 *$(id_E\times_Y h_g, g_h\pi_{fh_g}) : E\times_Y(X\times_Z V)\to(E\times_Y X)\times_Z V$ maps $(u,(x,v))\in E\times_Y(X\times_Z V)$ to $((u,x),v)\in(E\times_Y X)\times_Z V$.*

For an object $\boldsymbol{E}=((E,\mathscr{E})\xrightarrow{\pi}(G_0,\mathscr{G}_0))$ of $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(G_0,\mathscr{G}_0)}$ and a morphism $\boldsymbol{\xi}:\sigma^*(\boldsymbol{E})\to\tau^*(\boldsymbol{E})$ in $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(G_1,\mathscr{G}_1)}$, we denote by $\hat{\boldsymbol{\xi}}:\tau_*\sigma^*(\boldsymbol{E})\to\boldsymbol{E}$ the adjoint of $\boldsymbol{\xi}$ with respect to the adjunction $\tau_*\dashv\tau^*$.

Proposition 9.13 *([11] Proposition 3.4.2) $\boldsymbol{\xi}$ satisfies condition (A) of (9.3) if and only if $\hat{\boldsymbol{\xi}}$ makes the following diagram commute.*

$$\begin{array}{ccccc}
(\tau\mathrm{pr}_2)_*(\sigma\mathrm{pr}_1)^*(\boldsymbol{E}) & \xrightarrow{\theta_{\sigma,\tau,\sigma,\tau}(\boldsymbol{E})} & \tau_*\sigma^*\tau_*\sigma^*(\boldsymbol{E}) & \xrightarrow{\tau_*\sigma^*(\hat{\boldsymbol{\xi}})} & \tau_*\sigma^*(\boldsymbol{E}) \\
\| & & & & \downarrow{\scriptstyle\hat{\boldsymbol{\xi}}} \\
(\tau\mu)_*(\sigma\mu)^*(\boldsymbol{E}) & \xrightarrow{\boldsymbol{E}_\mu} & \tau_*\sigma^*(\boldsymbol{E}) & \xrightarrow{\hat{\boldsymbol{\xi}}} & \boldsymbol{E}
\end{array}$$

$\boldsymbol{\xi}$ satisfies condition (U) of (9.3) if and only if a composition $\boldsymbol{E}=(\tau\varepsilon)_*(\sigma\varepsilon)^*(\boldsymbol{E})\xrightarrow{\boldsymbol{E}_\varepsilon}\tau_*\sigma^*(\boldsymbol{E})\xrightarrow{\hat{\boldsymbol{\xi}}}\boldsymbol{E}$ coincides with the identity morphism of $\boldsymbol{E}$.

Remark 9.14 *We consider the following diagrams whose rectangles are all cartesian.*

$$\begin{array}{ccc}
E\times^{\sigma\mu}_{G_0}(G_1\times_{G_0}G_1) & \xrightarrow{(\sigma\mu)_\pi} & E \\
\downarrow{\scriptstyle\pi_{\sigma\mu}=\pi_{\sigma\mathrm{pr}_1}} & & \downarrow{\scriptstyle\pi} \\
G_1\times_{G_0}G_1 & \xrightarrow{\sigma\mu=\sigma\mathrm{pr}_1} & G_0
\end{array}
\qquad
\begin{array}{ccc}
E\times^{\tau}_{G_0}G_1 & \xrightarrow{\tau_\pi} & E \\
\downarrow{\scriptstyle\pi_\tau} & & \downarrow{\scriptstyle\pi} \\
G_1 & \xrightarrow{\tau} & G_0
\end{array}
\qquad
\begin{array}{ccccc}
(E\times^\sigma_{G_0}G_1)\times^\sigma_{G_0}G_1 & \xrightarrow{\sigma_{\tau\pi_\sigma}} & E\times^\sigma_{G_0}G_1 & \xrightarrow{\sigma_\pi} & E \\
\downarrow & & \downarrow{\scriptstyle\pi_\sigma} & & \downarrow{\scriptstyle\pi} \\
\downarrow{\scriptstyle(\tau\pi_\sigma)_\sigma} & & G_1 & \xrightarrow{\sigma} & G_0 \\
\downarrow & & \downarrow{\scriptstyle\tau} & & \\
G_1 & \xrightarrow{\sigma} & G_0 & &
\end{array}$$

Then, we have the following equalities.

$$\tau_*\sigma^*(\boldsymbol{E})=((E\times^\sigma_{G_0}G_1,\mathscr{E}^{\sigma_\pi}\cap\mathscr{G}_1^{\pi_\sigma})\xrightarrow{\tau\pi_\sigma}(G_0,\mathscr{G}_0))$$
$$(\tau\mathrm{pr}_2)_*(\sigma\mathrm{pr}_1)^*(\boldsymbol{E})=(\tau\mu)_*(\sigma\mu)^*(\boldsymbol{E})=((E\times^{\sigma\mu}_{G_0}(G_1\times_{G_0}G_1),\mathscr{E}^{(\sigma\mu)_\pi}\cap(\mathscr{G}_1^{\mathrm{pr}_1}\cap\mathscr{G}_1^{\mathrm{pr}_2})^{\pi_{\sigma\mu}})\xrightarrow{\tau\mu\pi_{\sigma\mu}}(G_0,\mathscr{G}_0))$$
$$\tau_*\sigma^*\tau_*\sigma^*(\boldsymbol{E})=((E\times^\sigma_{G_0}G_1)\times^\sigma_{G_0}G_1,(\mathscr{E}^{\sigma_\pi}\cap\mathscr{G}_1^{\pi_\sigma})^{\sigma_{\tau\pi_\sigma}}\cap\mathscr{G}_1^{(\tau\pi_\sigma)_\sigma})\xrightarrow{\tau(\tau\pi_\sigma)_\sigma}(G_0,\mathscr{G}_0))$$

If we put $\boldsymbol{\xi}=\langle\xi,id_{G_1}\rangle$ and $\hat{\boldsymbol{\xi}}=\langle\hat{\xi},id_{G_0}\rangle$ for morphisms $\xi:(E\times^\sigma_{G_0}G_1,\mathscr{E}^{\sigma_\pi}\cap\mathscr{G}_1^{\pi_\sigma})\to(E\times^\tau_{G_0}G_1,\mathscr{E}^{\tau_\pi}\cap\mathscr{G}_1^{\pi_\tau})$ and $\hat{\xi}:(E\times^\sigma_{G_0}G_1,\mathscr{E}^{\sigma_\pi}\cap\mathscr{G}_1^{\pi_\sigma})\to(E,\mathscr{E})$ in $\mathscr{P}_F(\mathcal{C},J)$, then $\hat{\xi}$ is a composition $E\times^\sigma_{G_0}G_1\xrightarrow{\xi}E\times^\tau_{G_0}G_1\xrightarrow{\tau_\pi}E$ and $\xi=(\hat{\xi},\pi_\sigma)$ holds. The diagram of (9.13) is commutative if and only if the following diagram is commutative.

$$\begin{array}{ccccc}
E\times^{\sigma\mathrm{pr}_1}_{G_0}(G_1\times_{G_0}G_1) & \xrightarrow{(id_E\times_Y\mathrm{pr}_1,\,\mathrm{pr}_2\pi_{\sigma\mathrm{pr}_1})} & (E\times^\sigma_{G_0}G_1)\times^\sigma_{G_0}G_1 & \xrightarrow{\hat{\xi}\times_{G_0}id_{G_1}} & E\times^\sigma_{G_0}G_1 \\
\| & & & & \downarrow{\scriptstyle\hat{\xi}} \\
E\times^{\sigma\mu}_{G_0}(G_1\times_{G_0}G_1) & \xrightarrow{id_E\times_{G_0}\mu} & E\times^\sigma_{G_0}G_1 & \xrightarrow{\hat{\xi}} & E
\end{array}$$

A composition $\boldsymbol{E}=(\tau\varepsilon)_*(\sigma\varepsilon)^*(\boldsymbol{E})\xrightarrow{\boldsymbol{E}_\varepsilon}\tau_*\sigma^*(\boldsymbol{E})\xrightarrow{\hat{\boldsymbol{\xi}}}\boldsymbol{E}$ coincides with the identity morphism of $\boldsymbol{E}$ if and only if a composition $E\xrightarrow{(id_E,\varepsilon\pi)}E\times^\sigma_{G_0}G_1\xrightarrow{\hat{\xi}}E$ coincides with the identity morphism of E.

The next result follows from the naturality of the adjointness.

Proposition 9.15 *Let $(\boldsymbol{E},\boldsymbol{\xi})$ and $(\boldsymbol{F},\boldsymbol{\zeta})$ be representations of $\boldsymbol{G}$. A morphism $\boldsymbol{\varphi}:\boldsymbol{E}\to\boldsymbol{F}$ in $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(G_0,\mathscr{G}_0)}$ makes the following left diagram commute if and only if it makes the following right diagram commute.*

$$\begin{array}{ccc}
\sigma^*(\boldsymbol{E}) & \xrightarrow{\boldsymbol{\xi}} & \tau^*(\boldsymbol{E}) \\
\downarrow{\scriptstyle\sigma^*(\boldsymbol{\varphi})} & & \downarrow{\scriptstyle\tau^*(\boldsymbol{\varphi})} \\
\sigma^*(\boldsymbol{F}) & \xrightarrow{\boldsymbol{\zeta}} & \tau^*(\boldsymbol{F})
\end{array}
\qquad
\begin{array}{ccc}
\tau_*\sigma^*(\boldsymbol{E}) & \xrightarrow{\hat{\boldsymbol{\xi}}} & \boldsymbol{E} \\
\downarrow{\scriptstyle\tau_*\sigma^*(\boldsymbol{\varphi})} & & \downarrow{\scriptstyle\boldsymbol{\varphi}} \\
\tau_*\sigma^*(\boldsymbol{F}) & \xrightarrow{\hat{\boldsymbol{\zeta}}} & \boldsymbol{F}
\end{array}$$

If a morphism $\hat{\boldsymbol{\xi}} : \tau_*\sigma^*(\boldsymbol{E}) \to \boldsymbol{E}$ in $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(G_0,\mathscr{G}_0)}$ satisfies both conditions of (9.14), we also call a pair $(\boldsymbol{E}, \hat{\boldsymbol{\xi}} : \tau_*\sigma^*(\boldsymbol{E}) \to \boldsymbol{E})$ a representation of $\boldsymbol{G}$ on $\boldsymbol{E}$.

Example 9.16 *For an object* $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (B, \mathscr{B}))$ *of* $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C}, J))$*, we consider the groupoid* $\boldsymbol{G}(\boldsymbol{E})$ *associated with* $\boldsymbol{E}$*. We define a morphism* $\hat{\boldsymbol{\xi}}_{\boldsymbol{E}} : \tau_{\boldsymbol{E}*}\sigma^*_{\boldsymbol{E}}(\boldsymbol{E}) \to \boldsymbol{E}$ *in* $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(B,\mathscr{B})}$ *by* $\hat{\boldsymbol{\xi}}_{\boldsymbol{E}} = \langle \hat{\xi}_{\boldsymbol{E}}, id_B\rangle$*. It follows from (6.5) and (9.14) that* $(\boldsymbol{E}, \hat{\boldsymbol{\xi}}_{\boldsymbol{E}})$ *is a representation of* $\boldsymbol{G}(\boldsymbol{E})$ *on* $\boldsymbol{E}$*. We call* $(\boldsymbol{E}, \hat{\boldsymbol{\xi}}_{\boldsymbol{E}})$ *the canonical representation of* $\boldsymbol{E}$*.*

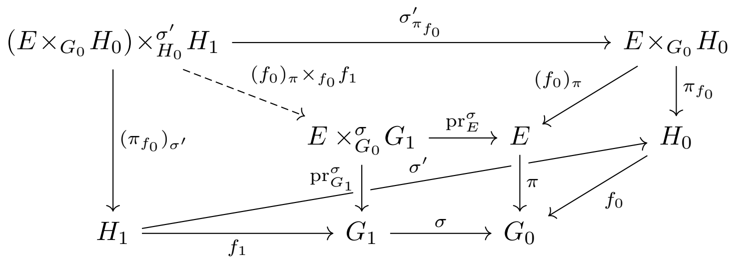
Let $\boldsymbol{G} = ((G_0, \mathscr{G}_0), (G_1, \mathscr{G}_1); \sigma, \tau, \varepsilon, \mu, \iota)$ and $\boldsymbol{H} = ((H_0, \mathscr{H}_0), (H_1, \mathscr{H}_1); \sigma', \tau', \varepsilon', \mu', \iota')$ be a groupoids in $\mathscr{P}_F(\mathcal{C}, J)$ and $\boldsymbol{E} = ((E, \mathscr{E}) \xrightarrow{\pi} (G_0, \mathscr{G}_0))$ an object of $\mathscr{P}_F(\mathcal{C}, J)^{(2)}_{(G_0,\mathscr{G}_0)}$. For a morphism $\boldsymbol{f} = (f_0, f_1) : \boldsymbol{H} \to \boldsymbol{G}$ of groupoids in $\mathscr{P}_F(\mathcal{C}, J)$, we consider the following diagram in $\mathscr{P}_F(\mathcal{C}, J)$ whose rectangles are cartesian.

$$\begin{array}{ccccc}
\big((E\times_{G_0}H_0)\times^{\sigma'}_{H_0}H_1, (\mathscr{E}^{(f_0)_\pi}\cap\mathscr{H}_0^{\pi_{f_0}})^{\sigma'_{\pi_{f_0}}}\cap\mathscr{H}_1^{(\pi_{f_0})_{\sigma'}}\big) & \xrightarrow{\sigma'_{\pi_{f_0}}} & (E\times_{G_0}H_0, \mathscr{E}^{(f_0)_\pi}\cap\mathscr{H}_0^{\pi_{f_0}}) & \xrightarrow{(f_0)_\pi} & (E, \mathscr{E}) \\
\downarrow{\scriptstyle (\pi_{f_0})_{\sigma'}} & & \downarrow{\scriptstyle \pi_{f_0}} & & \downarrow{\scriptstyle \pi} \\
(H_1, \mathscr{H}_1) & \xrightarrow{\sigma'} & (H_0, \mathscr{H}_0) & \xrightarrow{f_0} & (G_0, \mathscr{G}_0)
\end{array}$$

There exists unique morphism

$$(f_0)_\pi\times_{f_0}f_1 : \big((E\times_{G_0}H_0)\times^{\sigma'}_{H_0}H_1, (\mathscr{E}^{(f_0)_\pi}\cap\mathscr{H}_0^{\pi_{f_0}})^{\sigma'_{\pi_{f_0}}}\cap\mathscr{H}_1^{(\pi_{f_0})_{\sigma'}}\big) \to \big(E\times^{\sigma}_{G_0}G_1, \mathscr{E}^{\mathrm{pr}^\sigma_E}\cap\mathscr{G}_1^{\mathrm{pr}^\sigma_{G_1}}\big)$$

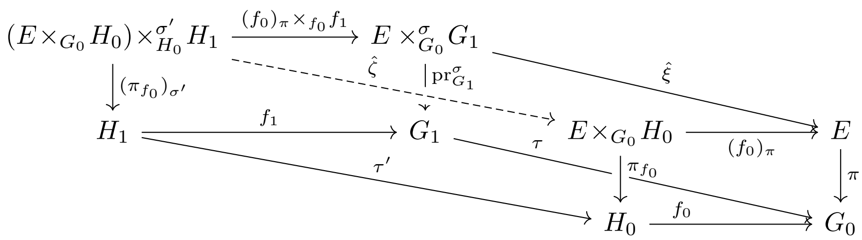
in $\mathscr{P}_F(\mathcal{C}, J)$ that makes the following diagram commute.



Consider a representation $(\boldsymbol{E}, \hat{\boldsymbol{\xi}})$ of $\boldsymbol{G}$ on $\boldsymbol{E}$ and put $\hat{\boldsymbol{\xi}} = \langle\hat{\xi}, id_{G_0}\rangle$. There exists unique morphism

$$\hat{\zeta} : \big((E\times_{G_0}H_0)\times^{\sigma'}_{H_0}H_1, (\mathscr{E}^{(f_0)_\pi}\cap\mathscr{H}_0^{\pi_{f_0}})^{\sigma'_{\pi_{f_0}}}\cap\mathscr{H}_1^{(\pi_{f_0})_{\sigma'}}\big) \to (E\times_{G_0}H_0, \mathscr{E}^{(f_0)_\pi}\cap\mathscr{H}_0^{\pi_{f_0}})$$

in $\mathscr{P}_F(\mathcal{C}, J)$ that makes the following diagram commute.



Define a morphism $\hat{\boldsymbol{\zeta}} : \tau'_*\sigma'^*(f_0^*(\boldsymbol{E})) \to f_0^*(\boldsymbol{E})$ by $\hat{\boldsymbol{\zeta}} = \langle\hat{\zeta}, id_{H_0}\rangle$.

Proposition 9.17 $(f_0^*(\boldsymbol{E}), \hat{\boldsymbol{\zeta}})$ *coincides with the restriction of the representation* $(\boldsymbol{E}, \hat{\boldsymbol{\xi}})$ *of* $\boldsymbol{G}$ *on* $\boldsymbol{E}$ *along* $\boldsymbol{f}$*.*

Proof. Let $(f_0^*(\boldsymbol{E}), \boldsymbol{\xi_f})$ be the restriction of $(\boldsymbol{E}, \boldsymbol{\xi})$ along $\boldsymbol{f} : \boldsymbol{H} \to \boldsymbol{G}$ and put $\boldsymbol{\xi_f} = \langle\xi_{\boldsymbol{f}}, id_{H_0}\rangle$. We denote by $\hat{\boldsymbol{\xi}}_{\boldsymbol{f}} = \langle\hat{\xi}_{\boldsymbol{f}}, id_{H_0}\rangle : \tau'_*\sigma'^*(f_0^*(\boldsymbol{E})) \to \boldsymbol{E}$ the adjoint of $\boldsymbol{\xi_f}$ with respect to the adjunction $\tau'_* \dashv \tau'^*$. It follows from (9.8) that $\hat{\xi}_{\boldsymbol{f}}$ maps $((u, x), y) \in (E\times_{G_0}H_0)\times^{\sigma'}_{H_0}H_1$ to $(\hat{\xi}(u, f_1(y)), \tau'(y)) \in E\times_{G_0}H_0$. On the other hand, $\hat{\zeta}$ also maps $((u, x), y) \in (E\times_{G_0}H_0)\times^{\sigma'}_{H_0}H_1$ to $(\hat{\xi}(u, f_1(y)), \tau'(y)) \in E\times_{G_0}H_0$ by the definition of $\hat{\zeta}$. Thus we have $\hat{\boldsymbol{\xi}}_{\boldsymbol{f}} = \hat{\zeta}$. □

Proposition 9.18 *Let $\boldsymbol{E} = ((E,\mathscr{E}) \xrightarrow{\pi} (G_0,\mathscr{G}_0))$ be an object $\mathrm{Epi}_c(\mathscr{P}_F(\mathcal{C},J))$ and $(\boldsymbol{E},\hat{\boldsymbol{\xi}} : \tau_*\sigma^*(\boldsymbol{E}) \to \boldsymbol{E})$ a representation of $\boldsymbol{G} = ((G_0,\mathscr{G}_0),(G_1,\mathscr{G}_1);\sigma,\tau,\varepsilon,\mu,\iota)$ on $\boldsymbol{E}$. There exists a morphism $\boldsymbol{f} : \boldsymbol{G} \to \boldsymbol{G}(\boldsymbol{E})$ of groupoids in $\mathscr{P}_F(\mathcal{C},J)$ such that $(\boldsymbol{E},\hat{\boldsymbol{\xi}})$ coincides with the restriction of the canonical representation $(\boldsymbol{E},\hat{\boldsymbol{\xi}}_{\boldsymbol{E}})$ along $\boldsymbol{f}$. Moreover, if $\boldsymbol{g} = (id_{G_0}, g_1) : \boldsymbol{G} \to \boldsymbol{G}(\boldsymbol{E})$ is a morphism of groupoids in $\mathscr{P}_F(\mathcal{C},J)$ such that $(\boldsymbol{E},\hat{\boldsymbol{\xi}})$ coincides with the restriction of the canonical representation $(\boldsymbol{E},\hat{\boldsymbol{\xi}}_{\boldsymbol{E}})$ along $\boldsymbol{g}$, then $\boldsymbol{g} = \boldsymbol{f}$ holds.*

Proof. We put $\hat{\boldsymbol{\xi}} = \langle \hat{\xi}, id_{G_0}\rangle$. Here, $\hat{\xi}$ is a morphism in $\mathscr{P}_F(\mathcal{C},J)$ from $(E \times^{\sigma}_{G_0} G_1, \mathscr{E}^{\sigma_\pi} \cap \mathscr{G}_1^{\pi_\sigma})$ to $(E,\mathscr{E})$. By the commutativity of the following diagram, $\hat{\xi}(e,g) \in \pi^{-1}(\tau(g))$ holds for $g \in G_1$ and $e \in \pi^{-1}(\sigma(g))$.

$$\begin{array}{ccc} E \times^{\sigma}_{G_0} G_1 & \xrightarrow{\ \hat{\xi}\ } & E \\ \downarrow{\scriptstyle \pi_\sigma} & & \downarrow{\scriptstyle \pi} \\ G_1 & \xrightarrow{\ \tau\ } & G_0 \end{array}$$

For $g \in G_1$, $U \in \mathrm{Ob}\,\mathcal{C}$, $\lambda \in F_{\pi^{-1}(\sigma(g))}(U) \cap \mathscr{E}^{i_{\pi^{-1}(\sigma(g))}}$, we denote by $c_g : F(U) \to G_1$ the constant map to g and define a map $\lambda_g : F(U) \to E \times^{\sigma}_{G_0} G_1$ by $\lambda_g = (i_{\pi^{-1}(\sigma(g))}\lambda, c_g)$. Since $\sigma_\pi\lambda_g = i_{\pi^{-1}(\sigma(g))}\lambda = (F_{i_{\pi^{-1}(\sigma(g))}})_U(\lambda) \in \mathscr{E}$ and $\pi_\sigma\lambda = c_g \in \mathscr{G}_1$, λ_g belongs to $\mathscr{E}^{\sigma_\pi} \cap \mathscr{G}_1^{\pi_\sigma}$. We define a map $\varphi_g : \pi^{-1}(\sigma(g)) \to \pi^{-1}(\tau(g))$ by $\varphi_g(e) = \hat{\xi}(e,g)$. If $\lambda \in F_{\pi^{-1}(\sigma(g))}(U) \cap \mathscr{E}^{i_{\pi^{-1}(\sigma(g))}}$, then we have $(F_{i_{\pi^{-1}(\tau(g))}\varphi_g})_U(\lambda) = \hat{\xi}\lambda_g = (F_{\hat{\xi}})_U(\lambda_g) \in \mathscr{E}$, which shows that φ_g defines a morphism $\varphi_g : (\pi^{-1}(\sigma(g)), \mathscr{E}^{i_{\pi^{-1}(\sigma(g))}}) \to (\pi^{-1}(\tau(g)), \mathscr{E}^{i_{\pi^{-1}(\sigma(g))}})$. For $(g,h) \in G_1 \times^{\sigma}_{G_0} G_1$, it follows from the commutativity of the diagram of (9.14) that we have $\varphi_h\varphi_g(e) = \hat{\xi}(\hat{\xi}(e,g),h) = \hat{\xi}(e,\mu(g,h)) = \varphi_{\mu(g,h)}(e)$. This implies that $\varphi_{\iota(g)} : \pi^{-1}(\tau(g)) \to \pi^{-1}(\sigma(g))$ is the inverse of φ_g, hence $\varphi_g \in G_1(\boldsymbol{E})(\sigma(g),\tau(g)) \subset G_1(\boldsymbol{E})$.

We define a map $f_1 : G_1 \to G_1(\boldsymbol{E})$ by $f_1(g) = \varphi_g$. Then, f_1 makes the following diagrams commute.

$$\begin{array}{ccc} E \times^{\sigma}_{G_0} G_1 & \xrightarrow{\ \hat{\xi}\ } & E \\ \downarrow{\scriptstyle id_E \times^{\sigma}_{G_0} f_1} & \nearrow{\scriptstyle \hat{\xi}_{\boldsymbol{E}}} & \\ E \times^{\sigma}_{G_0} G_1(\boldsymbol{E}) & & \end{array} \qquad \begin{array}{ccccc} G_0 & \xleftarrow{\ \sigma\ } & G_1 & \xrightarrow{\ \tau\ } & G_0 \\ & {\scriptstyle \sigma_{\boldsymbol{E}}}\nwarrow & \downarrow{\scriptstyle f_1} & \nearrow{\scriptstyle \tau_{\boldsymbol{E}}} & \\ & & G_1(\boldsymbol{E}) & & \end{array} \qquad \begin{array}{ccccc} G_1 \times_{G_0} G_1 & \xrightarrow{\ \mu\ } & G_1 & \xleftarrow{\ \varepsilon\ } & G_0 \\ \downarrow{\scriptstyle f_1 \times_{G_0} f_1} & & \downarrow{\scriptstyle f_1} & \swarrow{\scriptstyle \varepsilon_{\boldsymbol{E}}} & \\ G_1(\boldsymbol{E}) \times_{G_0} G_1(\boldsymbol{E}) & \xrightarrow{\ \mu_{\boldsymbol{E}}\ } & G_1(\boldsymbol{E}) & & \end{array}$$

For $U \in \mathrm{Ob}\,\mathcal{C}$ and $\gamma \in F_{G_1}(U) \cap \mathscr{G}_1$, we verify $(F_{f_1})(\gamma) = f_1\gamma \in F_{G_1(\boldsymbol{E})}(U) \cap \mathscr{G}_{\boldsymbol{E}}$ below. It follows from the commutativity of the above middle diagram that the following compositions belong to $\mathscr{G}_0 \cap F_{G_0}(U)$.

$$F(U) \xrightarrow{f_1\gamma} G_1(\boldsymbol{E}) \xrightarrow{\sigma_{\boldsymbol{E}}} G_0, \quad F(U) \xrightarrow{f_1\gamma} G_1(\boldsymbol{E}) \xrightarrow{\tau_{\boldsymbol{E}}} G_0$$

Assume that $V, W \in \mathrm{Ob}\,\mathcal{C}$, $j \in \mathcal{C}(W,U)$, $k \in \mathcal{C}(W,V)$ and $\lambda \in \mathscr{E} \cap F_E(V)$ satisfy $\pi\lambda F(k) = \sigma_{\boldsymbol{E}} f_1\gamma F(j)$. Then, $\pi\lambda F(k) = \sigma\gamma F(j)$ holds by the commutativity of the above middle diagram, there exists a morphism $(\lambda F(k), \gamma F(j)) : F(W) \to E \times^{\sigma}_{G_0} G_1$ which makes the following diagram commute. It follows that a composition $F(W) \xrightarrow{(\lambda F(k),\, f_1\gamma F(j))} E \times^{\sigma_{\boldsymbol{E}}}_{G_0} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E} \cap F_E(W)$.

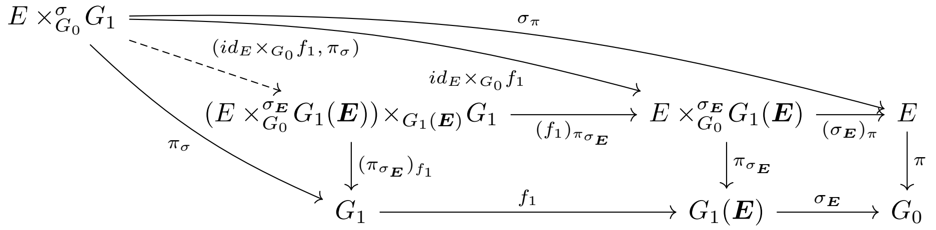
$$\begin{array}{ccccc} & & E \times^{\sigma}_{G_0} G_1 & \xrightarrow{\ \hat{\xi}\ } & E \\ & {\scriptstyle (\lambda F(k), \gamma F(j))}\nearrow & \downarrow{\scriptstyle id_E \times^{\sigma}_{G_0} f_1} & \nearrow{\scriptstyle \hat{\xi}_{\boldsymbol{E}}} & \\ F(W) & \xrightarrow{(\lambda F(k),\, f_1\gamma F(j))} & E \times^{\sigma}_{G_0} G_1(\boldsymbol{E}) & & \end{array}$$

Assume that $V, W \in \mathrm{Ob}\,\mathcal{C}$, $j \in \mathcal{C}(W,U)$, $k \in \mathcal{C}(W,V)$ and $\lambda \in \mathscr{E} \cap F_E(V)$ satisfy $\pi\lambda F(k) = \tau_{\boldsymbol{E}} f_1\gamma F(j)$. Then, $\pi\lambda F(k) = \sigma\iota\gamma F(j)$ holds by the commutativity of the above middle diagram, there exists a morphism $(\lambda F(k), \iota\gamma F(j)) : F(W) \to E \times^{\sigma}_{G_0} G_1$ which makes the following diagram commute. We note that $f_1\iota = \iota_{\boldsymbol{E}} f_1$ holds. It follows that a composition $F(W) \xrightarrow{(\lambda F(k),\, \iota_{\boldsymbol{E}} f_1\gamma F(j))} E \times^{\sigma_{\boldsymbol{E}}}_{G_0} G_1(\boldsymbol{E}) \xrightarrow{\hat{\xi}_{\boldsymbol{E}}} E$ belongs to $\mathscr{E} \cap F_E(W)$.

$$\begin{array}{ccccc} & & E \times^{\sigma}_{G_0} G_1 & \xrightarrow{\ \hat{\xi}\ } & E \\ & {\scriptstyle (\lambda F(k), \iota\gamma F(j))}\nearrow & \downarrow{\scriptstyle id_E \times^{\sigma}_{G_0} f_1} & \nearrow{\scriptstyle \hat{\xi}_{\boldsymbol{E}}} & \\ F(W) & \xrightarrow{(\lambda F(k),\, f_1\iota\gamma F(j))} & E \times^{\sigma}_{G_0} G_1(\boldsymbol{E}) & & \end{array}$$

Thus we conclude that $f_1\gamma$ belongs to $F_{G_1(\boldsymbol{E})}(U) \cap \mathscr{G}_{\boldsymbol{E}}$ by the definition of $\mathscr{G}_{\boldsymbol{E}}$ and that we have a morphism $\boldsymbol{f} = (id_{G_0}, f_1) : \boldsymbol{G} \to \boldsymbol{G}(\boldsymbol{E})$ of groupoids in $\mathscr{P}_F(\mathcal{C},J)$.

We define $\xi\colon E\times^{\sigma}_{G_0}G_1\to E\times^{\tau}_{G_0}G_1$ and $\xi_{\boldsymbol{E}}\colon E\times^{\sigma_{\boldsymbol{E}}}_{G_0}G_1(\boldsymbol{E})\to E\times^{\tau_{\boldsymbol{E}}}_{G_0}G_1(\boldsymbol{E})$ by $\xi=(\hat{\xi},\pi_\sigma)$ and $\xi_{\boldsymbol{E}}=(\hat{\xi}_{\boldsymbol{E}},\pi_{\sigma_{\boldsymbol{E}}})$, respectively. Consider a morphism $\boldsymbol{\xi}_{\boldsymbol{E}}:\sigma^*_{\boldsymbol{E}}(\boldsymbol{E})\to\tau^*_{\boldsymbol{E}}(\boldsymbol{E})$ in $\mathscr{P}_F(\mathcal{C},J)^{(2)}_{(G_1(\boldsymbol{E}),\mathscr{G}_{\boldsymbol{E}})}$ given by $\boldsymbol{\xi}_{\boldsymbol{E}}=\langle\xi_{\boldsymbol{E}},id_{G_1(\boldsymbol{E})}\rangle$. Note that $(\boldsymbol{\xi}_{\boldsymbol{E}})_{\boldsymbol{f}}=(\boldsymbol{\xi}_{\boldsymbol{E}})_{f_1}:\sigma^*(\boldsymbol{E})=(\sigma_{\boldsymbol{E}}f_1)^*(\boldsymbol{E})\to(\tau_{\boldsymbol{E}}f_1)^*(\boldsymbol{E})=\tau^*(\boldsymbol{E})$ and put $(\boldsymbol{\xi}_{\boldsymbol{E}})_{\boldsymbol{f}}=\langle(\xi_{\boldsymbol{E}})_{\boldsymbol{f}},id_{G_1}\rangle$. We consider the following diagrams whose rectangles are all cartesian.



$$
\begin{array}{ccccc}
(E\times^{\tau_{\boldsymbol{E}}}_{G_0}G_1(\boldsymbol{E}))\times_{G_1(\boldsymbol{E})}G_1 & \xrightarrow{(f_1)_{\pi_{\tau_{\boldsymbol{E}}}}} & E\times^{\tau_{\boldsymbol{E}}}_{G_0}G_1(\boldsymbol{E}) & \xrightarrow{(\tau_{\boldsymbol{E}})_\pi} & E\\
\downarrow{\scriptstyle(\pi_{\tau_{\boldsymbol{E}}})_{f_1}} & & \downarrow{\scriptstyle\pi_{\tau_{\boldsymbol{E}}}} & & \downarrow{\scriptstyle\pi}\\
G_1 & \xrightarrow{f_1} & G_1(\boldsymbol{E}) & \xrightarrow{\tau_{\boldsymbol{E}}} & G_0
\end{array}
$$

Then, $(\xi_{\boldsymbol{E}})_{\boldsymbol{f}}$ is the following composition.

$$
\begin{aligned}
&E\times^{\sigma}_{G_0}G_1\xrightarrow{(id_E\times_{G_0}f_1,\pi_\sigma)}(E\times^{\sigma_{\boldsymbol{E}}}_{G_0}G_1(\boldsymbol{E}))\times_{G_1(\boldsymbol{E})}G_1\xrightarrow{\xi_{\boldsymbol{E}}\times_{G_1(\boldsymbol{E})}id_{G_1}}(E\times^{\tau_{\boldsymbol{E}}}_{G_0}G_1(\boldsymbol{E}))\times_{G_1(\boldsymbol{E})}G_1\\
&\xrightarrow{((\tau_{\boldsymbol{E}})_\pi(f_1)_{\pi_{\tau_{\boldsymbol{E}}}},(\pi_{\tau_{\boldsymbol{E}}})_{f_1})}E\times^{\tau}_{G_0}G_1
\end{aligned}
$$

Since $\hat{\xi}_{\boldsymbol{E}}(id_E\times_{G_0}f_1)=\hat{\xi}$, we have the following equalities by the commutativity of the above diagrams.

$$
\begin{aligned}
\tau_\pi(\xi_{\boldsymbol{E}})_{\boldsymbol{f}}&=\tau_\pi((\tau_{\boldsymbol{E}})_\pi(f_1)_{\pi_{\tau_{\boldsymbol{E}}}},(\pi_{\tau_{\boldsymbol{E}}})_{f_1})(\xi_{\boldsymbol{E}}\times_{G_1(\boldsymbol{E})}id_{G_1})(id_E\times_{G_0}f_1,\pi_\sigma)=(\tau_{\boldsymbol{E}})_\pi(f_1)_{\pi_{\tau_{\boldsymbol{E}}}}(\xi_{\boldsymbol{E}}(id_E\times_{G_0}f_1),\pi_\sigma)\\
&=(\tau_{\boldsymbol{E}})_\pi(f_1)_{\pi_{\tau_{\boldsymbol{E}}}}((\hat{\xi}_{\boldsymbol{E}},\pi_{\sigma_{\boldsymbol{E}}})(id_E\times_{G_0}f_1),\pi_\sigma)=(\tau_{\boldsymbol{E}})_\pi(f_1)_{\pi_{\tau_{\boldsymbol{E}}}}((\hat{\xi}_{\boldsymbol{E}}(id_E\times_{G_0}f_1),\pi_{\sigma_{\boldsymbol{E}}}(id_E\times_{G_0}f_1)),\pi_\sigma)\\
&=(\tau_{\boldsymbol{E}})_\pi(f_1)_{\pi_{\tau_{\boldsymbol{E}}}}((\hat{\xi},f_1\pi_\sigma),\pi_\sigma)=(\tau_{\boldsymbol{E}})_\pi(\hat{\xi},f_1\pi_\sigma)=\hat{\xi}=\tau_\pi\xi\\
\pi_\tau(\xi_{\boldsymbol{E}})_{\boldsymbol{f}}&=\pi_\tau((\tau_{\boldsymbol{E}})_\pi(f_1)_{\pi_{\tau_{\boldsymbol{E}}}},(\pi_{\tau_{\boldsymbol{E}}})_{f_1})(\xi_{\boldsymbol{E}}\times_{G_1(\boldsymbol{E})}id_{G_1})(id_E\times_{G_0}f_1,\pi_\sigma)=(\pi_{\tau_{\boldsymbol{E}}})_{f_1}(\xi_{\boldsymbol{E}}(id_E\times_{G_0}f_1),\pi_\sigma)\\
&=\pi_\sigma=\pi_\tau\xi
\end{aligned}
$$

Hence we have $(\xi_{\boldsymbol{E}})_{\boldsymbol{f}}=\xi$, equivalently $(\boldsymbol{\xi}_{\boldsymbol{E}})_{\boldsymbol{f}}=\langle\xi,id_{G_1}\rangle$, which shows that $(\boldsymbol{E},\hat{\boldsymbol{\xi}})$ coincides with the restriction of the canonical representation $(\boldsymbol{E},\hat{\boldsymbol{\xi}}_{\boldsymbol{E}})$ along $\boldsymbol{f}$.

For a morphism $\boldsymbol{g}=(id_{G_0},g_1):\boldsymbol{G}\to\boldsymbol{G}(\boldsymbol{E})$ of groupoids in $\mathscr{P}_F(\mathcal{C},J)$, we consider the restriction $(\boldsymbol{E},(\boldsymbol{\xi}_{\boldsymbol{E}})_{\boldsymbol{g}})$ of the canonical representation $(\boldsymbol{E},\boldsymbol{\xi}_{\boldsymbol{E}})$ along $\boldsymbol{g}$. We denote by $(\hat{\boldsymbol{\xi}}_{\boldsymbol{E}})_{\boldsymbol{g}}=\langle(\hat{\xi}_{\boldsymbol{E}})_{\boldsymbol{g}},id_{G_0}\rangle:\tau_*\sigma^*(\boldsymbol{E})\to\boldsymbol{E}$ the adjoint of $(\boldsymbol{\xi}_{\boldsymbol{E}})_{\boldsymbol{G}}=\langle(\xi_{\boldsymbol{E}})_{\boldsymbol{G}},id_{G_1}\rangle:\sigma^*(\boldsymbol{E})\to\tau^*(\boldsymbol{E})$ with respect to the adjunction $\tau_*\dashv\tau^*$. It follows from (9.8) that $(\hat{\xi}_{\boldsymbol{E}})_{\boldsymbol{g}}$ maps $(e,u)\in E\times^{\sigma}_{G_0}G_1$ to $\hat{\xi}_{\boldsymbol{E}}(e,g_1(u))=g_1(u)(e)\in E$. Assume that $(\boldsymbol{E},(\boldsymbol{\xi}_{\boldsymbol{E}})_{\boldsymbol{g}})$ coincides with $(\boldsymbol{E},\hat{\boldsymbol{\xi}})$. Since $(\boldsymbol{E},\hat{\boldsymbol{\xi}})$ coincides with the restriction $(\hat{\boldsymbol{\xi}}_{\boldsymbol{E}})_{\boldsymbol{f}}=\langle(\hat{\xi}_{\boldsymbol{E}})_{\boldsymbol{f}},id_{G_0}\rangle$ of the canonical representation of $\boldsymbol{E}$ along $\boldsymbol{f}$ and $\hat{\xi}_{\boldsymbol{E}})_{\boldsymbol{f}}$ maps $(e,u)\in E\times^{\sigma}_{G_0}G_1$ to $\hat{\xi}_{\boldsymbol{E}}(e,f_1(u))=f_1(u)(e)\in E$, it follows that $g_1(u)(e)=f_1(u)(e)$ holds for any $e\in\pi^{-1}(\sigma(u))$ and $u\in G_1$. Thus $g_1(u)=f_1(u)$ holds for any $u\in G_1$, which shows $g_1=f_1$, equivalently $\boldsymbol{g}=\boldsymbol{f}$. □

Remark 9.19 *If the groupoid $\boldsymbol{G}$ in (9.18) is fibrating, so is $\boldsymbol{G}(\boldsymbol{E})$ by (7.3) hence $\boldsymbol{E}$ is a fibration.*

10 Concrete presheaves

Let $\mathcal{C}$ be a category. For an object X of $\mathcal{C}$, we denote by $h^X:\mathcal{C}\to\mathcal{S}et$ a functor defined by $h^X(U)=\mathcal{C}(X,U)$ and $h^X(f:U\to V)=(f_*:\mathcal{C}(X,U)\to\mathcal{C}(X,V))$. For a morphism $\varphi:X\to Y$ of $\mathcal{C}$, let $h^\varphi:h^Y\to h^X$ be a natural transformation defined by $h^\varphi_U=\varphi^*:\mathcal{C}(Y,U)\to\mathcal{C}(X,U)$.

For a natural transformation $T:G\to F$ between functors $F,G:\mathcal{C}\to\mathcal{S}et$, define a morphism $T_X:F_X\to G_X$ of presheaves by $(T_X)_U=T^*_U:F_X(U)=\mathcal{S}et(F(U),X)\to\mathcal{S}et(G(U),X)=G_X(U)$.

Definition 10.1 *Assume that a category $\mathcal{C}$ has a terminal object $1_{\mathcal{C}}$.*

(1) Let $*$ be an element of $F(1_{\mathcal{C}})$. For an object U of $\mathcal{C}$, let $(e_F)_U : h^{1_{\mathcal{C}}}(U) \to F(U)$ be a map defined by $(e_F)_U(\alpha) = F(\alpha)(*)$. Then, $(e_F)_U$ is natural in U and we have a natural transformation $e_F : h^{1_{\mathcal{C}}} \to F$. For a set X, we denote by $e_{F,X} : F_X \to h^{1_{\mathcal{C}}}_X$ the natural transformation $(e_F)_X$ defined from e_F.

(2) For a presheaf $P : \mathcal{C}^{op} \to \mathcal{S}et$ on $\mathcal{C}$, we define a map $\hat{P}_U : P(U) \to \mathcal{S}et(h^{1_{\mathcal{C}}}(U), P(1_{\mathcal{C}})) = h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}(U)$ by $(\hat{P}_U(x))(\alpha) = P(\alpha)(x)$ for $U \in \operatorname{Ob}\mathcal{C}$. Then, $\hat{P}_U$ is natural in U and we have a morphism $\hat{P} : P \to h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}$ of presheaves.

For a category $\mathcal{C}$, we denote by $\widehat{\mathcal{C}}$ the category of presheaves on $\mathcal{C}$.

Remark 10.2 *Let P be a presheaf on $\mathcal{C}$ which has a terminal object $1_{\mathcal{C}}$.*

(1) For an object U of $\mathcal{C}$, let $\theta_U : P(U) \to \widehat{\mathcal{C}}(h_U, P)$ be the map defined as follows. For $x \in P(U)$, let $\theta_U(x) : h_U \to P$ be a natural transformation defined by $(\theta_U(x))_V(\alpha) = P(\alpha)(x)$ if $\alpha \in h_U(V)$. Then, θ_U is bijective by Yoneda's lemma. Define a map $\Phi : \widehat{\mathcal{C}}(h_U, P) \to \mathcal{S}et(h_U(1_{\mathcal{C}}), P(1_{\mathcal{C}}))$ by $\Phi(\varphi) = \varphi_{1_{\mathcal{C}}}$. Then, the following diagram is commutative.

$$\begin{array}{ccc} P(U) & \xrightarrow{\hat{P}_U} & \mathcal{S}et(h^{1_{\mathcal{C}}}(U), P(1_{\mathcal{C}})) \\ \downarrow{\scriptstyle \theta_U} & & \| \\ \widehat{\mathcal{C}}(h_U, P) & \xrightarrow{\Phi} & \mathcal{S}et(h_U(1_{\mathcal{C}}), P(1_{\mathcal{C}})) \end{array}$$

(2) Since $h^{1_{\mathcal{C}}}(1_{\mathcal{C}})$ consists of a single element $id_{1_{\mathcal{C}}}$ and $\hat{P}_{1_{\mathcal{C}}} : P(1_{\mathcal{C}}) \to \mathcal{S}et(h^{1_{\mathcal{C}}}(1_{\mathcal{C}}), P(1_{\mathcal{C}}))$ maps $x \in P(1_{\mathcal{C}})$ to a map which maps $id_{1_{\mathcal{C}}}$ to x, $\hat{P}_{1_{\mathcal{C}}}$ is bijective.

It is easy to verify the following fact.

Proposition 10.3 *For a morphism $\varphi : P \to Q$ of presheaves on $\mathcal{C}$, the following diagram is commutative for any $U \in \operatorname{Ob}\mathcal{C}$.*

$$\begin{array}{ccc} P(U) & \xrightarrow{\hat{P}_U} & \mathcal{S}et(h^{1_{\mathcal{C}}}(U), P(1_{\mathcal{C}})) \\ \downarrow{\scriptstyle \varphi_U} & & \downarrow{\scriptstyle \varphi_{1_{\mathcal{C}}*}} \\ Q(U) & \xrightarrow{\hat{Q}_U} & \mathcal{S}et(h^{1_{\mathcal{C}}}(U), Q(1_{\mathcal{C}})) \end{array}$$

For a set X, define a map $ev_X : h^{1_{\mathcal{C}}}_X(1_{\mathcal{C}}) = \mathcal{S}et(h^{1_{\mathcal{C}}}(1_{\mathcal{C}}), X) \to X$ by $ev_X(\alpha) = \alpha(id_{1_{\mathcal{C}}})$. We can verify that $h^{1_{\mathcal{C}}}_{ev_X} : h^{1_{\mathcal{C}}}_{h^{1_{\mathcal{C}}}_X(1_{\mathcal{C}})}(U) \to h^{1_{\mathcal{C}}}_X(U)$ is the inverse of $(\widehat{h^{1_{\mathcal{C}}}_X})_U : h^{1_{\mathcal{C}}}_X(U) \to h^{1_{\mathcal{C}}}_{h^{1_{\mathcal{C}}}_X(1_{\mathcal{C}})}(U)$. Hence (10.3) implies the following.

Corollary 10.4 *For a morphism $\varphi : P \to h^{1_{\mathcal{C}}}_X$ of presheaves and $U \in \operatorname{Ob}\mathcal{C}$, a map $\varphi_U : P(U) \to h^{1_{\mathcal{C}}}_X(U)$ coincides with a composition $P(U) \xrightarrow{\hat{P}_U} h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}(U) \xrightarrow{h^{1_{\mathcal{C}}}_{\varphi_{1_{\mathcal{C}}}}} h^{1_{\mathcal{C}}}_{h^{1_{\mathcal{C}}}_X(1_{\mathcal{C}})}(U) \xrightarrow{h^{1_{\mathcal{C}}}_{ev_X}} h^{1_{\mathcal{C}}}_X(U)$.*

Definition 10.5 *A presheaf $P : \mathcal{C}^{op} \to \mathcal{S}et$ on $\mathcal{C}$ is called a concrete presheaf if $\hat{P}_U : P(U) \to h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}(U)$ is injective for any object U of $\mathcal{C}$.*

Remark 10.6 *Let P and Q be presheaves on $\mathcal{C}$ and $f : P(1_{\mathcal{C}}) \to Q(1_{\mathcal{C}})$ a map. If Q is a concrete presheaf, it follows from (10.3) that there exists at most one morphism $\varphi : P \to Q$ of presheaves such that $\varphi_{1_{\mathcal{C}}} = f$. Moreover, if Q is a concrete presheaf and φ is a monomorphism, P is also a concrete presheaf. Hence a subpresheaf of a concrete presheaf is a concrete presheaf.*

Example 10.7 *For a set X, define a constant presheaf C_X on a category $\mathcal{C}$ by $C_X(U) = X$ for $U \in \operatorname{Ob}\mathcal{C}$ and $C_X(\varphi) = id_X$ for $\varphi \in \operatorname{Mor}\mathcal{C}$. For $U \in \operatorname{Ob}\mathcal{C}$, $(\widehat{C_X})_U : C_X(U) = X \to \mathcal{S}et(h^{1_{\mathcal{C}}}(U), X)$ maps $x \in X$ to a constant map $h^{1_{\mathcal{C}}}(U) \to X$ whose image is $\{x\}$. Hence C_X is a concrete presheaf on $\mathcal{C}$. For a map $f : X \to Y$, we define a morphism $C_f : C_X \to C_Y$ of presheaves by $(C_f)_U = f$ for any $U \in \operatorname{Ob}\mathcal{C}$.*

Proposition 10.8 *Let $F : \mathcal{C} \to \mathcal{S}et$ be a functor. Suppose that $\mathcal{C}$ has a terminal object $1_{\mathcal{C}}$ and that $*$ is an element of $F(1_{\mathcal{C}})$. For a set X, we define a map $ev_X : F_X(1_{\mathcal{C}}) = \mathcal{S}et(F(1_{\mathcal{C}}), X) \to X$ by $ev_X(c) = c(*)$. Then a composition $F_X(U) \xrightarrow{(\widehat{F_X})_U} h^{1_{\mathcal{C}}}_{F_X(1_{\mathcal{C}})}(U) \xrightarrow{h^{1_{\mathcal{C}}}_{ev_X}} h^{1_{\mathcal{C}}}_X(U)$ coincides with $(e_{F,X})_U : F_X(U) \to h^{1_{\mathcal{C}}}_X(U)$. Hence F_X is a concrete presheaf on $\mathcal{C}$ if $(e_F)_U : h^{1_{\mathcal{C}}}(U) \to F(U)$ is surjective for any $U \in \mathrm{Ob}\,\mathcal{C}$.*

Proof. $(\widehat{F_X})_U$ maps $t \in F_X(U)$ to a map $(\widehat{F_X})_U(t) : \mathcal{C}(1_{\mathcal{C}}, U) \to F_X(1_{\mathcal{C}})$ which is defined by $((\widehat{F_X})_U(t))(\alpha) = F_X(\alpha)(t) = tF(\alpha)$ for $\alpha \in \mathcal{C}(1_{\mathcal{C}}, U)$. Hence $h^{1_{\mathcal{C}}}_{ev_X}(\widehat{F_X})_U : F_X(U) \to h^{1_{\mathcal{C}}}_X(U)$ maps $t \in F_X(U)$ to a map which maps $\alpha \in \mathcal{C}(1_{\mathcal{C}}, U)$ to $tF(\alpha)(*) \in X$. On the other hand, $(e_{F,X})_U$ maps $t \in F_X(U)$ to a map which maps $\alpha \in \mathcal{C}(1_{\mathcal{C}}, U)$ to $t(e_F)_U(\alpha) = tF(\alpha)(*) \in X$. $\square$

Remark 10.9 *(1) Since $(e_{h^{1_{\mathcal{C}}}})_U : h^{1_{\mathcal{C}}}(U) \to h^{1_{\mathcal{C}}}(U)$ is the identity map, $h^{1_{\mathcal{C}}}_X : \mathcal{C}^{op} \to \mathcal{S}et$ is a concrete presheaf.*

(2) Let $\mathscr{F} : \mathcal{C}^\infty \to \mathcal{S}et$ be the forgetful functor. Then, the natural transformation $e_{\mathscr{F}} : h^{\boldsymbol{R}^0} \to \mathscr{F}$ defined in (10.1) is an equivalence. Hence, for a set X, $e_{\mathscr{F}}$ induces a natural equivalence $e(X) : \mathscr{F}_X \to h^{\boldsymbol{R}^0}_X$ of presheaves on $\mathcal{C}^\infty$.

Proposition 10.10 *For a set X, a concrete presheaf P on a category $\mathcal{C}$ such that $P(1_{\mathcal{C}})$ is a subset of X is a subpresheaf of $h^{1_{\mathcal{C}}}_X$. Conversely, a subpresheaf of $h^{1_{\mathcal{C}}}_X$ is a concrete presheaf.*

Proof. Let $i : P(1_{\mathcal{C}}) \to X$ be the inclusion map. For $U \in \mathrm{Ob}\,\mathcal{C}$, we define a map $\psi_U : P(U) \to h^{1_{\mathcal{C}}}_X(U)$ to be a composition $P(U) \xrightarrow{\hat{P}_U} \mathcal{S}et(\mathcal{C}(1_{\mathcal{C}}, U), P(1_{\mathcal{C}})) \xrightarrow{i_*} \mathcal{S}et(\mathcal{C}(1_{\mathcal{C}}, U), X) = h^{1_{\mathcal{C}}}_X(U)$. Since $\hat{P}_U$ is injective by the assumption, ψ_U is a natural injection. Since $h^{1_{\mathcal{C}}}_X$ is a concrete presheaf by (10.9), it follows from (10.6) that a subpresheaf of $h^{1_{\mathcal{C}}}_X$ is a concrete presheaf. $\square$

We denote by $\widehat{\mathcal{C}}^c$ a full subcategory of $\widehat{\mathcal{C}}$ consisting of concrete presheaves.

Proposition 10.11 *$\widehat{\mathcal{C}}^c$ is complete.*

Proof. For a family $(P_i)_{i \in I}$ of concrete presheaves and $U \in \mathrm{Ob}\,\mathcal{C}$, $\prod\limits_{i \in I} \hat{P}_{iU} : \prod\limits_{i \in I} P_i(U) \to \prod\limits_{i \in I} h^{1_{\mathcal{C}}}_{P_i(1_{\mathcal{C}})}(U)$ is injective. Let $\prod\limits_{i \in I} P_i$ be the product of P_i's defined by $\Big(\prod\limits_{i \in I} P_i\Big)(U) = \prod\limits_{i \in I} P_i(U)$. Then, we have a monomorphism $\prod\limits_{i \in I} \hat{P}_i : \prod\limits_{i \in I} P_i \to \prod\limits_{i \in I} h^{1_{\mathcal{C}}}_{P_i(1_{\mathcal{C}})}$ in $\widehat{\mathcal{C}}$. On the other hand, the projections $\mathrm{pr}_i : \prod\limits_{i \in I} P_i(1_{\mathcal{C}}) \to P_i(1_{\mathcal{C}})$ induce a bijection

$$(\mathrm{pr}_{i*})_{i \in I} : h^{1_{\mathcal{C}}}_{\prod\limits_{i \in I} P_i(1_{\mathcal{C}})}(U) = \mathcal{S}et\Big(\mathcal{C}(1_{\mathcal{C}}, U), \prod_{i \in I} P_i(1_{\mathcal{C}})\Big) \to \prod_{i \in I} \mathcal{S}et(\mathcal{C}(1_{\mathcal{C}}, U), P_i(1_{\mathcal{C}})) = \prod_{i \in I} h^{1_{\mathcal{C}}}_{P_i(1_{\mathcal{C}})}(U)$$

which is natural in U. We denote by $\Pi_U : \prod\limits_{i \in I} h^{1_{\mathcal{C}}}_{P_i(1_{\mathcal{C}})}(U) \to h^{1_{\mathcal{C}}}_{\prod\limits_{i \in I} P_i(1_{\mathcal{C}})}(U)$ the inverse of the above map. Thus we have an isomorphism $\Pi : \prod\limits_{i \in I} h^{1_{\mathcal{C}}}_{P_i(1_{\mathcal{C}})} \to h^{1_{\mathcal{C}}}_{\prod\limits_{i \in I} P_i(1_{\mathcal{C}})}$ of presheaves. Hence $\prod\limits_{i \in I} P_i$ is regarded as a subpresheaf of $h^{1_{\mathcal{C}}}_{\prod\limits_{i \in I} P_i(1_{\mathcal{C}})}$ and it is a concrete presheaf by (10.10). Since a subpresheaf of a concrete presheaf is also a concrete presheaf by (10.6), an equalizer of a parallel pair of morphisms between concrete presheaves is a concrete presheaf. Therefore $\widehat{\mathcal{C}}^c$ is complete. $\square$

For a presheaf P on $\mathcal{C}$ and an object U of $\mathcal{C}$, let $P^c(U)$ be the image of $\hat{P}_U : P(U) \to h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}(U)$. Note that $P^c(1_{\mathcal{C}}) = h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}(1_{\mathcal{C}})$ by (2) of (10.2). Let $f : U \to V$ be a morphism in $\mathcal{C}$. It follows from the naturality of $\hat{P}_U$ that $h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}(f) : h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}(V) \to h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}(U)$ maps $P^c(V)$ to $P^c(U)$. Thus we have a subpresheaf P^c of $h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}$. We denote by $\iota_P : P^c \to h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}$ a morphism of presheaves induced by the inclusion maps $P^c(U) \to h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}(U)$.

For a morphism $\varphi : P \to Q$ of presheaves, it follows from (10.3) that $(h^{1_{\mathcal{C}}}_{\varphi_{1_{\mathcal{C}}}})_U : h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}(U) \to h^{1_{\mathcal{C}}}_{Q(1_{\mathcal{C}})}(U)$ maps $P^c(U)$ to $Q^c(U)$. Hence we have a morphism $\varphi^c : P^c \to Q^c$ of presheaves. Since P^c is a concrete presheaf by (10.10), we define a functor $\mathscr{C} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}^c$ by $\mathscr{C}(P) = P^c$ and $\mathscr{C}(\varphi) = \varphi^c$.

Proposition 10.12 *$\mathscr{C} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}^c$ is a left adjoint of the inclusion functor $i^c : \widehat{\mathcal{C}}^c \to \widehat{\mathcal{C}}$.*

Proof. For a presheaf P on $\mathcal{C}$ and $U \in \mathrm{Ob}\,\mathcal{C}$, let $(\eta_P)_U : P(U) \to P^c(U) = i^c\mathscr{C}(P)(U)$ be a map defined by $(\eta_P)_U(x) = \hat{P}_U(x)$. Then we have a morphism $\eta_P : P \to i^c\mathscr{C}(P)$ of presheaves and $\hat{P} : P \to h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}$ is a composition of $\eta_P : P \to i^c\mathscr{C}(P)$ and inclusion morphism $\iota_P : i^c\mathscr{C}(P) \to h^{1_{\mathcal{C}}}_{P(1_{\mathcal{C}})}$. For a concrete presheaf Q on $\mathcal{C}$ and a morphism $\varphi : P \to i^c(Q)$, the following diagram is commutative.

$$\begin{array}{ccccc} & & \overset{\hat{P}}{\frown} & & \\ P & \xrightarrow{\eta_P} & i^c\mathscr{C}(P) & \overset{\iota_P}{\hookrightarrow} & h^{1_\mathcal{C}}_{P(1_\mathcal{C})} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle i^c\mathscr{C}(\varphi)} & & \downarrow{\scriptstyle\varphi_{1_\mathcal{C}*}} \\ i^c(Q) & \underset{\cong}{\xrightarrow{\eta_{i^c(Q)}}} & i^c\mathscr{C}i^c(Q) & \overset{\iota_{i^c\mathscr{C}i^c(Q)}}{\hookrightarrow} & h^{1_\mathcal{C}}_{Q(1_\mathcal{C})} \\ & & \underset{\hat{Q}}{\smile} & & \end{array}$$

Since Q is a concrete presheaf, $\eta_{i^c(Q)} : i^c(Q) \to i^c\mathscr{C}i^c(Q)$ is an isomorphism of presheaves. It follows that $\eta_P^* : \widehat{\mathcal{C}}(i^c\mathscr{C}(P), i^c(Q)) \to \widehat{\mathcal{C}}(P, i^c(Q))$ is surjective. Since η_P is an epimorphism, $\eta_P^* : \widehat{\mathcal{C}}(i^c\mathscr{C}(P), i^c(Q)) \to \widehat{\mathcal{C}}(P, i^c(Q))$ is injective. Therefore $\eta_P^* : \widehat{\mathcal{C}}(i^c\mathscr{C}(P), i^c(Q)) \to \widehat{\mathcal{C}}(P, i^c(Q))$ is bijective. Since $\widehat{\mathcal{C}}^c$ is a full subcategory of $\widehat{\mathcal{C}}$, $i^c : \widehat{\mathcal{C}}^c(\mathscr{C}(P), Q) \to \widehat{\mathcal{C}}(i^c\mathscr{C}(P), i^c(Q))$ is bijective. Hence a composition $\widehat{\mathcal{C}}^c(\mathscr{C}(P), Q) \xrightarrow{i^c} \widehat{\mathcal{C}}(i^c\mathscr{C}(P), i^c(Q)) \xrightarrow{\eta_P^*} \widehat{\mathcal{C}}(P, i^c(Q))$ is a natural bijection and the assertion follows. □

Remark 10.13 *(1)* $(\eta_P)_{1_\mathcal{C}} : P(1_\mathcal{C}) \to P^c(1_\mathcal{C})$ *is bijective.*
(2) P *is a concrete presheaf if and only if* $\eta_P : P \to i^c\mathscr{C}(P)$ *is an isomorphism.*

Proposition 10.14 $\mathscr{C} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}^c$ *preserves products.*

Proof. Let $(P_i)_{i\in I}$ be a family of presheaves on $\mathcal{C}$. We denote by $\mathrm{pr}_j : \prod_{i\in I} P_i \to P_j$ the projection to j-th factor. Then pr_j's define a bijection $((\mathrm{pr}_i)_{1_\mathcal{C}})_{i\in I} : \mathcal{S}et\Big(h^{1_\mathcal{C}}(U), \prod_{i\in I} P_i(1_\mathcal{C})\Big) \to \prod_{i\in I} \mathcal{S}et(h^{1_\mathcal{C}}(U), P_i(1_\mathcal{C}))$ which is natural in $U \in \mathrm{Ob}\,\mathcal{C}$. Since a product of surjections is also a surjection and a product of injections is also an injection, we have a bijection $((\mathrm{pr}_i^c)_U)_{i\in I} : \Big(\prod_{i\in I} P_i\Big)^c(U) \to \prod_{i\in I} P_i^c(U)$.

$$\begin{array}{ccccc} \Big(\prod_{i\in I} P_i\Big)(U) & \xrightarrow{\left(\eta_{\prod_{i\in I} P_i}\right)_U} & \Big(\prod_{i\in I} P_i\Big)^c(U) & \xrightarrow{\left(\iota_{\prod_{i\in I} P_i}\right)_U} & \mathcal{S}et\Big(h^{1_\mathcal{C}}(U), \prod_{i\in I} P_i(1_\mathcal{C})\Big) \\ \| & & \downarrow{\scriptstyle((\mathrm{pr}_i^c)_U)_{i\in I}} & & \cong\downarrow{\scriptstyle((\mathrm{pr}_i)_{1_\mathcal{C}})_{i\in I}} \\ \prod_{i\in I} P_i(U) & \xrightarrow{\prod_{i\in I}(\eta_{P_i})_U} & \prod_{i\in I} P_i^c(U) & \xrightarrow{\prod_{i\in I}(\iota_{P_i})_U} & \prod_{i\in I} \mathcal{S}et(h^{1_\mathcal{C}}(U), P_i(1_\mathcal{C})) \end{array}$$

□

11 Concrete site and concrete sheaves

Definition 11.1 *Let* $(\mathcal{C}, J)$ *be a site and* $F : \mathcal{C} \to \mathcal{S}et$ *a functor. If* $(\mathcal{C}, J)$ *and* F *satisfies the following condition,* $(\mathcal{C}, J)$ *is called an* F*-preconcrete site. Moreover, if* $F : \mathcal{C} \to \mathcal{S}et$ *is faithful,* $(\mathcal{C}, J)$ *is called an* F*-concrete site.*

(PCS) For every covering $(U_i \xrightarrow{f_i} U)_{i\in I}$, $(F(U_i) \xrightarrow{F(f_i)} F(U))_{i\in I}$ *is an epimorphic family in* $\mathcal{S}et$.

Assume that $\mathcal{C}$ *has a terminal object* $1_\mathcal{C}$*. A* $h^{1_\mathcal{C}}$*-preconcrete site is called a preconcrete site and an* $h^{1_\mathcal{C}}$*-concrete site is called a concrete site.*

Remark 11.2 *Let* X *be a set and* $(\mathcal{C}, J)$ *an* F*-preconcrete site. For a covering* $(U_i \xrightarrow{f_i} U)_{i\in I}$ *in* $(\mathcal{C}, J)$*, since* $(F(U_i) \xrightarrow{F(f_i)} F(U))_{i\in I}$ *is an epimorphic family in* $\mathcal{S}et$*, the map* $(F_X(f_i))_{i\in I} : F_X(U) \to \prod_{i\in I} F_X(U_i)$ *induced by* $F_X(f_i) = F(f_i)^* : F_X(U) \to F_X(U_i)$*'s is injective. Hence* F_X *is a separated presheaf on* $\mathcal{C}$ *and* $F_\mathscr{D}$ *is also a separated presheaf for a the-ology* $\mathscr{D}$ *on* X.

Proposition 11.3 *Let* $(\mathcal{C}, J)$ *be a preconcrete site. If* $R \in J(1_\mathcal{C})$ *is not an empty subfunctor of* $h_{1_\mathcal{C}}$*, then* $R = h_{1_\mathcal{C}}$.

Proof. It follows from (11.1) that there exist $(o_V : V \to 1_\mathcal{C}) \in R$ and $\alpha \in h^{1_\mathcal{C}}(V) = \mathcal{C}(1_\mathcal{C}, V)$ which satisfy $o_V\alpha = id_{1_\mathcal{C}}$. This implies that $R(1_\mathcal{C}) = \{id_{1_\mathcal{C}}\}$. For any $U \in \mathrm{Ob}\,\mathcal{C}$, since the unique morphism $o_U : U \to 1_\mathcal{C}$ induces a map $R(o_U) : R(1_\mathcal{C}) \to R(U)$, $R(U)$ is not an empty set. Since $R(U)$ is a subset of $h_{1_\mathcal{C}}(U) = \{o_U : U \to 1_\mathcal{C}\}$, we have $R(U) = h_{1_\mathcal{C}}(U)$. □

Proposition 11.4 *$(\mathcal{C}^\infty, J_\infty)$ given in (7.12) is a concrete site.*

Proof. $\boldsymbol{R}^0 = \{0\}$ is a terminal object of $\mathcal{C}^\infty$. For $U, V \in \operatorname{Ob}\mathcal{C}^\infty$ and morphisms $f, g : U \to V$, suppose that $f_* = g_* : \mathcal{C}^\infty(\boldsymbol{R}^0, U) \to \mathcal{C}^\infty(\boldsymbol{R}^0, V)$ holds. For $x \in U$, let $c_x : \boldsymbol{R}^0 \to U$ be the map defined by $c_x(0) = x$. Then we have $fc_x = f_*(c_x) = g_*(c_x) = gc_x$ which implies $f(x) = g(x)$. Thus $f = g$ and $h^{\boldsymbol{R}^0}$ is faithful.

Let $(U_i \xrightarrow{f_i} U)_{i\in I}$ be a covering in $\mathcal{C}^\infty$ and $c \in \mathcal{C}^\infty(\boldsymbol{R}^0, U)$. There exists $i \in I$ such that $c(0) \in f_i(U_i)$. Hence $c(0) = f_i(x)$ for some $x \in U_i$. Define a map $c_x : \boldsymbol{R}^0 \to U_i$ by $c_x(0) = x$. Then, $f_{i*} : \mathcal{C}^\infty(\boldsymbol{R}^0, U_i) \to \mathcal{C}^\infty(\boldsymbol{R}^0, U)$ maps c_x to c. It follows that $(\mathcal{C}^\infty, J_\infty)$ is a concrete site. □

Definition 11.5 *If $(\mathcal{C}, J)$ is a site, a concrete presheaf on $\mathcal{C}$ which is a sheaf is called a concrete sheaf. We denote by $\operatorname{CSh}(\mathcal{C}, J)$ a full subcategory of the category $\operatorname{Sh}(\mathcal{C}, J)$ of sheaves on $(\mathcal{C}, J)$ consisting of concrete sheaves.*

Proposition 11.6 *If $(\mathcal{C}, J)$ is a preconcrete site, $h^{1_\mathcal{C}}_X$ is a concrete sheaf on $(\mathcal{C}, J)$.*

Proof. We note that $h^{1_\mathcal{C}}(1_\mathcal{C})$ consists of single element $id_{1_\mathcal{C}}$ and that $(e_{h^{1_\mathcal{C}}})_U : \mathcal{C}(1_\mathcal{C}, U) \to h^{1_\mathcal{C}}(U)$ is the identity map for $U \in \operatorname{Ob}\mathcal{C}$. Hence $h^{1_\mathcal{C}}_X$ is a concrete presheaf by (10.8).

For an object U of $\mathcal{C}$ and $R \in J(U)$, let $(U_i \xrightarrow{f_i} U)_{i\in I}$ be a family of morphisms in $\mathcal{C}$ which generates R. Let $(h^{1_\mathcal{C}}_X(f_i))_{i\in I} : h^{1_\mathcal{C}}_X(U) = \mathcal{S}et(h^{1_\mathcal{C}}(U), X) \to \prod_{i\in I} \mathcal{S}et(h^{1_\mathcal{C}}(U_i), X) = \prod_{i\in I} h^{1_\mathcal{C}}_X(U_i)$ be the map induced by $h^{1_\mathcal{C}}_X(f_i) = h^{1_\mathcal{C}}(f_i)^* : h^{1_\mathcal{C}}_X(U) \to h^{1_\mathcal{C}}_X(U_i)$'s. Since $(h^{1_\mathcal{C}}(f_i) : h^{1_\mathcal{C}}(U_i) \to h^{1_\mathcal{C}}(U))_{i\in I}$ is an epimorphic family by the assumption, $(h^{1_\mathcal{C}}_X(f_i))_{i\in I}$ is injective. It remains to verify that the image of $(h^{1_\mathcal{C}}_X(f_i))_{i\in I} : h^{1_\mathcal{C}}_X(U) \to \prod_{i\in I} h^{1_\mathcal{C}}_X(U_i)$ is $\left\{(x_i)_{i\in I} \in \prod_{i\in I} h^{1_\mathcal{C}}_X(U_i) \,\middle|\, h^{1_\mathcal{C}}_X(g)(x_i) = h^{1_\mathcal{C}}_X(h)(x_j) \text{ if } f_i g = f_j h \text{ for } i, j \in I \text{ and } g : Z \to U_i,\ h : Z \to U_j\right\}$ which we denote by M below. For $t \in h^{1_\mathcal{C}}_X(U) = \mathcal{S}et(h^{1_\mathcal{C}}(U), X)$, we claim that $(h^{1_\mathcal{C}}_X(f_i)(t))_{i\in I}$ belongs to M. For $i, j \in I$ and morphisms $g : Z \to U_i$, $h : Z \to U_j$ of $\mathcal{C}$ which satisfy $f_i g = f_j h$, we have the following.

$$h^{1_\mathcal{C}}_X(g)(h^{1_\mathcal{C}}_X(f_i)(t)) = h^{1_\mathcal{C}}_X(gf_i)(t) = h^{1_\mathcal{C}}_X(hf_j)(t) = h^{1_\mathcal{C}}_X(h)(h^{1_\mathcal{C}}_X(f_j)(t))$$

Thus $(h^{1_\mathcal{C}}_X(f_i)(t))_{i\in I}$ belongs to M. For $(x_i)_{i\in I} \in M$, we define $x \in h^{1_\mathcal{C}}_X(U) = \mathcal{S}et(h^{1_\mathcal{C}}(U), X)$ as follows. For $\alpha \in h^{1_\mathcal{C}}(U)$, since $(h^{1_\mathcal{C}}(f_i) : h^{1_\mathcal{C}}(U_i) \to h^{1_\mathcal{C}}(U))_{i\in I}$ is an epimorphic family in $\mathcal{S}et$, we can choose $i \in I$ and $g \in h^{1_\mathcal{C}}(U_i)$ such that $f_i g = \alpha$. We define $x \in h^{1_\mathcal{C}}_X(U)$ by $x(\alpha) = x_i(g)$. If $j \in I$ and $h \in h^{1_\mathcal{C}}(U_j)$ satisfy $f_j h = \alpha$, then we have $x_i(g) = x_i g_*(id_{1_\mathcal{C}}) = h^{1_\mathcal{C}}_X(g)(x_i)(id_{1_\mathcal{C}}) = h^{1_\mathcal{C}}_X(h)(x_j)(id_{1_\mathcal{C}}) = x_j h_*(id_{1_\mathcal{C}}) = x_j(h)$. Hence $x(\alpha)$ does not depend on the choice of $i \in I$ and $g \in h^{1_\mathcal{C}}(U_i)$ such that $f_i g = \alpha$. For $i \in I$ and $g \in h^{1_\mathcal{C}}(U_i)$, put $\alpha = f_i g$. Then we have $(h^{1_\mathcal{C}}_X(f_i)(x))(g) = (xf_{i*})(g) = x(\alpha) = x_i(g)$ which shows $h^{1_\mathcal{C}}_X(f_i)(x) = x_i$, that is, $(x_i)_{i\in I}$ belongs to the image of $(h^{1_\mathcal{C}}_X(f_i))_{i\in I} : h^{1_\mathcal{C}}_X(U) \to \prod_{i\in I} h^{1_\mathcal{C}}_X(U_i)$. □

Proposition 11.7 *Let $(\mathcal{C}, J)$ be a preconcrete site. A concrete presheaf on $\mathcal{C}$ is a separated presheaf.*

Proof. Let F be a concrete presheaf on $\mathcal{C}$. For a covering $(U_i \xrightarrow{f_i} U)_{i\in I}$, the following diagram is commutative by (10.3).

$$\begin{array}{ccc}
F(U) & \xrightarrow{(F(f_i))_{i\in I}} & \prod_{i\in I} F(U_i) \\
\big\downarrow {\scriptstyle \hat{F}_U} & & \big\downarrow {\scriptstyle \prod_{i\in I} \hat{F}_{U_i}} \\
h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(U) & \xrightarrow{(h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(f_i))_{i\in I}} & \prod_{i\in I} h^{1_\mathcal{C}}_{F(1_\mathcal{C})} F(U_i)
\end{array}$$

Since the vertical maps and lower horizontal map of the above diagram are injective by (11.6), so is the upper horizontal map. □

Proposition 11.8 *Let $(\mathcal{C}, J)$ be a preconcrete site and F a concrete presheaf on $\mathcal{C}$. Then the sheafification $a(F)$ of F is a concrete sheaf such that $a(F)(1_\mathcal{C}) = F(1_\mathcal{C})$.*

Proof. For $U \in \operatorname{Ob}\mathcal{C}$, we regard $J(U)$ as a subcategory of $\widehat{\mathcal{C}}$ whose morphisms are inclusion functors. We denote by $\iota^S_R : S \to R$ the inclusion functor if S is a subfunctor of R. Define a functor $D_{F,U} : J(U)^{op} \to \mathcal{S}et$ by $D_{F,U}(R) = \widehat{\mathcal{C}}(R, F)$ and $D_{F,U}(\iota^S_R) = \iota^{S*}_R$. Let $(\widehat{\mathcal{C}}(R, F) \xrightarrow{i_{R,U}} LF(U))_{R\in J(U)}$ be a colimiting cone of $D_{F,U}$. Then, a correspondence $U \mapsto LF(U)$ defines a presheaf LF on $\mathcal{C}$. Since F is a separated presheaf by (11.7), LF is a sheaf. Hence LF is the sheafification $a(F)$ of F.

The following diagram is commutative. Here we put $F(1_\mathcal{C}) = X$.

$$\begin{array}{ccccc}
\widehat{\mathcal{C}}(h_U,F) & \xrightarrow{\iota_{h_U}^{R*}} & \widehat{\mathcal{C}}(R,F) & \xrightarrow{\iota_R^{S*}} & \widehat{\mathcal{C}}(S,F) \\
\downarrow{\scriptstyle \hat{F}_*} & & \downarrow{\scriptstyle \hat{F}_*} & & \downarrow{\scriptstyle \hat{F}_*} \\
\widehat{\mathcal{C}}(h_U,h_X^{1_\mathcal{C}}) & \xrightarrow{\iota_{h_U}^{R*}} & \widehat{\mathcal{C}}(R,h_X^{1_\mathcal{C}}) & \xrightarrow{\iota_R^{S*}} & \widehat{\mathcal{C}}(S,h_X^{1_\mathcal{C}})
\end{array}$$

Since $\hat{F}: F \to h_X^{1_\mathcal{C}}$ is a monomorphism, the vertical maps of the above diagram are injective. Since $h_X^{1_\mathcal{C}}$ is a sheaf by (11.6), the lower horizontal maps are bijective. It follows that if $(\widehat{\mathcal{C}}(R,h_X^{1_\mathcal{C}}) \xrightarrow{j_{R,U}} Lh_X^{1_\mathcal{C}}(U))_{R\in J(U)}$ is a colimiting cone of $D_{h_X^{1_\mathcal{C}},U}$, $j_{R,U}$ is bijective for any $R \in J(U)$. Hence the upper horizontal maps of the above diagram are injective and this implies that $i_{R,U}: \widehat{\mathcal{C}}(R,F) \to LF(U)$ is injective.

$$\begin{array}{ccc}
\widehat{\mathcal{C}}(R,F) & \xrightarrow{i_{R,U}} & LF(U) \\
\downarrow{\scriptstyle \hat{F}_*} & & \downarrow{\scriptstyle L\hat{F}_U} \\
\widehat{\mathcal{C}}(R,h_X^{1_\mathcal{C}}) & \xrightarrow[\cong]{j_{R,U}} & Lh_X^{1_\mathcal{C}}(U)
\end{array}
\qquad
\begin{array}{ccccc}
F(U) & \xrightarrow[\cong]{\theta_U} & \widehat{\mathcal{C}}(h_U,F) & \xrightarrow{i_{h_U,U}} & LF(U) \\
\downarrow{\scriptstyle \hat{F}_U} & & \downarrow{\scriptstyle \hat{F}_*} & & \downarrow{\scriptstyle L\hat{F}_U} \\
h_X^{1_\mathcal{C}}(U) & \xrightarrow[\cong]{\theta_U} & \widehat{\mathcal{C}}(h_U,h_X^{1_\mathcal{C}}) & \xrightarrow[\cong]{j_{h_U,U}} & Lh_X^{1_\mathcal{C}}(U)
\end{array}$$

Since $LF(U)$ is the union of the images of $i_{R,U}$, it follows from the commutativity of the above left diagram that $L\hat{F}_U : LF(U) \to Lh_X^{1_\mathcal{C}}(U)$ is injective. Since $j_{h_U,U}\theta_U : h_X^{1_\mathcal{C}}(U) \to Lh_X^{1_\mathcal{C}}(U)$ defines a natural equivalence $h_X^{1_\mathcal{C}} \to Lh_X^{1_\mathcal{C}}$, LF is a subfunctor of $h_X^{1_\mathcal{C}}$. Therefore LF is a concrete sheaf by (10.10). Finally, $LF(1_\mathcal{C}) = F(1_\mathcal{C})$ follows from (11.3). $\square$

Let $(\mathcal{C}, J)$ be a preconcrete site and F a concrete presheaf on $\mathcal{C}$. For an object U of $\mathcal{C}$ and a sieve $R \in J(U)$, let M_R be a subset of $\prod\limits_{f\in R} F(\mathrm{dom}(f))$ consisting of elements $(x_f)_{f\in R}$ which satisfy the following condition.

($*$) If $f, g \in R$ and $p : Z \to \mathrm{dom}(f)$, $q : Z \to \mathrm{dom}(g)$ satisfy $fp = gq$, then $F(p)(x_f) = F(q)(x_g)$ holds.

We denote by $\bar{M}_R$ the image of M_R by a map $\prod\limits_{f\in R} \hat{F}_{\mathrm{dom}(f)} : \prod\limits_{f\in R} F(\mathrm{dom}(f)) \to \prod\limits_{f\in R} h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(\mathrm{dom}(f))$. We also denote by $\bar{F}_R(U)$ the inverse image of $\bar{M}_R$ by $(h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(f))_{f\in R} : h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(U) \to \prod\limits_{f\in R} h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(\mathrm{dom}(f))$ and put $\bar{F}(U) = \bigcup\limits_{R\in J(U)} \bar{F}_R(U)$.

Proposition 11.9 *A correspondence $U \mapsto \bar{F}(U)$ defines a subsheaf $\bar{F}$ of $h^{1_\mathcal{C}}_{F(1_\mathcal{C})}$ and $\bar{F}$ is isomorphic to the sheafification of F.*

Proof. Let $\rho : U \to V$ be a morphism in $\mathcal{C}$ and x an element of $\bar{F}(V)$. There exists a sieve $R \in J(V)$ such that $x \in \bar{F}_R(V)$. Thus we have $(h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(f)(x))_{f\in R} \in \bar{M}_R(V)$, which implies that there exists $(x_f)_{f\in R} \in M_R(V)$ such that $\hat{F}_{\mathrm{dom}(f)}(x_f) = h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(f)(x)$ for any $f \in R$. We put $h_\rho^{-1}(R) = \{g \in \mathrm{Ob}\,(\mathcal{C}/U)\,|\,\rho g \in R\}$ and $y_g = x_{\rho g}$. Then $\prod\limits_{g\in h_\rho^{-1}(R)} \hat{F}_{\mathrm{dom}(g)} : \prod\limits_{g\in h_\rho^{-1}(R)} F(\mathrm{dom}(g)) \to \prod\limits_{g\in R} h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(\mathrm{dom}(g))$ maps $(y_g)_{g\in h_\rho^{-1}(R)}$ to $(h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(\rho g)(x))_{g\in h_\rho^{-1}(R)}$. Since $(x_f)_{f\in R} \in M_R$, if $g, h \in h_\rho^{-1}(R)(U)$ and $p : Z \to \mathrm{dom}(g)$, $q : Z \to \mathrm{dom}(h)$ satisfy $gp = hq$, then $F(p)(y_g) = F(p)(x_{\rho g}) = F(q)(x_{\rho h}) = F(q)(y_h)$ holds. Therefore we have $(y_g)_{g\in h_\rho^{-1}(R)} \in M_{h_\rho^{-1}(R)}$ and $(h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(\rho g)(x))_{g\in h_\rho^{-1}(R)} \in \bar{M}_{h_\rho^{-1}(R)}$, which is the image of $h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(\rho)(x) \in h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(U)$ by $\left(h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(g)\right)_{g\in h_\rho^{-1}(R)} : h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(U) \to \prod\limits_{g\in h_\rho^{-1}(R)} h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(\mathrm{dom}(g))$. Thus we see that $h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(\rho)(x) \in \bar{F}_{h_\rho^{-1}(R)}(U)$. Since $h_\rho^{-1}(R) \in J(U)$, it follows that $h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(\rho)(x)$ belongs to $\bar{F}(U)$. This shows that $h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(\rho) : h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(V) \to h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(U)$ maps $\bar{F}(V)$ into $\bar{F}(U)$ and a correspondence $U \mapsto \bar{F}(U)$ defines a subpresheaf $\bar{F}$ of $h^{1_\mathcal{C}}_{F(1_\mathcal{C})}$.

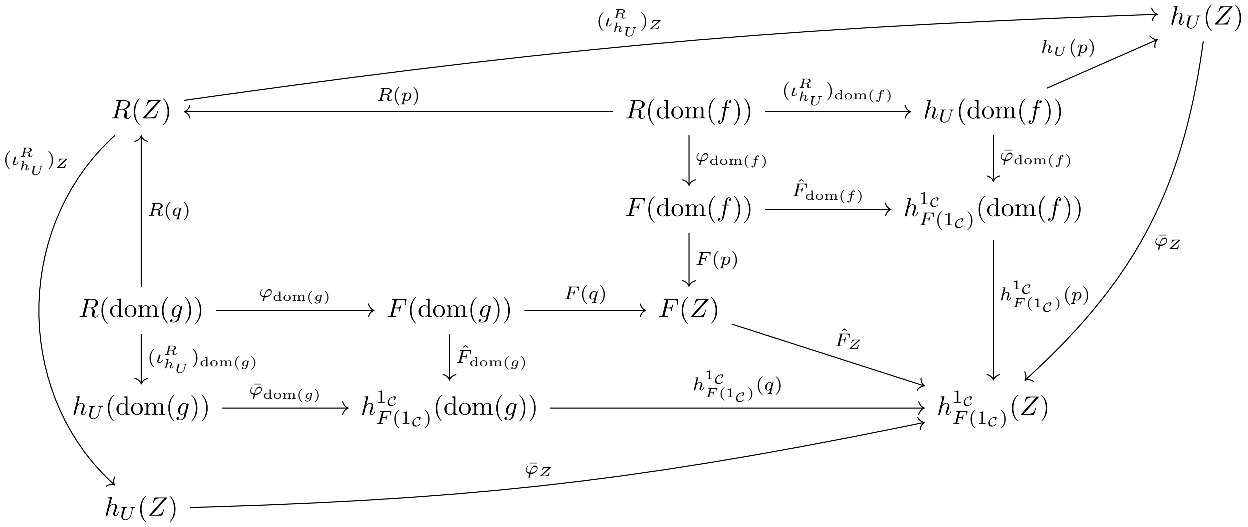
For an object U of $\mathcal{C}$ and a sieve $R \in J(U)$, the map $\iota_{h_U}^{R*} : \widehat{\mathcal{C}}(h_U, h^{1_\mathcal{C}}_{F(1_\mathcal{C})}) \to \widehat{\mathcal{C}}(R, h^{1_\mathcal{C}}_{F(1_\mathcal{C})})$ induced by the inclusion functor $\iota_{h_U}^R : R \to h_U$ is bijective since $h^{1_\mathcal{C}}_{F(1_\mathcal{C})}$ is a sheaf on $(\mathcal{C}, J)$ by (11.6). By Yoneda's lemma, a map $\theta_U^{-1} : \widehat{\mathcal{C}}(h_U, h^{1_\mathcal{C}}_{F(1_\mathcal{C})}) \to h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(U)$ defined by $\theta_U^{-1}(\psi) = \psi_U(id_U)$ is bijective. We consider the following composition of maps.

$$\widehat{\mathcal{C}}(R,F) \xrightarrow{\hat{F}_*} \widehat{\mathcal{C}}(R,h^{1_\mathcal{C}}_{F(1_\mathcal{C})}) \xrightarrow{(\iota_{h_U}^{R*})^{-1}} \widehat{\mathcal{C}}(h_U,h^{1_\mathcal{C}}_{F(1_\mathcal{C})}) \xrightarrow{\theta_U^{-1}} h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(U) \quad \cdots (**)$$

For $\varphi \in \widehat{\mathcal{C}}(R,F)$, we put $(\iota_{h_U}^{R*})^{-1}(\hat{F}\varphi) = \bar{\varphi}$. Then $\bar{\varphi} \in \widehat{\mathcal{C}}(h_U, h^{1_\mathcal{C}}_{F(1_\mathcal{C})})$ makes the following diagrams commute.

$$
\begin{array}{ccc}
R & \xrightarrow{\iota^R_{h_U}} & h_U \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\bar\varphi} \\
F & \xrightarrow{\hat F} & h^{1_{\mathcal C}}_{F(1_{\mathcal C})}
\end{array}
$$

For $f \in R$, let y_f be the image of f by a map $\varphi_{\mathrm{dom}(f)} : R(\mathrm{dom}(f)) \to F(\mathrm{dom}(f))$. Suppose that $f, g \in R$ and $p : Z \to \mathrm{dom}(f)$, $q : Z \to \mathrm{dom}(g)$ satisfy $fp = gq$. We note that the following diagram is commutative.



The commutativity of the above diagram implies the following equalities.

$$
\begin{aligned}
\hat F_Z(F(p)(y_f)) &= \hat F_Z(F(p)(\varphi_{\mathrm{dom}(f)}(f))) = h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(p)(\bar\varphi_{\mathrm{dom}(f)}((\iota^R_{h_U})_{\mathrm{dom}(f)}(f))) \\
&= \bar\varphi_Z(h_U(p)((\iota^R_{h_U})_{\mathrm{dom}(f)}(f))) = \bar\varphi_Z((\iota^R_{h_U})_Z(R(p)(f))) = \bar\varphi_Z((\iota^R_{h_U})_Z(fp)) \\
&= \bar\varphi_Z((\iota^R_{h_U})_Z(gq)) = \bar\varphi_Z((\iota^R_{h_U})_Z(R(q)(g))) = \bar\varphi_Z(h_U(q)((\iota^R_{h_U})_{\mathrm{dom}(g)}(g))) \\
&= h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(q)(\bar\varphi_{\mathrm{dom}(g)}((\iota^R_{h_U})_{\mathrm{dom}(g)}(g))) = \hat F_Z(F(q)(\varphi_{\mathrm{dom}(g)}(g))) = \hat F_Z(F(q)(y_g))
\end{aligned}
$$

Since $\hat F_Z$ is injective, we have $F(p)(y_f) = F(q)(y_g)$ which shows $(y_f)_{f\in R} \in M_R$. It follows that

$$
\Bigl(\prod_{f\in R} \hat F_{\mathrm{dom}(f)}\Bigr)((y_f)_{f\in R}) = (\hat F_{\mathrm{dom}(f)}(\varphi_{\mathrm{dom}(f)}(f)))_{f\in R} = (\bar\varphi_{\mathrm{dom}(f)}((\iota^R_{h_U})_{\mathrm{dom}(f)}(f)))_{f\in R} = (\bar\varphi_{\mathrm{dom}(f)}(f))_{f\in R}
$$

belongs to $\bar M_R$. On the other hand, $(h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(f))_{f\in R} : h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(U) \to \prod_{f\in R} h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(\mathrm{dom}(f))$ maps $\bar\varphi_U(id_U)$ to $(\bar\varphi_{\mathrm{dom}(f)}(f))_{f\in R}$. Hence we have $\bar\varphi_U(id_U) \in \bar F_R(U)$ and the image of the composition $(**)$ is contained in $\bar F_R(U)$.

For $x \in \bar F_R(U)$, then we have $(h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(f)(x))_{f\in R} \in \bar M_R$ and there exists unique $(x_f)_{f\in R} \in M_R$ such that $\hat F_{\mathrm{dom}(f)}(x_f) = h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(f)(x)$ for any $f \in R$. For $V \in \mathrm{Ob}\,\mathcal C$, we define a map $\varphi_{xV} : R(V) \to F(V)$ by $\varphi_{xV}(f) = x_f$ for $f \in R(V)$. Let $\alpha : V \to W$ be a morphism in $\mathcal C$. Then, the right rectangle of the following diagram is commutative by the naturality of $\hat F$.

$$
\begin{array}{ccccc}
R(W) & \xrightarrow{\varphi_{xW}} & F(W) & \xrightarrow{\hat F_W} & h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(W) \\
\downarrow{\scriptstyle R(\alpha)} & & \downarrow{\scriptstyle F(\alpha)} & & \downarrow{\scriptstyle h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(\alpha)} \\
R(V) & \xrightarrow{\varphi_{xV}} & F(V) & \xrightarrow{\hat F_V} & h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(V)
\end{array}
$$

For $g \in R(W)$, the following equality holds.

$$
\begin{aligned}
\hat F_V(F(\alpha)(\varphi_{xW}(g))) &= h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(\alpha)(\hat F_W(x_g)) = h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(\alpha)(h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(g)(x)) = h^{1_{\mathcal C}}_{F(1_{\mathcal C})}(g\alpha)(x) \\
&= \hat F_V(x_{g\alpha}) = \hat F_V(\varphi_{xV}(g\alpha)) = \hat F_V(\varphi_{xV}(R(\alpha)(g)))
\end{aligned}
$$

Since $\hat{F}_V$ is injective, it follows that $F(\alpha)(\varphi_{xW}(g)) = \varphi_{xV}(R(\alpha)(g))$. Thus we have a natural transformation $\varphi_x : R \to F$. On the other hand, since $\theta_U(x) \in \widehat{\mathcal{C}}(h_U, h^{1_\mathcal{C}}_{F(1_\mathcal{C})})$ is given by

$$\theta_U(x)_V(f) = (xf_* : \mathcal{C}(1_\mathcal{C}, V) \to F(1_\mathcal{C})) = h^{1_\mathcal{C}}_{F(1_\mathcal{C})}(f)(x)$$

for $V \in \mathrm{Ob}\,\mathcal{C}$ and $f \in h_U(V)$, $\theta_U(x)_V(f) = \hat{F}_V(x_f) = \hat{F}_V(\varphi_{xV}(f)) = (\hat{F}\varphi_x)_V(f)$ holds if $f \in R(V)$. Hence we have $\theta_U(x) = \iota^{R*}_{h_U}\hat{F}\varphi_x \in \widehat{\mathcal{C}}(R, F)$, which implies that x belongs to the image of the composition $(**)$. Therefore $\bar{F}_R(U)$ coincides with the image of the composition $(**)$ and the assertion follows from the proof of (11.8). $\square$

Remark 11.10 *For $(x_f)_{f\in R} \in M_R$, since $F(p)(x_f) = F(q)(x_g)$ holds for any $f, g \in R$ and $p : Z \to \mathrm{dom}(f)$, $q : Z \to \mathrm{dom}(g)$ which satisfy $fp = gq$, then it follows from (11.6) that $(\hat{F}_{\mathrm{dom}(f)}(x_f))_{f\in R}$ belong to the image of $(F_X(f))_{f\in R} : F_X(U) \to \prod_{f\in R} F_X(\mathrm{dom}(f))$. Therefore $\bar{M}_R$ is contained in the image of $(F_X(f))_{f\in R}$ and $(F_X(f))_{f\in R}$ maps $\bar{F}_R(U)$ bijectively onto $\bar{M}_R$.*

Define a functor $\tilde{\Gamma} : \mathrm{CSh}(\mathcal{C}, J) \to \mathcal{S}et$ by $\tilde{\Gamma}(F) = F(1_\mathcal{C})$ and $\tilde{\Gamma}(\varphi : F \to G) = (\varphi_{1_\mathcal{C}} : F(1_\mathcal{C}) \to G(1_\mathcal{C}))$. It follows from (10.3) that $\tilde{\Gamma}$ is faithful.

Proposition 11.11 *If $(\mathcal{C}, J)$ is a preconcrete site, $\tilde{\Gamma}$ has right and left adjoints.*

Proof. Since $h^{1_\mathcal{C}}_X$ is an object of $\mathrm{CSh}(\mathcal{C}, J)$ for a set X by (11.6), we define a functor $\mathcal{R} : \mathcal{S}et \to \mathrm{CSh}(\mathcal{C}, J)$ by $\mathcal{R}(X) = h^{1_\mathcal{C}}_X$ and $\mathcal{R}(\varphi : X \to Y) = (h^{1_\mathcal{C}}_\varphi : h^{1_\mathcal{C}}_X \to h^{1_\mathcal{C}}_Y)$. For a concrete sheaf F, we define a morphism of sheaves $\eta_F : F \to h^{1_\mathcal{C}}_{F(1_\mathcal{C})} = \mathcal{R}\tilde{\Gamma}(F)$ by $\eta_F = \hat{F}$. Then, η_F is natural in F by (10.3). For a set X, we define a map $\varepsilon_X : \tilde{\Gamma}\mathcal{R}(X) = \mathcal{S}et(\mathcal{C}(1_\mathcal{C}, 1_\mathcal{C}), X) \to X$ by $\varepsilon_X(t) = t(id_{1_\mathcal{C}})$. Then, ε_X is a bijection and $\tilde{\Gamma}(\eta_F) = \hat{F}_{1_\mathcal{C}} : \tilde{\Gamma}(F) = F(1_\mathcal{C}) \to \mathcal{S}et(\mathcal{C}(1_\mathcal{C}, 1_\mathcal{C}), F(1_\mathcal{C})) = \tilde{\Gamma}\mathcal{R}\tilde{\Gamma}(F)$ is the inverse of $\varepsilon_{F(1_\mathcal{C})}$. Hence a composition $\tilde{\Gamma}(F) \xrightarrow{\tilde{\Gamma}(\eta_F)} \tilde{\Gamma}\mathcal{R}\tilde{\Gamma}(F) \xrightarrow{\varepsilon_{\tilde{\Gamma}(F)}} \tilde{\Gamma}(F)$ is the identity map of $\tilde{\Gamma}(F)$.

We have $\mathcal{R}(X)(U) = h^{1_\mathcal{C}}_X(U) = \mathcal{S}et(\mathcal{C}(1_\mathcal{C}, U), X)$ and $\mathcal{R}\tilde{\Gamma}\mathcal{R}(X)(U) = h^{1_\mathcal{C}}_{\tilde{\Gamma}\mathcal{R}(X)}(U) = \mathcal{S}et(\mathcal{C}(1_\mathcal{C}, U), \tilde{\Gamma}\mathcal{R}(X))$ for a set X and $U \in \mathrm{Ob}\,\mathcal{C}$. $(\eta_{\mathcal{R}(X)})_U = \widehat{(h^{1_\mathcal{C}}_X)}_U : \mathcal{R}(X)(U) \to \mathcal{R}\tilde{\Gamma}\mathcal{R}(X)(U)$ maps $t \in \mathcal{R}(X)(U) = \mathcal{S}et(\mathcal{C}(1_\mathcal{C}, U), X)$ to a map $f_t : \mathcal{C}(1_\mathcal{C}, U) \to \mathcal{S}et(\mathcal{C}(1_\mathcal{C}, 1_\mathcal{C}), X) = \tilde{\Gamma}\mathcal{R}(X)$ given by $f_t(\alpha) = t\alpha_*$. Since $\varepsilon_X f_t : \mathcal{C}(1_\mathcal{C}, U) \to X$ maps α to $\varepsilon_X f_t(\alpha) = \varepsilon_X(t\alpha_*) = t\alpha_*(id_{1_\mathcal{C}}) = t(\alpha)$, we have $\varepsilon_X f_t = t$ which implies that $\mathcal{R}(\varepsilon_X)_U = (h^{1_\mathcal{C}}_{\varepsilon_X})_U : \mathcal{R}\tilde{\Gamma}\mathcal{R}(X)(U) \to \mathcal{R}(X)(U)$ is the inverse of $(\eta_{\mathcal{R}(X)})_U$. Hence a composition $\mathcal{R}(X) \xrightarrow{\eta_{\mathcal{R}(X)}} \mathcal{R}\tilde{\Gamma}\mathcal{R}(X) \xrightarrow{\mathcal{R}(\varepsilon_X)} \mathcal{R}(X)$ is the identity morphism of $\mathcal{R}(X)$. Thus $\mathcal{R}$ is a right adjoint of $\tilde{\Gamma}$.

For a set X, let $\mathcal{L}(X)$ be the sheafification $a(C_X)$ of the constant presheaf C_X on $\mathcal{C}$. For a map $f : X \to Y$, let $\mathcal{L}(f) : \mathcal{L}(X) \to \mathcal{L}(Y)$ be the morphism $a(C_f) : a(C_X) \to a(C_Y)$ induced by $C_f : C_X \to C_Y$. Hence we have a functor $\mathcal{L} : \mathcal{S}et \to \mathrm{CSh}(\mathcal{C}, J)$. We denote by $i : \mathrm{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ be the inclusion functor. Then, the sheafification functor $a : \widehat{\mathcal{C}} \to \mathrm{Sh}(\mathcal{C}, J)$ is a left adjoint of i. Let $\bar{\Gamma} : \widehat{\mathcal{C}} \to \mathcal{S}et$ a functor defined by $\bar{\Gamma}(F) = F(1_\mathcal{C})$ and $\bar{\Gamma}(f : F \to G) = (f_{1_\mathcal{C}} : F(1_\mathcal{C}) \to G(1_\mathcal{C}))$. For a set X and a concrete sheaf F, we claim that $\bar{\Gamma} : \widehat{\mathcal{C}}(C_X, i(F)) \to \mathcal{S}et(C_X(1_\mathcal{C}), F(1_\mathcal{C}))$ is bijective. In fact, for a map $\varphi : X \to F(1_\mathcal{C})$, define a morphism $\bar{\Gamma}^{-1}(\varphi) : C_X \to i(F)$ of sheaves by $\bar{\Gamma}^{-1}(\varphi)_U = F(o_U)\varphi$ for $U \in \mathrm{Ob}\,\mathcal{C}$. For $f \in \widehat{\mathcal{C}}(C_X, i(F))$, $U \in \mathrm{Ob}\,\mathcal{C}$ and $x \in X$, we have $\bar{\Gamma}^{-1}(\bar{\Gamma}(f))_U(x) = \bar{\Gamma}^{-1}(f_{1_\mathcal{C}})_U(x) = F(o_U)(f_{1_\mathcal{C}}(x)) = C_X(o_U)f_U(x) = f_U(x)$. It follows that $\bar{\Gamma}^{-1}(\bar{\Gamma}(f)) = f$. For a map $\varphi : X \to F(1_\mathcal{C})$, $\bar{\Gamma}(\bar{\Gamma}^{-1}(\varphi)) = \bar{\Gamma}^{-1}(\varphi)_{1_\mathcal{C}} = F(o_{1_\mathcal{C}})\varphi = \varphi$. Therefore $\bar{\Gamma}^{-1}$ is the inverse of $\bar{\Gamma}$. Hence a composition

$$\mathrm{CSh}(\mathcal{L}(X), F) = \mathrm{Sh}(a(C_X), F) \xrightarrow{\cong} \widehat{\mathcal{C}}(C_X, i(F)) \xrightarrow{\bar{\Gamma}} \mathcal{S}et(C_X(1_\mathcal{C}), F(1_\mathcal{C})) = \mathcal{S}et(X, \tilde{\Gamma}(F))$$

is a natural bijection. Thus $\mathcal{L}$ is a left adjoint of $\tilde{\Gamma}$. $\square$

Proposition 11.12 *Let $(\mathcal{C}, J)$ be a preconcrete site. $\mathrm{CSh}(\mathcal{C}, J)$ has limits and colimits.*

Proof. Since $\{0\}$ is a terminal object of $\mathcal{S}et$, it follows from (11.11) that $\mathcal{R}(\{0\}) = h^{1_\mathcal{C}}_{\{0\}}$ is a terminal object of $\mathrm{CSh}(\mathcal{C}, J)$. Since empty set $\emptyset$ is an initial object of $\mathcal{S}et$, it follows from (11.11) that $\mathcal{L}(\emptyset)$ is an initinal object of $\mathrm{CSh}(\mathcal{C}, J)$.

For a family of objects $(F_i)_{i\in I}$ of $\mathrm{CSh}(\mathcal{C}, J)$, we define a presheaf $\prod_{i\in I} F_i$ on $\mathcal{C}$ by $\left(\prod_{i\in I} F_i\right)(U) = \prod_{i\in I} F_i(U)$ and $\left(\prod_{i\in I} F_i\right)(f) = \prod_{i\in I} F_i(f)$ for $U \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathrm{Mor}\,\mathcal{C}$. We put $F_i(1_\mathcal{C}) = X_i$ and let $\mathrm{pr}_j : \prod_{i\in I} X_i \to X_j$ be the projection. Then, for any object U of $\mathcal{C}$, $(\mathrm{pr}_{i*})_{i\in I} : \mathcal{S}et\Big(h^{1\varepsilon}(U), \prod_{i\in I} X_i\Big) \to \prod_{i\in I} \mathcal{S}et(h^{1\varepsilon}(U), X_i)$ is a bijection. There are monomorphisms $\hat{F}_i : F_i \to F_{X_i}$ for $i \in I$ and the following diagram is commutative.

$$\begin{array}{ccccc}
\Big(\prod_{i\in I} F_i\Big)(U) & \xrightarrow{\big(\widehat{\prod_{i\in I} F_i}\big)_U} & h^{1\varepsilon}_{\prod_{i\in I} X_i}(U) & = & \mathcal{S}et\Big(h^{1\varepsilon}(U), \prod_{i\in I} X_i\Big) \\
\| & & & & \cong\big\downarrow (\mathrm{pr}_{i*})_{i\in I} \\
\prod_{i\in I} F_i(U) & \xrightarrow{\prod_{i\in I} \hat{F}_{iU}} & \prod_{i\in I} h^{1\varepsilon}_{X_i}(U) & = & \prod_{i\in I} \mathcal{S}et(h^{1\varepsilon}(U), X_i)
\end{array}$$

It follows that $\prod_{i\in I} F_i$ is a concrete presheaf. It is clear that $\prod_{i\in I} F_i$ is a sheaf. Hence $\mathrm{CSh}(\mathcal{C}, J)$ has products.

Define a presheaf $\coprod_{i\in I} F_i$ on $\mathcal{C}$ by $\Big(\coprod_{i\in I} F_i\Big)(U) = \coprod_{i\in I} F_i(U)$ and $\Big(\coprod_{i\in I} F_i\Big)(f) = \coprod_{i\in I} F_i(f)$ for $U \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathrm{Mor}\,\mathcal{C}$. Let $\iota_j : X_j \to \coprod_{i\in I} X_i$ be the inclusion. Then $\iota_{j*} : \mathcal{S}et(h^{1\varepsilon}(U), X_j) \to \mathcal{S}et\Big(h^{1\varepsilon}(U), \coprod_{i\in I} X_i\Big)$ induces an injection $\coprod_{i\in I} \mathcal{S}et(h^{1\varepsilon}(U), X_i) \rightarrowtail \mathcal{S}et\Big(h^{1\varepsilon}(U), \coprod_{i\in I} X_i\Big)$. Since $\coprod_{i\in I} \hat{F}_{iU} : \coprod_{i\in I} F_i(U) \to \coprod_{i\in I} \mathcal{S}et(h^{1\varepsilon}(U), X_i)$ is injective and the following diagram is commutative, $\big(\widehat{\coprod_{i\in I} F_i}\big)_U : \Big(\coprod_{i\in I} F_i\Big)(U) \to h^{1\varepsilon}_{\coprod_{i\in I} X_i}(U)$ is also injective.

$$\begin{array}{ccccc}
\Big(\coprod_{i\in I} F_i\Big)(U) & \xrightarrow{\big(\widehat{\coprod_{i\in I} F_i}\big)_U} & h^{1\varepsilon}_{\coprod_{i\in I} X_i}(U) & = & \mathcal{S}et\Big(h^{1\varepsilon}(U), \coprod_{i\in I} X_i\Big) \\
\| & & & & \big\uparrow \\
\coprod_{i\in I} F_i(U) & \xrightarrow{\coprod_{i\in I} \hat{F}_{iU}} & \coprod_{i\in I} h^{1\varepsilon}_{X_i}(U) & = & \coprod_{i\in I} \mathcal{S}et(h^{1\varepsilon}(U), X_i)
\end{array}$$

Hence $\coprod_{i\in I} F_i$ is a concrete presheaf. Since the sheafification functor is a left adjoint of the inclusion functor, the sheafification functor preserves coproducts. Hence $\coprod_{i\in I} F_i$ is a sheaf since F_i is a sheaf for any $i \in I$. Thus $\mathrm{CSh}(\mathcal{C}, J)$ has coproducts.

Let $f, g : F \to G$ be morphisms of $\mathrm{CSh}(\mathcal{C}, J)$. For $U \in \mathrm{Ob}\,\mathcal{C}$, put $E(U) = \{x \in F(U) \,|\, f_U(x) = g_U(x)\}$ and let $e_U : E(U) \to F(U)$ be the inclusion map. Let $p_U : G(U) \to \bar{C}(U)$ be a coequalizer of f and g in $\mathcal{S}et$, namely $\bar{C}(U)$ is the quotient set of $G(U)$ by an equivalence relation $\sim$ generated by $f_U(x) \sim g_U(x)$ for $x \in F(U)$. For a morphism $\varphi : U \to V$ in $\mathcal{C}$, $F(\varphi) : F(V) \to F(U)$ maps $E(V)$ into $E(U)$ by the naturality of f and g. Hence if we define a map $E(\varphi) : E(V) \to E(U)$ by $E(\varphi)(x) = F(\varphi)(x)$, we have a presheaf E on $\mathcal{C}$ and a monomorphism $e : E \to F$ of presheves. Again by the naturality of f and g, there exists a unique map $\bar{C}(\varphi) : \bar{C}(V) \to \bar{C}(U)$ that satisfies $\bar{C}(\varphi)p_V = p_U G(\varphi)$, thus we have a presheaf $\bar{C}$ and a morphism $p : G \to \bar{C}$ of presheaves. It follows from (10.3) that E is a concrete presheaf. It can be verified that E is a sheaf on (C, J) and $e : E \to F$ is an equalizer of f and g. Therefore, $\mathrm{CSh}(C, J)$ has equalizers. We apply the functor $\mathscr{C} : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}^c$ to a diagram $F \overset{f}{\underset{g}{\rightrightarrows}} G \xrightarrow{p} \bar{C}$. Since F and G are concrete presheaves and $\mathscr{C}$ has a right adjoint and preserves colimits, there is a diagram $F \overset{f}{\underset{g}{\rightrightarrows}} G \xrightarrow{p'} \mathscr{C}(\bar{C})$ in $\widehat{\mathcal{C}}^c$ of coequalizer of f and g. We apply the sheafification functor to this disgram. Since F and G are sheaves and the sheafification functor also has a right adjoint, we have a diagram $F \overset{f}{\underset{g}{\rightrightarrows}} G \xrightarrow{p''} a\mathscr{C}(\bar{C})$ in $\mathrm{CSh}(\mathcal{C}, J)$ of coequalizer of f and g. We conclude that $\mathrm{CSh}(C, J)$ has coequalizers. □

Proposition 11.13 *Let $(\mathcal{C}, J)$ be a preconcrete site and X a set. If a subset $\mathscr{D}$ of $\coprod_{U\in \mathrm{Ob}\,\mathcal{C}} h^{1c}_X(U)$ satisfies conditions (ii) and (iii) of (1.2), then $h^{1c}_{\mathscr{D}}$ is a concrete sheaf on $(\mathcal{C}, J)$.*

Proof. It follows from (10.10) that $h^{1c}_{\mathscr{D}}$ is a concrete presheaf. Hence $h^{1c}_{\mathscr{D}}$ is a separated presheaf by (11.7). For an object U of $\mathcal{C}$ and $R \in J(U)$, let $(U_i \xrightarrow{f_i} U)_{i\in I}$ be a family of morphisms in $\mathcal{C}$ which generates R. Let

$(\check{h}^{1c}_{\mathscr{D}}(f_i))_{i\in I} : h^{1c}_{\mathscr{D}}(U) \to \prod_{i\in I} h^{1c}_{\mathscr{D}}(U_i)$ be the map induced by $\check{h}^{1c}_{\mathscr{D}}(f_i) : h^{1c}_{\mathscr{D}}(U) \to h^{1c}_{\mathscr{D}}(U_i)$'s. Put

$$M = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} h^{1c}_{\mathscr{D}}(U_i) \,\middle|\, h^{1c}_{\mathscr{D}}(g)(x_i) = h^{1c}_{\mathscr{D}}(h)(x_j) \text{ if } f_i g = f_j h \text{ for } i, j \in I \text{ and } g : Z \to U_i,\ h : Z \to U_j \right\}.$$

We verify that the image of $(h^{1c}_{\mathscr{D}}(f_i))_{i\in I} : h^{1c}_{\mathscr{D}}(U) \to \prod_{i\in I} h^{1c}_{\mathscr{D}}(U_i)$ coincides with M. For $t \in h^{1c}_{\mathscr{D}}(U) \subset h^{1c}_X(U)$, we claim that $(h^{1c}_{\mathscr{D}}(f_i)(t))_{i\in I} = (tf_{i*})_{i\in I}$ belongs to M. For $i, j \in I$ and morphisms $g : Z \to U_i$, $h : Z \to U_j$ of $\mathcal{C}$ which satisfy $f_i g = f_j h$, we have the following.

$$h^{1c}_{\mathscr{D}}(g)(tf_{i*}) = h^{1c}_X(g)(tf_{i*}) = tf_{i*}g_* = t(f_i g)_* = t(f_j h)_* = tf_{j*}h_* = h^{1c}_X(h)(tf_{j*}) = h^{1c}_{\mathscr{D}}(h)(tf_{j*})$$

Thus $(h^{1c}_{\mathscr{D}}(f_i)(t))_{i\in I}$ belongs to M. For $(x_i)_{i\in I} \in M$, we define $x \in h^{1c}_X(U)$ as follows. For $\alpha \in \mathcal{C}(1_{\mathcal{C}}, U)$, since $(f_{i*} : \mathcal{C}(1_{\mathcal{C}}, U_i) \to \mathcal{C}(1_{\mathcal{C}}, U))_{i\in I}$ is an epimorphic family in $\mathscr{S}et$, we can choose $i \in I$ and $g \in \mathcal{C}(1_{\mathcal{C}}, U_i)$ such that $f_i g = \alpha$. We define $x \in h^{1c}_X(U)$ by $x(\alpha) = x_i(g)$. If $j \in I$ and $h \in \mathcal{C}(1_{\mathcal{C}}, U_j)$ satisfy $f_j h = \alpha$, then we have $x_i(g) = x_i g_* = h^{1c}_{\mathscr{D}}(g)(x_i) = h^{1c}_{\mathscr{D}}(h)(x_j) = x_j h_* = x_j(h)$. Hence $x(\alpha)$ does not depend on the choice of $i \in I$ and $g \in \mathcal{C}(1_{\mathcal{C}}, U_i)$ such that $f_i g = \alpha$. For $i \in I$ and $g \in \mathcal{C}(1_{\mathcal{C}}, U_i)$, it follows from the definition of x that we have $(h^{1c}_X(f_i))(x))(g) = (xf_{i*})(g) = x(f_i g) = x_i(g)$ which shows $h^{1c}_X(f_i)(x) = x_i \in h^{1c}_{\mathscr{D}}(U_i)$. Hence $x \in h^{1c}_{\mathscr{D}}(U)$ by (iii) and $(x_i)_{i\in I}$ belongs to the image of $(h^{1c}_{\mathscr{D}}(f_i))_{i\in I} : h^{1c}_{\mathscr{D}}(U) \to \prod_{i\in I} h^{1c}_{\mathscr{D}}(U_i)$. □

We consider the-ology with respect to h^{1c} and $(\mathcal{C}, J)$ below.

Proposition 11.14 *For a concrete sheaf P on a preconcrete site $(\mathcal{C}, J)$ which is a subfunctor of h^{1c}_X for some set X, we put $\mathscr{D} = \coprod_{U\in \mathrm{Ob}\,\mathcal{C}} P(U)$. If $P(1_{\mathcal{C}}) = h^{1c}_X(1_{\mathcal{C}})$, then $\mathscr{D}$ is a the-ological object on X.*

Proof. The condition (i) of (1.2) follows from the assumption $P(1_{\mathcal{C}}) = h^{1c}_X(1_{\mathcal{C}})$. It follows from the definition of $\mathscr{D}$ that $h^{1c}_{\mathscr{D}}(U) = P(U)$ holds for any $U \in \mathrm{Ob}\,\mathcal{C}$ and that $h^{1c}_{\mathscr{D}}(f) = P(f)$ is a restriction of $h^{1c}_X(f)$ for any $f \in \mathrm{Mor}\,\mathcal{C}$. Hence $\mathscr{D}$ satisfies (ii). For $x \in h^{1c}_X(U)$, suppose that there exists a covering $(U_i \xrightarrow{f_i} U)_{i\in I}$ of U such that $h^{1c}_X(f_i) : h^{1c}_X(U) \to h^{1c}_X(U_i)$ maps x into $h^{1c}_{\mathscr{D}}(U_i)$ for any $i \in I$. For $i, j \in I$ and morphisms $g : Z \to U_i$, $h : Z \to U_j$ of $\mathcal{C}$ which satisfy $f_i g = f_j h$, the following equality holds.

$$P(g)(h^{1c}_X(f_i)(x)) = h^{1c}_X(g)(h^{1c}_X(f_i)(x)) = h^{1c}_X(f_i g)(x) = h^{1c}_X(f_j h)(x) = h^{1c}_X(h)(h^{1c}_X(f_j)(x)) = P(h)(h^{1c}_X(f_j)(x))$$

Since P is a sheaf, there exists a unique $y \in P(U)$ such that $h^{1c}_X(f_i)(y) = h^{1c}_X(f_i)(x)$ for any $i \in I$. Since $(f_{i*} : \mathcal{C}(1_{\mathcal{C}}, U_i) \to \mathcal{C}(1_{\mathcal{C}}, U))_{i\in I}$ is an epimorphic family in $\mathscr{S}et$, $(h^{1c}_X(f_i) : h^{1c}_X(U) \to h^{1c}_X(U_i))_{i\in I}$ is a monomorphic family in $\mathscr{S}et$. Thus we have $x = y \in P(U) = h^{1c}_{\mathscr{D}}(U)$ and $\mathscr{D}$ satisfies (iii). □

Recall from (10.13) that, for a concrete sheaf P on a site $(\mathcal{C}, J)$ and $U \in \mathrm{Ob}\,\mathcal{C}$, $(\eta_P)_U : P(U) \to \mathscr{C}(P)(U) = P^c(U)$ is bijective and that $P^c(1_{\mathcal{C}}) = \mathscr{S}et(\mathcal{C}(1_{\mathcal{C}}, 1_{\mathcal{C}}), P(1_{\mathcal{C}}))$ holds. We put $\mathscr{D}(P) = \coprod_{U\in \mathrm{Ob}\,\mathcal{C}} P^c(U)$. Then $\mathscr{D}(P)$ is a the-ology on $P(1_{\mathcal{C}})$ by (11.14).

Proposition 11.15 *Let $\xi : P \to Q$ be a morphism in $\mathrm{CSh}(\mathcal{C}, J)$. Then, $\xi_{1_{\mathcal{C}}} : P(1_{\mathcal{C}}) \to Q(1_{\mathcal{C}})$ defines a morphism $(P(1_{\mathcal{C}}), \mathscr{D}(P)) \to (Q(1_{\mathcal{C}}), \mathscr{D}(Q))$ of the-ological objects.*

Proof. The following diagram is commutative by (10.3)

$$\begin{array}{ccc} P(U) & \xrightarrow{\quad \xi_U \quad} & Q(U) \\ \downarrow{\scriptstyle \hat{P}_U} & & \downarrow{\scriptstyle \hat{Q}_U} \\ h^{1c}_{P(1_{\mathcal{C}})}(U) & \xrightarrow{\left(h^{1c}_{\xi_{1_{\mathcal{C}}}}\right)_U} & h^{1c}_{Q(1_{\mathcal{C}})}(U) \end{array}$$

It follows that $\left(h^{1c}_{\xi_{1_{\mathcal{C}}}}\right)_U$ maps the image $P^c(U)$ of $\hat{P}_U$ into the image $Q^c(U)$ of $\hat{Q}_U$, which implies the assertion. □

For a set X, define a map $ev_X : h^{1c}_X(1_{\mathcal{C}}) = \mathscr{S}et(\mathcal{C}(1_{\mathcal{C}}, 1_{\mathcal{C}}), X) \to X$ by $ev_X(\alpha) = \alpha(id_{1_{\mathcal{C}}})$. Then, ev_X is bijective and natural in X. For sets X and Y, we define a map $\sigma : \mathscr{S}et(h^{1c}_X(1_{\mathcal{C}}), h^{1c}_Y(1_{\mathcal{C}})) \to \mathscr{S}et(X, Y)$ to be a composition $\mathscr{S}et(h^{1c}_X(1_{\mathcal{C}}), h^{1c}_Y(1_{\mathcal{C}})) \xrightarrow{(ev_X^*)^{-1}} \mathscr{S}et(X, h^{1c}_Y(1_{\mathcal{C}})) \xrightarrow{ev_{Y*}} \mathscr{S}et(X, Y)$. We note that the inverse $\sigma^{-1} : \mathscr{S}et(X, Y) \to \mathscr{S}et(h^{1c}_X(1_{\mathcal{C}}), h^{1c}_Y(1_{\mathcal{C}}))$ of σ is given by $\sigma^{-1}(\varphi) = (h^{1c}_{\varphi})_{1_{\mathcal{C}}}$.

For the-ology $\mathscr{D}$ on a set X and $U \in \mathrm{Ob}\,\mathcal{C}$, let us denote by $(\iota_{\mathscr{D}})_U : h^{1c}_{\mathscr{D}}(U) \to h^{1c}_X(U)$ the inclusion map, which is natural in U. Thus we have a morphism of sheaves $\iota_{\mathscr{D}} : h^{1c}_{\mathscr{D}} \to h^{1c}_X$.

Proposition 11.16 *Let $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ be the-ological objects. For a morphism of sheaves $\xi : h^{1_{\mathcal{C}}}_{\mathscr{D}} \to h^{1_{\mathcal{C}}}_{\mathscr{E}}$, put $\varphi = \sigma(\xi_{1_{\mathcal{C}}}) : X \to Y$. Then $\coprod_{U \in \operatorname{Ob}\mathcal{C}} (h^{1_{\mathcal{C}}}_{\varphi})_U : \coprod_{U \in \operatorname{Ob}\mathcal{C}} h^{1_{\mathcal{C}}}_X(U) \to \coprod_{U \in \operatorname{Ob}\mathcal{C}} h^{1_{\mathcal{C}}}_Y(U)$ maps $\mathscr{D}$ to $\mathscr{E}$ and ξ coincides with the morphism $\check{h}^{1_{\mathcal{C}}}_{\varphi} : h^{1_{\mathcal{C}}}_{\mathscr{D}} \to h^{1_{\mathcal{C}}}_{\mathscr{E}}$ induced by $h^{1_{\mathcal{C}}}_{\varphi} : h^{1_{\mathcal{C}}}_X \to h^{1_{\mathcal{C}}}_Y$. Moreover, $\varphi : X \to Y$ is unique map that satisfies $\check{h}^{1_{\mathcal{C}}}_{\varphi} = \xi$.*

Proof. Since $(h^{1_{\mathcal{C}}}_{\varphi})_{1_{\mathcal{C}}} = \sigma^{-1}(\varphi) = \xi_{1_{\mathcal{C}}}$, the following diagram is commutative.

$$\begin{array}{ccccc}
h^{1_{\mathcal{C}}}_{\mathscr{D}}(1_{\mathcal{C}}) & \xrightarrow[=]{(\iota_{\mathscr{D}})_{1_{\mathcal{C}}}} & h^{1_{\mathcal{C}}}_X(1_{\mathcal{C}}) & \xrightarrow{ev_X} & X \\
\downarrow{\scriptstyle \xi_{1_{\mathcal{C}}}} & & \downarrow{\scriptstyle (h^{1_{\mathcal{C}}}_{\varphi})_{1_{\mathcal{C}}}} & & \downarrow{\scriptstyle \varphi} \\
h^{1_{\mathcal{C}}}_{\mathscr{E}}(1_{\mathcal{C}}) & \xrightarrow[=]{(\iota_{\mathscr{E}})_{1_{\mathcal{C}}}} & h^{1_{\mathcal{C}}}_Y(1_{\mathcal{C}}) & \xrightarrow{ev_Y} & Y
\end{array}$$

For $U \in \operatorname{Ob}\mathcal{C}$, it follows from (10.3) that the left rectangle of the following diagram is commutative and the middle and right diagram is commutative by the commutativity of the above diagrams.

$$\begin{array}{ccccccc}
h^{1_{\mathcal{C}}}_{\mathscr{D}}(U) & \xrightarrow{(\hat{h}^{1_{\mathcal{C}}}_{\mathscr{D}})_U} & h^{1_{\mathcal{C}}}_{h^{1_{\mathcal{C}}}_{\mathscr{D}}(1_{\mathcal{C}})}(U) & \xrightarrow[=]{(h^{1_{\mathcal{C}}}_{(\iota_{\mathscr{D}})_{1_{\mathcal{C}}}})_U} & h^{1_{\mathcal{C}}}_{h^{1_{\mathcal{C}}}_X(1_{\mathcal{C}})}(U) & \xrightarrow[\cong]{(h^{1_{\mathcal{C}}}_{ev_X})_U} & h^{1_{\mathcal{C}}}_X(U) \\
\downarrow{\scriptstyle \xi_U} & & \downarrow{\scriptstyle (h^{1_{\mathcal{C}}}_{\xi_{1_{\mathcal{C}}}})_U} & & \downarrow{\scriptstyle \left(h^{1_{\mathcal{C}}}_{(h^{1_{\mathcal{C}}}_{\varphi})_{1_{\mathcal{C}}}}\right)_U} & & \downarrow{\scriptstyle (h^{1_{\mathcal{C}}}_{\varphi})_U} \\
h^{1_{\mathcal{C}}}_{\mathscr{E}}(U) & \xrightarrow{(\hat{h}^{1_{\mathcal{C}}}_{\mathscr{E}})_U} & h^{1_{\mathcal{C}}}_{h^{1_{\mathcal{C}}}_{\mathscr{E}}(1_{\mathcal{C}})}(U) & \xrightarrow[=]{(h^{1_{\mathcal{C}}}_{(\iota_{\mathscr{E}})_{1_{\mathcal{C}}}})_U} & h^{1_{\mathcal{C}}}_{h^{1_{\mathcal{C}}}_Y(1_{\mathcal{C}})}(U) & \xrightarrow[\cong]{(h^{1_{\mathcal{C}}}_{ev_Y})_U} & h^{1_{\mathcal{C}}}_Y(U)
\end{array}$$

Thus the following diagram is commutative by (10.4).

$$\begin{array}{ccc}
h^{1_{\mathcal{C}}}_{\mathscr{D}}(U) & \xrightarrow{\xi_U} & h^{1_{\mathcal{C}}}_{\mathscr{E}}(U) \\
\downarrow{\scriptstyle (\iota_{\mathscr{D}})_U} & & \downarrow{\scriptstyle (\iota_{\mathscr{E}})_U} \\
h^{1_{\mathcal{C}}}_X(U) & \xrightarrow{h^{1_{\mathcal{C}}}_{\varphi}} & h^{1_{\mathcal{C}}}_Y(U)
\end{array}$$

This shows that $\coprod_{U \in \operatorname{Ob}\mathcal{C}} (h^{1_{\mathcal{C}}}_{\varphi})_U : \coprod_{U \in \operatorname{Ob}\mathcal{C}} h^{1_{\mathcal{C}}}_X(U) \to \coprod_{U \in \operatorname{Ob}\mathcal{C}} h^{1_{\mathcal{C}}}_Y(U)$ maps $\mathscr{D}$ to $\mathscr{E}$. Since a diagram

$$\begin{array}{ccc}
h^{1_{\mathcal{C}}}_{\mathscr{D}}(U) & \xrightarrow{(\check{h}^{1_{\mathcal{C}}}_{\varphi})_U} & h^{1_{\mathcal{C}}}_{\mathscr{E}}(U) \\
\downarrow{\scriptstyle (\iota_{\mathscr{D}})_U} & & \downarrow{\scriptstyle (\iota_{\mathscr{E}})_U} \\
h^{1_{\mathcal{C}}}_X(U) & \xrightarrow{(h^{1_{\mathcal{C}}}_{\varphi})_U} & h^{1_{\mathcal{C}}}_Y(U)
\end{array}$$

is also commutative and $(\iota_{\mathscr{E}})_U$ is injective, we have $\xi_U = (\check{h}^{1_{\mathcal{C}}}_{\varphi})_U$ for any $U \in \operatorname{Ob}\mathcal{C}$. Since $h^{1_{\mathcal{C}}}_{\mathscr{D}}(1_{\mathcal{C}}) = h^{1_{\mathcal{C}}}_X(1_{\mathcal{C}})$ and $h^{1_{\mathcal{C}}}_{\mathscr{E}}(1_{\mathcal{C}}) = h^{1_{\mathcal{C}}}_Y(1_{\mathcal{C}})$, we have $(h^{1_{\mathcal{C}}}_{\varphi})_{1_{\mathcal{C}}} = (\check{h}^{1_{\mathcal{C}}}_{\varphi})_{1_{\mathcal{C}}}$ by the definition of $\check{h}^{1_{\mathcal{C}}}_{\varphi}$. Hence $\sigma^{-1}(\varphi) = (h^{1_{\mathcal{C}}}_{\varphi})_{1_{\mathcal{C}}} = (\check{h}^{1_{\mathcal{C}}}_{\varphi})_{1_{\mathcal{C}}} = \xi_{1_{\mathcal{C}}}$ holds which implies the uniqueness of φ. □

It follows from (11.15) that we can define a functor $\Delta : \operatorname{CSh}(\mathcal{C}, J) \to \mathscr{P}_{h^{1_{\mathcal{C}}}}(\mathcal{C}, J)$ by $\Delta(P) = (P(1_{\mathcal{C}}), \mathscr{D}(P))$ for $P \in \operatorname{Ob}\operatorname{CSh}(\mathcal{C}, J)$ and $\Delta(\xi) = (\xi_{1_{\mathcal{C}}} : (P(1_{\mathcal{C}}), \mathscr{D}(P)) \to (Q(1_{\mathcal{C}}), \mathscr{D}(Q)))$ for a morphism $\xi : P \to Q$ of concrete sheaves. If $(\mathcal{C}, J)$ is a preconcrete site, it follows from (11.13) and (1.3) that we can also define a functor $\Delta^{-1} : \mathscr{P}_{h^{1_{\mathcal{C}}}}(\mathcal{C}, J) \to \operatorname{CSh}(\mathcal{C}, J)$ by $\Delta^{-1}(X, \mathscr{D}) = h^{1_{\mathcal{C}}}_{\mathscr{D}}$ for $(X, \mathscr{D}) \in \operatorname{Ob}\mathscr{P}_{h^{1_{\mathcal{C}}}}(\mathcal{C}, J)$ and $\Delta^{-1}(\varphi) = \check{h}^{1_{\mathcal{C}}}_{\varphi}$ for $\varphi \in \mathscr{P}_{h^{1_{\mathcal{C}}}}(\mathcal{C}, J)((X, \mathscr{D}), (Y, \mathscr{E}))$. We note that the following diagrams is commutative and that the bijection $ev_X : \tilde{\Gamma}\Delta^{-1}(X, \mathscr{D}) = h^{1_{\mathcal{C}}}_X(1_{\mathcal{C}}) \to X = \Gamma_{h^{1_{\mathcal{C}}}}(X, \mathscr{D})$ defines a natural equivalence $ev : \tilde{\Gamma}\Delta^{-1} \to \Gamma_{h^{1_{\mathcal{C}}}}$.

$$\begin{array}{ccc}
\operatorname{CSh}(\mathcal{C}, J) & \xrightarrow{\Delta} & \mathscr{P}_{h^{1_{\mathcal{C}}}}(\mathcal{C}, J) \\
& {\scriptstyle \tilde{\Gamma}}\searrow \quad \swarrow{\scriptstyle \Gamma_{h^{1_{\mathcal{C}}}}} & \\
& \mathcal{S}et &
\end{array}$$

Proposition 11.17 *If $(\mathcal{C}, J)$ is a preconcrete site, $\Delta : \operatorname{CSh}(\mathcal{C}, J) \to \mathscr{P}_{h^{1_{\mathcal{C}}}}(\mathcal{C}, J)$ is an equivalence of categories.*

Proof. For $P \in \operatorname{Ob}\mathrm{CSh}(\mathcal{C}, J)$ and $U \in \operatorname{Ob}\mathcal{C}$, we have the following equality which shows $\varDelta^{-1}(\varDelta(P)) = P^c$.

$$\varDelta^{-1}(\varDelta(P))(U) = \varDelta^{-1}(P(1_{\mathcal{C}}), \mathscr{D}(P))(U) = \mathscr{D}(P)_{P(1_{\mathcal{C}})}(U) = \mathscr{D}(P) \cap \mathcal{S}et(\mathcal{C}(1_{\mathcal{C}}, U), P(1_{\mathcal{C}})) = P^c(U)$$

Thus $\eta_P : P \to P^c = \varDelta^{-1}(\varDelta(P))$ is an isomorphism in $\mathrm{CSh}(\mathcal{C}, J)$ by (10.13) since P is a concrete sheaf.

For $(X, \mathscr{D}) \in \operatorname{Ob}\mathscr{P}_{h^{1_{\mathcal{C}}}}(\mathcal{C}, J)$, $U \in \operatorname{Ob}\mathcal{C}$ and $x \in h^{1_{\mathcal{C}}}_{\mathscr{D}}(U) = \mathscr{D} \cap h^{1_{\mathcal{C}}}_X(U)$, since

$$(\hat{h}^{1_{\mathcal{C}}}_{\mathscr{D}})_U(x) : \mathcal{C}(1_{\mathcal{C}}, U) \to h^{1_{\mathcal{C}}}_X(1_{\mathcal{C}}) = h^{1_{\mathcal{C}}}_{\mathscr{D}}(1_{\mathcal{C}})$$

maps $\alpha \in \mathcal{C}(1_{\mathcal{C}}, U)$ to a map $\mathcal{C}(1_{\mathcal{C}}, 1_{\mathcal{C}}) \to X$ given by $id_{1_{\mathcal{C}}} \mapsto h^{1_{\mathcal{C}}}_{\mathscr{D}}(\alpha)(x) = x\alpha_*$, $ev_{X*}(\hat{h}^{1_{\mathcal{C}}}_{\mathscr{D}})_U(x) : \mathcal{C}(1_{\mathcal{C}}, U) \to X$ is a map given by $\alpha \mapsto x\alpha_*(id_{1_{\mathcal{C}}}) = x(\alpha)$, which shows that the following diagram is commutative.

$$\begin{array}{ccc} h^{1_{\mathcal{C}}}_{\mathscr{D}}(U) & = & \mathscr{D} \cap h^{1_{\mathcal{C}}}_X(U) \\ \downarrow{\scriptstyle (\hat{h}^{1_{\mathcal{C}}}_{\mathscr{D}})_U} & & \downarrow{\scriptstyle \text{inclusion}} \\ \mathcal{S}et(\mathcal{C}(1_{\mathcal{C}}, U), \mathcal{S}et(\mathcal{C}(1_{\mathcal{C}}, 1_{\mathcal{C}}), X)) & \xrightarrow[\cong]{ev_{X*}} & h^{1_{\mathcal{C}}}_X(U) \end{array}$$

Since ${h^{1_{\mathcal{C}}}_{\mathscr{D}}}^c(U)$ is the image of $(\hat{h}^{1_{\mathcal{C}}}_{\mathscr{D}})_U : h^{1_{\mathcal{C}}}_{\mathscr{D}}(U) \to \mathcal{S}et(\mathcal{C}(1_{\mathcal{C}}, U), \mathcal{S}et(\mathcal{C}(1_{\mathcal{C}}, 1_{\mathcal{C}}), X)) = h^{1_{\mathcal{C}}}_{h^{1_{\mathcal{C}}}_{\mathscr{D}}(1_{\mathcal{C}})}(U)$, the commutativity of the above diagram implies that $ev_{X*} : \mathcal{S}et(\mathcal{C}(1_{\mathcal{C}}, U), \mathcal{S}et(\mathcal{C}(1_{\mathcal{C}}, 1_{\mathcal{C}}), X)) \to h^{1_{\mathcal{C}}}_X(U)$ maps ${h^{1_{\mathcal{C}}}_{\mathscr{D}}}^c(U)$ bijectively onto $\mathscr{D} \cap h^{1_{\mathcal{C}}}_X(U)$. This shows that $ev_X : h^{1_{\mathcal{C}}}_{\mathscr{D}}(1_{\mathcal{C}}) = h^{1_{\mathcal{C}}}_X(1_{\mathcal{C}}) \to X$ defines an isomorphism $\varDelta(\varDelta^{-1}(X, \mathscr{D})) = \varDelta(h^{1_{\mathcal{C}}}_{\mathscr{D}}) = \Bigl(h^{1_{\mathcal{C}}}_{\mathscr{D}}(1_{\mathcal{C}}), \coprod_{U \in \operatorname{Ob}\mathcal{C}} {h^{1_{\mathcal{C}}}_{\mathscr{D}}}^c(U)\Bigr) \to (X, \mathscr{D})$ in $\mathscr{P}_{h^{1_{\mathcal{C}}}}(\mathcal{C}, J)$. □

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